The Order Barrier for Strong Approximation of Rough Volatility Models

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Abstract

We study the strong approximation of a rough volatility model, in which the log-volatility is given by a fractional Ornstein-Uhlenbeck process with Hurst parameter H < 1/2. Our methods are based on an equidistant discretization of the volatility process and of the driving Brownian motions, respectively. For the root mean-square error at a single point the optimal rate of convergence that can be achieved by such methods is n^{-H} , where n denotes the number of subintervals of the discretization. This rate is in particular obtained by the Euler method and an Euler-trapezoidal type scheme.

Keywords: optimal approximation, lower error bounds, fractional Ornstein Uhlenbeck process, asset models with rough volatility, Euler and trapezoidal methods

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1. Introduction and Main Results

Let $B = \{B_t, t \in \mathbb{R}\}$ be a fractional Brownian motion (fBm in what follows) with Hurst parameter $H \in (0, 1/2)$, i.e. B is a centered Gaussian processes with continuous sample paths, $B_0 = 0$ and mean square smoothness

$$\mathbf{E}|B_t - B_s|^2 = |t - s|^{2H}, \quad s, t \in \mathbb{R}.$$

Moreover, let $V = \{V_t, t \ge 0\}$, $W = \{W_t, t \ge 0\}$ be two independent Brownian motions, $\mu \in \mathbb{R}, \lambda, \theta, s_0 > 0$, $\rho \in (-1, 1)$ and consider

$$S_t = s_0 e^{X_t},$$

$$X_t = -\frac{1}{2} \int_0^t e^{2Y_s} ds + \rho \int_0^t e^{Y_s} dV_s + \sqrt{1 - \rho^2} \int_0^t e^{Y_s} dW_s,$$

$$Y_t = \mu + \theta e^{-\lambda t} \int_{-\infty}^t e^{\lambda s} dB_s.$$
(1)

Here $X = \{X_t, t \ge 0\}$ models the log-price of an asset, whose log-volatility $Y = \{Y_t, t \ge 0\}$ is given by the stationary solution of the Langevin equation

$$dY_t = \lambda(\mu - Y_t)dt + \theta dB_t.$$

The fractional Brownian motion B and the Brownian motion V are correlated, i.e.

$$\mathbf{E}B_tV_s=\gamma(t,s), \qquad t\in\mathbb{R}, s\geq 0,$$

for some suitable, i.e. in particular positive definite, function $\gamma : \mathbb{R} \times [0, \infty) \to \mathbb{R}$, while B and W are independent, i.e.

$$\mathbf{E}B_tW_s=0, \qquad t\in\mathbb{R}, s>0.$$

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Such a model has been proposed by Gatheral, Jaisson and Rosenbaum based on striking empirical evidence that the log-volatility of assets behaves essentially as fBm with with $H \approx 0.1$, see Gatheral et al. (2014). This model has been further analysed in Bayer et al. (2015).

In this manuscript, we will study the optimal mean square approximation of X_T based on

$$V_0, V_{T/n}, \dots, V_T, W_0, W_{T/n}, \dots, W_T, Y_0, Y_{T/n}, \dots, Y_T.$$
 (2)

The fractional Ornstein-Uhlenbeck process (fOUp) Y is a Gaussian process with known mean covariance function and thus exact joint simulation of V, W, Y at a finite number of time points is possible. Clearly, the optimal mean square approximation of X_T using (2) is given by

$$X_n^{\text{opt}} = \mathbb{E}(X_T | V_{kT/n}, W_{kT/n}, Y_{kT/n}, k = 0, ..., n)$$
 (3)

and the corresponding minimal errors are

$$e(n) = (\mathbf{E}|X_T - X_n^{\text{opt}}|^2)^{1/2}.$$
 (4)

We will show that rough volatility models are numerically tough in the sense that they admit only low convergence rates for the mean square approximation based on the information given by (2). More precisely, we will show that

$$\liminf_{n \to \infty} \left(\frac{n}{T} \right)^{2H} e(n)^2 \ge (1 - \rho^2) \frac{1}{(2H + 1)(2H + 2)} T \theta^2 \mathbf{E} |e^{Y_0}|^2,$$

see Theorem 2. Moreover, the optimal convergence rate n^{-H} is obtained by the Euler method

$$X_n^{\mathsf{E}} = -\frac{1}{2} \sum_{k=0}^{n-1} e^{2Y_{k\Delta}} \Delta + \rho \sum_{k=0}^{n-1} e^{Y_{k\Delta}} \Delta_k V + \sqrt{1 - \rho^2} \sum_{k=0}^{n-1} e^{Y_{k\Delta}} \Delta_k W, \tag{5}$$

and the trapezoidal scheme

$$X_n^{\mathsf{Tr}} = -\frac{1}{4} \sum_{k=0}^{n-1} \left(e^{2Y_{k\Delta}} + e^{2Y_{(k+1)\Delta}} \right) \Delta + \rho \sum_{k=0}^{n-1} e^{Y_{k\Delta}} \Delta_k V + \frac{1}{2} \sqrt{1 - \rho^2} \sum_{k=0}^{n-1} \left(e^{Y_{k\Delta}} + e^{Y_{(k+1)\Delta}} \right) \Delta_k W, \quad (6)$$

where $\Delta = T/n$ and

$$\Delta_k V = V_{(k+1)\Delta} - V_{k\Delta}, \quad \Delta_k W = W_{(k+1)\Delta} - W_{k\Delta}, \qquad k = 0, \dots, n-1.$$

For these schemes we have

$$\lim_{n\to\infty} \left(\frac{n}{T}\right)^{2H} \mathbf{E} |X_T - X_n^{\mathsf{E}}|^2 = \frac{1}{2H+1} T \theta^2 \mathbf{E} |e^{Y_0}|^2,$$

and

$$\lim_{n \to \infty} \left(\frac{n}{T} \right)^{2H} \mathbf{E} |X_T - X_n^{\mathsf{Tr}}|^2 = \left(\frac{1}{2H+1} - \frac{1-\rho^2}{4} \right) T \theta^2 \mathbf{E} |e^{Y_0}|^2,$$

see Theorem 1. Note that

$$\mathbf{E}|e^{Y_t} - e^{Y_s}|^2 = \theta^2 \mathbf{E}|e^{Y_0}|^2 \cdot |t - s|^{2H} + o(|t - s|^{2H})$$
 for $|t - s| \to 0$,

i.e. the limiting constants on the right hand side of the above expressions depend on the Hölder constant of the mean square smoothness of the volatility process $\{e^{Y_t}, t \ge 0\}$.

The remainder of the manuscript is structured as follows. In the next section, we collect several properties of the stationary fractional Ornstein-Uhlenbeck process and other auxiliary results. Section 3 is devoted to the analysis of the Euler and the trapezoidal method, while in Section 4 we establish the lower bound for the minimal errors. Finally, in Section 5, we discuss the joint simulation of the fractional Ornstein-Uhlenbeck process and Brownian motion.

