Efficient option pricing for the Rough Bergomi model

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Abstract

One of the recent rough volatility models that showed a promising potential in quantitative finance is the rough Bergomi model, introduced in [3]. This new model exhibits consistent results with the empirical fact of implied volatility surfaces being essentially time-invariant, and ability to capture the term structure of skew observed in equity markets. In the absence of analytical European option pricing methods for the model, the prevalent option is to use Monte Carlo (MC) simulation for efficient pricing. We design a novel alternative hierarchical approach, based on adaptive sparse grids quadrature, specifically multi-index stochastic collocation (MISC) as in [18], coupled with Brownian bridge construction and Richardson extrapolation. This hierarchical method demonstrates substantial computational gains with respect to the standard Monte Carlo method, assuming a sufficiently small error tolerance in the price estimates, across different parameter constellations.

Keywords Rough volatility, Monte Carlo, multi-index stochastic collocation, Brownian bridge construction, Richardson extrapolation.
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1   Introduction

Modeling the volatility to be stochastic, rather than deterministic as in the Black-Scholes model, enabled quantitative analysts to explain certain phenomena observed in option price data, in particular the implied volatility smile. However, this family of models has a main drawback consisting in failing to capture the true steepness of the implied volatility smile close to maturity. To overcome this undesired feature, one may add jumps to stock price models, for instance modeling the stock price process as an exponential Lévy process. Unfortunately, the presence of jumps in stock price processes remains controversial [11, 2], and is one of the major criticisms of such models.

Motivated by the statistical analysis of realized volatility by Gatheral, Jaisson and Rosenbaum [16] and the theoretical results on implied volatility by Fukasawa [14], rough stochastic volatility has emerged as a new paradigm in quantitative finance, overcoming the observed limitations of diffusive stochastic volatility models. In these models, the trajectories of the volatility has lower

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Hölder regularity than those of the standard Brownian motion [16, 3]. In fact, they are based on fractional Brownian motion (fBm), which is a centered Gaussian process, whose covariance structure depends on the Hurst parameter $0 < H < 1/2$ (we refer to [22, 12, 9] for more details regarding the fBm processes). In this case, the fBm has negatively correlated increments and rough sample paths. Gatheral, Jaisson, and Rosenbaum [16] empirically demonstrate the advantages of such models. For instance, they show that the log-volatility in practice has a similar behavior to fBm with the Hurst exponent $H \approx 0.1$ at any reasonable time scale (see also [15]). These results were confirmed by Bennedsen, Lunde and Pakkanen [6], who studied over a thousand individual US equities and showed that the Hurst parameter H lies in $(0, 1/2)$ for each equity. Other works showed further benefits of such rough volatility models over standard stochastic volatility ones, in terms of explaining crucial phenomena observed in financial markets. For instance, from a statistical point of view, we mention [16, 6], and from an option-pricing point of view, we mention [3].

One of the recent rough volatility models is the rough Bergomi (rBergomi) model, developed by Bayer, Friz and Gatheral [3]. This model showed consistent behavior with the stylized fact of implied volatility surfaces being essentially time-invariant. It was also proven that this model is able to capture the term structure of skew observed in equity markets. The construction of the rBergomi model was performed by moving from physical to pricing measure and simulating prices under that model to fit the implied volatility surface well in the case of the S&P 500 index with few parameters. The model's name is inspired by the Bergomi variance curve model [8], and may be seen as a non-Markovian extension of the latter.

Despite the promising features of the rBergomi model, pricing and hedging under such model still constitutes a challenging task due to the non-Markovian nature of the fractional driver. In fact, the standard numerical pricing methods, such as: Monte Carlo (MC) estimators, PDE discretization schemes, asymptotic expansions and transform methods, being efficient in the case of diffusion, are not easily carried over to the rough setting. Furthermore, due to the lack of Markovianity or affine structure, conventional analytical pricing methods do not apply. To the best of our knowledge, the only prevalent method for pricing options under such models is MC simulation. In particular, recent advances in simulation methods for the rBergomi model and different variants of pricing methods based on MC under such model have been proposed in [3, 4, 23, 7, 20]. For instance, in [23], the authors employ a novel composition of variance reduction methods. When pricing under the rBergomi model, they got substantial computational gains over the standard MC method. More attempts to have analytical understanding of option pricing and implied volatility under this model have been achieved in [21, 5, 13]. We should point out that pricing and model calibration under rough volatility models still remains a time consuming task.

In this paper, we design a novel hierarchical fast option pricer, based on a hierarchical adaptive sparse grids quadrature, specifically multi-index stochastic collocation (MISC) as in [18], coupled with Brownian bridge construction and Richardson extrapolation, for options whose underlyings follow the rBergomi model as in [3]. In order to use adaptive sparse grids quadrature for our purposes, we solve two main issues, that constitute the two stages of our new designed method. In the first stage, we smoothen the integrand by using the conditional expectation as was proposed by [26] in the context of Markovian stochastic volatility models. In a second stage, we apply MISC as in [18], to solve the integration problem. In this stage, we apply two transformations before using the MISC solver, in order to overcome the issue of facing a high dimensional integrand due to the discretization scheme used for simulating the rBergomi dynamics. Given that MISC profits from anisotropy, the first transformation consists of applying a hierarchical path generation method,

based on Brownian bridge (Bb) construction, with the aim of reducing the effective dimension. The second transformation consists of applying Richardson extrapolation to reduce the bias, which in turn reduces the needed number of time steps in the coarsest level to achieve a certain error tolerance, and therefore, the maximum number of dimensions needed for the integration problem.

Compared to the works that we mentioned above, mainly [23], our first contribution is that we design an alternative approach based on a variant of adaptive sparse grid quadrature. Given that the only prevalent option in this context is to use different variants of MC method, our work opens a new research direction in this field to investigate the performance of other methods besides MC to solve pricing and calibration problems related to the rBergomi model. Our second contribution is that we reduce the computational cost by using the conditional expectation as in [23] but also through bias reduction through using Richardson extrapolation. Finally, assuming one targets price estimates with a sufficiently small error tolerance, our third contribution is manifested by the observed substantial computational gains over standard MC method. We show these gains through our numerical experiments for different parameter constellations. In this paper, we limit ourselves to compare our novel proposed method against standard MC. A more systematic comparison against the variant of MC proposed in [23] can be carried over but this is left for a future work. Another eventual direction of research may also investigate the performance of quasi-Monte Carlo (QMC) for such problems.

The outline of this paper is as follows: We start in Section 2 by introducing the pricing framework that we are considering. We provide some details about the rBergomi model, option pricing under this model and simulation scheme used to simulate asset prices following the rBergomi dynamics. Then, in Section 3, we explain the different building blocks that constitute our proposed method, which are basically MISC, Brownian bridge construction and Richardson extrapolation. Finally, in Section 4, we show the results, the results obtained through different numerical experiments, across different parameter constellations for the rBergomi model. Assuming one targets European call option prices estimates with a sufficiently small error tolerance, the reported results illustrate the substantial computational gains of our novel hierarchical method over standard MC, under different settings.

2   Problem setting

In this section, we introduce the pricing framework that we are considering in this project. We start in Section 2.1 by giving some details for the rBergomi model proposed in [3]. We then derive the formula of the price of a European call option under the rBergomi model, in Section 2.2. This Section corresponds basically to the first stage of our approach, that is the analytical smoothing step. Finally, we explain in Section 2.3, some details about the hybrid scheme that we used to simulate the dynamics of asset prices under the rBergomi model.

2.1   The rBergomi model

We use the rBergomi model for the price process S_t as defined in [3], normalized to $r = 0$, which is defined by

$$dS_t = \sqrt{v_t(\widetilde{W}^H)}S_t dZ_t,$$

(2.1)
$$v_t = \xi_0(t)\exp\left(\eta\widetilde{W}_t^H - \frac{1}{2}\eta^2 t^{2H}\right),$$

where $0 < H < 1$ (Hurst parameter) and $\eta > 0$. We refer to v_t as the variance process, where $\xi_0(t) = \mathrm{E}\,[v_t]$ is the forward variance curve. Here, $\widetilde{W}^H$ is a certain Riemann-Liouville fBm process, defined by

(2.2)
$$\widetilde{W}_t^H = \int_0^t K^H(t,s)dW_s^1, \quad t \geq 0$$

where the kernel $K^H : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$ reads

$$K^H(t,s) = \sqrt{2H}(t-s)^{H-1/2}, \quad \forall\, 0 \leq s \leq t.$$

We note that the map $s \to K^H(s,t)$ belongs to L^2, so that the stochastic integral (2.2) is well defined. $\widetilde{W}^H$ is a centered, locally $(H-\epsilon)$- Hölder continuous, Gaussian process with $\mathrm{var}\,[\widetilde{W}_t^H] = t^{2H}$.

W^1, Z denote two *correlated* standard Brownian motions with correlation $\rho \in [-1,1]$, so that we can write

$$Z := \rho W^1 + \bar{\rho}W^\perp \equiv \rho W^1 + \sqrt{1-\rho^2}W^\perp,$$

where $(W^1, W^\perp)$ are two independent standard Brownian motions. Therefore, (2.1) can be written as

$$S_t = S_0 \exp\left(\int_0^t \sqrt{v(s)}dZ(s) - \frac{1}{2}\int_0^t v(s)ds\right), \quad S_0 > 0$$

(2.3)
$$v_u = \xi_0(u)\exp\left(\eta\widetilde{W}_u^H - \frac{\eta^2}{2}u^{2H}\right), \quad \xi_0 > 0.$$

The filtration $(\mathcal{F}_t)_{t\geq 0}$ can here be taken as the one generated by the two-dimensional Brownian motion $(W^1, W^\perp)$ under the risk neutral measure $\mathbb{Q}$, resulting in a filtered probability space $(\Omega, \mathcal{F}, \mathcal{F}_t, \mathbb{Q})$. The stock price process S is clearly then a local $(\mathcal{F}_t)_{t\geq 0}$-martingale and a super-martingale, therefore integrable. We shall henceforth use the notation $\mathrm{E}\,[.] = E^{\mathbb{Q}}\,[.\,|\,\mathcal{F}_0]$ unless we state otherwise.

2.2 Option pricing under the rBergomi model

In our project, we are interested in pricing European call options under the rBergomi model. Assuming $S_0 = 1$, and using the conditioning argument on the σ-algebra generated by W^1 (argument first used by [26] in the context of Markovian stochastic volatility models), we can show that the call price is given by

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$$C_{RB}(T, K) = E\left[(S_T - K)^+\right]$$
$$= \mathrm{E}\left[\mathrm{E}\left[(S_T - K)^+ \mid \sigma(W^1(t), t \leq T)\right]\right]$$

(2.4)

$$= E\left[C_{BS}\left(S_0 = \exp\left(\rho \int_0^T \sqrt{v_t} dW_t^1 - \frac{1}{2}\rho^2 \int_0^T v_t dt\right), \ K = K, \ T = 1, \ \sigma^2 = (1 - \rho^2)\int_0^T v_t dt\right)\right],$$

where C_{BS} denotes the Black-Scholes price.

In fact, if we use the orthogonal decomposition of S_t into S_t^1 and S_t^2, where

$$S_t^1 := \mathcal{E}\{\rho \int_0^t \sqrt{v_s} dW_s^1\}, \ S_t^2 := \mathcal{E}\{\sqrt{1 - \rho^2}\int_0^t \sqrt{v_s} dW_s^\perp\},$$

where $\mathcal{E}(.)$ denotes the stochastic exponential, then we obtain by conditional log-normality

$$\log S_t \mid \mathcal{F}_t^1 \sim \mathcal{N}\left(\log S_t^1 - \frac{1}{2}(1 - \rho^2)\int_0^t v_s ds, (1 - \rho^2)\int_0^t v_s ds\right),$$

where $\mathcal{F}_t^1 = \sigma\{W_s^1 : s \leq t\}$. Therefore, we obtain (2.4).

We point out that the analytical smoothing, based on conditioning, performed in (2.4) enables us to get a smooth analytic integrand inside the expectation. Therefore, applying sparse quadrature techniques becomes an adequate option for computing the call price as we will investigate later.

