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# Forward variance dynamics : Bergomi's model revisited

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## **Abstract**

In this article, we propose an arbitrage-free modeling framework for the joint dynamics of forward variance along with the underlying index, which can be seen as a combination of the two approaches proposed by Bergomi. The difference between our modeling framework and the Bergomi models (2008), is mainly the ability to compute the prices of VIX futures and options by using semi-analytic formulas. Also, we can express the sensitivities of the prices of VIX futures and options with respect to the model parameters, which enables us to propose an efficient and easy calibration to the VIX futures and options. The calibrated model allows to Delta-hedge VIX options by trading in VIX futures, the corresponding hedge ratios can be computed analytically<sup>1</sup>.

## **1 Introduction**

Several recent studies pointed out some of the limitations of classical models used for equity derivatives. For example, Bergomi (cf [1]) showed that Dupire's formula gives un-

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realistic smile dynamics, causing significant pricing errors for path-dependant or forward-starting options; traditional stochastic volatility and jump/Lévy models impose structural constraints on the relationship between the forward skew. To alleviate this problem, a new modeling approach proposed by Bergomi [1], [2], [3] (see also related work of Bühler [8] and Gatheral [16]) in which, instead of modelling “instantaneous” volatility, one starts by specifying the dynamics of the entire curve of (forward) variance as a random variable, just as HJM-type interest rate models start from the forward rate curve. The additional complication in the case of forward variance is that we do not only want to model the variance swap price process, but we need to derive the dynamics of a stock price process which is consistent with the modeled variance. One of the strengths of this modelling approach is that it provides two levels of calibration. In the first step, we calibrate the parameters that generate the forward variance curve to match the prices of volatility derivatives, which allows a better control of the term structure of the volatility of volatility (For example, by calibrating VIX futures and the implied volatility of its options). At the second step, we use the resulting parameters, from the first step, and calibrate the correlation coefficients between the Brownian motion deriving the stock price and the factors to control the term structure of the skew of the vanilla smiles.

Before describing our contribution, we will recall a few facts about variance modelling and Bergomi’s modelling framework.

## 1.1 Variance Swaps and Forward Variance Curve

A variance swap with maturity  $T$  is a contract which pays out the realized variance of the logarithmic total returns up to  $T$  less a strike called the variance swap rate  $V_0^T$ , determined in such a way that the contract has zero value today.

The annualized realized variance of a stock price process  $S$  for the period  $[0, T]$  with business days  $0 = t_0 < \dots < t_n = T$  is usually defined as

$$RV^{0,T} := \frac{d}{n} \sum_{i=1}^n \left( \log \frac{S_{t_i}}{S_{t_{i-1}}} \right)^2.$$

The constant  $d$  denotes the number of trading days per year and is usually fixed to 252 so that  $\frac{d}{n} \approx \frac{1}{T}$ . We assume the market is arbitrage-free and prices of traded instruments are represented as conditional expectations with respect to an equivalent pricing measure

$\mathbb{Q}$ . A standard result gives that as  $\sup_{i=1,\dots,n} |t_i - t_{i-1}| \rightarrow 0$ , we have

$$\sum_{i=1}^n \left( \log \frac{S_{t_i}}{S_{t_{i-1}}} \right)^2 \rightarrow \langle \log S \rangle_T \quad \text{in probability} \quad (1.1)$$

when  $(S_t)_{t \geq 0}$  is a continuous semimartingale.

Approximating the realized variance by the quadratic variation of the log returns works very well for variance swaps, but care should be taken in practise if we price short dated non-linear payoffs on realized variance. Denote by  $V_t^T$ , the price at time  $t$  of a variance swap with maturity  $T < \infty$ . It is given under  $\mathbb{Q}$  by

$$V_t^T = \mathbb{E}_t^{\mathbb{Q}} [RV^{0,T}] = \mathbb{E}_t^{\mathbb{Q}} [\langle \log S \rangle_T].$$

We define the forward variance curve  $(\xi^T)_{T \geq 0}$  as

$$\xi_t^T := \partial_T V_t^T, \quad T \geq t \geq 0.$$

Note that, if we assume that the S&PX index follows a diffusion process,  $dS_t = \mu_t S_t dt + \sigma_t S_t dW_t$  with a general stochastic volatility process,  $\sigma$ , the forward variance is given by

$$\xi_t^T = \mathbb{E}_t^{\mathbb{Q}} (\sigma_T^2).$$

It can be seen as the forward instantaneous variance for date  $T$ , observed at  $t$ . In particular

$$\xi_t^t = \sigma_t^2, \quad \forall t \geq 0.$$

The current price of a variance swap,  $V_t^T$ , is given in terms of the forward variances as

$$V_t^T = \langle \log S \rangle_t + \int_t^T \xi_t^u du$$

The models used in practice are based on diffusion dynamics where forward variance curves are given as a functional of a finite-dimensional Markov-process:

$$\xi_t^T = G(T; t, Z_t), \quad (1.2)$$

where the function  $G$  and the  $m$ -dimensional Markov-process  $Z$  satisfy some consistency

condition, which essentially ensures that for every fixed maturity  $T > 0$ , the forward variance  $(\xi_t^T)_{t \leq T}$  is a martingale.

## 1.2 Bergomi's model(s)

In two articles [2], [3] published in 2005 and 2008 in *Risk*, L. Bergomi proposed a new model based on the direct modelling of the forward variance curve. He proposed two versions of this model: a continuous and a discrete one. The discrete version can be seen as the analog of the LIBOR market model for volatility modeling, since it aimed to model forward variance swaps for a discrete tenor of maturities while the second one is Markovian and time homogeneous, where the volatilities of forward variance depend on time-to-maturity only.

Both models allow to match any specified term-structure of the at-the-money implied volatility skew for the short maturities, while being consistent with the market prices for variance swaps and fitting with a good approximation the at-the-money implied volatilities and the at-the-money skews. They also provide better control on the smile of forward variance by calibrating the VIX futures and smiles.

Denote by  $T_0 < T_1 < \dots < T_n$  the tenor structure of the VIX futures. The forward variance curve in the continuous version of the model is given by

$$\xi_t^T = \xi_0^T f^T(x_t^T, t) = \xi_0^T \left( (1 - \gamma_T) e^{\omega_T x_t^T - \frac{\omega_T^2}{2} \mathbb{E}(x_t^T)^2} + \gamma_T e^{\beta_T \omega_T x_t^T - \frac{\beta_T^2 \omega_T^2}{2} \mathbb{E}(x_t^T)^2} \right), \quad (1.3)$$

where

$$x_t^T := \sum_n \theta_n e^{-\kappa_n(T-t)} \int_0^t e^{-\kappa_n(t-s)} dW_s^n \quad (1.4)$$

The Brownian motions  $W^n$  are correlated with correlation coefficients  $\rho_{i,j}$ . The curves  $\gamma$ ,  $\beta$  and  $\omega$  are taken to be piece-wise constant over the interval  $[T_i, T_{i+1}[$ .

The main drawback for this model is the inability to express explicitly the prices of VIX futures and options in terms of the model parameters ( $\gamma$ ,  $\beta$  and  $\omega$ ). Therefore, the calibration of the model parameters to a set of market option prices becomes very difficult and sometimes impossible.

The forward variance, in the discrete version, is defined over the time interval  $[T_{i-1}, T_i[$  as

$$\xi_t^T = \xi_t^i = \xi_0^i f^i(t, x_t^{T_i}), \quad t \leq T \quad \text{and} \quad T \in [T_{i-1}, T_i[.$$

The calibration to the VIX smiles is exact and fully detailed in Bergomi (2008). However, the functional of the model,  $f^i(t, x_t^{T_i})$ , is not given in advance, but obtained after calibration on a finite number of points, which makes the use of the model difficult to price other types of products. Furthermore, conditionally to  $\mathcal{F}_{T_i}$ ,  $S$  is lognormal over  $[T_i, T_{i+1}[$ : the spot process in the discrete version follows the dynamics

$$\frac{dS}{S} = (r - q)dt + \sqrt{\xi_{T_i}^{i+1}} dW \quad T_i \leq t < T_{i+1},$$

which does not allow much flexibility to match the market prices of the S&P options with 30-days maturity starting from  $T_i$ .

### 1.3 Our contribution

In this article, we propose an arbitrage-free modeling framework for the joint dynamics of the forward variance curve along with the underlying index, based on the modelling of the forward variance.

Our model combines features of the discrete and continuous versions of Bergomi's model, without being reduced to either of them. Indeed, as in (1.3), we have an explicit form of the forward variance in terms of the state variables. On the other hand, we have a piecewise constant dependence with respect to maturity, as in (1.4). This dependence with respect to maturity will allow us to express the VIX futures payoff as a deterministic function of a normal random variable (cf (2.3)).

One of the strengths of this modelling approach is that it provides two levels of calibration. In the first step, we calibrate the parameters that generate the forward variance curve to match the VIX futures and the implied volatility of its options. In particular, our model reproduces the stylised features of the skew of the implied volatility of VIX. At the second step, we use the resulting parameters, from the first step, and calibrate the correlation coefficients between the Brownian motion driving the stock price and the factors to control the term structure of the skew of the vanilla smiles. The first step of the calibration problem is reduced to the inversion of some monotonic functions, thanks to the explicit dependence of the prices of VIX futures and Puts with respect to the model

parameters. The second step is performed by using an efficient minimization algorithm.

This paper is organized as follows: Section 2 presents a description of the model for the forward variance curve and the pricing of the VIX futures and options. In section 3, we specify the term structure of the curve  $\omega$  (The scale factor of the volatility of the forward variance). The specification of the model is reduced to three parameters for each tenor date and analytic formulas are derived for the prices of VIX futures and options in terms of these parameters. In section 4 we show how to calibrate the model parameters to fit the market prices of VIX futures and options. We also give numerical examples which demonstrate the performances of the model for calibration to market data. Section 5 shows the hedging of VIX options. We will show that any European option on VIX can be hedged by taking position on the VIX future, a variance swap, a Put on VIX and on the skew. The corresponding hedge ratio can be computed analytically. In section 6 we study the dynamics of the underlying asset and show how the model can be calibrated to fit implied volatility of the European options on the S&P index, near-the-money, and finally, we conclude in section 7.