Remark 1. Bayer et al. (2015) proposed that the correlation between B and V is introduced using the Mandelbrot-van Ness representation of fBm (Mandelbrot and Van Ness (1968)), i.e. V is in fact a two-sided Brownian motion $V = \{V_t, t \in \mathbb{R}\}$ and

$$B_t = G(H) \int_{\mathbb{R}} ((t-s)_+^{H-1/2} - (-s)_+^{H-1/2}) dV_s, \qquad t \in \mathbb{R},$$

with

$$G(H)^2 = \frac{2H\Gamma(3/2 - H)}{\Gamma(H + 1/2)\Gamma(2 - 2H)}.$$

This leads to a correlation structure with

$$\gamma(t,s) = \frac{G(H)}{H+1/2} \left(t^{H+1/2} - (t - \min\{t,s\})^{H+1/2} \right) \mathbf{1}_{[0,\infty) \times [0,\infty)}(t,s), \qquad t,s \ge 0.$$

Remark 2. For Itō stochastic differential equations (SDEs) driven by Brownian motion the minimal errors e(n) have been studied in detail, also for more general discretizations of the Brownian motion, see Müller-Gronbach and Ritter (2008) for a survey of the respective results. For stochastic differential equations driven by fractional Brownian motion minimal errors have been studied for H > 1/2 in Neuenkirch (2008) for the scalar case, respectively in Neuenkirch and Shalaiko (2016) for the fractional Lévy area.

Remark 3. In Section 5 we will point out that simulating

$$V_0, V_{T/n}, \dots, V_T, W_0, W_{T/n}, \dots, W_T, Y_0, Y_{T/n}, \dots, Y_T$$

exactly has a computational cost (number of random numbers and number of arithmetic operations) of order n^2 , up to the best of our knowledge. This makes the barrier of order H even worse. In our future work, we will therefore study the approximation of Y and B via the Mandelbrot-van Ness representation of fBm similar to the recent work of Bennedsen et al. (2015) and also weak approximation methods.

2. Preliminaries

2.1. Fractional Brownian motion

Let $(\Omega, \mathscr{F}, \mathbf{P})$ be a probability space and recall that a centered Gaussian process $B = \{B_t, t \in \mathbb{R}\}$ is called a fractional Brownian motion with Hurst index $H \in (0, 1)$ if its covariance function equals

$$K(s,t) = \frac{1}{2}(t^{2H} + s^{2H} - |t - s|^{2H}), \quad s, t \in \mathbb{R}.$$

Since $\mathbb{E}|B_t - B_s|^2 = |t - s|^{2H}$ there exists a modification with γ -Hölder trajectories for any $\gamma < H$, which we consider in what follows. The process B^H is moreover H-self-similar, i.e. for all c > 0 we have

$$\left\{c^{-H}B_{ct}, t \in \mathbb{R}\right\} \stackrel{\mathsf{law}}{=} \left\{B_t, t \in \mathbb{R}\right\},$$

and is shift invariant, i.e. for any $s \in \mathbb{R}$ the process $\{B_{t+s} - B_s, t \in \mathbb{R}\}$ is a again an fBm. Furthermore, fBm has polynomial growth as $|t| \to \infty$, i.e. there exists a set $\mathscr{A} \in \mathscr{F}$ with P(A) = 1 and a random variable K such that

$$|B_t(\omega)| \le K(\omega)(1+|t|^2), \qquad t \in \mathbb{R}, \quad \omega \in \mathscr{A},$$
 (7)

see Maslowski and Schmalfuss (2004). In the sequel we will change Ω such that $B_{\cdot}(\omega) = 0$ for $\omega \notin \mathscr{A}$.

2.2. Young integration

Let $\alpha, \beta \in (0,1)$ such that $\alpha + \beta > 1$ and consider two Hölder functions $f \in \mathscr{C}^{\alpha}([0,T];\mathbb{R})$, $g \in \mathscr{C}^{\beta}([0,T];\mathbb{R})$. Then the Young integral $\int_0^T f(t) dg(t)$ is defined as the limit of the corresponding Riemann-Stieltjes sums, see e.g. Young (1936).

Therefore, due to the Hölder smoothness of the sample paths of fBm the pathwise Riemann-Stieltjes integrals

$$\int_{s}^{t} a(\tau) dB_{\tau}, \qquad -T \leq s < t \leq T,$$

exist if $a \in C^{\alpha}([-T,T];\mathbb{R})$ with $\alpha + H > 1$. Moreover, if $-T \le s_1 < t_1 \le s_2 < t_2 < T$, we have the fractional Itō isometry

$$\mathbf{E} \int_{s_1}^{t_1} a(\tau_1) dB_{\tau_1} \int_{s_2}^{t_2} a(\tau_2) dB_{\tau_2} = H(2H - 1) \int_{s_1}^{t_1} \int_{s_2}^{t_2} a(\tau_1) a(\tau_2) |\tau_1 - \tau_2|^{2H - 2} d\tau_2 d\tau_1.$$
 (8)

Property (7) implies that the improper Riemann-Stieltjes integrals

$$\int_{-\infty}^t e^{\lambda s} \mathsf{d}B_s(\omega), \qquad t \in \mathbb{R}, \quad \omega \in \Omega,$$

are well-defined and that the integration by parts relation

$$\int_{-\infty}^{t} e^{\lambda s} dB_{s}(\omega) = e^{\lambda t} B_{t}(\omega) - \lambda \int_{-\infty}^{t} e^{\lambda u} B_{u}(\omega) du, \qquad t \in \mathbb{R}, \quad \omega \in \Omega,$$
(9)

holds.

Moreover, for these integrals the isometry (10) is still valid, see Cheridito et al. (2003):

$$\mathbf{E} \int_{-\infty}^{t_1} e^{\lambda \tau_1} dB_{\tau_1} \int_{t_2}^{t_3} e^{\lambda \tau_2} dB_{\tau_2} = H(2H - 1) \int_{-\infty}^{t_1} \int_{t_2}^{t_3} e^{\lambda (\tau_1 + \tau_2)} |\tau_1 - \tau_2|^{2H - 2} d\tau_2 d\tau_1$$
 (10)

for $-\infty < t_1 \le t_2 \le t_3 < \infty$.

2.3. Itō integration

Throughout this manuscript, we assume that $(B_t, W_t, V_t)_{t\geq 0}$ are $(\mathscr{F}_t)_{t\geq 0}$ -adapted, where this filtration is constructed from the canonical filtration by the usual extension procedure, see e.g. Chapter 2.7 in Karatzas and Shreve (1991), and that $(B_t)_{t<0}$ is \mathscr{F}_0 -measurable. Consequently, we must have that

$$\gamma(t,s) = \gamma(t,t)$$
 for all $s \ge t$, (11)

since the adaptedness implies that B_t and $V_{t+h} - V_t$ with h > 0 are independent, and in particular

$$\gamma(t,s) = 0 \qquad \text{for all} \quad t < 0 < s. \tag{12}$$

Under the above adaptedness assumption the stochastic integrals

$$\int_0^t e^{Y_s} \mathrm{d}V_s, \qquad \int_0^t e^{Y_s} \mathrm{d}W_s, \qquad t \in [0,T],$$

are standard Itō integrals and we can use all classical tools as the Itō isometry, Burkholder-Davis-Gundy inequality etc.

