جامعة الملك عبدالله للعلوم والتقنية
King Abdullah University of Science and Technology

Center for Uncertainty Quantification

Hierarchical adaptive sparse grids and Quasi Monte Carlo for option pricing under the rough Bergomi model

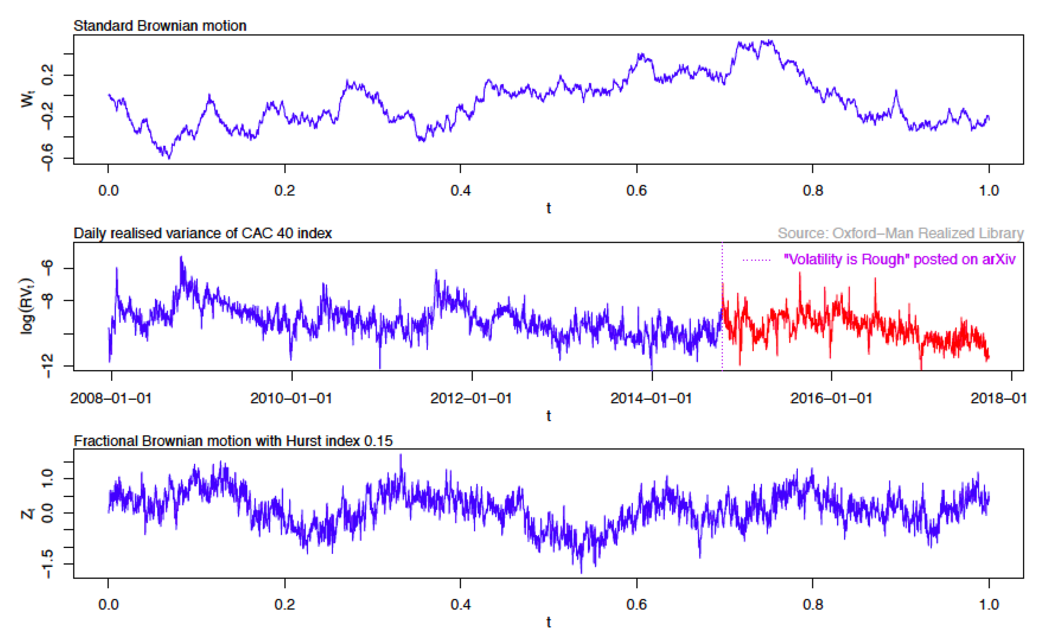
Christian Bayer, Chiheb Ben Hammouda and Raul Tempone

King Abdullah University of Science and Technology (KAUST), Computer, Electrical and Mathematical Sciences & Engineering Division (CEMSE), Saudi Arabia

Abstract

The rough Bergomi (rBergomi) model, introduced recently in [1], is a promising rough volatility model in quantitative finance. This new model exhibits consistent results with the empirical fact of implied volatility surfaces being essentially time-invariant. This model also has the ability to capture the term structure of skew observed in equity markets. In the absence of analytical European option pricing methods for the model, and due to the non-Markovian nature of the fractional driver, the prevalent option is to use Monte Carlo (MC) simulation for pricing. Despite recent advances in the MC method in this context, pricing under the rBergomi model is still a time-consuming task. To overcome this issue, we design a novel, alternative, hierarchical approach, based on i) adaptive sparse grids quadrature (ASGQ), specifically using the same construction in [6], and ii) Quasi Monte Carlo (QMC). Both techniques are coupled with Brownian bridge construction and Richardson extrapolation. By uncovering the available regularity, our hierarchical methods demonstrates substantial computational gains with respect to the standard MC method, when reaching a sufficiently small error tolerance in the price estimates across different parameter constellations, even for very small values of the Hurst parameter. Our work opens a new research direction in this field, i.e. to investigate the performance of methods other than Monte Carlo for pricing and calibrating under the rBergomi model.

