Hierarchical adaptive sparse grids for option pricing under the rough Bergomi model

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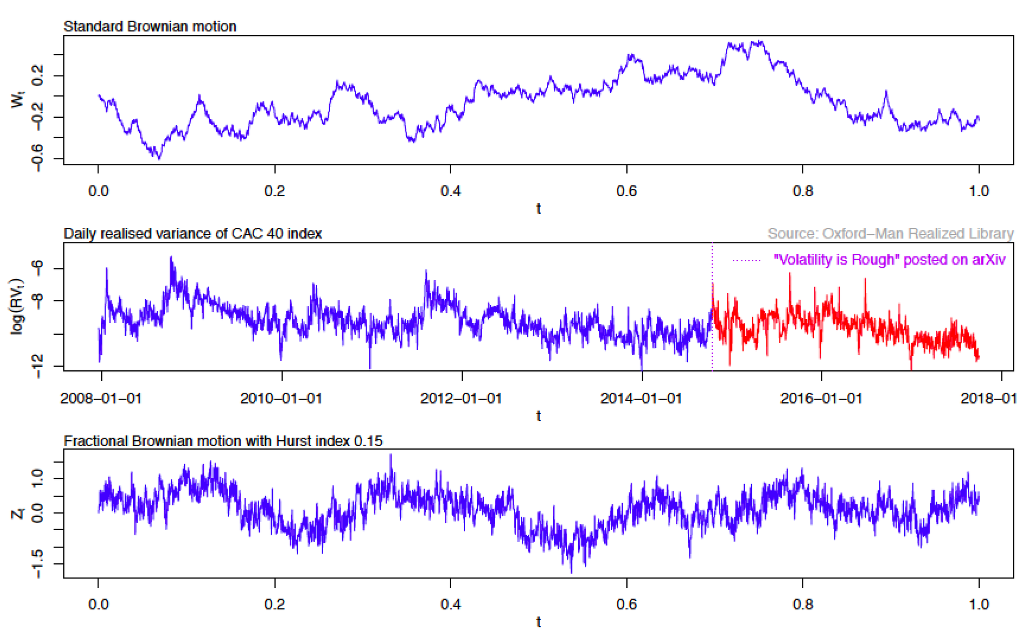
Abstract

The rough Bergomi (rBergomi) model, introduced recently in [1], is a promising rough volatility model in quantitative finance. This new model exhibits consistent results with the empirical fact of implied volatility surfaces being essentially time-invariant. This model also has the ability to capture the term structure of skew observed in equity markets. In the absence of analytical European option pricing methods for the model, and due to the non-Markovian nature of the fractional driver, the prevalent option is to use Monte Carlo (MC) simulation for pricing. Despite recent advances in the MC method in this context, pricing under the rBergomi model is still a time-consuming task. To overcome this issue, we design a novel, alternative, hierarchical approach, based on adaptive sparse grids quadrature (ASGQ), specifically using the same construction in [5], coupled with Brownian bridge construction and Richardson extrapolation. By uncovering the available regularity, our hierarchical method demonstrates substantial computational gains with respect to the standard MC method, when reaching a sufficiently small error tolerance in the price estimates across different parameter constellations, even for very small values of the Hurst parameter. Our work opens a new research direction in this field, i.e. to investigate the performance of methods other than Monte Carlo for pricing and calibrating under the rBergomi model.

Rough volatility



The rough Bergomi model [1]

This model, under a pricing measure, is given by

$$\begin{cases} dS_t &= \sqrt{v_t} S_t dZ_t, \\ v_t &= \xi_0(t) \exp\left(\eta \widetilde{W}_t^H - \frac{1}{2}\eta^2 t^{2H}\right), \\ Z_t &:= \rho W_t^1 + \bar{\rho} W_t^{\perp} \equiv \rho W^1 + \sqrt{1-\rho^2} W^{\perp}, \end{cases} \tag{1}$$

- $(W^1, W^{\perp})$: two independent standard Brownian motions
- $\widetilde{W}^H$ is Riemann-Liouville process, defined by

$$\widetilde{W}_t^H = \int_0^t K^H(t-s) dW_s^1, \quad t \geq 0,$$
$$K^H(t-s) = \sqrt{2H}(t-s)^{H-1/2}, \quad \forall\, 0 \leq s \leq t.$$

- $H \in (0, 1/2]$ ($H = 1/2$ for Brownian motion): controls the roughness of paths, , $\rho \in [-1, 1]$ and $\eta > 0$.
- $t \mapsto \xi_0(t)$: forward variance curve, known at time 0.

