MATHEMATICS OF COMPUTATION
Volume 86, Number 306, July 2017, Pages 1847–1854
http://dx.doi.org/10.1090/mcom/3172
Article electronically published on October 7, 2016

NOTE ON "THE SMOOTHING EFFECT OF INTEGRATION IN $\mathbb{R}^d$ AND THE ANOVA DECOMPOSITION"

MICHAEL GRIEBEL, FRANCES Y. KUO, AND IAN H. SLOAN

Abstract. This is a note on Math. Comp. **82** (2013), 383–400. We first report a mistake, in that the main result Theorem 3.1, though correct, does not as claimed apply to the Asian option pricing problem. This is because assumption (3.3) in the theorem is not satisfied by the Asian option pricing problem. In this note we present a strengthened theorem, which removes that assumption. The new theorem is immediately applicable to the Asian option pricing problem with the standard and Brownian bridge constructions. Thus the option pricing conclusions of our original paper stand.

1. Background

In [3] we studied a d-variate integration problem of the form

$$I_d(f) := \int_{\mathbb{R}^d} f(\boldsymbol{x})\, \rho_d(\boldsymbol{x})\, \mathrm{d}\boldsymbol{x},$$

where ρ_d is a product of univariate $\mathcal{N}(0,1)$ Gaussian probability densities, and

$$f(\boldsymbol{x}) := \max(\phi(\boldsymbol{x}), 0),$$

with ϕ a smooth function of all variables.

The main theorem, Theorem 3.1, states that under certain assumptions the ANOVA decomposition of f has every term smooth except for the very highest term, the one that depends on all the variables.

Though the theorem is correct as stated, it does not as claimed apply to the Asian option pricing problem because one of the assumptions in the theorem, equation (3.3), is not satisfied for that problem.

The purpose of this note is first to point out the mistake in the option pricing application in [3], and then to present a strengthened form of the main theorem which does not need assumption (3.3). The new result (Theorem 1) below is immediately applicable to the Asian option pricing problem in the standard and Brownian bridge formulations, thus the conclusions of paper [3] stand.

2. The option pricing mistake

In [3, pages 396–397], we considered integrands arising from the Asian option pricing problem, which take the form $f(\boldsymbol{x}) = \phi(\boldsymbol{x})_+ := \max(\phi(\boldsymbol{x}), 0)$, and in particular, in [3, equation (4.2)]

$$\phi(\boldsymbol{x}) = \frac{S_0}{d} \sum_{\ell=1}^{d} \exp\left(\left(\mu - \frac{\sigma^2}{2}\right) \ell \Delta t + \sigma \sum_{i=1}^{d} A_{\ell,i}\, x_i \right) - K. \tag{1}$$

Received by the editor June 23, 2015 and, in revised form, November 29, 2015.
2010 *Mathematics Subject Classification.* Primary 41A63, 41A99; Secondary 65D30.

We claimed erroneously that, in the case of the standard and Brownian bridge constructions, we have for each j and each fixed $\boldsymbol{x}_{\mathfrak{D}\setminus\{j\}}$, where $\mathfrak{D} = \{1, 2, \ldots, d\}$, that

$$\phi(\boldsymbol{x}) \;=\; \phi(x_j, \boldsymbol{x}_{\mathfrak{D}\setminus\{j\}}) \;\to\; \begin{cases} +\infty & \text{as } x_j \to +\infty, \\ -K & \text{as } x_j \to -\infty. \end{cases}$$

The correct observation is that

$$\phi(\boldsymbol{x}) \;=\; \phi(x_j, \boldsymbol{x}_{\mathfrak{D}\setminus\{j\}}) \;\to\; \begin{cases} +\infty & \text{as } x_j \to +\infty, \\ B_j(\boldsymbol{x}_{\mathfrak{D}\setminus\{j\}}) & \text{as } x_j \to -\infty, \end{cases} \tag{2}$$

where

$$B_j(\boldsymbol{x}_{\mathfrak{D}\setminus\{j\}}) \;:=\; \frac{S_0}{d} \sum_{\substack{\ell=1 \\ A_{\ell,j}=0}}^{d} \exp\left(\left(\mu - \frac{\sigma^2}{2}\right) \ell \Delta t + \sigma \sum_{\substack{i=1 \\ i \neq j}}^{d} A_{\ell,i}\, x_i \right) - K. \tag{3}$$

If j is such that the set $\{\ell \in \mathfrak{D} : A_{\ell,j} = 0\}$ is not empty, then $B_j(\boldsymbol{x}_{\mathfrak{D}\setminus\{j\}})$ can take all values between $-K$ and $+\infty$, from which it follows that the condition [3, equation (3.3)], namely

$$\text{for each } \boldsymbol{x}_{\mathfrak{D}\setminus\{j\}} \in \mathbb{R}^{\mathfrak{D}\setminus\{j\}} \text{ there exists } x_j \in \mathbb{R} \text{ such that } \phi(x_j, \boldsymbol{x}_{\mathfrak{D}\setminus\{j\}}) = 0, \tag{4}$$

does *not* hold in general. Hence Theorem 3.1 as it stands does *not* apply to the Asian option pricing problem.

