A New Rate-Optimal Series Expansion of Fractional Brownian Motion

Mohamed Ndaoud

A New Rate-Optimal Series Expansion of Fractional Brownian Motion

Mohamed Ndaoud

Department of Applied Mathematics, École Polytechnique de Paris

emails: `mohamed.ndaoud@polytechnique.edu`

Abstract

In this paper, we give a new series expansion to simulate B an fBm based on harmonic analysis of the auto-covariance function. We prove that the convergence holds in L^2 and uniformly, with a rate-optimal decay of the norm of the rest of the series in both senses. We also give a general framework of rate-optimal series expansion for a class of Gaussian processes. Finally we apply this expansion to functional quantization.

Keywords: fractional Brownian motion, Karhunen-Loève, Fourier series, fractional Orstein-Uhlenbeck, functional quantization

1 Introduction

Let $B = (B_t)_{t\in\mathbb{R}}$ be a centered Gaussian process. B is called a fractional Brownian motion (fBm) with Hurst exponent $H \in \big(0,1\big)$ if it has the following covariance structure:

$$\forall t, s \in \mathbb{R}, \quad \mathbb{E}B_sB_t = \frac{1}{2}\bigg(|t|^{2H} + |s|^{2H} - |t-s|^{2H}\bigg)$$

fBm is a self-similar process i.e $\forall c > 0,\ (B_{ct})_{t\in\mathbb{R}} \sim (c^H B_t)_{t\in\mathbb{R}}$, and has stationary increments. When H=1/2, it coincides with standard Brownian motion. Sample paths of fBm are Hölder-continuous of any order strictly less than H. One of the main challenges with fBm is simulation. The most efficient algorithms (in particular circulant embedding method in [1]) have complexity of $\big(N\log N\big)$ where N is the number of time-steps, to be compared with the linear complexity for standard Brownian motion. Besides, local refinement requires all the already-simulated dates. Alternative approximative methods involve the Karhunen-Loève expansion that we know explicitly for some processes, such as the Brownian motion, the Brownian bridge [2] and the Orstein-Uhlenbeck process [3]... Unfortunately, this expansion is not explicit for fBm.

In [4], Dzhaparidze and van Zanten discovered the following series expansion for fBm

$$B_t = \sum_{n=1}^{\infty} \frac{\sin x_n t}{x_n} X_n + \sum_{n=1}^{\infty} \frac{1-\cos y_n t}{y_n} Y_n, \quad t \in [0,1]$$

where $(X_n)_{n\geq 1}$ and $(Y_n)_{n\geq 1}$ are i.i.d centered Gaussian random variables, $(x_n)_{n\geq 1}$ the positive roots of the Bessel function J_{-H}, and $(y_n)_{n\geq 1}$ the positive roots of the Bessel function J_{1-H}. The variance of the Gaussian variables is given by: $VarX_n = 2c_H^2 x_n^{-2H} J_{1-H}^{-2}(x_n)$, $VarY_n = 2c_H^2 y_n^{-2H} J_{-H}^{-2}(y_n)$, where $c_H^2 = \pi^{-1}\Gamma(1+2H)\sin\pi H$. In their paper, they prove that this expansion is rate-optimal in the following sense:

*This work was conducted while the author was doing internship at Bloomberg LP New York.

Definition 1. *Let B^H be an fBm with Hurst exponent H. The series expansion*

$$B^H = \sum_{i=0}^{\infty} Z_i e_i$$

where $(Z_i)_{i\in\mathbb{N}}$ are independent Gaussian random variables and $(e_i)_{i\in\mathbb{N}}$ continuous deterministic functions, is said to be uniformly rate-optimal if

$$\mathbb{E} \sup_{t\in[0,T]} \left| \sum_{i=N}^{\infty} Z_i e_i(t) \right| \underset{N\to\infty}{\sim} AN^{-H}\sqrt{\log N}$$

for some $A > 0$.

The rate-optimality also means that there can not be another series expansion of fBm with a faster rate of convergence. We show further how the rate-optimality implies uniform and almost-sure convergence of the series.

In [5], Igloi gives another rate-optimal series expansion for fBm in the case H>1/2 which is similar to our representation in that it is based on the same frequencies. This expansion is of the form

$$B_t = a_0 t X_0 + \sum_{k=1}^{\infty} a_k \bigg(\sin(k\pi t) X_k + (1 - \cos(k\pi t)) X_{-k} \bigg), \quad t \in [0,1]$$

where

$$a_0 = \sqrt{\frac{\Gamma(2-2H)}{B(H-\frac{1}{2}, \frac{3}{2}-H)(2H-1)}},$$

$$\forall k \in \mathbb{N}^*, \quad a_k = \sqrt{\frac{\Gamma(2-2H)}{B(H-\frac{1}{2}, \frac{3}{2}-H)(2H-1)} 2\Re(i \exp^{-i\pi H} \gamma(2H-1, ik\pi))(k\pi)^{-H-\frac{1}{2}}},$$

and $(X_k)_{k\in\mathbb{Z}}$ are i.i.d standard Gaussian random variables.

Even if this representation is easier to evaluate than the previous one, it still requires special functions. In this paper, we give a constructive representation of fBm for all H which is only based on Fourier series. Our approach is inspired from the Karhunen-Loève expansion where it replaces the eigenvalues given in this expansion by some adapted positive coefficients. It is of the form

$$B_t = \sqrt{c_0} t Z_0 + \sum_{k=1}^{\infty} \sqrt{\frac{-c_k}{2}} \left(\sin\frac{k\pi t}{T} Z_k + \left(1 - \cos\frac{k\pi t}{T}\right) Z_{-k} \right), \quad t \in [0,T]$$

where

$$\begin{cases} c_0 := 0, \quad H < 1/2 \\ c_0 := HT^{2H-2}, \quad H > 1/2 \end{cases} \tag{1}$$

and

$$\forall k \geq 1 \begin{cases} c_k := \frac{2}{T}\int_0^T t^{2H} \cos\frac{k\pi t}{T} \mathrm{d}t, \quad H < 1/2 \\ c_k := -\frac{4H(2H-1)T}{(k\pi)^2}\int_0^T t^{2H-2} \cos\frac{k\pi t}{T} \mathrm{d}t, \quad H > 1/2 \end{cases} \tag{2}$$

The First section is devoted to showing some useful lemmas. In the second section, we will present our series expansion, where we will prove both uniform convergence and rate-optimality. Finally, we will generalize this series expansion to a class of auto-covariance functions, before applying it to functional quantization.

2 Preliminaries

For $T > 0$, let us define

$$c_k := \frac{2}{T}\int_0^T \gamma(t)\cos\frac{k\pi t}{T}\mathrm{d}t \tag{3}$$

In this section we consider γ a continuously differentiable, increasing and concave function in (0,T]. We will also assume that $\gamma'(x) \sim_{x\to 0^+} \frac{A}{x^\delta}$ for some $\delta \in (0,2)$ and $A > 0$.

