5

LIBOR and swap market models

In the previous chapters we presented models based on the instantaneous short rate and the instantaneous forward rate. These models suffer from a number of drawbacks. Firstly, calibration to the prices of commonly traded vanilla instruments such as caps, floors or swaptions can be quite involved. Exotic derivatives depending on the volatilities of many different rates may need to be calibrated to a large set of market instruments, which is difficult when using a short-rate model. Secondly, although instantaneous rates are mathematically convenient, they are not directly observable in the market, nor are they related in a straightforward manner to the prices of any traded instruments. It can be difficult to relate the model parameters, such as mean-reversion in the Hull–White model, to a market-observable quantity.

In the LIBOR market model (LMM) we are going to use market rates, namely the forward LIBOR rates, as state variables modelled by a set of stochastic differential equations. For a suitable choice of numeraire we will

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express the drifts in these SDEs as functions of the volatilities and correlations among the forward rates.

A remarkable feature of the LMM is that the model prices are consistent with Black's formula. For a given forward LIBOR rate setting at time S and maturing at time $T > S$, the forward-rate dynamics is driftless under the forward measure P_T. This is consistent with Black's formula for caplets, where the LIBOR rate underlying each caplet is a log-normal process. Using these facts, we shall see how Black's formula arises naturally in the LMM framework. This is a major advantage of the LMM. It means that we can calibrate to implied (at-the-money) cap volatilities automatically.

The swap market model (SMM) makes it possible to derive Black's formula for swaptions. Moreover, it is possible to apply the LMM to obtain an analytic approximation (known as Rebonato's formula) for the volatility in Black's swaption formula. This facilitates efficient calibration of the LMM.

5.1 LIBOR market model

Consider a set of dates $0 \leq T_0 < T_1 < \cdots < T_n$ with accrual periods $\tau_i = T_i - T_{i-1}$ for $i = 1, \ldots, n$. The forward LIBOR rate $F(t; T_{i-1}, T_i)$ associated with each accrual period τ_i is a simply compounded rate parameterised by three time indices, the present time t, the start of a spot LIBOR rate T_{i-1} and the maturity T_i, where $t \leq T_{i-1} < T_i$. In Section 1.2 we saw that the forward rate can be expressed in terms of zero-coupon bonds as

$$F(t; T_{i-1}, T_i) = \frac{1}{\tau_i}\left(\frac{B(t, T_{i-1})}{B(t, T_i)} - 1\right).$$

For notational convenience we put

$$F_i(t) = F(t; T_{i-1}, T_i)$$

for $i = 1, \ldots, n$. In this notation $F_i(T_{i-1}) = L(T_{i-1}, T_i)$ is the spot LIBOR rate starting at time T_{i-1} and maturing at T_i (see Figure 5.1). It is common practice to refer to T_{i-1} as the reset date or expiry of the forward rate.

We fix any $j = 1, \ldots, n$ and take $Z_1^j(t), \ldots, Z_n^j(t)$ to be correlated Brownian motions under the forward measure P_{T_j} such that

$$dZ_i^j(t) dZ_k^j(t) = \rho_{i,k} dt \tag{5.1}$$

for each $i, k = 1, \ldots, n$, where $\rho = (\rho_{i,k})_{i,k=1}^n$ is a positive definite symmetric matrix of correlations.

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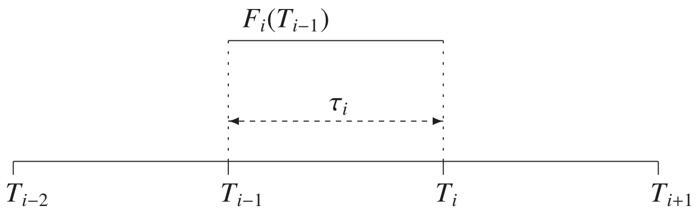

Figure 5.1 Schematic of $F_i(T_{i-1}) = L(T_{i-1}, T_i)$, the spot LIBOR rate between times T_{i-1} and T_i.

In the **LIBOR market model** (LMM) the forward rates $F_i(t)$ are assumed to satisfy an SDE of the form

$$dF_i(t) = \mu_i^j(t)F_i(t)dt + \sigma_i(t)F_i(t)dZ_i^j(t), \tag{5.2}$$

where $\sigma_i(t)$ is a bounded deterministic function for each $i = 1, \ldots, n$. The $\sigma_i(t)$ are called the **instantaneous volatilities** of the forward rates $F_i(t)$, and the $\rho_{i,k}$ the **instantaneous correlations** between $F_i(t)$ and $F_k(t)$.

In Section 5.3 we are going to establish a formula determining the drifts $\mu_i^j(t)$. For the time being, we just consider the particularly simple case when $i = j$. From Section 2.4 we know that $F_i(t)$ is a martingale under the forward measure P_{T_i}. Because $Z_i^i(t)$ is a Brownian motion under P_{T_i}, this means that $\mu_i^i(t) = 0$ and

$$dF_i(t) = \sigma_i(t)F_i(t)dZ_i^i(t). \tag{5.3}$$

Orthogonal Brownian motions

In Section 5.3 we are going to apply the Girsanov theorem to derive a formula for the drifts $\mu_i^j(t)$. One slight difficulty will be that the standard version of the Girsanov theorem (see [BSM]) allows for orthogonal (independent) Brownian motions only, whereas $Z_1^j(t), \ldots, Z_n^j(t)$ are correlated Brownian motions under the forward measure P_{T_j}. To deal with this difficulty we can take $W^j(t) = (W_1^j(t), \ldots, W_n^j(t))$ to be an n-dimensional Brownian motion under P_{T_j} (hence $W_1^j(t), \ldots, W_n^j(t)$ are independent) and put

$$Z_i^j(t) = \sum_{l=1}^{n} \eta_{i,l} W_l^j(t) \quad \text{for each } i = 1, \ldots, n, \tag{5.4}$$

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where η is a square root of the correlation matrix ρ, that is, a symmetric matrix η such that

$$\sum_{l=1}^{n} \eta_{i,l}\eta_{k,l} = \rho_{i,k} \quad \text{for each } i, k = 1, \ldots, n. \tag{5.5}$$

It follows that $Z_1^j(t), \ldots, Z_n^j(t)$ are Brownian motions under the forward measure P_{T_j} correlated as in (5.1).

Exercise 5.1 Show that $Z_1^j(t), \ldots, Z_n^j(t)$ given by (5.4) are indeed Brownian motions under the forward measure P_{T_j} such that (5.1) holds.

5.2 Black's caplet formula

The fact that each forward rate $F_i(t)$ can be modelled as a driftless log-normal process under the forward measure P_{T_i} allows us to derive Black's pricing formula for caplets stated in Section 2.9. The ability of the LMM framework to reproduce in a natural way one of the most important market formulae is one of its key features.

Assuming a notional amount $N = 1$, at each T_i for $i = 1, \ldots, n$ the holder of an interest rate cap receives the payoff

$$\tau_i(L(T_{i-1}, T_i) - K)^+ = \tau_i(F_i(T_{i-1}) - K)^+,$$

where K is the strike. Hence the price of the ith caplet at time t is

$$\begin{aligned}\mathbf{Cpl}_i(t) &= \tau_i B(t, T_i)\mathbb{E}_{P_{T_i}}\left((F_i(T_{i-1}) - K)^+ \middle| \mathcal{F}_t \right), \\ &= \tau_i B(t, T_i)\left(\mathbb{E}_{P_{T_i}}\left(F_i(T_{i-1})\mathbf{1}_{\{F_i(T_{i-1})\geq K\}} \middle| \mathcal{F}_t \right) - K\mathbb{E}_{P_{T_i}}\left(\mathbf{1}_{\{F_i(T_{i-1})\geq K\}} \middle| \mathcal{F}_t \right) \right).\end{aligned}$$

The forward rate $F_i(t)$ satisfies the SDE (5.3). This can be solved to give

$$F_i(T_{i-1}) = F_i(t)\exp\left(\int_t^{T_{i-1}} \sigma_i(s)dZ_i^i(s) - \frac{1}{2}\int_t^{T_{i-1}} \sigma_i(s)^2 ds \right).$$

Since $\sigma_i(t)$ is a bounded deterministic function and $Z_i^i(t)$ is a Brownian motion under P_{T_i}, the stochastic integral $\int_t^{T_{i-1}} \sigma_i(s)dZ_i^i(s)$ is independent of $\mathcal{F}_t$ and normally distributed with mean 0 and variance $\int_t^{T_{i-1}} \sigma_i(s)^2 ds$. It follows that

$$\mathbb{E}_{P_{T_i}}\left(F_i(T_{i-1})\mathbf{1}_{\{F_i(T_{i-1})\geq K\}} \middle| \mathcal{F}_t \right) = F_i(t)N(d_+) \tag{5.6}$$

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and

$$\mathbb{E}_{P_{T_i}}\left(\mathbf{1}_{\{F_i(T_{i-1})\geq K\}}\middle|\mathcal{F}_t\right) = N(d_-), \tag{5.7}$$

where

$$d_\pm = \frac{\ln\frac{F_i(t)}{K} \pm \frac{1}{2}\int_t^{T_{i-1}}\sigma_i(s)^2 ds}{\sqrt{\int_t^{T_{i-1}}\sigma_i(s)^2 ds}}$$

and

$$N(x) = \int_{-\infty}^{x}\frac{1}{\sqrt{2\pi}}e^{-\frac{y^2}{2}}dy,$$

is the standard normal distribution function.

