Smoothing the Payoff for Efficient Computation of Option Pricing in Time-Stepping Setting

1 Problem Setting:

We aim at approximating $E[g(X(t))]$ given $g : \mathbb{R}^d \to \mathbb{R}$, where $X \in \mathbb{R}^d$ solves

$$X(t) = X(0) + \int_0^t a(s, X(s))ds + \sum_{\ell=1}^{\ell_0} \int_0^t b^\ell(s, X(s))dW^\ell(s) \tag{1}$$

Let us decompose the Wiener process in the interval $[0, T]$ as

$$W(t) = W(T)\frac{t}{T} + B(t) \tag{2}$$

with $B(t)$ a Brownian bridge with zero end value. Then, for each $t \in [0, T]$ we have

$$\begin{aligned} X(t) &= X(0) + \int_0^t b(X(s))dB(s) + \frac{W(t)}{t}\int_0^t b(X(s))ds \\ &= X(0) + \int_0^t b(X(s))dB(s) + \frac{Y}{\sqrt{t}}\int_0^t b(X(s))ds, \end{aligned} \tag{3}$$

Where $Y \sim \mathcal{N}(0, 1)$ and B and Y are independent.

As a consequence,

$$\begin{aligned} E[g(X(T))] &= E^B[E^Y[g(X(T)) \mid B]] \\ &= \frac{1}{\sqrt{2\pi}}E^B[H(B)], \end{aligned} \tag{4}$$

where $H(B) = \int g(X(T; y, B)) \exp(-y^2/2)dy$.

We note that $H(B)$ has for many practical cases, a smooth dependence wrt to X due to the smoothness of the pdf of Y.

For illustration, we have $W_t = \frac{t}{T}W_T + B_t$ and

$$\begin{aligned} \Delta W_i &= (B_{t_{i+1}} - B_{t_i}) + \Delta t\frac{Y}{\sqrt{T}} \\ &= \Delta B_i + \Delta t\frac{Y}{\sqrt{T}}, \end{aligned} \tag{5}$$

implying that the numerical approximation of $X(T)$ satisfies

$$\overline{X}_T = \Phi(\Delta t, W_1(y), \Delta B_0, \dots, \Delta B_{N-1}), \tag{6}$$

for some path function Φ.

2 Numerical Approaches

2.1 First approach

- Use sparse grid $\mathcal{D}$ for $\Delta B_0, \dots, \Delta B_{N-1}$.
- Given $(\mathbf{X}^0, \dots, \mathbf{X}^{N-1}) := \mathcal{X} \in \mathcal{D}$ with weights $(\boldsymbol{\omega}^0, \dots, \boldsymbol{\omega}^{N-1})$, add grid points $(y_1(\mathcal{X}), \dots, y_K(\mathcal{X})) = \mathbf{y}$ with weights $(W_1, \dots, W_K)$ such that the mapping $y \to g(\Phi(\Delta t, y, X^0, \dots, X^{N-1}))$ is smooth outside the kink point. Mainly here we will use the **Newton iteration** to determine the kink point.
- Construct our estimator for $E[g(X(T))]$ by looping over step 1 and 2 such that we choose the optimal indices of sparse grids that achieves a global error of order TOL.

$$E[g(X(T))] = \sum_{n=0}^{N-1} \sum_{j} \sum_{i=1}^{K} W_i g(\Phi(\Delta t, \mathbf{y}, \mathcal{X})) \omega_j^n$$

2.1.1 Some discussion on the complexity and errors

- We expect that the global error of our procedure will be bounded by the weak error which is in our case of order $O(\Delta t)$. In this case, the overall complexity of our procedure will be of order $O(TOL^{-1})$. We note that this rate can be improved up to $O(TOL^{-\frac{1}{2}})$ if we use **Richardson extrapolation**. Another way that can improve the complexity could be based on **Cubature on Wiener Space** (This is left for a future work). The aimed complexity rate illustrates the contribution of our procedure which outperforms Monte Carlo forward Euler (MC-FE) and multi-level MC-FE, having complexity rates of order $O(TOL^{-3})$ and $O(TOL^{-2}log(TOL)^2)$ respectively.
- We need to check the impact of the error caused by the Newton iteration on the integration error. In the worst case, we expect that if the error in the Newton iteration is of order $O(\epsilon)$ than the integration error will be of order $\log(\epsilon)$. But we need to check that too.

