

Hierarchical adaptive sparse grids and Quasi Monte Carlo for option pricing under the rough Bergomi model

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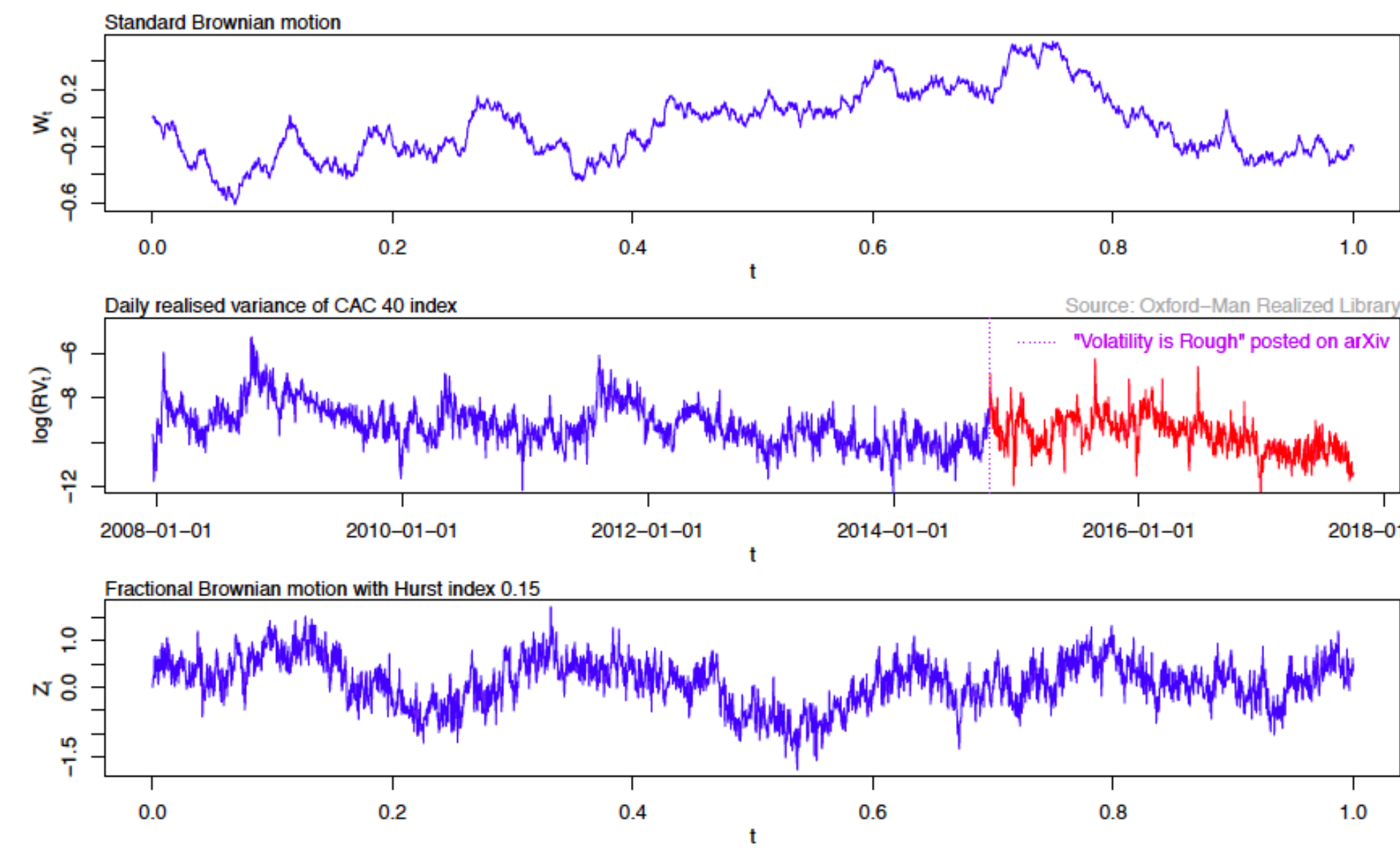
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Abstract

The **rough Bergomi (rBergomi) model**, introduced in [1], is a promising rough volatility model in quantitative finance. In the absence of analytical European option pricing methods for the model, and due to **the non-Markovian nature of the fractional driver**, the prevalent option is to use Monte Carlo (MC) simulation for pricing. Despite recent advances in the MC method in this context, pricing under the rBergomi model is still a time-consuming task. To overcome this issue, **we design a novel, alternative, hierarchical approach, based on i) adaptive sparse grids quadrature (ASGQ)**, specifically using the same construction in [6], and **ii) Quasi Monte Carlo (QMC)**. Both techniques are coupled with **Brownian bridge** construction and **Richardson extrapolation**. By uncovering the available regularity, our hierarchical methods demonstrates **substantial computational gains with respect to the standard MC method**, when reaching a sufficiently small error tolerance in the price estimates across different parameter constellations, even for very small values of the Hurst parameter.