Exercise 5.2 Verify formulae (5.6) and (5.7).

It follows that the price at time t of the ith caplet is

$$\mathbf{Cpl}_i(t) = \tau_i B(t,T_i)(F(t;T_{i-1},T_i)N(d_+) - KN(d_-)). \tag{5.8}$$

Setting

$$v_i = \sqrt{\frac{1}{T_{i-1}-t}\int_t^{T_{i-1}}\sigma_i(s)^2 ds}, \tag{5.9}$$

we recover Black's formula (2.24) for caplets with v_i substituted for the volatility σ, that is, we have

$$\mathbf{Cpl}_i(t) = \mathbf{Cpl}_i^{\mathrm{Black}}(t;v_i).$$

We can think of v_i as the model implied caplet volatility. The caplet price in the LMM will be consistent with the market price when $v_i = \hat{\sigma}_i^{\mathrm{caplet}}$; see Section 2.9.

5.3 Drifts and change of numeraire

In this section we establish formulae for the drifts $\mu_i^j(t)$ in the SDE (5.2) for the forward rate $F_i(t)$. We have already seen in (5.3) that $\mu_i^i(t) = 0$. Hence, using the change of numeraire technique, we can compute $\mu_i^j(t)$ when $i \neq j$. To calculate the drift there are a number of related approaches we can take. Here we apply the Girsanov theorem.

Let $Z_1^j(t),\dots,Z_n^j(t)$ be correlated Brownian motions under the forward

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measure P_{T_j} such that (5.1) holds. Suppose that $i \leq j$. We then have $T_i \leq T_j$. By Exercise 2.2, the Radon–Nikodym derivative of P_{T_i} with respect to P_{T_j} is

$$\frac{dP_{T_i}}{dP_{T_j}} = \frac{B(0,T_j)}{B(0,T_i)} \frac{1}{B(T_i,T_j)}.$$

This is related to the change of numeraire from $B(t,T_j)$ to $B(t,T_i)$. The corresponding Radon–Nikodym density process is

$$\begin{aligned} \xi_j^i(t) &= \mathbb{E}_{P_{T_j}}\left(\frac{dP_{T_i}}{dP_{T_j}} \middle| \mathcal{F}_t \right) \\ &= \mathbb{E}_{P_{T_j}}\left(\frac{B(0,T_j)}{B(0,T_i)} \frac{B(T_i,T_i)}{B(T_i,T_j)} \middle| \mathcal{F}_t \right) = \frac{B(0,T_j)}{B(0,T_i)} \frac{B(t,T_i)}{B(t,T_j)} \end{aligned}$$

for any $t \in [0,T_i]$. It can be written as

$$\xi_j^i(t) = \frac{B(0,T_j)}{B(0,T_i)} \prod_{k=i+1}^{j} \frac{B(t,T_{k-1})}{B(t,T_k)} = \frac{B(0,T_j)}{B(0,T_i)} \prod_{k=i+1}^{j} (1+\tau_k F_k(t)) \,.$$

Applying the Itô formula, we get

$$d\xi_j^i(t) = \xi_j^i(t) \sum_{k=i+1}^{j} \frac{\tau_k dF_k(t)}{1+\tau_k F_k(t)} + (\cdots)dt.$$

The explicit expressions for the terms with dt will not be needed. We substitute

$$dF_k(t) = \mu_k^j(t)F_k(t)dt + \sigma_k(t)F_k(t)dZ_k^j(t)$$

using the SDE (5.2) and collect the terms with $dZ_k^j(t)$ and dt separately. Because $\xi_j^i(t)$ is a martingale under P_{T_j}, the terms with dt cancel out, leaving only those with $dZ_k^j(t)$, so

$$d\xi_j^i(t) = \xi_j^i(t) \sum_{k=i+1}^{j} \frac{\tau_k \sigma_k(t) F_k(t)}{1+\tau_k F_k(t)} dZ_k^j(t). \tag{5.10}$$

Next, substituting for $Z_k^j(t)$ from (5.4), we get

$$d\xi_j^i(t) = \xi_j^i(t) \sum_{k=i+1}^{j} \sum_{l=1}^{n} \frac{\tau_k \sigma_k(t) F_k(t)}{1+\tau_k F_k(t)} \eta_{k,l} dW_l^j(t),$$

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and solve this SDE to obtain

$$\xi_j^i(t) = \exp\Bigg(\int_0^t \sum_{k=i+1}^{j} \sum_{l=1}^{n} \frac{\tau_k\sigma_k(s)F_k(s)}{1+\tau_k F_k(s)} \eta_{k,l} dW_l^j(s) \\ - \frac{1}{2}\int_0^t \sum_{k=i+1}^{j} \sum_{l=i+1}^{j} \frac{\tau_k\sigma_k(s)F_k(s)}{1+\tau_k F_k(s)} \frac{\tau_l\sigma_l(s)F_l(s)}{1+\tau_l F_l(s)} \rho_{k,l} ds \Bigg).$$

Given that $W_l^j(t)$ for $l = 1, \ldots, n$ are the components of an n-dimensional Brownian motion under the forward measure P_{T_j}, we can apply the Girsanov theorem (see [BSM]) to conclude that

$$W_l^i(t) = W_l^j(t) - \int_0^t \sum_{k=i+1}^{j} \frac{\tau_k\sigma_k(s)F_k(s)}{1+\tau_k F_k(s)} \eta_{k,l} ds$$

for $l = 1, \ldots, n$ are the components of an n-dimensional Brownian motion under P_{T_i}. Now we apply (5.4) once again together with (5.5) to finally find using Exercise 5.1 that

$$\begin{aligned} Z_l^i(t) &= \sum_{m=1}^{n} \eta_{l,m} W_m^i(t) = \sum_{m=1}^{n} \eta_{l,m} \left(W_m^j(t) - \int_0^t \sum_{k=i+1}^{j} \frac{\tau_k\sigma_k(s)F_k(s)}{1+\tau_k F_k(s)} \eta_{k,m} ds \right) \\ &= Z_l^j(t) - \int_0^t \sum_{k=i+1}^{j} \frac{\tau_k\sigma_k(s)F_k(s)}{1+\tau_k F_k(s)} \rho_{k,l} ds \end{aligned}$$

for $l = 1, \ldots, n$ are Brownian motions under P_{T_i} correlated so that

$$dZ_k^i(t) dZ_l^i(t) = \rho_{k,l} dt. \tag{5.11}$$

We have proved the following proposition.

Proposition 5.1
Let $Z_1^j(t), \ldots, Z_n^j(t)$ be correlated Brownian motions under P_{T_j} satisfying (5.1). *Then for any $i \leq j$*

$$Z_l^i(t) = Z_l^j(t) - \int_0^t \sum_{k=i+1}^{j} \frac{\tau_k\sigma_k(s)F_k(s)}{1+\tau_k F_k(s)} \rho_{k,l} ds,$$

where $l = 1, \ldots, n$, are correlated Brownian motions under P_{T_i} such that (5.11) *holds.*

As a consequence of Proposition 5.1, we obtain the important result that the drifts $\mu_i^j(t)$ of the forward rates $F_i(t)$ are uniquely determined by the following formulae.