2.3 Simulation of the rBergomi model

One of the numerical challenges encountered in the simulation of the rBergomi dynamics is the computation of the terms $\int_0^T \sqrt{v_t} dW_t^1$ and $V = \int_0^T v_t dt$ as in (2.4), mainly because of the singularity of the Volterra kernel $K^H(s, t)$ at the diagonal $s = t$. In fact, one needs to jointly simulate the two-dimensional Gaussian process $(W_t^1, \widetilde{W}_t^H : 0 \leq t \leq T)$, resulting in $W_{t_1}^1, \ldots, W_{t_N}^1$ and $\widetilde{W}_{t_1}^H, \ldots, \widetilde{W}_{t_N}^H$ along a given time grid $t_1 < \cdots < t_N$. In the literature, there are essentially three possible ways to achieve this:

i) Euler discretization of the integral (2.2), defining $\widetilde{W}^H$, together with classical simulation of increments of W^1. This is inefficient because the integral is singular and adaptivity probably does not help, as the singularity moves with time. For this method, we need an N-dimensional random Gaussian input vector to produce one (approximate, inaccurate) sample of $W_{t_1}^1, \ldots, W_{t_N}^1, \widetilde{W}_{t_1}^H, \ldots, \widetilde{W}_{t_N}$.

ii) Given that $W_{t_1}^1, \ldots, W_{t_N}^1, \widetilde{W}_{t_1}^H, \ldots, \widetilde{W}_{t_N}$ together forms a $(2N)$-dimensional Gaussian random vector with computable covariance matrix. One can use Cholesky decomposition of the covariance matrix to produce exact samples of $W_{t_1}^1, \ldots, W_{t_N}^1, \widetilde{W}_{t_1}^H, \ldots, \widetilde{W}_{t_N}$, but unlike the first way, we need $2 \times N$-dimensional Gaussian random vectors as input. This method is exact but slow (See [3] and Section 4 in [5] for more details about this scheme). The simulation requires $\mathcal{O}\left(N^3\right)$ flops.

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iii) The hybrid scheme of [7] uses a different approach, which is essentially based on Euler discretization as the first way but crucially improved by moment matching for the singular term in the left point rule. It is also inexact in the sense that samples produced here do not exactly have the distribution of $W^1_{t_1}, \ldots, W^1_{t_N}, \widetilde{W}^H_{t_1}, \ldots, \widetilde{W}_{t_N}$, however they are much more accurate then samples produced from method (i), but much faster than method (ii). As in method (ii), in this case, we need a $2 \times N$-dimensional Gaussian random input vector to produce one sample of $W^1_{t_1}, \ldots, W^1_{t_N}, \widetilde{W}^H_{t_1}, \ldots, \widetilde{W}_{t_N}$.

In this project, we adopt approach (iii) for the simulation of the rBergomi asset price. We utilize the first order variant ($\kappa = 1$) of the hybrid scheme [7], which is based on the approximation

$$\widetilde{W}^H_{\frac{i}{N}} \approx \overline{W}_{\frac{i}{N}} := \sqrt{2H} \left(\int_{\frac{i-1}{N}}^{\frac{i}{N}} \left(\frac{i}{N} - s \right)^{H-\frac{1}{2}} dW^1_u + \sum_{k=2}^{i} \left(\frac{b_k}{N} \right)^{H-\frac{1}{2}} \left(W^1_{\frac{i-(k-1)}{N}} - W^1_{\frac{i-k}{N}} \right) \right),$$

where N is the number of time steps and

$$b_k := \left(\frac{k^{H+\frac{1}{2}} - (k-1)^{H+\frac{1}{2}}}{H + \frac{1}{2}} \right)^{\frac{1}{H-\frac{1}{2}}}.$$

Employing the fast Fourier transform to evaluate the sum in (2.3), which is a discrete convolution, a skeleton $\overline{W}^H_0, \overline{W}^H_1, \ldots, \overline{W}^H_{\frac{[Nt]}{N}}$ can be generated in $\mathcal{O}(N \log N)$ floating point operations.

The variates $\overline{W}^H_0, \overline{W}^H_1, \ldots, \overline{W}^H_{\frac{[Nt]}{N}}$ are generated by sampling $[nt]$ i.i.d draws from a $(\kappa + 1)$-dimensional Gaussian distribution and computing a discrete convolution. We denote these pairs of Gaussian random variables from now on by (W^1, W^2).

3 Details of our approach

We recall that our goal is to compute the expectation in (2.4). In fact, as seen in Section 2.3, we need $2N$ dimensional Gaussian inputs for the used hybrid scheme (N is the number of time steps in the time grid), namely

- $\{W^1_i\}_{i=1}^N$: The N Gaussian random variables that are defined in Section 2.1.

- $\{W^2_j\}_{j=1}^N$: An artificial introduced N Gaussian random variables that are used for left-rule points in the hybrid scheme, as explained in Section 2.3.

Thus, we can rewrite (2.4) as

$$C_{RB}(T, K) = E\left[C_{BS}\left(S_0 = \exp\left(\rho \int_0^T \sqrt{v_t} dW^1_t - \frac{1}{2}\rho^2 \int_0^T v_t dt \right), \ K = K, \ T = 1, \ \sigma^2 = (1 - \rho^2) \int_0^T v_t dt \right) \right],$$

(3.1)

$$\approx \int_{\mathbb{R}^{2N}} C_{BS}\left(G(\mathbf{W}^1, \mathbf{W}^2) \right) \rho_N(\mathbf{W}^1) \rho_N(\mathbf{W}^2) d\mathbf{W}^1 \mathbf{W}^2$$

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where G maps $2N$ independent standard Gaussian random inputs to the parameters fed to Black-Scholes formula as in (3.1), and ρ_N is the multivariate Gaussian density, given by

$$\rho_N(\mathbf{z}) = \frac{1}{(2\pi)^{N/2}} e^{-\frac{1}{2}\mathbf{z}^T\mathbf{z}}.$$

Therefore, the initial integration problem that we are solving lives in $2N$-dimensional space, which becomes very large as the number of time steps N, used in the hybrid scheme, increases.

Our approach of approximating the expectation in (3.1) is based on multi-index stochastic collocation (MISC), proposed in [18]. We describe the MISC solver in our context in Section 3.1. To make an effective use of MISC, we first apply two transformations to overcome the issue of facing a high dimensional integrand due to the discretization scheme used for simulating the rBergomi dynamics. The first transformation consists of applying a hierarchical path generation method, based on Brownian bridge (Bb) construction, with the aim of reducing the effective dimension as described in Section 3.2. The second transformation consists of applying Richardson extrapolation to reduce the bias, resulting in reducing the maximum number of dimensions needed for the integration problem. Details about Richardson extrapolation are provided in Section 3.3.

If we denote by $\mathcal{E}_{\text{tot}}$ the total error of computing the expectation in (2.4) using MISC solver, then we have a natural error decomposition, giving by

$$(3.2) \qquad \mathcal{E} \leq \mathcal{E}_Q(\text{TOL}_{\text{MISC}}, N) + \mathcal{E}_B(N),$$

where $\mathcal{E}_Q$ is the quadrature error, a function of the MISC tolerance: TOL_{MISC} and N: the number of time steps, and $\mathcal{E}_B$ is the bias, function of N.

3.1 The MISC solver

We assume that we want to approximate the expected value $\mathrm{E}[f(Y)]$ of an analytic function $f : \Gamma \to \mathbb{R}$ using a tensorization of quadrature formula over Γ.

To introduce simplified notations, we start with the one-dimensional case. Let us denote by $\beta \geq 1$ an integer referred to as a "stochastic discretization level", and by $m : \mathbb{N} \to \mathbb{N}$ a strictly increasing function with $m(0) = 0$ and $m(1) = 1$, that we call "level-to-nodes function". At level β, we consider a set of $m(\beta)$ distinct quadrature points in $\mathbb{R}$, $\mathcal{H}^{m(\beta)} = \{y_\beta^1, y_\beta^2, \ldots, y_\beta^{m(\beta)}\} \subset \mathbb{R}$, and a set of quadrature weights, $\boldsymbol{\omega}^{m(\beta)} = \{\omega_\beta^1, \omega_\beta^2, \ldots, \omega_\beta^{m(\beta)}\}$. We also let $C^0(\mathbb{R})$ be the set of real-valued continuous functions over $\mathbb{R}$. We then define the quadrature operator as

$$(3.3) \qquad Q(m(\beta)) : C^0(\mathbb{R}) \to \mathbb{R}, \quad Q(m(\beta))[f] = \sum_{j=1}^{m(\beta)} f(y_\beta^j)\omega_\beta^j.$$

In our case, and following (3.1), we have a multi-variate integration problem and we have, given the previous notations, $f := C_{\text{BS}} \circ G$, $\mathbf{Y} = (\mathbf{W}^1, \mathbf{W}^2)$, and $\Gamma = \mathbb{R}^{2N}$. Therefore, we define for any multi-index $\boldsymbol{\beta} \in \mathbb{N}^{2N}$

$$Q^{m(\boldsymbol{\beta})} : \Gamma \to \mathbb{R}, \quad Q^{m(\boldsymbol{\beta})} = \bigotimes_{n=1}^{2N} Q^{m(\beta_n)}$$

where the n-th quadrature operator is understood to act only on the n-th variable of f. Practically, we obtain the value of $Q^{m(\boldsymbol{\beta})}[f]$ by considering the tensor grid $\mathcal{T}^{m(\boldsymbol{\beta})} = \prod_{n=1}^{2N} \mathcal{H}^{m(\beta_n)}$ with cardinality $\#\mathcal{T}^{m(\boldsymbol{\beta})} = \prod_{n=1}^{2N} m(\beta_n)$ and computing

$$Q^{\mathcal{T}^{m(\boldsymbol{\beta})}}[f] = \sum_{j=1}^{\#\mathcal{T}^{m(\boldsymbol{\beta})}} f(\widehat{y}_j)\overline{\omega}_j$$

where $\widehat{y}_j \in \mathcal{T}^{m(\boldsymbol{\beta})}$ and $\overline{\omega}_j$ are products of weights of the univariate quadrature rules.

Remark 3.1. We note that the quadrature points are chosen to optimize the convergence properties of the quadrature error. For instance, in our context, since we are dealing with Gaussian densities, using Gauss-Hermite quadrature points is the appropriate choice.

A direct approximation $\mathrm{E}\left[f[\mathbf{Y}]\right] \approx Q^{m(\boldsymbol{\beta})}[f]$ is not an appropriate option due to the well-known "curse of dimensionality" effect. We use MISC as it was suggested in [18]. MISC is a hierarchical adaptive sparse grids quadrature strategy that uses stochastic discretizations and classic sparsification approach to obtain an effective approximation scheme for $\mathrm{E}\left[f\right]$.

For the sake of concreteness, in our setting from (3.1), we are left with a $2N$-dimensional Gaussian random inputs, which are chosen independently, resulting in $2N$ numerical parameters for MISC, which we use as the basis of the multi-index construction, reflecting the fact that W_i^1 and W_j^2 can vary independently of each other regardless of $i \neq j$ or $i = j$. Let $l \in \{1, \ldots, 2N\}$ and set

(3.4)
$$p_l := \begin{cases} W_l^1, & 1 \leq l \leq N, \\ W_{l-N}^2, & N+1 \leq l \leq 2N. \end{cases}$$

For a multi-index $\ell = (l_i)_{i=1}^{2N} \in \mathbb{N}^{2N}$, we denote by $Q_N^\ell := Q^{N,m(\ell)}(p_\ell)$ the result of a discretized integral (we emphasize that the "integral" here is the time integral, not the expectation integral) using N time steps and with parameters $p_\ell := (p_{l_i})_{i=1}^{2N}$, and with a number of quadrature points $m(l_i)$ in the dimension p_{l_i}. We further define the set of differences ΔQ_ℓ^N as follows: for a single index $1 \leq i \leq 2N$, let

(3.5)
$$\Delta_i Q_N^\ell := \begin{cases} Q_N^\ell - Q_N^{\ell'} \text{ with } \ell' = \ell - e_i, & \text{if } \ell_i > 0 \\ Q_N^\ell, & \text{otherwise} \end{cases}$$

where e_i denotes the ith $2N$-dimensional unit vector. Then, ΔQ_N^ℓ is defined as

(3.6)
$$\Delta Q_N^\ell := \left(\prod_{i=1}^{2N} \Delta_i\right) Q_N^\ell.$$

For instance, when $N = 1$, then

$$\Delta Q_1^\ell = \Delta_2\Delta_1 Q_1^{(l_1,l_2)} = \Delta_2\left(Q_1^{(l_1,l_2)} - Q_1^{(l_1-1,l_2)}\right) = \Delta_2 Q_1^{(l_1,l_2)} - \Delta_2 Q_1^{(l_1-1,l_2)}$$
$$= Q_1^{(l_1,l_2)} - Q_1^{(l_1,l_2-1)} - Q_1^{(l_1-1,l_2)} + Q_1^{(l_1-1,l_2-1)}.$$

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We have the telescoping property

$$(3.7) \qquad Q_N^\infty = \sum_{l_1=0}^\infty \cdots \sum_{l_{2N}=0}^\infty \Delta Q_N^{(l_1,\ldots,l_{2N})} = \sum_{\ell \in \mathbb{N}^{2N}} \Delta Q_N^\ell,$$

where Q_N^∞ is the biased option price computed with N time steps.

