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CONTINGENT CLAIMS AND MARKET COMPLETENESS IN A STOCHASTIC VOLATILITY MODEL

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In an incomplete market framework, contingent claims are of particular interest since they improve the market efficiency. This paper addresses the problem of market completeness when trading in contingent claims is allowed. We extend recent results by Bajeux and Rochet (1996) in a stochastic volatility model to the case where the asset price and its volatility variations are correlated. We also relate the ability of a given contingent claim to complete the market to the convexity of its price function in the current asset price. This allows us to state our results for general contingent claims by examining the convexity of their "admissible arbitrage prices."

KEY WORDS: incomplete market, partial differential equations, maximum principle

1. INTRODUCTION

In a complete (frictionless) market framework, contingent claims written on the existing primitive assets are redundant. Assuming that their introduction in the economy does not affect equilibrium prices,[1] the agent utility cannot be increased by trading in such redundant assets. However in incomplete markets, contingent claims can improve the market efficiency by reducing the dimensionality of unhedgeable risks (see Hart 1975). In this paper we address this problem in a continuous-time incomplete market model.

In a one period model with finite state space, Ross (1976) provided necessary and sufficient conditions in order to achieve completeness of the market by allowing trading on multiple or simple contingent claims.[2] In such a framework, options are of particular interest since they span the same payoff space as general simple contingent claims. In a recent paper, Bajeux and Rochet (1996) showed that the static model results generally do not hold in a multiperiod model. The necessary and sufficient conditions for an option to be nonredundant constrain heavily the dynamics of prices. Notice that, in contrast with the one period model, the treatment of the multiperiod model requires the definition of a pricing rule for the option in order to check its redundancy at any date. As is well known, this is

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[1] Detemple and Selden (1991) showed that trading in options does affect the equilibrium prices since the agents' demand is changed.

[2] A multiple contingent claim is defined by a terminal payoff which depends on terminal payoffs of all existing primitive assets. The terminal payoff of a simple contingent claim depends only on the terminal payoff of a single primitive asset.

a serious problem in an incomplete market framework. Bajeux and Rochet used the price induced by a single agent equilibrium model; the state price process is thus given by the agent marginal rate of substitution at equilibrium. Nevertheless, their results do not depend on the specification of the utility function.

Bajeux and Rochet (1996) also examined the problem in a continuous time model. Considering the stochastic volatility model studied by Hull and White (1987), they showed that any European option completes the market. As in the multiperiod model, a pricing rule of the option is required in order to check its redundancy at any time. Since the market is incomplete, any choice of an equivalent martingale measure induces an admissible arbitrage price of the option. The precise result of Bajeux and Rochet can be stated as follows. Fix an equivalent martingale measure under which the volatility process is Markov and define the price process of the option as the admissible price process induced by such an equivalent martingale measure. Then the option completes the market in the sense that the derivative of its price functional with respect to the current value of the volatility process does not vanish at any time before the maturity of the option and for any current value of the underlying asset price and its volatility. Therefore, if the undetermined option price is induced by an equivalent martingale measure under which the volatility process is Markov, then the option completes the market since the volatility risk is duplicated by the option. In this paper we consider this notion of market completeness which is very close to Harrison and Pliska's (1981) one, as it will be discussed in Section 3. However, empirical evidence shows important (negative) correlation between the asset price variations and its volatility variations (see Black 1976), and Bajeux and Rochet's proof cannot be extended naturally to the case where the two risks are correlated. Moreover, since their proof relies heavily on the Hull and White (1987) option pricing formula, they did not consider general contingent claims.

The major contribution of this paper is to prove the Bajeux and Rochet result with another approach which relies on the PDE (partial differential equations) characterization of the price. Our approach allows for the presence of a correlation process between the price variations and its volatility depending on the volatility process. We show that a sufficient condition for a contingent claim to complete the market is the strict convexity of its price function in the current underlying asset price at any time (strictly) before maturity. Such a sufficient condition is satisfied by European call and put option. In general, given a contingent claim with convex payoff function (satisfying a logarithmic growth condition), such a sufficient condition is satisfied by any admissible arbitrage price induced by an equivalent martingale measure under which the volatility process is Markov (under additional technical conditions). Thus under the same restriction as Bajeux and Rochet (1996), this paper extends their result to a large class of contingent claims and allows for a possible correlation between the asset price and its volatility variations.

This result is very important from at least three points of view.

1. From a hedging point of view, it justifies the well-known *delta-sigma hedging strategy* (see, e.g., Scott 1991), which is widely used by practitioners.
2. From an equilibrium point of view, it justifies the work of Karatzas, Lehockzky, and Shreve (1990) who assumed the existence of contingent claims, in zero net supply, which complete the market in order to ensure that the multiagent intertemporal equilibrium can be reduced to a single representative agent one (see Huang 1987) and to solve an incomplete market intertemporal equilibrium problem.
3. From an econometric point of view, this result suggests the use of option price

data in order to "filter" the unobservable volatility process from options prices (see Renault and Touzi 1992).

The paper is organized as follows. Section 2 presents the general framework and recalls some general results on European contingent claims valuation. In Section 3, we introduce a definition of completeness, following Bajeux and Rochet (1996), and we discuss its connection to the completeness in the sense of Harrison and Pliska (1981); then we give the basic result relating market completeness to the convexity of the price function. Finally Section 4 presents the different extensions of Bajeux and Rochet's (1996) result by examining the convexity of admissible arbitrage prices.