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Theorem 5.2
The drifts $\mu_i^j(t)$ of the forward rates $F_i(t)$ in (5.2) are given by

$$\mu_i^j(t) = \begin{cases} -\displaystyle\sum_{k=i+1}^{j} \frac{\tau_k \rho_{k,i} \sigma_i(t) \sigma_k(t) F_k(t)}{1+\tau_k F_k(t)} & \text{when } i < j, \\ 0 & \text{when } i = j, \\ \displaystyle\sum_{k=j+1}^{i} \frac{\tau_k \rho_{k,i} \sigma_i(t) \sigma_k(t) F_k(t)}{1+\tau_k F_k(t)} & \text{when } i > j. \end{cases}$$

Proof The case when $i = j$ was covered in (5.3). If $i < j$, we can apply Proposition 5.1 with $l = i$ to get

$$Z_i^i(t) = Z_i^j(t) - \int_0^t \sum_{k=i+1}^{j} \frac{\tau_k \sigma_k(s) F_k(s)}{1+\tau_k F_k(s)} \rho_{k,i} ds.$$

We substitute this into (5.3) and obtain

$$\begin{aligned} dF_i(t) &= \sigma_i(t) F_i(t) dZ_i^i(t) \\ &= \sigma_i(t) F_i(t) dZ_i^j(t) - \sigma_i(t) F_i(t) \sum_{k=i+1}^{j} \frac{\tau_k \sigma_k(t) F_k(t)}{1+\tau_k F_k(t)} \rho_{k,i} dt. \end{aligned}$$

Comparing this with (5.2), we arrive at the formula for $\mu_i^j(t)$. The case when $i > j$ is left as an exercise. □

Exercise 5.3 Let $Z_1^j(t), \ldots, Z_n^j(t)$ be correlated Brownian motions under P_{T_j} satisfying (5.1). Show that for any $i \geq j$

$$Z_l^i(t) = Z_l^j(t) + \int_0^t \sum_{k=j+1}^{i} \frac{\tau_k \sigma_k(s) F_k(s)}{1+\tau_k F_k(s)} \rho_{k,l} ds,$$

where $l = 1, \ldots, n$, are correlated Brownian motions under P_{T_i} such that (5.11) holds.

Exercise 5.4 Verify the formula for $\mu_i^j(t)$ in Theorem 5.2 when $i > j$.

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5.4 Terminal measure

A popular choice of numeraire is the zero-coupon bond $B(t, T_n)$ maturing at time T_n. The associated measure P_{T_n} is known as the **terminal measure**. It follows from Theorem 5.2 that

$$dF_i(t) = \mu_i^n(t)F_i(t)dt + \sigma_i(t)F_i(t)dZ_i^n(t) \tag{5.12}$$

with

$$\mu_i^n(t) = -\sum_{k=i+1}^{n} \frac{\tau_k\rho_{k,i}\sigma_i(t)\sigma_k(t)F_k(t)}{1+\tau_kF_k(t)} \tag{5.13}$$

for $i = 1, \ldots, n$, where $Z_1^n(t), \ldots, Z_n^n(t)$ are correlated Brownian motions under the terminal measure P_{T_n} such that for all $i, j = 1, \ldots, n$

$$dZ_i^n(t)dZ_j^n(t) = \rho_{i,j}dt. \tag{5.14}$$

5.5 Spot LIBOR measure

We are going to construct a discrete analogue of the money market account. At time 0 we begin by investing one dollar to buy an amount $B(0, T_0)^{-1}$ of T_0-bonds. At time T_0 the bonds will be worth $B(0, T_0)^{-1}$, which we reinvest to buy an amount $B(0, T_0)^{-1}B(T_0, T_1)^{-1}$ of T_1-bonds. We continue in this way, reinvesting the proceeds at each date T_{i-1} into zero-coupon bonds maturing at the next date T_i for $i = 1, \ldots, n$. For any $t \in [0, T_n]$, let $\alpha(t)$ denote the index of the next reset date at time t, that is,

$$\alpha(t) = \min\{j : t \le T_j,\ j = 0, \ldots, n\}. \tag{5.15}$$

The value of our bond portfolio at time t will be

$$L(t) = B(t, T_{\alpha(t)}) \prod_{k=0}^{\alpha(t)} \frac{1}{B(T_{k-1}, T_k)},$$

where we put $T_{-1} = 0$ for notational simplicity. We call $L(t)$ the **discrete money market account**.

> **Exercise 5.5** Show that $L(t)$ discounted by $B(t, T_n)$ is a martingale under the terminal measure P_{T_n}, and $L(t)$ discounted by $B(t)$ is a martingale under the risk-neutral measure Q.

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The measure corresponding to the choice of $L(t)$ as numeraire is called the **spot LIBOR measure** and will be denoted by P_L. The Radon–Nikodym derivative of P_L with respect to the terminal measure P_{T_n} is (see Section 2.1)

$$\frac{dP_L}{dP_{T_n}} = \frac{B(0,T_n)}{L(0)}\frac{L(T_n)}{B(T_n,T_n)} = B(0,T_n)L(T_n).$$

Observe that $B(0,T_n)L(T_n)$ is an $\mathcal{F}_{T_{n-1}}$-measurable random variable, hence P_L is defined on the σ-field $\mathcal{F}_{T_{n-1}}$.

We are going to show that the forward rates satisfy an SDE of the form

$$dF_i(t) = \mu_i^L(t)F_i(t)dt + \sigma_i(t)F_i(t)dZ_i^L(t), \tag{5.16}$$

where $Z_1^L(t),\ldots,Z_n^L(t)$ are correlated Brownian motions under P_L such that for each $i,j = 1,\ldots,n$

$$dZ_i^L(t)dZ_j^L(t) = \rho_{i,j}dt,$$

and we shall derive a formula for the drift $\mu_i^L(t)$.

We proceed in a similar fashion as in Section 5.3. Take $Z_1^n(t),\ldots,Z_n^n(t)$ to be correlated Brownian motions under the terminal measure P_{T_n} that satisfy (5.14) and consider the Radon–Nikodym density process

$$\begin{aligned}\xi_n^L(t) &= \mathbb{E}_{P_{T_n}}\left(\frac{dP_L}{dP_{T_n}}\middle|\mathcal{F}_t\right) = \mathbb{E}_{P_{T_n}}(B(0,T_n)L(T_n)|\mathcal{F}_t)\\ &= B(0,T_n)\mathbb{E}_{P_{T_n}}\left(\frac{L(T_n)}{B(T_n,T_n)}\middle|\mathcal{F}_t\right) = B(0,T_n)\frac{L(t)}{B(t,T_n)}.\end{aligned}$$

The last equality holds since $L(t)$ discounted by $B(t,T_n)$ is a martingale under P_{T_n} according to Exercise 5.5. Observe that for each $t \in [0,T_{n-1}]$

$$L(t) = B(t,T_{\alpha(t)})L(T_{\alpha(t)}),$$

and so

$$\begin{aligned}\xi_n^L(t) &= B(0,T_n)L(T_{\alpha(t)})\frac{B(t,T_{\alpha(t)})}{B(t,T_n)} = B(0,T_n)L(T_{\alpha(t)})\prod_{k=\alpha(t)+1}^{n}\frac{B(t,T_{k-1})}{B(t,T_k)}\\ &= B(0,T_n)L(T_{\alpha(t)})\prod_{k=\alpha(t)+1}^{n}(1+\tau_kF_k(t)).\end{aligned}$$

Just like in Section 5.3, it follows that

$$d\xi_n^L(t) = \xi_n^L(t)\sum_{k=\alpha(t)+1}^{n}\frac{\tau_k dF_k(t)}{1+\tau_kF_k(t)} + (\cdots)dt.$$

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Expressing $dF_k(t)$ by means of (5.12) and using the fact that $\xi_i^L(t)$ is a martingale under P_L, we find that all terms in dt will cancel. This gives

$$d\xi_n^L(t) = \xi_n^L(t) \sum_{k=\alpha(t)+1}^{n} \frac{\tau_k \sigma_k(t) F_k(t)}{1 + \tau_k F_k(t)} dZ_k^n(t).$$

The same argument as in Section 5.3, involving orthogonal Brownian motions and the Girsanov theorem, then shows that

$$Z_i^L(t) = Z_i^n(t) - \int_0^t \sum_{k=\alpha(t)+1}^{n} \frac{\tau_k \sigma_k(s) F_k(s)}{1 + \tau_k F_k(s)} \rho_{k,i} ds \tag{5.17}$$

for $i = 1, \dots, n$ are correlated Brownian motions under the spot LIBOR measure P_L such that

$$dZ_i^L(t) dZ_j^L(t) = \rho_{i,j} dt.$$

Hence we arrive at the following result.

Proposition 5.3
The forward rates $F_i(t)$ satisfy the SDE (5.16) *with drift*

$$\mu_i^L(t) = \sum_{k=\alpha(t)+1}^{i} \frac{\tau_k \rho_{k,i} \sigma_i(t) \sigma_k(t) F_k(t)}{1 + \tau_k F_k(t)}$$

for any $i = 1, \dots, n$ and $t \in [0, T_{i-1}]$, where $\alpha(t)$ denotes the index of the next reset date at time t given by (5.15).