2.2 Second approach

An alternative approach could be achieved by tensorizing all the quadrature rules (this is not clear to me how to do it yet). The advantage of this procedure is that the additional cost that we will pay by using fine quadrature in the dimension of y will be rewarded by the ability of using coarser quadratures in the remaining dimensions.

2.3 Choice of functional

We should restrict ourselves to a few possible choices g such as:

- hockey-stick function, i.e., put or call payoff functions;
- indicator functions (both relevant in finance and in other applications of estimation of probabilities of certain events);
- delta-functions for density estimation (and derivatives thereof for estimation of derivatives of the density).

More specifically, g should be the composition of one of the above with a smooth function. (For instance, the basket option payoff as a function of the log-prices of the underlying.)

3 Plan of work and miscancellous observations

We recall the discussion between Raul and Christian on June 1st.

Given we want to compute

$$E\left[g\left(\Phi(Z_1,\ldots,Z_N)\right]\right.$$

for some non-smooth function g and a Gaussian vector Z. (Here, the discretization of the SDE is in the function Φ.) We assume that Z is already rotated such that $h(Z_{-1}) \coloneqq E\left[g\left(\Phi(Z_1,\ldots,Z_N)\mid\mid\mid Z_1\right]\right.$ is as smooth as possible, where $Z_{-1} \coloneqq (Z_2,\ldots,Z_N)$.

3.1 Smoothing

A crucial element of the smoothing property is that the "location of irregularity" $y = y(z_{-1})$ such that g is not smooth at the point $\Phi(y, z_{-1})$. Generally, there might be (for given z_{-1}

- no solution, i.e., the integrand in the definition of $h(z_{-1})$ above is smooth (*best case*);
- a unique solution;
- multiple solutions.

Generally, we need to assume that we are in the first or second case. Specifically, we need that

$$z_{-1} \mapsto h(z_{-1}) \text{ and } z_{-1} \mapsto \widehat{h}(z_{-1})$$

are smooth, where $\widehat{h}$ denotes the numerical approximation of h based on a grid containing $y(z_{-1})$. In particular, y itself should be smooth in z_{-1}. This would already be challenging in practice in the third case. Moreover, in the general situation we expect the number of solutions y to increase when the discretization of the SDE gets finer.

In many situations, case 2 (which is thought to include case 1) can be guaranteed by monotonicity. For instance, in the case of one-dimensional SDEs with Z_1 representing the terminal value of the underlying Brownian motion (and Z_{-1} representing the Brownian bridge), this can often be seen from the SDE itself. Specifically, if each increment "dX" is increasing in Z_1, no matter the value of X, then the solution X_T must be increasing in Z_1. This is easily seen to be true in examples such as the Black-Scholes model and the CIR process. (Strictly speaking, we have to distinguish between the continuous and discrete time solutions. In these examples, it does not matter.) On the other hand, it is also quite simple to construct counter examples, where monotonicity fails, for instance SDEs for which the "volatility" changes sign, such as a trigonometric function.[1]

Even in multi-dimensional settings, such monotonicity conditions can hold in specific situations. For instance, in case of a basket option in a multivariate Black Scholes framework, we can choose a linear combination Z_1 of the terminal values of the driving Bm, such that the basket is a monotone function of Z_1. (The coefficients of the linear combination will depend on the correlations and the weights of the basket.) However, in that case this may actually not correspond to the optimal "rotation" in terms of optimizing the smoothing effect.

