Learning Rough Volatility

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Hammamet, 29th October 2018

International Conference on Control, Games and Stochastic Analysis
Hammamet, Oct. 29 to Nov. 01, 2018, Tunisia

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- These models have repeatedly proven to be superior to standard models in many areas: in volatility forecasting, in option pricing, close fits to the implied vol surface, . . .

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- These models have repeatedly proven to be superior to standard models in many areas: in volatility forecasting, in option pricing, close fits to the implied vol surface, . . .
- Relaxing the assumption of independence of volatility increments was crucial for the superior performance of rough volatility models $\Rightarrow$ but: several standard pricing methods no longer available & naive Monte Carlo methods slow
- Calibration time has been a bottleneck for rough volatility several advances have been made to speed up the calibration process [BLP '15, MP '17, HJM '17].

Today's talk:
Speedups for rough volatility models along two lines:

1. in pricing of vanilla options based on faster Monte Carlo approximations for a family of rough stochastic volatility models. [H-Jacquier-Muguruza '17])
2. in calibration by means of machine learning (ongoing with A. Muguruza and with M. Tomas).

Digression: Rough Volatility

Suppose a generic Itô process framework for the stock price $(S_t)_{t\geq 0}$:

$$\frac{dS_t}{S_t} = \mu_t dt + \sigma_t dB_t, \quad t \geq 0.$$

The phrase "**rough volatility**" refers to the idea that sample paths of the log volatility $\log(\sigma_t)$, $t \geq 0$ are rougher than the sample paths of Brownian motion.

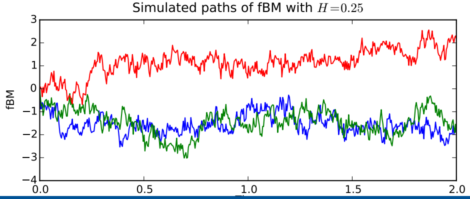
Volatility is Rough

Gatheral, Jaisson and Rosenbaum (2014) suggested that volatility is rough. The slogan "**volatility is rough**" refers to the idea that sample paths of the log volatility $\log(\sigma_t)$, $t \geq 0$ are rougher than the sample paths of Brownian motion (in terms of Hölder regularity).

Fractional Brownian motion

A fractional Brownian motion with Hurst parameter $H \in (0,1)$ is a continuous centered Gaussian process $(B_t^H)_{t\in\mathbb{R}}$ with covariance function

$$Cov(B_t^H, B_s^H) = \frac{1}{2}\left(|t|^{2H} + |s|^{2H} - |t-s|^{2H}\right), \quad s,t \in \mathbb{R}. \tag{1}$$



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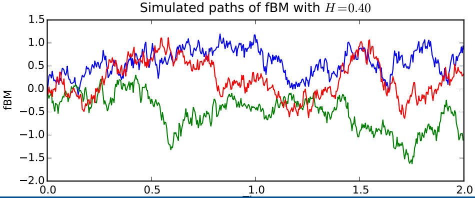

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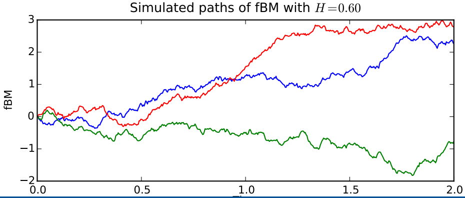

Under the physical measure, $\mathbb{P}$:

Gatheral, Jaisson and Rossenbaum proposed the following rough/fractional volatility model:

$$\begin{cases} dS_t = S_t \mu_t dt + S_t \sigma_t dW_t, & S_0 > 0 \\ \sigma_t = \sigma_0 \exp(W_t^H), & \sigma_0 > 0. \end{cases}$$

where W^H is a fractional Brownian motion with Hurst parameter $H \in (0, 1/2)$.