Proof Using (5.17) to replace $Z_i^n(t)$ in (5.12) by $Z_i^L(t)$, we get

$$\begin{aligned}
dF_i(t) &= -\sum_{k=i+1}^{n} \frac{\tau_k \rho_{k,i} \sigma_i(t) \sigma_k(t) F_i(t) F_k(t)}{1 + \tau_k F_k(t)} dt + \sigma_i(t) F_i(t) dZ_i^n(t) \\
&= -\sum_{k=i+1}^{n} \frac{\tau_k \rho_{k,i} \sigma_i(t) \sigma_k(t) F_i(t) F_k(t)}{1 + \tau_k F_k(t)} dt \\
&\quad + \sigma_i(t) F_i(t) \left(\sum_{k=\alpha(t)+1}^{n} \frac{\tau_k \sigma_k(t) F_k(t)}{1 + \tau_k F_k(t)} \rho_{k,i} dt + dZ_i^L(t) \right) \\
&= \sum_{k=\alpha(t)+1}^{i} \frac{\tau_k \rho_{k,i} \sigma_i(t) \sigma_k(t) F_i(t) F_k(t)}{1 + \tau_k F_k(t)} dt + \sigma_i(t) F_i(t) dZ_i^L(t).
\end{aligned}$$

This shows that $F_i(t)$ satisfies (5.16) with drift $\mu_i^L(t)$ as stated in the proposition. □

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Remark 5.4
In the LMM we need to discretise the continuous-time dynamics if we wish to model a given derivative. This will introduce a bias in the forward rates, which depends on the number of terms in the drift summation. From (5.12) we can see that, under the terminal measure P_{T_n}, the drift of a given forward rate $F_i(t)$ contains $n - i$ terms in the summation. However, under the spot LIBOR measure P_L, the number of terms in the summation is time dependent. In fact for $T_{i-2} \leq t < T_{i-1}$ there is just one term. Therefore, under the spot LIBOR measure the bias is more evenly spread between the different rates.

5.6 Brace–Gątarek–Musiela approach

The approach due to Brace, Gątarek and Musiela is to write the SDE for the forward rates $F_i(t)$ in terms of Brownian motions under the risk-neutral measure Q and to derive a formula for the drifts by working within the HJM framework.

To this end it is convenient to write the SDE (5.2) for the forward rates $F_i(t)$ in terms of orthogonal rather than correlated Brownian motions. Substituting (5.4) into (5.2), we get

$$dF_i(t) = \mu_i^j(t)F_i(t)dt + F_i(t)\sum_{k=1}^{n}\lambda_{i,k}(t)dW_k^j(t), \tag{5.18}$$

where

$$\lambda_{i,k}(t) = \sigma_i(t)\eta_{i,k}$$

for $i, k = 1, \ldots, n$, and where $(W_1^j(t), \ldots, W_n^j(t))$ is an n-dimensional Brownian motion under the forward measure P_{T_j}.

In Section 2.6, formula (2.14), we related a one-dimensional Brownian motion under the risk-neutral measure Q to a one-dimensional Brownian motion under the forward measure P_S by means of the Girsanov theorem. This argument can be extended to multi-dimensional Brownian motions.

Exercise 5.6 Let $(W_1(t), \ldots, W_n(t))$ be an n-dimensional Brownian motion under the risk-neutral measure Q. Show that

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$(W_1^S(t), \ldots, W_n^S(t))$, where

$$W_k^S(t) = W_k(t) - \int_0^t \Sigma_k(u, S)du \quad \text{for } k = 1, \ldots, n \text{ and } t \in [0, S],$$

is an n-dimensional Brownian motion under the forward measure P_S.

It follows from this exercise that $(W_1^j(t), \ldots, W_n^j(t))$, where

$$W_k^j(t) = W_k(t) - \int_0^t \Sigma_k(u, T_j)du \quad \text{for } k = 1, \ldots, n \text{ and } t \in [0, T_j],$$

in an n-dimensional Brownian motion under P_{T_j}. Substituting this into (5.18) where we set $j = i$ and use the fact that $\mu_i^i(t) = 0$, we find that

$$dF_i(t) = \mu_i(t)F_i(t)dt + F_i(t)\sum_{k=1}^n \lambda_{i,k}(t)dW_k(t), \tag{5.19}$$

where

$$\mu_i(t) = -\sum_{k=1}^n \lambda_{i,k}(t)\Sigma_k(t, T_i),$$

and where (by an argument similar to that in Exercise 5.1)

$$Z_i(t) = \sum_{l=1}^n \eta_{i,l} W_l(t)$$

for $i = 1, \ldots, n$ are correlated Brownian motions under the risk-neutral measure Q such that

$$dZ_i(t)dZ_j(t) = \rho_{i,j}dt.$$

Now recall that the zero-coupon bond price process in a multi-factor HJM model evolves according to

$$dB(t, T) = r(t)B(t, T)dt + \sum_{k=1}^n \Sigma_k(t, T)B(t, T)dW_k(t),$$

where $(W_1(t), \ldots, W_n(t))$ is an n-dimensional Brownian motion under the risk-neutral measure Q and

$$\Sigma_k(t, T) = -\int_t^T \sigma_k(t, u)du,$$

with $\sigma_k(t, T)$ for $k = 1, \ldots, n$ being the volatilities of the instantaneous forward rate $f(t, T)$; see Theorem 4.6.

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Applying the Itô formula to

$$F_i(t) = \frac{1}{\tau_i}\left(\frac{B(t,T_{i-1})}{B(t,T_i)} - 1\right),$$

we can find that

$$\begin{aligned} dF_i(t) &= \frac{1}{\tau_i}\frac{dB(t,T_{i-1})}{B(t,T_i)} + \frac{1}{\tau_i}B(t,T_{i-1})d\left(\frac{1}{B(t,T_i)}\right) + \frac{1}{\tau_i}dB(t,T_{i-1})d\left(\frac{1}{B(t,T_i)}\right) \\ &= \frac{1+\tau_i F_i(t)}{\tau_i}\sum_{k=1}^{n}\Sigma_k(t,T_i)\left(\Sigma_k(t,T_i) - \Sigma_k(t,T_{i-1})\right)dt \\ &\quad + \frac{1+\tau_i F_i(t)}{\tau_i}\sum_{k=1}^{n}\left(\Sigma_k(t,T_{i-1}) - \Sigma_k(t,T_i)\right)dW_k(t). \end{aligned}$$

By comparing the coefficients in front of $dW_k(t)$ in the above expressions for $dF_i(t)$, we conclude that

$$\Sigma_k(t,T_i) = \Sigma_k(t,T_{i-1}) - \frac{\tau_i F_i(t)}{1+\tau_i F_i(t)}\lambda_{i,k}(t)$$

for each i such that $t \le T_{i-1}$. The last relationship can be iterated until we reach the smallest index j such that $t \le T_j$, which as before will be denoted by $\alpha(t)$; see (5.15). Namely,

$$\begin{aligned} \Sigma_k(t,T_i) &= \Sigma_k(t,T_{i-1}) - \frac{\tau_i F_i(t)}{1+\tau_i F_i(t)}\lambda_{i,k}(t) \\ &= \Sigma_k(t,T_{i-2}) - \frac{\tau_i F_i(t)}{1+\tau_i F_i(t)}\lambda_{i,k}(t) - \frac{\tau_{i-1}F_{i-1}(t)}{1+\tau_{i-1}F_{i-1}(t)}\lambda_{i-1,k}(t) \\ &\;\;\vdots \\ &= \Sigma_k(t,T_{\alpha(t)}) - \sum_{l=\alpha(t)+1}^{i}\frac{\tau_l F_l(t)}{1+\tau_l F_l(t)}\lambda_{l,k}(t). \end{aligned} \tag{5.20}$$

Substituting this into the formula for $\mu_i(t)$, we get

$$\begin{aligned} \mu_i(t) &= -\sum_{k=1}^{n}\lambda_{i,k}(t)\Sigma_k(t,T_i) \\ &= -\sum_{k=1}^{n}\lambda_{i,k}(t)\Sigma_k(t,T_{\alpha(t)}) + \sum_{l=\alpha(t)+1}^{i}\frac{\tau_l F_l(t)}{1+\tau_l F_l(t)}\sum_{k=1}^{n}\lambda_{i,k}(t)\lambda_{l,k}(t) \\ &= \sum_{k=1}^{n}\lambda_{i,k}(t)\int_t^{T_{\alpha(t)}}\sigma_k(t,u)du + \sum_{l=\alpha(t)+1}^{i}\frac{\tau_l\rho_{i,l}\sigma_i(t)\sigma_l(t)F_l(t)}{1+\tau_l F_l(t)}. \end{aligned} \tag{5.21}$$

We have proved the following result.