2. EUROPEAN OPTIONS PRICING

In this paper we study the general framework which includes the stochastic volatility model introduced by Hull and White (1987) and Scott (1987):

$$\frac{dS_t}{S_t} = \mu(t, S_t, Y_t)dt + \sigma(Y_t)\sqrt{1-\rho^2(t, S_t, Y_t)}dW_t^1 + \sigma(Y_t)\rho(t, S_t, Y_t)dW_t^2 \tag{2.1}$$

$$dY_t = \eta(t, S_t, Y_t)dt + \gamma(t, S_t, Y_t)dW_t^2, \tag{2.2}$$

where $\{S_t, 0 \le t \le T\}$ is the price process of a primitive asset supposed to pay no dividends, $\{\sigma_t = \sigma(Y_t),\ 0 \le t \le T\}$ is the volatility process, and $\{\rho_t = \rho(t, S_t, Y_t),\ 0 \le t \le T\}$ is the correlation process between the asset price and its volatility taking values in the interval $(-1, 1)$. $W = (W^1, W^2)'$ is a standard Brownian motion on a probability space $(\Omega, \mathcal{F}, P)$ and $\{\mathcal{F}_t,\ 0 \le t \le T\}$ is the P-augmentation of the filtration generated by W.[3] In the remainder of this paper, we shall assume the following:

(i) $\sigma : \mathbb{R} \longrightarrow \mathbb{R}_+$ is Lipschitz, C^1-diffeomorphism and there exist $\underline{\sigma}, \overline{\sigma} > 0$ such that $\forall y \in \mathbb{R}, \underline{\sigma} \le \sigma(y) \le \overline{\sigma}$.

(ii) There exists $\varepsilon > 0$ such that $\forall (t, x, y) \in [0, T] \times \mathbb{R}_+^* \times \mathbb{R}$, $\min\{1-\rho^2(t, e^x, y), \gamma(t, e^x, y)\} \ge \varepsilon$.

(iii) The functions $(t, x, y) \longmapsto \mu(t, e^x, y)$, $(t, x, y) \longmapsto \rho(t, e^x, y)$, $(t, x, y) \longmapsto \eta(t, e^x, y)$, and $(t, x, y) \longmapsto \gamma(t, e^x, y)$ are bounded, Lipschitz in (x, y) uniformly in t.

(iv) The function $(t, x, y) \longmapsto \eta(t, e^x, y)$ is locally Lipschitz in (t, x, y).

Let Σ_t be the dispersion matrix of the diffusion (S, Y) and $A_t = \Sigma_t \Sigma_t'$; conditions (i) and (ii) ensure that the infinitesimal generator associated with the process (S, Y) is uniformly elliptic; i.e., there exists $\varepsilon > 0$, such that $z' A_t z \ge \varepsilon \|z\|^2$, $\forall z \in \mathbb{R}^2$, and $0 \le t \le T$ a.s. Condition (iii) guarantees the existence and the uniqueness of a strong Markov solution to (B.1) and (B.2) (see Karatzas and Shreve 1988, theorem 2.9, p. 289),[4] and (iv) is a technical condition.

We denote by r the instantaneous interest rate supposed to be constant, so that the time t-price of a zero-coupon bond maturing at time T is given by $B(t, T) = e^{-r(T-t)}$.

[3] We denote by X' the transpose matrix of X. Also the kth component of any vector x will be denoted x^k.

[4] Note that from condition (ii), if the function $(t, e^x) \longmapsto \rho(t, e^x)$ is Lipschitz, then $(t, e^x) \longmapsto \sqrt{1-\rho^2(t, e^x)}$ is also Lipschitz.

It is well known that the absence of arbitrage opportunities is "essentially" equivalent to the existence of a probability Q, equivalent to the initial probability P,[5] under which the discounted prices process is an $\{\mathcal{F}_t\}$-adapted martingale (see Stricker 1990 and Delbaen and Schachermayer 1994); such a probability will be called *equivalent martingale measure*. Any equivalent martingale measure Q is characterized by a continuous version of its density process with respect to P, which can be written from the integral form of martingale representation (see Karatzas and Shreve 1988, problem 4.16, p. 184):

$$M_t = \left.\frac{dQ}{dP}\right|_{\mathcal{F}_t} = \exp\left(-\int_0^t \lambda_u dW_u^1 - \int_0^t \nu_u dW_u^2 - \frac{1}{2}\int_0^t \lambda_u^2 du - \frac{1}{2}\int_0^t \nu_u^2 du\right), \tag{2.3}$$

where $(\lambda, \nu)'$ is adapted to $\{\mathcal{F}_t\}$ and satisfies the integrability conditions $\int_0^T \lambda_u^2 du < \infty$ and $\int_0^T \nu_u^2 du < \infty$ a.s. By the martingale property under Q of the discounted underlying asset prices, we have:

$$\left(\lambda_t\sqrt{1-\rho_t^2} + \nu_t\rho_t\right)\sigma_t = \mu_t - r \quad 0 \le t \le T \quad a.s. \tag{2.4}$$