Rough Volatility



The rough Bergomi Model [1]

This model, under a pricing measure, is given by

$$\begin{cases} dS_t = \sqrt{v_t} S_t dZ_t, \\ v_t = \xi_0(t) \exp\left(\eta \widetilde{W}_t^H - \frac{1}{2} \eta^2 t^{2H}\right), \\ Z_t := \rho W_t^1 + \bar{\rho} W_t^\perp \equiv \rho W^1 + \sqrt{1 - \rho^2} W^\perp, \end{cases} \quad (1)$$

- (W^1, W^\perp) : two independent standard Brownian motions
- \widetilde{W}^H is **Riemann-Liouville process**, defined by

$$\begin{aligned} \widetilde{W}_t^H &= \int_0^t K^H(t-s) dW_s^1, \quad t \geq 0, \\ K^H(t-s) &= \sqrt{2H}(t-s)^{H-1/2}, \quad \forall 0 \leq s \leq t. \end{aligned}$$

- $H \in (0, 1/2]$ ($H = 1/2$ for Brownian motion): controls the **roughness** of paths, $\rho \in [-1, 1]$ and $\eta > 0$.
- $t \mapsto \xi_0(t)$: forward variance curve, known at time 0.

Challenges

- **Numerically:**
 - The model is **non-affine** and **non-Markovian** \Rightarrow Standard numerical methods (PDEs, characteristic functions) seem inapplicable.
 - The only prevalent pricing method for mere **vanilla options** is **Monte Carlo** [1, 2, 7], still a **time consuming task**.
 - Discretization methods have **poor behavior of the strong error**, that is the convergence rate is of order of $H \in [0, 1/2]$ [8] \Rightarrow Variance reduction methods, such as MLMC, are inefficient for **very small values of H** .
- **Theoretically:**
 - No proper weak error analysis done in the rough volatility context.

Contributions

1. We design an **alternative hierarchical efficient pricing method** based on:
 - i) **Analytic smoothing** to uncover available regularity.
 - ii) Approximating the option price using a **deterministic quadrature method (ASGQ and QMC)** coupled with **Brownian bridges** and **Richardson Extrapolation**.
2. Our **hierarchical** methods demonstrate **substantial** computational gains with respect to the standard MC method, assuming a **sufficiently small relative error tolerance** in the price estimates, even for **small values of H** .

