Smoothing the Payoff for Efficient Computation of Option Pricing in Time-Stepping Setting

1 Motivation

To motivate our purposes, we consider the basket option under multi-dimensional GBM model where the process $\mathbf{X}$ is the discretized d-dimensional Black-Scholes model and the payoff function g is given by

$$g(\mathbf{X}(T)) = \max\left(\sum_{j=1}^{d} c_j X^{(j)}(T) - K, 0\right). \tag{1.1}$$

Precisely, we are interested in the d-dimensional lognormal example where the dynamics of the stock are given by

$$dX_t^{(j)} = \sigma^{(j)} X_t^{(j)} dB_t^{(j)}, \tag{1.2}$$

where $\{B^{(1)}, \dots, B^{(d)}\}$ are correlated Brownian motions with correlations ρ_{ij}.

We denote by $(z_1^{(j)}, \dots, z_N^{(j)})$ the N Gaussian independent rdvs that will be used to construct the path of the j-th asset $\overline{X}^{(j)}$, where $1 \leq j \leq d$ (d denotes the number of underlyings considered in the basket). We denote $\psi : (z_1^{(j)}, \dots, z_N^{(j)}) \rightarrow (B_1, \dots, B_N)$ the mapping of Brownian bridge construction and by $\Phi : (\Delta t, B_1^{(j)}, \dots, B_N^{(j)}) \rightarrow \overline{X}_T^{(j)}$, the mapping consisting of the time-stepping scheme. Then, we can express the option price as

This is unclear! First \psi and \Phi act only on component j, then on all of them! Please clarify the notation!

$$\begin{aligned} \mathrm{E}\left[g(\mathbf{X}(T))\right] &\approx \mathrm{E}\left[g\left(\Phi \circ \psi\right)\left(z_1^{(1)}, \dots, z_N^{(1)}, \dots, z_1^{(d)}, \dots, z_N^{(d)}\right)\right] \\ &= \int_{\mathbb{R}^{d\times N}} G(z_1^{(1)}, \dots, z_N^{(1)}, \dots, z_1^{(d)}, \dots, z_N^{(d)}))\rho_{d\times N}(\mathbf{z}) dz_1^{(1)} \dots dz_N^{(1)} \dots z_1^{(d)} \dots dz_N^{(d)}, \end{aligned} \tag{1.3}$$

where $G = g \circ \Phi \circ \psi$ and

$$\rho_{d\times N}(\mathbf{z}) = \frac{1}{(2\pi)^{d\times N/2}} e^{-\frac{1}{2}\mathbf{z}^T\mathbf{z}}.$$

The Brownian motions appear to be uncorrelated at this point?

In the discrete case, the numerical approximation of $X^{(j)}(T)$ satisfies

$$\begin{aligned} \overline{X}_T^{(j)} &= \Phi(\Delta t, z_1^{(j)}, \Delta B_0^{(j)}, \dots, \Delta B_{N-1}^{(j)}), \quad 1 \leq j \leq d, \\ &= \Phi(\Delta t, \psi(z_1^{(j)}, \dots, z_N^{(j)})), \quad 1 \leq j \leq d, \end{aligned} \tag{1.4}$$

So B denotes the Brownian motion as well as the Brownian bridge?
This is unfortunate, I think.

and precisely, we have

$$\overline{X}^{(j)}(T) = X_0^{(j)} \prod_{i=0}^{N-1} \left[1 + \frac{\sigma^{(j)}}{\sqrt{T}} z_1^{(j)} \Delta t + \sigma^{(j)} \Delta B_i^{(j)}\right], \quad 1 \leq j \leq d$$

$$= \prod_{i=0}^{N-1} f_i^{(j)}(z_1^{(j)}), \quad 1 \leq j \leq d. \tag{1.5}$$

1.1 Step 1: Numerical smoothing

The first step of our idea is to smoothen the problem by solving the root finding problem in one dimension after using a sub-optimal linear mapping for the coarsest factors of the Brownian increments $\mathbf{z}_1 = (z_1^{(1)}, \dots, z_1^{(d)})$. In fact, let us define for a certain $d \times d$ matrix $\mathcal{A}$, the linear mapping

$$\mathbf{W} = \mathcal{A}\mathbf{z}_1. \tag{1.6}$$

The notation is unfortunate, as W usually denotes a Brownian motion. In general, we should distinguish between random vaiiables (captal letters), and deterministic numbers (lowercase letters). E.g., z is lowercase as a real variable, since we use it to do integration against the Gauss kernel. The corresponding r.v. should be denoted by Z.

Then from (1.5), we have

$$\overline{X}^{(j)}(T) = \prod_{i=0}^{N-1} f_i^{(j)}(\mathcal{A}^{-1}\mathbf{W})_j, \quad 1 \leq j \leq d,$$

$$= \prod_{i=0}^{N-1} g_i^{(j)}(W_1, \mathbf{W}_{-1}) \quad 1 \leq j \leq d \tag{1.7}$$

where, with defining $\mathcal{A}^{\text{inv}} = \mathcal{A}^{-1}$, we have

$$g_i^{(j)}(W_1, \mathbf{W}_{-1}) = X_0^{(j)} \left[1 + \frac{\sigma^{(j)}}{\sqrt{T}} \left(\sum_{i=1}^{d} A_{ji}^{\text{inv}} W_i\right) \Delta t + \sigma^{(j)} \Delta B_i^{(j)}\right]$$

$$= X_0^{(j)} \left[1 + \frac{\sigma^{(j)} \Delta t}{\sqrt{T}} A_{j1}^{\text{inv}} W_1 - \frac{\sigma^{(j)}}{\sqrt{T}} \left(\sum_{i=2}^{d} A_{ji}^{\text{inv}} W_i\right) \Delta t + \sigma^{(j)} \Delta B_i^{(j)}\right] \tag{1.8}$$