Rough volatility



The rough Bergomi model [1]

This model, under a pricing measure, is given by

$$\begin{cases} dS_t &= \sqrt{v_t} S_t dZ_t, \\ v_t &= \xi_0(t) \exp\left(\eta \widetilde{W}_t^H - \frac{1}{2}\eta^2 t^{2H}\right), \\ Z_t &:= \rho W_t^1 + \bar{\rho} W_t^{\perp} \equiv \rho W^1 + \sqrt{1-\rho^2} W^{\perp}, \end{cases} \tag{1}$$

- $(W^1, W^{\perp})$: two independent standard Brownian motions
- $\widetilde{W}^H$ is Riemann-Liouville process, defined by

$$\widetilde{W}_t^H = \int_0^t K^H(t-s) dW_s^1, \quad t \geq 0,$$
$$K^H(t-s) = \sqrt{2H}(t-s)^{H-1/2}, \quad \forall\, 0 \leq s \leq t.$$

- $H \in (0, 1/2]$ ($H = 1/2$ for Brownian motion): controls the roughness of paths, , $\rho \in [-1, 1]$ and $\eta > 0$.
- $t \mapsto \xi_0(t)$: forward variance curve, known at time 0.

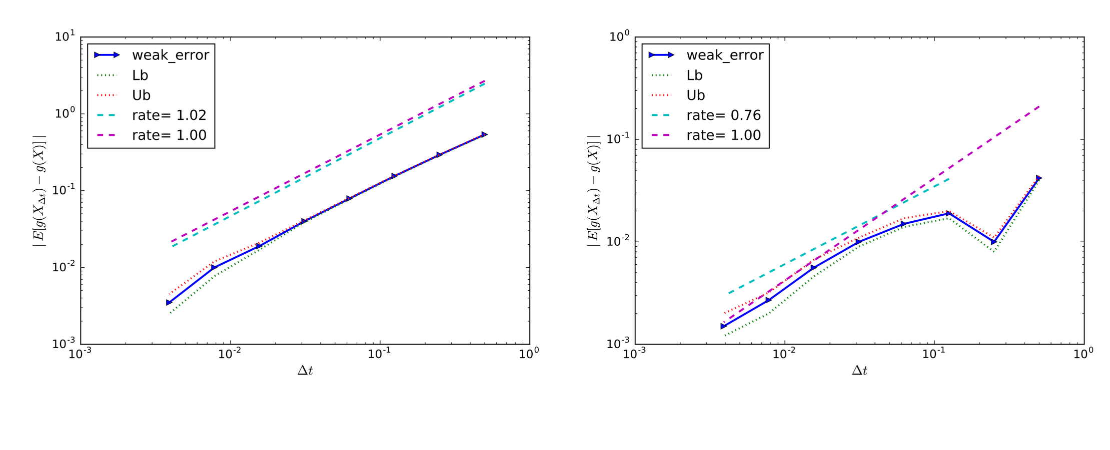
Challenges

- **Numerically:**
  - The model is non-affine and non-Markovian $\Rightarrow$ Standard numerical methods (PDEs, characteristic functions) seem inapplicable.
  - The only prevalent pricing method for mere vanilla options is Monte Carlo [1, 2, 7], still a time consuming task.
  - Discretization methods have poor behavior of the strong error, that is the convergence rate is of order of $H \in [0, 1/2]$ [8] $\Rightarrow$ Variance reduction methods, such as MLMC, are inefficient for very small values of H.
- **Theoretically:**
  - No proper weak error analysis done in the rough volatility context.

Contributions

1. We design an alternative hierarchical efficient pricing method based on:
   i) Analytic smoothing to uncover available regularity.
   ii) Approximating the option price using a deterministic quadrature method (ASGQ and QMC) coupled with Brownian bridges and Richardson Extrapolation.
2. Our hierarchical methods demonstrate substantial computational gains with respect to the standard MC method, assuming a sufficiently small error tolerance in the price estimates, even for very small values of the Hurst parameter, H.

On the choice of the simulation scheme



(a) (b)

Figure 2: *The convergence of the weak error $\mathcal{E}_B$, using MC with 6×10^6 samples, for Set 1 parameter in Table 1. The upper and lower bounds are 95% confidence intervals. a) With the hybrid scheme b) With the exact scheme.*

The Hybrid Scheme [4]

$$\widetilde{W}_t^H = \int_0^t K^H(t-s) dW_s^1, \quad t \geq 0,$$
$$K^H(t-s) = \sqrt{2H}(t-s)^{H-1/2}, \quad \forall\, 0 \leq s \leq t.$$

- The hybrid scheme discretizes the $\widetilde{W}^H$ process into Wiener integrals of power functions and a Riemann sum, appearing from approximating the kernel by power functions near the origin and step functions elsewhere.

$$\widetilde{W}_{\frac{i}{N}}^H \approx \overline{W}_{\frac{i}{N}}^H = \sqrt{2H}\left(W_i^2 + \sum_{k=2}^{i}\left(\frac{b_k}{N}\right)^{H-\frac{1}{2}}\left(W_{\frac{i-(k-1)}{N}}^1 - W_{\frac{i-k}{N}}^1\right)\right),$$

where
- N is the number of time steps
- $\{W_j^2\}_{j=1}^N$: Artificially introduced N Gaussian random variables that are used for left-rule points in the hybrid scheme.
- $b_k = \left(\frac{k^{H+\frac{1}{2}} - (k-1)^{H+\frac{1}{2}}}{H+\frac{1}{2}}\right)^{\frac{1}{H-\frac{1}{2}}}$.