Lemma 1. *Considering the coefficient c_k in (3), the following properties hold:*

i. $(c_k)_{k\geq 0}$ *is well defined*

ii. $\forall k \in \mathbb{N}^*,\ \ c_k < 0$

iii. $c_k \sim_{k\to\infty} \frac{C}{k^{2-\delta}}$, $C < 0$

Proof. i. We first show that γ is integrable. Since $\gamma'(x) \sim_{x\to 0^+} \frac{A}{x^\delta}$, there exists $M > 0$ and $\epsilon > 0$ such that

$$\forall x \in (0,\epsilon),\ |\gamma'(x)| \leq \frac{M}{x^\delta} \quad . \tag{4}$$

By integrating (4), we have $\gamma(x) = \underset{x\to 0^+}{O}\left(x^{1-\delta}\right)$. Moreover γ is continuous in (0,T], it comes out that γ is integrable on (0,T]. It is then immediate that $(c_k)_{k\in\mathbb{N}^*}$ is well defined.

ii. Before showing the second result, one may first notice that γ' is positive and decreasing since γ is concave and increasing. By a change of variable in (3), we get

$$c_k = \frac{2}{T}\frac{T}{k\pi}\int_0^{k\pi} \gamma\left(\frac{Tu}{k\pi}\right)\cos(u)\mathrm{d}u \quad ,$$

and by integrating by parts we get

$$\begin{aligned} c_k &= -\frac{2}{T}\left(\frac{T}{k\pi}\right)^2 \int_0^{k\pi} \gamma'\left(\frac{Tu}{k\pi}\right)\sin(u)\mathrm{d}u \\ &= -\frac{2}{T}\left(\frac{T}{k\pi}\right)^2 \sum_{n=0}^{k-1}(-1)^n \int_0^{\pi} \gamma'\left(\frac{T(u+n\pi)}{k\pi}\right)\sin(u)\mathrm{d}u \quad . \end{aligned} \tag{5}$$

Let us define $v_{k,n} := \int_0^\pi \gamma'\left(\frac{T(u+n\pi)}{k\pi}\right)\sin(u)\mathrm{d}u$. It is immediate that $\forall k \in \mathbb{N}^*,\ (v_{k,n})_{n<k}$ is positive and decreasing with n. By regrouping each pair of elements in the sum we get

$$c_k = -\frac{2}{T}\left(\frac{T}{k\pi}\right)^2\left(\sum_{n=0}^{\lfloor\frac{k}{2}\rfloor - 1}(v_{k,2n} - v_{k,2n+1}) + \frac{1-(-1)^k}{2}v_{k,k-1}\right)$$

As a consequence $\forall k \in \mathbb{N}^*,\ \ c_k < 0$

iii. For the last point, it is sufficient to show that

$$\sum_{n=0}^{\lfloor\frac{k}{2}\rfloor}(v_{k,2n} - v_{k,2n+1}) \underset{k\to\infty}{\sim} Ck^\delta,\ C > 0 \quad , \tag{6}$$

because γ is differentiable in T^- and

$$\left|\frac{1-(-1)^k}{2}v_{k,k-1}\right| \leq 2\gamma'\left(\frac{T(k-1)}{k}\right) \underset{k\to\infty}{\rightarrow} 2\gamma'(T) \quad .$$

Now to prove (6), we can prove that

$$\lim_{k\to\infty}\sum_{n=0}^{\infty}\frac{(v_{k,2n}-v_{k,2n+1})}{k^\delta}1_{(n\le\lfloor\frac{k}{2}\rfloor)}=C,\ C>0\quad.$$

Let $n\in\mathbb{N}$,

$$\begin{aligned}\lim_{k\to\infty}\frac{v_{k,n}}{k^\delta}&=\lim_{k\to\infty}\int_0^\pi\frac{\sin(u)}{k^\delta}\gamma'\left(\frac{T(u+n\pi)}{k\pi}\right)\mathrm{d}u\\&=\left(\frac{\pi}{T}\right)^\delta\lim_{k\to\infty}\int_0^\pi\frac{\sin(u)}{(u+n\pi)^\delta}\left(\frac{T(u+n\pi)}{k\pi}\right)^\delta\gamma'\left(\frac{T(u+n\pi)}{k\pi}\right)\mathrm{d}u\\&=A\left(\frac{\pi}{T}\right)^\delta\int_0^\pi\frac{\sin(u)}{(u+n\pi)^\delta}\mathrm{d}u=(-1)^nA\left(\frac{\pi}{T}\right)^\delta\int_{n\pi}^{(n+1)\pi}\frac{\sin(u)}{u^\delta}\mathrm{d}u\end{aligned}\tag{7}$$

Where the last equality holds because the integrand is positive and its limit is still integrable since $\delta\in(0,2)$. It follows that

$$\lim_{k\to\infty}\sum_{n=0}^{\infty}\frac{(v_{k,2n}-v_{k,2n+1})}{k^\delta}1_{(n\le\lfloor\frac{k}{2}\rfloor)}=A\left(\frac{\pi}{T}\right)^\delta\int_0^\infty\frac{\sin(u)}{u^\delta}\mathrm{d}u$$ □

Remark 1. *If γ is only continuously differentiable, increasing and concave in (0,T] then we only have that*

$$\forall k\in\mathbb{N}^*,\quad c_k\le 0$$

If moreover γ' has a finite limit in 0^+ (i.e $\delta=0$), then

$$c_k=O\left(\frac{1}{k^2}\right)$$

The last point comes from (5) since the terms in the sum are alternating signs, decreasing in norm and uniformly bounded.

When $\delta\in(0,1)$, γ has a finite limit in 0^+, we will consider then $\gamma(0):=\lim_{x\to0^+}\gamma(x)$

Lemma 2. *If $\delta\in[0,1)$, we have*

$$\forall t\in[-T,T],\ \gamma(|t|)=\gamma(0)+\sum_{k=1}^{\infty}c_k\left(\cos\frac{k\pi t}{T}-1\right)$$

Proof. Let g: $\mathbb{R}\to\mathbb{R}$ a function such that: $\forall t\in[-T,T],g(t)=\gamma(|t|)$ and let us extend g into a 2T-periodic function. Since g is an even function, its Fourier series is even and

$$\forall t\in[-T,T],\ g(t)=\sum_{k=0}^{\infty}c_k\cos\frac{k\pi t}{T}\tag{8}$$

where $c_0=\frac{1}{T}\int_0^T\gamma(t)\mathrm{d}t$ and $\forall k>0,\ c_k=\frac{2}{T}\int_0^T\gamma(t)\cos\frac{k\pi t}{T}\mathrm{d}t$.