As stated before, our goal is to approximate Q_N^∞. The MISC estimator computed using a set of multi-indices $\mathcal{I} \subset \mathbb{N}^{2N}$ is given by

$$Q_N^\mathcal{I} := \sum_{\ell \in \mathcal{I}} \Delta Q_N^\ell.$$

The quadrature error in this case is given by

$$(3.8) \qquad \mathcal{E}_Q(TOL_{\mathrm{MISC}}, N) = \left| Q_N^\infty - Q_N^\mathcal{I} \right| \le \sum_{\ell \in \mathbb{N}^{2N} \setminus \mathcal{I}} \left| \Delta Q_N^\ell \right|.$$

If we denote the computational work at level $\ell = (l_1, \ldots, l_{2N})$ for adding an increment ΔQ_N^ℓ in the telescoping sum by $\mathcal{W}_N^\ell$, then the construction of the optimal $\mathcal{I}$ will be done by profit thresholding, i.e., for a certain threshold value $\overline{T}$, we add a multi-index ℓ to $\mathcal{I}$ provided that

$$\log \left(\frac{|\Delta Q_N^\ell|}{\mathcal{W}_N^\ell} \right) \le \overline{T}.$$

Remark 3.2. The choice of the hierarchy of quadrature points, $m(\ell)$, is flexible in the MISC solver and can be fixed by the user, depending on the convergence properties of the problem at hand. For instance, for the sake of reproducibility, in our numerical experiments we used a linear hierarchy: $m(l) = 4(l-1) + 1$, $1 \le l$, for results of parameter sets 1 and 2 in table 4.1. For the remaining parameter sets in table 4.1, we used a geometric hierarchy: $m(l) = 2^{l-1} + 1$, $1 \le l$.

Remark 3.3. As emphasized in [18], one important requirement to get the optimal performance of MISC solver is to check the convergence of first and mixed differences operators (See [18] for details). We checked this requirement in all our numerical experiments, and for illustration, we show in figures (3.1, 3.2), the convergence of first and second order differences for the case of parameter set 3 in table 4.1.
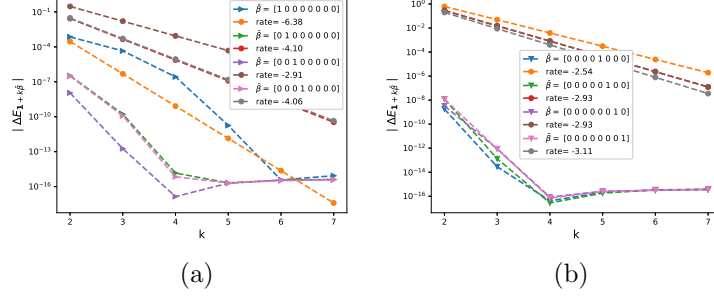
(a)   (b)

Figure 3.1: The rate of convergence of first order differences $|\Delta E_\beta|$ ($\beta = \mathbf{1} + k\overline{\beta}$) with respect to W^1 (a) and with respect to W^2 (b), for parameter set 3 in table 4.1. The number of quadrature points used in the i-th dimension is $N_i = 2^{\beta_i - 1} + 1$.

(a)   (b)

Figure 3.2: The rate of convergence of second order differences $|\Delta E_\beta|$ ($\beta = \mathbf{1} + k\overline{\beta}$) with respect to W^1 (a) and with respect to W^2 (b), for parameter set 3 in table 4.1. The number of quadrature points used in the i-th dimension is $N_i = 2^{\beta_i - 1} + 1$.
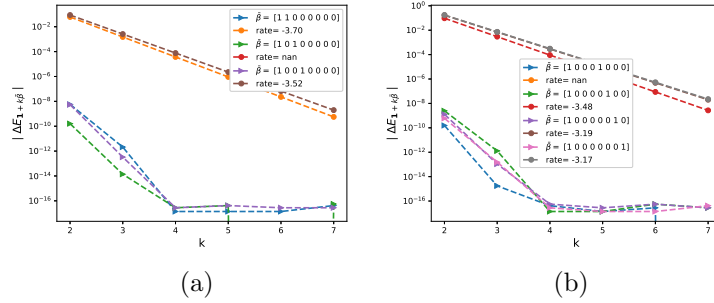
Remark 3.4. In this paper, we limited ourselves to designing a new alternative method based on hierarchical adaptive sparse grids quadrature for computing option prices under the rBergomi model. Giving the significant performance gains of our novel designed algorithm, we expect that designing a method based on QMC can bring similar or more gains as our approach. An investigation of the performance of QMC in a similar context is left for a future work.

3.2   Brownian bridge construction

In the literature of adaptive sparse grids and QMC, several hierarchical path generation methods (PGMs) or transformation methods have been proposed to reduce the effective dimension. Among these transformations, we cite the Brownian bridge (Bb) construction [24, 25, 10], the principal component analysis (PCA) [1] and the linear transformation (LT) [19].

In our context, the Brownian motion can be constructed either sequentially using a standard random walk construction or hierarchically using other hierarchical PGM as listed above. For our purposes, to make an effective use of MISC, which profits from anisotropy, we use the Bb construction since it produces dimensions with different importance for MISC, contrary to random walk procedure for which all the dimension of the stochastic space have equal importance. This

transformation reduces the effective dimension of the problem and as a consequence accelerates the MISC procedure by reducing the computational cost.

Let us denote $\{t_i\}_{i=0}^N$ the grid of time steps, then the Bb construction [17] consists of the following: given a past value B_{t_i} and a future value B_{t_k}, the value B_{t_j} (with $t_i < t_j < t_k$) can be generated according to the formula:

$$(3.9) \qquad B_{t_j} = (1 - \rho)B_{t_i} + \rho B_{t_k} + \sqrt{\rho(1-\rho)(k-i)\Delta t}\, z, \; z \sim \mathcal{N}(0,1),$$

where $\rho = \frac{j-i}{k-i}$. In particular, if N is a power of 2, then given $B_0 = 0$, Bb generates the Brownian motion at times $T, T/2, T/4, 3T/4, \ldots$ according

$$B_T = \sqrt{T} z_1$$

$$B_{T/2} = \frac{1}{2}(B_0 + B_T) + \sqrt{T/4}\, z_2 = \frac{\sqrt{T}}{2} z_1 + \frac{\sqrt{T}}{2} z_2$$

$$B_{T/4} = \frac{1}{2}(B_0 + B_{T/2}) + \sqrt{T/8}\, z_3 = \frac{\sqrt{T}}{4} z_1 + \frac{\sqrt{T}}{4} z_2 + \sqrt{T/8}\, z_3$$

$$\vdots$$

$$(3.10)$$

where $\{z_j\}_{j=1}^N$ are independent standard normal variables. In Bb construction scheme given by (3.10), the most important values that determine the large scale structure of Brownian motion are the first components of $\mathbf{z} = (z_1, \ldots, z_N)$.

3.3 Richardson extrapolation

Another transformation that we coupled with MISC is Richardson extrapolation [27]. In fact, applying level ℓ of Richardson extrapolation reduces dramatically the bias and as a consequence reduces the needed number of time steps N used in the coarsest level to achieve a certain error tolerance. This means basically that Richardson extrapolation reduces directly the total dimension of the integration problem for achieving some error tolerance.

We recall that the Euler scheme has weak order one so that

$$(3.11) \qquad \left| \mathrm{E}\left[f(\widehat{X}_T^h) \right] - \mathrm{E}\left[f(X_T) \right] \right| \leq Ch$$

for some constant C, all sufficiently small h and suitably smooth f. It can be easily shown that (3.11) can be improved to

$$(3.12) \qquad \mathrm{E}\left[f(\widehat{X}_T^h) \right] = \mathrm{E}\left[f(X_T) \right] + ch + \mathcal{O}\left(h^2 \right),$$

where c depends on f.

Applying (3.12) with discretization step $2h$, we obtain

$$(3.13) \qquad \mathrm{E}\left[f(\widehat{X}_T^{2h}) \right] = \mathrm{E}\left[f(X_T) \right] + 2ch + \mathcal{O}\left(h^2 \right),$$

implying

$$(3.14) \qquad 2\mathrm{E}\left[f(\widehat{X}_T^{2h})\right] - \mathrm{E}\left[f(\widehat{X}_T^h)\right] = \mathrm{E}\left[f(X_T)\right] + \mathcal{O}\left(h^2\right),$$

For higher levels extrapolations, we use the following: Let us denote by $h_J = h_0 2^{-J}$ the grid sizes (where h_0 is the coarsest grid size), by K_R the level of the Richardson extrapolation, and by $I(J, K_\mathrm{R})$ the approximation of $\mathrm{E}\left[f^{(X_T)}\right]$ by terms up to level K_R (leading to a weak error of order K_R), then we have the following recursion

$$(3.15) \qquad I(J, K_\mathrm{R}) = \frac{2^{K_\mathrm{R}}\left[I(J, K_\mathrm{R} - 1) - I(J - 1, K_\mathrm{R} - 1)\right]}{2^{K_\mathrm{R}} - 1}, \quad J = 1, 2, \ldots, K_\mathrm{R} = 1, 2, \ldots$$

Remark 3.5. Although we report later in Section 4 some plots of the weak error (Bias) rates for cases with and without Richardson extrapolation, we emphasize that our results are purely experimental, and hence we cannot be sure what will happen in the asymptotic regime. In fact, in this work, we target price estimates with a sufficiently small error tolerance, by using few number of time steps, that is, in the pre-asymptotic regime.

4 Numerical experiments

4.1 Summary of the numerical results

We conduct our experiments for 5 different parameter sets as presented in tables 4.1.

In Section 4.2, we estimate the weak error (Bias) for the different parameter constellations, for two scenarios involving with and without Richardson extrapolation. The conclusions of this section are:

- Without Richardson extrapolation: For all cases except parameter set 4, we get a weak error of order Δt, with different constants. Interestingly, we see that the case of parameter set 4 (see table 4.1) has a weak error rate of order around 0.4 but with a small constant.

- With Richardson extrapolation: For all reported cases (sets $1, 2, 3$ in table 4.1), we get an improvement for weak error in terms of the rate and constants compared to the case without Richardson extrapolation.

Remark 4.1. We emphasize that the reported weak rates correspond to the pre-asymptotic regime that we are interested in. We are not interested on estimating the rates specifically but rather a sufficient precise estimate of the weak error (Bias), $\mathcal{E}_B(N)$, for different time steps N, in order to get the biased MC solution for a given discretization, that we denoted Q_N^∞ in Section 3.1. For a fixed discretization, the corresponding biased solution will be set as a reference solution to the MISC solver in order to estimate the quadrature error $\mathcal{E}_Q(TOL_\mathrm{MISC}, N)$.

In Section 4.3, we show tables and plots reporting the different errors involved in MC method (bias and statistical error), related to the plots in Section 4.2, and in MISC (Quadrature error). We do this for each parameter set. The quadrature error (see (3.8)) is computed by subtracting the MISC solution from the biased solution, computed with sufficiently large number of samples.

Given that both methods, MC and MISC, have the same bias, the computational time of MC and MISC is compared such that the statistical error is almost equal to the stable quadrature error produced by MISC.