Challenges

- **Numerically:**
  - The model is non-affine and non-Markovian $\Rightarrow$ Standard numerical methods (PDEs, characteristic functions) seem inapplicable.
  - The only prevalent pricing method for mere vanilla options is Monte Carlo [1, 2, ?], still a time consuming task.
  - Discretization methods have poor behavior of the strong error, that is the convergence rate is of order of $H \in [0, 1/2]$ [6] $\Rightarrow$ Variance reduction methods, such as MLMC, are inefficient for very small values of H.
- **Theoretically:**
  - No proper weak error analysis done in the rough volatility context.

Contributions

1. We design an alternative hierarchical efficient pricing method based on:
   i) **Analytic smoothing** to uncover available regularity.
   ii) Approximating the option price using **ASGQ** coupled with **Brownian bridges** and **Richardson Extrapolation**.
2. Our hierarchical method demonstrates substantial computational gains with respect to the standard MC method, assuming a sufficiently small error tolerance in the price estimates, even for very small values of the Hurst parameter, H.

The Hybrid Scheme [4]

$$\widetilde{W}_t^H = \int_0^t K^H(t-s) dW_s^1, \quad t \geq 0,$$
$$K^H(t-s) = \sqrt{2H}(t-s)^{H-1/2}, \quad \forall\, 0 \leq s \leq t.$$

- The hybrid scheme discretizes the $\widetilde{W}^H$ process into Wiener integrals of power functions and a Riemann sum, appearing from approximating the kernel by power functions near the origin and step functions elsewhere.

$$\widetilde{W}_{\frac{i}{N}}^H \approx \overline{W}_{\frac{i}{N}}^H = \sqrt{2H}\left(W_i^2 + \sum_{k=2}^{i}\left(\frac{b_k}{N}\right)^{H-\frac{1}{2}}\left(W_{\frac{i-(k-1)}{N}}^1 - W_{\frac{i-k}{N}}^1\right)\right),$$

where

- N is the number of time steps
- $\{W_j^2\}_{j=1}^N$: Artificially introduced N Gaussian random variables that are used for left-rule points in the hybrid scheme.
- $b_k = \left(\frac{k^{H+\frac{1}{2}} - (k-1)^{H+\frac{1}{2}}}{H+\frac{1}{2}}\right)^{\frac{1}{H-\frac{1}{2}}}$.

The rough Bergomi Model: Analytic Smoothing

We show that the call price is given by

$$\begin{aligned} C_{RB}(T,K) &= E\left[(S_T - K)^+\right] \\ &= \mathrm{E}\left[\mathrm{E}\left[(S_T-K)^+ \mid \sigma(W^1(t), t \leq T)\right]\right] \\ &= E\left[C_{BS}\left(S_0 = \exp\left(\rho\int_0^T \sqrt{v_t} dW_t^1 - \frac{1}{2}\rho^2 \int_0^T v_t dt\right),\right.\right. \\ &\qquad \left.\left. k = K,\ \sigma^2 = (1-\rho^2)\int_0^T v_t dt\right)\right] \\ &\approx \int_{\mathbb{R}^{2N}} C_{BS}\left(G(\mathbf{w}^{(1)}, \mathbf{w}^{(2)})\right) \rho_N(\mathbf{w}^{(1)}) \rho_N(\mathbf{w}^{(2)}) d\mathbf{w}^{(1)} d\mathbf{w}^{(2)}, \end{aligned} \tag{2}$$

where

- $C_{\mathrm{BS}}(S_0, k, \sigma^2)$ denotes the Black-Scholes call price, for initial spot price S_0, strike price k, and volatility σ^2.
- ρ_N: the multivariate Gaussian density, N: number of time steps.