3. New theorem in place of Theorem 3.1

The following theorem strengthens Theorem 3.1, in that the condition (4), or [3, equation (3.3)], is no longer required. We show that integration with respect to x_j can have a smoothing effect. We prove that

$$(P_j f)(\boldsymbol{x}) \;:=\; \int_{-\infty}^{\infty} f(x_1, \ldots, x_{j-1}, t_j, x_{j+1}, \ldots, x_d)\, \rho(t_j)\, \mathrm{d}t_j$$

belongs to the Sobolev space $\mathcal{W}^r_{\mathfrak{D}\setminus\{j\}, p, \rho_{\mathfrak{D}\setminus\{j\}}}$ provided that a number of conditions on ϕ are satisfied:

(i) $\phi \in \mathcal{W}^r_{d,p,\rho_d} \cap \mathcal{C}^\infty(\mathbb{R}^d)$.
(ii) $D_j \phi := \partial\phi/\partial x_j$ is always positive or always negative.
(iii) Special conditions on ϕ hold; see (6) and (7) below.

Here $r \geq 1$, $p \in [1, \infty)$, ρ is a strictly positive univariate probability density function, and $\rho_d(\boldsymbol{x}) := \prod_{j=1}^d \rho(x_j)$. Some discussion on Sobolev spaces can be found in [3]; for more details see [2]. Note that if $g \in \mathcal{W}^r_{d,p,\rho_d}$, then for any multi-index $\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_d)$ with $|\boldsymbol{\alpha}| := \alpha_1 + \cdots + \alpha_d \leq r$, the weak derivative

$$(D^{\boldsymbol{\alpha}} g)(\boldsymbol{x}) \;:=\; (D_1^{\alpha_1} \cdots D_d^{\alpha_d} g)(\boldsymbol{x}) \;:=\; \frac{\partial^{|\boldsymbol{\alpha}|} g}{\partial x_1^{\alpha_1} \cdots \partial x_d^{\alpha_d}}(\boldsymbol{x})$$

satisfies

$$\int_{\mathbb{R}^d} |(D^{\boldsymbol{\alpha}} g)(\boldsymbol{x})|^p\, \rho_d(\boldsymbol{x})\, \mathrm{d}\boldsymbol{x} \;<\; \infty.$$

For $\mathbf{u} \subseteq \mathfrak{D}$, the space $\mathcal{W}^r_{\mathbf{u},p,\rho_{\mathbf{u}}}$ is a subspace of $\mathcal{W}^r_{d,p,\rho_d}$ containing functions that are constant with respect to the components whose indices are outside of $\mathbf{u}$, that is, functions that depend only the variables $\boldsymbol{x}_{\mathbf{u}} := (x_j)_{j \in \mathbf{u}}$ and $\rho_{\mathbf{u}}(\boldsymbol{x}_{\mathbf{u}}) := \prod_{j \in \mathbf{u}} \rho(x_j)$.

The proof builds upon the original proof of Theorem 3.1, but it requires several additional elements. For clarity we provide a complete proof here. The proof makes use of the inheritance and implicit function theorems; see [3, Theorems 2.2 and 2.3]. Note that all occurrences of the closure of U_j in [3, Theorem 2.3], denoted there by $\overline{U_j}$, should be replaced by just the set U_j itself.

Theorem 1. *Let $r \geq 1$, $p \in [1,\infty)$, and let $\rho \in \mathcal{C}^\infty(\mathbb{R})$ be a strictly positive probability density function. Let*

$$f(\boldsymbol{x}) \;=\; \phi(\boldsymbol{x})_+, \qquad \textit{where} \quad \phi \in \mathcal{W}^r_{d,p,\rho_d} \cap \mathcal{C}^\infty(\mathbb{R}^d).$$

Let $j \in \mathfrak{D}$ be fixed and suppose that

$$(D_j\phi)(\boldsymbol{x}) \neq 0 \qquad \textit{for all} \quad \boldsymbol{x} \in \mathbb{R}^d.$$

Denoting $\boldsymbol{y} := \boldsymbol{x}_{\mathfrak{D}\setminus\{j\}}$ so that $\boldsymbol{x} = (x_j, \boldsymbol{y})$, let

$$\begin{aligned} U_j \;&:=\; \{\boldsymbol{y} \in \mathbb{R}^{\mathfrak{D}\setminus\{j\}} \;:\; \phi(x_j,\boldsymbol{y}) = 0 \textit{ for some } x_j \in \mathbb{R}\},\\ U_j^+ \;&:=\; \{\boldsymbol{y} \in \mathbb{R}^{\mathfrak{D}\setminus\{j\}} \;:\; \phi(x_j,\boldsymbol{y}) > 0 \textit{ for all } x_j \in \mathbb{R}\},\\ U_j^- \;&:=\; \{\boldsymbol{y} \in \mathbb{R}^{\mathfrak{D}\setminus\{j\}} \;:\; \phi(x_j,\boldsymbol{y}) < 0 \textit{ for all } x_j \in \mathbb{R}\}. \end{aligned}$$