The processes λ and ν satisfying (2.4) (and the appropriate integrability conditions) are interpreted as the risk premia relative respectively to the two sources of uncertainty W^1 and W^2. Since S is the only traded asset, the risk premia λ and ν are not fixed by the last relation, which explains the nonuniqueness of the martingale measure Q in this incomplete market context (see Harrison and Pliska 1981, 1983). For each choice of the volatility risk premium process $\{\nu_t,\ 0 \le t \le T\}$ (satisfying the appropriate integrability conditions), the risk premium process $\{\lambda_t,\ 0 \le t \le T\}$ is fixed by (2.4) and we can define an *admissible equivalent martingale measure* $Q(\nu)$ characterized by its density process $\{M_t(\nu),\ 0 \le t \le T\}$ with respect to P. By Girsanov's theorem, the process $\widetilde{W}(\nu) = (\widetilde{W}^1(\nu), \widetilde{W}^2(\nu))'$ defined by:

$$\widetilde{W}_t^1(\nu) = W_t^1 + \int_0^t \lambda_u\, du \quad \text{and} \quad \widetilde{W}_t^2(\nu) = W_t^2 + \int_0^t \nu_u\, du \tag{2.5}$$

is a two-dimensional Brownian motion under $Q(\nu)$ adapted to the filtration $\{\mathcal{F}_t,\ 0 \le t \le T\}$. We denote by $\beta_t^\nu = \eta(t, S_t, Y_t) - \nu_t\gamma(t, S_t, Y_t)$ the drift term of the volatility diffusion under the admissible equivalent martingale measure $Q(\nu)$; the dynamics of the model under an admissible equivalent martingale measure $Q(\nu)$ is described by:

$$\frac{dS_t}{S_t} = r\,dt + \sigma(Y_t)\left(\sqrt{1-\rho^2(t, S_t, Y_t)}\,d\widetilde{W}_t^1(\nu) + \rho(t, S_t, Y_t)\,d\widetilde{W}_t^2(\nu)\right) \tag{2.6}$$

$$dY_t = \beta_t^\nu dt + \gamma(t, S_t, Y_t)\,d\widetilde{W}_t^2(\nu). \tag{2.7}$$

[5]The probability measures P and Q are said to be equivalent if they agree on the null sets. Any property which holds a.s. with respect to P also holds a.s. with respect to Q. Therefore the reference to the probability measure will be omitted.

In the sequel, we focus on a European contingent claim written on the underlying asset S, maturing at time T, and with final payoff $\psi(S_T)$ satisfying the following technical condition:

ASSUMPTION 2.1. *The terminal payoff function $\psi(.)$ is continuous and satisfies the logarithmic growth condition:*

$$|\psi(s)| \leq K\left(1 + (\ln s)^{\theta}\right), \quad s \in \mathbb{R}_+^*$$

for some positive constants K and θ.

An important example of such a contingent claim is the European put option P, characterized by its final payoff $\psi_P(S_T) = (K - S_T)^+$, where K is the exercise price of the option. Notice that the terminal payoff $(S_T - K)^+$ corresponding to a European call option C does not satisfy Assumption B.1; nevertheless, from the parity relation between call and put options,

$$P_t = C_t - S_t + Ke^{-r(T-t)}, \tag{2.8}$$

the results proved for put options in this paper can be extended trivially to call options.

Now for each admissible equivalent martingale measure $Q(\nu)$ define *an admissible price* of the contingent claim by:

$$U_t^{\nu} = e^{-r(T-t)} E_t^{Q(\nu)} \left[\psi(S_T)\right], \tag{2.9}$$

where $E_t^{Q(\nu)}$ stands for the conditional expectation given $\mathcal{F}_t$ under the probability measure $Q(\nu)$. The determination of a pricing rule for nonattainable contingent claims in incomplete markets is not clearly solved in the existing literature. However, if the pricing rule for contingent claims is a linear extension of the continuous linear pricing functional defined for attainable assets then, by the Riez representation theorem, it is necessarily induced by some admissible equivalent martingale measure. Therefore, the price of the contingent claim under consideration coincides with some U^{ν} and, since ν is unknown, any U^{ν} is an *admissible arbitrage price*. For example, the pricing rule induced by the minimal quadratic error criterion of Föllmer and Schweizer (1991) corresponds to a particular choice of the volatility risk premium (which induces the so-called minimal martingale measure, under appropriate conditions). In a recent paper, Davis (1994) suggested a utility-based pricing rule that also corresponds to a particular choice of the volatility risk premium (which induces the “mini-max” martingale measure). Our approach is to prove some results for a large class of “admissible” pricing rules assumed to contain the true undetermined pricing rule.

An alternative characterization of the admissible option prices is based on partial differential equations (PDE). However, this requires the following restriction on the set of admissible equivalent martingale measures.

ASSUMPTION 2.2. *The volatility risk premium depends only on the contemporaneous values of the state variables: $\nu_t = \nu(t, S_t, Y_t)$, $\forall\ 0 \leq t \leq T$. Moreover, ν is bounded and the function $(t, x, y) \longmapsto \nu(t, e^x, y)\gamma(t, e^x, y)$ is locally Lipschitz in (t, x, y).*

