A regularity structures for rough volatility

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Black–Scholes (BS)

- Given $\sigma > 0$ and scalar Brownian motion B,

$$dS_t / S_t = \sigma dB_t.$$

  Dynamics under pricing measure, zero rates, for option pricing.
- Call prices

$$\begin{aligned} C(S_0, K, T) &:= \mathbb{E}(S_T - K)^+ \\ &= \mathbb{E}\left(S_0 \exp\left(\sigma B_T - \frac{\sigma^2}{2} T\right) - K\right)^+ \\ &=: C_{BS}\left(S_0, K; \sigma^2 T\right) \end{aligned}$$

- Time-inhomogenous BS: $\sigma \rightsquigarrow \sigma_t$ determistic

$$C(S_0, K, T) = C_{BS}\left(S_0, K; \int_0^T \sigma^2 dt\right)$$

Stochastic volatility

Make volatility

$$\sigma \rightsquigarrow \sigma_t(\omega) \equiv \sqrt{v_t(\omega)} \quad \text{stochastic, adapted}$$

so that

$$dS_t / S_t = \sigma_t(\omega) dB_t = \sqrt{v_t(\omega)} dB_t$$

defines a martingale.

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$$dS_t / S_t = \sigma_t(\omega) dB_t = \sqrt{v_t(\omega)} dB_t$$

defines a martingale.

Examples:

- Dupire local volatility, 1-factor model, ∞ parameters

$$\sigma_t(\omega) = \sigma_{loc}(S_t(\omega), t)$$

More (classical) stochastic volatility

More examples:

- Heston, 2-factors: W, $\bar{W}$ indep. BMs, 5 parameters

$$
\begin{aligned}
dS_t / S_t &= \sqrt{v} dB_t \equiv \sqrt{v} \left(\rho dW_t + \bar{\rho} d\bar{W}_t \right) \\
v_t &= v_0 + \int_0^t (a - bv) dt + \int_0^t c\sqrt{v} dW
\end{aligned}
$$

  with correlation parameter ρ and $\rho^2 + \bar{\rho}^2 = 1$

- SABR, as above but 4 parameters

$$
\begin{aligned}
dS_t / S_t^{\beta} &= \sigma dB_t \equiv \sigma \left(\rho dW_t + \bar{\rho} d\bar{W}_t \right) \\
\sigma_t &= \sigma_0 \exp(\alpha W_t - \frac{1}{2}\alpha^2 t)
\end{aligned}
$$

  with correlation parameter ρ and $\rho^2 + \bar{\rho}^2 = 1$

Pricing under classical stochvol (selection)

Numerics:

- PDE pricing
- Monte Carlo pricing
- Strong / weak rates, Multi level MC
- (Kusuoka-Lyons-Victoir) Cubature methods
- Ninomia-Victoir splitting scheme

Analytics:

- Malliavin calculus, asymptotic expansions
- Large deviations, e.g. Freidlin-Wentzell

Famously used to understand smile: SABR formula etc

Rough fractional stochastic volatility

- Some history [ALV07], [Fuk11,15] .... Now much interest in

$$v_t \approx W_t^H \equiv \hat{W}_t$$

a **fractional Browniation motion** (fBm) in the rough regime of Hurst paramter $H < 1/2$: worse than Brownian regularity

- Recall: Volterra (a.k.a. Riemann-Liouville) fBm

$$\hat{W}_t = \int_0^t K^H(t,s)dW_s = \sqrt{2H}\int_0^t |t-s|^{H-1/2}dW_s$$

- Popular "simple" form (also in [GJR14x], [BFG16])

$$\sigma_t := f(\hat{W}_t, t)$$

EFR volatility dynamics

In [EFR16], El Euch, Fukasawa and Rosenbaum show that stylized facts of modern market microstructure naturally give rise to fractional dynamics and leverage effects. Specifically, they construct a sequence of Hawkes processes suitably rescaled in time and space that converges in law to a rough volatility model of rough Heston form

$$
\begin{aligned}
dS_t/S_t &= \sqrt{v}dB_t \equiv \sqrt{v}\left(\rho dW_t + \bar{\rho} d\bar{W}_t\right) \\
v_t &= v_0 + \int_0^t \frac{a - bv}{(t-s)^{1/2-H}} dt + \int_0^t \frac{c\sqrt{v}}{(t-s)^{1/2-H}} dW
\end{aligned}
$$

(As earlier, W, $\bar{W}$ independent Brownians.)

Pricing under rough volatility

Numerics:

- PDE pricing NO!
- Monte Carlo pricing ✓ [BFG16], [BLP15x], ...
- Strong / weak rates, Multilevel MC: Partial answers [NS16x], ...
- Cubature, Ninomia-Victoir ? Even classical Wong-Zakai ?