Proposition 5.5
The forward rates $F_i(t)$ satisfy the SDE

$$dF_i(t) = \mu_i(t)F_i(t)dt + \sigma_i(t)F_i(t)dZ_i(t),$$

where $Z_1(t), \ldots, Z_n(t)$ are correlated Brownian motions under the risk-neutral measure Q such that

$$dZ_i(t)dZ_j(t) = \rho_{i,j}dt,$$

and where the drifts $\mu_i(t)$ are given by (5.21).

Remark 5.6
Note that the first term in (5.21) contains the instantaneous forward-rate volatilities $\sigma_k(t, u)$. The need to model these volatilities complicates the formula for the drift. Brace, Gątarek and Musiela make the simplifying assumption that $\Sigma_k(t, T_{\alpha(t)}) = 0$. Under this assumption the SDE for the forward rate under the risk-neutral measure Q can be written as

$$dF_i(t) = \sum_{l=\alpha(t)+1}^{i} \frac{\tau_l \rho_{i,l} \sigma_i(t) \sigma_l(t) F_l(t)}{1 + \tau_l F_l(t)} F_i(t)dt + \sigma_i(t)F_i(t)dZ_i(t).$$

The following exercises show that the same formula for the drift applies without the need for the above simplifying assumption when working under the spot LIBOR measure P_L instead of the risk-neutral measure Q.

Exercise 5.7 Show that for all $k = 1, \ldots, n$ and $t \in [0, T_i]$

$$W_k^i(t) = W_k^L(t) - \int_0^t \left(\Sigma_k(s, T_i) - \Sigma_k(s, T_{\alpha(t)})\right) ds, \tag{5.22}$$

where $(W_1^L(t), \ldots, W_n^L(t))$ and $(W_1^i(t), \ldots, W_n^i(t))$ are n-dimensional Brownian motions under the spot LIBOR measure P_L and under the forward measure P_{T_i}, respectively.

Exercise 5.8 Using (5.22), show that the correlated Brownian motions

$$Z_j^i(t) = \sum_{k=1}^{n} \eta_{j,k} W_k^i(t), \quad Z_j^L(t) = \sum_{k=1}^{n} \eta_{j,k} W_k^L(t)$$

under the spot LIBOR measure P_L and under the forward measure P_{T_i},

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respectively, are related by

$$Z_j^i(t) = Z_j^L(t) + \int_0^t \sum_{k=\alpha(t)+1}^{i} \frac{\tau_k \rho_{j,k} \sigma_k(s) F_k(s)}{1 + \tau_k F_k(s)} ds \tag{5.23}$$

for all $j = 1, \ldots, n$ and $t \in [0, T_i]$.

Exercise 5.9 Using (5.23), show that the forward rates satisfy the SDE

$$dF_i(t) = \sum_{k=\alpha(t)+1}^{i} \frac{\tau_k \rho_{i,k} \sigma_i(t) \sigma_k(t) F_k(t)}{1 + \tau_k F_k(t)} F_i(t) dt + \sigma_i(t) F_i(t) dZ_i^L(t),$$

which is consistent with Proposition 5.3.

5.7 Instantaneous volatility

Much like the HJM model, the LMM is really a modelling framework rather than a fully formed model. To specify a concrete model we need to choose the form of both the instantaneous volatility and a correlation structure between the different forward rates. Then we need to calibrate the model to market data. In Section 5.2 we saw a direct link between the implied volatility in Black's formula for a caplet and the instantaneous volatility of the LMM formulation.

Piecewise constant form

For simplicity, we can assume that the instantaneous volatility $\sigma_i(t)$ of the forward rate $F_i(t)$ is constant over each accrual period and depends exclusively on the number of reset dates in the time interval $[t, T_{i-1}]$. This gives

$$\sigma_i(t) = \sigma_{i-\alpha(t)}, \tag{5.24}$$

where $\alpha(t)$ denotes the index of the next reset date defined by (5.15).

In Table 5.1 we can see that the volatility structure of the forward rates $F_1(t), \ldots, F_n(t)$ is described by n constant parameters $\sigma_1, \ldots, \sigma_n$. Note that the forward LIBOR rate $F_i(t)$ will have already expired for $t > T_{i-1}$.

Table 5.1 *Instantaneous volatilities for the time-homogeneous piecewise constant formulation* (5.24).

	$(0, T_0]$	$(T_0, T_1]$	$(T_1, T_2]$	$\cdots$	$(T_{n-2}, T_{n-1}]$
$F_1(t)$	σ_1	expired		$\cdots$	
$F_2(t)$	σ_2	σ_1	expired	$\cdots$	
$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	
$F_n(t)$	σ_n	σ_{n-1}	σ_{n-2}	$\cdots$	σ_1

Because the instantaneous volatility depends on time t only via the remaining number of reset dates, the choice (5.24) is referred to as a time-homogeneous form for the volatility. There is a sound reason why we would wish to avoid having an explicit dependence on t. A financially plausible model should allow for a stationary volatility term structure.

Another feature we need to account for is that in 'normal' market conditions the graph of the volatility term structure is typically hump shaped. The implied volatility is upward sloping for maturities out to two–three years and then falls gradually for caps with a longer time to maturity. However, in general, a time-homogeneous form is too inflexible to allow for an accurate fit to a typical hump-shaped term structure. With this in mind, we modify (5.24) by introducing constant parameters κ_i depending solely on the maturity of the ith forward rate,

$$\sigma_i(t) = \kappa_i \sigma_{i-\alpha(t)}.$$

If the κ_i are chosen so that they are close to one, the above form for the instantaneous volatility is approximately time homogenous.

Parametric form

An alternative to the piecewise constant form is to choose some parametric form for the instantaneous volatility. A popular parametric form that is both time homogenous (in the sense that it depends on t via the time to expiry $T_{i-1} - t$) and allows for a hump-shaped volatility is

$$\sigma_i(t) = (a + b(T_{i-1} - t))\mathrm{e}^{-c(T_{i-1}-t)} + d, \tag{5.25}$$

where a, b, c and d are constants and $c > 0$.

An appropriate choice of parameters leads to a curve with low volatility

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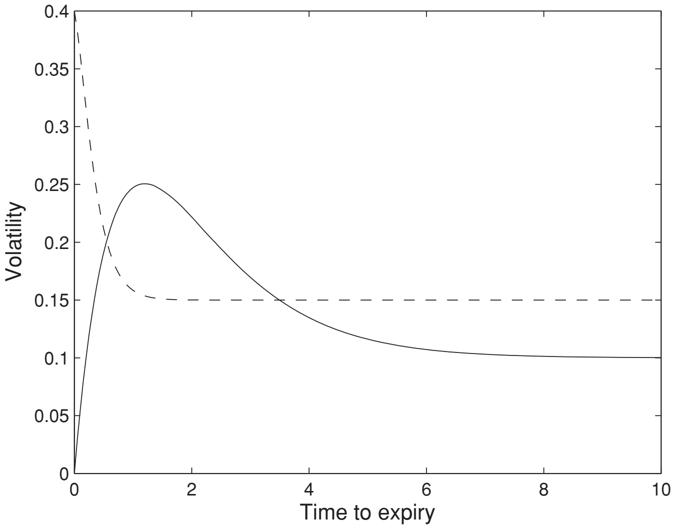

Figure 5.2 The hump-shaped curve (solid line) is given by (5.25) with parameters $a = -0.1$, $b = 0.5$, $c = 1$, $d = 0.1$. The dashed line is obtained with $a = 0.25$, $b = 1$, $c = 5$, $d = 0.15$.

for forward rates with a long time to expiry, peaking at around two years to expiry, and then falling again as the rates are about to reset. Examining the behaviour of (5.25) as $T_{i-1} - t$ tends to zero and as $T_{i-1} - t$ tends to infinity, we can see that $a + d$ is the instantaneous volatility of a forward rate for very short expiry times, while d is the volatility for very long expiry times. We must choose parameters such that $a + d > 0$ and $d > 0$. The extremum of (5.25) (when written as a function of time to expiry $T_{i-1} - t$) is given by $\frac{1}{c} - \frac{a}{b}$, and it is a maximum if $b > 0$ (see Exercise 5.10). In 'normal' market conditions the parameters should give rise to a hump-shaped curve as depicted in Figure 5.2, where the maximum occurs between one and two years to expiry.

Exercise 5.10 Show that the extremum of the instantaneous volatility (5.25) is attained when the time to expiry is $T_{i-1} - t = \frac{1}{c} - \frac{a}{b}$, and that it is a maximum if $b > 0$.

Remark 5.7

What constitutes normal market conditions is open to debate, particularly

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since the credit crisis, where the implied volatility could be at its highest for short expiries. Fortunately, for a certain range of parameters, (5.25) is monotonically decaying (see the dashed line in Figure 5.2), so is capable of reproducing the volatility term structure observed since the credit crisis.