The first part of this assumption means that the true unknown pricing rule is supposed to be induced by an admissible equivalent martingale measure which preserves the Markov property of the diffusion process (S, Y). The drift function under $Q(\nu)$ of the volatility process will be denoted $\beta^\nu(t, s, y)$ under Assumption B.2. Now consider the PDE:

$$\text{(2.10)} \qquad \begin{cases} \mathcal{L}^\nu V(t, s, y) = 0 & \forall (t, s, y) \in [0, T) \times \mathbb{R}_+^* \times \mathbb{R} \\ V(T, s, y) = \psi(s) & \forall (s, y) \in \mathbb{R}_+^* \times \mathbb{R}, \end{cases}$$

where

$$\text{(2.11)} \qquad \begin{aligned} \mathcal{L}^\nu . &= r. - \frac{\partial .}{\partial t} - rs\frac{\partial .}{\partial s} - \beta^\nu(t, s, y)\frac{\partial .}{\partial y} \\ &\quad - \frac{1}{2}s^2\sigma^2(y)\frac{\partial^2 .}{\partial s^2} - \frac{1}{2}\gamma^2(t, s, y)\frac{\partial^2 .}{\partial y^2} - \rho(t, s, y)\gamma(t, s, y)\sigma(y)s\frac{\partial^2 .}{\partial s \partial y} \end{aligned}$$

From Friedman (1975, theorem 5.3, p. 148), under Assumptions B.1 and B.2, the PDE (2.10) has a unique $C^{1,2}\left([0, T], \mathbb{R}_+^* \times \mathbb{R}\right)$ solution satisfying a polynomial growth condition.[6] Therefore, from the Feynman–Kac formula (see Karatzas and Shreve 1988, theorem 7.6, p. 366), this solution admits the stochastic representation (2.9).

In the following sections, we discuss the completeness of the market once the contingent claim is introduced, and we provide a generalization of the results of Bajeux and Rochet (1996) to the case where the correlation coefficient is allowed to be different from zero; to this end, we will use both the stochastic characterization (2.9) and the PDE one (2.10).

3. EUROPEAN OPTIONS AND MARKET COMPLETENESS

In a stochastic volatility model, the market is incomplete and the underlying asset is not sufficient to hedge a given contingent claim against the two risk factors.

DEFINITION 3.1. Let $\{U(t, S_t, Y_t),\ 0 \le t \le T\}$ be the price process of a contingent claim U on the asset S, where $(t, s, y) \longmapsto U(t, s, y)$ is a $C^{1,2}([0, T), \mathbb{R}_+^* \times \mathbb{R})$ function. We say that the contingent claim U completes the market in the time interval $[0, T)$ if and only if:

(i) there exists a risk premia process ν^0 such that the process $\{U_t e^{-rt}, 0 \le t \le T\}$ is a martingale under the admissible equivalent martingale measure $Q(\nu^0)$,

(ii) $\forall (t, s, y) \in [0, T) \times \mathbb{R}_+^* \times \mathbb{R}$, $U_y(t, s, y) = \dfrac{\partial U}{\partial y}(t, s, y) \neq 0$.

Let us first relate this definition to market completeness in the sense of Harrison and Pliska (1981). Applying Itô's lemma to the price process $U_t = U(t, S_t, Y_t)$ we get the following bivariate diffusion for the process $\{Z_t = (S_t e^{-rt}, U_t e^{-rt})', 0 \le t \le T\}$:

$$dZ_t = \begin{pmatrix} Z_t^1 & 0 \\ 0 & Z_t^2 \end{pmatrix} \zeta_t' d\widetilde{W}_t(\nu_0) \quad 0 \le t \le T \quad \text{a.s.}$$

[6] In order to check that Assumptions B.1 and B.2 together with the conditions (i), (ii), (iii), and (iv) ensure that the sufficient conditions of Friedman (1975) are satisfied, one has to make the change of variable $x = \ln s$.

where

$$\zeta_t = \begin{pmatrix} \sigma_t\sqrt{1-\rho_t^2} & \dfrac{\partial \ln U}{\partial s}(t, S_t, Y_t)\sigma_t S_t\sqrt{1-\rho_t^2} \\ \sigma_t\rho_t & \dfrac{\partial \ln U}{\partial s}(t, S_t, Y_t)\sigma_t S_t\rho_t + \dfrac{\partial \ln U}{\partial y}(t, S_t, Y_t)\gamma_t \end{pmatrix}.$$

Direct computation shows that the determinant of the matrix ζ_t is given by:

$$\det(\zeta_t) = \sigma_t\gamma_t\sqrt{1-\rho_t^2}\frac{\partial \ln U}{\partial y}(t, S_t, Y_t) \quad 0 \le t \le T \quad \text{a.s.}$$

Therefore condition (ii) of Definition C.1 is necessary and sufficient for the diffusion matrix ζ_t to be invertible (remember that the functions $\sigma(.)$, $\gamma(., ., .)$, and $1-\rho^2(., ., .)$ are bounded from below away from zero). This allows us to invert the bivariate stochastic differential equation:

$$d\widetilde{W}_t(\nu_0) = \zeta_t'^{-1}\begin{pmatrix} 1/Z_t^1 & 0 \\ 0 & 1/Z_t^2 \end{pmatrix} dZ_t \quad 0 \le t \le T \quad \text{a.s.} \tag{3.1}$$

