Adaptive sparse grids and quasi Monte Carlo for option pricing under the rough Bergomi model

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PhD Proposal Defense
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Outline

1 Part I: Adaptive Sparse Grids (SG) for Option Pricing
- Introduction
- Option pricing under the rough Bergomi model (Article 1)

Numerical integration methods

- **Plain Monte Carlo (MC)**
  - $\varepsilon(M) = \mathcal{O}\left(M^{-1/2}\right)$
  - (+) insensitive to d, (−) slow convergence, no profit from regularity.
- **Classical Quasi-Monte Carlo (QMC)**
  - $\varepsilon(M) = \mathcal{O}\left(M^{-1}\log(M)^{d-1}\right)$
  - (+) better convergence, (−) sensitive to d, no profit from regularity.
- **Quadrature based on product approaches**
  - $\varepsilon(M) = \mathcal{O}\left(M^{-r/d}\right)$
  - (+) profits from regularity, faster than QMC if $r > d$, (−) highly sensitive to d.
- **Sparse grids quadrature (SGQ)**
  - $\varepsilon(M) = \mathcal{O}\left(M^{-s}\log(M)^{(d-1)(s+1)}\right)$
  - (+) profits from regularity, faster than QMC if $s > 1$, less sensitive to d.

ε: prescribed accuracy, M: the amount of work, d: dimension of problem, r, s: smoothness indices (bounded mixed (total) derivatives up to order $s(r)$).

Motivation

In quantitative finance, the integration problem is usually challenging

- Issue 1: S often takes values in a high-dimensional space $\Rightarrow$ Curse of dimensionality when using numerical integration methods.
  - Case 1: Time-discretization of a stochastic differential equation (large N (number of time steps)).
  - Case 2: A large number of underlying assets (large d).
- Issue 2: The payoff function g is typically not smooth $\Rightarrow$ low regularity (small s) $\Rightarrow$ slow convergence of **SGQ**.

⚠ Curse of dimensionality: An integration error of order ε requires M function evaluations

$$M \geq c_{\varepsilon} \bar{d}^{-c \log \varepsilon},$$

where $\bar{d}$ depends on d and N.

Publications Plan

- **Article 1**: Bayer, C., Hammouda, C.B. and Tempone, R., 2018. **Hierarchical adaptive Sparse grids for option pricing under the rough Bergomi model**. arXiv preprint arXiv:1812.08533. ✓

Sparse Grids I

Goal: Given $F : \mathbb{R}^d \to \mathbb{R}$ and a multi-index $\boldsymbol{\beta} \in \mathbb{N}_+^d$, approximate

$$\mathrm{E}\left[F\right] \approx Q^{m(\boldsymbol{\beta})}[F],$$

where $Q^{m(\boldsymbol{\beta})}$ a Cartesian quadrature grid with $m(\beta_n)$ points along y_n.
Idea: Denote $Q^{m(\boldsymbol{\beta})}[F] = F_{\boldsymbol{\beta}}$ and introduce the first difference

$$\Delta_i F_{\boldsymbol{\beta}} \begin{cases} F_{\boldsymbol{\beta}} - F_{\boldsymbol{\beta}-e_i}, & \text{if } \beta_i > 1 \\ F_{\boldsymbol{\beta}} & \text{if } \beta_i = 1 \end{cases} \tag{1}$$

where e_i denotes the ith d-dimensional unit vector, and mixed difference operators

$$\Delta[F_{\boldsymbol{\beta}}] = \otimes_{i=1}^{d} \Delta_i F_{\boldsymbol{\beta}} \tag{2}$$

Sparse Grids II

A quadrature estimate of $E[F]$ is

$$\mathcal{M}_{\mathcal{I}_\ell}[F] = \sum_{\boldsymbol{\beta} \in \mathcal{I}_\ell} \Delta[F_{\boldsymbol{\beta}}], \tag{3}$$

- Product approach: $\mathcal{I}_\ell = \{\max\{\beta_1, \ldots, \beta_d\} \le \ell;\ \boldsymbol{\beta} \in \mathbb{N}_+^d\}$
- Regular SG: $\mathcal{I}_\ell = \{\mid \boldsymbol{\beta} \mid_1 \le \ell + d - 1;\ \boldsymbol{\beta} \in \mathbb{N}_+^d\}$

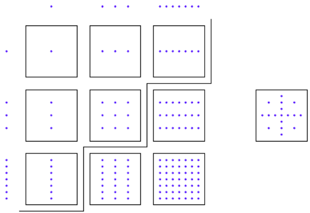


Figure 1.1: Left are product grids $\Delta_{\beta_1} \otimes \Delta_{\beta_2}$ for $1 \le \beta_1, \beta_2 \le 3$. Right is the corresponding SG construction.