The conclusions of this section are:

i) For the case of set 1 (See Section 4.3.1), MISC coupled with Richardson extrapolation is 3 times faster than MC coupled with Richardson extrapolation, to achieve a total relative error around 8% (see tables(4.8, 4.9)). This gain is improved when applying level 2 Richardson extrapolation. In fact, MISC coupled with Richardson extrapolation (level 2) is 4 times faster than MC coupled with Richardson extrapolation (level 2), to achieve a total relative error around 0.5% (see tables (4.12, 4.13)). Applying Richardson extrapolation brought a significant improvement for MISC (compare tables (4.4,4.5) (no Richardson), tables (4.8, 4.9) (Richardson (level 1)) and tables ((4.12, 4.13)) (Richardson (level 2)), also see figure 4.9).

ii) For the case of set 2 (See Section 4.3.2), MISC coupled with Richardson extrapolation is 19 times faster than MC coupled with Richardson extrapolation, to achieve a total relative error around 8% and 4 times faster than MC, to achieve total relative error around 2% (see tables (4.20,4.21)). This gain is improved when applying level 2 Richardson extrapolation. In fact, MISC coupled with Richardson extrapolation (level 2) is 17 times faster than MC coupled with Richardson extrapolation (level 2), to achieve a total relative error below 1% (see tables (4.24, 4.25)). Applying Richardson extrapolation brought a significant improvement for MISC (compare tables (4.16, 4.17) (no Richardson), tables (4.20,4.21) (Richardson (level 1)) and tables (4.24, 4.25) (Richardson (level 2)), also see figure 4.13).

iii) For the case of set 3 (See Section 4.3.3), MISC is 35 times faster than MC, to achieve a total relative error below 1% and 8 times faster than MC, to achieve a total relative error around 0.1% (see figure 4.15 and tables (4.29, 4.28)). Using Richardson extrapolation brought an improvement in terms of the complexity constant compared to using simple MISC (See figure 4.17 and tables (4.32, 4.33)).

iv) For the case of set 4 (See Section 4.3.4), MISC is 141 times faster than MC, to achieve a total relative error below 1% (see figure 4.19 and tables (4.37, 4.36)).

v) For the case of set 5 (See Section 4.3.5), MISC is 220 times faster than MC, to achieve a total relative error below 6% and 16 times faster than MC, to achieve total relative error around 2% (see figure 4.21 and tables (4.41, 4.40)).

4.2   Weak error plots

In this section, we include the results of weak errors for the different parameter sets as in table 4.1, with and without Richardson extrapolation. We considered a number of time steps $N \in \{2, 4, 8, 16\}$. The options are priced in terms of the moneyness K, where K is the strike price. The reference solution was computed with $N = 500$ time steps (reported in table 4.1). We note that the weak errors plotted here correspond to relative errors. We show in table 4.1 the different parameter constellations that we consider to report our results for MC and MISC.

Parameters	Reference solution
Set 1: $H = 0.43, K = 1, S_0 = 1, \rho = -0.9, \eta = 1.9, \xi = 0.235^2$	0.0712 $(5.7e-05)$
Set 2: $H = 0.07, K = 1, S_0 = 1, \rho = -0.9, \eta = 1.9, \xi = 0.235^2$	0.0791 $(7.9e-05)$
Set 3: $H = 0.02, K = 1, , S_0 = 1, \rho = -0.7, \eta = 0.4, \xi = 0.1$	0.1248 $(1.3e-04)$
Set 4: $H = 0.02, K = 0.8, S_0 = 1, \rho = -0.7, \eta = 0.4, \xi = 0.1$	0.2407 $(5.6e-04)$
Set 5: $H = 0.02, K = 1.2, S_0 = 1, \rho = -0.7, \eta = 0.4, \xi = 0.1$	0.0568 $(2.5e-04)$

Table 4.1: Reference solution, using MC with 500 time steps and $M = 10^6$, of call option price under the rBergomi model, for different parameter constellation. The numbers between parentheses correspond to the statistical errors.

4.2.1 Without Richardson extrapolation

From figures (4.1,4.2,4.3), we see that for all cases except parameter set 4, we get a weak error of order Δt, with different constants. Interestingly, we see that the case of parameter set 4 (see table 4.1) has a weak error rate of order around 0.4 but with a small constant. The upper and lower bounds are 95% confidence intervals.

(a)   (b)

Figure 4.1: The rate of convergence of the weak error $|\mathrm{E}\left[g(X_{\Delta t})\right] - g(X)|$ without Richardson extraploation, using MC with $M = 10^6$: a) Set 1 parameter in table 4.1. b) Set 2 parameter in table 4.1.

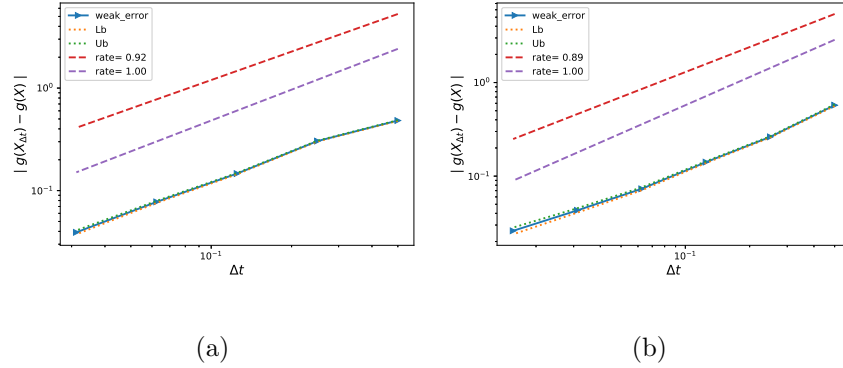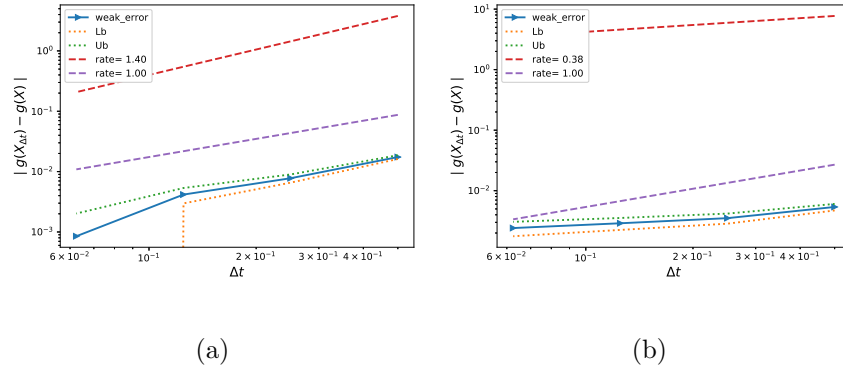(a)   (b)

Figure 4.2: The rate of convergence of the weak error $|\mathrm{E}\left[g(X_{\Delta t})\right] - g(X)|$ without Richardson extraploation, using MC with $M = 5.10^6$: a) Set 3 parameters in table 4.1, b) Set 4 parameters in table 4.1.

Figure 4.3: The rate of convergence of the weak error $|\mathrm{E}\left[g(X_{\Delta t})\right] - g(X)|$, for set 5 parameters in table 4.1, without Richardson extraploation, using MC with $M = 5.10^6$.

4.2.2   With Richardson extrapolation (level 1)

Figures (4.4, 4.5) illustrate the weak errors estimates for parameter sets $1, 2, 3$ in table 4.1. The upper and lower bounds are 95% confidence intervals.

(a)	(b)

Figure 4.4: The rate of convergence of the weak error $\left| \mathrm{E}\left[2g(X_{\Delta t/2}) - g(X_{\Delta t}) \right] - g(X) \right|$ with Richardson extraploation, using MC with $M = 10^6$: a) Set 1 parameters in table 4.1. b) Set 2 parameters in table 4.1,
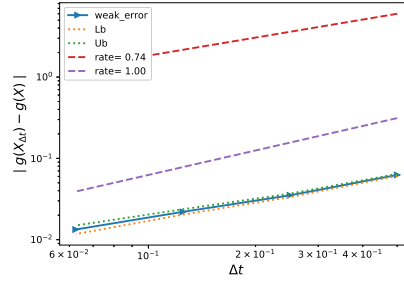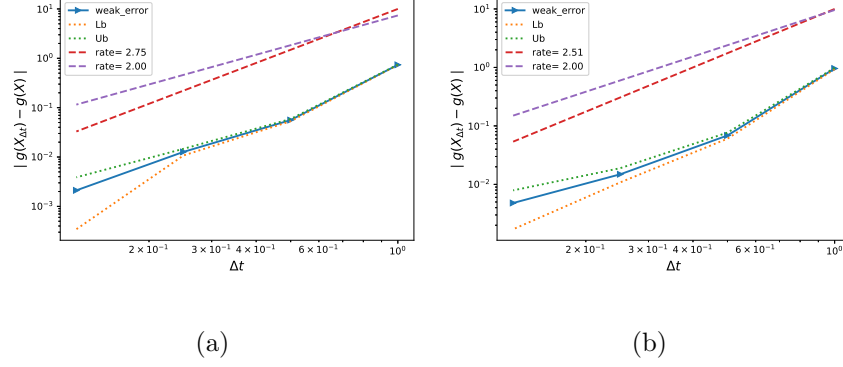

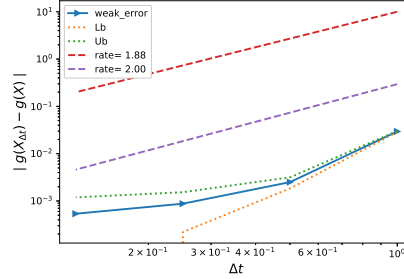Figure 4.5: The rate of convergence of the weak error $\left| \mathrm{E}\left[2g(X_{\Delta t/2}) - g(X_{\Delta t}) \right] - g(X) \right|$, for set 3 parameters in table 4.1, with Richardson extraploation, using MC with $M = 10^7$.

4.3 Comparing different errors and complexity for MC and MISC

The results were reported for the different sets of parameters defined in table 4.1. We considered a number of time steps $N \in \{2, 4, 8, 16\}$. The options are priced in terms of the moneyness K, where K is the strike price.

For each set, we report the results for 2 scenarios: i) Without using Richardson extrapolation and ii) Using level 1 Richardson extrapoaltion.

In each case, we show a table of computed values using MISC with different tolerances as well the biased MC value, using large number of samples to kill the statistical error. Then, in a second table, we show the bias as well the statistical error for MC method, related to Section 4.2. After that, a plot shows the behavior of the relative quadrature error which is computed as the normalized difference between the biased MC solution and MISC solution. Then, we provide the total relative error which is the sum of the bias and statistical error for MC, and the quadrature error for MISC. We note that we used a number of samples for MC such that the statistical error is almost equal to the stable quadrature error, in order to have a fair complexity comparison between the two methods. Finally, we show the CPU time needed for each solver. We note that in all cases the

16

actual work (runtime) was obtained using a 40-core Intel(R) Xeon(R) CPU E5-268 architecture.

4.3.1 Case of set 1 parameters in table 4.1

Without Richardson extrapolation

Method \Steps	2	4	8	16
MISC ($TOL_{\mathrm{MISC}} = 10^{-1}$)	0.1140	0.0961	0.0871	0.0802
MISC ($TOL_{\mathrm{MISC}} = 10^{-2}$)	0.1077	0.0944	0.0838	0.0772
MISC ($TOL_{\mathrm{MISC}} = 10^{-3}$)	0.1077	0.0921	0.0819	0.0762
MISC ($TOL_{\mathrm{MISC}} = 10^{-4}$)	0.1079	0.0921	0.0822	–
MC method ($M = 8.10^6$)	0.1078	0.0921	0.0822	0.0767

Table 4.2: Call option price of the different methods for different number of time steps. Case of set 1 parameters in table 4.1, without Richardson extrapolation.

Method \Steps	2	4	8	16
MC Bias ($M = 8.10^6$)	**0.5142** (0.0366)	**0.2933** (0.0209)	**0.1551** (0.0110)	**0.0777** (0.0055)
MC Statistical error ($M = 8.10^6$)	**8.4e − 04** (6.0e−05)	**4.8e − 04** (3.4e−05)	**3.8e − 04** (2.7e−05)	**3.3e − 04** (2.35e−05)

Table 4.3: Bias and statistical errors of MC for computing call option price for different number of time steps. Case set 1, without Richardson extrapolation. The numbers between parentheses are the corresponding absolute errors.
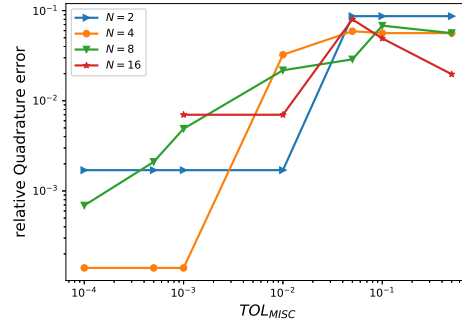

Figure 4.6: Quadrature error of MISC, with different tolerances, to compute call option price for different number of time steps. Case set 1 parameters, without Richardson extrapolation. See detailed values in table A.1

Method \Steps	2	4	8	16
MISC ($TOL_{\text{MISC}} = 10^{-1}$)	**0.6010**	**0.3496**	**0.2232**	**0.1269**
MISC ($TOL_{\text{MISC}} = 10^{-2}$)	0.5159	0.3257	0.1769	0.0847
MISC ($TOL_{\text{MISC}} = 10^{-3}$)	0.5159	0.2934	0.1600	0.0847
MISC ($TOL_{\text{MISC}} = 10^{-4}$)	0.5159	0.2934	0.1558	–
MC	0.5159	0.2938	0.1555	0.0817

Table 4.4: Total relative error of MISC, with different tolerances, and MC to compute call option price for different number of time steps. Case set 1 parametrs of table 4.1, without Richardson extrapolation. The values marked in red, for MISC method, correspond to the total relative errors associated with stable quadrature errors for MISC, and will be used for complexity comparison against MC.