4 Numerical examples

4.1 The discretized Black-Scholes model

The first example is the discretized Black-Scholes model. Precisely, we are interested in the 1-D lognormal example where the dynamics of the stock are given by

$$dX_t = \sigma X_t dW_t. \tag{7}$$

In this case, we want to compare different ways for identifying the location of the kink.

4.1.1 Exact location of the kink for the continuous problem

Let us denote y_* an invertible function that satisfies

$$X(T; y_*(x), B) = x. \tag{8}$$

We can easily prove that the expression of y_* for model given by (7) is given by

$$y_*(x) = \left(\log(x/x_0) + T\sigma^2/2\right) \frac{1}{\sqrt{T}\sigma}, \tag{9}$$

and since the kink for Black-Scholes model occurs at $x = K$, where K is the strike price then the exact location of the continuous problem is given by

$$y_*(K) = \left(\log(K/x_0) + T\sigma^2/2\right) \frac{1}{\sqrt{T}\sigma}. \tag{10}$$

4.1.2 Exact location of the kink for the discrete problem

The discrete problem of model (7) is solved by simulating

$$\begin{aligned} \Delta X_{t_i} &= \sigma X_{t_i} \Delta W_i,\ 0 < i < N-1 \\ X_{t_{i+1}} - X_{t_i} &= \sigma X_{t_i} \left(W_{t_{i+1}} - W_{t_i}\right),\ 0 < i < N \end{aligned} \tag{11}$$

[1] Actually, in every such case the simple remedy is to replace the volatility by its absolute value, which does not change the law of the solution. Hence, there does not seem to be a one-dimensional counter-example.

where $X(T_0) = X_0$ and $X(t_N) = X(T)$.

Using Brownian bridge construction given by (5), we have

$$
\begin{aligned}
X_{t_1} &= X_{t_0}\left[1 + \frac{\sigma}{\sqrt{T}} Y \Delta t + \sigma \Delta B_0\right] \\
X_{t_2} &= X_{t_1}\left[1 + \frac{\sigma}{\sqrt{T}} Y \Delta t + \sigma \Delta B_1\right] \\
\vdots &= \vdots \\
X_{t_N} &= X_{t_{N-1}}\left[1 + \frac{\sigma}{\sqrt{T}} Y \Delta t + \sigma \Delta B_{N-1}\right],
\end{aligned} \tag{12}
$$

implying that

$$
\overline{X}(T) = X_0 \prod_{i=0}^{N-1}\left[1 + \frac{\sigma}{\sqrt{T}} Y \Delta t + \sigma \Delta B_i\right]. \tag{13}
$$

Therefore, in order to determine y_*, we need to solve

$$
x = \overline{X}(T; y_*, B) = X_0 \prod_{i=0}^{N-1}\left[1 + \frac{\sigma}{\sqrt{T}} y_*(x) \Delta t + \sigma \Delta B_i\right], \tag{14}
$$

which implies that the location of the kink point for the approximate problem is equivalent to finding the roots of the polynomial $P(y_*(K))$, given by

$$
P(y_*(K)) = \prod_{i=0}^{N-1}\left[1 + \frac{\sigma}{\sqrt{T}} y_*(K) \Delta t + \sigma \Delta B_i\right] - \frac{K}{X_0}. \tag{15}
$$

The exact location of the kink can be obtained exactly by solving exactly $P(y_*(K)) = 0$.

4.1.3 Approximate location of the discrete problem

Here, we try to find the roots of polynomial $P(y_*(K))$, given by (15), by using **Newton iteration method**

In this case, we need the expression $P' = \frac{dP}{dy_*}$. If we denote $f_i(y) = 1 + \frac{\sigma}{\sqrt{T}} y \Delta t + \sigma \Delta B_i$, then we can easily show that

$$
P'(y) = \frac{\sigma \Delta t}{\sqrt{T}} \left(\prod_{i=0}^{N-1} f_i(y)\right) \left[\sum_{i=0}^{N-1} \frac{1}{f_i(y)}\right] \tag{16}
$$