Implied volatility

- Asset price process: $(S_t = \mathrm{e}^{X_t})_{t\geq 0}$, with $X_0 = 0$.
- Black-Scholes-Merton (BSM) framework:

$$C_{\mathrm{BS}}(\tau, k, \sigma) := \mathbb{E}_0\left(\mathrm{e}^{X_\tau} - \mathrm{e}^k\right)_+ = \mathcal{N}(d_+) - \mathrm{e}^k \mathcal{N}(d_-),$$

$$d_\pm := -\frac{k}{\sigma\sqrt{\tau}} \pm \frac{1}{2}\sigma\sqrt{\tau}.$$

- Spot implied volatility $\sigma_\tau(k)$: the unique (non-negative) solution to

$$C_{\mathrm{observed}}(\tau, k) = C_{\mathrm{BS}}(\tau, k, \sigma_\tau(k)).$$

- Implied volatility: unit-free measure of option prices.

At the money skew

Let $\sigma_{BS}(k,\tau)$ denote the Black-Scholes implied volatility ($\tau := T - t$ and $k = \log\left(\frac{K}{S}\right)$) for an asset S. Then the at-the-money volatility skew is defined as

$$\psi(\tau) = \left|\frac{\partial}{\partial k}\sigma_{BS}(k,\tau)\right|_{k=0} \quad ; \quad \tau \geq 0.$$

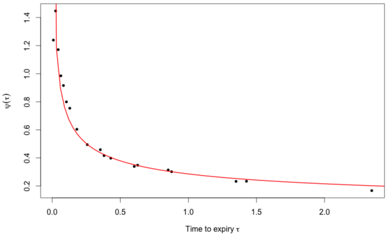

Figure 1.2: The black dots are non-parametric estimates of the S&P at-the-money (ATM) volatility skews as of August 14, 2013; the red curve is the power-law fit $\psi(\tau) = A\,\tau^{-0.407}$, τ measured in years.

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Our general framework

$$
\begin{aligned}
\mathrm{d}X_t &= -\frac{1}{2}V_t\mathrm{d}t + \sqrt{V_t}\mathrm{d}W_t, \quad && X_0 = 0, \\
V_t &= \Phi\left(\int_0^t g(t-s)\mathrm{d}Y_s\right), \quad && V_0 > 0,\ \alpha \in (-1/2, 1/2), \\
dY_t &= b(Y_t)\mathrm{d}t + \sigma(Y_t)\mathrm{d}Z_t, \quad && \mathrm{d}Z_t\mathrm{d}W_t = \rho\mathrm{d}t.
\end{aligned}
$$

where $\Phi \in \mathcal{C}^1$, $g \in \mathcal{L}^\alpha := \left\{u^\alpha L(u) : L \in \mathcal{C}_b^1([0,T]), \alpha \in \left(-\frac{1}{2}, \frac{1}{2}\right)\right\}$ and Y satisfies Yamada-Watanabe conditions for path-wise uniqueness.

Examples in this framework

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\mathrm{d}X_t &= -\frac{1}{2}V_t\mathrm{d}t + \sqrt{V_t}\mathrm{d}W_t, & X_0 &= 0,\\
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Rough Bergomi:

$$
\begin{aligned}
\mathrm{d}X_t &= -\frac{1}{2}V_t\mathrm{d}t + \sqrt{V_t}\mathrm{d}W_t, && X_0 = 0\\
V_t &= \xi_0(t)\mathcal{E}\left(2\nu C_H\int_0^t \frac{dZ_u}{(t-u)^{1/2-H}}\right), && \nu, \xi_0(\cdot) > 0\\
dZ_t dW_t &= \rho dt, && \rho \in (0,1)
\end{aligned}
$$

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$\Downarrow$ $g(u) = u^{\alpha}$, $\Phi(x) = \eta + Id.$, $\mathrm{d}Y_t = \kappa(\theta - Y_t)\mathrm{d}t + \xi\sqrt{Y_t}\mathrm{d}Z_t$