Now fix the volatility risk premium ν^0 of Definition C.1(i) and let the associated admissible equivalent martingale measure $Q(\nu^0)$ be *a reference measure* as in Harrison and Pliska (1981). Consider a contingent claim $X \in \mathcal{F}_T$ which is $Q(\nu^0)$-integrable and define the $Q(\nu^0)$-martingale $\{X_t = E_t^{Q(\nu^0)}(X),\ 0 \le t \le T\}$. Then the process $\{Y_t = M_t(\nu^0)X_t,\ 0 \le t \le T\}$ is a martingale under the probability measure P adapted to the filtration $\{\mathcal{F}_t,\ 0 \le t \le T\}$, and the martingale representation theorem (see Karatzas and Shreve 1988, problem 4.16, p. 184) ensures the existence of a bivariate adapted process $\{\phi_t = (\phi_t^1, \phi_t^2)',\ 0 \le t \le T\}$ satisfying $\int_0^T \|\phi_t\|^2\, dt < \infty$ a.s. such that:

$$Y_t = x + \int_0^t \phi_u' dW_u \quad 0 \le t \le T \ \text{ a.s.} \quad \text{with} \quad x = E_0(M_T(\nu^0)X) \ = \ E_0^{Q(\nu^0)}(X).$$

Applying Itô's lemma provides the following integral representation for the process $\{X_t,\ 0 \le t \le T\}$:

$$X_t = x + \int_0^t \eta_u' d\widetilde{W}_u(\nu^0) \quad 0 \le t \le T \quad \text{a.s.}$$

with

$$\eta_t^1 = \frac{\phi_t^1}{M_t(\nu^0)} + \lambda_t X_t \quad \text{and} \quad \eta_t^2 = \frac{\phi_t^2}{M_t(\nu^0)} + \nu_t^0 X_t.$$

Using equation (3.1) which follows from Definition C.1(ii) we get:

$$X_t = x + \int_0^t H_u' dZ_u \quad \text{with} \quad H_t \ = \ \zeta_t^{-1}\eta_t\begin{pmatrix} 1/Z_t^1 & 0 \\ 0 & 1/Z_t^2 \end{pmatrix}.$$

Following Harrison and Pliska (1981) the predictable process H defines an admissible trading strategy if the increasing process

$$\left(\int_0^t (H_u^k)^2 d\langle Z^k, Z^k\rangle_u\right)^{1/2}, \quad 0 \le t \le T,$$

is locally integrable under $Q(\nu^0)$ for each $k = 1, 2$. Therefore, in order for the notion of completeness introduced in Definition C.1 to imply completeness in the sense of Harrison and Pliska, more technical restrictions (which are beyond the scope of this paper) should be added. However, Definition C.1 can be seen as a necessary condition of completeness in the sense of Harrison and Pliska since the invertibility of the diffusion matrix ζ_t is a crucial step to represent the contingent claim X as a stochastic integral with respect to the traded assets price process $\{Z_t, 0 \le t \le T\}$.

In the context of the Hull and White model (see the following Assumptions D.1 and D.2 with a zero correlation coefficient $\rho \equiv 0$), Bajeux and Rochet (1996) proved that any European option completes the market (in the sense of Definition C.1) as follows. Since the no-arbitrage argument does not induce a unique arbitrage price, only a set of admissible arbitrage prices is available. If the (continuous and linear) pricing rule, defined on the set of attainable assets, extends in a linear way to all contingent claims, then the "true price" is necessarily contained in the set of admissible prices. Therefore, a sufficient condition for the contingent claim to complete the market is that any admissible price satisfies the requirements of Definition C.1.[7] Given an admissible price for the option, condition (i) of Definition C.1 is obviously satisfied and Bajeux and Rochet proved condition (ii) by a simple differentiation through the expectation operator. However, when the correlation coefficient is nonzero, such a simple technique does not allow the required result to be proved. We therefore provide an alternative proof which can handle the presence of a correlation coefficient and can be extended to a general European contingent claim. To this end, we need the following technical conditions:

ASSUMPTION 3.1. *The functions β^ν, γ, and ρ are differentiable with respect to y, and their derivatives β^ν_y, γ_y and ρ_y are bounded and locally Lipschitz in (t, s, y).*

THEOREM 3.1. *Consider a contingent claim with terminal payoff function ψ satisfying Assumption B.1 and a volatility risk premium ν satisfying Assumption B.2. Suppose that Assumption C.1 holds and let U^ν be the contingent claim price induced by the admissible martingale measure $Q(\nu)$. If $U^\nu(t, s, y)$ is strictly convex in s for all $y \in \mathbb{R}$ and $t < T$, then it is strictly increasing with respect to the volatility $\sigma(y)$, i.e.:*

$$\forall t \in [0, T),\ \forall (s, y) \in \mathbb{R}_+^* \times \mathbb{R}, \quad \sigma'(y)\, \frac{\partial}{\partial y} U^\nu(t, s, y) > 0. \tag{3.2}$$

Since the set of all admissible price functions contains the true undetermined price of the contingent claim, the last proposition shows that, if $U^\nu(t, s, y)$ is strictly convex in s for

[7] Notice that Bajeux and Rochet's proof holds for admissible prices induced by equivalent martingale measures under which the volatility process is Markov (see the following Assumptions D.1 and D.2).

any admissible equivalent martingale measure $Q(\nu)$, then the contingent claim completes the market in the sense of Definition C.1.