If U_j is not empty, then U_j is open, and there exists a unique function $\psi \equiv \psi_j \in \mathcal{C}^r(U_j)$ such that $\phi(x_j,\boldsymbol{y}) = 0$ if and only if $x_j = \psi(\boldsymbol{y})$ for $\boldsymbol{y} \in U_j$. In this case we assume that every function of the form

$$(5)\qquad \begin{cases} g(\boldsymbol{y}) \;=\; \beta\, \dfrac{\prod_{i=1}^a [(D^{\boldsymbol{\alpha}^{(i)}}\phi)(\psi(\boldsymbol{y}),\boldsymbol{y})]}{[(D_j\phi)(\psi(\boldsymbol{y}),\boldsymbol{y})]^b}\, \rho^{(c)}(\psi(\boldsymbol{y})), \quad \boldsymbol{y} \in U_j,\\ \textit{where } \beta, a, b, c \textit{ are integers and } \boldsymbol{\alpha}^{(i)} \textit{ are multi-indices with the constraints}\\ 2 \leq a \leq 2r-2, \quad 1 \leq b \leq 2r-3, \quad 0 \leq c \leq r-2, \quad |\boldsymbol{\alpha}^{(i)}| \leq r-1, \end{cases}$$

satisfies both

$$(6)\qquad g(\boldsymbol{y}) \to 0 \quad \textit{as} \quad \boldsymbol{y} \textit{ approaches a boundary point of } U_j \textit{ lying in } U_j^+ \textit{ or } U_j^-$$

and

$$(7)\qquad \int_{U_j} |g(\boldsymbol{y})|^p\, \rho_{\mathfrak{D}\setminus\{j\}}(\boldsymbol{y})\,\mathrm{d}\boldsymbol{y} \;<\; \infty.$$

Then

$$P_j f \in \mathcal{W}^r_{\mathfrak{D}\setminus\{j\},p,\rho_{\mathfrak{D}\setminus\{j\}}}.$$

Since $\psi(\boldsymbol{y}) \to \pm\infty$ as $\boldsymbol{y}$ approaches a boundary point of U_j lying in U_j^- and U_j^+, respectively, a sufficient condition for (6) to hold is that

$$(8)\qquad \begin{gathered} \frac{\prod_{i=1}^a [(D^{\boldsymbol{\alpha}^{(i)}}\phi)(x_j,\boldsymbol{y})]}{[(D_j\phi)(x_j,\boldsymbol{y})]^b}\, \rho^{(c)}(x_j) \;\leq\; E(x_j)\\ \text{for some } E(x_j) \text{ independent of } \boldsymbol{y}, \text{ where } E(x_j) \to 0 \text{ as } x_j \to \pm\infty. \end{gathered}$$

Proof. The case $r = 1$ is easy to prove; see [3, equation (3.2)]. We therefore assume below that $r \geq 2$.

Given that $\phi \in \mathcal{C}^\infty(\mathbb{R}^d)$, $(D_j\phi)(\boldsymbol{x}) \neq 0$ for all $\boldsymbol{x} \in \mathbb{R}^d$, and that U_j is not empty, it follows from the implicit function theorem [3, Theorem 2.3] that there exists a unique function $\psi \in \mathcal{C}^r(U_j)$ for which

$$\phi(x_j, \boldsymbol{y}) \,=\, 0 \iff \psi(\boldsymbol{y}) \,=\, x_j \quad \text{for all } \ \boldsymbol{y} \in U_j. \tag{9}$$

This justifies the existence of the function ψ as stated in the theorem.

For the function $f(\boldsymbol{x}) = \phi(x_j, \boldsymbol{y})_+$ we can write $P_j f$ as

$$(P_j f)(\boldsymbol{y}) \,=\, \int_{x_j \in \mathbb{R}\,:\,\phi(x_j,\boldsymbol{y}) \geq 0} \phi(x_j, \boldsymbol{y})\, \rho(x_j)\, \mathrm{d}x_j. \tag{10}$$

Note that the condition $(D_j\phi)(\boldsymbol{x}) \neq 0$, when combined with the continuity of $D_j\phi$, means that $D_j\phi$ is either everywhere positive or everywhere negative. For definiteness we assume that

$$(D_j\phi)(\boldsymbol{x}) \,>\, 0 \quad \text{for all} \quad \boldsymbol{x} \in \mathbb{R}^d;$$

the other case is similar. It follows that, for fixed $\boldsymbol{y}$, $\phi(x_j, \boldsymbol{y})$ is a strictly increasing function of x_j.