Method \Steps	2	4	8	16
MISC ($TOL_{\text{MISC}} = 10^{-1}$)	0.1	0.1	0.6	6
MISC ($TOL_{\text{MISC}} = 10^{-2}$)	0.2	1	9	215
MISC ($TOL_{\text{MISC}} = 10^{-3}$)	2	11	243	4650
MISC ($TOL_{\text{MISC}} = 10^{-4}$)	6	96	5760	–
MC	50	344	637	8
Ratio of (MC/MISC)	250	31	0.1	0.04

Table 4.5: Comparison of the computational time (in Seconds) of MC and MISC, used to compute call option price of the rBergomi model for different number of time steps. Case set 1 parametrs of table 4.1. The average MC CPU time is computed over 10 runs.

With Richardson extrapolation (level 1)

Method \Steps	$1-2$	$2-4$	$4-8$
MISC ($TOL_{\text{MISC}} = 10^{-1}$)	0.1357	0.0783	0.0785
MISC ($TOL_{\text{MISC}} = 10^{-2}$)	0.1237	0.0781	0.0745
MISC ($TOL_{\text{MISC}} = 10^{-3}$)	0.1224	0.0766	0.0720
MC method ($M = 10^6$)	0.1237	0.0752	0.0721

Table 4.6: Call option price of the different methods for different number of time steps. Case set 1 parameters of table 4.1, using Richardson extrapolation (level 1).

Method \Steps	$1-2$	$2-4$	$4-8$	$8-16$
MC Bias ($M = 10^6$)	**0.7378** (0.0525)	**0.0561** (0.0040)	**0.0127** (0.0009)	**0.0021** (0.0001)
MC Statistical error ($M = 10^6$)	**3.3e − 03** (2.3e−04)	**1.6e − 03** (1.1e−04)	**1.0e − 03** (7.1e−05)	**9.0e − 04** (6.4e−05)

Table 4.7: Bias and statistical errors of MC for computing call option price for different number of time steps. Case set 1 parameters in tabel 4.1, with Richardson extrapolation (level 1). The numbers between parentheses are the corresponding absolute errors.
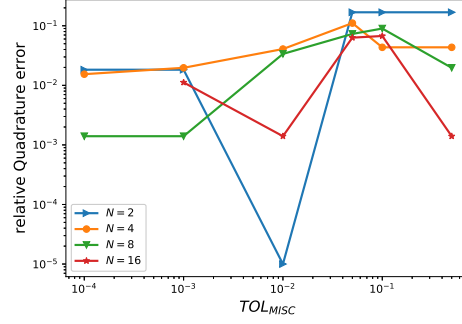
Figure 4.7: Quadrature error of MISC, with different tolerances, to compute call option price for different number of time steps. Case set 1 parameters, with Richardson extrapolation. See detailed values in table A.2.

Method \Steps	$1-2$	$2-4$	$4-8$
MISC $(TOL_{\mathrm{MISC}} = 10^{-1})$	**0.9063**	**0.0996**	**0.1026**
MISC $(TOL_{\mathrm{MISC}} = 10^{-2})$	**0.7378**	**0.0968**	**0.0464**
MISC $(TOL_{\mathrm{MISC}} = 10^{-3})$	0.7561	0.0758	0.0141
MC	0.7561	0.0758	0.0141

Table 4.8: Total relative error of MISC and MC, with different tolerances, to compute call option price for different number of time steps. Case set 1 parameters in table 4.1, with Richardson extrapolation(level 1). The values marked in red, for MISC method, correspond to the total relative errors associated with stable quadrature errors for MISC, and will be used for complexity comparison against MC.

Method \Steps	$1-2$	$2-4$	$4-8$
MISC $(TOL_{\mathrm{MISC}} = 10^{-1})$	0.1	0.1	0.6
MISC $(TOL_{\mathrm{MISC}} = 10^{-2})$	1	2	18
MISC $(TOL_{\mathrm{MISC}} = 10^{-3})$	4	12	520
MC	34.7	37	532
Ratio of (MC/MISC)	8.7	3.1	1

Table 4.9: Comparison of the computational time (in Seconds) of MC and MISC, using Richardson extrapolation (level 1), used to compute call option price of the rBergomi model for different number of time steps. Case set 1 parameters in table 4.1. The average MC CPU time is computed over 10 runs.
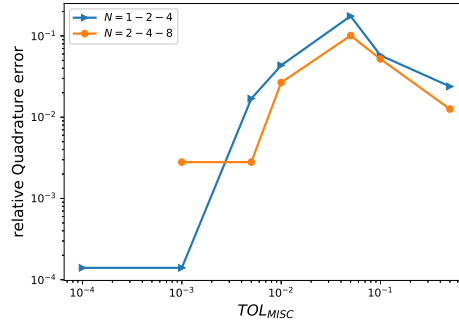
With Richardson extrapolation (level 2)

Method \Steps	$1-2-4$	$2-4-8$
MISC ($TOL_{\text{MISC}} = 10^{-1}$)	0.0567	0.0747
MISC ($TOL_{\text{MISC}} = 10^{-2}$)	0.0639	0.0729
MISC ($TOL_{\text{MISC}} = 5.10^{-3}$)	0.0620	0.0708
MC ($M = 3.10^{6}$)	0.0608	0.0710

Table 4.10: Call option price of the different methods for different number of time steps. Case set 1 parameters in table 4.1, using Richardson extrapolation (level 2).

Method \Steps	$1-2-4$	$2-4-8$
MC Bias ($M = 3.10^{6}$)	**0.1459** (0.0104)	**0.0024** (1.7e−04)
MC Statistical error ($M = 3.10^{6}$)	**1.5e − 03** (1.0e−04)	**6.8e − 04** (4.8e−05)

Table 4.11: Bias and statistical errors of MC for computing call option price for different number of time steps. Case set 1 parameters in tabel 4.1, with Richardson extrapolation (level 2). The numbers between parentheses are the corresponding absolute errors.

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Figure 4.8: Quadrature error of MISC, with different tolerances, to compute call option price of the different tolerances for different number of time steps. Case set 1 parameters, with Richardson extrapolation (level 2). See detailed values in table A.3.

Method \Steps	$1-2-4$	$2-4-8$
MISC ($TOL_{\mathrm{MISC}} = 10^{-1}$)	**0.2035**	**0.0544**
MISC ($TOL_{\mathrm{MISC}} = 10^{-2}$)	**0.1894**	**0.0291**
MISC ($TOL_{\mathrm{MISC}} = 5.10^{-3}$)	**0.1628**	**0.0052**
MC	**0.1628**	**0.0052**

Table 4.12: Total relative error of MISC, with different tolerances, and MC to compute call option price for different number of time steps. Case set 1 parameters in table 4.1, with Richardson extrapolation(level 2). The values marked in red, for MISC method, correspond to the total relative errors associated with stable quadrature errors for MISC, and will be used for complexity comparison against MC.

Method \Steps	$1-2-4$	$2-4-8$
MISC ($TOL_{\mathrm{MISC}} = 10^{-1}$)	0.25	2
MISC ($TOL_{\mathrm{MISC}} = 10^{-2}$)	2	24
MISC ($TOL_{\mathrm{MISC}} = 5.10^{-3}$)	5	64
MC	20	231
Ratio of (MC/MISC)	4	3.6

Table 4.13: Comparison of the computational time (in Seconds) of MC and MISC, using Richardson extrapolation (level 2), used to compute call option price of the rBergomi model for different number of time steps. Case set 1 parameters in table 4.1. The average MC CPU time is computed over 10 runs.
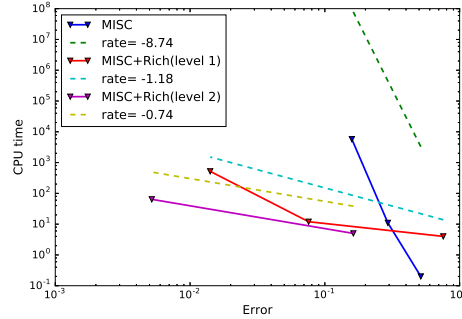

Figure 4.9: Complexity plot for MISC (with and without) Richardson extrapolation for case set 1 parameters of table 4.1.

4.3.2   Case of set 2 parameters in table 4.1

Without Richardson extrapolation

Method \Steps	2	4	8
MISC ($TOL_{\mathrm{MISC}} = 10^{-1}$)	0.1097	0.0926	0.0791
MISC ($TOL_{\mathrm{MISC}} = 10^{-2}$)	0.1119	0.1023	0.0910
MISC ($TOL_{\mathrm{MISC}} = 10^{-3}$)	0.1195	0.1023	0.0910
MISC ($TOL_{\mathrm{MISC}} = 10^{-4}$)	0.1218	0.1023	–
MC method ($M = 8.10^6$)	0.1218	0.1024	0.0914

Table 4.14: Call option price of the different methods for different number of time steps. Case of set 2 parameters in table 4.1, without Richardson extrapolation.

Method \Steps	2	4	8	16
MC Bias ($M = 8.10^6$)	**0.5375** (0.0426)	**0.2922** (0.0208)	**0.1542** (0.0122)	**0.0731** (0.0058)
MC Statistical error ($M = 8.10^6$)	**2.5e − 03** (2.0e−04)	**1.3e − 03** (1.0e−04)	**6.3e − 04** (5.0e−05)	**1.33e − 03** (1.1e−04)

Table 4.15: Bias and statistical errors of MC for computing call option price for different number of time steps. Case set 2 parameters in table 4.1, without Richardson extrapolation. The numbers between parentheses are the corresponding absolute errors.
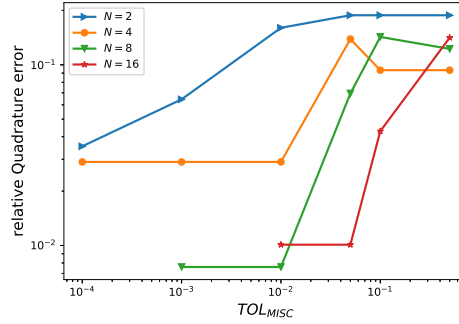

Figure 4.10: Quadrature error of MISC, with different tolerances, to compute call option price of time steps. Case set 2 parameters, without Richardson extrapolation. See detailed values in table A.4.

Method \Steps	2	4	8
MISC ($TOL_{\mathrm{MISC}} = 10^{-1}$)	**0.6900**	**0.4153**	**0.3097**
MISC ($TOL_{\mathrm{MISC}} = 10^{-2}$)	**0.6622**	**0.2928**	**0.1595**
MISC ($TOL_{\mathrm{MISC}} = 10^{-3}$)	**0.6371**	**0.2928**	**0.1595**
MISC ($TOL_{\mathrm{MISC}} = 10^{-4}$)	**0.5378**	**0.2928**	–
MC	**0.5400**	**0.2935**	**0.1593**

Table 4.16: Total relative error of MISC, with different tolerances, and MC to compute call option price for different number of time steps. Case set 2 parameters in table 4.1, without Richardson extrapolation. The numbers between parentheses are the corresponding absolute errors. The values marked in red, for MISC method, correspond to the total relative errors associated with stable quadrature errors for MISC, and will be used for complexity comparison against MC.

Method \Steps	2	4	8
MISC ($TOL_{\mathrm{MISC}} = 10^{-1}$)	0.08	0.13	0.7
MISC ($TOL_{\mathrm{MISC}} = 10^{-2}$)	0.2	5	333
MISC ($TOL_{\mathrm{MISC}} = 10^{-3}$)	2	73	3650
MISC ($TOL_{\mathrm{MISC}} = 10^{-4}$)	43	1240	–
MC method	220	358	9
Ratio of (MC/MISC)	5	72	0.03

Table 4.17: Comparison of the computational time (in Seconds) of MC and MISC, used to compute call option price of the rBergomi model for different number of time steps. Case set 2 parameters in table 4.1. The average MC CPU time is computed over 10 runs.