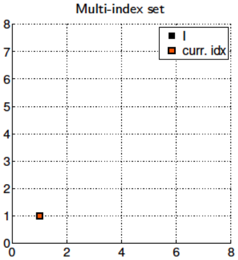
- ASGQ: $\mathcal{I}_\ell = \mathcal{I}^{\mathrm{ASGQ}}$.

ASGQ in Practice

- The construction of $\mathcal{I}^{\text{ASGQ}}$ is done by profit thresholding

$$\mathcal{I}^{\text{ASGQ}} = \{\boldsymbol{\beta} \in \mathbb{N}_+^d : P_{\boldsymbol{\beta}} \geq \overline{T}\}.$$

- A posteriori, adaptive construction: Given an index set $\mathcal{I}_k$, compute the profits of the neighbor indices and select the most profitable one



- **Profit of a hierarchical surplus** $P_{\boldsymbol{\beta}} = \frac{|\Delta E_{\boldsymbol{\beta}}|}{\Delta \mathcal{W}_{\boldsymbol{\beta}}}$.
- **Error contribution**: $\Delta E_{\boldsymbol{\beta}} = \left|\mathcal{M}^{\mathcal{I} \cup \{\boldsymbol{\beta}\}} - \mathcal{M}^{\mathcal{I}}\right|$.
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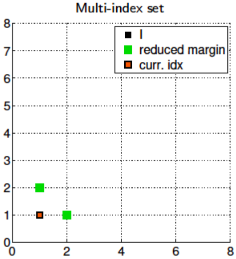

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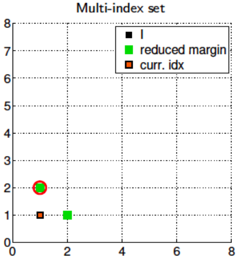

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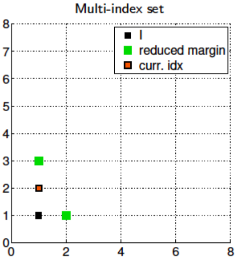

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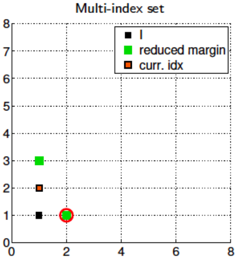

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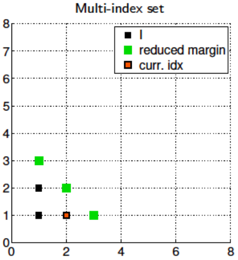

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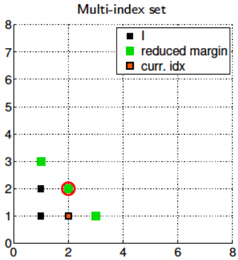

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Path Generation Methods

$\{t_i\}_{i=0}^N$: Grid of time steps, $\{B_{t_i}\}_{i=0}^N$: Brownian motion increments

- **Random Walk**:
  - Proceeds incrementally, given B_{t_i},

$$B_{t_{i+1}} = B_{t_i} + \sqrt{\Delta t} z_i, \; z_i \sim \mathcal{N}(0,1).$$

  - All components of $\mathbf{z} = (z_1, \ldots, z_N)$ have the same scale of importance: isotropic.
- **Hierarchical Brownian Bridge** [Ciesielski, 1961]:
  - Given a past value B_{t_i} and a future value B_{t_k}, the value B_{t_j} (with $t_i < t_j < t_k$) can be generated according to ($\rho = \frac{j-i}{k-i}$)

$$B_{t_j} = (1-\rho) B_{t_i} + \rho B_{t_k} + \sqrt{\rho(1-\rho)(k-i)\Delta t} z_j, \; z_j \sim \mathcal{N}(0,1). \quad (4)$$

  - The most important values (determine the large scale structure of Brownian motion) are the first components of $\mathbf{z} = (z_1, \ldots, z_N)$.
  - $\searrow$ the effective dimension (# important dimensions) and $\nearrow$ anisotropy between different directions $\Rightarrow$ Faster ASGQ convergence.