We now determine the limits of integration in (10). If $\boldsymbol{y} \in U_j^+$, then we integrate x_j from $-\infty$ to ∞. On the other hand, if $\boldsymbol{y} \in U_j^-$, then the integral is 0. The remaining scenario is that $\boldsymbol{y} \in U_j$, in which case $\phi(x_j, \boldsymbol{y})$ changes sign once as x_j goes from $-\infty$ to ∞, thus there exists a unique $x_j^* = \psi(\boldsymbol{y}) \in \mathbb{R}$ for which $\phi(x_j^*, \boldsymbol{y}) = 0$; in this case we integrate x_j from $\psi(\boldsymbol{y})$ to ∞. Hence we can write (10) as

$$(P_j f)(\boldsymbol{y}) \,=\, \begin{cases} \displaystyle\int_{-\infty}^{\infty} \phi(x_j, \boldsymbol{y})\, \rho(x_j)\, \mathrm{d}x_j & \text{if } \boldsymbol{y} \in U_j^+, \\ \displaystyle\int_{\psi(\boldsymbol{y})}^{\infty} \phi(x_j, \boldsymbol{y})\, \rho(x_j)\, \mathrm{d}x_j & \text{if } \boldsymbol{y} \in U_j, \\ 0 & \text{if } \boldsymbol{y} \in U_j^-. \end{cases}$$

Note that $P_j f$ is continuous across the boundaries between U_j and U_j^+ and between U_j and U_j^-, since $\psi(\boldsymbol{y})$ goes to $-\infty$ as the value of $\boldsymbol{y}$ approaches a boundary point of U_j lying in U_j^+, while $\psi(\boldsymbol{y})$ goes to $+\infty$ as $\boldsymbol{y}$ approaches a boundary point of U_j lying in U_j^-.

Below we will use repeatedly a multivariate extension of a result from classical 1-variable differential calculus: that if a real-valued function of a single variable is continuous in a neighborhood of $c \in \mathbb{R}$ and has continuous pointwise derivatives for $x > c$ and $x < c$ separately, with the property that the derivatives as $x \to c$ from above and below have a common finite limit, say λ, then (as a simple consequence of the mean-value theorem) the function is differentiable at c, and its derivative at c is λ (i.e., its derivative is continuous at c).

Now we differentiate $P_j f$ with respect to x_k for any $k \neq j$. We obtain from the Leibniz rule in the classical context that for $\boldsymbol{y} \in U_j$, we have

$$(D_k P_j f)(\boldsymbol{y}) \,=\, \int_{\psi(\boldsymbol{y})}^{\infty} (D_k\phi)(x_j, \boldsymbol{y})\, \rho(x_j)\, \mathrm{d}x_j - \phi(\psi(\boldsymbol{y}), \boldsymbol{y}) \cdot \rho(\psi(\boldsymbol{y})) \cdot (D_k\psi)(\boldsymbol{y}). \tag{11}$$

Note that all of the derivatives on the right-hand side of (11) are classical derivatives. The second term on the right-hand side of (11) is zero, since it follows from (9) that $\phi(\psi(\boldsymbol{y}),\boldsymbol{y})=0$. On the other hand, the first term on the right-hand side of (11) is continuous across the boundaries between U_j and U_j^+ and between U_j and U_j^-, because for $\boldsymbol{y}\in\text{interior}(U_j^+)$ we have $(D_kP_jf)(\boldsymbol{y})=\int_{-\infty}^{\infty}(D_k\phi)(x_j,\boldsymbol{y})\,\rho(x_j)\,\mathrm{d}x_j$, while for $\boldsymbol{y}\in\text{interior}(U_j^-)$ we have $(D_kP_jf)(\boldsymbol{y})=0$. Thus we conclude that D_kP_jf is continuous across the boundaries between U_j and U_j^+ and between U_j and U_j^-, and therefore that $P_jf\in C^1(\mathbb{R}^{\mathfrak{D}\setminus\{j\}})$.

Differentiating again with respect to x_ℓ for any $\ell\neq j$ (allowing the possibility that $\ell=k$), we obtain for $\boldsymbol{y}\in U_j$,

$$
\begin{aligned}
&(D_\ell D_kP_jf)(\boldsymbol{y})\\
&\quad=\int_{\psi(\boldsymbol{y})}^{\infty}(D_\ell D_k\phi)(x_j,\boldsymbol{y})\,\rho(x_j)\,\mathrm{d}x_j\;-\;(D_k\phi)(\psi(\boldsymbol{y}),\boldsymbol{y})\cdot\rho(\psi(\boldsymbol{y}))\cdot(D_\ell\psi)(\boldsymbol{y}),
\end{aligned}
\tag{12}
$$

and we see from [3, equation (2.14)] that $D_\ell\psi$ can be substituted by

$$(D_\ell\psi)(\boldsymbol{y})\;=\;-\frac{(D_\ell\phi)(\psi(\boldsymbol{y}),\boldsymbol{y})}{(D_j\phi)(\psi(\boldsymbol{y}),\boldsymbol{y})}.$$