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Rough Heston:

$$
\begin{aligned}
\mathrm{d}X_t &= -\frac{1}{2}V_t\mathrm{d}t + \sqrt{V_t}\mathrm{d}W_t, && X_0 = 0,\\
Y_t &= \int_0^t \kappa(\theta - Y_s)\mathrm{d}t + \int_0^t \xi\sqrt{Y_s}\mathrm{d}Z_s && V_0, \kappa, \xi, \theta > 0,\ 2\kappa\theta > \xi^2\\
V_t &= \eta + \int_0^t (t-s)^\alpha \mathrm{d}Y_s, && \eta > 0,\ \alpha \in (-1/2, 1/2).
\end{aligned}
$$

FCLT for Hölder cont. processes:

$$
\begin{tikzcd}
(\mathcal{C}^{1/2-}([0,T]), \|\cdot\|_{1/2}) \arrow[r, bend left, "\mathcal{G}^\alpha"] & (\mathcal{C}^{\alpha+1/2}([0,T]), \|\cdot\|_{\alpha+1/2}) \arrow[r, bend left, "\Phi"] & (\mathcal{C}^{\alpha+1/2}([0,T]), \|\cdot\|_{\alpha+1/2}) \\
Y_n(t) \arrow[r, "\mathcal{G}^\alpha"] \arrow[d, Rightarrow, "{(\mathcal{L}, \|\cdot\|_{1/2})}"'] & \mathcal{B}_n(t) = \mathcal{G}^\alpha(Y_n(t)) \arrow[r, "\Phi"] \arrow[d, Rightarrow, "{(\mathcal{L}, \|\cdot\|_{\alpha+1/2})}"'] & \Phi\left(\mathcal{B}_n(t)\right) \arrow[d, Rightarrow, "{(\mathcal{L}, \|\cdot\|_{\alpha+1/2})}"'] \\
Y(t) \arrow[r, "\mathcal{G}^\alpha"] & \mathcal{B}(t) = \int_0^t g(t-s)\mathrm{d}Y_s \arrow[r, "\Phi"] & \Phi\left(\mathcal{B}(t)\right)
\end{tikzcd}
$$

FCLT for Hölder continuous processes

Theorem (rough Donsker theorem)

Consider the sequence $(W_n(t))_{n\geq 1}$ *and* W *its weak limit in* $\left(\mathcal{C}^{1/2}([0,T]), \|\cdot\|_{1/2}\right)$.
Then $(\mathcal{G}^{\alpha} W_n)_{n\geq 1}$ *converges weakly to* $\int_0^{\cdot} g(\cdot - s)\mathrm{d}W_s$ *in* $\left(\mathcal{C}^{\alpha+1/2}([0,T]), \|\cdot\|_{\alpha+1/2}\right)$ *for* $\alpha \in (-\frac{1}{2}, \frac{1}{2})$.

FCLT for rough volatility models

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Define recursively in time, for any $n \geq 1$, $t \in [0, T]$, $t_k = \frac{k}{N}$

$$X_n(t) := -\frac{1}{2}\frac{T}{n}\sum_{k=1}^{\lfloor nt \rfloor} \Phi\left(\left(\mathcal{G}^{\alpha} Y_n\right)(t_k)\right) + \sqrt{\frac{T}{\sigma n}}\sum_{k=1}^{\lfloor nt \rfloor} \sqrt{\Phi\left(\left(\mathcal{G}^{\alpha} Y_n\right)(t_k)\right)}\left(W_n(t_{k+1}) - W_n(t_k)\right)$$

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Theorem (rDonsker for rough volatility models)

$(X_n)_{n\geq 1}$, converges weakly to X in $(\mathcal{C}^{1/2-}(\mathbb{T}), \|\cdot\|_{1/2_-})$,

$$\begin{aligned}
\mathrm{d}X_t &= -\frac{1}{2}V_t\mathrm{d}t + \sqrt{V_t}\mathrm{d}W_t, & X_0 = 0,\\
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\end{aligned}$$