As was the case with the time-homogeneous piecewise constant function (5.24), we need to modify (5.25) if we are to provide an accurate fit to market data. Issues relating to calibration are discussed in Chapter 6, where we focus our attention on the parametric form of the instantaneous volatility.

5.8 Instantaneous correlation

In the LMM (5.2), the instantaneous correlation between the forward rates $F_i(t)$ and $F_j(t)$ is denoted by $\rho_{i,j}$. The correlation matrix for the set of forward rates $F_1(t), \ldots, F_n(t)$ is $\rho = (\rho_{i,j})_{i,j=1}^n$.

For reasons of tractability it is desirable to describe ρ in terms of some parsimonious parametric form. We then need to fit the model parameters to a sample correlation matrix estimated using historical data. This is significantly different from the approach taken for the instantaneous volatilities, where we calibrate to the prices (or equivalently to the implied volatilities) of actively traded options. In general, it is difficult to extract information about the correlation function from the market prices of European swaptions. Swaptions are far more dependent on the functional form of the instantaneous volatilities than the qualitative shape of the correlation matrix.

We are going to examine the steps involved in estimating the correlation matrix ρ from historical data and then use the results to motivate some simple parametric form for ρ.

Empirical estimation of the correlation matrix

To begin estimating a correlation structure we first need a daily time series of forward LIBOR rates.

Example 5.8

For the US dollar the forward LIBOR rates can be derived by bootstrapping a set of mid-market spot-starting swap rates for maturities

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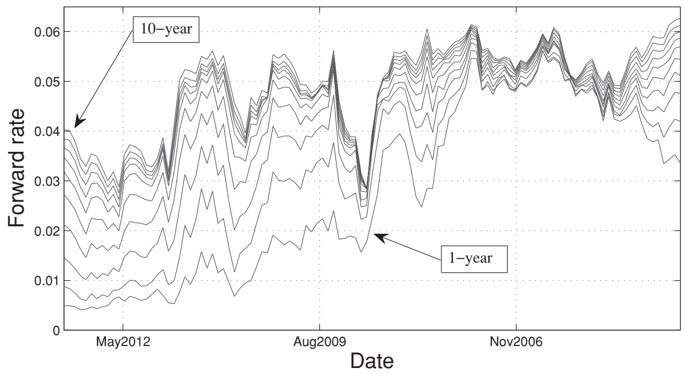

Figure 5.3 USD forward rates between 31 July 2004 and 15 Mar 2013. The time series labelled '1-year' and '10-year' represent the forward rates expiring in one year and 10 years, respectively. The forward rates are derived from mid-market spot-starting swap rates downloaded from the US Federal Reserve website.

1, 2, 3, 4, 5, 6, 7, 10 and 30 years. Before we perform the bootstrapping procedure, however, it is necessary to apply cubic spline interpolation between market spot rates to get an estimate of intermediate rates not present in the data.

The next step is to use the time series of LIBOR forward rates to create a sample correlation matrix. The matrix presented in Figure 5.4 is calculated by taking the correlation between log-changes in the forward-rate time series. In the forward-rate time series in Figure 5.3 used to calculate the correlation matrix the time-to-expiry is fixed (constant residual time to expiry) while the expiry date varies. In Figure 5.3 we plot the USD forward rates (with one-year tenor) for expiries of one, two, out to 10 years. For clarity we do not plot the USD rates for expiries of 11 out to 20 years.

Another approach is to use a forward-rate time series where the expiry of each forward rate is fixed and the time-to-expiry decreases while moving through time. In this approach only one year of data is used to construct the correlation matrix since the first forward rate expires in one year.

The sample correlation matrix is difficult to interpret at first glance. Owing to statistical noise the matrix is bound to contain some spurious correlations. Also, the market data we use to derive the forward LIBOR

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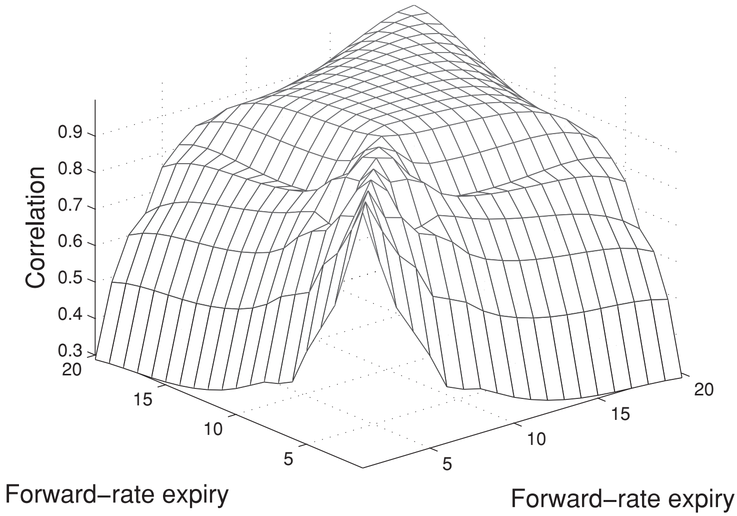

Figure 5.4 Sample correlation matrix for forward rate data in Figure 5.3.

rates should consist only of liquid instruments (those with a tight bid/ask spread). However, the one-year swap rate is not actively traded, and this will affect the correlations between the first forward rate and the later ones.

There are, nonetheless, a number of qualitative features we can observe in Figure 5.4. We would expect all correlations to be positive, and this is confirmed by Figure 5.4. Secondly, it is sensible to assume that the correlation between the first forward rate (expiring in one year) and the forward rates expiring in two, three, ... years should be a monotonically decreasing function. Again, this is broadly supported by the data. Moreover, as we can see in Figure 5.4, this function has a convex shape. We would also expect the correlation between the forward rate expiring in one year and the rate expiring in two years to be less than the correlation between the rate expiring in 19 years and that expiring in 20 years. Returning to Figure 5.4, we can see that for later expiring rates, for example the correlation between the forward rate expiring in 20 years and the rates expiring in $19, 18, \ldots$ years, the correlation is a monotonically decreasing function, but this time with an (approximately) concave shape.

In summary, we conclude that for a fixed i the correlation $\rho_{i,j}$ is a de-

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creasing function of j, where $j > i$. For $i = 1$ this function should have a convex shape, and as i increases the function should become less convex.

Remark 5.9

We should be mindful of the fact that, during a major market event such as the credit crisis, the correlation between the forward rates can change markedly. Unfortunately, estimates based on historical data can be slow to respond to such sudden changes.

Parametric form for the correlation matrix

There are a number of approaches one could take to modelling the above empirical correlation structure in a parsimonious way. A standard approach is to use a decaying exponential to model the correlation matrix with the assumption that the correlation between LIBOR rates $F_i(t)$ and $F_j(t)$ is a function of the time between resets $|T_{i-1} - T_{j-1}|$,

$$\rho_{i,j} = \exp(-\beta|T_{i-1} - T_{j-1}|), \tag{5.26}$$

where $\beta > 0$. In (5.26) the correlation between the first forward rate and the later expiring rates tends to zero asymptotically. Typically, we want to assume some non-zero finite value $\rho_\infty > 0$ for this asymptotic correlation. We can incorporate this assumption by rewriting (5.26) as

$$\rho_{i,j} = \rho_\infty + (1 - \rho_\infty)\exp(-\beta|T_{i-1} - T_{j-1}|).$$

For any positive β the corresponding correlation matrix ρ is symmetric and positive definite. It satisfies the first two empirical observations, namely the correlations are positive, and for a given forward rate $F_i(t)$ with fixed expiry T_{i-1} the correlation decreases as we increase the expiry T_{j-1} of the second rate. However, it does not capture the changes in convexity we observe empirically. In other words, (5.26) does not satisfy the requirement that the correlation between a one-year rate and a two-year rate should be less than that between, say, a 19-year rate and a 20-year rate. For vanilla instruments such as European swaptions (5.26) this is adequate. However, many exotic options are sensitive to the actual shape of the correlation. We need to introduce some dependence on the expiry dates T_{i-1}, T_{j-1} rather than just on $|T_{i-1} - T_{j-1}|$.

With the above issue in mind, practitioners use the following extension to (5.26):

$$\rho_{i,j} = \rho_\infty + (1 - \rho_\infty)\exp(-\beta|(T_{i-1} - t)^\alpha - (T_{j-1} - t)^\alpha|), \tag{5.27}$$

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where $t \leq \min(T_{i-1}, T_{j-1})$ and α is a positive constant. However, for a given set of parameters α, β and ρ_∞ this is not guaranteed to lead to a positive definite correlation matrix.