Note that, unlike the second term in (11), the second term in (12) does *not* vanish in general. Hence we have $(D_\ell D_kP_jf)(\boldsymbol{y})=\int_{-\infty}^{\infty}(D_\ell D_k\phi)(x_j,\boldsymbol{y})\,\rho(x_j)\,\mathrm{d}x_j$ for $\boldsymbol{y}\in\text{interior}(U_j^+)$, while $(D_\ell D_kP_jf)(\boldsymbol{y})=0$ for $\boldsymbol{y}\in\text{interior}(U_j^-)$, and by (6) we have

$$\frac{(D_k\phi)(\psi(\boldsymbol{y}),\boldsymbol{y})\,(D_\ell\phi)(\psi(\boldsymbol{y}),\boldsymbol{y})}{(D_j\phi)(\psi(\boldsymbol{y}),\boldsymbol{y})}\,\rho(\psi(\boldsymbol{y}))\to 0$$

as $\boldsymbol{y}\in U_j$ approaches a boundary point of U_j lying in U_j^+ or U_j^-. Thus $D_\ell D_kP_jf$ exists on the boundaries between U_j, U_j^+ and U_j^- and is continuous, and hence $P_jf\in C^2(\mathbb{R}^{\mathfrak{D}\setminus\{j\}})$.

In general, for every multi-index $\boldsymbol{\alpha}=(\alpha_1,\ldots,\alpha_d)$ with $|\boldsymbol{\alpha}|\le r$ and $\alpha_j=0$, we claim that $(D^{\boldsymbol{\alpha}}P_jf)(\boldsymbol{y})=\int_{-\infty}^{\infty}(D^{\boldsymbol{\alpha}}\phi)(x_j,\boldsymbol{y})\,\rho(x_j)\,\mathrm{d}x_j$ if $\boldsymbol{y}\in\text{interior}(U_j^+)$, and $(D^{\boldsymbol{\alpha}}P_jf)(\boldsymbol{y})=0$ if $\boldsymbol{y}\in\text{interior}(U_j^-)$. On the other hand, for $\boldsymbol{y}\in U_j$ we claim that

$$(D^{\boldsymbol{\alpha}}P_jf)(\boldsymbol{y})\;=\;\int_{\psi(\boldsymbol{y})}^{\infty}(D^{\boldsymbol{\alpha}}\phi)(x_j,\boldsymbol{y})\,\rho(x_j)\,\mathrm{d}x_j\;+\;\sum_{m=1}^{M_{|\boldsymbol{\alpha}|}}g_{\boldsymbol{\alpha},m}(\boldsymbol{y}),\tag{13}$$

where $M_{|\boldsymbol{\alpha}|}$ is a non-negative integer, and each function $g_{\boldsymbol{\alpha},m}$ is of the form (5), with integers β,a,b,c and multi-indices $\boldsymbol{\alpha}^{(i)}$ satisfying

$$2\le a\le 2|\boldsymbol{\alpha}|-2,\quad 1\le b\le 2|\boldsymbol{\alpha}|-3,\quad 0\le c\le|\boldsymbol{\alpha}|-2,\quad |\boldsymbol{\alpha}^{(i)}|\le|\boldsymbol{\alpha}|-1.\tag{14}$$

Moreover, $D^{\boldsymbol{\alpha}}P_jf$ is continuous across the boundaries between U_j, U_j^+ and U_j^-, given that by (6) each $g_{\boldsymbol{\alpha},m}(\boldsymbol{y})\to 0$ as $\boldsymbol{y}\in U_j$ approaches a boundary point of U_j lying in U_j^+ or U_j^-. Since $\boldsymbol{\alpha}$ is arbitrary, with $|\boldsymbol{\alpha}|\le r$, this yields that $P_jf\in\mathcal{C}^r(\mathbb{R}^{\mathfrak{D}\setminus\{j\}})$.

We will prove (13)–(14) by induction on $|\boldsymbol{\alpha}|$. The case $|\boldsymbol{\alpha}|=1$ is shown in (11); there we have $M_1=0$. The case $|\boldsymbol{\alpha}|=2$ is shown in (12); there we have $M_2=1$, and the function $g_{\boldsymbol{\alpha},1}$ is of the form (5), with $a=2$, $b=1$, $c=0$, $\beta=1$, $D^{\boldsymbol{\alpha}^{(1)}}=D_k$, $D^{\boldsymbol{\alpha}^{(2)}}=D_\ell$, and $|\boldsymbol{\alpha}^{(1)}|=|\boldsymbol{\alpha}^{(2)}|=1$.