Since $c_k=\mathrm{O}_{k\to\infty}\left(\frac{1}{k^{2-\delta}}\right)$ with $0\le\delta<1$, the series converges normally and therefore also uniformly. Replacing t by 0 in (8) we get : $c_0=\gamma(0)-\sum_{k=1}^{\infty}c_k$, and finally

$$g(t)=\gamma(0)+\sum_{k=1}^{\infty}c_k\left(\cos\frac{k\pi t}{T}-1\right)$$

The main result follows immediately. □

Lemma 3. *Let $(\lambda_k)_{k\in\mathbb{N}}$ be a sequence of real numbers, $(Z_k)_{k\in\mathbb{N}}$ centered standard Gaussian variables, and $(e_k)_{k\in\mathbb{N}}$ a family of continuous functions on [0,T]. Under the conditions*

- $\lambda_k = \underset{k\to\infty}{O}\left(\frac{1}{k^{H+1/2}}\right)$, *for some* $H > 0$
- $\exists L > 0, \forall k \in \mathbb{N}, \forall s,t \in [0,T], \quad |e_k(t) - e_k(s)| \leq L\,|t-s|$

we get that

$$\mathbb{E}\sup_{t\in[0,T]}\left|\sum_{k=N}^{\infty}\lambda_k e_k\left(\frac{k\pi t}{T}\right)Z_k\right| = \underset{N\to\infty}{O}\left(N^{-H}\sqrt{\log N}\right)$$

and the series $\sum_{k=0}^{N}\lambda_k e_k\left(\frac{k\pi\cdot}{T}\right)Z_k$ converges almost surely, uniformly in the space of continuous functions on [0,T].

Proof. For the proof see **Appendix A.** □

3 New rate-optimal series expansion

Since our approach does not hold for both cases, we will give separately the expansion for both fBm with $H < 1/2$ and $H > 1/2$ assuming that the series converge. We will prove after the convergence and rate-optimality of these series.

3.1 The series expansion

For the following Theorem we denote $c_k := \frac{2}{T}\int_0^T t^{2H}\cos\frac{k\pi}{T}\mathrm{d}t$ where $H < 1/2$

Theorem 1. *Let $H \in \left(0,\frac{1}{2}\right)$. B is the random process given by*

$$\forall t \in [0,T], \quad B_t = \sum_{k=1}^{\infty}\sqrt{-\frac{c_k}{2}}\left(\sin\frac{k\pi t}{T}Z_k + \left(1-\cos\frac{k\pi t}{T}\right)Z_{-k}\right) \quad .$$

where $(Z_k)_{k\in\mathbb{Z}}$ are independent standard Gaussian variables, then

$$\forall (s,t) \in [0,T]^2,\ \mathbb{E}B_sB_t = \frac{1}{2}\left(t^{2H} + s^{2H} - |t-s|^{2H}\right)$$

.

Proof. For H $\in \left(0,\frac{1}{2}\right)$, $\gamma(t) = |t|^{2H}$ satisfies all the assumptions enumerated in the previous section. The previous lemmas apply then, and the series is well-defined since $\forall k \geq 1,\ c_k < 0$. Because of the orthonormality of $((Z_k)_{k>0}\cup(Z_k)_{k<0})$ in the probability space, it follows immediately that

$$\begin{aligned}\mathbb{E}B_sB_t &= \sum_{k=1}^{\infty}-\frac{c_k}{2}\left(\sin\frac{k\pi s}{T}\sin\frac{k\pi t}{T} + \left(1-\cos\frac{k\pi s}{T})(1-\cos\frac{k\pi t}{T}\right)\right)\\ &= \sum_{k=1}^{\infty}-\frac{c_k}{2}\left(1-\cos\frac{k\pi s}{T} - \cos\frac{k\pi t}{T} + \cos\frac{k\pi(t-s)}{T}\right)\\ &= \sum_{k=1}^{\infty}\frac{c_k}{2}\left((\cos\frac{k\pi s}{T}-1) + (\cos\frac{k\pi t}{T}-1) - (\cos\frac{k\pi(t-s)}{T}-1)\right)\end{aligned} \tag{9}$$

And we can conclude using **Lemma 2**. □

We now give a new series expansion for fBm with $H > 1/2$ since the previous series expansion doesn't hold because Fourier coefficients in this case have alternating signs. When looking more in detail into this change in sign we figure out that it is due to the fact that the derivative of the auto-covariance function is not continuous on the borders -T and T which is partially due to the fact that the initial function is not periodic. In order to deal with this problem, we are adding a parabola that we set in order to compensate the discontinuity of the function. If we consider γ a function twice differentiable such as $\gamma'(0) \neq \gamma'(T)$ and γ'' integrable on (0,T), and a function f defined as follows:

$$\forall t \in [0,T], \quad f(t) = \gamma(t) - \frac{\gamma'(T)-\gamma'(0)}{2T}\left(t - \frac{T\gamma'(0)}{\gamma'(T)-\gamma'(0)}\right)^2$$

It comes directly that $f'(0) = f'(T) = 0$. We then get that $\forall k \in \mathbb{N}^*$:

$$\begin{aligned}\int_0^T f(t)\cos\left(\frac{k\pi t}{T}\right)\mathrm{d}t &= \frac{T}{k\pi}\left[f(t)\sin\left(\frac{k\pi t}{T}\right)\right]_0^T - \frac{T}{k\pi}\int_0^T f'(t)\sin\left(\frac{k\pi t}{T}\right)\mathrm{d}t \\ &= \left(\frac{T}{k\pi}\right)^2\left[f'(t)\cos\left(\frac{k\pi t}{T}\right)\right]_0^T - \left(\frac{T}{k\pi}\right)^2\int_0^T f''(t)\cos\left(\frac{k\pi t}{T}\right)\mathrm{d}t \\ &= -\left(\frac{T}{k\pi}\right)^2\int_0^T \gamma''(t)\cos\left(\frac{k\pi t}{T}\right)\mathrm{d}t\end{aligned} \tag{10}$$

where the last equality derives from the orthogonality between constants and harmonics. We can now derive a new series expansion for fBm with $H > 1/2$. For the next theorem, we will denote by $c_k := \frac{2}{T}2H(2H-1)\int_0^T t^{2H-2}\cos\left(\frac{k\pi t}{T}\right)\mathrm{d}t$ where $H > 1/2$.