Furthermore, we have the following Lemma:

Lemma 1. Let $R = (R_t)_{t \in [0,T]}$ be a process, which is $(\mathscr{F}_t)_{t \in [0,T]}$ adapted, independent of $W = (W_t)_{t \in [0,T]}$, and has root mean-square smoothness of order $\alpha \in (0,1)$, i.e. there exist C > 0, $\alpha \in (0,1)$, such that

$$\left(\mathbf{E}|R_t - R_s|^2\right)^{1/2} \le C|t - s|^{\alpha}, \quad s, t \in [0, T].$$

Moreover, let $t_k = kT/n$, k = 0, ..., n, and

$$\overline{W}_{t}^{n} = W_{t_{k}} + \left(\frac{nt}{T} - k\right) \left(W_{t_{k+1}} - W_{t_{k}}\right), \qquad t \in [t_{k}, t_{k+1}], \ k = 0, \dots, n-1.$$

Then, it holds

$$\mathbf{E} \left| \int_0^T R_s dW_s - \frac{1}{2} \sum_{k=0}^{N-1} (R_{t_k} + R_{t_{k+1}}) \Delta_k W \right|^2 = \sum_{k=0}^{n-1} \int_{t_k}^{t_{k+1}} \mathbf{E} \left| R_t - \frac{1}{2} (R_{t_k} + R_{t_{k+1}}) \right|^2 dt$$

and

$$\mathbf{E} \left| \int_0^T R_s dW_s - \int_0^T R_s d\overline{W}_s^n \right|^2 = \sum_{k=0}^{n-1} \int_{t_k}^{t_{k+1}} \mathbf{E} \left| R_t - \frac{n}{T} \int_{t_k}^{t_{k+1}} R_s ds \right|^2 dt.$$

Proof. To simplify our notation we put T = 1. We only proof the second assertion, the other proof is similar. First note that

$$\int_{t_k}^{t_{k+1}} R_s d\overline{W}_s^n = n\Delta_k W \int_{t_k}^{t_{k+1}} R_s ds.$$

Therefore we have

$$\mathbf{E}\left(\int_{t_k}^{t_{k+1}} R_s d\overline{W}_s^n \int_{t_l}^{t_{l+1}} R_s d\overline{W}_s^n\right) = n^2 \mathbf{E}\left(\int_{t_k}^{t_{k+1}} R_s ds \int_{t_l}^{t_{l+1}} R_s ds\right) \mathbf{E}\left(\Delta_k W \Delta_l W\right) = 0$$

for $k \neq l$ by independence of R and W. Moreover, using additionally the adaptedness of R and the properties of the Itō integral it holds

$$\mathbf{E}\left(\Delta_k W \int_{t_k}^{t_{k+1}} R_s \mathrm{d}s \int_{t_l}^{t_{l+1}} R_s \mathrm{d}W_s\right) = \mathbf{E}\left(\Delta_k W \int_{t_k}^{t_{k+1}} R_s \mathrm{d}s \mathbf{E}\left(\int_{t_l}^{t_{l+1}} R_s \mathrm{d}W_s \middle| \mathscr{F}_{t_{k+1}}\right)\right) = 0$$

and

$$\mathbf{E}\left(\int_{t_k}^{t_{k+1}} R_s \mathsf{d}W_s \,\Delta_l W \int_{t_l}^{t_{l+1}} R_s \mathsf{d}s\right) = \mathbf{E}\left(\int_{t_k}^{t_{k+1}} R_s \mathsf{d}W_s \,\mathbf{E}\left(\Delta_l W \int_{t_l}^{t_{l+1}} R_s \mathsf{d}s \middle| \mathscr{F}_{t_{k+1}}\right)\right) = 0$$

for k < l. So we end up with

$$\mathbf{E}\left(\int_{t_k}^{t_{k+1}}R_s\mathsf{d}W_s-\int_{t_k}^{t_{k+1}}R_s\mathsf{d}\overline{W}_s^n\right)\left(\int_{t_l}^{t_{l+1}}R_s\mathsf{d}W_s-\int_{t_l}^{t_{l+1}}R_s\mathsf{d}\overline{W}_s^n\right)=0.$$

for $k \neq l$.

Hence it remains to study

$$\mathbf{E} \left| \int_{t_k}^{t_{k+1}} R_s dW_s - \int_{t_k}^{t_{k+1}} R_s d\overline{W}_s^n \right|^2$$

$$= \int_{t_k}^{t_{k+1}} \mathbf{E} |R_s|^2 ds + n \mathbf{E} \left| \int_{t_k}^{t_{k+1}} R_s ds \right|^2 - 2n \mathbf{E} \left(\int_{t_k}^{t_{k+1}} R_s dW_s \int_{t_k}^{t_{k+1}} R_s ds \Delta_k W \right).$$

For the last term consider the approximation $\int_{t_k}^{t_{k+1}} R_s dW_s = \mathsf{L}^2 - \lim_{N \to \infty} S_N^{(k)}$ with

$$S_N^{(k)} = \sum_{l=0}^{N-1} R_{s_l^N} \Delta_{l^N} W,$$

where $\{s_l^N\}_{l=0}^N$ is a sequence of partitions of $[t_k, t_{k+1}]$ with meshsize going to zero and $\Delta_{l^N}W = W_{s_{l+1}^N} - W_{s_l^N}$. (The L²-convergence holds due to the mean-square smoothness assumption.) Then we have

$$\begin{split} \mathbf{E} \left(\int_{t_k}^{t_{k+1}} R_s \mathrm{d}W_s \int_{t_k}^{t_{k+1}} R_s \mathrm{d}s \Delta_k W \right) \\ &= \lim_{N \to \infty} \sum_{l=0}^{N-1} \mathbf{E} \left(R_{s_l^N} \int_{t_k}^{t_{k+1}} R_s \mathrm{d}s \right) \mathbf{E} (\Delta_{l^N} W \Delta_k W) = \lim_{N \to \infty} \sum_{l=0}^{N-1} \mathbf{E} \left(R_{s_l^N} \int_{t_k}^{t_{k+1}} R_s \mathrm{d}s \right) (s_{l+1}^N - s_l^N) \\ &= \int_{t_k}^{t_{k+1}} \mathbf{E} \left(R_t \int_{t_k}^{t_{k+1}} R_s \mathrm{d}s \right) \mathrm{d}t. \end{split}$$

Hence we obtain

$$\mathbf{E} \left| \int_{t_k}^{t_{k+1}} R_s dW_s - \int_{t_k}^{t_{k+1}} R_s d\overline{W}_s^n \right|^2$$

$$= \int_{t_k}^{t_{k+1}} \mathbf{E} |R_s|^2 ds + n \mathbf{E} \left| \int_{t_k}^{t_{k+1}} R_s ds \right|^2 - 2n \int_{t_k}^{t_{k+1}} \mathbf{E} \left(R_t \int_{t_k}^{t_{k+1}} R_s ds \right) dt$$

and summing over the subintervals yields the assertion.

2.4. Stationary fractional Ornstein-Uhlenbeck process

As already mentioned

$$Y_t = \mu + \theta e^{-\lambda t} \int_{-\infty}^t e^{\lambda u} \mathrm{d}B_u, \qquad t \in \mathbb{R},$$

is the stationary solution of the Langevin SDE

$$dY_t = \lambda(\mu - Y_t)dt + \theta dB_t$$

see e.g. Cheridito et al. (2003); Garrido-Atienza et al. (2009). The stationarity is a simple consequence of the shift invariance of fBm which gives in particular

$$(Y_{t}, Y_{s}) = \mu + \left(e^{-\lambda t} \int_{-\infty}^{t} e^{\lambda u} dB_{u}, e^{-\lambda s} \int_{-\infty}^{s} e^{\lambda u} dB_{u}\right)$$

$$\stackrel{\text{law}}{=} \mu + \left(e^{-\lambda t} \int_{-\infty}^{t} e^{\lambda u} d(B_{u-s} - B_{-s}), e^{-\lambda s} \int_{-\infty}^{s} e^{\lambda u} d(B_{u-s} - B_{-s})\right)$$

$$= \mu + \left(e^{-\lambda(t-s)} \int_{-\infty}^{t-s} e^{\lambda u} dB_{u}, \int_{-\infty}^{0} e^{\lambda u} dB_{u}\right) = (Y_{t-s}, Y_{0})$$

$$(13)$$

for any $s, t \in \mathbb{R}$.