## 2 A dynamic model for forward variance dynamics

In this section, we introduce the model which is designed to capture the market prices of volatility derivatives alongside the stock price. Our initial approach is very similar to Bergomi's approach for the forward variance modeling. We will assume that a set of settlement dates is given

$$T_0 < T_1 < \dots < T_n < \dots$$

and referred to as the tenor structure (we use especially the tenor structure of the VIX futures, but it can be generalized to any tenor structure). Consider an underlying asset whose price  $S$  is modeled as a stochastic process  $(S_t)_{t \geq 0}$  on a filtered probability space  $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, \mathbb{Q})$ , where  $\{\mathcal{F}_t\}_{t \geq 0}$  represents the history of the market.

We first specify the dynamics of the forward variance using a log normal specification which allows analytical pricing of European-type VIX derivatives. The dynamics of the forward variance under  $\mathbb{Q}$  is assumed to be

$$\xi_t^T = \xi_0^T e^{\omega_T x_t^{T_i} - \frac{\omega_T^2}{2} \mathbb{E}(x_t^{T_i})^2}, \quad t \leq T \quad \text{and} \quad T \in ]T_{i-1}, T_i] \quad (2.1)$$

where  $x_t^T$  is defined by (1.4), and the initial values of forward  $\xi_0^T$  are inputs of the model, deduced from the curve of variance-swap prices.

The main difference between our modelling approach and Bergomi's continuous model (1.3), in addition to the piecewise dependence with respect to maturity, is that it gives the S&P index as a diffusion process with stochastic-lognormal volatility process, while in Bergomi's model, the variance process is given as a sum of two lognormal processes. However, our model does not reduce to Bergomi's model with  $\gamma_T = 0$ . Indeed, we will see that by choosing a specific parametrization  $\omega$ , we can reproduce the positive skew observed in the VIX options.

The number of factors introduced in this dynamics is the number of degrees of freedom that will be available to calibrate S&P's smiles, therefore a single-factor model would not be precise enough. On the other hand, the computation time in a Monte-Carlo method increases proportionally to the number of factors, therefore a two-factor model offers a good quality/time ratio. Anyway, all the following formulas do not depend on the number of factors.

The parameters of the dynamics (1.4),  $\kappa_i$ ,  $\theta_i$  and  $\rho_{i,j}$ , are chosen and will not be calibrated to market data, because they are not directly involved in the pricing of volatility derivatives. We follow here the approach of Bergomi [3], where he proposes some parameter sets which can be chosen. For example, in the case of 2 factors, Bergomi proposed to set  $\frac{1}{\kappa_1}$  in the order of a few months, which corresponds to  $\kappa_1 \approx 8$ ,  $\frac{1}{\kappa_2}$  in the order of a few years:  $\kappa_2 \approx 0.3$ . The curve  $\omega$  is a deterministic function of  $T$ . It is a scale factor for the volatility of  $\xi$  and it allows to control the term structure of the volatility of volatility by calibrating VIX futures and options.

## 2.1 The VIX Index

The VIX Futures maturing at time  $T$  quotes the expected volatility for the next 30 days. So  $VIX^2$  represents 30-day S&P 500 variance swap rate, it is given under the risk neutral measure by

$$VIX_T = \sqrt{\mathbb{E}_T^{\mathbb{Q}} \left[ \frac{1}{\delta} RV^{T, T+\delta} \right]}, \quad (2.2)$$



where  $\delta = \frac{30}{365}$  and  $RV^{T,T+\delta} = RV^{0,T+\delta} - RV^{0,T}$ . In terms of the forward variance curve, the VIX is given by

$$VIX_T = \sqrt{\frac{1}{\delta} \int_T^{T+\delta} \xi_T^u du} = \sqrt{\frac{1}{\delta} \int_T^{T+\delta} \xi_0^u e^{\omega_u x_T^u - \frac{\omega_u^2}{2} \mathbb{E}(x_T^u)^2} du},$$

where  $\tau_u = \sum_{i \geq 1} T_i \mathbb{1}_{u \in [T_{i-1}, T_i]}$ . Note that  $T_{i+1} - T_i = \delta$

We are only interested in maturities on which VIX is traded. This corresponds to the special (typically useful) case where  $T = T_i$  for some  $i = 1, \dots, n$ . Denote by  $VIX_i := VIX_{T_i}$ . Assumed that  $\omega$  is Borel measurable and locally bounded. We have

$$VIX_i = \sqrt{\frac{1}{\delta} \int_{T_i}^{T_{i+1}} \xi_0^u e^{\omega_u x_{T_i}^{T_{i+1}} - \frac{\omega_u^2}{2} \mathbb{E}(x_{T_i}^{T_{i+1}})^2} du} \equiv \sqrt{g_i(Z)}, \quad (2.3)$$

where  $Z$  has the standard normal distribution and the function  $g_i$  is defined as

$$g_i(z) = \frac{1}{\delta} \int_{T_i}^{T_{i+1}} \xi_0^u e^{z \bar{\omega}_i(u) - \frac{\bar{\omega}_i(u)^2}{2}} du \quad (2.4)$$

and

$$\bar{\omega}_i(u) := \omega_u \sqrt{\mathbb{E}(x_{T_i}^{T_{i+1}})^2}. \quad (2.5)$$

## 2.2 Pricing VIX Futures and Options

In this section, we derive closed form expressions for the prices of the VIX futures and call options. By using (2.3), we can evaluate any given European-like claim on  $VIX_i$ , with pay-off function  $f$ , as

$$\mathbb{E}^Q f(VIX_i) = \int_{\mathbb{R}} f(\sqrt{g_i(x)}) \frac{e^{-\frac{x^2}{2}}}{\sqrt{2\pi}} dx.$$

We can also express the prices of VIX options in terms of Calls and Puts on VIX2, by using the following useful representation, which is valid for any twice-differentiable function  $G$  and for any  $k \in \mathbb{R}^+$

$$G(X^2) = G(k) + G'(k)(X^2 - k) + \int_k^\infty G''(K)(X^2 - K)_+ dK + \int_0^k G''(K)(K - X^2)_+ dK$$

This gives in particular the price of Calls and Put on VIX in terms of Calls and Puts on  $VIX^2$ , by extending this formula to the functions  $x \mapsto (\sqrt{x} - k)_+$  and  $x \mapsto (k - \sqrt{x})_+$

$$\mathbb{E}(VIX_i - k)_+ = \frac{1}{2k} \mathbb{E}(VIX_i^2 - k^2)_+ - \int_{k^2}^{\infty} \frac{1}{4K\sqrt{K}} \mathbb{E}(VIX_i^2 - K)_+ dK \quad (2.6)$$

$$\mathbb{E}(k - VIX_i)_+ = \frac{1}{2k} \mathbb{E}(k^2 - VIX_i^2)_+ + \int_0^{k^2} \frac{1}{4K\sqrt{K}} \mathbb{E}(K - VIX_i^2)_+ dK \quad (2.7)$$

and by call-put parity, one can express the VIX future price in terms of calls and puts on  $VIX^2$ . For every  $k > 0$ , we have

$$\mathbb{E}(VIX_i) = \frac{k + \mathbb{E}VIX_i^2}{2\sqrt{k}} - \int_0^k \frac{\mathbb{E}(K - VIX_i^2)_+ dK}{4K\sqrt{K}} - \int_k^{\infty} \frac{\mathbb{E}(VIX_i^2 - K)_+ dK}{4K\sqrt{K}} \quad (2.8)$$

In particular, for  $k = \mathbb{E}VIX_i^2$ , we have

$$\mathbb{E}(VIX_i) = \sqrt{\mathbb{E}VIX_i^2} - \int_0^{\mathbb{E}VIX_i^2} \frac{\mathbb{E}(K - VIX_i^2)_+ dK}{4K\sqrt{K}} - \int_{\mathbb{E}VIX_i^2}^{\infty} \frac{\mathbb{E}(VIX_i^2 - K)_+ dK}{4K\sqrt{K}} \quad (2.9)$$

Note that  $\mathbb{E}VIX_i^2 = \frac{1}{\delta} \int_{T_i}^{T_{i+1}} \xi_0^u du$ .

Now, the price of Calls and Puts on  $VIX_i$  are given by the next proposition

**Proposition 2.1.** *If the function  $\bar{\omega}_i$  is positive, then for any nonnegative strike  $K$ , the price of a call on  $VIX_i^2$  with strike  $K$  is given by*

$$\mathbb{E}(VIX_i^2 - K)_+ = \frac{1}{\delta} \int_{T_i}^{T_{i+1}} \xi_0^u N(-z_i^*(K) + \bar{\omega}_i(u)) du - KN(-z_i^*(K)) \quad (2.10)$$

and the price of a Put on  $VIX_i^2$  with strike  $K$  is given by

$$\mathbb{E}(K - VIX_i^2)_+ = KN(z_i^*(K)) - \frac{1}{\delta} \int_{T_i}^{T_{i+1}} \xi_0^u N(z_i^*(K) - \bar{\omega}_i(u)) du \quad (2.11)$$

where  $N$  denotes the standard normal cumulative distribution function and  $z_i^*$  is defined as

$$z_i^*(K) = \inf \left\{ z \in \mathbb{R} \mid g_i(z) \geq K \right\} = g_i^{-1}(K)$$

## Proof

We can write

$$\begin{aligned}
\mathbb{E}(VIX_i^2 - K)_+ &= \mathbb{E}(g_i(Z) - K)\mathbb{1}_{g_i(Z) \geq K} \\
&= \int_{z_i^*(K)}^{\infty} (g_i(x) - K) \frac{e^{-x^2/2} dx}{\sqrt{2\pi}} \\
&= \frac{1}{\delta} \int_{T_i}^{T_{i+1}} \xi_0^u \int_{z_i^*(K)}^{\infty} \frac{e^{-\frac{x^2}{2} + x\bar{\omega}_i(u) - \frac{\bar{\omega}_i^2(u)}{2}} dx}{\sqrt{2\pi}} du - KN(-z_i^*(K)) \\
&= \frac{1}{\delta} \int_{T_i}^{T_{i+1}} \xi_0^u N(-z_i^*(K) + \bar{\omega}_i(u)) du - KN(-z_i^*(K)) \quad \square
\end{aligned}$$

## 3 Specifying $\bar{\omega}_i$

The prices of VIX futures and options depend only on the volatility of volatility specification controlled by the parameter  $\bar{\omega}_i$ . So every assumption made on the structure of  $\bar{\omega}_i$  gives a new modelling approach.