With Richardson extrapolation (level 1)

Method \Steps	$1-2$	$2-4$	$4-8$
MISC ($TOL_{\mathrm{MISC}} = 10^{-1}$)	0.1260	0.0756	0.0702
MISC ($TOL_{\mathrm{MISC}} = 5.10^{-2}$)	0.1260	0.0716	0.0796
MISC ($TOL_{\mathrm{MISC}} = 10^{-2}$)	0.1456	0.0838	0.0796
MISC ($TOL_{\mathrm{MISC}} = 10^{-3}$)	0.1497	0.0838	–
MC method ($M = 10^6$)	0.1552	0.0846	0.0804

Table 4.18: Call option price of the different methods for different number of time steps. Case set 2 parameters of table 4.1, using Richardson extrapolation (level 1)

Method \Steps	$1-2$	$2-4$	$4-8$	$8-16$
MC Bias ($M = 10^6$)	**0.9594** (0.0760)	**0.0686** (0.0054)	**0.0149** (0.0012)	**0.0048** (0.0004)
MC Statistical error ($M = 10^6$)	**1.3e − 02** (1e−03)	**4.1e − 03** (3.2e−04)	**2.1e − 03** (1.7e−04)	**1.6e − 03** (1.3e−04)

Table 4.19: Bias and statistical errors of MC for computing call option price for different number of time steps. Case set 2 parameters in table 4.1, with Richardson extrapolation (level 1). The numbers between parentheses are the corresponding absolute errors.
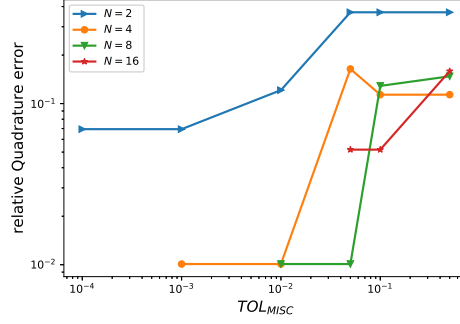
Figure 4.11: Quadrature error of MISC, with different tolerances, to compute call option price for different number of time steps. Case set 2 parameters, with Richardson extrapolation. See detailed values in table A.5.

Method \Steps	$1 - 2$	$2 - 4$	$4 - 8$
MISC $(TOL_{\text{MISC}} = 10^{-1})$	**1.3281**	**0.1822**	**0.1437**
MISC $(TOL_{\text{MISC}} = 5.10^{-2})$	**1.3281**	**0.2327**	**0.0250**
MISC $(TOL_{\text{MISC}} = 10^{-2})$	**1.0806**	**0.0787**	**0.0250**
MISC $(TOL_{\text{MISC}} = 10^{-3})$	**1.0288**	**0.0787**	–
MC	**1.0288**	**0.0787**	**0.0250**

Table 4.20: Total relative error of MISC, with different tolerances, and MC to compute call option price for different number of time steps. Case set 2 parameters in table 4.1, with Richardson extrapolation(level 1). The numbers between parentheses are the corresponding absolute errors. The values marked in red, for MISC method, correspond to the total relative errors associated with stable quadrature errors for MISC, and will be used for complexity comparison against MC.

Method \Steps	$1 - 2$	$2 - 4$	$4 - 8$
MISC $(TOL_{\text{MISC}} = 10^{-1})$	0.1	0.18	1.6
MISC $(TOL_{\text{MISC}} = 5.10^{-2})$	0.1	0.6	37
MISC $(TOL_{\text{MISC}} = 10^{-2})$	1.3	6	2382
MISC $(TOL_{\text{MISC}} = 10^{-3})$	3.5	244	–
MC	12	113	130
Ratio of (MC/MISC)	3.4	18.8	3.5

Table 4.21: Comparison of the computational time (in Seconds) of MC and MISC, using Richardson extrapolation (level 1), used to compute call option price of the rBergomi model for different number of time steps. Case set 2 parameters in table 4.1. The average MC CPU time is computed over 10 runs.

With Richardson extrapolation (level 2)

Method \Steps	$1 - 2 - 4$	$2 - 4 - 8$
MISC ($TOL_{\mathrm{MISC}} = 10^{-1}$)	0.0587	0.0702
MISC ($TOL_{\mathrm{MISC}} = 5.10^{-2}$)	0.0401	0.0790
MISC ($TOL_{\mathrm{MISC}} = 10^{-2}$)	0.0623	0.0784
MC ($M = 3.10^6$)	0.0601	0.0787

Table 4.22: Call option price of the different methods for different number of time steps. Case set 2 parameters in tabel 4.1, using Richardson extrapolation (level 2).

Method \Steps	$1 - 2 - 4$	$2 - 4 - 8$
MC Bias ($M = 3.10^6$)	**0.2411** (0.0191)	**0.0058** (4.6e−04)
MC Statistical error ($M = 3.10^6$)	**3.5e − 03** (2.8e−04)	**1.8e − 03** (1.4e−04)

Table 4.23: Bias and statistical errors of MC for computing call option price for different number of time steps. Case set 2 parameters in tabel 4.1, with Richardson extrapolation (level 2). The numbers between parentheses are the corresponding absolute errors.
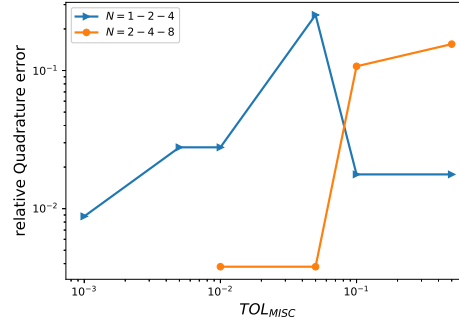

Figure 4.12: Quadrature error of MISC to compute call option price of the different tolerances for different number of time steps. Case set 2 parameters, with Richardson extrapolation (level 2). See detailed values in table A.6.

Method \Steps	$1 - 2 - 4$	$2 - 4 - 8$
MISC ($TOL_{\mathrm{MISC}} = 10^{-1}$)	**0.2588**	**0.1131**
MISC ($TOL_{\mathrm{MISC}} = 5.10^{-2}$)	**0.4936**	**0.0096**
MISC ($TOL_{\mathrm{MISC}} = 10^{-2}$)	**0.2689**	**0.0096**
MC	**0.2689**	**0.0096**

Table 4.24: Total relative error of MISC, with different tolerances, and MC to compute call option price for different number of time steps. Case set 2 parameters in table 4.1, with Richardson extrapolation(level 2). The values marked in red, for MISC method, correspond to the total relative errors associated with stable quadrature errors for MISC, and will be used for complexity comparison against MC.

Method \Steps	$1 - 2 - 4$	$2 - 4 - 8$
MISC ($TOL_{\mathrm{MISC}} = 10^{-1}$)	0.2	2
MISC ($TOL_{\mathrm{MISC}} = 5.10^{-2}$)	0.5	74
MISC ($TOL_{\mathrm{MISC}} = 10^{-2}$)	9	3455
MC	118	1274
Ratio of (MC/MISC)	13	17

Table 4.25: Comparison of the computational time (in Seconds) of MC and MISC, using Richardson extrapolation (level 2), used to compute call option price of the rBergomi model for different number of time steps. Case set 2 parameters in table 4.1. The average MC CPU time is computed over 10 runs.
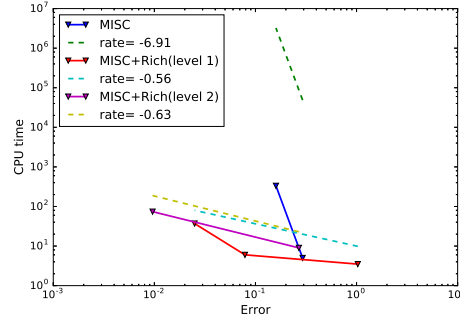

Figure 4.13: Complexity plot for MISC (with and without) Richardson extrapolation for case set 2 parameters of table 4.1.

4.3.3 Case of set 3 parameters in table 4.1

Without Richardson extrapolation

Method \Steps	2	4	8	16
MISC ($TOL_{\mathrm{MISC}} = 10^{-1}$)	0.1258	0.1239	0.1231	0.1229
MISC ($TOL_{\mathrm{MISC}} = 10^{-2}$)	0.1258	0.1246	0.1248	0.1250
MISC ($TOL_{\mathrm{MISC}} = 10^{-3}$)	0.1271	0.1259	0.1252	0.1249
MISC ($TOL_{\mathrm{MISC}} = 10^{-4}$)	0.1270	0.1258	0.1252	–
MC method ($M = 3.10^{6}$)	0.1269	0.1257	0.1253	0.1249

Table 4.26: Call option price of the different methods for different number of time steps. Case of set 3 parameters in table 4.1, without Richardson extrapolation.

Method \Steps	2	4	8	16
MC Bias ($M = 3.10^6$)	**0.0174** (0.0022)	**0.0078** (0.001)	**0.0042** (0.0005)	**0.0008** (0.0001)
MC Statistical error ($M = 3.10^6$)	**5.0e − 04** (6.2e−05)	**4.7e − 04** (5.9e−05)	**4.6e − 04** (5.8e−05)	**4.6e − 04** (5.8e−05)

Table 4.27: Bias and statistical errors of MC for computing call option price for different number of time steps. Case set 3, without Richardson extrapolation. The numbers between parentheses are the corresponding absolute errors.
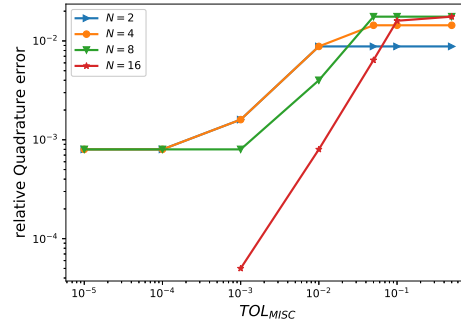

Figure 4.14: Quadrature error of MISC, with different tolerances, to compute call option price for different number of time steps. Case set 3 parameters, without Richardson extrapolation. See detailed values in table A.7.

Method \Steps	2	4	8	16
MISC ($TOL_{\mathrm{MISC}} = 10^{-1}$)	**0.0262**	**0.0222**	**0.0218**	**0.0168**
MISC ($TOL_{\mathrm{MISC}} = 10^{-2}$)	**0.0262**	**0.0166**	**0.0082**	**0.0016**
MISC ($TOL_{\mathrm{MISC}} = 10^{-3}$)	**0.0190**	**0.0094**	**0.0050**	**0.0008**
MISC ($TOL_{\mathrm{MISC}} = 10^{-4}$)	**0.0182**	**0.0086**	**0.0050**	–
MC	**0.0179**	**0.0083**	**0.0047**	**0.0013**

Table 4.28: Total relative error of MISC, with different tolerances, and MC to compute call option price for different number of time steps. Case set 3 parametrs of table 4.1, without Richardson extrapolation. The numbers between parentheses are the corresponding absolute errors. The values marked in red, for MISC method, correspond to the total relative errors associated with stable quadrature errors for MISC, and will be used for complexity comparison against MC.

Method \Steps	2	4	8	16
MISC ($TOL_{\mathrm{MISC}} = 10^{-1}$)	0.1	0.1	0.2	0.8
MISC ($TOL_{\mathrm{MISC}} = 10^{-2}$)	0.1	0.5	8	92
MISC ($TOL_{\mathrm{MISC}} = 10^{-3}$)	0.5	3	24	2226
MISC ($TOL_{\mathrm{MISC}} = 10^{-4}$)	1	6	80	–
MC method	122	211	427	766
Ratio of ($MC/MISC$)	122	35	18	8

Table 4.29: Comparison of the computational time (in Seconds) of MC and MISC, used to compute call option price of the rBergomi model for different number of time steps. Case set 3 parametrs of table 4.1. The average MC CPU time is computed over 10 runs.
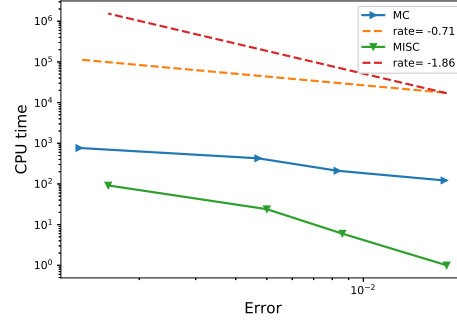

Figure 4.15: Complexity plot for MC and MISC for case set 3 parameters of table 4.1.