Proof. Differentiating the PDE (2.10), satisfied by the contingent claim price, with respect to y, we see that $U_y^\nu = \partial U^\nu/\partial y$ satisfies the differential system:

$$\begin{cases} \widetilde{\mathcal{L}}^\nu V = s^2\sigma(y)\sigma'(y)U_{ss}^\nu & \forall(t,s,y) \in [0,T) \times \mathbb{R}_+^* \times \mathbb{R} \\ V(T,s,\sigma) = 0, \quad \forall(s,y) \in \mathbb{R}_+^* \times \mathbb{R} \end{cases}$$

where $U_{ss}^\nu = \dfrac{\partial^2 U^\nu}{\partial s^2}$ and:

$$\begin{aligned} \widetilde{\mathcal{L}}^\nu. = {} & \left(r - \frac{\partial}{\partial y}\beta^\nu(t,s,y)\right). - \frac{\partial .}{\partial t} - \left(r + \frac{\partial}{\partial y}(\rho(t,s,y)\gamma(t,s,y)\sigma(y))\right) s\frac{\partial .}{\partial s} \\ & - \left(\beta^\nu(t,s,y) + \gamma(t,s,y)\frac{\partial}{\partial y}\gamma(t,s,y)\right)\frac{\partial .}{\partial y} \\ & - \frac{1}{2}s^2\sigma^2(y)\frac{\partial^2 .}{\partial s^2} - \frac{1}{2}\gamma^2(t,s,y)\frac{\partial^2 .}{\partial y^2} - \rho(t,s,y)\gamma(t,s,y)\sigma(y)s\frac{\partial^2 .}{\partial s\partial y}. \end{aligned}$$

Now, from the convexity of the contingent claim price with respect to s, we have that $U_{ss}^\nu(t,s,y) > 0$ for $(t,s,y) \in [0,T) \times \mathbb{R}_+^* \times \mathbb{R}$. Since σ' has a constant sign, $U_y(t,s,y)$ is a solution to:

$$\begin{cases} \widetilde{\mathcal{L}}^\nu V(t,s,y) & > 0 & \text{if } \sigma' > 0 & \\ & < 0 & \text{if } \sigma' < 0, & \forall(t,s,y) \in [0,T) \times \mathbb{R}_+^* \times \mathbb{R} \\ V(T,s,y) & = 0, & & \forall(s,y) \in \mathbb{R}_+^* \times \mathbb{R} \end{cases}$$

where $\widetilde{\mathcal{L}}^\nu$ is a differential operator satisfying the sufficient conditions given in Friedman (1975), by Assumptions B.2 and C.1. Consequently, by a direct application of the strong maximum principle, we deduce that U_y^ν has the same sign as σ', which provides the required result. □

4. CONVEXITY OF ADMISSIBLE PRICES

In this section, given a European contingent claim, we study the convexity of its admissible arbitrage prices in order to check whether it completes the market in the sense of Definition C.1, as suggested by Theorem C.1. We shall first focus on European options.

4.1. The Case of European Options

Assuming that the volatility process is Markov under the initial probability measure P and independent of W^1, Hull and White expressed the European option price, for a zero volatility risk premium ($\nu = 0$), as the expectation of the Black and Scholes (1973) formula, where the volatility parameter is replaced by the average future volatility $\frac{1}{T-t}\int_t^T \sigma_u^2\,du$. It is easily seen that the Hull–White formula is obtained for any admissible price induced by an equivalent martingale measure under which the volatility process is Markov (see Renault and Touzi 1992). We first provide an extension of the Hull–White formula to the case where the asset price and its volatility variations are allowed to be correlated.

ASSUMPTION 4.1. *The correlation coefficient ρ as well as the volatility (of the volatility process) coefficient γ do not depend on the primitive asset price, i.e., $\rho_t = \rho(t, Y_t)$ and $\gamma_t = \gamma(t, Y_t)$.*

ASSUMPTION 4.2. *The volatility risk premium ν is such that the drift coefficient of the volatility process under the equivalent martingale measure $Q(\nu)$ does not depend on the primitive asset price, i.e., $\beta_t^\nu = \beta^\nu(t, Y_t)$.*

The last assumption restricts the set of admissible arbitrage prices assumed to contain the true undetermined price, since it requires that the volatility risk premium is such that $\eta(t, s, y) - \nu(t, s, y)\gamma(t, y)$ does not depend on s. Under Assumption D.1, given a volatility risk premium process ν satisfying Assumption D.2, the volatility process is Markov under $Q(\nu)$. Unfortunately, we are not able to weaken the requirements of the last assumptions. Notice however that, if the volatility process is Markov under the initial probability P, as in the HW model, then these assumptions have been shown to be consistent with a large class of intertemporal additive equilibrium models (see Pham and Touzi 1996).

PROPOSITION 4.1. *Suppose that Assumption D.1 holds and consider a volatility risk premium ν satisfying Assumption D.2. Then the associated admissible arbitrage price of the European call option is given by:*

$$C_t^\nu = E_t^{Q(\nu)}\left[C_t^{BS}\left(S_t e^{Z_{t,T}^\nu}; \frac{1}{T-t}\int_t^T (1-\rho_u^2)\sigma_u^2 du\right)\right] \tag{4.1}$$

$$= S_t E_t^{Q(\nu)}\left[e^{Z_{t,T}^\nu}\Phi\left(\frac{x_t + Z_{t,T}^\nu}{V_{t,T}} + \frac{V_{t,T}}{2}\right) - e^{-x_t}\Phi\left(\frac{x_t + Z_{t,T}^\nu}{V_{t,T}} - \frac{V_{t,T}}{2}\right)\right], \tag{4.2}$$