On the other hand, the VIX dynamics is given by

$$\begin{aligned}
dVIX_t^2 &= \frac{\xi_t^{t+\delta} - \xi_t^t}{\delta} dt + \frac{1}{\delta} \int_t^{t+\delta} du \xi_0^u e^{\omega_u x_t^{\tau_u} - \frac{1}{2} \omega_u^2 \mathbb{E}(x_t^{\tau_u})^2} dx_t^{\tau_u} \\
&= \frac{\xi_t^{t+\delta} - \xi_t^t}{\delta} dt + \sum_n \left[ \frac{\theta_n}{\delta} \int_t^{t+\delta} \xi_0^u e^{\omega_u x_t^{\tau_u} - \frac{1}{2} \omega_u^2 \mathbb{E}(x_t^{\tau_u})^2} e^{-\kappa_n(\tau_u - t)} du \right] dW_t^n.
\end{aligned}$$

It is easy to check that if the Brownian motions  $W^n$  are positively correlated, the volatility of  $VIX^2$  dynamics is stochastic and positively correlated with the VIX dynamics, for any form of  $\bar{\omega}_i$ . So by Durrleman [15] the model will generate the positive skew observed in the VIX options. Note that in the particular case where  $\bar{\omega}_i$  is constant, the VIX skew generated by the model will be equal to 0.

In this work, we assume that  $\bar{\omega}_i$  takes only two values within interval  $[T_i, T_{i+1}]$ . We will show that in addition to modeling the positive skew observed in the VIX options, we can calibrate, exactly, VIX future as well as "at least" one Put option by maturity.

**Assumption 3.1.** *The function  $\bar{\omega}_i$  is decreasing and does not take more than two values over the time interval  $[T_i, T_{i+1}]$ .*

Denote by  $L_i$  the point where it changes its value. The curve  $\bar{\omega}_i$  can then be parametrized as follows :

$$\bar{\omega}_i(t) = \zeta_i \mathbf{1}_{t \in [T_i, L_i]} + \beta_i \zeta_i \mathbf{1}_{t \in [L_i, T_{i+1}]} \quad (3.1)$$

where  $\beta_i \in [0, 1]$ .

Under this assumption,  $F^{i2}$  takes the form

$$VIX_i^2 = m_i \left[ (1 - \gamma_i) e^{\zeta_i Z_i - \frac{\zeta_i^2}{2}} + \gamma_i e^{\beta_i \zeta_i Z_i - \frac{\beta_i^2 \zeta_i^2}{2}} \right]$$

where  $m_i := \frac{1}{\delta} \int_{T_i}^{T_{i+1}} \xi_0^u du$ ,  $\gamma_i = \frac{\frac{1}{\delta} \int_{L_i}^{T_{i+1}} \xi_0^u du}{m_i}$  and the random variable  $Z_i := \frac{1}{\sqrt{\mathbb{E}(x_{T_i}^{T_{i+1}})^2}} x_{T_i}^{T_{i+1}}$

has the standard normal distribution.

The price of any VIX future contract is then given as a function of the triplet  $(\gamma_i, \beta_i, \zeta_i)$ . This is given as function of calls and puts on  $VIX_i^2$  by using proposition 2.11 and the equalities (2.6), (2.7) and (2.8). Noting that, with this special parametrization of the function  $\bar{\omega}_i$ , the price of call on  $VIX_i^2$  with strike  $K$  is given by

$$\mathbb{E}(VIX_i^2 - K)_+ = m_i [(1 - \gamma_i) N(-z_i^*(K) + \zeta_i) + \gamma_i N(-z_i^*(K) + \beta_i \zeta_i)] - K N(-z_i^*(K)) \quad (3.2)$$

and the price of put on  $VIX_i^2$  with strike  $K$  is given by

$$\mathbb{E}(K - VIX_i^2)_+ = K N(-z_i^*(K)) - m_i [(1 - \gamma_i) N(z_i^*(K) - \zeta_i) + \gamma_i N(z_i^*(K) - \beta_i \zeta_i)] \quad (3.3)$$

Prices of VIX futures and options are then given by explicit formulas in terms of parameters  $\gamma_i$ ,  $\beta_i$  and  $\zeta_i$ .

We will study the particular case of VIX future and Put prices. First, to simplify the notation, denote

$$V_i^{\gamma, \beta, \zeta} := m_i \left[ (1 - \gamma) e^{\zeta Z_i - \frac{\zeta^2}{2}} + \gamma e^{\beta \zeta Z_i - \frac{\beta^2 \zeta^2}{2}} \right],$$

for  $(\zeta, \beta, \gamma) \in [0, 1] \times [0, 1] \times \mathbb{R}^+$ . Also, let

$$F_{VIX}(\gamma, \beta, \zeta) = \mathbb{E} \sqrt{V_i^{\gamma, \beta, \zeta}} \quad \text{and} \quad p_k(\gamma, \beta, \zeta) = \mathbb{E}(k - \sqrt{V_i^{\gamma, \beta, \zeta}})_+,$$

the prices of VIX future and Put on VIX with strike  $k$  respectively.

The next result gives more information about the function giving the price of VIX future and put in terms of the parameters  $\gamma$ ,  $\beta$  and  $\zeta$ . The proof can be found in the appendix

**Proposition 3.2.** *The functions  $p_k$  and  $F_{VIX}$  are differentiable and their first partial derivatives are given by*

- $\partial_\zeta F_{VIX}(\gamma, \beta, \zeta) = -\int_0^\infty \frac{1}{4K\sqrt{K}} [m_i(1-\gamma)N'(z_i^*(K) - \zeta) + m_i\gamma\beta N'(z_i^*(K) - \beta\zeta)] dK,$
- $\partial_\beta F_{VIX}(\gamma, \beta, \zeta) = -m_i\omega\gamma \int_0^\infty N'(z_i^*(K) - \beta\zeta) \frac{dK}{4K\sqrt{K}},$
- $\partial_\gamma F_{VIX}(\gamma, \beta, \zeta) = m_i \int_0^\infty [N(z_i^*(K) - \beta\zeta) - N(z_i^*(K) - \zeta)] \frac{dK}{4K\sqrt{K}},$
- $\partial_\zeta p_k(\gamma, \beta, \zeta) = -m_i(1-\gamma) \int_{-\infty}^{z_i^*(k^2)-\zeta} \frac{KN'(K)dK}{2\sqrt{g_i(K+\zeta)}} - m_i\beta\gamma \int_{-\infty}^{z_i^*(k^2)-\beta\zeta} \frac{KN'(K)dK}{2\sqrt{g_i(K+\beta\zeta)}},$
- $\partial_\beta p_k(\gamma, \beta, \zeta) = -m_i\zeta\gamma \int_{-\infty}^{z_i^*(k^2)-\beta\zeta} \frac{KN'(K)dK}{2\sqrt{g_i(K+\beta\zeta)}},$
- $\partial_\gamma p_k(\gamma, \beta, \zeta) = m_i \int_{-\infty}^{z_i^*(k^2)-\zeta} \frac{N'(K)dK}{2\sqrt{g_i(K+\zeta)}} - m_i \int_{-\infty}^{z_i^*(k^2)-\beta\omega} \frac{N'(K)dK}{2\sqrt{g_i(K+\beta\omega)}}.$

In particular, we have

1.  $\partial_\omega F_{VIX}$  and  $\partial_\beta F_{VIX}$  are negative.
2.  $\partial_\gamma F_{VIX}$ ,  $\partial_\zeta p_k$  and  $\partial_\beta p_k$  are positive.
3. If  $k \leq \sqrt{m_i}$ , then  $\partial_\gamma p_k$  is negative.

## 4 Calibrating $\gamma$ , $\beta$ and $\zeta$

A model cannot be used in practise without a reliable and reasonably quick calibration scheme. We therefore describe here how the model can be calibrated using the "explicit dependence" between VIX futures and options prices and the model parameters  $\gamma$ ,  $\beta$  and  $\zeta$  for each maturity  $T_i$ , given by proposition 3.2.

## Data

Assume that we observe the Variance-Swap market prices for all maturities. We deduce the initial variance curve  $(\xi_0^T)_{T \geq 0}$  from the market prices of Variance-Swap.

Let us also assume that we observe the VIX future price and a series of Put options on VIX, for each maturity  $T_i$ . Obviously, we will not pretend to be able to calibrate VIX futures and all European options on VIX, nevertheless we will show that for each maturity we can calibrate "exactly" both VIX future price and one Put by leaving free the parameter  $\gamma$  along some interval. This parameter, left free, will serve to calibrate other options and/or to reproduce the VIX skew.

## Phase 1: Calibrating VIX future

The hedging of VIX options is typically done with trading in VIX futures contracts, we want the model to reproduce the VIX futures prices for each maturity  $T_i$ .

Let's denote by  $F$  the market price of VIX future for some maturity  $T_i$  and denote by  $m = \frac{1}{T_{i+1}-T_i} \int_{T_i}^{T_{i+1}} \xi_0^u du$ . Note that  $F$  and  $m$  must satisfy the following (no-arbitrage) condition :

$$F \leq \sqrt{m}.$$

The calibration problem of the VIX future is to find a triplet  $(\gamma, \beta, \zeta) \in [0, 1] \times [0, 1] \times \mathbb{R}_+$  such that

$$F_{VIX}(\gamma, \beta, \zeta) = F.$$

In what follows, we will show that for every  $(\gamma, \zeta)$  belonging to some subset of  $[0, 1] \times \mathbb{R}_+$ , there exists a unique  $\beta \in [0, 1]$ , such that  $F_{VIX}(\gamma, \beta, \zeta) = F$ .