With Richardson extrapolation (level 1)

Method \Steps	$1 - 2$	$2 - 4$	$4 - 8$	$8 - 16$
MISC ($TOL_{\mathrm{MISC}} = 10^{-1}$)	0.1261	0.1220	0.1222	0.1226
MISC ($TOL_{\mathrm{MISC}} = 10^{-2}$)	0.1267	0.1230	0.1245	0.1247
MISC ($TOL_{\mathrm{MISC}} = 10^{-3}$)	0.1285	0.1247	0.1247	0.1247
MC method ($M = 10^7$)	0.1284	0.1251	0.1249	0.1248

Table 4.30: Call option price of the different methods for different number of time steps. Case set 3 parameters of table 4.1, using Richardson extrapolation (level 1).

Method \Steps	$1-2$	$2-4$	$4-8$	$8-16$
MC Bias ($M = 10^7$)	**0.0295**	**0.0025**	**0.0009**	**0.0005**
	(0.0037)	(0.0003)	(0.0001)	(6.2e−05)
MC Statistical error ($M = 10^7$)	**3.5e − 04**	**3.4e − 04**	**3.3e − 04**	**3.3e − 04**
	(4.4e−05)	(4.2e−05)	(4.1e−05)	(4.1e−05)

Table 4.31: Bias and statistical errors of MC for computing call option price for different number of time steps. Case set 3 parameters in table 4.1, with Richardson extrapolation (level 1). The numbers between parentheses are the corresponding absolute errors.
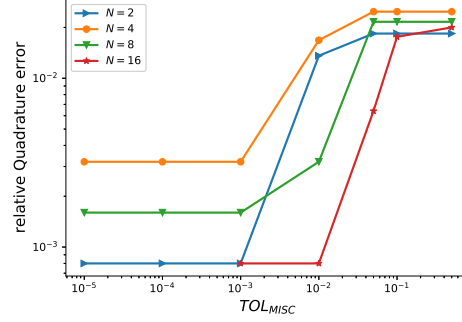

Figure 4.16: Quadrature error of MISC, with different tolerances, to compute call option price, for different number of time steps. Case set 3 parameters, with Richardson extrapolation. See detailed values in table A.8.

Method \Steps	$2-4$	$4-8$	$8-16$
MISC ($TOL_{\mathrm{MISC}} = 10^{-1}$)	0.0273	0.0225	0.0181
MISC ($TOL_{\mathrm{MISC}} = 10^{-2}$)	0.0193	0.0041	0.0013
MISC ($TOL_{\mathrm{MISC}} = 10^{-3}$)	0.0057	0.0025	0.0013
MC	**0.0073**	**0.0025**	**0.0013**

Table 4.32: Total relative error of MISC, with different tolerances, and MC to compute call option price for different number of time steps. Case set 3 parameters in table 4.1, with Richardson extrapolation(level 1). The numbers between parentheses are the corresponding absolute errors. The values marked in red, for MISC method, correspond to the total relative errors associated with stable quadrature errors for MISC, and will be used for complexity comparison against MC.

Method \Steps	$2-4$	$4-8$	$8-16$
MISC ($TOL_{\mathrm{MISC}} = 10^{-1}$)	0.15	0.25	1
MISC ($TOL_{\mathrm{MISC}} = 10^{-2}$)	0.6	10	112
MISC ($TOL_{\mathrm{MISC}} = 10^{-3}$)	3.5	34	3150
MC		45	438	2240
Ratio of (MC/MISC)		13	13	20

Table 4.33: Comparison of the computational time (in Seconds) of MC and MISC, using Richardson extrapolation (level 1), used to compute call option price of the rBergomi model for different number of time steps. Case set 3 parameters in table 4.1. The average MC CPU time is computed over 10 runs.
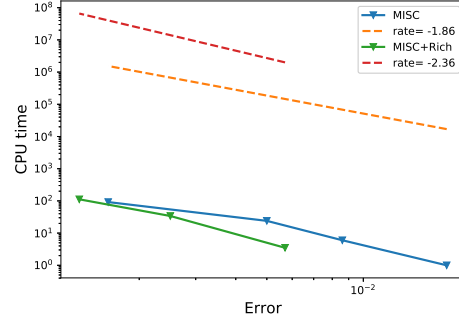

Figure 4.17: Complexity plot for MISC (with and without) Richardson extrapolation for case set 3 parameters of table 4.1.

4.3.4 Case of set 4 parameters in table 4.1

Method \Steps	2	4	8	16
MISC ($TOL_{\mathrm{MISC}} = 10^{-1}$)	0.2413	0.2403	0.2403	0.2397
MISC ($TOL_{\mathrm{MISC}} = 10^{-2}$)	0.2413	0.2403	0.2409	0.2413
MISC ($TOL_{\mathrm{MISC}} = 10^{-3}$)	0.2413	0.2411	0.2414	0.2413
MISC ($TOL_{\mathrm{MISC}} = 10^{-4}$)	0.2421	0.2416	0.2414	$-$
MC method ($M = 5.10^6$)	0.2420	0.2416	0.2414	0.2413

Table 4.34: Call option price of the different methods for different number of time steps. Case set 4 parameters in table 4.1, without Richardson extrapolation.

Method \Steps	2	4	8	16
MC Bias ($M = 5.10^6$)	**0.0054** (0.0013)	**0.0035** (0.0008)	**0.0029** (0.0007)	**0.0024** (0.0006)
MC Statistical error ($M = 5.10^6$)	**3.4e − 04** (8.3e−05)	**3.4e − 04** (8.1e−05)	**3.3e − 04** (8.0e−05)	**3.3e − 04** (8.0e−05)

Table 4.35: Bias and statistical errors of MC for computing call option price for different number of time steps. Case set 4, without Richardson extrapolation. The numbers between parentheses are the corresponding absolute errors.
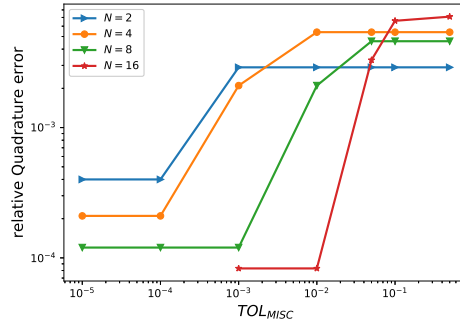

Figure 4.18: Quadrature error of MISC, with different tolerances, to compute call option price for different number of time steps. Case set 4 parameters, without Richardson extrapolation. See detailed values in table A.9

Method \Steps	2	4	8	16
MISC ($TOL_{\text{MISC}} = 10^{-1}$)	**0.0083**	**0.0089**	**0.0075**	**0.0090**
MISC ($TOL_{\text{MISC}} = 10^{-2}$)	**0.0083**	**0.0089**	**0.0050**	**0.0025**
MISC ($TOL_{\text{MISC}} = 10^{-3}$)	**0.0083**	**0.0056**	**0.0030**	**0.0025**
MISC ($TOL_{\text{MISC}} = 10^{-4}$)	**0.0058**	**0.0037**	**0.0030**	–
MC	**0.0057**	**0.0038**	**0.0032**	**0.0027**

Table 4.36: Total relative error of MISC,different tolerances, and MC to compute call option price for different number of time steps. Case set 4 parametrs of table 4.1, without Richardson extrapolation. The numbers between parentheses are the corresponding absolute errors. The values marked in red, for MISC method, correspond to the total relative errors associated with stable quadrature errors for MISC, and will be used for complexity comparison against MC.

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Method \Steps	2	4	8	16
MISC ($TOL_{\mathrm{MISC}} = 10^{-1}$)	0.1	0.1	0.1	1
MISC ($TOL_{\mathrm{MISC}} = 10^{-2}$)	0.1	0.15	9	112
MISC ($TOL_{\mathrm{MISC}} = 10^{-3}$)	0.2	2	27	2226
MISC ($TOL_{\mathrm{MISC}} = 10^{-4}$)	1	6	136	–
MC method	141	246	461	820
Ratio of ($MC/MISC$)	141	41	17	7

Table 4.37: Comparison of the computational time (in Seconds) of MC and MISC, used to compute call option price of the rBergomi model for different number of time steps. Case set 4 parameters of table 4.1. The average MC CPU time is computed over 10 runs.
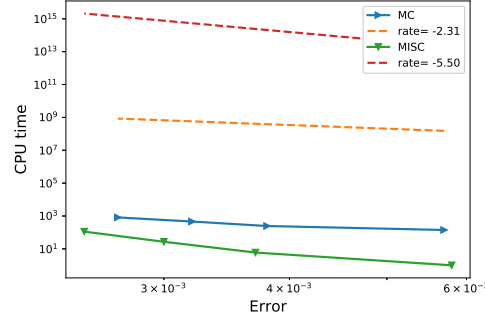

Figure 4.19: Complexity plot for MC and MISC for case set 4 parameters of table 4.1.

4.3.5 Case of set 5 parameters in table 4.1

Method \Steps	2	4	8	16
MISC ($TOL_{\mathrm{MISC}} = 10^{-1}$)	0.0590	0.0564	0.0552	0.0546
MISC ($TOL_{\mathrm{MISC}} = 10^{-2}$)	0.0590	0.0564	0.0574	0.0572
MISC ($TOL_{\mathrm{MISC}} = 10^{-3}$)	0.0605	0.0587	0.0579	0.0575
MC method ($M = 5.10^6$)	0.0605	0.0587	0.0579	0.0576

Table 4.38: Call option price of the different methods for different number of time steps. Case of set 5 parameters in table 4.1, without Richardson extrapolation.

Method \Steps	2	4	8	16
MC Bias ($M = 5.10^6$)	**0.0650** (0.0037)	**0.0330** (0.0019)	**0.0202** (0.0012)	**0.0130** (0.0007)
MC Statistical error ($M = 5.10^6$)	**7.0e − 04** (4.0e−05)	**6.7e − 04** (3.8e−05)	**6.5e − 04** (3.7e−05)	**6.3e − 04** (3.6e−05)

Table 4.39: Bias and statistical errors of MC for computing call option price for different number of time steps. Case set 5, without Richardson extrapolation. The numbers between parentheses are the corresponding absolute errors.
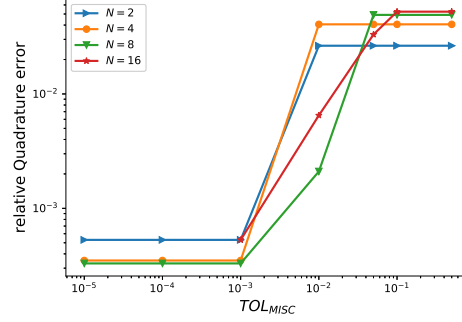

Figure 4.20: Quadrature error of MISC, with different tolerances, to compute call option price for different number of time steps. Case set 5 parameters, without Richardson extrapolation. See detailed values in table A.10

Method \Steps	2	4	8	16
MISC ($TOL_{\mathrm{MISC}} = 10^{-1}$)	**0.0914**	**0.0736**	**0.0693**	**0.0654**
MISC ($TOL_{\mathrm{MISC}} = 10^{-2}$)	**0.0914**	**0.0736**	**0.0223**	**0.0195**
MISC ($TOL_{\mathrm{MISC}} = 10^{-3}$)	**0.0655**	**0.0334**	**0.0205**	**0.0135**
MC	**0.0657**	**0.0337**	**0.0209**	**0.0136**

Table 4.40: Total relative error of MISC, with different tolerances, and MC to compute call option price for different number of time steps. Case set 5 parametrs of table 4.1, without Richardson extrapolation. The numbers between parentheses are the corresponding absolute errors. The values marked in red, for MISC method, correspond to the total relative errors associated with stable quadrature errors for MISC, and will be used for complexity comparison against MC.

Method \Steps	2	4	8	16
MISC ($TOL_{\mathrm{MISC}} = 10^{-1}$)	0.1	0.1	0.2	0.5
MISC ($TOL_{\mathrm{MISC}} = 10^{-2}$)	0.1	0.1	8	97
MISC ($TOL_{\mathrm{MISC}} = 10^{-3}$)	0.7	4	26	1984
MC method	154	229	420	938
Ratio of ($MC/MISC$)	220	57	16	0.5

Table 4.41: Comparison of the computational time (In seconds) of MC and MISC, used to compute call option price of rBergomi model for different number of time steps. Case set 5 parametrs of table 4.1. The average MC CPU time is computed over 10 runs.

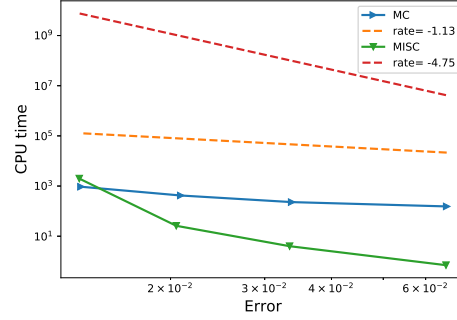

Figure 4.21: Complexity plot for MC and MISC for case set 5 parameters of table 4.1.