To establish the inductive step we now differentiate $D^{\boldsymbol{\alpha}} P_j f$ once more: for $\ell \neq j$ we have from (13) that

$$(D_\ell D^{\boldsymbol{\alpha}} P_j f)(\boldsymbol{y}) \;=\; \int_{\psi(\boldsymbol{y})}^{\infty} (D_\ell D^{\boldsymbol{\alpha}} \phi)(x_j, \boldsymbol{y})\, \rho(x_j)\, \mathrm{d}x_j$$

$$- \;(D^{\boldsymbol{\alpha}}\phi)(\psi(\boldsymbol{y}), \boldsymbol{y}) \cdot \rho(\psi(\boldsymbol{y})) \cdot (D_\ell \psi)(\boldsymbol{y}) \;+\; \sum_{m=1}^{M_{|\boldsymbol{\alpha}|}} (D_\ell g_{\boldsymbol{\alpha},m})(\boldsymbol{y}). \tag{15}$$

Clearly, the first term in (15) has the desired form. The second term in (15) is of the form (5), with $a = 2$, $b = 1$, $c = 0$, $\beta = 1$, $|\boldsymbol{\alpha}^{(1)}| = |\boldsymbol{\alpha}|$, and $|\boldsymbol{\alpha}^{(2)}| = 1$. For the remaining terms in (15), we have from (5) that

$$\begin{aligned}
(D_\ell g)(\boldsymbol{y}) \;=\; & \beta\, \frac{D_\ell\big(\prod_{i=1}^a [(D^{\boldsymbol{\alpha}^{(i)}}\phi)(\psi(\boldsymbol{y}), \boldsymbol{y})]\big)}{[(D_j\phi)(\psi(\boldsymbol{y}), \boldsymbol{y})]^b}\, \rho^{(c)}(\psi(\boldsymbol{y})) \\
& + \beta\, \frac{\prod_{i=1}^a [(D^{\boldsymbol{\alpha}^{(i)}}\phi)(\psi(\boldsymbol{y}), \boldsymbol{y})]}{[(D_j\phi)(\psi(\boldsymbol{y}), \boldsymbol{y})]^b}\, \rho^{(c+1)}(\psi(\boldsymbol{y})) \cdot (D_\ell\psi)(\boldsymbol{y}) \\
& - \beta b\, \frac{\prod_{i=1}^a [(D^{\boldsymbol{\alpha}^{(i)}}\phi)(\psi(\boldsymbol{y}), \boldsymbol{y})]}{[(D_j\phi)(\psi(\boldsymbol{y}), \boldsymbol{y})]^{b+1}}\, \rho^{(c)}(\psi(\boldsymbol{y})) \\
& \qquad \cdot \Big[(D_\ell D_j \phi)(\psi(\boldsymbol{y}), \boldsymbol{y}) + (D_j D_j \phi)(\psi(\boldsymbol{y}), \boldsymbol{y}) \cdot (D_\ell\psi)(\boldsymbol{y})\Big],
\end{aligned}$$

where

$$\begin{aligned}
& D_\ell\bigg(\prod_{i=1}^a [(D^{\boldsymbol{\alpha}^{(i)}}\phi)(\psi(\boldsymbol{y}), \boldsymbol{y})]\bigg) \\
& \quad = \sum_{t=1}^a \Bigg(\Big[(D_\ell D^{\boldsymbol{\alpha}^{(t)}}\phi)(\psi(\boldsymbol{y}), \boldsymbol{y}) + (D_j D^{\boldsymbol{\alpha}^{(t)}}\phi)(\psi(\boldsymbol{y}), \boldsymbol{y}) \cdot (D_\ell\psi)(\boldsymbol{y})\Big] \\
& \qquad\qquad \cdot \prod_{\substack{i=1\\ i\neq t}}^a (D^{\boldsymbol{\alpha}^{(i)}}\phi)(\psi(\boldsymbol{y}), \boldsymbol{y})\Bigg).
\end{aligned}$$

Thus we conclude that $D_\ell g$ is a sum of functions of the form (5), but with a increased by at most 2, b increased by at most 2, c increased by at most 1, $|\beta|$ multiplied by a factor of at most b, and with each $|\boldsymbol{\alpha}^{(i)}|$ increased by at most 1.

Hence, $D_\ell D^{\boldsymbol{\alpha}} P_j f$ consists of the first term in (15), plus a sum of functions of the form (5). This completes the induction proof for (13)–(14). In particular, the bounds in (14) can be deduced from the induction step.

We are now ready to consider

$$\begin{aligned}
& \int_{\mathbb{R}^{\mathfrak{D}\setminus\{j\}}} |(D^{\boldsymbol{\alpha}} P_j f)(\boldsymbol{y})|^p\, \rho_{\mathfrak{D}\setminus\{j\}}(\boldsymbol{y})\, \mathrm{d}\boldsymbol{y} \\
& = \int_{U_j^+} |(D^{\boldsymbol{\alpha}} P_j f)(\boldsymbol{y})|^p\, \rho_{\mathfrak{D}\setminus\{j\}}(\boldsymbol{y})\, \mathrm{d}\boldsymbol{y} + \int_{U_j} |(D^{\boldsymbol{\alpha}} P_j f)(\boldsymbol{y})|^p\, \rho_{\mathfrak{D}\setminus\{j\}}(\boldsymbol{y})\, \mathrm{d}\boldsymbol{y},
\end{aligned} \tag{16}$$

where we have split the integral noting that U_j is open, and the disjoint sets U_j^+ and U_j^- are closed. Using the special form of $D^{\boldsymbol{\alpha}} P_j f$ in (13), we have for $\boldsymbol{y} \in U_j$,