Analytics:

- Malliavin calculus, asymptotic expansions ✓
  [ALV07], [Fuk11,Fuk15], ...
- Large deviations: Partial answers [FZ17], ...

Moderate and large deviations

In recent joint work with **S. Gerhold** and **A. Pinter** we introduced [FGP17] moderate deviation analysis in option pricing. In [BFGHS17x] we extend this to simple rough volatility of the form $\sigma_t = f(\hat{W})$. In particular, we refine Forde–Zhang rough large deviations [FZ17] while regaining analytical tracktability. To state the result, set $\sigma_0 := f(0)$ (spotvol) and also $\sigma_0' := f'(0)$.

Theorem (Rough vol skew, "moderate" regime)

Let $0 < \beta < H < 1/2$ and fix $y, z > 0$ with $y \neq z$. Set $\hat{y} := y t^{\frac{1}{2} - H + \beta}$ and similarly for z. Then, as $t \to 0$,

$$\frac{\sigma_{\mathrm{impl}}(\hat{y}, t) - \sigma_{\mathrm{impl}}(\hat{z}, t)}{\hat{y} - \hat{z}} \sim \frac{\rho}{(H + 1/2)(H + 3/2)} \frac{\sigma_0'}{\sigma_0} t^{H - \frac{1}{2}}.$$

Extends CLT regime of [ALV07, Fuk11].

A SV call price formula

- Any SV model of form

$$dS_t / S_t = \sqrt{v} dB_t \equiv \sqrt{v}(\rho dW_t + \bar{\rho} d\bar{W}_t)$$
$$v_t \text{ adapted to } (W)$$

allows to price calls as averages of Black–Scholes calls

$$\mathbb{E}\left[C_{BS}\left(S_0 \exp\left(\rho \int_0^T \sqrt{v} dW - \frac{\rho^2}{2} \int_0^T v dt \right), K, \frac{\bar{\rho}^2}{2} \int_0^T v dt \right) \right]$$

(Romano–Touzi proof: condition payoff with respect to W.)

- Even for “simple” rough volatility: $\sqrt{v} = f(\hat{W})$

$$\text{Question: efficient simulation} \int_0^T f(\hat{W}) dW \quad ?$$

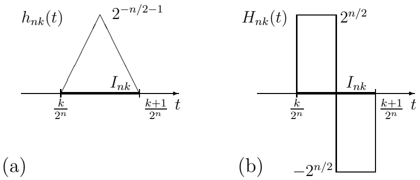
- Also need to simulate $\int_0^T v dt$, but this one harmless.

Intermezzo on approximating Brownian motion

In terms of (Z_{nk}), i.i.d. standard Gaussians, can expand Bm

$$W_t = Z_0(\omega)t + \sum_{n=0}^{\infty} \sum_{k=0}^{2^n-1} Z_{nk}(\omega) h_{nk}(t),$$

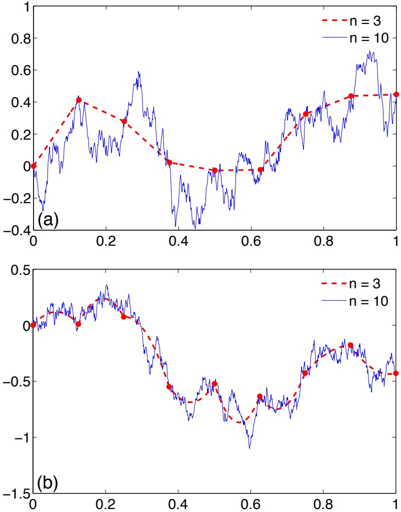
where h_{nk} is the integrated (here: Haar) wavelet basis H_{nk}.



Truncation at level n: **natural approximation** W^ε at scale $\varepsilon = 2^{-n}$.

Brownian approximations cont'd

Figure: Based on (a) Haar (b) Alpert-Rokhlin wavelets



Approximation theory: simple rough vol

We encountered the (Itô) integral

$$\int_0^T f(\hat{W})\,dW$$

where $\hat{W}$ is rough, namely a fBm with, say, $H \approx 0.05 << 1/2$

$$\hat{W}_t = \int_0^t K^H(t,s)\,dW_s$$

- Badly behaved even under classical Wong-Zakai approximations! Problem: ∞ Itô-Stratonovich correction whenever $H < 1/2$,

$$\begin{aligned} \int f(\hat{W}^\varepsilon)\,dW^\varepsilon \quad &\to \quad \int f(\hat{W}) \circ dW \\ &= \quad \int f(\hat{W})\,dW + \frac{1}{2}[f(\hat{W}), W] \; = +\infty \end{aligned}$$

$\Longrightarrow$ Itô $\checkmark$ but Stratonovich $\nexists$ ....