5   Conclusions and future work

In this work, we propose a novel hierarchical fast option pricer, based on a hierarchical adaptive sparse grids quadrature, specifically MISC as in [18], coupled with Brownian bridge construction and Richardson extrapolation, for options whose underlyings follow the rBergomi model as in [3]. In order to use adaptive sparse grids quadrature for our purposes, we solve two main issues, that constitutes the two stages of our new designed method. In the first stage, we smoothen the integrand by using the conditional expectation as was proposed by [26], in the context of Markovian stochastic volatility models. In a second stage, we apply MISC as in [18], to solve the integration problem. In this stage, we apply two transformations before using the MISC solver, in order to overcome the issue of facing a high dimensional integrand, due to the discretization scheme used for simulating the rBergomi dynamics. Given that MISC profits from anisotropy, the first transformation consists of applying a hierarchical path generation method, based on Brownian bridge construction, with the aim of reducing the effective dimension. The second transformation consists of applying Richardson extrapolation to reduce the bias, resulting in reducing the needed number of time steps in the coarsest level to achieve a certain error tolerance, and therefore, the maximum number of dimensions needed for the integration problem.

Given that the only prevalent option, in this context, is to use different variants of MC method, our first contribution is that we design an alternative approach based on adaptive sparse grid

quadrature, which opens a new research direction in this field to investigate the performance of other methods besides MC, to solve pricing and calibration problems related to the rBergomi model. Our second contribution is that we reduce the computational cost by using the conditional expectation as in [23] but also through bias reduction through using Richardson extrapolation. Finally, assuming one targets prices estimates with a sufficiently small error tolerance, our third contribution is manifested by the observed substantial computational gains over standard MC method, when pricing under the rBergomi model. We show these gains through our numerical experiments for different parameters constellations.

In this paper, we limit ourselves to compare our novel proposed method against standard MC. A more systematic comparison against the variant of MC proposed in [23] can be carried over but this is left for a future work. Another eventual direction of research may also investigate the performance of QMC for such problems. Finally, accelerating our novel designed method can be reached by coupling MISC with a more optimal hierarchical path generation method than Brownian bridge construction, such as PCA or LT transformations.

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A   Numerical comparison between MISC and MC

A.1   Case of set 1 parameters in table 4.1

Method \Steps	2	4	8	16
MISC ($TOL_{\mathrm{MISC}} = 10^{-1}$)	**0.0868** 0.0062	**0.0563** 0.0040	**0.0681** 0.0049	**0.0492** 0.0035
MISC ($TOL_{\mathrm{MISC}} = 10^{-2}$)	**0.0017** 0.0001	**0.0324** 0.0023	**0.0218** 0.0016	**0.0070** 0.0005
MISC ($TOL_{\mathrm{MISC}} = 10^{-3}$)	**0.0017** 0.0001	**1.4e − 04** $1.0e{-}05$	**0.0049** $3.5e{-}04$	**0.0070** 0.0005
MISC ($TOL_{\mathrm{MISC}} = 10^{-4}$)	**0.0017** 0.0001	**1.4e − 04** $1.0e{-}05$	**6.9e − 04** $(4.9e{-}05)$	− −

Table A.1: Quadrature error of MISC, with different tolerances, to compute call option price for different number of time steps. Case set 1 parameters in table 4.1, without Richardson extrapolation. The numbers between parentheses are the corresponding absolute errors. The values marked in red correspond to stable quadrature errors for MISC, and will be used for complexity comparison against MC.

Method \Steps	$1 − 2$	$2 − 4$	$4 − 8$
MISC ($TOL_{\mathrm{MISC}} = 10^{-1}$)	**0.1685** (0.0120)	**0.0435** (0.0031)	**0.0899** (0.0064)
MISC ($TOL_{\mathrm{MISC}} = 10^{-2}$)	**1e − 05** $(7e{-}07)$	**0.0407** (0.0029)	**0.0337** (0.0024)
MISC ($TOL_{\mathrm{MISC}} = 10^{-3}$)	**0.0183** (0.0013)	**0.0197** (0.0014)	**0.0014** (0.0001)

Table A.2: Quadrature error of MISC, with different tolerances, to compute call option price for different number of time steps. Case set 1 parameters in table 4.1, with Richardson extrapolation(level 1). The numbers between parentheses are the corresponding absolute errors. The values marked in red correspond to stable quadrature errors for MISC, and will be used for complexity comparison against MC.

Method \Steps	$1-2-4$	$2-4-8$
MISC ($TOL_{\text{MISC}} = 10^{-1}$)	**0.0576** (0.0041)	**0.0520** (0.0037)
MISC ($TOL_{\text{MISC}} = 10^{-2}$)	**0.0435** (0.0031)	**0.0267** (0.0019)
MISC ($TOL_{\text{MISC}} = 5.10^{-3}$)	**0.0169** (0.0012)	**0.0028** (0.0002)

Table A.3: Quadrature error of MISC, with different tolerances, to compute call option price for different number of time steps. Case set 1 parameters in table 4.1, with Richardson extrapolation(level 2). The numbers between parentheses are the corresponding absolute errors. The values marked in red correspond to stable quadrature errors for MISC, and will be used for complexity comparison against MC.

A.2   Case of set 2 parameters in table 4.1

Method \Steps	2	4	8
MISC ($TOL_{\text{MISC}} = 10^{-1}$)	**0.1525** (0.0121)	**0.1231** (0.0097)	**0.1555** (0.0123)
MISC ($TOL_{\text{MISC}} = 10^{-2}$)	**0.1247** (0.0099)	**6.3e − 04** (5.0e−05)	**0.0053** (0.0004)
MISC ($TOL_{\text{MISC}} = 10^{-3}$)	**0.0288** (0.0023)	**6.3e − 04** (5.0e−05)	**0.0053** (0.0004)
MISC ($TOL_{\text{MISC}} = 10^{-4}$)	**2.5e − 04** (2.0e−05)	**6.3e − 04** (5.0e−05)	–

Table A.4: Quadrature error of MISC, with different tolerances, to compute call option price for different number of time steps. Case set 2 parameters in table 4.1, without Richardson extrapolation. The numbers between parentheses are the corresponding absolute errors. The values marked in red correspond to stable quadrature errors for MISC, and will be used for complexity comparison against MC.

Method \Steps	$1-2$	$2-4$	$4-8$
MISC ($TOL_{\text{MISC}} = 10^{-1}$)	**0.3687** 0.0292	**0.1136** 0.0090	**0.1288** 0.0102
MISC ($TOL_{\text{MISC}} = 5.10^{-2}$)	**0.3687** 0.0292	**0.1641** 0.0130	**0.0101** 0.0008
MISC ($TOL_{\text{MISC}} = 10^{-2}$)	**0.1212** 0.0096	**0.0101** 0.0008	**0.0101** 0.0008
MISC ($TOL_{\text{MISC}} = 10^{-3}$)	**0.0694** 0.0055	**0.0101** 0.0008	– –

Table A.5: Quadrature error of MISC, with different tolerances, to compute call option price for different number of time steps. Case set 2 parameters in table 4.1, with Richardson extrapolation(level 1). The numbers between parentheses are the corresponding absolute errors. The values marked in red correspond to stable quadrature errors for MISC, and will be used for complexity comparison against MC.

Method \Steps	$1-2-4$	$2-4-8$
MISC ($TOL_{\text{MISC}} = 10^{-1}$)	**0.0177** (0.0014)	**0.1073** (0.0085)
MISC ($TOL_{\text{MISC}} = 5.10^{-2}$)	**0.2525** (0.0200)	**0.0038** (0.0004)
MISC ($TOL_{\text{MISC}} = 10^{-2}$)	**0.0278** (0.0022)	**0.0038** (0.0004)

Table A.6: Quadrature error of MISC, with different tolerances, to compute Call option price for different number of time steps. Case set 2 parameters in table 4.1, with Richardson extrapolation(level 2). The numbers between parentheses are the corresponding absolute errors. The values marked in red correspond to stable quadrature errors for MISC, and will be used for complexity comparison against MC.

A.3 Case of set 3 parameters in table 4.1

Method \Steps	2	4	8	16
MISC ($TOL_{\text{MISC}} = 10^{-1}$)	**0.0088** (0.0011)	**0.0144** (0.0018)	**0.0176** (0.0022)	**0.0160** (0.0020)
MISC ($TOL_{\text{MISC}} = 10^{-2}$)	**0.0088** (0.0011)	**0.0088** (0.0011)	**0.0040** (0.0005)	**0.0008** 0.0001
MISC ($TOL_{\text{MISC}} = 10^{-3}$)	**0.0016** (0.0002)	**0.0016** (0.0002)	**0.0008** 0.0001	**0.0004** 0.00005
MISC ($TOL_{\text{MISC}} = 10^{-4}$)	**0.0008** (0.0001)	**0.0008** 0.0001	**0.0008** 0.0001	− −

Table A.7: Quadrature error of MISC, with different tolerances, to compute call option price for different number of time steps. Case set 3 parameters in table 4.1, without Richardson extrapolation. The numbers between parentheses are the corresponding absolute errors. The values marked in red correspond to stable quadrature errors for MISC, and will be used for complexity comparison against MC.

Method \Steps	$1-2$	$2-4$	$4-8$	$8-16$
MISC ($TOL_{\text{MISC}} = 10^{-1}$)	**0.0184** (0.0023)	**0.0248** (0.0031)	**0.0216** (0.0027)	**0.0176** (0.0022)
MISC ($TOL_{\text{MISC}} = 10^{-2}$)	**0.0136** (0.0017)	**0.0168** (0.0021)	**0.0032** (0.0004)	**0.0008** (0.0001)
MISC ($TOL_{\text{MISC}} = 10^{-3}$)	**0.0008** (0.0001)	**0.0032** (0.0006)	**0.0016** (0.0002)	**0.0008** (0.0001)

Table A.8: Quadrature error of MISC, with different tolerances, to compute Call option price for different number of time steps. Case set 3 parameters in table 4.1, with Richardson extrapolation(level 1). The numbers between parentheses are the corresponding absolute errors. The values marked in red correspond to stable quadrature errors for MISC, and will be used for complexity comparison against MC.

A.4 Case of set 4 parameters in table 4.1

Method \Steps	2	4	8	16
MISC ($TOL_{\text{MISC}} = 10^{-1}$)	**0.0029** (0.0007)	**0.0054** (0.0013)	**0.0046** (0.0011)	**0.0066** (0.0016)
MISC ($TOL_{\text{MISC}} = 10^{-2}$)	**0.0029** (0.0007)	**0.0054** (0.0013)	**0.0021** (0.0005)	**8.3e − 05** 2.0e−05
MISC ($TOL_{\text{MISC}} = 10^{-3}$)	**0.0029** (0.0007)	**0.0021** (0.0005)	**1.2e − 04** 3.0e−05	**8.3e − 05** 2.0e−05
MISC ($TOL_{\text{MISC}} = 10^{-4}$)	**0.0004** (0.0001)	**2.1e − 04** 5.0e−05	**1.2e − 04** 3.0e−05	− −

Table A.9: Quadrature error of MISC, with different tolerances, to compute call option price for different number of time steps. Case set 4 parameters in table 4.1, without Richardson extrapolation. The numbers between parentheses are the corresponding absolute errors. The values marked in red correspond to stable quadrature errors for MISC, and will be used for complexity comparison against MC.

A.5 Case of set 5 parameters in table 4.1

Method \Steps	2	4	8	16
MISC ($TOL_{\text{MISC}} = 10^{-1}$)	**0.0264** (0.0015)	**0.0406** (0.0023)	**0.0491** (0.0028)	**0.0524** (0.0030)
MISC ($TOL_{\text{MISC}} = 10^{-2}$)	**0.0264** (0.0015)	**0.0406** (0.0023)	**0.0021** (0.0005)	**0.0065** 0.0004
MISC ($TOL_{\text{MISC}} = 10^{-3}$)	**5.3e − 04** (3.0e−05)	**3.5e − 04** 2.0e−05	**3.3e − 04** 1.9e−05	**5.3e − 04** 3.0e−05

Table A.10: Quadrature error of MISC, with different tolerances, to compute call option price for different number of time steps. Case set 5 parameters in table 4.1, without Richardson extrapolation. The numbers between parentheses are the corresponding absolute errors. The values marked in red correspond to stable quadrature errors for MISC, and will be used for complexity comparison against MC.