Theorem 2. *Let $H \in \left(\frac{1}{2},1\right)$. $(B_t)_{t\in[0,T]}$ is a stochastic process defined by the series expansion:*

$$\forall t \in [0,T], \quad B_t = \sqrt{HT^{2H-2}}tZ_0 + \sum_{k=1}^{\infty}\frac{T}{k\pi}\sqrt{\frac{c_k}{2}}\left(\sin\frac{k\pi t}{T}Z_k + \left(1-\cos\frac{k\pi t}{T}\right)Z_{-k}\right)$$

where $(Z_k)_{k\in\mathbb{Z}}$ independent standard Gaussian variables. We have

$$\forall (s,t) \in [0,T]^2,\ \mathbb{E}B_sB_t = \frac{1}{2}\left(t^{2H} + s^{2H} - |t-s|^{2H}\right)$$

Proof. By considering $\gamma(t) = t^{2H}$, we have that γ is convex and decreasing on (0,T), and moreover $\gamma'(t) \underset{t\to 0^+}{\sim} \frac{-A}{t^{3-2H}}$ for some $A > 0$. Since $3-2H < 2$ we get using **Lemma 1.** that $\forall k \geq 1,\ c_k > 0$ and $c_k \underset{k\to\infty}{\sim} \frac{C}{k^{2H-1}}$ for some $C > 0$. Using (10) we get that

$$\frac{2}{T}\int_0^T \left(t^{2H} - HT^{2H-2}t^2\right)\cos\left(\frac{k\pi t}{T}\right)\mathrm{d}t = -\left(\frac{T}{k\pi}\right)^2 c_k$$

Since $\frac{c_k}{k^2} \underset{k\to\infty}{\sim} \frac{C}{k^{2H+1}}$ the Fourier series converges uniformly and we can then use **Lemma 2.** and obtain

$$\forall t \in [-T,T], \quad |t|^{2H} = HT^{2h-2}t^2 - \sum_{k=1}^{\infty}\left(\frac{T}{k\pi}\right)^2 c_k\left(\cos\frac{k\pi t}{T} - 1\right)$$

Since $c_k > 0$ the series is well defined and

$$
\begin{aligned}
\mathbb{E}B_tB_s &= HT^{2H-2}ts + \frac{1}{2}\sum_{k=1}^{\infty}\left(\frac{T}{k\pi}\right)^2 c_k \left(\sin\frac{k\pi t}{T}\sin\frac{k\pi s}{T} + \left(1-\cos\frac{k\pi t}{T}\right)\left(1-\cos\frac{k\pi s}{T}\right)\right)\\
&= HT^{2H-2}ts + \frac{1}{2}\sum_{k=1}^{\infty}\left(\frac{T}{k\pi}\right)^2 c_k \left(1-\cos\frac{k\pi t}{T} - \cos\frac{k\pi s}{T} + \cos\frac{k\pi(t-s)}{T}\right)\\
&= \frac{1}{2}\left(HT^{2H-2}(t^2+s^2-(t-s)^2) - \sum_{k=1}^{\infty}\left(\frac{T}{k\pi}\right)^2 c_k \left(\cos\frac{k\pi t}{T} + \cos\frac{k\pi s}{T} - \cos\frac{k\pi(t-s)}{T} - 1\right)\right)\\
&= \frac{|t|^{2H}+|s|^{2H}-|t-s|^{2H}}{2}
\end{aligned}
\tag{11}
$$

□

3.2 Convergence and rate-optimality

After giving an explicit representation of fBm, we will prove that it converges uniformly and in the mean square sense. We will also show its uniform rate-optimality. We now denote more precisely

$$
\begin{cases}
c_0 := 0, \quad H < 1/2\\
c_0 := HT^{2H-2}, \quad H > 1/2
\end{cases}
\tag{12}
$$

and

$$
\forall k \geq 1 \begin{cases}
c_k := \frac{2}{T}\int_0^T t^{2H}\cos\frac{k\pi t}{T}\mathrm{d}t, \quad H < 1/2\\
c_k := -\frac{4H(2H-1)T}{(k\pi)^2}\int_0^T t^{2H-2}\cos\frac{k\pi t}{T}\mathrm{d}t, \quad H > 1/2
\end{cases}
\tag{13}
$$

One may first notice that $c_k \underset{k\to\infty}{\sim} \frac{C^H}{k^{2H+1}}$. We will now consider the series expansion of the last section

$$\forall t \in [0,T],\ B_t = \sqrt{c_0}tZ_0 + \sum_{k=1}^{\infty}\sqrt{-\frac{c_k}{2}}\left(\sin\frac{k\pi t}{T}Z_k + \left(1-\cos\frac{k\pi t}{T}\right)Z_{-k}\right)$$

The following theorems show the uniform convergence of the series in mean square and almost surely. In this section we define the truncated series of B as follows:

$$B_t^N = \sqrt{c_0}tZ_0 + \sum_{k=1}^{N}\sqrt{-\frac{c_k}{2}}\left(\sin\frac{k\pi t}{T}Z_k + \left(1-\cos\frac{k\pi t}{T}\right)Z_{-k}\right)$$

Theorem 3. *B_t^N converges in mean square, and the rate of convergence is given by*

$$\sup_{t\in[0,T]}\sqrt{\mathbb{E}\big(B_t - B_t^N\big)^2} = \underset{N\to\infty}{\mathrm{O}}\left(N^{-H}\right)$$

Proof. As we did previously, we will use the orthonormality of the Gaussian random variables (Z_k), it comes that $\forall$ t $\in [0,T]$

$$\mathbb{E}\big(B_t - B_t^N\big)^2 = \sum_{k>N} -\frac{c_k}{2}\left((\sin\frac{k\pi t}{T})^2 + (1-\cos\frac{k\pi t}{T})^2\right) = \sum_{k>N} -c_k\left(1-\cos\frac{k\pi t}{T}\right)$$

then

$$\sup_{t\in[0,T]}\sqrt{\mathbb{E}\big(B_t - B_t^N\big)^2} \leq \sqrt{\sum_{k>N} -c_k}$$

Since for some C > 0

$$-c_k \underset{k\to\infty}{\sim} \frac{C}{k^{2H+1}} \underset{k\to\infty}{\sim} 2HC\left(\frac{1}{k^{2H}} - \frac{1}{(k+1)^{2H}}\right)$$

It comes by the comparison of the residuals of positive convergent series that

$$\sqrt{\sum_{k>N} -c_k} \underset{N\to\infty}{\sim} \frac{\sqrt{2HC}}{N^H} \quad .$$

$\square$

Theorem 4. *Almost surely, B_t^N converges uniformly, and the rate of convergence is given by*

$$\mathbb{E} \sup_{t\in[0,T]} \left|B_t - B_t^N\right| \underset{N\to\infty}{\sim} AN^{-H}\sqrt{\log(N)},\ A > 0$$