Lessons from SPDE theory

- Universal model for fluctuations of interface growth: the Kardar–Parisi–Zhang (KPZ) equation

$$\partial_t u = \partial_x^2 u + |\partial_x u|^2 + \xi$$

  with space-time white noise $\xi = \xi(x, t; \omega)$.
- **Fact:** Itô-solution ✓ ("Cole-Hopf") but

$$\text{with mollified noise } \xi^\varepsilon \implies \text{get: } u^\varepsilon \to +\infty$$

  hence $\nexists$ Stratonovich solution ...

What is going on?

- Formally at least, with $u_0 \equiv 0$ for simplicity,

$$u = H * \left(|\partial_x u|^2 + \xi \right)$$

with space-time convolution $\star$ and heat-kernel

$$H(t,x) = \frac{1}{\sqrt{4\pi t}} \exp\left(-\frac{x^2}{4t} \right) 1_{\{t>0\}}$$

- Naiv Picard iteration

$$u = H \star \xi + H \star ((H' \star \xi)^2) + \dots$$

does not work because

$$\nexists (H' \star \xi)^2, \quad \nexists \lim_{\varepsilon \to 0} (H' \star \xi^\varepsilon)^2 \dots$$

KPZ analysis cont'd

- However, there exists a diverging sequence (C_ε) such that

$$\exists \lim_{\varepsilon\to 0} (H' \star \xi^\varepsilon)^2 - C_\varepsilon \to (\text{new object}) =: (H' \star \xi)^{\diamond 2}.$$

- Rough path inspired idea (Hairer): Accept

$$H \star \xi, (H' \star \xi)^{\diamond 2} \text{ (and a few more)}$$

as enhanced noise ("**model**") upon which solution depends in pathwise robust fashion.

- This unlocks the seemingly fixed relation

$$H \star \xi \to \xi \to (H' \star \xi)^2.$$

(NB: last term does not make analytical sense and in fact diverges upon approximation)

The Hairer result

- **Theorem** [Hairer] There exist diverging constants C_ε such that

$$\tilde{u}^\varepsilon \to (\text{Itô-solution})$$

in terms of the renormalized approximating equation

$$\partial_t \tilde{u}^\varepsilon = \partial_x^2 \tilde{u}^\varepsilon + |\partial_x \tilde{u}^\varepsilon|^2 - C_\varepsilon + \xi^\varepsilon.$$

(Same approach works e.g. for Φ_3^4 from QFT.)

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(Same approach works e.g. for Φ_3^4 from QFT.)

- This can be traced back to the **Milstein-scheme** for SDEs (and then "rough paths"). Take

$$dY = f(Y)dW,$$

with $Y_0 = 0$ for simplicity, and consider

$$Y \approx f(0)W + ff'(0) \int WdW$$

SDEs vs. SPDEs

- SDE case cont'd: unlock the seemingly fixed relation

$$W \to \dot{W} \to \int W \dot{W} \, ,$$

  for there is a choice to be made (e.g. Itô / Stratonovich) !
- Take this one step further (rough path): accept $\int W dW \equiv \mathbb{W}$ as new object,

$$\text{SDE theory = analysis based on } (W, \mathbb{W})$$

  Similarly

$$\begin{aligned} &\text{SPDE theory à la Hairer} \\ = \quad &\text{analysis based on (renormalized) enhanced noise.} \end{aligned}$$

Inside Hairer's theory

Motivation: The Taylor-expansion (at x) of a smooth function,

$$f(y) = f(x) + f'(x)(y-x) + \frac{1}{2}f''(x)(y-x)^2 + \dots ,$$

can be written as abstract polynomial ("jet") at x,

$$F(x) := f(x)\,\mathbf{1} + g(x)X + h(x)X^2 + \dots \tag{1}$$

$$[\text{necessarily: } g = f',\ h = f''/2, \dots] \tag{2}$$

If we "realize" these abstract symbols again as honest monomials

$$\Pi_x : X^k \mapsto (. - x)^k$$

and extend Π_x linearly, then we recover the full Taylor expansion:

$$\Pi_x[F(x)](.) = f(x) + g(x)(. - x) + \frac{1}{2}h(x)(. - x)^2 + \dots$$

Inside Hairer's theory cont'd

Hairer looks for solution of this form: at every space-time point a jet is attached, which in case of KPZ turns out to be of the form

$$U(x,s) = u(x,s)\,\mathbf{1} + \mathsf{i} + \Upsilon + v(x,s)\,X + 2\,\Upsilon' + v(x,s)\,\angle\,.$$