The process *Y* is Gaussian with mean

$$\mathbf{E}Y_0 = \mu$$

and covariance

$$\mathbf{E}(Y_s - \mathbf{E}Y_s)(Y_t - \mathbf{E}Y_t) = R_Y(|t - s|), \quad s, t \in \mathbb{R},$$

where

$$R_Y(\tau) = \theta^2 \left(\frac{\Gamma(2H+1)\cosh(\lambda \tau)}{2\lambda^{2H}} - H \int_0^{\tau} \cosh(\lambda(\tau - u)) u^{2H-1} du \right), \quad \tau \ge 0.$$
 (14)

In particular we have

$$\mathbf{V}(Y_t) = R_Y(0) = \theta^2 \frac{\Gamma(2H+1)}{2\lambda^{2H}}, \qquad t \in \mathbb{R}.$$

For a derivation see the Appendix. Note that another representation of the covariance function in terms of the confluent hypergeometric function $_1F_2$ has been given in Proposition 4.1.2 in Schöchtel (2013), starting from the Fourier representation

$$R_Y(\tau) = \theta^2 \frac{\Gamma(2H+1)\sin(\pi H)}{2\pi} \int_{-\infty}^{\infty} e^{i\tau x} \frac{|x|^{1-2H}}{\lambda^2 + x^2} \mathrm{d}x, \quad \tau \ge 0.$$

By straightforward calculations using (14) and $2\cosh(\lambda(\tau - u)) = e^{\lambda \tau}e^{-\lambda u} + e^{-\lambda \tau}e^{\lambda u}$ we have:

Lemma 2. We have $R_Y \in \mathscr{C}^{\infty}((0,\infty);\mathbb{R})$ and

$$R_Y(au) = R_Y(0) - rac{ heta^2}{2} au^{2H} + o(au^{2H}), \qquad au
ightarrow 0.$$

The process

$$Z_t^{(a)} = e^{a(Y_t - \mu)}, \qquad t \in \mathbb{R},$$

is again stationary with mean

$$\mathbb{E}Z_0^{(a)} = \exp\left(\frac{a^2}{2}R_Y(0)\right)$$

and covariance, again in terms of R_Y ,

$$R_{Z(a)}(|t-s|) = \exp(a^2 R_Y(0)) \left(\exp(a^2 R_Y(|t-s|)) - 1 \right).$$

Here we have exploited that the moment generating function of a d-dimensional Gaussian random variable ξ with mean m and covariance matrix C is given by

$$\mathbf{E}e^{\langle a,\xi\rangle}=\exp\left(\langle a,m\rangle+rac{1}{2}\langle a,Ca
angle
ight),\qquad a\in\mathbb{R}^d.$$

We have the following asymptotic expansion:

Lemma 3. Let $a \neq 0$. We have $R_{\mathbf{Z}^{(a)}} \in \mathscr{C}^{\infty}((0,\infty);\mathbb{R})$ and

$$R_{Z^{(a)}}(\tau) = c_0 - c_1 \tau^{2H} + o(\tau^{2H}), \qquad \tau \to 0,$$

with

$$c_0 = c_0(a, R_Y(0)) = \exp(a^2 R_Y(0)) \left(\exp(a^2 R_Y(0)) - 1\right),$$

$$c_1 = c_1(a, \theta, R_Y(0)) = \frac{a^2 \theta^2}{2} \exp(2a^2 R_Y(0)).$$

The previous Lemma allows to use results from Benhenni (1998) for the approximation of $\int_0^T Z_s^{(a)} ds$. We need another Lemma, which follows again by straightforward computations:

Lemma 4. *Let the notation of the Lemma* (3) *prevail and* $s \le u \le t$. We have

$$\mathbf{E}|Z_t^{(a)} - Z_s^{(a)}|^2 = 2c_1|t - s|^{2H} + o(|t - s|^{2H}), \qquad |t - s| \to 0,$$

and

$$\mathbf{E} \left| Z_u^{(a)} - \frac{1}{2} (Z_s^{(a)} + Z_t^{(a)}) \right|^2 = c_1 \left(|t - u|^{2H} + |s - u|^{2H} - \frac{1}{2} |t - s|^{2H} \right) + o(|t - s|^{2H}), \qquad |t - s| \to 0.$$

It also holds

$$\mathbf{E} \big(Z_t^{(a)} - Z_u^{(a)} \big) \big(Z_t^{(a)} - Z_s^{(a)} \big) = c_1 \big(|t - s|^{2H} + |t - u|^{2H} - |u - s|^{2H} \big) + o(|t - s|^{2H}), \qquad |t - s| \to 0$$

3. Analysis of the Euler- and Trapezoidal scheme

Let

$$X_T^{RS} = \int_0^T e^{2Y_s} ds = e^{2\mu} \int_0^T Z_s^{(2)} ds.$$

Since $Z^{(a)}$ is stationary and the covariance function is infinitely differentiable away from zero and admits the expansion given in Lemma 3, one can apply Theorem 1 in Benhenni (1998) to obtain

$$\mathbf{E} \left| X_T^{RS} - \frac{1}{2} \sum_{k=0}^{n-1} \left(e^{2Y_{k\Delta}} + e^{2Y_{(k+1)\Delta}} \right) \Delta \right|^2 = O(\Delta^{1+2H}). \tag{15}$$

Since

$$\frac{1}{2} \sum_{k=0}^{n-1} \left(e^{2Y_{k\Delta}} + e^{2Y_{(k+1)\Delta}} \right) \Delta = \sum_{k=0}^{n-1} e^{2Y_{k\Delta}} \Delta + \frac{1}{2} \left(e^{2Y_T} - e^{2Y_0} \right) \Delta$$

and H < 1/2, we also have

$$\mathbf{E} \left| X_T^{RS} - \sum_{k=0}^{n-1} e^{2Y_{k\Delta}} \Delta \right|^2 = O(\Delta^{1+2H}).$$

Theorem 1. Suppose X_n^{E} and X_n^{Tr} , $n \geq 1$, are given by (5) and (6), respectively. It holds

$$\mathbf{E} \left| X_T - X_n^{\mathsf{E}} \right|^2 = C_{\mathsf{E}} \cdot n^{-2H} + o(n^{-2H}),$$

$$\mathbf{E} \left| X_T - X_n^{\mathsf{Tr}} \right|^2 = C_{\mathsf{Tr}} \cdot n^{-2H} + o(n^{-2H}),$$

with

$$\begin{split} C_{\mathsf{E}} &= C_{\mathsf{E}}(\mu, \lambda, \theta, H, T) = \frac{2e^{2\mu}c_1(1, \theta, R_Y(0))T^{2H+1}}{2H+1}, \\ C_{\mathsf{Tr}} &= C_{\mathsf{Tr}}(\mu, \lambda, \theta, \rho, H, T) = C_{\mathsf{E}} - (1-\rho^2)\frac{e^{2\mu}c_1(1, \theta, R_Y(0))T^{2H+1}}{2}. \end{split}$$

Note that

$$\theta^2 \mathbf{E} |e^{Y_0}|^2 = \theta^2 e^{2\mu} \mathbf{E} |Z_0^{(1)}|^2 = \theta^2 e^{2\mu} e^{2R_Y(0)} = 2e^{2\mu} c_1(1, \theta, R_Y(0)).$$