The rough Bergomi Model: Analytic Smoothing

We show that the call price is given by

$$\begin{aligned} C_{RB}(T,K) &= E\left[(S_T - K)^+\right] \\ &= \mathrm{E}\left[\mathrm{E}\left[(S_T-K)^+ \mid \sigma(W^1(t), t \leq T)\right]\right] \\ &= E\left[C_{BS}\left(S_0 = \exp\left(\rho\int_0^T \sqrt{v_t} dW_t^1 - \frac{1}{2}\rho^2\int_0^T v_t dt\right),\right.\right. \\ &\qquad \left.\left. k = K,\ \sigma^2 = (1-\rho^2)\int_0^T v_t dt\right)\right] \\ &\approx \int_{\mathbb{R}^{2N}} C_{BS}\left(G(\mathbf{w}^{(1)}, \mathbf{w}^{(2)})\right) \rho_N(\mathbf{w}^{(1)}) \rho_N(\mathbf{w}^{(2)}) d\mathbf{w}^{(1)} d\mathbf{w}^{(2)}, \end{aligned} \tag{2}$$

where
- $C_{\text{BS}}(S_0, k, \sigma^2)$ denotes the Black-Scholes call price, for initial spot price S_0, strike price k, and volatility σ^2.
- ρ_N: the multivariate Gaussian density, N: number of time steps.

Sparse Grids

A quadrature estimate of $E[F]$ is

$$\mathcal{M}_{\mathcal{I}_\ell}[F] = \sum_{\boldsymbol{\beta} \in \mathcal{I}_\ell} \Delta[F_{\boldsymbol{\beta}}], \tag{3}$$

- Product approach: $\mathcal{I}_\ell = \{\max\{\beta_1, \ldots, \beta_d\} \leq \ell;\ \boldsymbol{\beta} \in \mathbb{N}_+^d\}$
- Regular SG: $\mathcal{I}_\ell = \{\mid \boldsymbol{\beta} \mid_1 \leq \ell + d - 1;\ \boldsymbol{\beta} \in \mathbb{N}_+^d\}$

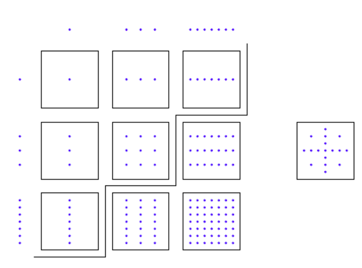


Figure 3: *Left are product grids $\Delta_{\beta_1} \otimes \Delta_{\beta_2}$ for $1 \leq \beta_1, \beta_2 \leq 3$. Right is the corresponding SG construction.*