where $\Phi(.) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{.} e^{-u^2/2}du$ is the cumulative probability distribution function of the normal $\mathcal{N}(0, 1)$ distribution, $C_t^{BS}(S, \sigma^2)$ is the classical Black–Scholes formula, and:

$$x_t = \ln\left(\frac{S_t}{Ke^{-r(T-t)}}\right), \tag{4.3}$$

$$Z_{t,T}^\nu = \int_t^T \rho_u\sigma_u d\widetilde{W}_u^2(\nu) - \frac{1}{2}\int_t^T \rho_u^2\sigma_u^2\, du, \tag{4.4}$$

$$V_{t,T} = \sqrt{\int_t^T (1-\rho_u^2)\sigma_u^2\, du}. \tag{4.5}$$

Proof. Integrating the diffusion (B.6), we obtain

$$S_T = S_t e^{Z_{t,T}^\nu}\exp\left(r(T-t) - \frac{1}{2}\int_t^T \sigma_u^2(1-\rho_u^2)du + \int_t^T \sigma_u\sqrt{1-\rho_u^2}d\widetilde{W}_u^1(\nu)\right),$$

where $Z^{\nu}_{t,T}$ is given in (D.4). Now, from Assumptions D.1 and D.2, the volatility and the correlation processes are independent of $\{\widetilde{W}^1_t(\nu),\ 0 \le t \le T\}$ under $Q(\nu)$, and the $Q(\nu)$-distribution of S_T conditionally on the volatility path and the information set up to time t is thus:

$$\ln\left(\frac{S_T}{S_t e^{Z^{\nu}_{t,T}}}\right) \Bigg| \ \mathcal{F}_t, \{\sigma_u, t \le u \le T\} \ \overset{Q(\nu)}{\rightsquigarrow} \ \mathcal{N}\left(r(T-t) - \frac{1}{2}V^2_{t,T}\ ,\ V^2_{t,T}\right),$$

where $V_{t,T}$ is given in (D.5). Now, computing the conditional expectation (2.9) by first conditioning on the volatility process, it is easily checked that the European call option price is the expectation of the Black–Scholes formula where the underlying asset price is replaced by $S_t e^{Z^{\nu}_{t,T}}$ and the volatility parameter is replaced by $(1/\sqrt{T-t})V_{t,T}$. □

Notice again that the analogue of equations (D.1) and (D.2) for a European put option is deduced from the parity relation (2.8). From (D.4), if the volatility is deterministic, $\gamma = \rho \equiv 0$ then $Z^{\nu}_{t,T} = 0$ for any t and equation (D.1) reduces to the classical Merton (1973) formula, while the Black–Scholes formula is obtained if the volatility process is constant. If the correlation coefficient $\rho \equiv 0$ (W^1 and σ are noncorrelated under P) then, from (D.4), $Z^{\nu}_{t,T} = 0$ for all t and equation (D.1) reduces to the Hull–White formula. The option price (D.1) shows that the correlation between the asset price and its volatility variation induces two effects: a reduction of the average future volatility and a random perturbation of the current asset price by a factor $\exp(Z^{\nu}_{t,T})$.

We now use the extension of the Hull–White formula to check the convexity of the European option price function with respect to the underlying asset price.

PROPOSITION 4.2. *Suppose that Assumption D.1 holds and consider a volatility risk premium ν satisfying Assumption D.2. Then the associated admissible arbitrage price for a European (call or put) option is a strictly convex function of the current underlying asset price:*

$$C^{\nu}_{ss}(t,s,y) = P^{\nu}_{ss}(t,s,y) > 0, \quad \forall (t,s,y) \in [0,T) \times \mathbb{R}^*_+ \times \mathbb{R}.$$

Thus, if the set of admissible arbitrage prices, assumed to contain the true unknown price, is restricted by Assumption D.2, then any European option completes the market in the sense of Definition C.1 (under the assumptions of Theorem C.1).

Proof. From the parity relation between European call and put options (2.8), we have $P^{\nu}_{ss}(t,s,y) = C^{\nu}_{ss}(t,s,y)$; therefore it suffices to prove the required result for call options. Fix an equivalent martingale measure $Q(\nu)$ satisfying Assumption D.2. Then, under Assumption D.1, the associated admissible call option price is given by (D.2) and the result is obtained by differentiating this expression twice through the expectation operator:

$$C^{\nu}_{ss}(t,s,y) = \frac{1}{S_t} E^{Q(\nu)}_t \left[e^{Z^{\nu}_{t,T}} \frac{1}{V_{t,T}} \varphi\left(\frac{x_t + Z^{\nu}_{t,T}}{V_{t,T}} + \frac{V_{t,T}}{2}\right)\right] > 0;$$

the differentiation through the expectation operator is justified by Lebesgue's theorem and the fact that $V_{t,T}$ is bounded away from zero. □

4.2. The General Case

The proof of Proposition D.2 relies heavily on Proposition D.1, which expresses the admissible option prices as expectations of the Black–Scholes formula. For a general contingent claim with terminal payoff function ψ, it is not always possible to derive an explicit closed-form analogue of (D.2). The following result exploits again the PDE chacterization of the contingent claim price (2.10) in order to check the convexity with respect to the underlying asset price. This requires a stronger assumption on the terminal payoff function ψ.