By Proposition 3.2, we know that for  $(\gamma, \zeta) \in [0, 1] \times \mathbb{R}_+$ , the function  $\beta \mapsto F_{VIX}(\gamma, \beta, \zeta)$  is continuous, decreasing over  $[0, 1]$ . It is then bijective from  $[0, 1]$  to  $\left[ F_{VIX}(\gamma, \beta = 1, \zeta), F_{VIX}(\gamma, \beta = 0, \zeta) \right]$ . (Note that  $V^{\beta, 0, \zeta}$  and  $V^{\beta, 1, \zeta}$  are lognormales)

Now, it becomes clear that if the pair  $(\gamma, \zeta)$  is such that  $F_{VIX}(\gamma, 1, \zeta) \leq F$  and  $F_{VIX}(\gamma, 0, \zeta) \geq F$ , then there exists  $\beta \in [0, 1]$  such that  $F_{VIX}(\gamma, \beta, \zeta) = F$ .

So, for  $\gamma \geq 0$ , consider the function  $\zeta \mapsto F_{VIX}(\gamma, 0, \zeta)$ . From Proposition 3.2, we know that it is continuous and decreasing over  $\mathbb{R}^+$  satisfying  $F_{VIX}(\gamma, 0, \zeta = 0) = \sqrt{m\gamma}$

and  $\lim_{\zeta \rightarrow \infty} F_{VIX}(\gamma, 0, \zeta) = 0$ . It allows us to define the function

$$\bar{\zeta}_F : \gamma \in [0, \frac{F^2}{m}) \mapsto \bar{\zeta}_F(\gamma) := F_{VIX}(\gamma, 0, \cdot)^{-1}(F) \quad (4.1)$$

This function,  $\bar{\zeta}_F$ , is continuous, increasing over  $[0, \frac{F^2}{m})$  and satisfies

$$\begin{cases} \bar{\zeta}_F(0) = \zeta_F, \\ \lim_{\gamma \rightarrow \frac{F^2}{m}} \bar{\zeta}_F(\gamma) = +\infty. \end{cases}$$

where

$$\zeta_F := 2\sqrt{\log(\frac{m}{F^2})} \quad (4.2)$$

Denote

$$\Omega_F = \left\{ (\gamma, \zeta) \in [0, \frac{F^2}{m}) \times \mathbb{R}^+ \mid \zeta \in [\zeta_F, \bar{\zeta}_F(\gamma)] \right\}. \quad (4.3)$$

It is easy to check that for every  $(\gamma, \zeta) \in \Omega_F$ , we have  $F_{VIX}(\gamma, 1, \zeta) \leq F$  and  $F_{VIX}(\gamma, 0, \zeta) \geq F$ . This means that the mapping

$$\beta_F : (\gamma, \zeta) \in \Omega_F \mapsto \beta_F(\gamma, \zeta) := F_{VIX}(\gamma, \cdot, \zeta)^{-1}(F) \quad (4.4)$$

is well defined. In particular, for every  $(\gamma, \zeta) \in \Omega_F$ , we have  $F_{VIX}(\gamma, \beta_F(\gamma, \zeta), \zeta) = F$   $\square$

## Phase 2: Calibrating VIX Put

As mentioned above, we choose to calibrate one Put option for each maturity  $T_i$ . Denote by  $P$  the market price of this option and by  $K_0$  its strike. Here we will try to find a family of pairs  $(\gamma, \zeta) \in \Omega_F$ , such that

$$p_{k_0}(\gamma, \beta_F(\gamma, \zeta), \zeta) = P.$$

The map  $(\gamma, \zeta) \mapsto p_{k_0}(\gamma, \beta_F(\gamma, \zeta), \zeta)$  is neither monotonic in  $\gamma$ , nor in  $\zeta$  because of  $\beta_F$ , then we cannot obtain its inverse easily. To address this problem, we proceed as follows.

By using proposition 3.2 and in the same way as before we can define

$$\bar{\zeta}_P : \gamma \in [0, \frac{(k_0 - P)^2}{m}) \mapsto \bar{\zeta}_P(\gamma) := p_{k_0}(\gamma, 0, \cdot)^{-1}(P) \quad (4.5)$$

In particular,  $\bar{\zeta}_P$  is continuous, increasing over  $[0, \frac{(P-k_0)^2}{m})$  and satisfies

$$\begin{cases} \bar{\zeta}_P(0) = \zeta_P, \\ \lim_{\gamma \rightarrow \frac{(P-k_0)^2}{m}} \bar{\zeta}_P(\gamma) = +\infty. \end{cases}$$

where

$$\zeta_P = \sup \left\{ \zeta > 0; \mathbb{P}_{BS}(\sqrt{m}e^{-\frac{\zeta^2}{8}}, k_0, \frac{\zeta}{2}) \leq P \right\} \quad (4.6)$$

and

$$\mathbb{P}_{BS}(S, k, \sigma) = -SN \left( \frac{-\log(\frac{S}{k}) - \frac{\sigma^2}{2}}{\sigma} \right) + kN \left( \frac{-\log(\frac{S}{k}) + \frac{\sigma^2}{2}}{\sigma} \right).$$

Denote

$$\Omega_P = \left\{ (\gamma, \zeta) \in [0, \frac{(k_0 - P)^2}{m}) \times [\zeta_P, \bar{\zeta}_P(\gamma)] \right\} \bigvee \left\{ [\frac{(k_0 - P)^2}{m}, \frac{F^2}{m}) \times [\zeta_P, \infty) \right\}$$

We can define the map

$$\beta_P : (\gamma, \zeta) \in \Omega_P \longmapsto \beta_P(\gamma, \zeta) := p_{k_0}(\gamma, \cdot, \zeta)^{-1}(P) \quad (4.7)$$

Now to solve the "double" calibration problem of the parameters  $\gamma$ ,  $\beta$  and  $\zeta$  to F and P, it suffices to find  $(\gamma, \zeta) \in \Omega_F \cap \Omega_P$  such that

$$\beta_F(\gamma, \zeta) = \beta_P(\gamma, \zeta).$$

The proof of the next theorem can be found in the appendix.

**Theorem 4.1.** *Assume  $\zeta_P \leq \zeta_F$  (where  $\zeta_P$  is defined by (4.6) and  $\zeta_F$  by (4.2)), then there exists  $\gamma^* < \frac{(k_0 - P)^2}{m}$  such that for every  $\gamma \in [\gamma^*, \frac{F^2}{m})$ , there exists  $\zeta_\gamma = \zeta^*(\gamma, k_0, F, P) \geq 0$  such that  $\beta_F(\gamma, \zeta_\gamma) = \beta_P(\gamma, \zeta_\gamma)$ . Furthermore, the map  $\zeta^*$  is differentiable and its derivatives are given by*

- $\partial_\gamma \zeta^*(\gamma, k_0, F, P) = \frac{\partial_\gamma p_{k_0} \times \partial_\beta F_{VIX} - \partial_\beta p_{k_0} \times \partial_\gamma F_{VIX}}{\partial_\zeta F_{VIX} \times \partial_\beta p_{k_0} - \partial_\beta F_{VIX} \times \partial_\zeta p_{k_0}}(\gamma, \beta_F(\gamma, \zeta_\gamma), \zeta_\gamma),$
- $\partial_k \zeta^*(\gamma, k_0, F, P) = \frac{p_{k_0} \times \partial_\beta F_{VIX}}{\partial_\zeta F_{VIX} \times \partial_\beta p_{k_0} - \partial_\beta F_{VIX} \times \partial_\zeta p_{k_0}}(\gamma, \beta_F(\gamma, \zeta_\gamma), \zeta_\gamma),$
- $\partial_F \zeta^*(\gamma, k_0, F, P) = \frac{\partial_\beta p_{k_0}}{\partial_\zeta F_{VIX} \times \partial_\beta p_{k_0} - \partial_\beta F_{VIX} \times \partial_\zeta p_{k_0}}(\gamma, \beta_F(\gamma, \zeta_\gamma), \zeta_\gamma),$



- $\partial_P \zeta^*(\gamma, k_0, F, P) = \frac{-\partial_\beta F_{VIX}}{\partial_\zeta F_{VIX} \times \partial_\beta p_{k_0} - \partial_\beta F_{VIX} \times \partial_\zeta p_{k_0}}(\gamma, \beta_F(\gamma, \zeta_\gamma), \zeta_\gamma).$

**Remark 4.1.** With all market data that we have dealt with, the condition  $\zeta_p \leq \zeta_F$  is satisfied for  $k_0 = F$ . For general case, we note that

$$\zeta_p \leq \zeta_F \iff \sigma_{IMP}(k_0) \leq \frac{\zeta_P}{2} \iff \sigma_{IMP}(k_0) \leq \frac{\zeta_F}{2}$$

where  $\sigma_{IMP}(k) = \mathbb{P}_{BS}(F, k_0, \cdot)^{-1}(P)$ . Since "in practice", the implied volatility of VIX options are increasing with respect to the strike, then if  $k_0$  is such that  $\sigma_{IMP}(k_0) \leq \frac{\zeta_F}{2}$ , so the condition is still satisfied if  $k_0$  is replaced by  $k \leq k_0$ .

**Remark 4.2.** Thanks to the monotonicity properties of all the functions we have defined, the calculation of  $\gamma^*$  and  $\zeta_\gamma$  are made by using a "special" binary search algorithm. This algorithm will be detailed in the appendix ( see Remark B.1).

### Phase 3: Calibrating $\gamma$

We can do without this calibration step if we only want to fit the future price and the Put price by choosing any value of  $\gamma$  between  $\gamma^*$  and  $\frac{F^2}{m}$ . Noting that  $\frac{(k_0 - P)^2}{m} \in [\gamma^*, \frac{F^2}{m}]$ . Otherwise, we can calibrate the VIX skew or another Put option on VIX.

