

# Smoothing the Payoff for Efficient Computation of Option Pricing in Time-Stepping Setting

## 1 Motivation

To motivate our purposes, we consider the basket option under multi-dimensional GBM model where the process  $\mathbf{X}$  is the discretized  $d$ -dimensional Black-Scholes model and the payoff function  $g$  is given by

$$(1.1) \quad g(\mathbf{X}(T)) = \max \left( \sum_{j=1}^d c_j X^{(j)}(T) - K, 0 \right).$$

Precisely, we are interested in the  $d$ -dimensional lognormal example where the dynamics of the stock are given by

$$(1.2) \quad dX_t^{(j)} = \sigma^{(j)} X_t^{(j)} dB_t^{(j)},$$

where  $\{B^{(1)}, \dots, B^{(d)}\}$  are correlated Brownian motions with correlations  $\rho_{ij}$ .

We denote by  $(z_1^{(j)}, \dots, z_N^{(j)})$  the  $N$  Gaussian independent rdvs that will be used to construct the path of the  $j$ -th asset  $\bar{X}^{(j)}$ , where  $1 \leq j \leq d$  ( $d$  denotes the number of underlyings considered in the basket). We keep the same notations by denoting  $\psi : (z_1^{(j)}, \dots, z_N^{(j)}) \rightarrow (B_1, \dots, B_N)$  the mapping of Brownian bridge construction and by  $\Phi : (\Delta t, B_1^{(j)}, \dots, B_N^{(j)}) \rightarrow \bar{X}_T^{(j)}$ , the mapping consisting of the time-stepping scheme. Then, we can express the option price as

$$(1.3) \quad \begin{aligned} \mathbb{E}[g(\mathbf{X}(T))] &\approx \mathbb{E} \left[ g(\Phi \circ \psi)(z_1^{(1)}, \dots, z_N^{(1)}, \dots, z_1^{(d)}, \dots, z_N^{(d)}) \right] \\ &= \int_{\mathbb{R}^{d \times N}} G(z_1^{(1)}, \dots, z_N^{(1)}, \dots, z_1^{(d)}, \dots, z_N^{(d)}) \rho_{d \times N}(\mathbf{z}) dz_1^{(1)} \dots dz_N^{(1)} \dots dz_1^{(d)} \dots dz_N^{(d)}, \end{aligned}$$

where  $G = g \circ \Phi \circ \psi$  and

$$\rho_{d \times N}(\mathbf{z}) = \frac{1}{(2\pi)^{d \times N/2}} e^{-\frac{1}{2} \mathbf{z}^T \mathbf{z}}.$$

In the discrete case, the numerical approximation of  $X^{(j)}(T)$  satisfies

$$(1.4) \quad \begin{aligned} \bar{X}_T^{(j)} &= \Phi(\Delta t, z_1^{(j)}, \Delta B_0^{(j)}, \dots, \Delta B_{N-1}^{(j)}), \quad 1 \leq j \leq d, \\ &= \Phi(\Delta t, \psi(z_1^{(j)}, \dots, z_N^{(j)})), \quad 1 \leq j \leq d, \end{aligned}$$

and precisely, we have

$$\begin{aligned}
\overline{X}^{(j)}(T) &= X_0^{(j)} \prod_{i=0}^{N-1} \left[ 1 + \frac{\sigma^{(j)}}{\sqrt{T}} z_1^{(j)} \Delta t + \sigma^{(j)} \Delta B_i^{(j)} \right], \quad 1 \leq j \leq d \\
(1.5) \quad &= \prod_{i=0}^{N-1} f_i^{(j)}(z_1^{(j)}), \quad 1 \leq j \leq d.
\end{aligned}$$

### 1.1 Step 1: Numerical smoothing

The first step of our idea is to smoothen the problem by solving the root finding problem in one dimension after using a sub-optimal linear mapping for the coarsest factors of the Brownian increments  $\mathbf{z}_1 = (z_1^{(1)}, \dots, z_1^{(d)})$ . In fact, let us define for a certain  $\boldsymbol{\alpha} = (\alpha_1, \dots, \alpha_d)$ , the linear transformation

$$\begin{aligned}
\omega &= L(\mathbf{z}_1) \\
(1.6) \quad &= \sum_{i=1}^d \alpha_i z_1^{(i)}
\end{aligned}$$