ASSUMPTION 4.3. *The terminal payoff function ψ is convex and twice continuously differentiable with a second derivative satisfying the logarithmic growth condition:*

$$|\psi_{ss}(s)| \leq K\left(1 + \ln s)^{\theta}\right), \quad s \in \mathbb{R}_+^*$$

for some positive constants K and θ.

PROPOSITION 4.3. *Suppose that Assumption D.1 holds and consider a European contingent claim whose payoff function ψ satisfies Assumptions B.1 and D.3. Let ν be a volatility risk premium process satisfying Assumption D.2. Then the associated admissible arbitrage price function U^{ν} for the contingent claim is a strictly convex function of the current underlying asset price:*

$$U_{ss}^{\nu}(t, s, y) > 0, \quad (t, s, y) \in [0, T) \times \mathbb{R}_+^* \times \mathbb{R}.$$

Thus, if the set of admissible arbitrage prices, assumed to contain the true unknown price, is restricted by Assumption D.2, then any European contingent claim with terminal payoff function ψ satisfying Assumptions B.1 and D.3 completes the market in the sense of Definition C.1 (under the assumptions of Theorem C.1).

Proof. Under the assumptions of the proposition, differentiating the PDE (2.10) twice with respect to s, we find that U_{ss}^{ν} is the unique $C^{1,2}([0, T), \mathbb{R}_+^* \times \mathbb{R})$ solution to:

$$\begin{cases} \widehat{\mathcal{L}}^{\nu} V(t, s, y) = 0 & \forall (t, s, y) \in [0, T) \times \mathbb{R}_+^* \times \mathbb{R} \\ V(T, s, y) = \psi_{ss}(s) \geq 0, & \forall (s, y) \in \mathbb{R}_+^* \times \mathbb{R}, \end{cases}$$

where

$$\begin{aligned} \widehat{\mathcal{L}}^{\nu}. \;=\; & -(r + \sigma^2(y)). - \frac{\partial .}{\partial t} - (r + 2\sigma^2(y))s\frac{\partial .}{\partial s} - (\beta^{\nu}(t, y) + 2\rho(t, y)\gamma(t, y)\sigma(y))\frac{\partial .}{\partial y} \\ & -\frac{1}{2}s^2\sigma(y)^2\frac{\partial^2 .}{\partial s^2} - \frac{1}{2}\gamma^2(t, y)\frac{\partial^2 .}{\partial y^2} - \rho(t, y)\gamma(t, y)\sigma(y)s\frac{\partial^2 .}{\partial s \partial y} \end{aligned}$$

is a differential operator satisfying the conditions of Friedman (1975). This proves that $(t, x, y) \longmapsto U_{ss}^{\nu}(t, e^x, y)$ satisfies a polynomial growth condition and that $U^{\nu}(t, s, y)$ is a convex function of s, by a direct application of the maximum principle.

To conclude the proof, we have to show U^{ν} is strictly convex in s. Suppose that $U^{\nu}_{ss}(t_0, s_0, y_0) = 0$ and consider any open subset $\mathcal{O}$ of $[0, T) \times \mathbb{R}^*_+ \times \mathbb{R}$ containing (t_0, s_0, y_0). Then the restriction of U^{ν}_{ss} to $\mathcal{O}$ is the unique classical solution of the PDE:

$$\begin{cases} \widehat{\mathcal{L}}^{\nu} V(t, s, y) = 0, & \forall (t, s, y) \in \mathcal{O} \\ V(t, s, y) \;=\; U^{\nu}_{ss}(t, s, y), & \forall (t, s, y) \in \partial\mathcal{O}. \end{cases}$$

Since $U^{\nu}_{ss}(t, s, y) \geq 0$ on $\partial\mathcal{O}$ and $U^{\nu}_{ss}(t_0, s_0, y_0) = 0$, the strong maximum principle implies that $U^{\nu}_{ss} = 0$ a.s. on $\partial\mathcal{O}$ for any open subset $\mathcal{O}$ of $[0, T) \times \mathbb{R}^*_+ \times \mathbb{R}$ containing (t_0, s_0, y_0). This proves that U^{ν} is a linear function in s on $[0, T) \times \mathbb{R}^*_+ \times \mathbb{R}$. By the joint continuity property of the solution in $(t, s, y) \in [0, T) \times \mathbb{R}^*_+ \times \mathbb{R}$, this implies that the terminal payoff function ψ is linear in s which contradicts the logarithmic growth condition in Assumption B.1. $\square$

Notice that the regularity of ψ required in Assumption D.3 can be weakened by assuming that ψ can be approximated uniformly by a sequence $\psi^{(n)}$ of C^2 convex functions. Such an approximation can be achieved for the European put option terminal payoff and Proposition D.3 can be seen in some sense as an extension of Proposition D.2.

The conditions of Assumptions D.1 and D.2 are very restrictive since they restrict the set of admissible arbitrage prices and they do not allow for a general correlation function. Unfortunately, they are crucial in order to prove the convexity of the contingent claim price function in the underlying asset. In a complete market model where the asset price process is driven by a Markov scalar diffusion, Bergman, Grundy, and Wiener (1995) and El Karoui, Jeanblanc-Piqué, and Shreve (1995)[8] proved that the price of a European contingent claim with convex payoff is a convex function of the current underlying asset price (under some technical conditions). In a stochastic volatility model, Bergman, Grundy and Wiener prove the partial convexity of the contingent claim price, with respect to the current underlying asset price, under the same assumptions as ours.

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