Proof. We will only need to prove that the rate of convergence of the series given here is faster than the second part since the later is the optimal rate for fBm as shown in [6]. By truncating the series, we have

$$B_t - B_t^N = \sum_{k=N+1}^{\infty} \sqrt{-\frac{c_k}{2}} \left(\sin\frac{k\pi t}{T} Z_k + \left(1 - \cos\frac{k\pi t}{T}\right) Z_{-k}\right)$$

Since $\sqrt{-\frac{c_k}{2}} = \underset{k\to\infty}{O}\left(\frac{1}{k^{H+1/2}}\right)$ and the fact that $t \to \sin(t)$ and $t \to 1 - \cos(t)$ are both 1-Lipschitz functions we can directly use **Lemma 3.** to conclude the proof. $\square$

The last theorem gives the rate-optimality of our expansion

4 Generalization to a class of Gaussian processes

In this section we give an optimal-rate series expansion for a class of Gaussian processes. We will denote $c_k = \frac{2}{T}\int_0^T \gamma(t)\cos\frac{k\pi t}{T}\mathrm{d}t$. We will first need a proposition given in [7]. Since the trigonometric sequence we use in our expansion is admissible in the sense given by the author, then according to **Proposition 4.** in [7] if

$$|c_k| \underset{k\to\infty}{\sim} \frac{A}{k^\zeta}, \quad \zeta > 1, A > 0 \tag{14}$$

then the series is rate-optimal.

Theorem 5. *Let $(X_t)_{t\in[0,T]}$ be a centered Gaussian process with stationnary increments and γ the auto-covariance function of the increments. Assume that γ is continuously differentiable, concave and increasing on (0,T]. If $\gamma'(x) \sim_{x\to 0} \frac{A}{x^\delta}$ with $\delta \in (0,1)$ and $A > 0$, then the series*

$$Y_t = \sum_{k=1}^{\infty} \sqrt{\frac{-c_k}{2}} \left(\sin\frac{k\pi t}{T} Z_k + \left(1 - \cos\frac{k\pi t}{T}\right) Z_{-k}\right), \quad t \in [0,T]$$

converges uniformly in [0,T], almost surely. It is moreover a rate-optimal expansion for X.

Proof. We may first notice that $\gamma(0) = 0$ since γ is the auto-covariance function of the increments. by using **Lemma 1.** again we obtain that

$$-c_k \underset{k\to\infty}{\sim} \frac{A}{k^{2-\delta}}, \quad , A > 0$$

since $\delta \in (0,1)$ we can use **Lemma 3.** to get that

$$\mathbb{E} \sup_{t\in[0,T]} \left|\sum_{k=N+1}^{\infty} \sqrt{\frac{-c_k}{2}} \left(\sin\frac{k\pi t}{T} Z_k + \left(1 - \cos\frac{k\pi t}{T}\right) Z_{-k}\right)\right| \underset{k\to\infty}{\sim} \frac{C\log(N)}{N^{\frac{1+\delta}{2}}}, \quad C > 0$$

and that the series converges almost surely and uniformly in [0,T], and

$$\forall s,t \in [0,T], \quad \mathbb{E}Y_sY_t = \sum_{k=1}^{\infty} \frac{-c_k}{2}\left(1 - \cos\frac{k\pi t}{T} - \cos\frac{k\pi s}{T} + \cos\frac{k\pi(t-s)}{T}\right)$$

by using **Lemma 2.** we can conclude that

$$\forall s,t \in [0,T], \quad \mathbb{E}Y_sY_t = \frac{1}{2}\left(\gamma(t) + \gamma(s) - \gamma(|t-s|)\right)$$

The rate-optimality comes directly from (14) □

For the rest of this section we consider γ a continuously differentiable, decreasing and convex function in (0,T] such that $\gamma'(x) \sim_{x\to 0} \frac{-A}{x^\delta}$ with $\delta \in (0,1)$, $A > 0$. We get the following theorem

Theorem 6. *Let $(X_t)_{t\in[0,T]}$ be a centered stationary Gaussian process and γ its auto-covariance function. Assume that $c_0 \geq 0$, then the series*

$$Y_t = \sqrt{c_0}Z_0 + \sum_{k=1}^{\infty} \sqrt{c_k}\left(\sin\frac{k\pi t}{T}Z_k + \cos\frac{k\pi t}{T}Z_{-k}\right), \quad t \in [0,T]$$

converges uniformly in [0,T], almost surely. It is moreover a rate-optimal expansion for X.

Proof. Applying **Lemma 1.** to $-\gamma$ we get that:

- $\forall k \in \mathbb{N}^*, \ c_k > 0$
- $c_k \sim_{k\to\infty} \frac{C}{k^{2-\delta}}$, $C > 0$

now that the series is well defined, we use again **Lemma 3.** to get the uniform and almost sure convergence of the series.

$$\forall s,t \in [0,T], \quad \mathbb{E}Y_sY_t = \sum_{k=0}^{\infty} c_k \cos\frac{k\pi t}{T} = \gamma(t-s)$$

Again the rate-optimality comes from (14) □

One immediate consequence is a series expansion for the stationary fractional Ornstein-Uhlenbeck with $H < 1/2$, where a stationary fOU is a centered gaussian process such as:

$$\forall s,t \in [0,T], \quad \mathbb{E}X_sX_t = e^{-|t-s|^{2H}}$$

This expansion is already given in [7].

The method we are proposing here can be applied also for stochastic processes that are neither stationary nor with stationary increments. As an example we will apply it to another class of Gaussian processes (X_t) where:

$$\forall t,s \in [0,T], \ \mathbb{E}X_sX_t = \frac{1}{2}\left(\gamma(|t-s|) - \gamma(t+s)\right) \tag{15}$$

In this case we give another rate optimal expansion

Theorem 7. *Let $(X_t)_{t\in[0,T]}$ be a centered Gaussian process with a covariance structure given in (15), then the expansion*

$$X_t = \sum_{k=1}^{\infty} \sqrt{c_k}\sin\frac{k\pi t}{2T}Z_k, \quad t \in [0,T]$$

is almost surely uniformly convergent. It is also a rate-optimal series expansion of the process.