Sparse Grids I

Goal: Given $F: \mathbb{R}^d \to \mathbb{R}$ and a multi-index $\boldsymbol{\beta} \in \mathbb{N}_+^d$, approximate

$$\mathrm{E}[F] \approx Q^{m(\boldsymbol{\beta})}[F],$$

where $Q^{m(\boldsymbol{\beta})}$ a Cartesian quadrature grid with $m(\beta_n)$ points along y_n.

Idea: Denote $Q^{m(\boldsymbol{\beta})}[F] = F_{\boldsymbol{\beta}}$ and introduce the first difference

$$\Delta_i F_{\boldsymbol{\beta}} \begin{cases} F_{\boldsymbol{\beta}} - F_{\boldsymbol{\beta}-e_i}, & \text{if } \beta_i > 1 \\ F_{\boldsymbol{\beta}} & \text{if } \beta_i = 1 \end{cases} \tag{3}$$

where e_i denotes the ith d-dimensional unit vector, and mixed difference operators

$$\Delta[F_{\boldsymbol{\beta}}] = \otimes_{i=1}^d \Delta_i F_{\boldsymbol{\beta}} \tag{4}$$

Sparse Grids II

A quadrature estimate of $E[F]$ is

$$\mathcal{M}_{\mathcal{I}_\ell}[F] = \sum_{\boldsymbol{\beta} \in \mathcal{I}_\ell} \Delta[F_{\boldsymbol{\beta}}], \tag{5}$$

- Product approach: $\mathcal{I}_\ell = \{\max\{\beta_1, \dots, \beta_d\} \leq \ell;\ \boldsymbol{\beta} \in \mathbb{N}_+^d\}$
- Regular SG: $\mathcal{I}_\ell = \{|\ \boldsymbol{\beta}\ |_1 \leq \ell + d - 1;\ \boldsymbol{\beta} \in \mathbb{N}_+^d\}$

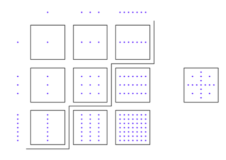


Figure 2: *Left are product grids $\Delta_{\beta_1} \otimes \Delta_{\beta_2}$ for $1 \leq \beta_1, \beta_2 \leq 3$. Right is the corresponding SG construction.*

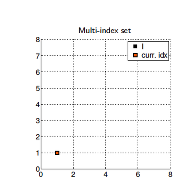
- ASGQ based on same construction as in [5]: $\mathcal{I}_\ell = \mathcal{I}^{\text{ASGQ}}$.

ASGQ in Practice

- The construction of $\mathcal{I}^{\text{ASGQ}}$ is done by profit thresholding

$$\mathcal{I}^{\text{ASGQ}} = \{\boldsymbol{\beta} \in \mathbb{N}_+^d : P_{\boldsymbol{\beta}} \geq \overline{T}\}.$$

- A posteriori, adaptive construction: Given an index set $\mathcal{I}_k$, compute the profits of the neighbor indices and select the most profitable one



- **Profit of a hierarchical surplus** $P_{\boldsymbol{\beta}} = \frac{|\Delta E_{\boldsymbol{\beta}}|}{\Delta \mathcal{W}_{\boldsymbol{\beta}}}$.
- **Error contribution**: $\Delta E_{\boldsymbol{\beta}} = \left|\mathcal{M}^{\mathcal{I}\cup\{\boldsymbol{\beta}\}} - \mathcal{M}^{\mathcal{I}}\right|$.
- **Work contribution**: $\Delta \mathcal{W}_{\boldsymbol{\beta}} = \text{Work}[\mathcal{M}^{\mathcal{I}\cup\{\boldsymbol{\beta}\}}] - \text{Work}[\mathcal{M}^{\mathcal{I}}]$