As before, every symbol is given concrete meaning by "realizing" it as honest function (or Schwartz distribution). In particular,

$$\mathsf{i} \mapsto \begin{cases} H \star \xi^{\epsilon}, & \text{mollified noise; \textbf{or}} \\ H \star \xi & \text{noise} \end{cases} \tag{3}$$

and then, more interestingly,

$$\Upsilon \mapsto \begin{cases} H \star (H' \star \xi^{\epsilon})^2, & \text{canonically enhanced mollified noise; \textbf{or}} \\ H \star \left[(H' \star \xi^{\epsilon})^2 - C_{\epsilon}\right], & \text{renormalized} \sim \textbf{or} \\ H \star (H' \star \xi)^{\diamond 2}, & \text{renormalized enhanced noise (REM)} \end{cases} \tag{4}$$

etc ... This realization map is called "model".

Inside Hairer's theory cont'd

Hence, the model captures (in a pathwise fashion) your choice of noise. Writing $\Pi_{x,s}$ for the realization map for REM have e.g.

$$\Pi_{x,s}[U(x,s)](.) = u(x,s) + H \star \xi|_{(*)} + H \star (H' \star \xi)^{\diamond 2}|_{(*)} + \ldots$$

where $(*)$ indicates suitable centering at (x,s). Mind that U takes values in a (finite) linear space spanned by (sufficiently many) symbols,

$U(x,s) \in \langle \ldots, 1,$ $,$  $, X,$  $,$  $, \ldots \rangle =: T$

Together with some regularity, $(x,s) \mapsto U(x,s)$ is an example of a modelled distribution. Also need to keep track of regularity, better: degree, of each symbol (e.g. $|X^2| = 2$, $|$ $| = 1/2 - \kappa$), collected in index set. Last not least, in order to compare jets at different points (think $(X - \delta 1)^3 = \ldots$), use a group of linear maps on $\mathcal{T}$, called structure group. Last not least, the reconstruction map uniquely maps modelled distributions to function / Schwartz distributions.

Back to rough volatility

We can apply these ideas in the context of rough volatility. Recall the basic problem

$$\int f(\hat{W}^\varepsilon)dW^\varepsilon \quad \to \quad \int f(\hat{W}) \circ dW$$
$$= \quad \int f(\hat{W})dW + \frac{1}{2}[f(\hat{W}), W] \; = +\infty$$

The (wavelet) expansion of white noise

$$\dot{W}^\varepsilon(t) = Z_0(\omega) + \sum_{n=0}^{N} \sum_{k=0}^{2^n-1} Z_{nk}(\omega) H_{nk}(t)$$

naturally induced fBm approximation, namely

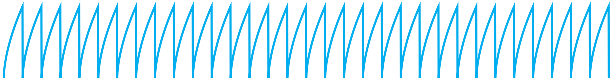
$$\hat{W}^\varepsilon(t) = \int_0^t K(t,s)\, dW^\varepsilon(s) = Z_0 \hat{1}(t) + \sum_{n=0}^{N} \sum_{k=0}^{2^n-1} Z_{nk}(\omega) \hat{H}_{nk}(t)$$

where $\hat{f}(t) = \sqrt{2H} \int_0^t |t-s|^{H-1/2} f(s) ds$.

Back to rough volatility

On scale $\varepsilon \equiv 2^{-N}$, set

$$\color{red}{\mathscr{C}^{\varepsilon}(t)} = \frac{\sqrt{2H}\,2^N}{1+H-1/2}\left|t - \lfloor t2^N \rfloor 2^{-N}\right|^{1+H-1/2}, \tag{5}$$



which, at least when $H > 1/4$, can be replaced (below) by diverging local average

$$\frac{\sqrt{2H}}{(H+1/2)(H+3/2)}\varepsilon^{H-1/2} =: C_{\varepsilon}.$$

Theorem (BFGMS17)

For any $H \le 1/2$ and nice function f, with **strong rate** *H,*

$$\lim_{\varepsilon \to 0} \int_0^T f(\hat{W}^{\varepsilon})\,dW^{\varepsilon} - \int_0^T \mathscr{C}^{\varepsilon}(t) f'(\hat{W}^{\varepsilon}_t)\,dt = \int_0^T f(\hat{W})\,dW$$