Proof. As for the previous theorem, we still have

- $\forall k \in \mathbb{N}^*,\ c_k > 0$
- $c_k \sim_{k\to\infty} \frac{C}{k^{2-\delta}}\ ,\ C > 0$

As a consequence **Lemma 3.** give that the series converges almost surely uniformly. Moreover we obtain

$$\forall t, s \in [0,T],\ \mathbb{E}X_sX_t = \sum_{k=1}^{\infty} c_k\Bigg(\sin\frac{k\pi t}{2T}\sin\frac{k\pi s}{2T}\Bigg) = \frac{1}{2}\sum_{k=1}^{\infty} c_k\Bigg(\cos(t-s) - cos(t+s)\Bigg)$$

and get directly that

$$\forall t, s \in [0,T],\ \mathbb{E}X_sX_t = \frac{1}{2}\Bigg(\gamma(|t-s|) - \gamma(t+s)\Bigg)$$

rate-optimality is just a consequence of (14) $\square$

Remark 2. *We may notice that here we consider a 4T-periodic function instead of 2T because* $\forall 0 < t, s < T, \quad -2T \leq t-s\ ,\ s+t \leq 2T$.
*The previous expansions still apply when $\gamma'(0)$ is finite where we get the uniform and absolute convergence from **Remark 1.** but we can not conclude about the rate-optimality.*

Example 1. *Karhunen Loeve expansion of Brownian motion.*
*In this example we consider the auto-covariance function $\gamma(t) = -|t|$. This function is convex, decreasing on $[0,2T]$ such as $\gamma'(0) = 0$, we can then apply **Theorem 7.** We should first notice that this process is a Brownian motion since*

$$\forall t, s \in \mathbb{R}^+, \quad \frac{1}{2}\left(-|t-s| + |t+s|\right) = \min(t,s)$$

In this case we have

$$\begin{aligned}\forall k \in \mathbb{N}^*, \quad c_k &= \frac{1}{T}\int_0^{2T} -t\cos\frac{k\pi t}{2T}\mathrm{d}t \\ &= \frac{2}{k\pi}\int_0^{2T}\sin\frac{k\pi t}{2T}\mathrm{d}t \\ &= \left(1-(-1)^k\right)\left(\frac{2}{k\pi}\right)^2 T\end{aligned} \tag{16}$$

We then get that

$$\forall t \in [0,T], \quad X_t = \sqrt{2}\sum_{k=1}^{\infty}\frac{\sqrt{T}}{(k-\frac{1}{2})\pi}\sin\frac{(k-\frac{1}{2})\pi t}{T}Z_k$$

Example 2. *A new rate-optimal series expansion of generalized Ornstein-Uhlenbeck.*
In this example we consider the non-stationary Ornstein-Uhlenbeck process $(X_t)_{t\geq 0}$ where X_0 is distributed such as $\mathcal{N}(\mu, \sigma_0^2)$. This process can be defined as follows:

$$\forall t \geq 0, \quad \mathbb{E}X_t = \mu e^{-\theta t} + \alpha(1 - e^{-\theta t})$$

and

$$\forall s, t \geq 0, \quad cov\left(X_s, X_t\right) = \sigma_0^2 e^{-\theta(t+s)} + \frac{\sigma^2}{2\theta}\left(e^{-\theta(|t-s|)} - e^{-\theta(t+s)}\right)$$

*for some $\theta > 0$ and $\alpha, \sigma \in \mathbb{R}$. By setting $\gamma(t) = \frac{\sigma^2}{\theta} e^{-\theta|t|}$, we have that γ is convex and decreasing on [0,T], and that $\gamma'(0)$ has a finite value. We can apply again **Theorem 7.** to get the following expansion*

$$\forall t \geq 0, \quad X_t = X_0 e^{-\theta t} + \alpha(1 - e^{-\theta t}) + \sum_{k=1}^{\infty} \sqrt{c_k} \sin \frac{k\pi t}{2T} Z_k$$

where X_0 is independent from $(Z_k)_{k \geq 1}$. We can also calculate c_k explicitly as follows:

$$\begin{aligned} \forall k \geq 1, \quad \frac{\theta}{\sigma^2} c_k = \frac{1}{T} \int_0^{2T} e^{-\theta t} \cos \frac{k\pi t}{2T} &= Re \left(\frac{1}{T} \int_0^{2T} e^{(-\theta + i\frac{k\pi}{2T})t} \mathrm{d}t \right) \\ &= Re \left(\frac{1 - (-1)^k e^{-2\theta T}}{\theta T - \frac{ik\pi}{2}} \right) = \frac{1}{1 + \left(\frac{k\pi}{2\theta T}\right)^2} \frac{1 - (-1)^k e^{-2\theta T}}{\theta T} \end{aligned} \tag{17}$$

This expansion is rate-optimal, and easier to use than the one known that includes zeros of Bessel functions.

5 Application: *Functional quantization*

The quantization of a random variable X taking values in $(E, |.|)$ consists in approximating it by the best discretized random variables Y taking finite values in E. If we set N to be the maximum number of values taken by Y, the problem is equivalent to minimizing the error defined below

$$\xi_N(X, \Gamma) = \{ \| |X - Proj_\Gamma(X)| \|_2, \ \Gamma \subset E \text{ such that } |\Gamma| \leq N \ \} \tag{18}$$

A solution of (18) is an L^2-optimal quantizer of X.

For a multidimensional Gaussian random variable X real optimal-quantization is expensive. One way to mitigate this cost, is to consider the product-quantization that is to use a cartesian product of one-dimensional optimal-quantizers of each marginale as in [8]. The resulting quantizer is stationnary when marginals of X are independent. In [9], it is shown that Karhunen-Love product-quantization, while it is sub-optimal, remains rate-optimal in the case of Gaussian processes.

We consider now a stochastic continuous process $(X_t)_{t \in [0,T]}$ such that $\int_0^T \mathbb{E}|X_t^2| \mathrm{d}t < \infty$, and its expansion:

$$\forall t \in [0, T], \quad X_t = \sum_{i=0}^{\infty} \lambda_i e_i(t) Z_i$$

where $(\lambda_i)_{i \in \mathbb{N}}$ is a sequence of real numbers such that $\sum_{i=0}^{\infty} \lambda_i^2 < \infty$, $(e_i)_{i \in \mathbb{N}}$ is an orthonormal sequence of continuous functions, and $(Z_i)_{i \in \mathbb{N}}$ independent standard gaussian variables. Notice that the Karhunen-Loeve expansion is a special case of what we are introducing. In this case the error induced by replacing the process by a rate-optimal quantizer of its truncation up to m is given by:

$$\xi_N(X)^2 = \int_0^T \mathbb{E} \left(X_t - \sum_{i=0}^{m} \lambda_i e_i(t) Y_i \right)^2 \mathrm{d}t$$

where $\forall\ 0 \leq i \leq N$, Y_i is an optimal quantizer of Z_i. More precisely we get that:

$$\xi_N(X)^2 = \sum_{i=m+1}^{\infty} \lambda_i^2 + \sum_{i=0}^{m} \xi_{N_i}(\mathcal{N}(0, \lambda_i^2))$$

where $\prod_{i=0}^{m} N_i \leq N$. If moreover $\lambda_N^2 \underset{N \to \infty}{\sim} \frac{1}{N^\delta}$, with $1 < \delta < 3$, it is shown in [10] that, the optimal product-quantization of level N is achieved when the dimension of the quantizer is of order $\log N$ and then it satisfies

$$\xi_N(X) \underset{N \to \infty}{\sim} A(\log N)^{\frac{1-\delta}{2}}, \ A > 0$$