By proposition 4.1, we know that by choosing any value of  $\gamma$  in  $[\gamma^*, \frac{F^2}{m})$ , we can find a couple  $(\beta_\gamma, \zeta_\gamma)$  such that the model price of VIX future and Put on VIX with strike  $k_0$  coincides with their market prices. There is therefore a possibility to calibrate  $\gamma$  to match another VIX future contract. Here we choose to calibrate with the aim to reproduce the skew of VIX at  $k_0$ . i.e the slope of the Put implied volatility of VIX at the point  $k_0$ .

In practice, the skew, at some point  $k$ , is measured as the difference of the implied volatilities of 95% and 105% strike. Now to compute the "skew" from the market data on VIX, we choose  $k_1$ : the nearest strike to  $k_0$ , on which the VIX put is available and we approximate the skew by the difference of the implied volatility of  $k_0$  and  $k_1$ .

This step of calibration reduces to finding  $\gamma$  such that

$$\partial_k p_{k_0}(\gamma, \beta_F(\gamma, \zeta_\gamma); \zeta_\gamma) = \frac{P_1 - P}{k_1 - k_0}.$$

The calibration is thus reduced to minimizing

$$\left\{ \left[ \frac{k_0^2}{m} - \left( (1 - \gamma) e^{\zeta_\gamma Z - \frac{1}{2} \zeta_\gamma^2} + \gamma e^{\zeta_\gamma \beta_F(\gamma, \zeta_\gamma) Z - \frac{1}{2} \zeta_\gamma^2 \beta_F(\gamma, \zeta_\gamma)^2} \right) \right]^2, \quad \gamma \in \left[ \gamma^*, \frac{F^2}{m} \right) \right\}$$

**Remark 4.3.** By differentiating the Black-Scholes formula giving the price of Put with strike  $k_0$  with respect to the strike, we can express  $\partial_k p_{k_0}$  in terms of the skew and the implied volatility of P at the point  $k_0$  as

$$\partial_k \mathbb{E}(K - \sqrt{V_i^{\gamma, \beta_P(\gamma, \zeta_\gamma), \zeta_\gamma}})_+ \Big|_{k=k_0} = N(-d_2) + \mathcal{S}_i \times k_0 \sqrt{T_i} N'(-d_2),$$

where  $d_2 = \frac{\log(\frac{F}{k_0}) - \frac{\sigma_{VIX}^2(k_0)}{2} T_i}{\sigma_{VIX}(k_0) \sqrt{T}}$  and  $\sigma_{VIX}(k_0)$  is the implied volatility of the Put on VIX with strike  $k_0$ . We can then synthesize  $\mathbb{E}_t \mathbf{1}_{k \geq VIX}$  by observing continuously the price of Put P, the future price of VIX and the skew.

To illustrate our model, we calibrate it to the VIX options data observed on November 2, 2010. Figure 1 shows the results of the calibration to the implied volatilities of VIX. Note that the VIX futures prices are perfectly reproduced by the model. Furthermore, as mentioned in Remark 4.1, the calibration is done by using a "special" binary search algorithm. The computation time is of the order of a few seconds.

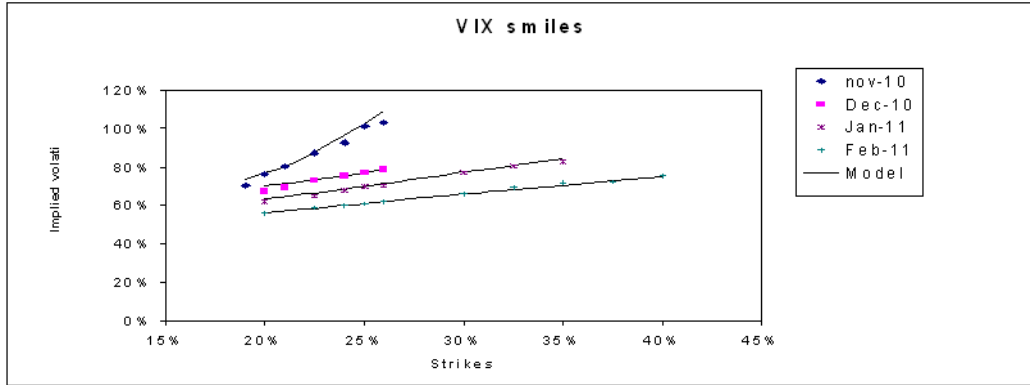


Figure 1: Model v.s. Market VIX implied volatility smiles on November 2, 2010

## 5 Hedging VIX options

We have seen in the previous sections, that, for each tenor date  $T_i$ , the price of VIX futures and options are given as a function of the parameters  $\gamma_i$ ,  $\beta_i$  and  $\zeta_i$ . Now, since those parameters are calibrated to fit VIX future price and one Put option (with strike  $k_0$ ) for each tenor date, we can hedge any "other" European contract in VIX futures by taking positions in the VIX future and the Put used for the calibration.

**Example:** Fix one maturity  $T = T_i$ , for some  $i \geq 1$ . Assume that until  $T$  we calibrate the model to VIX future price  $(F_t)_{t \leq T}$ , the put on VIX with strike  $(k_0 = k_0(F_t))$  and the skew of VIX at the point  $k_0$ . Consider the hedging strategies implied by this calibrated model for one Put option with strike  $k$ .

According to the previous sections, we know that the price of this options is given as

$$\mathbb{E}(k - VIX_T)_+ = p_k(\gamma, \beta_F(\gamma, \zeta_\gamma), \zeta_\gamma)$$

This option can be hedged by trading, continuously in F, P, D and M. Where  $D_t := \mathbb{E}_t \mathbf{1}_{k_0 \geq VIX_T}$  and  $M_t = \mathbb{E}_t VIX_T^2$ . The hedging strategy is given by the next result

**Theorem 5.1.** *For  $t \leq T$ , we have*

$$d[\mathbb{E}_t(k - VIX_T)_+] = \Delta_F(t) \times dF_t + \Delta_P(t) \times dP_t + \Delta_S(t) \times dD_t + \Delta_m \times dM_t \quad (5.1)$$

where

$$\begin{cases} \Delta_F = \frac{1}{\partial_\beta F_{VIX}} + \partial_F \gamma \left[ \partial_\gamma p_k - \partial_\beta p_k \frac{\partial_\gamma F_{VIX}}{\partial_\beta F_{VIX}} \right] + \Gamma^F \times \left[ \partial_\zeta p_k - \partial_\beta p_k \frac{\partial_\zeta F_{VIX}}{\partial_\beta F_{VIX}} \right], \\ \Delta_P = \partial_P \gamma \left[ \partial_\gamma p_k - \partial_\beta p_k \frac{\partial_\gamma F_{VIX}}{\partial_\beta F_{VIX}} \right] + \Gamma^P \times \left[ \partial_\zeta p_k - \partial_\beta p_k \frac{\partial_\zeta F_{VIX}}{\partial_\beta F_{VIX}} \right], \\ \Delta_S = \frac{\partial_\gamma p_{k_0} + \partial_\gamma \zeta^* \left( \partial_\zeta p_{k_0} - \frac{\partial_\zeta p_{k_0}}{\partial_\beta p_{k_0}} \partial_\beta p_{k_0} \right)}{\partial_{k\gamma} p_{k_0} + \partial_\gamma \zeta^* \left( \partial_{k\zeta} p_{k_0} - \frac{\partial_\zeta p_{k_0}}{\partial_\beta p_{k_0}} \partial_{k\beta} p_{k_0} \right)}, \\ \Delta_m = \frac{p_k - k \partial_k p_k - F \Delta_F - P \Delta_P - k_0 \Delta_{k_0}}{2M_t}. \end{cases}$$

With

$$\Gamma^F = \frac{\partial_\beta p_{k_0} + \partial_F \gamma \times (\partial_\gamma p_{k_0} \times \partial_\beta F_{VIX} - \partial_\beta p_{k_0} \times \partial_\gamma F_{VIX}) + k'_0 \partial_k p_{k_0} \times \partial_\beta F_{VIX}}{\partial_\zeta F_{VIX} \times \partial_\beta p_{k_0} - \partial_\beta F_{VIX} \times \partial_\zeta p_{k_0}},$$

$$\Gamma^P = \frac{\partial_\beta F_{VIX} + \partial_F \gamma \times (\partial_\gamma F_{VIX} \times \partial_\beta p_{k_0} - \partial_\beta F_{VIX} \times \partial_\gamma p_{k_0})}{\partial_\zeta p_{k_0} \times \partial_\beta F_{VIX} - \partial_\beta p_{k_0} \times \partial_\zeta F_{VIX}},$$

$$\Delta_{k_0} = \partial_{k_0} \gamma \left[ \partial_\gamma p_k - \partial_\beta p_k \frac{\partial_\gamma F_{VIX}}{\partial_\beta F_{VIX}} \right] + \partial_{k_0} \zeta^* \left[ \partial_\zeta p_k - \partial_\beta p_k \frac{\partial_\zeta F_{VIX}}{\partial_\beta F_{VIX}} \right].$$

And the derivatives of  $\gamma$  with respect to  $F$  and  $P$  are directly derived from

$$\partial_{kk} p_{k_0} + \partial_{k_0} \gamma \left[ \partial_{k\gamma} p_{k_0} - \frac{\partial_\gamma p_{k_0}}{\partial_\beta p_{k_0}} \partial_{k\beta} p_{k_0} \right] + \partial_{k_0} \zeta^* \left[ \partial_{k\zeta} p_{k_0} - \frac{\partial_\zeta p_{k_0}}{\partial_\beta p_{k_0}} \partial_{k\beta} p_{k_0} \right] = 0.$$

$$k'(F) \partial_{kk} p_{k_0} + \partial_F \gamma \left[ \partial_{k\gamma} p_{k_0} - \frac{\partial_\gamma p_{k_0}}{\partial_\beta p_{k_0}} \partial_{k\beta} p_{k_0} \right] + \Gamma^F \times \left[ \partial_{k\zeta} p_{k_0} - \frac{\partial_\zeta p_{k_0}}{\partial_\beta p_{k_0}} \partial_{k\beta} p_{k_0} \right] = 0.$$

and

$$\partial_P \gamma \left[ \partial_{k\gamma} p_{k_0} - \frac{\partial_\gamma F_{VIX}}{\partial_\beta F_{VIX}} \partial_{k\beta} p_{k_0} \right] + \Gamma^P \times \left[ \partial_{k\zeta} p_{k_0} - \frac{\partial_\zeta F_{VIX}}{\partial_\beta F_{VIX}} \partial_{k\beta} p_{k_0} \right] = 0.$$

Here  $\beta_F$  is evaluated in  $(\gamma_t, \zeta_{\gamma_t}^*)$  and the derivatives of  $p_k$  and  $F_{VIX}$  are evaluated in  $(\gamma, \beta_F(\gamma, \zeta^*), \zeta^*)$ .