1. Part I: Adaptive Sparse Grids (SG) for Option Pricing
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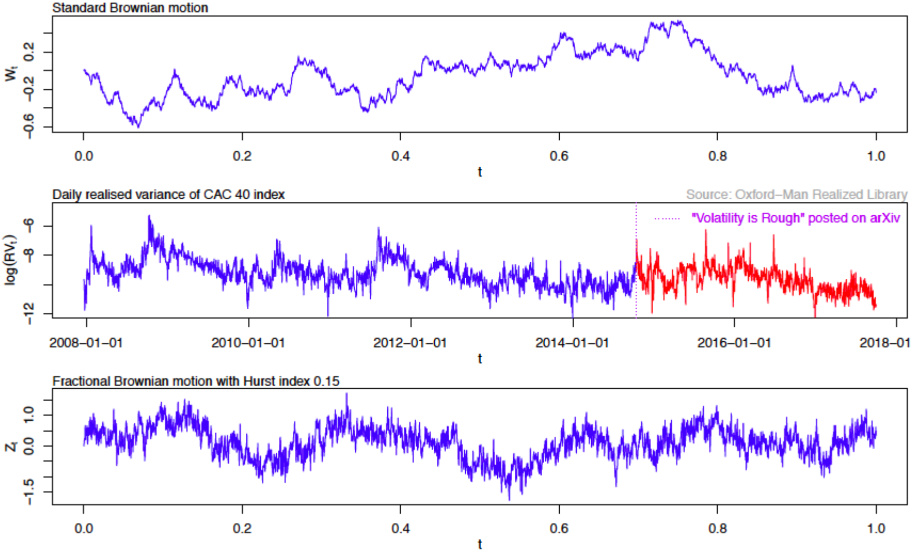
Rough volatility



The rough Bergomi model [Bayer et al., 2016]

This model, under a pricing measure, is given by

$$\begin{cases} dS_t &= \sqrt{v_t} S_t dZ_t, \\ v_t &= \xi_0(t) \exp\left(\eta \widetilde{W}_t^H - \frac{1}{2}\eta^2 t^{2H} \right), \\ Z_t &:= \rho W_t^1 + \bar{\rho} W_t^{\perp} \equiv \rho W^1 + \sqrt{1-\rho^2} W^{\perp}, \end{cases} \tag{5}$$

- $(W^1, W^{\perp})$: two independent standard Brownian motions
- $\widetilde{W}^H$ is Riemann-Liouville process, defined by

$$\widetilde{W}_t^H = \int_0^t K^H(t-s) dW_s^1, \quad t \geq 0,$$
$$K^H(t-s) = \sqrt{2H}(t-s)^{H-1/2}, \quad \forall\, 0 \leq s \leq t.$$

- $H \in (0, 1/2]$ ($H = 1/2$ for Brownian motion): controls the roughness of paths, , $\rho \in [-1, 1]$ and $\eta > 0$.
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Challenges

- **Numerically:**
  - The model is non-affine and non-Markovian $\Rightarrow$ Standard numerical methods (PDEs, characteristic functions) seem inapplicable.
  - The only prevalent pricing method for mere vanilla options is Monte Carlo [Bayer et al., 2016, Bayer et al., 2017, McCrickerd and Pakkanen, 2017], still a time consuming task.
  - Discretization methods have poor behavior of the strong error, that is the convergence rate is of order of $H \in [0, 1/2]$ [Neuenkirch and Shalaiko, 2016] $\Rightarrow$ Variance reduction methods, such as MLMC, are inefficient for very small values of H.
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   i) Analytic smoothing to uncover available regularity.
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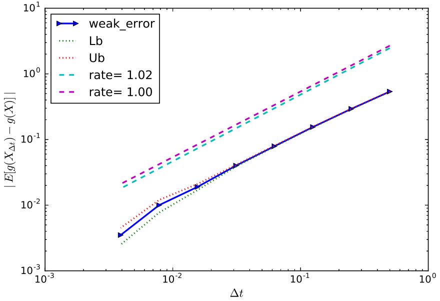
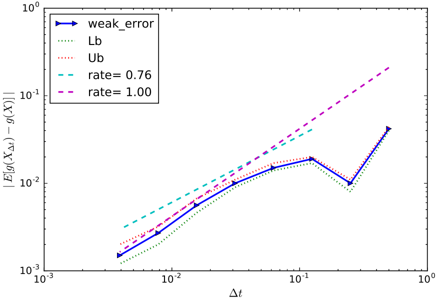
On the Choice of the Simulation Scheme



(a)



(b)

Figure 2.1: The convergence of the weak error $\mathcal{E}_B$, using MC with 6×10^6 samples, for Set 1 parameter in Table 1. The upper and lower bounds are 95% confidence intervals. a) With the hybrid scheme b) With the exact scheme.