When the basis chosen in the expansion in not orthonormal, Junglen gives a method to quantize in his paper [11]. The idea consists in truncating up to an order m and considering the covariance matrix K^m of the truncation. More specifically we consider H a linear L^2-subspace defined by $H = Span((e_i)_{0\le i\le m})$ and define the operator T_{K^m}

$$T_{K^m} : \begin{cases} H \to H \\ f \to \int_0^T K^m(s,.)f(s)\mathrm{d}s \end{cases}$$

T_{K^m} is clearly an endomorphism from the definition of K^m. Unlike the Karhunen-Love theorem, in this case we deal with a linear and symmetric operator in finite dimension. Hence there exists $(\mu_i^m)_{0\le i\le m}$ a positive sequence of real numbers and $(f_i^m)_{0\le i\le m}$ an orthonormal base of H such that

$$\forall t,s \in [0,T], \quad K^m(t,s) = \sum_{i=0}^{m} \mu_i^m f_i^m(t) f_i^m(s)$$

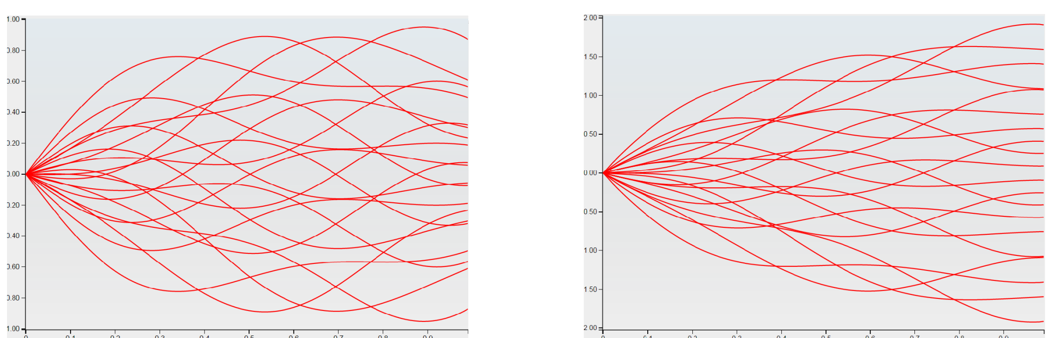
We can then assert that there exists $(Y_i^m)_{0\le i\le m}$ a sequence of standard random gaussian variables such that:

$$\forall t \in [0,T], \quad \sum_{i=0}^{m} \lambda_i e_i(t) Z_i = \sum_{i=0}^{m} \sqrt{\mu_i^m} f_i^m(t) Y_i^m \quad a.s$$

It is obvious now that if we replace the process by a rate-optimal quantizer of $\sum_{i=0}^{m} \sqrt{\mu_i^m} f_i^m(t) Y_i^m$ we will get a quadratic quantization error similar to the previous one

$$\xi_N(X)^2 = \sum_{i=m+1}^{\infty} \lambda_i^2 + \sum_{i=0}^{m} \xi_{N_i}(\mathcal{N}(0,\mu_i^2))$$

As a consequence, the quadratic quantization of level N would be optimal for $m \sim \log N$. To illustrate this, we give a rate-optimal quantization of both fBm and generalized Ornstein Uhlenbeck for T=1 and N=20.



(a) generalized OU $\theta = 2, \alpha = 0$

(b) fBm H=0.4

Figure 1: Product quantization of a centered Ornstein-Uhlenbeck process, starting from $X_0 = 0$ (left), and a fBm (right)

6 Conclusion

We have derived a new rate-optimal series expansion of fBm an far more. The advantage of this expansion is that the coefficients are easily calculated which can reduce the complexity of simulation, especially for the case $H < 1/2$ where no other trigonometric series expansion is known. We have shown that our approach can be generalized to a class of Gaussian processes, in particular to Ornstein-Uhlenbeck process. The application to quantization is interesting. Usually we need the Karhunen-Loève decomposition to have an optimal quantization because of the orthnormality of the base. In this case, we show how to deal with the non-orthonormality of our base and then get a rate-optimal quantization.

7 Acknowledgments

I would like to express my gratitude to Bruno Dupire, for giving me the opportunity to work on this subject. I also would like to thank warmly Sylvain Corlay for mentoring me during my research.

References

[1] CR Dietrich and Garry Neil Newsam. Fast and exact simulation of stationary gaussian processes through circulant embedding of the covariance matrix. *SIAM Journal on Scientific Computing*, 18(4):1088–1107, 1997.

[2] Paul Deheuvels. A karhunen–loève expansion for a mean-centered brownian bridge. *Statistics & probability letters*, 77(12):1190–1200, 2007.

[3] Sylvain Corlay et al. Functional quantization-based stratified sampling methods. *arXiv preprint arXiv:1008.4441*, 2010.

[4] Kacha Dzhaparidze and Harry Van Zanten. A series expansion of fractional brownian motion. *Probability theory and related fields*, 130(1):39–55, 2004.

[5] Endre Iglói. A rate-optimal trigonometric series expansion of the fractional brownian motion. *Electron. J. Probab*, 10:1381–1397, 2005.

[6] Thomas Kühn and Werner Linde. Optimal series representation of fractional brownian sheets. *Bernoulli*, pages 669–696, 2002.

[7] Harald Luschgy and Gilles Pagès. Expansions for gaussian processes and parseval frames. *Electron. J. Probab*, 14(42):1198–1221, 2009.

[8] Jacques Printems et al. Functional quantization for numerics with an application to option pricing. *Monte Carlo Methods and Applications mcma*, 11(4):407–446, 2005.

[9] Harald Luschgy, Gilles Pagès, et al. High-resolution product quantization for gaussian processes under sup-norm distortion. *Bernoulli*, 13(3):653–671, 2007.

[10] Harald Luschgy and Gilles Pagès. Functional quantization of gaussian processes. *Journal of Functional Analysis*, 196(2):486–531, 2002.

[11] Stefan Junglen and Harald Luschgy. A constructive sharp approach to functional quantization of stochastic processes. *Journal of Applied Mathematics*, 2010, 2010.

[12] Kiyosi Itô, Makiko Nisio, et al. On the convergence of sums of independent banach space valued random variables. *Osaka Journal of Mathematics*, 5(1):35–48, 1968.