**Remark 5.1.** More generally, any claim on a function  $G(VIX)$ , where  $G$  is given as difference of convex functions, can be synthesized using VIX puts and calls at all strikes by the so-called Carr-formula as

$$G(VIX) = G(k) + G'(k)(VIX - k) + \int_k^\infty G''(K)(VIX - K)_+ dK + \int_0^k G''(K)(K - VIX)_+ dK$$

Then, there exist  $g$  and  $\psi$  such that

$$\mathbb{E}G(VIX) = g(\gamma, \beta_F(\gamma, \zeta_\gamma), \zeta_\gamma) = \psi(F, P)$$

## 6 The dynamics of the underlying asset

Until now, we have only addressed issues concerning the modelling of the forward variance curve. But, once the dynamics of forward variance has been specified, we obtain the (risk neutral) dynamics of the underlying asset  $(S_t)_{t \geq 0}$  as

$$\frac{dS_t}{S_t} = rdt + \sqrt{\xi_t^S} dW_t^S, \quad (6.1)$$

where  $r$  is the annualized risk-free interest rate and  $\xi_t^t$  is given by (2.1) as

$$\xi_t^t = \xi_0^t e^{\omega_t x_t^{T_i} - \frac{\omega_t^2}{2} \mathbb{E}(x_t^{T_i})^2}, \quad t \in ]T_{i-1}, T_i]$$

and for  $t \leq T$ ,  $x_t^T$  is defined in (1.4) as

$$x_t^T = \sum_n \theta_n e^{-\kappa_n(T-t)} \int_0^t e^{-\kappa_n(t-s)} dW_s^n$$

The Brownian motion  $W^S$  is correlated with the factors  $X^n$ , denote by  $\rho_n^S = \frac{d\langle W^S, X^n \rangle_t}{dt}$ . The number of factors has been discussed in the beginning of this work, it corresponds to the number of degrees of freedom that will be available to fit many different smile shapes.

Now, thanks to lognormal form of the instantaneous variance, we can use the very robust approximations we obtain in [20] for the prices of the European options under this model. We can specify the correlations  $\rho_n^S$  to match the specified skew and to calibrate the ATM implied volatility

## 7 Conclusion

We have presented a new model for the joint dynamics of the forward variance curve along with the underlying index, which can be made consistent with both the market prices of the VIX futures and options, and options on the S&P500 index. This model leads to a tractable pricing framework for VIX futures and options, where the prices of such instruments are given by analytical formulas.

We demonstrate how the calibration of the model to VIX futures and options is reduced to a binary search algorithms. This tractability feature distinguishes our model from previous attempts [2], [3] which only allow for full Monte Carlo pricing of VIX options and calibration with a least-square minimisation. The model allows also to Hedge VIX options by trading on VIX future and one Put option (typically ATM Put), where the corresponding hedge ratio can be computed explicitly.

This model allows also to control the term structure the forward skew of the at-the-money implied volatility as well as the implied volatility near the money of the S&P index. It can therefore be useful in pricing path-dependent options that are sensitive to the forward smiles, such as Cliquet or forward start options, as well as volatility derivatives,

since it is consistent with the variance swap curve.

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## A Proof of proposition 3.2

By using (2.10) and (2.11), we can easily verify that for every  $(\gamma, \beta, \zeta) \in [0, 1] \times [0, 1] \times \mathbb{R}^+$ , we have

- $\partial_\zeta \mathbb{E}(K - V_i^{\gamma, \beta, \zeta})_+ = m_i(1 - \gamma)N'(z_i^*(K) - \zeta) + m_i\gamma\beta N'(z_i^*(K) - \beta\zeta) > 0$ ,
- $\partial_\beta \mathbb{E}(K - V_i^{\gamma, \beta, \zeta})_+ = m_i\zeta\gamma N'(z_i^*(K) - \beta\zeta) > 0$ ,
- $\partial_\gamma \mathbb{E}(K - V_i^{\gamma, \beta, \zeta})_+ = m_i(N(z_i^*(K) - \zeta) - N(z_i^*(K) - \beta\zeta)) < 0$ .

Now by using (2.7) and by performing an integration by parts of the integral part of (2.7), we obtain the partial derivatives of  $p_k$  as

- $\partial_\zeta p_k(\gamma, \beta, \zeta) = -m_i(1 - \gamma) \int_0^{k^2} \partial_K z_i^*(K)(z_i^*(K) - \zeta)N'(z_i^*(K) - \zeta) \frac{dK}{2\sqrt{K}} - m_i\gamma\beta \int_0^{k^2} \partial_K z_i^*(K)(z_i^*(K) - \beta\zeta)N'(z_i^*(K) - \beta\zeta) \frac{dK}{2\sqrt{K}}$ ,
- $\partial_\beta p_k(\gamma, \beta, \zeta) = -m_i\zeta\gamma \int_0^{k^2} \partial_K z_i^*(K)(z_i^*(K) - \beta\zeta)N'(z_i^*(K) - \beta\zeta) \frac{dK}{2\sqrt{K}}$ ,
- $\partial_\gamma p_k(\gamma, \beta, \zeta) = m_i \int_0^{k^2} \partial_K z_i^*(K)(N'(z_i^*(K) - \zeta) - N'(z_i^*(K) - \beta\zeta)) \frac{dK}{2\sqrt{K}}$ .

We obtain the results of the theorem by an appropriate change of variables. The partial derivatives of  $F_{VIX}$  can be found in the same way as for  $p_k$ . They are given by

- $\partial_\zeta F_{VIX}(\gamma, \beta, \zeta) = - \int_0^\infty \frac{1}{4K\sqrt{K}} [m_i(1 - \gamma)N'(z_i^*(K) - \zeta) + m_i\gamma\beta N'(z_i^*(K) - \beta\zeta)] dK$ ,
- $\partial_\beta F_{VIX}(\gamma, \beta, \zeta) = -m_i\zeta\gamma \int_0^\infty N'(z_i^*(K) - \beta\zeta) \frac{dK}{4K\sqrt{K}}$ ,
- $\partial_\gamma F_{VIX}(\gamma, \beta, \zeta) = m_i \int_0^\infty [N(z_i^*(K) - \beta\zeta) - N(z_i^*(K) - \zeta)] \frac{dK}{4K\sqrt{K}}$ .

Consider the derivative of  $p_k$  with respect to  $\beta$ . It is given by

$$\partial_\beta p_k(\gamma, \beta, \zeta) = -m_i\zeta\gamma \int_{-\infty}^{z_i^*(k^2) - \beta\zeta} \frac{KN'(K)dK}{2\sqrt{g_i(K + \beta\zeta)}}.$$

- **If**  $z_i^*(k^2) - \beta\zeta \leq 0$ , then  $\partial_\beta p_k(\gamma, \beta, \zeta) \geq 0$ .
- **If**  $z_i^*(k^2) - \beta\zeta > 0$ , we decompose  $\partial_\beta p_k(\gamma, \beta, \zeta)$  as follows

$$\partial_\beta p_k(\gamma, \beta, \zeta) = m_i\zeta\gamma \left[ - \int_{-\infty}^{-z_i^*(k^2) + \beta\zeta} \frac{KN'(K)dK}{2\sqrt{g_i(K + \beta\zeta)}} - \int_{-z_i^*(k^2) + \beta\zeta}^{z_i^*(k^2) - \beta\zeta} \frac{KN'(K)dK}{2\sqrt{g_i(K + \beta\zeta)}} \right]$$



Since  $z_i^*(k^2) - \beta\zeta > 0$ , so that  $-\int_{-\infty}^{-z_i^*(k^2)+\beta\zeta} \frac{KN'(K)dK}{2\sqrt{g_i(K+\beta\zeta)}} \geq 0$ . It follows

$$\partial_\beta p_k(\gamma, \beta, \zeta) \geq -m_i \zeta \gamma \int_{-z_i^*(k^2)+\beta\zeta}^{z_i^*(k^2)-\beta\zeta} \frac{KN'(K)dK}{2\sqrt{g_i(K+\beta\zeta)}}.$$

On the other hand, we can decompose this integral as follows

$$\int_{-z_i^*(k^2)+\beta\omega}^{z_i^*(k^2)-\beta\omega} \frac{KN'(K)dK}{2\sqrt{g_i(K+\beta\omega)}} = \int_{-z_i^*(k^2)+\beta\omega}^0 \frac{KN'(K)dK}{2\sqrt{g_i(K+\beta\omega)}} + \int_0^{z_i^*(k^2)-\beta\omega} \frac{KN'(K)dK}{2\sqrt{g_i(K+\beta\omega)}}.$$

Since  $g_i$  is increasing, we have  $\frac{1}{\sqrt{g_i(K+\beta\zeta)}} \leq \frac{1}{\sqrt{g_i(-K+\beta\zeta)}}$ ,  $\forall K \geq 0$ , so that

$$\int_0^{z_i^*(k^2)-\beta\zeta} \frac{KN'(K)dK}{2\sqrt{g_i(K+\beta\zeta)}} \leq - \int_{-z_i^*(k^2)+\beta\zeta}^0 \frac{KN'(K)dK}{2\sqrt{g_i(K+\beta\zeta)}}$$

Thus

$$\int_{-z_i^*(k^2)+\beta\omega}^{z_i^*(k^2)-\beta\omega} \frac{KN'(K)dK}{2\sqrt{g_i(K+\beta\omega)}} \leq 0.$$

Hence  $\partial_\beta p_k(\gamma, \beta, \zeta) \geq 0$ .