Proof. The Riemann integral part of X_T , i.e. X_T^{RS} , is considered above. Recalling $\Delta = T/n$, $Y_t = e^{\mu}Z_t^{(1)}$ and using the Itō isometry and Lemma 4 we have

$$\begin{split} \mathbf{E} \left| \int_{0}^{T} e^{Y_{s}} \mathrm{d}V_{s} - \sum_{k=0}^{n-1} e^{Y_{k\Delta}} \Delta_{k} V \right|^{2} &= \mathbf{E} \left| \sum_{k=0}^{n-1} \int_{k\Delta}^{(k+1)\Delta} (e^{Y_{s}} - e^{Y_{k\Delta}}) \mathrm{d}V_{s} \right|^{2} \\ &= e^{2\mu} \sum_{k=0}^{n-1} \int_{k\Delta}^{(k+1)\Delta} \mathbf{E} |Z_{s}^{(1)} - Z_{k\Delta}^{(1)}|^{2} \mathrm{d}s \\ &= 2c_{1}e^{2\mu} \sum_{k=0}^{n-1} \int_{k\Delta}^{(k+1)\Delta} \left((s - k\Delta)^{2H} + o(|s - k\Delta|^{2H}) \right) \mathrm{d}s \\ &= \frac{2c_{1}e^{2\mu}}{2H + 1} n\Delta^{2H + 1} (1 + o(1)), \end{split}$$

with $c_1 = c_1(1, \theta, R_Y(0))$ and analogously

$$\mathbf{E} \left| \int_0^T e^{Y_s} dW_s - \sum_{k=0}^{n-1} e^{Y_{k\Delta}} \Delta_k W \right|^2 = \frac{2c_1 e^{2\mu}}{2H+1} n \Delta^{2H+1} (1+o(1)).$$

Summing up both terms using the orthogonality of the stochastic integrals we end up with the statement for the Euler scheme.

For the analysis of the trapezoidal scheme we have that

$$\mathbf{E} \left| \int_0^T Z_s^{(1)} dW_s - \frac{1}{2} \sum_{k=0}^{N-1} (Z_{k\Delta}^{(1)} + Z_{(k+1)\Delta}^{(1)}) \Delta_k W \right|^2 = \sum_{k=0}^{n-1} \int_{k\Delta}^{(k+1)\Delta} \mathbf{E} \left| Z_t^{(1)} - \frac{1}{2} (Z_{k\Delta}^{(1)} + Z_{(k+1)\Delta}^{(1)}) \right|^2 dt$$

by Lemma 1. Due to Lemma 4 we have

$$\begin{split} \sum_{k=0}^{n-1} \int_{k\Delta}^{(k+1)\Delta} \mathbf{E} \left| Z_t^{(1)} - \frac{1}{2} (Z_{k\Delta}^{(1)} + Z_{(k+1)\Delta}^{(1)}) \right|^2 \mathrm{d}t \\ &= c_1 \sum_{k=0}^{n-1} \int_{k\Delta}^{(k+1)\Delta} \left(|t - k\Delta|^{2H} + |(k+1)\Delta - t|^{2H} - \frac{1}{2} \Delta^{2H} + o(\Delta^{2H}) \right) \mathrm{d}t \\ &= c_1 \left(\frac{2}{2H+1} - \frac{1}{2} \right) n \Delta^{2H+1} + o(\Delta^{2H}). \end{split}$$

Using this estimate, taking into account the correlation and the Euler estimate for the dV-integral, we obtain our assertion.

4. The order barrier

4.1. Conditional Expectations

Let

$$\mathscr{G}_n = \sigma(V_{kT/n}, W_{kT/n}, Y_{kT/n}, k = 0, ..., n), \qquad \mathscr{H}_n = \sigma(V_t, Y_t, t \ge 0, W_{kT/n}, k = 0, ..., n)$$
 (16)

and

$$X_T^V = \rho \int_0^T e^{Y_s} dV_s,$$
 $X_T^W = \sqrt{1 - \rho^2} \int_0^T e^{Y_s} dW_s.$ (17)

We start with some representation formulae for the involved conditional expectations.

Lemma 5. (i) We have

$$\mathbf{E}(X_T^W|\mathscr{G}_n) = \sqrt{1-\rho^2} \int_0^T \mathbf{E}(e^{Y_s} | V_{kT/n}, Y_{kT/n}, k = 0, \dots, n) d\overline{W}_s^n, \tag{18}$$

where

$$\overline{W}_{t}^{n} = W_{kT/n} + \left(\frac{nt}{T} - k\right) \left(W_{(k+1)T/n} - W_{kT/n}\right), \qquad t \in [kT/n, (k+1)T/n], \ k = 0, \dots, n-1,$$

and

$$\mathbf{E}(X_T^W|\mathcal{H}_n) = \sqrt{1-\rho^2} \int_0^T e^{Y_s} d\overline{W}_s^n. \tag{19}$$

(ii) It holds

$$\mathbf{E}(X_T^V|\mathscr{G}_n) = \rho V_T e^{Y_T} - \rho \lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \mathbf{E} \left(\int_0^T V_s e^{Y_s^{\varepsilon}} (Y_s - Y_{s-\varepsilon}) \, ds \middle| V_{kT/n}, Y_{kT/n}, k = 0, \dots, n \right)$$

with

$$Y_t^{arepsilon} = rac{1}{arepsilon} \int_{t-arepsilon}^t Y_s \mathrm{d}s, \qquad t \in \mathbb{R}, \quad arepsilon > 0.$$

Proof. The proof relies on the fact that for any sub- σ -algebra \mathscr{G} of \mathscr{F} and random variables Z, Z_n , $n \in \mathbb{N}$, we have that

$$\mathsf{L}^2 - \lim_{n \to \infty} Z_n = Z \quad \Longrightarrow \quad \mathsf{L}^2 - \lim_{n \to \infty} \mathbf{E}(Z_n | \mathscr{G}) = \mathbf{E}(Z | \mathscr{G}). \tag{20}$$

(i) We start with the first equality. Consider the L²-approximation $X_T^W/\sqrt{1-\rho^2}=\lim_{N\to\infty}S_N$ where

$$S_N = \sum_{l=0}^{N-1} e^{Y_{Tl/N}} (W_{(l+1)T/N} - W_{Tl/N}), \quad N \in \mathbb{N}.$$

Since W is independent of (V, Y), one can write

$$\mathbf{E}(S_N|\mathscr{G}_n) = \sum_{l=0}^{N-1} \mathbf{E}(e^{Y_{Tl/N}}|V_{kT/n}, Y_{kT/n}, k = 0, \dots, n) \mathbf{E}(W_{(l+1)T/N} - W_{Tl/N}|W_{kT/n}, k = 0, \dots, n).$$

Due to the normal correlation theorem we have

$$\mathbb{E}(W_{T(l+1)/N} - W_{Tl/N}|W_{kT/n}, k = 0, \dots, n) = \overline{W}_{T(l+1)/N}^n - \overline{W}_{Tl/N}^n$$

and hence

$$\mathbf{E}(S_N|\mathscr{G}_n) = \sum_{l=0}^{N-1} \mathbf{E}(e^{Y_{Tl/N}}|V_{kT/n}, Y_{kT/n}, k = 0, \dots, n) (\overline{W}_{(l+1)T/N}^n - \overline{W}_{Tl/N}^n).$$

Since $\overline{W}_{\cdot}^{n}(\omega)$ is piecewise differentiable, it follows

$$\lim_{N\to\infty} \mathbf{E}(S_N|\mathscr{G}_n) \stackrel{\mathbf{P}-a.s.}{=} \int_0^T \mathbf{E}(e^{Y_s}|V_{kT/n},Y_{kT/n},k=0,\ldots,n) d\overline{W}_s^n,$$

which finishes the proof of (19) using (20).