A Appendix

Lemma 4. *Let $(\lambda_k)_{k\in\mathbb{N}}$ be a sequence of real numbers, $(Z_k)_{k\in\mathbb{N}}$ centered standard Gaussian variables, and $(e_k)_{k\in\mathbb{N}}$ a family of continuous functions on [0,T]. Under the conditions*

- $\lambda_k = \underset{k\to\infty}{O}\left(\frac{1}{k^{H+1/2}}\right)$, *for some $H > 0$*
- $\exists L > 0, \forall k \in \mathbb{N}, \forall s,t \in [0,T], \quad |e_k(t) - e_k(s)| \le L\,|t-s|$

we get that

$$\mathbb{E} \sup_{t\in[0,T]} \left| \sum_{k=N}^{\infty} \lambda_k e_k(\frac{k\pi t}{T}) Z_k \right| = \underset{N\to\infty}{O}\left(N^{-H}\sqrt{\log N}\right)$$

and the series $\sum_{k=0}^{N} \lambda_k e_k Z_k$ converges almost surely, uniformly in the space of continuous functions on [0,T].

Proof. We will need the following result first. For $(X_i)_{0\le i\le M}$ a finite set of gaussian variables.

$$\mathbb{E} \max_{1\le i\le M} |X_i| \le c\sqrt{\log M} \max_{1\le i\le M} \sqrt{\mathbb{E}X_i^2}, \quad c > 0 \tag{19}$$

We denote by $v_k(t) := \lambda_k e_k(\frac{k\pi t}{T}) Z_k$ for $k \in \mathbb{N}$ and $t \in [0,T]$. To prove the lemma we will first show that for some $A > 0$,

$$\forall n \in \mathbb{N}, \quad \mathbb{E} \sup_{t\in[0,T]} \left| \sum_{k=2^n}^{2^{n+1}-1} v_k(t) \right| \le A\sqrt{n}2^{-nH} \tag{20}$$

Let $N \in \mathbb{N}$, we denote by $I_j = \left[j\frac{T}{N}, (j+1)\frac{T}{N}\right]$ and t_j the respective centers, $\forall 0 \le j \le N-1$. Let $n \in \mathbb{N}$

$$\begin{aligned}
\mathbb{E} \sup_{t\in[0,T]} \left| \sum_{k=2^n}^{2^{n+1}-1} v_k(t) \right| &= \mathbb{E} \sup_{0\le j<N} \sup_{t\in I_j} \left| \sum_{k=2^n}^{2^{n+1}-1} v_k(t) \right| \\
&\le \mathbb{E} \sup_{0\le j<N} \left| \sum_{k=2^n}^{2^{n+1}-1} v_k(t_j) \right| + \mathbb{E} \sup_{0\le j<N} \sup_{t\in I_j} \left| \sum_{k=2^n}^{2^{n+1}-1} (v_k(t) - v_k(t_j)) \right|
\end{aligned} \tag{21}$$

Using (19) we get that,

$$\begin{aligned}
\mathbb{E} \sup_{0\le j<N} \left| \sum_{k=2^n}^{2^{n+1}-1} v_k(t_j) \right| &\le c\sqrt{\log N} \sup_{0\le j<N} \sqrt{\mathbb{E}\left| \sum_{k=2^n}^{2^{n+1}-1} v_k(t_j) \right|^2} \\
&\le c\sqrt{\log N} \sup_{0\le j<N} \sqrt{\sum_{k=2^n}^{2^{n+1}-1} \mathbb{E}v_k(t_j)^2} \\
&\le C\sqrt{\log N} 2^{-nH}
\end{aligned} \tag{22}$$

where the last inequality comes from $\mathbb{E}v_k(t_j)^2 \le \lambda_k^2 ||e_k||_\infty^2 \le \frac{C}{k^{1+2H}}$, for some $C > 0$.
For the second part of (21),

$$\mathbb{E} \sup_{0\le j<N} \sup_{t\in I_j} \left| \sum_{k=2^n}^{2^{n+1}-1} (v_k(t) - v_k(t_j)) \right| \le \mathbb{E} \sup_{0\le j<N} \sum_{k=2^n}^{2^{n+1}-1} \sup_{t\in I_j} |v_k(t) - v_k(t_j)| \tag{23}$$

Since $\forall t \in I_j, |t - t_j| \leq \frac{T}{N}$ we get

$$\begin{aligned}\sup_{t\in I_j} |v_k(t) - v_k(t_j)| &\leq |\lambda_k||Z_k| \left| e_k\left(\frac{k\pi t}{T}\right) - e_k\left(\frac{k\pi t_j}{T}\right)\right| \\ &\leq C' k^{\frac{1}{2}-H} |Z_k| \frac{\pi}{N}\end{aligned} \tag{24}$$

By replacing in (23),

$$\begin{aligned}\mathbb{E} \sup_{0\leq j<N} \sup_{t\in I_j} \left| \sum_{k=2^n}^{2^{n+1}-1} (v_k(t) - v_k(t_j)) \right| &\leq \frac{C'}{N} \sum_{k=2^n}^{2^{n+1}-1} k^{\frac{1}{2}-H} \\ &\leq \frac{C^*}{N} 2^{n(\frac{3}{2}-H)}\end{aligned} \tag{25}$$

from (22) and (25) we get

$$\mathbb{E} \sup_{t\in[0,T]} \left| \sum_{k=2^n}^{2^{n+1}-1} v_k(t) \right| \leq C\sqrt{\log N} 2^{-nH} + \frac{C^*}{N} 2^{n(\frac{3}{2}-H)} \tag{26}$$

By taking $N = 2^{2n}$ we have proved (20). This result holds even if we replace $\left|\sum_{k=2^n}^{2^{n+1}-1} v_k(t)\right|$ by $\left|\sum_{k=M}^{2^{n+1}-1} v_k(t)\right|$ for some $M \in [2^n, 2^{n+1} - 1]$. Now we consider $n = \lfloor \log N/log2 \rfloor$

$$\mathbb{E} \sup_{t\in[0,T]} |v_k(t)| \leq \mathbb{E} \sup_{t\in[0,T]} \left| \sum_{k=N+1}^{2^{n+1}-1} v_k(t) \right| + \sum_{i=n+1}^{\infty} \mathbb{E} \sup_{t\in[0,T]} \left| \sum_{k=2^i}^{2^{i+1}-1} v_k(t) \right| \tag{27}$$

We can conclude by using (20),

$$\mathbb{E} \sup_{t\in[0,T]} \left| \sum_{k=n}^{\infty} v_k(t) \right| \leq A \sum_{k=n}^{\infty} \sqrt{k} 2^{-kH} \leq A' \sqrt{n} 2^{-nH} \tag{28}$$

and the fact that $2^n \leq N \leq 2^{n+1}$. The uniform tightness implies that $\sum_{k=0}^{N} v_k$ has a weak limit in C[0,T] the space of continuous functions on [0,T]. If we endow this space with the supremum metric, we get by the It-Nisio theorem, as in [12], that the process $\sum_{k=0}^{N} v_k$ converges in $C[0,T]$ almost surely. $\square$