Hybrid Scheme [Bennedsen et al., 2017]

$$\widetilde{W}_t^{H} = \int_0^t K^{H}(t-s)dW_s^1, \quad t \geq 0,$$
$$K^{H}(t-s) = \sqrt{2H}(t-s)^{H-1/2}, \quad \forall\, 0 \leq s \leq t.$$

- The hybrid scheme discretizes the $\widetilde{W}^{H}$ process into Wiener integrals of power functions and a Riemann sum, appearing from approximating the kernel by power functions near the origin and step functions elsewhere.

$$\widetilde{W}_{\frac{i}{N}}^{H} \approx \overline{W}_{\frac{i}{N}}^{H} = \sqrt{2H}\left(W_i^2 + \sum_{k=2}^{i}\left(\frac{b_k}{N}\right)^{H-\frac{1}{2}}\left(W_{\frac{i-(k-1)}{N}}^1 - W_{\frac{i-k}{N}}^1\right)\right),$$

  - N is the number of time steps
  - $\{W_j^2\}_{j=1}^N$: Artificially introduced N Gaussian random variables that are used for left-rule points in the hybrid scheme.
  - $b_k = \left(\frac{k^{H+\frac{1}{2}} - (k-1)^{H+\frac{1}{2}}}{H+\frac{1}{2}}\right)^{\frac{1}{H-\frac{1}{2}}}$.

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The rough Bergomi Model: Analytic Smoothing

We show that the call price is given by

$$\begin{aligned}
C_{RB}(T,K) &= E\left[(S_T-K)^+\right] \\
&= \mathrm{E}\left[\mathrm{E}\left[(S_T-K)^+ \mid \sigma(W^1(t), t \le T)\right]\right] \\
&= E\left[C_{BS}\left(S_0 = \exp\left(\rho\int_0^T \sqrt{v_t}\,dW_t^1 - \frac{1}{2}\rho^2\int_0^T v_t dt\right),\right.\right. \\
&\qquad\qquad \left.\left. k = K,\ \sigma^2 = (1-\rho^2)\int_0^T v_t dt\right)\right] \\
&\approx \int_{\mathbb{R}^{2N}} C_{BS}\left(G(\mathbf{w}^{(1)}, \mathbf{w}^{(2)})\right)\rho_N(\mathbf{w}^{(1)})\rho_N(\mathbf{w}^{(2)})d\mathbf{w}^{(1)}d\mathbf{w}^{(2)}, \qquad (7)
\end{aligned}$$

where

- $C_{\mathrm{BS}}(S_0, k, \sigma^2)$ denotes the Black-Scholes call price, for initial spot price S_0, strike price k, and volatility σ^2.
- ρ_N: the multivariate Gaussian density, N: number of time steps.

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&= C_{RB}^N, \qquad (8)
\end{aligned}
$$

- $C_{\mathrm{BS}}(S_0, k, \sigma^2)$ denotes the Black-Scholes call price, for initial spot price S_0, strike price k, and volatility σ^2.
- G maps $2N$ independent standard Gaussian random inputs to the parameters fed to Black-Scholes formula.
- ρ_N: the multivariate Gaussian density, N: number of time steps.

Numerical Experiments

$\mathcal{E}_{\text{tot}}$: the total error of approximating the expectation in (8)

- When using ASGQ estimator, Q_N

$$\mathcal{E}_{\text{tot}} \leq \left|C_{\text{RB}} - C_{\text{RB}}^N\right| + \left|C_{\text{RB}}^N - Q_N\right| \leq \mathcal{E}_B(N) + \mathcal{E}_Q(\text{TOL}_{\text{ASGQ}}, N), \tag{9}$$

where $\mathcal{E}_Q$ is the quadrature error, $\mathcal{E}_B$ is the bias, TOL_{ASGQ} is a user selected tolerance for ASGQ method.

- When using randomized QMC or MC estimator, $Q_N^{\text{MC (QMC)}}$

$$\mathcal{E}_{\text{tot}} \leq \left|C_{\text{RB}} - C_{\text{RB}}^N\right| + \left|C_{\text{RB}}^N - Q_N^{\text{MC (QMC)}}\right| \leq \mathcal{E}_B(N) + \mathcal{E}_S(M, N), \tag{10}$$

where $\mathcal{E}_S$ is the statistical error, M is the number of samples used for MC or randomized QMC method.