Numerical Experiments

Parameters	Reference solution
Set 1: $H = 0.07, K = 1, S_0 = 1, \rho = -0.9, \eta = 1.9, \xi = 0.235^2$	0.0791 (7.9e−05)
Set 2: $H = 0.02, K = 1, S_0 = 1, \rho = -0.7, \eta = 0.4, \xi = 0.1$	0.1248 (1.3e−04)
Set 3: $H = 0.02, K = 0.8, S_0 = 1, \rho = -0.7, \eta = 0.4, \xi = 0.1$	0.2407 (5.6e−04)
Set 4: $H = 0.02, K = 1.2, S_0 = 1, \rho = -0.7, \eta = 0.4, \xi = 0.1$	0.0568 (2.5e−04)

Table 1: *Reference solution, using MC with 500 time steps and number of samples, $M = 10^6$, of call option price under the rough Bergomi model, for different parameter constellations. The numbers between parentheses correspond to the statistical errors estimates.*

- The first set is the one that is closest to the empirical findings [**?**, 3], which suggest that $H \approx 0.1$. The choice of parameters values of $\nu = 1.9$ and $\rho = -0.9$ is justified by [1].
- For the remaining three sets, we wanted to test the potential of our method for a very rough case, where variance reduction methods, such as MLMC are inefficient.

Results

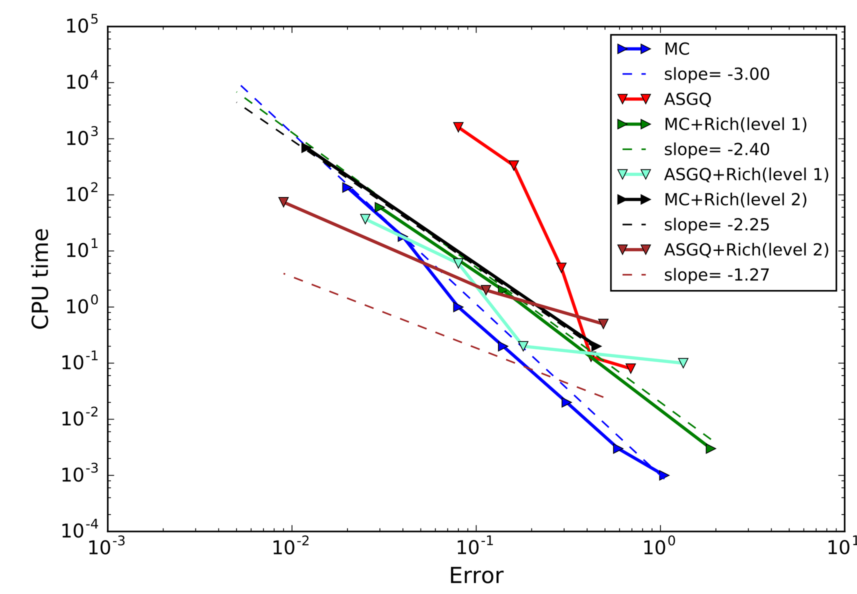


Figure 4: *Computational work comparison for ASGQ and MC (with and without) Richardson extrapolation, for the case of parameter set 1 in Table 1. In Figure 4, we consider relative errors.*

Conclusions

- Our proposed estimator is useful in systems with the presence of slow and fast timescales (stiff systems).
- Through our numerical experiments, we obtained substantial gains with respect to both the explicit MLMC and the drift-implicit, single-level tau-leap methods. We also showed that for large values of TOL the pure drift-implicit MLMC method has the same order of computational work as does the explicit MLMC tau-leap methods, which is of $\mathcal{O}\left(TOL^{-2}\log(TOL)^2\right)$ [**?**], but with a smaller constant.

Acknowledgements

C. Bayer gratefully acknowledges support from the German Research Foundation (DFG, grant BA5484/1). This work was supported by the KAUST Office of Sponsored Research (OSR) under Award No. URF/1/2584-01-01 and the Alexander von Humboldt Foundation. C. Ben Hammouda and R. Tempone are members of the KAUST SRI Center for Uncertainty Quantification in Computational Science and Engineering.

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