Example: rough Bergomi smiles

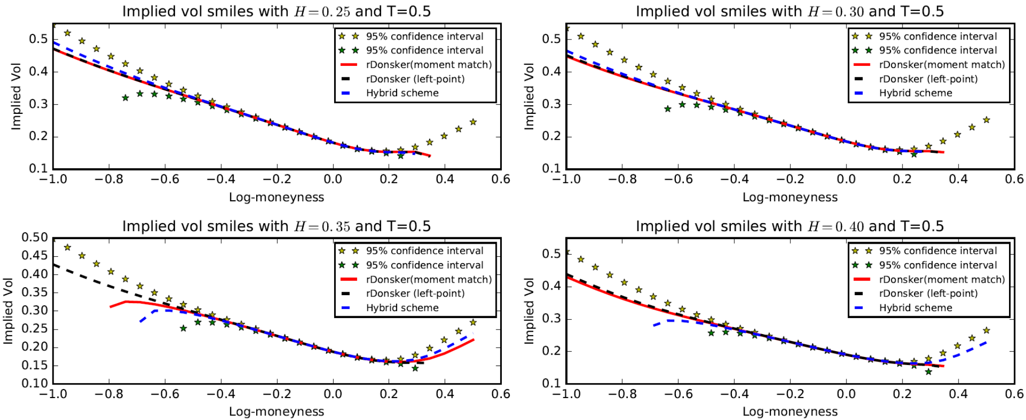
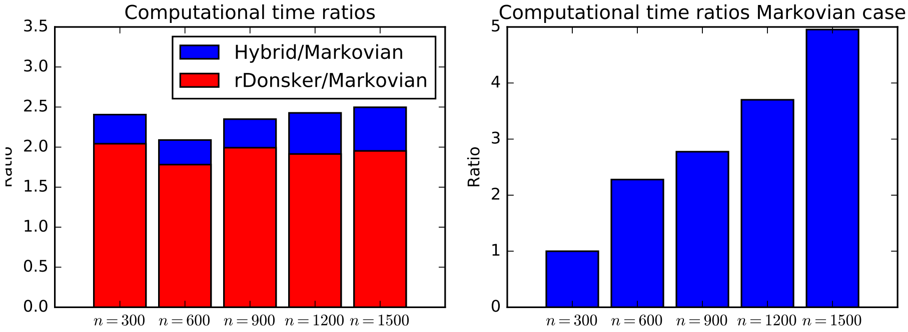


Figure 1: Parameters: $\nu = 1, \rho = -0.7, \xi_0 = 0.04$, $n = 468$ steps

Performance

- rDonsker is $1.25\times$ faster than Hybrid scheme (because we omit the Cholesky bit)



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Part 1: in pricing of vanilla options based on faster Monte Carlo approximations for a family of rough stochastic volatility models. [H-Jacquier-Muguruza '17])

Part 2: in calibration by means of machine learning techniques (ongoing with A. Muguruza and M. Tomas)

Part 2: Speed-ups on calibration

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- one step away from of-the-shelf optimizers to explore the parameter space more efficiently, limiting the number of function evaluations for calibration. Tests on this with Amir Sani and Aitor Muguruza.
- Main idea: prior to calibration, approximate the implied volatility function

$$\sigma : (\underbrace{\alpha, \beta, \rho, H}_{\text{parameters}}) \times (\tau, k) \mapsto \sigma_\tau(k;\ \alpha, \beta, \rho, H)$$

by a deterministic function, learned by a neural network.
Two parts of the neural network (i) Approximation network (2) Calibration network on top.
See also calibration by neural networks: Recent work of Bayer and Stemper:
Both works rely on the crucial observation of separation the **approximation** and the **calibration** networks.

Approximation by neural networks I

General setup: two parts of the network:

1. Generator: Input (parameters) Output (implied volatilities)
2. Calibrator: Input (implied volatilities) Output (*optimal* parameters).