Remark 5.10

Note that $\rho_{i,j}$ given by (5.27) depends on t and as such falls outside the scope of the LMM framework adopted in Section 5.1, where the correlations $\rho_{i,j}$ are assumed to be constant (independent of t). Nonetheless, it is possible to extend the LMM to include time-dependent correlations.

The following procedure for constructing a correlation matrix is consistent with the empirical observations, and the matrix is guaranteed to be positive definite and symmetric. The correlation matrix of the set of forward rates $F_1(t), \ldots, F_n(t)$ is parameterised by $n-1$ constants, denoted by $a_k \in [-1, 1]$ for $k = 1, \ldots, n-1$,

$$\rho = \begin{bmatrix} 1 & a_1 & a_1 a_2 & \cdots & \prod_{k=1}^{n-1} a_k \\ a_1 & 1 & a_2 & \cdots & \prod_{k=2}^{n-1} a_k \\ a_1 a_2 & a_2 & 1 & \cdots & \prod_{k=3}^{n-1} a_k \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \prod_{k=1}^{n-1} a_k & \prod_{k=2}^{n-1} a_k & \prod_{k=3}^{n-1} a_k & \cdots & 1 \end{bmatrix}. \tag{5.28}$$

To specify the correlation structure we need to choose some parametric form for a_k. A good choice is a decaying exponential of the form

$$a_k = \exp(-\beta_k(T_k - T_{k-1})), \tag{5.29}$$

where $\beta_k > 0$ for each k. This choice guarantees that $0 < a_k \leq 1$.

To get the correct empirical behaviour we choose a functional form for β_k that is decreasing in k. We also need to ensure that $\beta_k > 0$ for each k. To give a simple example we can take

$$\beta_k = \frac{\alpha}{k^\gamma}, \tag{5.30}$$

where α and γ are constants.

Example 5.11

Finally, we calibrate to the sample correlation matrix in Figure 5.4. The resulting matrix is shown in Figure 5.5. We get a smooth correlation surface,

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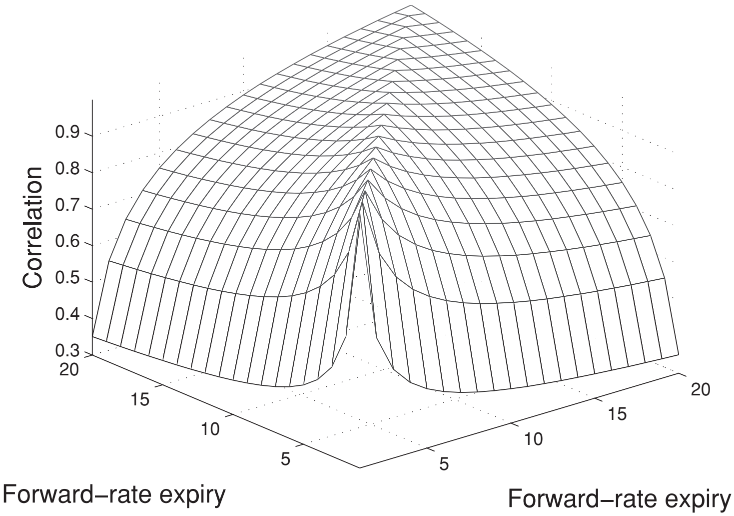

Figure 5.5 Correlation matrix (5.28) with a_k, β_k given by (5.29), (5.30), calibrated to the sample correlation matrix in Figure 5.4. The parameters are $\alpha = 0.441$ and $\gamma = 1.383$.

which retains the key empirical features. Note that the sample correlation matrix in Figure 5.4 may not necessarily be positive definite, but the one constructed in Figure 5.5 is guaranteed to be.

In Chapter 6 we will see that for reasons of numerical efficiency we would typically not use the LMM formulation (5.2) whose correlation structure is modelled by a rank-n correlation matrix. Instead, we use the formulation (5.18) in terms of orthogonal Brownian motions, but with the n forward rates driven by an m-dimensional rather than n-dimensional Brownian motion, with m much smaller than n.

5.9 Swap market model

Consider a unit notional amount and a set of dates $T_0 < T_1 < \cdots < T_n$ with accrual periods $\tau_i = T_i - T_{i-1}$ for $i = 1, \ldots, n$, and a forward interest rate

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swap (payer or receiver) at time $t < T_0$ with reset dates $T_0, \ldots, T_{n-1}$ and settlement dates $T_1, \ldots, T_n$.

Recall from Section 1.6 that the forward swap rate, denoted by $S_{0,n}(t)$, is the value of the fixed rate that makes the swap contract have zero value at time t. According to (1.15), it can be expressed as

$$S_{0,n}(t) = \frac{B(t, T_0) - B(t, T_n)}{A_{0,n}(t)}. \tag{5.31}$$

The denominator

$$A_{0,n}(t) = \sum_{k=1}^{n} \tau_k B(t, T_k) \tag{5.32}$$

in (5.31) is referred to as the **swap annuity** or **level**.

If we choose $A_{0,n}(t)$ as numeraire, the swap rate $S_{0,n}(t)$ discounted by $A_{0,n}(t)$ becomes a martingale under the associated measure P_A, called the the **forward swap measure** or simply the **swap measure**. This is because, according to (5.31), the swap rate $S_{0,n}(t)$ can be seen as the value $B(t, T_0) - B(t, T_n)$ of a portfolio of zero-coupon bonds discounted by the swap annuity $A_{0,n}(t)$.

In the **swap market model** (SMM) the forward swap rate $S_{0,n}(t)$ follows log-normal dynamics. Because $S_{0,n}(t)$ is a martingale under the swap measure P_A, this can be achieved by assuming that $S_{0,n}(t)$ satisfies the SDE

$$dS_{0,n}(t) = \sigma_{0,n}(t) S_{0,n}(t) dW^A(t), \tag{5.33}$$

where $\sigma_{0,n}(t)$ is a deterministic time-dependent volatility and $W^A(t)$ is a Brownian motion under P_A.

5.10 Black's formula for swaptions

In Section 5.2 we saw how the fact that each forward rate $F_i(t)$ can be modelled as a driftless log-normal process under the forward measure P_{T_i} leads to Black's formula for caplets. Analogously, Black's formula for swaptions arises naturally within the SMM.

By (1.14), (1.15) and (5.32), the value of a payer swap at time t is

$$\mathbf{PS}(t) = A_{0,n}(t)(S_{0,n}(t) - K).$$

Consider a payer swaption. The swaption payoff at time T_0 is

$$\mathbf{PSwpt}_{0,n}(T_0) = (\mathbf{PS}(T_0))^+ = A_{0,n}(T_0)(S_{0,n}(T_0) - K)^+.$$

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It follows by Proposition 2.2 that the value at time $t \leq T_0$ of the payer swaption can be written as

$$\mathbf{PSwpt}_{0,n}(t) = A_{0,n}(t)\mathbb{E}_{P_A}\Big((S_{0,n}(T_0) - K)^+ \Big| \mathcal{F}_t\Big).$$

We derive Black's formula for the swaption by computing this expectation.

Exercise 5.11 Show that

$$\mathbf{PSwpt}_{0,n}(t) = A_{0,n}(t)(S_{0,n}(t)N(d_+) - KN(d_-)),$$

where

$$d_{\pm} = \frac{\ln \frac{S_{0,n}(t)}{K} \pm \frac{1}{2}\int_t^{T_0} \sigma_{0,n}(s)^2 ds}{\sqrt{\int_t^{T_0} \sigma_{0,n}(s)^2 ds}}.$$

Setting

$$v_{0,n} = \sqrt{\frac{1}{T_0 - t}\int_t^{T_0} \sigma_{0,n}(s)^2 ds}, \tag{5.34}$$

we recover Black's formula (2.26) for swaptions with $v_{0,n}$ substituted for the volatility σ, that is, we have

$$\mathbf{PSwpt}_{0,n}(t) = \mathbf{PSwpt}^{\text{Black}}_{0,n}(t; v_{0,n}).$$

We can think of $v_{0,n}$ as the model-implied swaption volatility. The swaption price in the SMM will be consistent with the market price when $v_{0,n} = \hat{\sigma}^{\text{swpt}}_{0,n}$; see Section 2.9.

5.11 LMM versus SMM

In this section we compare the LMM and SMM and discover that they are incompatible with one another. The forward LIBOR rates $F_i(t)$ and the swap rate $S_{0,n}(t)$ cannot simultaneously satisfy the assumptions of these models.