Rough idea of proof

- Robust interpretation of Itô solution (here: integral) ?
  $\Longrightarrow$ Identify correct enhancement of noise

$$\mathbf{W}^{k,1}_{s,t} := \int_s^t \left(\hat{W}_r - \hat{W}_s\right)^k dW_r \qquad k = 0, 1, ..., ?$$

- Formally, for any partition P of $[0, T]$,

$$\begin{aligned}\int_0^T f\left(\hat{W}_r\right) dW_r &= \sum_{[s,t]\in P} \int_s^t f\left(\hat{W}_r\right) dW_r \\ &= \sum_{[s,t]\in P} \left\{ \sum_{0\le k<K} \frac{1}{k!} f^{(k)}\left(\hat{W}_s\right) \mathbf{W}^{k,1}_{s,t} + ... \right\}\end{aligned}$$

- Want $(...)$ to vanish, upon refinement of partitions.
- Approximation theory needed for building blocks $\mathbf{W}^{k,1}$, perform renormalization at this level.

Rough path type considerations

- **Degree of regularity of building blocks?** We can see a.s.

$$\left|\mathbf{W}^{k,1}_{s,t}\right| \lesssim |t-s|^{kH+1/2-\varepsilon}$$

- Terms with exponent > 1 may be dropped since $\left|\mathbf{W}^{k,1}_{s,t}\right| = o(t-s)$

$$\implies \text{keep } k < K := \min\{j : jH + 1/2 > 1\}$$

e.g. when $H = 1/2$, $K = 2$ and $K \sim 1/(2H)$ as $H \to 0$.

- Expect (from rough paths theory)

$$\int_0^T f\left(\hat{W}_r\right) dW_r = \lim_{\text{mesh}(P)\to 0} \sum_{[s,t]\in P} \left\{ \sum_{0\le k<K} \frac{1}{k!} f^{(k)}\left(\hat{W}_s\right) \mathbf{W}^{k,1}_{s,t} \right\}$$

such that Itô integral **continuous function of** enhanced noise

$$\left\{\mathbf{W}^{k,1} : 0 \le k < K\right\}$$

A regularity structure for rough volatility

Although previous heuristics are (hopefully) convincing, a direct proof would amount to (re)invent rough path in a non-geometric setting, with renormalization. Fortunately, Hairer's theory already provides a framework to all this!

- A regularity structures is a triplet (T, A, G) with T spanned by symbols, a set of degrees A and a structure group G.
- In case of rough volatility,

$$
\begin{aligned}
T &= \left\langle \dot{W}, \dot{W}\hat{W}, \dots, \dot{W}\hat{W}^{K-1}, 1, \hat{W} \dots, \hat{W}^{K-1} \right\rangle \\
A &= \left\{ -\frac{1}{2}, H - \frac{1}{2}, \dots, (K-1)H - \frac{1}{2}, 0, H, \dots, (K-1)H \right\}
\end{aligned}
$$

- Structure group ?

Structure group explicit

- Take $K = 3$ for better visibility,

$$T = \left\langle \dot{W}, \dot{W}\hat{W}, \dot{W}\hat{W}^2, 1, \hat{W}, \hat{W}^2 \right\rangle$$

- Then $G = \{\Gamma_h : h \in (\mathbb{R}, +)\}$ with $\Gamma_h \in Lin(T, T)$ given by block-matrix

$$\begin{pmatrix} 1 & h & h^2 & & & \\ 0 & 1 & 2h & & \mathbf{0} & \\ 0 & 0 & 1 & & & \\ & & & 1 & h & h^2 \\ & \mathbf{0} & & 0 & 1 & 2h \\ & & & 0 & 0 & 1 \end{pmatrix}$$

The Itô enhanced "model" for rough vol

- Unsurprisingly,

$$\Pi_s \hat{W}^k(r) := \left(\hat{W}_r - \hat{W}_s\right)^k$$

$$\Pi_s \hat{W}^k \dot{W} := \left(\hat{W}_\cdot - \hat{W}_s\right)^k \dot{W} := \frac{d}{dt}\mathbf{W}^{k,1}_{s,t} \text{ (in } \mathcal{D}')$$

  and also

$$\Gamma_{s,t} = \Gamma_h \text{ with } h = \hat{W}_{s,t}$$

- Defines a model in the sense of Hairer, call it Π^{Ito}.
- Gaussian (since $\mathbf{W}^{k,1}_{s,t} \in$ first k Wiener-Itô chaos')

Reconstructing the integral

- A modelled distribution of regularity γ is defined by

$$t \mapsto \sum_{0 \le k < K} \frac{1}{k!} f^{(k)}\left(\hat{W}_t\right) \hat{W}^k \dot{W}$$

where, recall $K := \min\{j : jH + 1/2 > 1\}$,

$$\gamma = KH - 1/2 > 0$$

- The (unique) reconstruction is precisely the Schwartz derivative of

$$\int f\left(\hat{W}\right) dW,$$

which in turn is recover by testing again indicator functions

- Best of all, continuous dependence of integrals as function of the model

Rough vol: Approximation and renormalization ...