Then from (1.5), we have

$$(1.7) \quad \overline{X}^{(j)}(T) = \prod_{i=0}^{N-1} g_i^{(j)}(w), \quad 1 \leq j \leq d,$$

where

$$\begin{aligned}
g_i^{(j)}(w) &= X_0^{(j)} \left[ 1 + \frac{\sigma^{(j)}}{\sqrt{T}} \left( \frac{w - \sum_{l=1, l \neq j} \alpha_l z_1^{(l)}}{\alpha_j} \right) \Delta t + \sigma^{(j)} \Delta B_i^{(j)} \right] \\
(1.8) \quad &= X_0^{(j)} \left[ 1 + \frac{\sigma^{(j)} \Delta t}{\alpha_j \sqrt{T}} w - \frac{\sigma^{(j)}}{\sqrt{T}} \left( \frac{\sum_{l=1, l \neq j} \alpha_l z_1^{(l)}}{\alpha_j} \right) \Delta t + \sigma^{(j)} \Delta B_i^{(j)} \right]
\end{aligned}$$

Therefore, in order to determine  $w^*$ , we need to solve

$$(1.9) \quad x = \sum_{j=1}^d c_j \prod_{i=0}^{N-1} g_i^{(j)}(w^*(x)),$$

which implies that the location of the kink point for the approximate problem is equivalent to finding the roots of the polynomial  $P(w_*(K))$ , given by

$$(1.10) \quad P(w^*(K)) = \left( \sum_{j=1}^d c_j \prod_{i=0}^{N-1} g_i^{(j)}(w^*) \right) - K.$$

Using **Newton iteration method**, we use the expression  $P' = \frac{dP}{dw^*}$ , and we can easily show that

$$(1.11) \quad P'(w) = \sum_{j=1}^d c_j \frac{\sigma^{(j)} \Delta}{\alpha_j \sqrt{T}} \left( \prod_{i=0}^{N-1} g_i^{(j)}(w) \right) \left[ \sum_{i=0}^{N-1} \frac{1}{g_i^{(j)}(w)} \right].$$

**Question 2:** One question that arises here: Do we have to optimize over  $\alpha$  to get the optimal linear transformation? If yes what will be the metric to be used to optimize with respect to it? If no, how do we check that choices of  $\alpha$  are good choices, at least not bad chosen directions?

## 1.2 Step 2: Integration

At this stage, we want to perform the pre-integrating step with respect to  $w^*$ . In fact, we have from (1.3)

$$(1.12) \quad \begin{aligned} \mathbb{E}[g(\mathbf{X}(T))] &= \int_{\mathbb{R}^{d \times N}} G(z_1^{(1)}, \dots, z_N^{(1)}, \dots, z_1^{(d)}, \dots, z_N^{(d)}) \rho_{d \times N}(\mathbf{z}) dz_1^{(1)} \dots dz_N^{(1)} \dots dz_1^{(d)} \dots dz_N^{(d)} \\ &= \int_{\mathbb{R}^{d \times (N-1)}} \left( \int_{\mathbb{R}} G(w, \mathbf{z}_{-1}^{(1)}, \dots, \mathbf{z}_{-1}^{(d)}) \rho_w(w) dw \right) \rho_{d \times (N-1)}(\mathbf{z}_{-1}^{(1)}, \dots, \mathbf{z}_{-1}^{(d)}) d\mathbf{z}_{-1}^{(1)} \dots d\mathbf{z}_{-1}^{(d)} \\ &= \int_{\mathbb{R}^{d \times (N-1)}} h(\mathbf{z}_{-1}^{(1)}, \dots, \mathbf{z}_{-1}^{(d)}) \rho_{d \times (N-1)}(\mathbf{z}_{-1}^{(1)}, \dots, \mathbf{z}_{-1}^{(d)}) d\mathbf{z}_{-1}^{(1)} \dots d\mathbf{z}_{-1}^{(d)}, \\ &= \mathbb{E} \left[ h(\mathbf{z}_{-1}^{(1)}, \dots, \mathbf{z}_{-1}^{(d)}) \right], \end{aligned}$$

where  $\rho_w \sim \mathcal{N}(0, \sum_{j=1}^d \alpha_j^2)$  and

$$(1.13) \quad \begin{aligned} h(\mathbf{z}_{-1}^{(1)}, \dots, \mathbf{z}_{-1}^{(d)}) &= \int_{\mathbb{R}} G(w, \mathbf{z}_{-1}^{(1)}, \dots, \mathbf{z}_{-1}^{(d)}) \rho_w(w) dw \\ &= \int_{-\infty}^{w^*} G(w, \mathbf{z}_{-1}^{(1)}, \dots, \mathbf{z}_{-1}^{(d)}) \rho_w(w) dw + \int_{w^*}^{+\infty} G(w, \mathbf{z}_{-1}^{(1)}, \dots, \mathbf{z}_{-1}^{(d)}) \rho_w(w) dw \end{aligned}$$