Suppose that the forward rates $F_i(t)$ satisfy the SDE (5.12) of the LMM under the terminal measure P_{T_n}. Switching from the terminal measure P_{T_n} to the swap measure P_A, we can write the SDE for $F_i(t)$ as

$$dF_i(t) = \mu_i^A(t)F_i(t)dt + \sigma_i(t)F_i(t)dZ_i^A(t), \tag{5.35}$$

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where $Z_i^A(t)$ for $i = 1, \ldots n$ are correlated Brownian motions under P_A such that

$$dZ_i^A(t)dZ_j^A(t) = \rho_{i,j}dt \tag{5.36}$$

for all $i, j = 1, \ldots, n$. We switch to the swap measure P_A because $S_{0,n}(t)$ is a martingale under P_A. Using this to compute the volatility $\sigma_{0,n}(t)$ of the swap rate will lead to the conclusion that $S_{0,n}(t)$ is not log-normally distributed in the LMM, contrary to the assumption underlying the SMM.

> **Exercise 5.12** Show that $F_i(t)$ satisfies the SDE (5.35) under the swap measure P_A and derive a formula for the drift $\mu_i^A(t)$.

Next we write the swap rate in terms of the forward rates as

$$S_{0,n}(t) = \frac{B(t, T_0) - B(t, T_n)}{\sum_{j=1}^{n} \tau_j B(t, T_j)} = \frac{1 - \prod_{i=1}^{n} (1 + \tau_i F_i(t))^{-1}}{\sum_{j=1}^{n} \tau_j \prod_{i=1}^{j} (1 + \tau_i F_i(t))^{-1}}.$$

By Itô's lemma,

$$dS_{0,n}(t) = \sum_{i=1}^{n} \frac{\partial S_{0,n}(t)}{\partial F_i} dF_i(t) + (\cdots)\, dt,$$

where for the sake of brevity we abuse the notation slightly, regarding $S_{0,n}(t)$ as a function of the forward rates. (Strictly speaking, we should write $S_{0,n}(F_1(t), \ldots, F_n(t))$ in place of $S_{0,n}(t)$.) We do not specify the terms with dt explicitly as they are not needed. Substituting for $dF_i(t)$ from (5.35), we obtain

$$dS_{0,n}(t) = \sum_{i=1}^{n} \frac{\partial S_{0,n}(t)}{\partial F_i} \sigma_i(t) F_i(t) dZ_i^A(t). \tag{5.37}$$

All the terms containing dt disappear because $S_{0,n}(t)$ is a martingale and the $Z_i^A(t)$ are Brownian motions under the swap measure P_A.

We are ready to compute the volatility $\sigma_{0,n}(t)$ of the swap rate $S_{0,n}(t)$. From (5.33) we have

$$\sigma_{0,n}(t)^2 dt = \frac{dS_{0,n}(t)}{S_{0,n}(t)} \frac{dS_{0,n}(t)}{S_{0,n}(t)}. \tag{5.38}$$

Substituting the right-hand side of (5.37) for $dS_{0,n}(t)$ and using (5.36), we get

$$\sigma_{0,n}(t)^2 dt = \frac{dS_{0,n}(t)}{S_{0,n}(t)} \frac{dS_{0,n}(t)}{S_{0,n}(t)}$$
$$= \frac{\sum_{i,j=1}^{n} \frac{\partial S_{0,n}(t)}{\partial F_i} \frac{\partial S_{0,n}(t)}{\partial F_j} \sigma_i(t)\sigma_j(t)F_i(t)F_j(t)\rho_{i,j}}{S_{0,n}(t)^2} dt.$$

In the LMM $\sigma_{0,n}(t)$ is clearly not a deterministic function, which shows that the swap rate $S_{0,n}(t)$ does not follow a log-normal process under P_A. This allows us to conclude that the LMM is incompatible with the SMM.

Remark 5.12
Although we have formally demonstrated that the swap rate $S_{0,n}(t)$ does not follow a log-normal process under P_A in the LMM, in many practical applications it turns out that $S_{0,n}(t)$ is approximately log-normally distributed, and the inconsistency between the LMM and SMM can often be ignored.

5.12 LMM approximation for swaption volatility

According to (5.34), the swap volatility $\sigma_{0,n}(s)$ in the SMM dynamics (5.33) enters Black's formula (2.26) for swaptions via an integral of $\sigma_{0,n}(s)^2$. We saw in Section 5.11 that, within the LMM, this integral is random. Nonetheless, we can approximate it by a deterministic quantity, making it possible to express Black's swaption volatility $\sigma_{0,n}(s)$ in the SMM via the parameters of the LMM. This simplifies calibration, when we often need to evaluate a wide range of swaptions quickly and accurately.

According to (1.16), we can express the swap rate as a linear combination of the forward rates $F_i(s) = F(s, T_{i-1}, T_i)$ for $i = 1, \ldots, n$, namely

$$S_{0,n}(s) = \sum_{i=1}^{n} w_i(s)F_i(s),$$

where the weights $w_i(s)$ are

$$w_i(s) = \frac{\tau_i B(s, T_i)}{A_{0,n}(s)}.$$

By Itô's lemma,

$$dS_{0,n}(s) = \sum_{i=1}^{n} w_i(s)dF_i(s) + \sum_{i=1}^{n} F_i(s)dw_i(s) + (\cdots)\, ds, \tag{5.39}$$

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where the terms with ds are not written explicitly as they will not be needed. Using (5.35), we get

$$dS_{0,n}(s) = \sum_{i=1}^{n} w_i(s)F_i(s)\sigma_i(s)dZ_i^A(s) + \sum_{i=1}^{n} F_i(s)dw_i(s).$$

All terms containing ds disappear on the right-hand side in the last expression because $S_{0,n}(s)$ is a martingale under the swap measure P_A, and so are $Z_i^A(s)$ and $w_i(s)$ for each $i = 1, \ldots, n$.

Since the weights $w_i(s)$ are martingales under P_A, it follows that the stochastic integrals $\int_t^s F_i(u)dw_i(u)$ are also martingales under P_A. In practice, their variability is often lower than that of $S_{0,n}(s)$. As the martingales do not vary much from their value at t, they can be approximated by this value, which is 0 and gives

$$dS_{0,n}(s) \approx \sum_{i=1}^{n} w_i(s)F_i(s)\sigma_i(s)dZ_i^A(s).$$

Consequently, substituting this into (5.38), we obtain

$$\begin{aligned}\sigma_{0,n}(s)^2 ds &= \frac{dS_{0,n}(s)}{S_{0,n}(s)} \frac{dS_{0,n}(s)}{S_{0,n}(s)} \\ &\approx \frac{\sum_{i,j=1}^{n} w_i(s)F_i(s)w_j(s)F_j(s)}{S_{0,n}(s)^2} \sigma_i(s)\sigma_j(s)\rho_{i,j} ds.\end{aligned}$$

In the last expression there are more martingales under P_A, namely

$$S_{0,n}(s) = \frac{B(s, T_0) - B(s, T_n)}{A_{0,n}(s)}$$

and

$$w_i(s)F_i(s) = \frac{B(s, T_{i-1}) - B(s, T_i)}{A_{0,n}(s)}.$$

For any $s \in [t, T_0]$ these martingales can be approximated by their values at t, yielding

$$\sigma_{0,n}(s)^2 ds \approx \frac{\sum_{i,j=1}^{n} w_i(t)F_i(t)w_j(t)F_j(t)}{S_{0,n}(t)^2} \sigma_i(s)\sigma_j(s)\rho_{i,j} ds.$$

It follows that the volatility (5.34) in Black's swaption formula can be ap-

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proximated as

$$
\begin{aligned}
v_{0,n}^2 &= \frac{1}{T_0 - t} \int_t^{T_0} \sigma_{0,n}(s)^2 ds \\
&\approx \frac{1}{T_0 - t} \sum_{i,j=1}^{n} \frac{w_i(t) w_j(t) F_i(t) F_j(t)}{S_{0,n}(t)^2} \int_t^{T_0} \sigma_i(s) \sigma_j(s) \rho_{i,j} ds. \qquad (5.40)
\end{aligned}
$$

This approximation is known as **Rebonato's formula**. Without the means of approximating Black's swaption volatility $v_{0,n}$ and thus calculating the swaption price, calibration would become time consuming, which is why the approximation is important.

Remark 5.13

The accuracy of Rebonato's formula can be improved by approximating the martingale

$$\int_t^s F_i(u) dw_i(u) = \sum_{j=1}^{n} \int_t^s F_i(u) \frac{\partial w_u(t)}{\partial F_j} F_j(u) \sigma_j(u) dZ_j^A(u)$$

by

$$\sum_{j=1}^{n} F_i(t) \frac{\partial w_i(t)}{\partial F_j} F_j(t) \int_t^s \sigma_j(u) dZ_j^A(u)$$

rather than by its value at $s = t$, which is 0. (Here we consider the weights as functions of the forward rates, and strictly speaking should write $w_i(F_1(s), \ldots, F_n(s))$ in place of $w_i(s)$.)

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