Therefore, in order to determine W_1^*, we need to solve

$$x = \sum_{j=1}^{d} c_j \prod_{i=0}^{N-1} g_i^{(j)}(W_1^*(x), \mathbf{W}_{-1}), \tag{1.9}$$

which implies that the location of the kink point for the approximate problem is equivalent to finding the roots of the polynomial $P(W_1^*(K))$, given by

$$P(W_1^*(K)) = \left(\sum_{j=1}^{d} c_j \prod_{i=0}^{N-1} g_i^{(j)}(W_1^*)\right) - K. \tag{1.10}$$

Using **Newton iteration method**, we use the expression $P' = \frac{dP}{dW_1^*}$, and we can easily show that

$$P'(W_1) = \sum_{j=1}^{d} c_j \frac{\sigma^{(j)} \Delta t A_{j1}^{\text{inv}}}{\sqrt{T}} \left(\prod_{i=0}^{N-1} g_i^{(j)}(W_1)\right) \left[\sum_{i=0}^{N-1} \frac{1}{g_i^{(j)}(W_1)}\right]. \tag{1.11}$$

Just to be clear, I think that a sufficiently good matrix A would be on which leads to W_1 = \sum_i z_1^{(i)} — up to rescaling.

Remark 1.1. For our purposes, we suggest already that the coarsest factors of the Brownian increments are the most important ones, compared to the remaining factors. However, one may expect that in case we want to optimize over the choice of the linear mapping $\mathcal{A}$, and which direction is the most important for the kink location, one needs then to solve

$$\sup_{\mathcal{A}\in\mathbb{R}^{d\times d}} \left(\max_{1\leq i\leq d} \frac{\partial g}{\partial W_i} \right),$$

which becomes hard to solve when d increases.

Remark 1.2. When choosing the linear mapping $\mathcal{A}$, one needs to make sure by scaling and construction that we end up with $\{W_i\}_{i=1}^d$ being independent standard normal variables.

1.2 Step 2: Integration

At this stage, we want to perform the pre-integrating step with respect to W_1^*. In fact, we have from (1.3)

$$\begin{aligned}
\mathrm{E}\left[g(\mathbf{X}(T))\right] &= \int_{\mathbb{R}^{d\times N}} G(z_1^{(1)},\dots,z_N^{(1)},\dots,z_1^{(d)},\dots,z_N^{(d)}))\rho_{d\times N}(\mathbf{z})dz_1^{(1)}\dots dz_N^{(1)}\dots z_1^{(d)}\dots dz_N^{(d)} \\
&= \int_{\mathbb{R}^{dN-1}} \left(\int_{\mathbb{R}} G(W_1,\mathbf{W}_{-1},\mathbf{z}_{-1}^{(1)},\dots,\mathbf{z}_{-1}^{(d)})\rho_{W_1}(W_1)dW_1 \right) \rho_{d-1}(\mathbf{W}_{-1})d\mathbf{W}_{-1}\rho_{d\times(N-1)}(\mathbf{z}_{-1}^{(1)},\dots,\mathbf{z}_{-1}^{(d)})d\mathbf{z}_{-1}^{(1)}\dots d\mathbf{z}_{-1}^{(d)} \\
&= \int_{\mathbb{R}^{dN-1}} h(\mathbf{W}_{-1},\mathbf{z}_{-1}^{(1)},\dots,\mathbf{z}_{-1}^{(d)})\rho_{d-1}(\mathbf{W}_{-1})d\mathbf{W}_{-1}\rho_{d\times(N-1)}(\mathbf{z}_{-1}^{(1)},\dots,\mathbf{z}_{-1}^{(d)})d\mathbf{z}_{-1}^{(1)}\dots d\mathbf{z}_{-1}^{(d)}, \\
&= \mathrm{E}\left[h(\mathbf{W}_{-1},\mathbf{z}_{-1}^{(1)},\dots,\mathbf{z}_{-1}^{(d)}) \right],
\end{aligned} \tag{1.12}$$

where

$$\begin{aligned}
h(\mathbf{W}_{-1},\mathbf{z}_{-1}^{(1)},\dots,\mathbf{z}_{-1}^{(d)}) &= \int_{\mathbb{R}} G(W_1,\mathbf{W}_{-1},\mathbf{z}_{-1}^{(1)},\dots,\mathbf{z}_{-1}^{(d)})\rho_{W_1}(W_1)dW_1 \\
&= \int_{-\infty}^{W_1^*} G(W_1,\mathbf{W}_{-1},\mathbf{z}_{-1}^{(1)},\dots,\mathbf{z}_{-1}^{(d)})\rho_{W_1}(W_1)dW_1 \\
&+ \int_{W_1^*}^{+\infty} G(W_1,\mathbf{W}_{-1},\mathbf{z}_{-1}^{(1)},\dots,\mathbf{z}_{-1}^{(d)})\rho_{W_1}(W_1)dW_1
\end{aligned} \tag{1.13}$$

2 The best call option case under GBM and Heston model

The second example that we consider for multi-dimension is the best call option under GBM and Heston model. I am in process of formulating this but somehow we agreed that the first potential directions of smoothing for the Heston model will be the first factors related to the asset prices.