Modelling Uncertainty 1D Project

SC06 Group 4
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Q0.

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$$p_0 = \frac{\frac{1005155+1005471+1005215+1004890+1005291}{5} - 10^6}{10^4}$$
$$= 0.52044$$

Q1.

a)

$$F(n,m,p) = \sum_{k=0}^{m-1} {}^{n-1+k}C_k \cdot p^n \left(1-p\right)^k$$

b)

$$F(8,10,p_0) = 0.743459444891$$

Q2.

a)

Claim 1:

Claim 2:

b)

$$F(n,m,p) = \sum_{k=n}^{n+m-1} {}^{n+m-1}C_k \cdot p^k \left(1-p\right)^{(n+m-1-k)}$$

Q3.

a)

$$
\begin{aligned}
F(n, m, p) &= p \cdot F(n-1, m, p) + (1-p) \cdot F(n, m-1, p) \\
&\text{Base cases:} \\
F(0, m, p) &= 1 \\
F(n, 0, p) &= 0
\end{aligned}
$$

b)

Refer to Excel.

Q4.

Let X_{player} be a random variable representing the probability that the game ends after (inclusive) x turns, with a win for that player.

$$
\begin{aligned}
P(X_A = x) &= {}^{x-1}C_{x-4} \cdot p^4 \, (1-p)^{(x-4)} \\
P(X_B = x) &= {}^{x-1}C_{x-4} \cdot p^{(x-4)} \, (1-p)^4 \\
X &= X_A + X_B \\
P(X = x) &= P(X_A = x) + P(X_B = x) \\
&= {}^{x-1}C_{x-4} \cdot \left(p^4 \, (1-p)^{(x-4)} + p^{(x-4)} \, (1-p)^4\right)
\end{aligned}
$$

x	4	5	6	7
$P(X = x)$	0.126253729903	0.250833492582	0.311976886701	0.310935890814

Q5.

$$
\begin{aligned}
X &\sim \text{Binomial}(2n-1, p) \\
\therefore E(X) &= (2n-1)p, \\
\text{Var}(X) &= (2n-1)p(1-p), \\
\sigma(X) &= \sqrt{\text{Var}(X)} = \sqrt{(2n-1)p(1-p)} \\
\text{Distance } D \text{ between } n \text{ and } E(X) &= \frac{n - E(X)}{\sigma(X)} \\
&= |\frac{n - (2n-1)p}{\sqrt{(2n-1)p(1-p)}}| \\
&= |\frac{n(1-2p) + p}{\sqrt{(2n-1)p(1-p)}}| \\
&= |\frac{n(1-2p)}{\sqrt{(2n-1)p(1-p)}} + \frac{p}{\sqrt{(2n-1)p(1-p)}}| \\
\text{As } n \text{ tends to infinity,} &\frac{p}{\sqrt{(2n-1)p(1-p)}} \to 0 \\
D &= |\frac{n(1-2p)}{\sqrt{(2n-1)p(1-p)}}| \\
&= |\frac{n(2p-1)}{\sqrt{(2n-1)p(1-p)}}| \\
&= |n \cdot \frac{(2p-1)}{\sqrt{(2n-1)p(1-p)}}|
\end{aligned}
$$

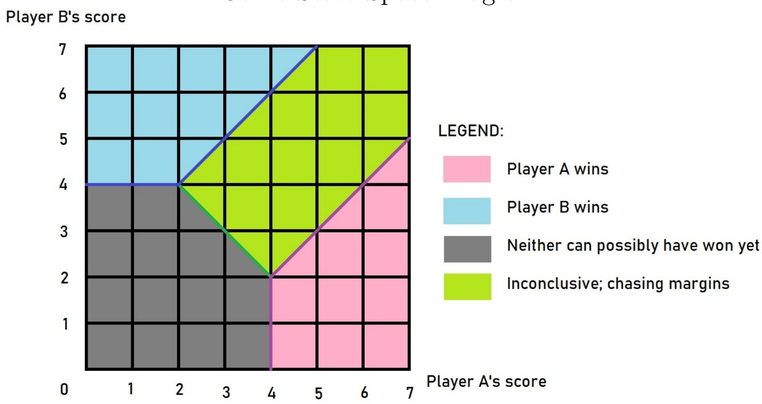
Since $\frac{(2p-1)}{\sqrt{(2n-1)p(1-p)}}$ is always positive for $0.5 < p < 1$, we see that, as n increases, D increases.

b)

From a picture, we can see that $E(X)$ is to the right of n. Since the distance D between the two is increasing, $E(X)$ is getting further away from n. In plain English, this means that the amount of rounds player A is expected to win is increasing faster than the minimum amount of rounds player A has to win n order to win the game. Therefore, as n increases, the probability that player A will win the game increases.

Q6.

Game State Space Diagram:



Probability of player A winning 4 points and player B winning $k \leq 2$ points (so player A wins immediately):

$$\sum_{k=0}^{2} {}^{3+k}C_3 \cdot p^4 \, (1-p)^k$$

Probability of both players A and B winning 3 points each:

$$^{3+3}C_3 \cdot p_0^3(1-p_0)^3$$

If both players are at 3 points each, what matters from then on is only the margin between the two. For each round, the probabilities of:

- A wins: p_0^2
- B wins: $(1-p_0)^2$
- Inconclusive: $^2C_1 \cdot p_0^1(1-p_0)^1 = 2p_0(1-p_0)$

Therefore, the probability of player A winning the game under these circumstances is:

$$\begin{aligned} p_0^2 + 2p_0(1-p_0)p_0^2 + [2p_0(1-p_0)]^2p_0^2 + [2p_0(1-p_0)]^3p_0^2 \cdots &= p_0^2\sum_{i=0}^{\infty}(2p_0(1-p_0))^i \\ &= \frac{p_0^2}{1-2p_0+2p_0^2} \end{aligned}$$

Therefore, the total probability that player A wins the game is:

$$\sum_{k=0}^{2} {}^{3+k}C_3 \cdot p^4 \, (1-p)^k + {}^{3+3}C_3 \cdot p_0^3(1-p_0)^3 \cdot \frac{p_0^2}{1-2p_0+2p_0^2}$$