Both feed-forward neural networks for the generator three hidden layers (1000-800-600)-nodes. Calibrator 1 layer on top.

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In order to train the network we first need to build a training set (supervised learning).

- For this we can use numerical valuation functions (Bergomi model, Rough Bergomi, Heston, ... Part 1): We generate 20,000 surfaces for each model, using a fixed grid of strikes and tenors.
- Though training time consuming, it can be done offline.
- We sample uniformly points in the parameter set $\theta \in \Theta$, then compute and save $f(\theta)$. Those samples will constitute our training set. We repeat this procedure until we reach enough samples for our surrogate function to be a good approximation.

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General setup: two parts of the network:

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In order to train the network we first need to build a training set. Conclusions:

- This can be done online, fast (within range of ˜ 1 second already unoptimized)
- Evaluation of parameters now more direct than via Monte Carlo. One minimizes now the distance between the (approximator) surrogate functions $\hat{f}(\theta^*)$ and the volatility surface.

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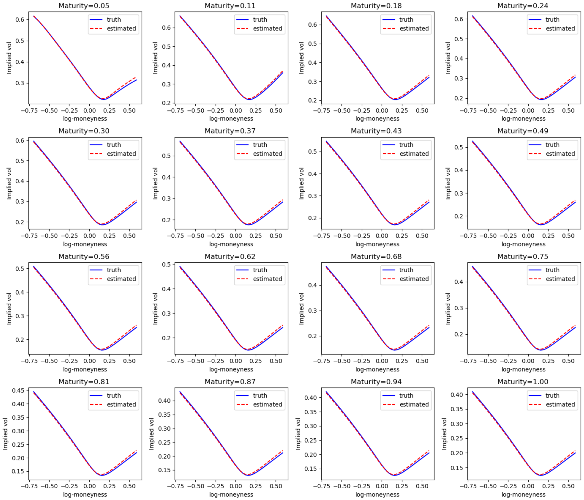
$\Rightarrow$ approximate several models at the same time.
New learning procedure:

- Train the generator on several models at the same time (here Parameters from Heston and Bergomi parameters) in Monte Carlo experiments as before.
- Calibrate several models at the same time $\Rightarrow$ determine the best-fit model to a given data (flag).
- Controlled experiments: train on both Bergomi and Heston $\Rightarrow$ test on data generated by Heston.

Approximation experiment via NN (Bergomi)



Thank you for your attention!

FCLT for Hölder continuous processes

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Define for any $\omega \in \Omega$, $n \geq 1$, $t \in [0, T]$, the approximating sequence

$$W_n(t,\omega) := \frac{1}{\sigma\sqrt{n}} \sum_{k=1}^{j} \xi_k(\omega) + \frac{nt - j}{\sigma\sqrt{n}} \xi_{j+1}(\omega), \quad \text{whenever } t \in \left[\frac{j}{n}, \frac{j+1}{n}\right), \text{ for } j = 0, \dots, n-1.$$

where the family $(\xi_i)_{i\geq 1}$ forms an iid sequence of centered random variables with finite moments of all orders and $\mathbb{E}(\xi_1^2) = \sigma^2 > 0$.

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Theorem (Donsker-Lamperti Theorem)

The sequence $(W_n)_{n\geq 1}$ converges weakly to a Brownian motion in $(\mathcal{C}^\alpha([0, T]), \|\cdot\|_\alpha)$ for all $\alpha < \frac{1}{2}$.

Monte-Carlo

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The left-point approximation may be modified e.g.

$$\int_0^{\frac{Ti}{n}} g\left(\frac{Ti}{n} - s\right) \mathrm{d}W_s \approx \frac{1}{\sqrt{n\sigma}} \sum_{k=1}^{j-1} g\left(t_k^*\right) \xi_k, \quad j = 0, \ldots, n$$

where t_k^* is chosen optimally to match first and second moments

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- The hybrid scheme also admits this trick