The second assertion can be shown analogously.

(ii) To prove the second equality introduce the following family of random variables

$$X_T^{V,arepsilon} =
ho \int_0^T e^{Y_t^arepsilon} \mathrm{d}V_t, \quad arepsilon > 0.$$

The Itō isometry, the mean value theorem and Hölder's inequality give that

$$\mathbf{E}|X_T^{V,\varepsilon}-X_T^V|^2 = \int_0^T \mathbf{E}|e^{Y_t^{\varepsilon}}-e^{Y_t}|^2 dt \leq \int_0^T \left(\mathbf{E}\left(\int_0^1 e^{\lambda Y_t^{\varepsilon}+(1-\lambda)Y_t} d\lambda\right)^4\right)^{1/2} \left(\mathbf{E}|Y_t^{\varepsilon}-Y_t|^4\right)^{1/2} dt.$$

Since

$$|Y_t^{\varepsilon}| \leq \sup_{t \in [-1,T]} |Y_t| \quad \text{for} \quad t \in [0,T], \ \varepsilon \in (0,1],$$

it follows that

$$\mathbf{E}|X_T^{V,\varepsilon} - X_T^V|^2 \le C \int_0^T \left(\mathbf{E}|Y_t^{\varepsilon} - Y_t|^4\right)^{1/2} \mathrm{d}t$$

for $\varepsilon \in (0,1]$ with

$$C^2 = \mathbf{E}e^{4\sup_{t\in[-1,T]}|Y_t|} < \infty$$

due to Fernique's theorem. Finally we have

$$\left(\mathbf{E}|Y_t^{\varepsilon}-Y_t|^4\right)^{1/2}=c\,\mathbf{E}|Y_t^{\varepsilon}-Y_t|^2\leq c\sup_{s\in[t-\varepsilon,t]}\mathbf{E}|Y_t-Y_s|^2,\qquad t\in[0,T],$$

for some constant c > 0 (again by Gaussianity of (Y, Y^{ε}) , respectively the definition of Y^{ε}) and so Lemma 2 implies that

$$\lim_{\varepsilon \to 0} \int_0^T \left(\mathbf{E} |Y_t^{\varepsilon} - Y_t|^4 \right)^{1/2} dt = 0,$$

which shows that

$$\lim_{\varepsilon \to 0} \mathbf{E} |X_T^{V,\varepsilon} - X_T^V|^2 = 0.$$

Consequently, we have

$$\mathbf{E}(X_T^V|\mathscr{G}_n) = \lim_{\varepsilon \to 0} \mathbf{E}(X_T^{V,\varepsilon}|\mathscr{G}_n).$$

For all $\varepsilon > 0$ the map $[0,T] \ni t \mapsto Y_t^{\varepsilon}(\omega) \in \mathbb{R}$ is differentiable, so partial integration gives

$$X_T^{V,arepsilon} =
ho e^{Y_T^{arepsilon}} V_T - rac{
ho}{arepsilon} \int_0^T V_t (Y_t - Y_{t-arepsilon}) e^{Y_t^{arepsilon}} \, \mathrm{d}t$$

and thus

$$\mathbf{E}(X_T^V|\mathscr{G}_n) = \rho V_T e^{Y_T} - \rho \lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \mathbf{E}\left(\int_0^T V_t(Y_t - Y_{t-\varepsilon}) e^{Y_t^{\varepsilon}} dt \left| V_{kT/n}, Y_{kT/n}, k = 0, \dots, n \right.\right),$$

since Y and V are independent of W.

The previous Lemma in particular implies that

$$\mathbf{E}(X_{T}^{W}|\mathscr{G}_{n}) = \sqrt{1 - \rho^{2}} \sum_{l=0}^{n-1} \frac{n}{T} \Delta_{l} W \int_{lT/n}^{(l+1)T/n} \mathbf{E}(e^{Y_{s}} | V_{kT/n}, Y_{kT/n}, k = 0, \dots, n) ds$$

and that $\mathbf{E}(X_T^V|\mathscr{G}_n) = \mathbf{E}(X_T^V|V_{kT/n}, Y_{kT/n}, k = 0, \dots, n)$. Consequently we obtain that

$$\mathbf{E}(\mathbf{E}(X_T^W|\mathcal{G}_n)X_T^V) = \mathbf{E}(\mathbf{E}(X_T^V|\mathcal{G}_n)X_T^W) = \mathbf{E}(\mathbf{E}(X_T^V|\mathcal{G}_n)\mathbf{E}(X_T^W|\mathcal{G}_n)) = 0,$$

since (V, Y) and W are independent. Since moreover $\mathbf{E}(X_T^W X_T^V) = 0$ by independence of W and V it follows that

$$\mathbf{E}(X_T^W - \mathbf{E}(X_T^W | \mathcal{G}_n))(X_T^V - \mathbf{E}(X_T^V | \mathcal{G}_n)) = 0.$$

This yields

$$\mathbf{E} \left| X_T^W + X_T^V - \mathbf{E} (X_T^W + X_T^V | \mathscr{G}_n) \right|^2 = \mathbf{E} \left| X_T^W - \mathbf{E} (X_T^W | \mathscr{G}_n) \right|^2 + \mathbf{E} \left| X_T^V - \mathbf{E} (X_T^V | \mathscr{G}_n) \right|^2$$

$$\geq \mathbf{E} \left| X_T^W - \mathbf{E} (X_T^W | \mathscr{G}_n) \right|^2, \tag{21}$$

i.e. we can establish a lower bound for the minimal error by considering only the Itō integral with respect to W. The optimal approximation of the dV-integral seems to be much harder to analyse due the dependence of Y and V.

After these preparations we can establish our lower error bound:

Theorem 2. In the notation above the following holds

$$\liminf_{n\to\infty} n^{2H} \mathbf{E}(X_T - \mathbf{E}(X_T|\mathcal{G}_n))^2 \ge (1-\rho^2) \frac{2}{(2H+1)(2H+2)} T^{2H+1} e^{2\mu} c_1(1,\theta,R_Y(0)).$$

Proof. First note that

$$\lim_{n\to\infty} n^{2H} \mathbf{E} |X_T^{RS} - \mathbf{E}(X_T^{RS}|\mathscr{G}_n)|^2 = 0$$

by (15). Using this, (21) and $\mathcal{G}_n \subset \mathcal{H}_n$ it follows that

$$\liminf_{n\to\infty} n^{2H} \mathbf{E} |X_T - \mathbf{E}(X_T|\mathscr{G}_n)|^2 \geq \liminf_{n\to\infty} n^{2H} \mathbf{E} |X_T^W - \mathbf{E}(X_T^W|\mathscr{G}_n)|^2 \geq \liminf_{n\to\infty} n^{2H} \mathbf{E} |X_T^W - \mathbf{E}(X_T^W|\mathscr{H}_n)|^2.$$

The Lemmata 5 and 1 imply

$$\mathbf{E}|X_T^W - \mathbf{E}(X_T^W|\mathcal{H}_n)|^2 = (1 - \rho^2) \frac{n^2}{T^2} \sum_{k=0}^{n-1} \int_{kT/n}^{(k+1)T/n} \mathbf{E} \left| \int_{kT/n}^{(k+1)T/n} (e^{Y_t} - e^{Y_s}) ds \right|^2 dt.$$