$$\begin{aligned}
|(D^{\boldsymbol{\alpha}} P_j f)(\boldsymbol{y})|^p &= \left| \int_{\psi(\boldsymbol{y})}^{\infty} (D^{\boldsymbol{\alpha}} \phi)(x_j, \boldsymbol{y})\, \rho(x_j)\, \mathrm{d}x_j + \sum_{m=1}^{M_{|\boldsymbol{\alpha}|}} g_{\boldsymbol{\alpha},m}(\boldsymbol{y}) \right|^p \\
&\le \left(\left| \int_{\psi(\boldsymbol{y})}^{\infty} (D^{\boldsymbol{\alpha}} \phi)(x_j, \boldsymbol{y})\, \rho(x_j)\, \mathrm{d}x_j \right| + \sum_{m=1}^{M_{|\boldsymbol{\alpha}|}} |g_{\boldsymbol{\alpha},m}(\boldsymbol{y})| \right)^p \\
&\le (M_{|\boldsymbol{\alpha}|} + 1)^{p-1} \left(\left| \int_{\psi(\boldsymbol{y})}^{\infty} (D^{\boldsymbol{\alpha}} \phi)(x_j, \boldsymbol{y})\, \rho(x_j)\, \mathrm{d}x_j \right|^p + \sum_{m=1}^{M_{|\boldsymbol{\alpha}|}} |g_{\boldsymbol{\alpha},m}(\boldsymbol{y})|^p \right) \\
&\le (M_{|\boldsymbol{\alpha}|} + 1)^{p-1} \left(\int_{\psi(\boldsymbol{y})}^{\infty} |(D^{\boldsymbol{\alpha}} \phi)(x_j, \boldsymbol{y})|^p\, \rho(x_j)\, \mathrm{d}x_j + \sum_{m=1}^{M_{|\boldsymbol{\alpha}|}} |g_{\boldsymbol{\alpha},m}(\boldsymbol{y})|^p \right),
\end{aligned}$$

where in the second to last step we used a generalized mean inequality (see [1, 3.2.4])

$$\frac{\sum_{i=1}^n a_i}{n} \le \left(\frac{\sum_{i=1}^n a_i^p}{n} \right)^{1/p}, \qquad a_i \ge 0, \quad p \in [1, \infty),$$

and in the last step we used Hölder's inequality as in [3, equation (2.11)]. Thus using (16) we find

$$\begin{aligned}
&\int_{\mathbb{R}^{\mathfrak{D} \setminus \{j\}}} |(D^{\boldsymbol{\alpha}} P_j f)(\boldsymbol{y})|^p\, \rho_{\mathfrak{D} \setminus \{j\}}(\boldsymbol{y})\, \mathrm{d}\boldsymbol{y} \le \int_{\mathbb{R}^d} |(D^{\boldsymbol{\alpha}} \phi)(\boldsymbol{x})|^p \rho_d(\boldsymbol{x})\, \mathrm{d}\boldsymbol{x} \\
&+ (M_{|\boldsymbol{\alpha}|} + 1)^{p-1} \left(\int_{\mathbb{R}^d} |(D^{\boldsymbol{\alpha}} \phi)(\boldsymbol{x})|^p \rho_d(\boldsymbol{x})\, \mathrm{d}\boldsymbol{x} + \sum_{m=1}^{M_{|\boldsymbol{\alpha}|}} \int_{U_j} |g_{\boldsymbol{\alpha},m}(\boldsymbol{y})|^p \rho_{\mathfrak{D} \setminus \{j\}}(\boldsymbol{y})\, \mathrm{d}\boldsymbol{y} \right),
\end{aligned}$$

which is finite, since $\phi \in \mathcal{W}^r_{d,p,\rho_d}$ and each integral involving $g_{\boldsymbol{\alpha},m}$ is finite due to the condition (7). This proves that $P_j f \in \mathcal{W}^r_{\mathfrak{D} \setminus \{j\}, p, \rho_{\mathfrak{D} \setminus \{j\}}}$ as claimed. ☐

In the following theorem, the property $(D_j \phi)(\boldsymbol{x}) \neq 0$ for all $\boldsymbol{x} \in \mathbb{R}^d$ and the conditions (6) and (7) are assumed to hold for all j in a subset $\mathbf{z} \subseteq \mathfrak{D}$. In the best case $\mathbf{z} = \mathfrak{D}$, we see that smoothing occurs for all ANOVA terms except for the term with the highest order. The proof follows that of [3, Theorem 3.2], but makes use of the new theorem above. It is based on the explicit formula $f_{\mathbf{u}} = \sum_{\mathbf{v} \subseteq \mathbf{u}} (-1)^{|\mathbf{u}| - |\mathbf{v}|} P_{\mathfrak{D} \setminus \mathbf{v}} f$; see [3, equation (2.3)].