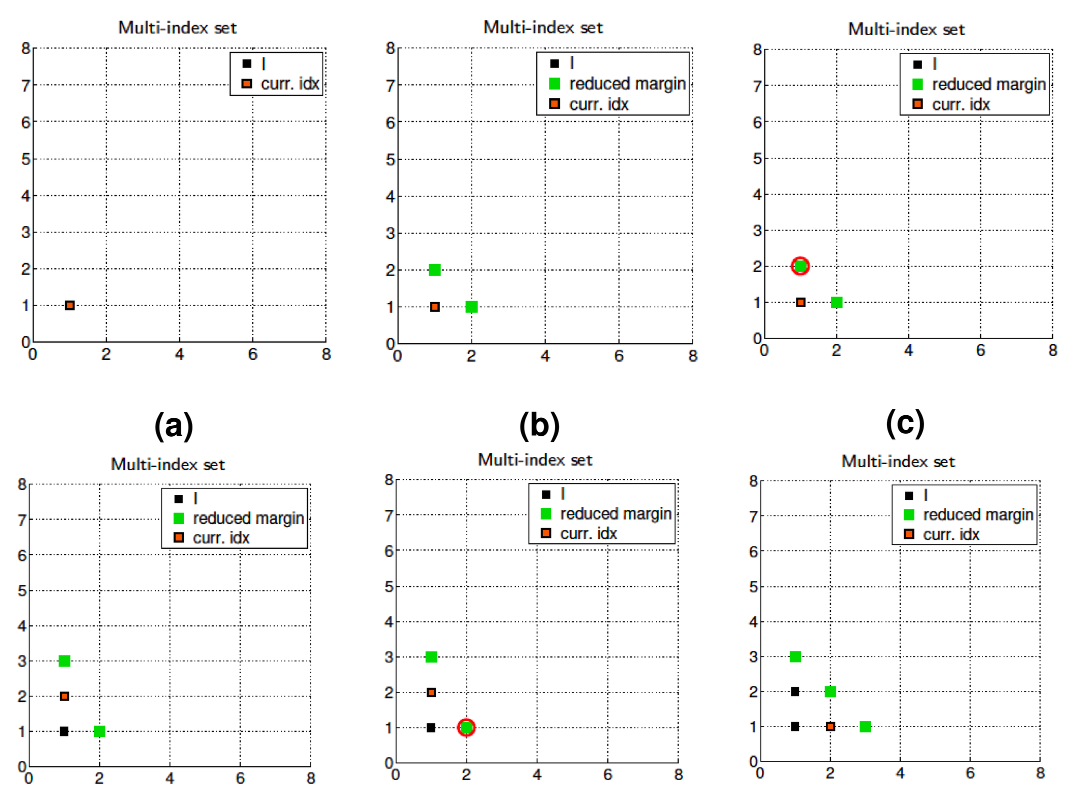
- ASGQ based on same construction as in [6]: $\mathcal{I}_\ell = \mathcal{I}^{\text{ASGQ}}$.

ASGQ in Practice

- The construction of $\mathcal{I}^{\text{ASGQ}}$ is done by profit thresholding

$$\mathcal{I}^{\text{ASGQ}} = \{\boldsymbol{\beta} \in \mathbb{N}_+^d : P_{\boldsymbol{\beta}} \geq \overline{T}\}.$$

- **Profit of a hierarchical surplus** $P_{\boldsymbol{\beta}} = \frac{|\Delta E_{\boldsymbol{\beta}}|}{\Delta \mathcal{W}_{\boldsymbol{\beta}}}$.
- **Error contribution**: $\Delta E_{\boldsymbol{\beta}} = \left|\mathcal{M}^{\mathcal{I}\cup\{\boldsymbol{\beta}\}} - \mathcal{M}^{\mathcal{I}}\right|$.
- **Work contribution**: $\Delta \mathcal{W}_{\boldsymbol{\beta}} = \text{Work}[\mathcal{M}^{\mathcal{I}\cup\{\boldsymbol{\beta}\}}] - \text{Work}[\mathcal{M}^{\mathcal{I}}]$
- A posteriori, adaptive construction: Given an index set $\mathcal{I}_k$, compute the profits of the neighbor indices and select the most profitable one



(a) (b) (c)

(d) (e) (f)

Figure 4: *Construction of the index set for ASGQ method*

Numerical Experiments

Table 1: *Reference solution, using MC with 500 time steps and number of samples, $M = 4 \times 10^6$, of call option price under the rough Bergomi model, for different parameter constellations. The numbers between parentheses correspond to the statistical errors estimates.*

Parameters	Reference solution
Set 1: $H = 0.07, K = 1, S_0 = 1, \rho = -0.9, \eta = 1.9, \xi = 0.235^2$	0.0791 $(7.9e{-}05)$
Set 2: $H = 0.02, K = 1, S_0 = 1, \rho = -0.7, \eta = 0.4, \xi = 0.1$	0.1248 $(1.3e{-}04)$
Set 3: $H = 0.02, K = 0.8, S_0 = 1, T = 1, \rho = -0.7, \eta = 0.4, \xi_0 = 0.1$	0.2412 $(1.8e{-}04)$
Set 4: $H = 0.02, K = 1.2, S_0 = 1, T = 1, \rho = -0.7, \eta = 0.4, \xi_0 = 0.1$	0.0570 $(8.0e{-}05)$