Since

$$\begin{split} \mathbf{E} \left| \int_{kT/n}^{(k+1)T/n} (e^{Y_t} - e^{Y_s}) \mathrm{d}s \right|^2 \\ &= e^{2\mu} \int_{kT/n}^{(k+1)T/n} \int_{kT/n}^{(k+1)T/n} \mathbf{E}(Z_t^{(1)} - Z_{s_1}^{(1)}) (Z_t^{(1)} - Z_{s_2}^{(1)}) \mathrm{d}s_1 \mathrm{d}s_2 \\ &= e^{2\mu} c_1(1, \theta, R_Y(0)) \int_{kT/n}^{(k+1)T/n} \int_{kT/n}^{(k+1)T/n} \left(|t - s_1|^{2H} + |t - s_2|^{2H} - |s_1 - s_2|^{2H} \right) \mathrm{d}s_1 \mathrm{d}s_2 + o(n^{-2H-2}) \end{split}$$

by Lemma 4, it follows

$$\begin{split} &\frac{1}{c_{1}e^{2\mu}}\int_{kT/n}^{(k+1)T/n}\mathbf{E}\left|\int_{kT/n}^{(k+1)T/n}(e^{Y_{t}}-e^{Y_{s}})\mathrm{d}s\right|^{2}\mathrm{d}t\\ &=\int_{kT/n}^{(k+1)T/n}\int_{kT/n}^{(k+1)T/n}\int_{kT/n}^{(k+1)T/n}\left(|t-s_{1}|^{2H}+|t-s_{2}|^{2H}-|s_{1}-s_{2}|^{2H}\right)\mathrm{d}s_{1}\mathrm{d}s_{2}\mathrm{d}t\\ &+o(n^{-2H-3})\\ &=\int_{0}^{T/n}\int_{0}^{T/n}\int_{0}^{T/n}|s_{1}-s_{2}|^{2H}\mathrm{d}s_{1}\mathrm{d}s_{2}\mathrm{d}t+o(n^{-2H-3})\\ &=\frac{2}{(2H+1)(2H+2)}\Delta^{2H+3}+o(n^{-2H-3}), \end{split}$$

and summing up yields the assertion.

5. Joint Simulation of fOUp and Brownian motion

Set $Y_t^c = Y_t - \mu$, $t \in \mathbb{R}$. Since integration by parts (9) gives

$$Y_t^c = \theta \left(B_t - \lambda e^{-\lambda t} \int_{-\infty}^t e^{\lambda s} B_u du \right),$$

it follows that

$$\mathbf{E}Y_t^c V_s = \theta \left(\gamma(t,s) - \lambda e^{-\lambda t} \int_0^t e^{\lambda u} \gamma(u,s) du \right), \qquad s,t \ge 0$$

In particular we have

$$\mathbf{E}Y_t^c(V_{s_2} - V_{s_1}) = \theta\left(\left(\gamma(t, s_2) - \gamma(t, s_1)\right) - \lambda e^{-\lambda t} \int_0^t e^{\lambda u} (\gamma(u, s_2) - \gamma(u, s_1)) du\right). \tag{22}$$

Hence for a given γ the covariance matrix $C \in \mathbb{R}^{2n+1,2n+1}$ of

$$(Y^c, \Delta V)_{\mathsf{disc}} = (Y_0^c, Y_{T/n}^c, \dots, Y_T^c, \Delta_0 V, \Delta_1 V, \dots, \Delta_{n-1} V),$$

can be computed explicitly and has the form

$$C = \begin{pmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{pmatrix} = \begin{pmatrix} (\mathbf{E}Y_{iT/n}^c Y_{jT/n}^c)_{i,j=0,\dots,n} & (\mathbf{E}Y_{iT/n}^c \Delta_j V)_{i=0,\dots,n,j=0,\dots,n-1} \\ (\mathbf{E}\Delta_i V Y_{jT/n}^c)_{i=0,\dots,n-1,j=0,\dots,n} & (\mathbf{E}\Delta_i V \Delta_j V)_{i,j=0,\dots,n-1} \end{pmatrix}.$$

Considering the increments rather than point evaluations of V has the advantage that

$$C_{22} = (\mathbf{E}\Delta_i V \Delta_j V)_{i,j=0,\dots,n-1} = \Delta \cdot I_n,$$

where I_n is the *n*-dimensional identity matrix. Additionally, by (11) and (22) we have that

$$\mathbf{E}Y_{iT/n}^{c}\Delta_{j}V=0 \quad \text{for} \quad i\leq j,$$

and therefore

$$C_{12} = (\mathbf{E}Y_{iT/n}^c \Delta_j V)_{i=0,\dots,n,j=0,\dots,n-1} = C'_{21}$$

is a lower triangular matrix. Consequently, C is a banded matrix.

We have

$$(Y^c, \Delta V)_{\mathsf{disc}}' \stackrel{\mathsf{law}}{=} L \left(egin{array}{c} \xi_1 \ \xi_2 \ dots \ \xi_{2n+1} \end{array}
ight),$$

where $L \in \mathbb{R}^{2n+1,2n+1}$ is the lower triangular matrix, which arises from the Cholesky decomposition of C, i.e. C = LL', and ξ_i , $i = 1, \ldots, 2n+1$, are iid standard Gaussian random variables. After precomputation of the matrix L, the computational cost (number of standard normal random numbers and arithmetic operations) to generate a sample of $(Y^c, \Delta V)_{\text{disc}}$ is $O(n^2)$.

Due to its stationarity the random vector $(Y_0^c, Y_{T/n}^c, \dots, Y_T^c)$ alone can be sampled with a computational cost of $O(n\log(n))$ using the Davis-Harte algorithm, see e.g. Davies and Harte (1987). This method relies on the fact that after embedding C_{11} in a circulant matrix C^{ce} of size $m = 2^{\lceil \log_2(n+1) \rceil + 1}$ this matrix can be decomposed as

$$C^{ce} = Q\Lambda Q^*$$

where Λ is the diagonal matrix of eigenvectors of C^{ce} , and Q is the unitary matrix given by

$$Q_{j,k} = \frac{1}{\sqrt{m}} \exp\left(-2i\pi \frac{jk}{m}\right), \qquad j,k = 0,...,m-1.$$

Here Q^* denotes the adjoint of Q, i.e. $(Q^*)_{j,k}$ is the complex conjugate of $Q_{k,j}$. Moreover the eigenvalues of C^{ce} are

$$\lambda_k = \sum_{j=0}^{m-1} C_{j,1}^{ce} \exp\left(2i\pi \frac{jk}{m}\right), \qquad k = 0, ..., m-1.$$

Thus, we have

$$Q\Lambda^{1/2}Q^*\xi \stackrel{\text{law}}{=} \begin{pmatrix} Y_0^c \\ Y_{T/n}^c \\ \vdots \\ Y_{(m-1)T/n}^c \end{pmatrix}, \tag{23}$$

where ξ is a vector of m independent standard normal random variables. The actual computation of the left hand side of the previous equation is carried out by using fast Fourier transformation, which leads to the computational cost of $O(n\log(n))$.