- Canonical model for ε-mollified noise (divergent!)

$$\Pi^{\varepsilon}_s \hat{W}^k(r) := \left(\hat{W}^{\varepsilon}_r - \hat{W}^{\varepsilon}_s\right)^k$$

$$\Pi^{\varepsilon}_s \hat{W}^k \dot{W} := \left(\hat{W}^{\varepsilon}_{\cdot} - \hat{W}^{\varepsilon}_s\right)^k \dot{W}^{\varepsilon}$$

- Renormalized model: define

$$\hat{\Pi}^{\varepsilon}_s \hat{W}^k(r) := \left(\hat{W}^{\varepsilon}_r - \hat{W}^{\varepsilon}_s\right)^k$$

$$\hat{\Pi}^{\varepsilon}_s \hat{W}^k \dot{W} := \left(\hat{W}^{\varepsilon}_{\cdot} - \hat{W}^{\varepsilon}_s\right)^k \dot{W}^{\varepsilon} - C_{\varepsilon} k \left(\hat{W}^{\varepsilon}_{\cdot} - \hat{W}^{\varepsilon}_s\right)^{k-1}$$

- **Theorem:** There exists a choice of C_{ε} (necessarily divergent) such that

$$\hat{\Pi}^{\varepsilon} \to \Pi^{Ito} \text{ as } \varepsilon \downarrow 0$$

in probablity and model distance. (This implies the rough vol approximation result stated several slides ago.)

So what?

So what?

True, (so far) we mostly talked about a scalar Itô integral. Of course, Euler / left-point approximation also works: with mesh P to zero, in every good sense,

$$\sum_{[s,t]\in P} f(\hat{W}_s) W_{s,t} \to \int f(\hat{W}) dW$$

Attention: Euler Simulation of W not good for $\hat{W} = \int (\text{singular}) dW$, but can simulate $(W, \hat{W})$ directly via known covariance structure.
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- Flexible wavelet approximations possible. Often highly effective in combination with QMC (Sobol); Alpert-Rokhlin wavelets expected to go well with fBm. (Numerics in progress.)
- Robustness, also as analytical tool!

Large deviation principle (reminder)

A family of random variables $\left(X^\delta : \delta > 0\right)$ satisfies a LDP iff

$$P\left(X^\delta \approx x\right) \sim \exp\left(-\frac{I(x)}{\delta^2}\right).$$

Formal definition: with rate function $I \geq 0$ and speed δ^2, have

$$\inf_{x \in \bar{A}} I(x) + o(1) \leq -\delta^2 \log P\left(X^\delta \in A\right) \leq \inf_{x \in A^\circ} I(x) + o(1).$$

Contraction principle: basic fact of large deviation theory - stability under continuous maps. LDP for $Y^\delta = \Phi\left(X^\delta\right)$ with rate function

$$J(y) = \inf\left\{I(x) : \Phi(x) = y\right\}.$$

Works well with rough paths / regularity structures!

Rough vol large deviations

Theorem

For nice f, a LDP for

$$X_1^\delta = \int_0^1 f(\delta \hat{W}_t)\delta dW_t$$

holds with speed δ^2 and rate function

$$J(x) = \inf\left\{\frac{1}{2}\|h\|_{L^2}^2 : x = \int_0^1 f\left(\int_0^t |t-s|^{H-1/2} h(s)ds\right) h(t)dt\right\}$$

By scaling, this gives also short time LDP for $t^{H-1/2} X_t^1$ with speed t^{2H} and same rate function.

"Simple" rough vol large deviations [FZ17] as a consequence ...

EFR type rough vol dynamics

Until now we considered "simple" rough vol of form $\sigma_t(\omega) = f(\hat{W}_t)$. Following Rosenbaum and coworkers, we consider

$$\sigma_t^2 \equiv v_t = v_0 + \int_0^t \frac{g(v_s)}{|t-s|^{1/2-H}} dW_s + \int_0^t \frac{h(v_s)}{|t-s|^{1/2-H}} ds$$

for general (but nice) coefficient functions g, h. This is a Volterra stochastic differential equation $\notin$ usual SDE theory.