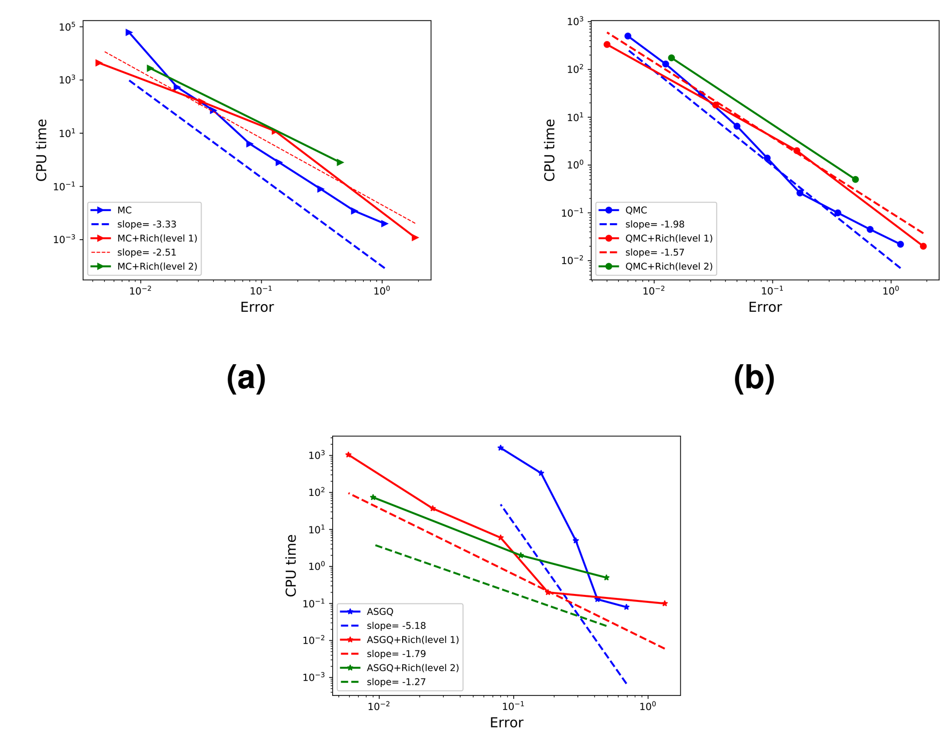
- The first set is the one that is closest to the empirical findings [5, 3], which suggest that $H \approx 0.1$. The choice of parameters values of $\nu = 1.9$ and $\rho = -0.9$ is justified by [1].
- For the remaining three sets, we wanted to test the potential of our method for a very rough case, where variance reduction methods, such as MLMC are inefficient.

Results I

Table 2: *Summary of relative errors and computational gains, achieved by the different methods. In this table, we highlight the computational gains achieved by ASGQ and QMC over MC method to meet a certain error tolerance. We note that the ratios are computed for the best configuration with Richardson extrapolation for each method.*

Parameter set	Total relative error	CPU time ratio (MC/ASGQ)	CPU time ratio (MC/QMC)
Set 1	1%	15	10
Set 2	0.2%	21.5	73.2
Set 3	0.4%	26.7	21.3
Set 4	2%	5	10

Results II



(a) (b)

(c)

Figure 5: *Comparing the numerical complexity of the different methods with the different configurations in terms of the level of Richardson extrapolation, for the case of parameter set 1 in Table 1. a) MC methods. b) QMC methods. d) ASGQ methods.*

Results III

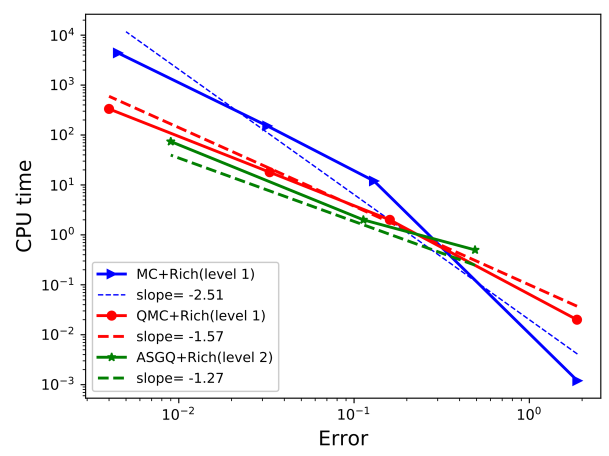


Figure 6: *Computational work comparison for the different methods with the best configurations concluded from Figure 5, for the case of parameter set 1 in Table 1. This plot shows that to achieve a relative error below 1%, ASGQ coupled with level 2 of Richardson extrapolation and QMC coupled with level 1 of Richardson extrapolation have the same performance. Furthermore, they outperform significantly MC method coupled with level 1 of Richardson extrapolation.*

Acknowledgements

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