Formulation: Weak Approximation of Diffusions

$\boxed{\text{Problem:}}$ Given $g : \mathbb{R}^d \to \mathbb{R}$, approximate $E\left[g(X(T))\right]$, where $X \in \mathbb{R}^d$ solves

$$\begin{aligned} X(t) =& X(0) + \int_0^t a(s, X(s))ds \\ &+ \sum_{\ell=1}^{\ell_0} \int_0^t b^\ell(s, X(s))dW^\ell(s) \end{aligned} \tag{120}$$

Decomposition towards a smoothing approach

Let us decompose the Wiener process in the interval $[0, T]$ as

$$W(t) = W(T)\frac{t}{T} + B(t),$$

with $B(t)$ a Brownian bridge with zero end value. Then, for each $t \in [0, T]$ we have

$$\begin{aligned} X(t) =& X(0) + \int_0^t b(X(s))dB(t) + \frac{W(T)}{T}\int_0^t b(X(s))dt \\ =& X(0) + \int_0^t b(X(s))dB(t) + \frac{Y}{\sqrt{T}}\int_0^t b(X(s))dt \end{aligned}$$

with $Y \sim N(0, 1)$.

Conditional expectation Monte Carlo:

Observe that Y and B are independent. Then

$$\begin{aligned} E[g(X(T))] =& E^B[E^Y[g(X(T))|B]] \\ =& \frac{1}{\sqrt{2\pi}} E^B[\int g(X(T;y,B)) \exp(-y^2/2) dy] \end{aligned}$$

Observe that

$$H(B) = \int g(X(T;y,B)) \exp(-y^2/2) dy$$

has, for many practical cases, a smooth dependence wrt X_0 due to the smoothness of the pdf of Y. Use integration by parts to check it!

Pdf Example Let us take $g(x) = \delta(x - K)$. Then

$$H(B) = \int \delta(X(T; y, B) - K) \exp(-y^2/2) dy$$
$$= \exp(-y_*^2(K)/2) \frac{dy_*}{dx}(K)$$

where $y_*(x)$, assumed here an invertible function, satisfies

$$X(T; y_*(x), B) = x.$$

Lognormal Example Let $dX = \sigma X dW$. Then y_* is deterministic (it does not depend on B) and reads

$$y_*(x) = (\log(x/x_0) + T\sigma^2/2) \frac{1}{\sqrt{T}\sigma}.$$

Binary Example Let us take $g(x) = \mathbf{1}_{x>K}$. Then

$$H(B) = \int \mathbf{1}_{X(T;y,B)>K} \exp(-y^2/2)dy$$
$$= \sqrt{2\pi} P(Y > y_*(K)) \frac{dy_*}{dx}(K)$$

where $y_*(x)$, assumed here an invertible function, satisfies

$$X(T; y_*(x), B) = x.$$

In general, we have via the implicit function theorem that

$$\frac{\partial}{\partial y} X(T; y_*(x), B) \frac{dy_*}{dx}(x) = 1$$

Here the Malliavin derivative $Z(T) = \frac{\partial}{\partial y} X(T; y, B)$ is given by the equation (which holds for $t \in [0, T]$)

$$Z(t) = \int_0^t b'(X(s))Z(s)dB(s) + \frac{1}{\sqrt{T}} \int_0^t b(X(s))dB(s) + \frac{y}{\sqrt{T}} \int_0^t b'(X(s))Z(s)ds$$