- The number of samples, M^{QMC} and M^{MC}, are chosen so that the statistical errors of QMC, $\mathcal{E}_{S,\text{QMC}}(M^{\text{QMC}})$, and MC, $\mathcal{E}_{S,\text{MC}}(M^{\text{MC}})$, satisfy

$$\mathcal{E}_{S,\text{QMC}}(M^{\text{QMC}}) = \mathcal{E}_{S,\text{MC}}(M^{\text{MC}}) = \mathcal{E}_B(N) = \frac{\mathcal{E}_{\text{tot}}}{2}, \tag{11}$$

Table 1: Reference solution, using MC with 500 time steps and number of samples, $M = 8 \times 10^6$, of call option price under the rough Bergomi model, for different parameter constellations. The numbers between parentheses correspond to the statistical errors estimates.

Parameters	Referen
Set 1: $H = 0.07, K = 1, S_0 = 1, T = 1, \rho = -0.9, \eta = 1.9, \xi_0 = 0.235^2$	0 (5
Set 2: $H = 0.02, K = 1, S_0 = 1, T = 1, \rho = -0.7, \eta = 0.4, \xi_0 = 0.1$	0 (9
Set 3: $H = 0.02, K = 0.8, S_0 = 1, T = 1, \rho = -0.7, \eta = 0.4, \xi_0 = 0.1$	0 (5
Set 4: $H = 0.02, K = 1.2, S_0 = 1, T = 1, \rho = -0.7, \eta = 0.4, \xi_0 = 0.1$	0 (8

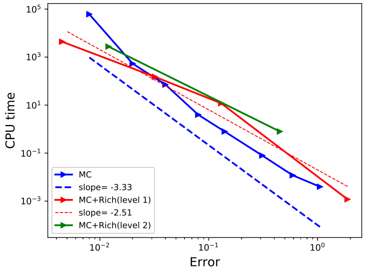
- The first set is the closest to the empirical findings [**?**, Bennedsen et al., 2016], suggesting that $H \approx 0.1$. The choice of values $\nu = 1.9$ and $\rho = -0.9$ is justified by [Bayer et al., 2016].
- For the remaining three sets, we wanted to test the potential of our method for a very rough case, where variance reduction methods are inefficient.

lative Errors and Computational Gains of the Different Me

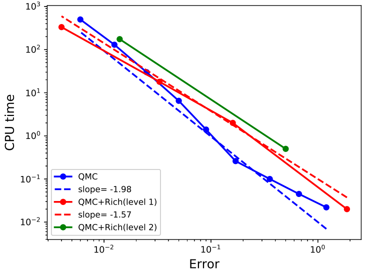
Table 2: In this table, we highlight the computational gains achieved by ASGQ and QMC over MC method to meet a certain error tolerance. We note that the ratios are computed for the best configuration with Richardson extrapolation for each method.

Parameter set	**Total relative error**	**CPU time ratio (MC/ASGQ)**
Set 1	1%	15
Set 2	0.2%	21.5
Set 3	0.4%	26.7
Set 4	2%	5

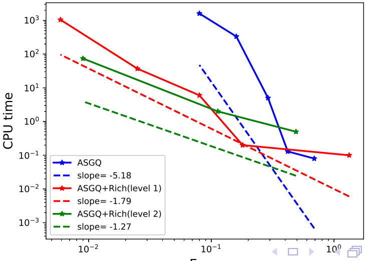
plexity of the Different Methods with the Different Configu



(a)



(b)



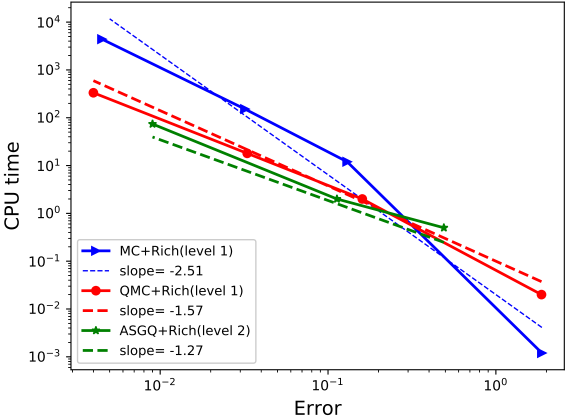


Figure 2.3: Computational work comparison for the different methods with the best configurations concluded from Figure 2.2, for the case of parameter set 1 in Table 1.

Thank you for your attention

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