In the same way, we can show that  $\partial_\zeta p_k(\gamma, \beta, \zeta) \geq 0$ .

Consider the derivative of  $p_k$  with respect to  $\gamma$ . It is given by

$$\partial_\gamma p_k(\gamma, \beta, \zeta) = m_i \int_0^{k^2} \partial_K z_i^*(K) (N'(z_i^*(K) - \zeta) - N'(z_i^*(K) - \beta\zeta)) \frac{dK}{2\sqrt{K}}$$

It is easy to check that, on  $[0, k^2]$ ,  $N'(z_i^*(K) - \zeta) - N'(z_i^*(K) - \beta\zeta) \leq 0$  if and only if

$$(z_i^*(K) - \zeta)^2 \geq (z_i^*(K) - \beta\zeta)^2.$$

which is equivalent to

$$z_i^*(K) \leq \frac{\zeta(1+\beta)}{2} \iff K \leq g_i\left(\frac{\zeta(1+\beta)}{2}\right) = m_i e^{\frac{\zeta^2\beta^2}{2}}.$$

In particular, if  $k \leq \sqrt{m_i}$ , we have  $\partial_\gamma p_k(\gamma, \beta, \zeta) \leq 0$ .

## B Proof of theorem 4.1 and theorem 5.1

**Lemma B.1.** *The mapping  $\beta_P$  is continuous over  $\Omega_P$ . Furthermore, for  $\gamma \in [0, \frac{(P-k_0)^2}{m})$  the function  $\zeta \mapsto \beta_P(\gamma, \zeta)$  is decreasing and continuously differentiable over  $(\zeta_P, \bar{\zeta}_P(\gamma))$  such that  $\beta_P(\zeta_P) = 1$ ,  $\beta_P(\bar{\zeta}_P(\gamma)) = 0$  and for  $\zeta \in (\zeta_P, \bar{\zeta}_P(\gamma))$ , we have*

$$\partial_\zeta \beta_P(\gamma, \zeta) = - \frac{\partial_\zeta p_{k_0}(\gamma, \beta_P(\zeta), \zeta)}{\partial_\beta p_{k_0}(\gamma, \beta_P(\zeta), \zeta)} \quad (\text{B.1})$$

*Also, for  $\zeta \in (\zeta_P, \bar{\zeta}_P(\gamma))$ , the function  $\gamma \mapsto \beta_P(\gamma, \zeta)$  is increasing and continuously differentiable over  $(0, \frac{(P-k_0)^2}{m})$  such that*

$$\partial_\gamma \beta_P(\gamma, \zeta) = - \frac{\partial_\gamma p_{k_0}(\gamma, \beta_P(\zeta), \zeta)}{\partial_\beta p_{k_0}(\gamma, \beta_P(\zeta), \zeta)} \quad (\text{B.2})$$

### Proof

Let  $(\gamma, \zeta_1), (\gamma, \zeta_2) \in \Omega_P$  such that  $\zeta_1 < \zeta_2$ , then by definition of  $\beta_P$ , we have

$$p_{k_0}(\gamma, \beta_P(\gamma, \zeta_2), \zeta_2) = P = p_{k_0}(\gamma, \beta_P(\gamma, \zeta_1), \zeta_1)$$

On the other hand, by proposition 3.2, we know that for  $(\gamma, \beta) \in [0, 1] \times [0, 1]$ , the function  $\zeta \mapsto p_{k_0}(\gamma, \beta, \zeta)$  is increasing. Therefore

$$p_{k_0}(\gamma, \beta_P(\gamma, \zeta_2), \zeta_1) \leq P.$$

This means that  $\beta_P(\gamma, \zeta_2) \in \{\beta \in [0, 1] ; p_{k_0}(\gamma, \beta, \zeta_1) \leq P\}$ . We deduce that  $\beta_P(\gamma, \zeta_2) \leq \beta_P(\gamma, \zeta_1)$ .

Now, for  $\epsilon$  small enough, we have

$$\begin{aligned} p_{k_0}(\gamma, \beta_P(\zeta + \epsilon), \zeta + \epsilon) &= p_{k_0}(\gamma, \beta_P(\zeta), \zeta) + \\ &\quad \epsilon [\partial_\zeta p_{k_0}(\gamma, \beta_P(\zeta), \zeta) + \partial_\zeta \beta_P(\gamma, \zeta) \partial_\beta p_{k_0}(\gamma, \beta_P(\zeta), \zeta)] + \mathcal{O}(\epsilon^2) \end{aligned}$$

Note that  $p_{k_0}(\gamma, \beta_P(\zeta + \epsilon), \zeta + \epsilon) = p_{k_0}(\gamma, \beta_P(\zeta), \zeta) = P$ , so by letting  $\epsilon$  go to 0 we find (B.1).

**Lemma B.2.** *The mapping  $\beta_F$  is continuous over  $\Omega_F$ . Furthermore, for  $\gamma \in [0, \frac{F^2}{m})$  the*

function  $\zeta \mapsto \beta_F(\gamma, \zeta)$  is decreasing and continuously differentiable over  $(\zeta_F, \bar{\zeta}_F(\gamma))$  such that  $\beta_F(\zeta_F) = 1$ ,  $\beta_F(\bar{\zeta}_F(\gamma)) = 0$  and for  $\zeta \in (\zeta_F, \bar{\zeta}_F(\gamma))$ , we have

$$\partial_\zeta \beta_F(\gamma, \zeta) = -\frac{\partial_\zeta F_{VIX}(\gamma, \beta_F(\zeta), \zeta)}{\partial_\beta F_{VIX}(\gamma, \beta_F(\zeta), \zeta)}. \quad (\text{B.3})$$

Also, for  $\zeta \in (\zeta_F, \bar{\zeta}_F(\gamma))$ , we have

$$\partial_\gamma \beta_F(\gamma, \zeta) = -\frac{\partial_\gamma F_{VIX}(\gamma, \beta_F(\zeta), \zeta)}{\partial_\beta F_{VIX}(\gamma, \beta_F(\zeta), \zeta)}. \quad (\text{B.4})$$

## Proof of theorem 4.1

Let's consider the maps  $\bar{\zeta}_P$  and  $\bar{\zeta}_F$  defined above and denote by

$$\gamma^* := \sup \left\{ \gamma \in [0, \frac{(P-k_0)^2}{m}) ; \bar{\zeta}_F(\gamma) \geq \bar{\zeta}_P(\gamma) \right\}.$$

Since  $k_0 - F < P$ , then  $\frac{(k_0-P)^2}{m} < \frac{F^2}{m}$ . In particular  $\bar{\zeta}_F$  is continuous at  $\frac{(P-k_0)^2}{m}$ . In the other hand, we know that  $\lim_{\gamma \rightarrow \frac{(P-k_0)^2}{m}} \bar{\zeta}_P(\gamma) = +\infty$ : This means that  $\exists \gamma_0 < \frac{(P-k_0)^2}{m}$ , such that

$$\forall \gamma \in [\gamma_0, \frac{(P-k_0)^2}{m}), \quad \bar{\zeta}_P(\gamma) > \bar{\zeta}_F(\frac{(P-k_0)^2}{m}) > \bar{\zeta}_F(\gamma).$$

Hence

$$\gamma^* \leq \gamma_0 < \frac{(P-k_0)^2}{m}.$$

In particular, we have

$$\bar{\zeta}_F(\gamma) \leq \bar{\zeta}_P(\gamma), \quad \forall \gamma \in [\gamma^*, \frac{(P-k_0)^2}{m}). \quad (\text{B.5})$$

Now let  $\gamma \in [\gamma^*, \frac{(K_0-F)^2}{m})$ . We have  $\bar{\zeta}_F(\gamma) \leq \bar{\zeta}_P(\gamma)$ . i.e  $[\zeta_F, \bar{\zeta}_F(\gamma)] \subset [\zeta_P, \bar{\zeta}_P(\gamma)]$ . This means that both  $\beta_P(\gamma, \cdot)$  and  $\beta_F(\gamma, \cdot)$  are well defined over  $[\zeta_F, \bar{\zeta}_F(\gamma)]$ . In particular, we have

$$\begin{cases} \beta_P(\gamma, \zeta_F) \leq 1 = \beta_F(\gamma, \zeta_F), \\ \beta_P(\gamma, \bar{\zeta}_F(\gamma)) \geq 0 = \beta_F(\gamma, \bar{\zeta}_F(\gamma)). \end{cases}$$

So there exists  $\zeta \in [\zeta_F, \bar{\zeta}_F(\gamma)]$  such that  $\beta_F(\gamma, \zeta) = \beta_P(\gamma, \zeta)$ .

On the other, let  $\gamma \in [\frac{(K_0-F)^2}{m}, \frac{F^2}{m})$ . Both  $\beta_P(\gamma, \cdot)$  and  $\beta_F(\gamma, \cdot)$  are well defined over  $[\zeta_F, \bar{\zeta}_F(\gamma)]$  and satisfy

$$\begin{cases} \beta_P(\gamma, \zeta_F) \leq 1 = \beta_F(\gamma, \zeta_F), \\ \beta_P(\gamma, \bar{\zeta}_F(\gamma)) \geq 0 = \beta_F(\gamma, \bar{\zeta}_F(\gamma)). \end{cases}$$

Hence, there exists  $\zeta = \zeta^*(F, P, \gamma) \in [\zeta_F, \bar{\zeta}_F(\gamma)]$  such that  $\beta_F(\gamma, \zeta) = \beta_P(\gamma, \zeta)$ .