Theorem (BFGMS17)

For any $H \in (0, 1/2]$, this is a subcritical equation and has a unique Itô solution. Naiv approximations based on W^ε diverge; but this is fixed by renormalization:

$$\tilde{v}_t^\varepsilon = v_0 + \int_0^t \frac{g(\tilde{v}_s^\varepsilon)}{|t-s|^{1/2-H}} dW_s^\varepsilon + \int_0^t \frac{(g - \mathscr{C}^\varepsilon(.) h\, h')(\tilde{v}_s^\varepsilon)}{|t-s|^{1/2-H}} ds$$

EFR type rough vol dynamics cont'd

- Solution theory à la Hairer identifies limiting (Itô) solution as robust image of the enhanced noise.
- At least when $H > 1/4$ can replace renormalization function $\mathscr{C}^{\varepsilon}(.)$ by a (diverging) constant C_ε, so that

$$\tilde{v}_t^\varepsilon = v_0 + \int_0^t \frac{g(\tilde{v}_s^\varepsilon)}{|t-s|^{1/2-H}} dW_s^\varepsilon + \int_0^t \frac{(g - C_\varepsilon h\, h')(\tilde{v}_s^\varepsilon)}{|t-s|^{1/2-H}} ds$$

- Immediate large deviations! But: rate function not explicit, more work along [BFGHS17] needed
- Everything works, though more involved, for $H \leq 1/4$.

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Appendix on regularity structures 1

Definition

A *regularity structure* $\mathscr{T} = (A, T, G)$ consists of the following elements:

- An index set $A \subset \mathbf{R}$ such that $0 \in A$, A is bounded from below, and A is locally finite.
- A *model space* T, which is a graded vector space $T = \bigoplus_{\alpha \in A} T_\alpha$, with each T_α a Banach space ; elements in T_α are said to have *homogeneity* (or *degree*) α. Furthermore $T_0 = \langle \mathbf{1} \rangle \cong \mathbf{R}$. Given $\tau \in T$, we will write $\|\tau\|_\alpha$ for the norm of its component in T_α.
- A *structure group* G of (continuous) linear operators acting on T such that, for every $\Gamma \in G$, every $\alpha \in A$, and every $\tau_\alpha \in T_\alpha$, one has

$$\Gamma \tau_\alpha - \tau_\alpha \in T_{<\alpha} \overset{\text{def}}{=} \bigoplus_{\beta < \alpha} T_\beta \ . \tag{6}$$

  Furthermore, $\Gamma \mathbf{1} = \mathbf{1}$ for every $\Gamma \in G$.

Appendix cont'd: rough path structure

Definition

Let $\alpha \in (1/3, 1/2]$. The *regularity structure for α-Hölder rough paths (over $\mathbf{R}^e$)* is given by

- The set of possible homogeneities is given by $A = \{\alpha - 1, 2\alpha - 1, 0, \alpha\}$.
- The model space T is given by $T = T_{\alpha-1} \oplus T_{2\alpha-1} \oplus T_0 \oplus T_\alpha \cong \mathbf{R}^{e+e^2+1+e}$ with

$$T_0 = \langle \mathbf{1} \rangle \,, \qquad T_\alpha = \langle W^1, \ldots, W^e \rangle \,,$$
$$T_{\alpha-1} = \langle \dot{W}^1, \ldots, \dot{W}^e \rangle \,, \qquad T_{2\alpha-1} = \langle \dot{\mathbb{W}}^{ij} : 1 \le i, j \le e \rangle \,.$$

- The group $G \sim (\mathbf{R}^e, +)$ acts on T via

$$\Gamma_h \mathbf{1} = \mathbf{1} \,, \qquad \Gamma_h W^i = W^i + h^i \mathbf{1} \,,$$
$$\Gamma_h \dot{W}^i = \dot{W}^i \,, \qquad \Gamma_h \dot{\mathbb{W}}^{ij} = \dot{\mathbb{W}}^{ij} + h^i \dot{W}^j \,. \tag{7}$$

Appendix cont'd: models

Given a test function ϕ on $\mathbf{R}^d$, we write $\phi_x^\lambda \equiv$ as a shorthand for

$$\phi_x^\lambda(y) = \lambda^{-d}\phi\big(\lambda^{-1}(y-x)\big) .$$

Given an integer $r > 0$, we also denote by $\mathcal{B}_r$ the set of all functions $\phi\colon \mathbf{R}^d \to \mathbf{R}$ such that $\phi \in \mathcal{C}_b^r$ with $\|\phi\|_{\mathcal{C}_b^r} \leq 1$ that are furthermore supported in the unit ball around the origin. We also write $\mathcal{D}'(\mathbf{R}^d)$ for the space of Schwartz distributions on $\mathbf{R}^d$.