Theorem 2. *Let $r \ge 1$, $p \in [1, \infty)$, and $\rho \in \mathcal{C}^\infty(\mathbb{R})$ be a strictly positive probability density function. Let $\mathbf{z}$ be a non-empty subset of $\mathfrak{D}$, and let*

$$f(\boldsymbol{x}) = \phi(\boldsymbol{x})_+, \quad \text{with} \quad \begin{cases} \phi \in \mathcal{W}^r_{d,p,\rho_d} \cap \mathcal{C}^\infty(\mathbb{R}^d), \\ (D_j \phi)(\boldsymbol{x}) \neq 0 \quad \text{for all } j \in \mathbf{z} \text{ and all } \boldsymbol{x} \in \mathbb{R}^d, \\ (6) \text{ and } (7) \text{ hold for all } j \in \mathbf{z}. \end{cases}$$

Then $f \in \mathcal{W}^1_{d,p,\rho_d} \cap \mathcal{C}(\mathbb{R}^d)$, and the ANOVA terms of f satisfy

$$f_{\mathbf{u}} \in \begin{cases} \mathcal{W}^1_{\mathbf{u},p,\rho_{\mathbf{u}}} & \text{if} \quad \mathbf{z} \subseteq \mathbf{u}, \\ \mathcal{W}^r_{\mathbf{u},p,\rho_{\mathbf{u}}} & \text{if} \quad \mathbf{z} \not\subseteq \mathbf{u}, \end{cases} \qquad \text{for all} \quad \mathbf{u} \subseteq \mathfrak{D}.$$

In particular, if $\mathbf{z} = \mathfrak{D}$, then $f_{\mathfrak{D}} \in \mathcal{W}^1_{d,p,\rho_d}$ and $f_{\mathbf{u}} \in \mathcal{W}^r_{\mathbf{u},p,\rho_{\mathbf{u}}}$ for all $\mathbf{u} \subsetneq \mathfrak{D}$.

4. Application of the new theorem to option pricing problems

The conditions (6) and (7) in their current form are not easy to check due to the presence of the function ψ_j. However, sufficient conditions that are easier to check can be obtained if (as in the case of the option pricing problem) we have precise information about the weight function ρ.

We have already explained that (8) is a sufficient condition for (6). In the case of the option pricing problem, ρ is the standard Gaussian density, whereas ϕ and its derivatives (see (1)) have only exponential dependence, thus (8) certainly holds. The condition (7) is weaker than the condition (3.4) in [3]. It was shown in [3, Section 4] that the latter condition holds for the option pricing problem.

As outlined in §1, we mistakenly claimed [3, pages 396–397] that ϕ always changes sign. From the fact that $\phi(x_j, \boldsymbol{x}_{\mathfrak{D}\setminus\{j\}}) \to +\infty$ as $x_j \to +\infty$, it follows that the set U_j^- is empty for the Asian option pricing problem. On the other hand, if j is such that the set $\{\ell \in \mathfrak{D} : A_{\ell,j} = 0\}$ is not empty, then $B_j(\boldsymbol{x}_{\mathfrak{D}\setminus\{j\}})$ in (3) can take all values between $-K$ and $+\infty$, from which it follows that the set U_j^+ will not be empty. Hence Theorem 1 holds, and in turn Theorem 2 applies with $\boldsymbol{z} = \mathfrak{D}$ for the standard and Brownian bridge constructions. The conclusion of our original manuscript stands.

Acknowledgements

We sincerely thank an anonymous referee who identified the mistake in §2 while reviewing a follow-up paper. We also thank Andreas Griewank for valuable discussions. The authors acknowledge the support of the Australian Research Council under the projects FT130100655 and DP150101770. Michael Griebel was partially supported by the Sonderforschungsbereich 1060 *The Mathematics of Emergent Effects* funded by the Deutsche Forschungsgemeinschaft.

References

[1] M. Abramowitz and I. A. Stegun (eds.), *Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables*, Dover Publications, Inc., New York, 1992. Reprint of the 1972 edition. MR1225604

[2] R. A. Adams, *Sobolev Spaces*, Pure and Applied Mathematics, Vol. 65, Academic Press [A subsidiary of Harcourt Brace Jovanovich, Publishers], New York-London, 1975. MR0450957

[3] M. Griebel, F. Y. Kuo, and I. H. Sloan, *The smoothing effect of integration in $\mathbb{R}^d$ and the ANOVA decomposition*, Math. Comp. **82** (2013), no. 281, 383–400, DOI 10.1090/S0025-5718-2012-02578-6. MR2983028

Institut für Numerische Simulation, Universität Bonn, Wegelerstrasse 6, 53115, Bonn, Germany – and – Fraunhofer Institute for Algorithms and Scientific Computing SCAI, Schloss Birlinghoven, 53754 Sankt Augustin, Germany

E-mail address: griebel@ins.uni-bonn.de

School of Mathematics and Statistics, University of New South Wales, Sydney, New South Wales 2052, Australia

E-mail address: f.kuo@unsw.edu.au

School of Mathematics and Statistics, University of New South Wales, Sydney, New South Wales 2052, Australia

E-mail address: i.sloan@unsw.edu.au