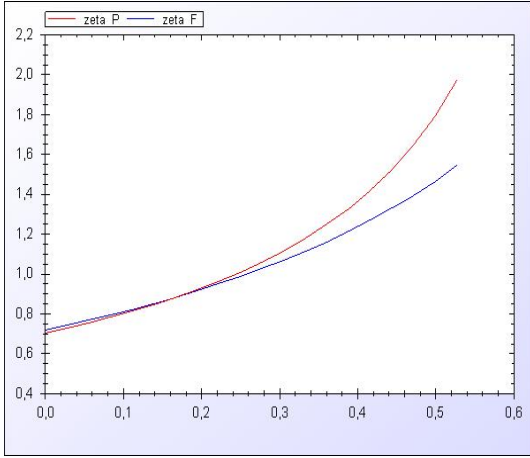


Figure 2:  $\bar{\zeta}_F$  vs  $\bar{\zeta}_P$

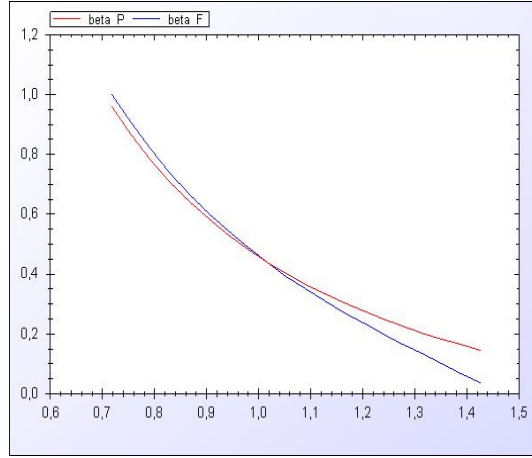


Figure 3:  $\beta_F$  vs  $\beta_P$ , where  $\gamma = 0.5 > \gamma^*$

Now consider the function  $F \mapsto \zeta^*(F, \gamma, \zeta)$ . First, it is easy to check that

$$\partial_F \beta_F(\gamma, \zeta) = \frac{1}{\partial_{\beta} F_{VIX}(\gamma, \beta_F(\gamma, \zeta), \zeta)},.$$

Let's take  $\epsilon$  small enough. The Taylor expansion of  $\beta_F(\zeta)$  with respect to  $F$  gives

$$\beta_{F+\epsilon}(\gamma_{F+\epsilon}, \zeta) = \beta_F(\gamma_F, \zeta) + \frac{\epsilon}{\partial_{\beta} F_{VIX}|_{(\gamma_F, \beta_F(\gamma_F, \zeta), \zeta)}} + \mathcal{O}_1(\epsilon^2)$$

Now denote by  $\zeta$  and  $\zeta_\epsilon$ , respectively the unique solution of

$$\beta_F(\gamma, \zeta) = \beta_P(\gamma, \zeta) \quad \text{and} \quad \beta_{F+\epsilon}(\gamma, \zeta_\epsilon) = \beta_P(\gamma, \zeta_\epsilon).$$

In particular, we have

$$\beta_P(\gamma, \zeta_\epsilon) = \beta_P + \partial_{\zeta} \beta_P(\gamma, \zeta) \times (\zeta_\epsilon - \zeta) + \mathcal{O}_2(\|(\epsilon, \zeta_\epsilon - \zeta)^T\|^2).$$

On the other hand, we have

$$\beta_{F+\epsilon}(\gamma, \zeta_\epsilon) = \beta_F(\gamma, \zeta) + \epsilon \partial_F \beta_F(\gamma, \zeta) + \partial_\zeta \beta_F(\gamma, \zeta) \times (\zeta_\epsilon - \zeta) + \mathcal{O}_3(\|(\epsilon, \zeta_\epsilon - \zeta)^T\|^2).$$

So by identifying both equalities, we obtain

$$\frac{\zeta_\epsilon - \zeta}{\epsilon} = \frac{\partial_F \beta_F}{\partial_\zeta \beta_P(\gamma, \zeta) - \partial_\zeta \beta_F(\gamma, \zeta)} + \frac{\zeta_\epsilon - \zeta}{\epsilon} \times \mathcal{O}_4(\zeta_\epsilon - \zeta) + \mathcal{O}_5(\epsilon)$$

Thus,  $\zeta^*$  is differentiable with respect to F and its derivative is given by

$$\partial_F \zeta_\gamma = \frac{\partial_F \beta_F}{\partial_\zeta \beta_P(\gamma, \zeta) - \partial_\zeta \beta_F(\gamma, \zeta)}. \quad (\text{B.6})$$

We can show the other equalities of the theorem in the same way.

**Remark B.1.** Denote by  $\gamma_* := \inf \left\{ \gamma \in [0, \frac{(P-k_0)^2}{m}) ; \bar{\zeta}_F(\gamma) \leq \bar{\zeta}_P(\gamma) \right\}$ . We conjecture that  $\gamma^* = \gamma_*$ : This means that the curves  $\bar{\zeta}_F$  and  $\bar{\zeta}_P$  intersect only one time in  $[0, \frac{(P-k_0)^2}{m})$ . In particular, for  $\gamma \in [\gamma^* st, \frac{F^2}{m})$  the calibration is performed by using the following algorithm ( noting first that  $\gamma^* st$  can be easily computed using a Binary search )

$$\zeta_{min} = \zeta_F, \zeta_{max} = \bar{\zeta}_F(\gamma)$$

- $\zeta = \frac{\zeta_{min} + \zeta_{max}}{2}$  and compute  $\beta_F(\gamma, \zeta)$  and  $\beta_P(\gamma, \zeta)$  .
- if  $\beta_F(\gamma, \zeta) \geq \beta_P(\gamma, \zeta)$ ,  $\zeta_{min} = \zeta$ , otherwise  $\zeta_{max} = \zeta$ .
- if  $|\beta_F(\gamma, \zeta) - \beta_P(\gamma, \zeta)| \leq \text{ERROR}$ , Break.

## Proof of theorem 5.1

The price of the Put is given by

$$\mathbb{E}_t(k - VIX)_+ = p_k(t, \gamma_t, \beta_t, \zeta_t)$$

Now, since the parameters  $\gamma$ ,  $\beta$  and  $\zeta$  are calibrated to fit the future price, the Put price and the skew, the Put price is then given as

$$\mathbb{E}_t(k - VIX)_+ = p_k(t, \gamma_t, \beta_{F_t}(\gamma_t, \zeta^*(\gamma_t)), \zeta^*(\gamma_t))$$

where  $\gamma_t = \gamma(t, F_t, P_t, Q_t)$  is the solution of

$$\partial_k p_{k_0}(t, \gamma_t, \beta_{F_t}(\gamma_t, \zeta^*(\gamma_t), \zeta^*(\gamma_t))) = Q_t.$$

That means that the price of the put is given as a function of the price of the Put with strike  $k_0$ , the future price and  $D$  as

$$\mathbb{E}_t(k - VIX)_+ = \varphi(t, M_t, F_t, P_t, D_t) \quad (\text{B.7})$$

where

$$\varphi(M, F, P, D) = p_k(t, M; \beta_F(\gamma, \zeta^*(\gamma(M, F, P, D), F, P)), \zeta^*(M; \gamma(M, F, P, D), F, P))$$

The derivatives of  $\varphi$  with respect to  $F$ ,  $P$  and  $D$  can be easily computed by using the theorem 4.1 as

$$\partial_F \varphi = k'_0(F) \partial_k p_k + \partial_F \zeta^* \partial_\zeta p_k + \partial_F \beta_F \partial_\beta p_k + \partial_F \gamma [\partial_\gamma p_k \partial_\gamma \zeta^* (\partial_\zeta p_k + \partial_\zeta \beta_F \partial_\beta p_k)],$$

$$\partial_P \varphi = \partial_P \zeta^* \partial_\zeta p_k + \partial_P \gamma [\partial_\gamma p_k \partial_\gamma \zeta^* (\partial_\zeta p_k + \partial_\zeta \beta_F \partial_\beta p_k)],$$

and the derivatives of  $\gamma$  are obtained by differentiating (B.7). We obtain

$$\partial_D \gamma [\partial_\gamma p_{k_0 \gamma} + \partial_\gamma \zeta^* (\partial_\zeta p_{k_0 \gamma} + \partial_\zeta \beta_F \partial_\beta p_{k_0 \gamma})] = 1,$$

$$k'_0(F) \partial_k p_{k_0 \gamma} + \partial_F \zeta^* \partial_\zeta p_{k_0 \gamma} + \partial_F \beta_F \partial_\beta p_{k_0 \gamma} + \partial_F \gamma [\partial_\gamma p_{k_0 \gamma} \partial_\gamma \zeta^* (\partial_\zeta p_{k_0 \gamma} + \partial_\zeta \beta_F \partial_\beta p_{k_0 \gamma})] = 0,$$

$$\partial_P \zeta^* \partial_\zeta p_{k_0 \gamma} + \partial_P \gamma [\partial_\gamma p_{k_0 \gamma} \partial_\gamma \zeta^* (\partial_\zeta p_{k_0 \gamma} + \partial_\zeta \beta_F \partial_\beta p_{k_0 \gamma})] = 0.$$

On the other hand, in addition to  $P$ ,  $F$  and  $D$ , the price of VIX Put depends also on  $M_t$  and  $k_0$ . We can easily check that

$$\varphi(M; k, F, P, D, k_0) = \sqrt{M} \varphi(M = 1; \frac{k}{\sqrt{M}}, \frac{F}{\sqrt{M}}, \frac{P}{\sqrt{M}}, D, \frac{k_0}{\sqrt{M}})$$

In particular, we have

$$\partial_M \varphi = \frac{1}{2M} [\varphi - k \partial_k - F \partial_F - P \partial_P - k_0 \partial_{k_0}] \varphi.$$

Finally, by Ito lemma, we know that

$$d[\mathbb{E}_t(k - VIX)_+] = (\dots)dt + \partial_F \varphi dF_t + \partial_P \varphi dP_t + \partial_D \varphi dD_t + \partial_M \varphi dM_t$$

and by the martingale properties of P, F and D, the term  $(\dots)$  is then zero.