Definition

Given a regularity structure $\mathscr{T}$ and an integer $d \geq 1$, a *model* $\mathrm{M} = (\Pi, \Gamma)$ for $\mathscr{T}$ on $\mathbf{R}^d$ consists of maps

$$\begin{aligned} \Pi\colon \mathbf{R}^d &\to \mathcal{L}\big(T, \mathcal{D}'(\mathbf{R}^d)\big) \qquad & \Gamma\colon \mathbf{R}^d \times \mathbf{R}^d &\to G \\ x &\mapsto \Pi_x & (x,y) &\mapsto \Gamma_{xy} \end{aligned}$$

such that $\Gamma_{xy}\Gamma_{yz} = \Gamma_{xz}$ and $\Pi_x\Gamma_{xy} = \Pi_y$. We then say that Π_x *realizes* an element of T as a Schwartz distribution.

Appendix cont'd: models

Definition (model, cont'd)

Furthermore, write r for the smallest integer such that $r > |\min A| \geq 0$. We then impose that for every compact set $\mathfrak{K} \subset \mathbf{R}^d$ and every $\gamma > 0$, there exists a constant $C = C(\mathfrak{K}, \gamma)$ such that the bounds

$$\left|(\Pi_x \tau)(\phi_x^\lambda)\right| \leq C\lambda^\alpha \|\tau\|_\alpha \,, \qquad \|\Gamma_{xy}\tau\|_\beta \leq C|x-y|^{\alpha-\beta}\|\tau\|_\alpha \,, \tag{8}$$

hold uniformly over $\phi \in \mathcal{B}_r$, $(x, y) \in \mathfrak{K}$, $\lambda \in (0, 1]$, $\tau \in T_\alpha$ with $\alpha \leq \gamma$ and $\beta < \alpha$.

Appendix on regularity structures: modelled distributions

Definition (modelled distribution)

Given a regularity structure $\mathscr{T}$ equipped with a model $\mathrm{M} = (\Pi, \Gamma)$ over $\mathbf{R}^d$, the space $\mathscr{D}^\gamma_{\mathrm{M}} = \mathscr{D}^\gamma_{\mathrm{M}}(\mathscr{T})$ is given by the set of functions $f \colon \mathbf{R}^d \to T_{<\gamma}$ such that, for every compact set $\mathfrak{K}$ and every $\alpha < \gamma$, there exists a constant C with

$$\|f(x) - \Gamma_{xy} f(y)\|_\alpha \leq C|x-y|^{\gamma-\alpha} \tag{9}$$

uniformly over $x, y \in \mathfrak{K}$. Such functions f are called *modelled distributions*. For fixed $\mathfrak{K}$, a semi-norm $\|f\|_{\mathrm{M},\gamma;\mathfrak{K}}$ is defined as the smallest constant C in the bound (9). The space $\mathscr{D}^\gamma_{\mathrm{M}}$ endowed with this family of seminorms is then a Fréchet space.

Distance between models: the smallest constant C in the bound

$$\|f(x) - \Gamma_{xy} f(y) - \bar{f}(x) + \bar{\Gamma}_{xy}\bar{f}(y)\|_\alpha \leq C|x-y|^{\gamma-\alpha} .$$

Appendix on regularity structures: reconstruction

The most fundamental result in the theory of regularity structures then states that given $f \in \mathscr{D}^\gamma$ with $\gamma > 0$, there exists a *unique* Schwartz distribution $\mathcal{R}f$ on $\mathbf{R}^d$ such that, for every $x \in \mathbf{R}^d$, $\mathcal{R}f$ "looks like $\Pi_x f(x)$ near x". More precisely, one has

Theorem (Reconstruction)

Let $\mathrm{M} = (\Pi, \Gamma)$ *be a model for a regularity structure* $\mathscr{T}$ *on* $\mathbf{R}^d$*. Assume* $f \in \mathscr{D}^\gamma_{\mathrm{M}}$ *with* $\gamma > 0$*. Then, there exists a unique linear map*

$$\mathcal{R} = \mathcal{R}_{\mathrm{M}} \colon \mathscr{D}^\gamma_{\mathrm{M}} \to \mathcal{D}'(\mathbf{R}^d)$$

such that

$$\left|\left(\mathcal{R}f - \Pi_x f(x)\right)(\phi_x^\lambda)\right| \lesssim \lambda^\gamma \,, \tag{10}$$

uniformly over $\phi \in \mathcal{B}_r$ *and* λ *as before, and locally uniformly in* x*.*
Without the positivity assumption on γ*, everything remains valid but uniqueness of* $\mathcal{R}$*.*