Since

$$\sqrt{\Delta}QI_{n}Q^{*}\xi = \sqrt{\Delta}\xi \stackrel{\text{law}}{=} \begin{pmatrix} \Delta_{0}V \\ \Delta_{1}V \\ \vdots \\ \Delta_{m-1}V, \end{pmatrix}, \tag{24}$$

it might be tempting to use

$$(Y^c, \Delta V)'_{\sf disc} \stackrel{\sf law}{=} \left(egin{array}{c} Q \Lambda^{1/2} Q^* \xi \ \sqrt{\Delta} \xi \end{array}
ight),$$

which would have a computational cost of $O(n \log(n))$. However, this sampling procedure yields the covariance structure

$$C_{12} = \sqrt{\Delta}Q\Lambda^{1/2}Q^*$$

which neither incorporates the covariance structure γ nor is an upper triangular matrix, which is required by the adaptedness assumption for B, V, W.

6. Appendix: Covariance function of the fOUp

Recall that $Y_t^c = Y_t - \mu$, $t \in \mathbb{R}$. By stationarity, i.e. (13), we only have to compute $\mathbf{E} Y_0^c Y_t^c$, $t \ge 0$. (i) First note that

$$\mathbf{E}Y_0^c Y_t^c = \mathbf{E}Y_0^c (Y_t^c - Y_0^c) + \mathbf{E}|Y_0^c|^2.$$

and moreover

$$\mathbf{E}|Y_0^c|^2 = \theta^2 \frac{\Gamma(2H+1)}{2\lambda^{2H}},\tag{25}$$

see e.g. Gatheral et al. (2014).

(ii) So it remains to consider $\mathbf{E}Y_0^c(Y_t^c-Y_0^c)$. Since

$$Y_t^c - Y_0^c = \theta \exp(-\lambda t) \int_0^t \exp(\lambda \tau) dB_\tau + (\exp(-\lambda t) - 1) Y_0^c$$

the fractional Itō isometry (10) now gives

$$\begin{split} \mathbf{E} Y_0^c(Y_t^c - Y_0^c) &= \theta^2 H(2H - 1) \exp(-\lambda t) \int_0^t \int_{-\infty}^0 \exp(\lambda(\tau_1 + \tau_2)) |\tau_1 - \tau_2|^{2H - 2} \mathrm{d}\tau_1 \mathrm{d}\tau_2 \\ &+ \theta^2 (\exp(-\lambda t) - 1) \frac{\Gamma(2H + 1)}{2\lambda^{2H}}. \end{split}$$

Hence, using step (i), we have

$$\mathbf{E}Y_{0}^{c}Y_{t}^{c} = \theta^{2}H(2H - 1)\exp(-\lambda t)\int_{0}^{t}\int_{-\infty}^{0}\exp(\lambda(\tau_{1} + \tau_{2}))|\tau_{1} - \tau_{2}|^{2H - 2}d\tau_{1}d\tau_{2}$$

$$+ \theta^{2}\exp(-\lambda t)\frac{\Gamma(2H + 1)}{2\lambda^{2H}}.$$
(26)

(iii) Using $\tau_1 = u + \tau_2$ we have

$$\int_0^t \int_{-\infty}^0 \exp(\lambda(\tau_1 + \tau_2)) |\tau_1 - \tau_2|^{2H - 2} d\tau_1 d\tau_2 = \int_0^t \int_{-\infty}^{-\tau_2} \exp(\lambda(u + 2\tau_2)) |u|^{2H - 2} du d\tau_2.$$

Exchanging the order of integration and using the transformation $u \mapsto -u$ we obtain

$$\begin{split} \int_{0}^{t} \int_{-\infty}^{-\tau_{2}} \exp(\lambda(u+2\tau_{2}))|u|^{2H-2} du d\tau_{2} \\ &= \int_{-\infty}^{-t} \int_{0}^{t} \exp(\lambda(u+2\tau_{2}))|u|^{2H-2} d\tau_{2} du \\ &+ \int_{-t}^{0} \int_{0}^{-u} \exp(\lambda(u+2\tau_{2}))|u|^{2H-2} d\tau_{2} du \\ &= \frac{1}{2\lambda} \left(\exp(2\lambda t) - 1 \right) \int_{t}^{\infty} \exp(-\lambda u)|u|^{2H-2} du \\ &+ \frac{1}{2\lambda} \int_{0}^{t} \left(\exp(\lambda u) - \exp(-\lambda u) \right) |u|^{2H-2} du \end{split}$$

and therefore

$$\begin{split} \exp(-\lambda t) \int_0^t \int_{-\infty}^0 & \exp(\lambda(\tau_1 + \tau_2)) |\tau_1 - \tau_2|^{2H - 2} \mathrm{d}\tau_1 \mathrm{d}\tau_2 \\ &= \frac{1}{\lambda} \sinh(\lambda t) \int_t^\infty \exp(-\lambda u) |u|^{2H - 2} \mathrm{d}u \\ &+ \frac{1}{\lambda} \exp(-\lambda t) \int_0^t \sinh(\lambda u) |u|^{2H - 2} \mathrm{d}u \end{split}$$

Integration by parts gives

$$\frac{2H-1}{\lambda}\sinh(\lambda t)\int_{t}^{\infty}\exp(-\lambda u)|u|^{2H-2}du$$

$$=\sinh(\lambda t)\int_{t}^{\infty}\exp(-\lambda u)|u|^{2H-1}du-\frac{\sinh(\lambda t)\exp(-\lambda t)}{\lambda}t^{2H-1}$$

and

$$\begin{split} \frac{2H-1}{\lambda} \exp(-\lambda t) \int_0^t \sinh(\lambda u) |u|^{2H-2} \mathrm{d}u \\ &= \frac{1}{\lambda} \exp(-\lambda t) \sinh(\lambda t) t^{2H-1} \\ &- \exp(-\lambda t) \int_0^t \cosh(\lambda u) |u|^{2H-1} \, \mathrm{d}u, \end{split}$$

since H < 1/2. Thus, we have

$$\begin{split} H(2H-1)\exp(-\lambda t)\int_0^t \int_{-\infty}^0 \exp(\lambda(\tau_1+\tau_2))|\tau_1-\tau_2|^{2H-2}\mathrm{d}\tau_1\mathrm{d}\tau_2 \\ &= H\sinh(\lambda t)\int_t^\infty \exp(-\lambda u)|u|^{2H-1}\mathrm{d}u - H\exp(-\lambda t)\int_0^t \cosh(\lambda u)|u|^{2H-1}\mathrm{d}u. \end{split}$$

So, using

$$\int_0^\infty t^b \exp(-at) \, \mathrm{d}t = \frac{\Gamma(b+1)}{a^{b+1}} \tag{27}$$

for b > -1, a > 0 it follows

$$\begin{split} H(2H-1)\exp(-\lambda t)\int_0^t \int_{-\infty}^0 \exp(\lambda(\tau_1+\tau_2))|\tau_1-\tau_2|^{2H-2} \mathrm{d}\tau_1 \mathrm{d}\tau_2 \\ = \sinh(\lambda t) \left(\frac{\Gamma(2H+1)}{2\lambda^{2H}} - H \int_0^t \exp(-\lambda u)|u|^{2H-1} \mathrm{d}u\right) \\ - H \exp(-\lambda t) \int_0^t \cosh(\lambda u)|u|^{2H-1} \mathrm{d}u. \end{split}$$

Plugging this into (26) we have derived that

$$\mathbf{E}Y_0^c Y_t^c = \theta^2 \cosh(\lambda t) \frac{\Gamma(2H+1)}{2\lambda^{2H}} - H\theta^2 \int_0^t \cosh(\lambda (t-u)) |u|^{2H-1} du. \tag{28}$$

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