On the Choice of the Simulation Scheme

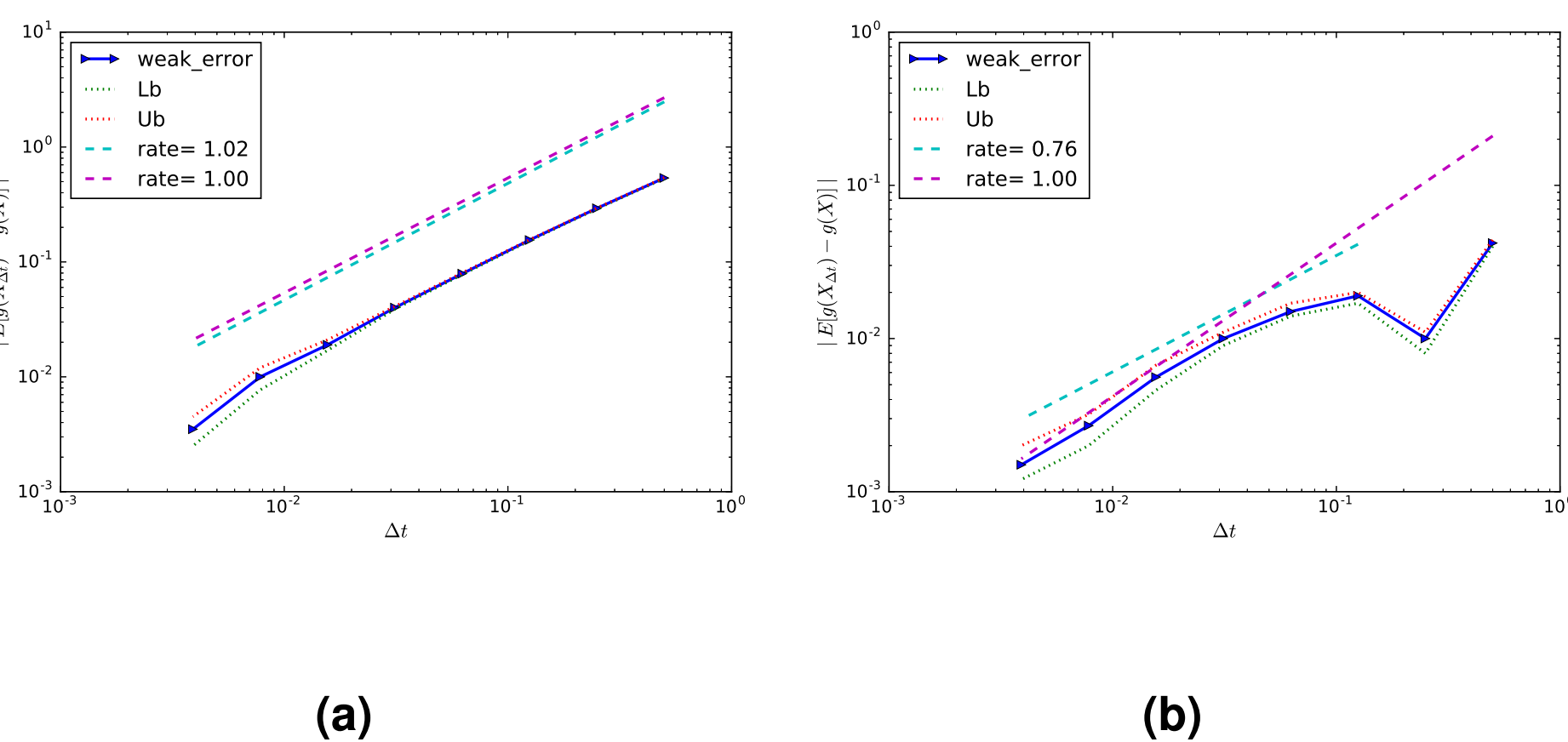


Figure 2: The convergence of the weak error \mathcal{E}_B , using MC with 6×10^6 samples, for **Set 1 parameter in Table 1**. The upper and lower bounds are 95% confidence intervals. a) With **the hybrid scheme** b) With **the exact scheme**.

The Hybrid Scheme [4]

The rough Bergomi Model: Analytic Smoothing

$$\begin{aligned} C_{RB}(T, K) &= E[(S_T - K)^+] \\ &= E[E[(S_T - K)^+ | \sigma(W^1(t), t \leq T)]] \\ &= E\left[C_{BS}\left(S_0 = \exp\left(\rho \int_0^T \sqrt{v_t} dW_t^1 - \frac{1}{2} \rho^2 \int_0^T v_t dt\right), \right. \right. \\ &\quad \left. \left. k = K, \sigma^2 = (1 - \rho^2) \int_0^T v_t dt\right)\right] \\ &\approx \int_{\mathbb{R}^{2N}} C_{BS}(G(\mathbf{w}^{(1)}, \mathbf{w}^{(2)})) \rho_N(\mathbf{w}^{(1)}) \rho_N(\mathbf{w}^{(2)}) d\mathbf{w}^{(1)} d\mathbf{w}^{(2)} \\ &= C_{RB}^N. \end{aligned} \quad (2)$$

- $C_{BS}(S_0, k, \sigma^2)$ denotes the Black-Scholes call price, for initial spot price S_0 , strike price k , and volatility σ^2 .
- G maps $2N$ independent standard Gaussian random inputs to the parameters fed to Black-Scholes formula.
- ρ_N : the multivariate Gaussian density, N : number of time steps.

Sparse Grids

A quadrature estimate of $E[F]$ is

$$\mathcal{M}_{\mathcal{I}_\ell}[F] = \sum_{\beta \in \mathcal{I}_\ell} \Delta[F_\beta], \quad (\Delta : \text{mixed difference operator}) \quad (3)$$

- **Product approach:** $\mathcal{I}_\ell = \{\max\{\beta_1, \dots, \beta_d\} \leq \ell; \beta \in \mathbb{N}_+^d\}$
- **Regular SG:** $\mathcal{I}_\ell = \{\|\beta\|_1 \leq \ell + d - 1; \beta \in \mathbb{N}_+^d\}$
- **ASGQ** based on same construction as in [6]: $\mathcal{I}_\ell = \mathcal{I}^{\text{ASGQ}}$.

ASGQ in Practice

- The construction of $\mathcal{I}^{\text{ASGQ}}$ is done by profit thresholding

$$\mathcal{I}^{\text{ASGQ}} = \{\beta \in \mathbb{N}_+^d : P_\beta \geq \bar{T}\}.$$

- **Profit of a hierarchical surplus** $P_\beta = \frac{|\Delta F_\beta|}{\Delta W_\beta}$.
- **Error contribution:** $\Delta \mathcal{E}_\beta = |\mathcal{M}^{\text{IU}(\beta)} - \mathcal{M}^{\mathcal{I}}|$.
- **Work contribution:** $\Delta W_\beta = \text{Work}[\mathcal{M}^{\text{IU}(\beta)}] - \text{Work}[\mathcal{M}^{\mathcal{I}}]$

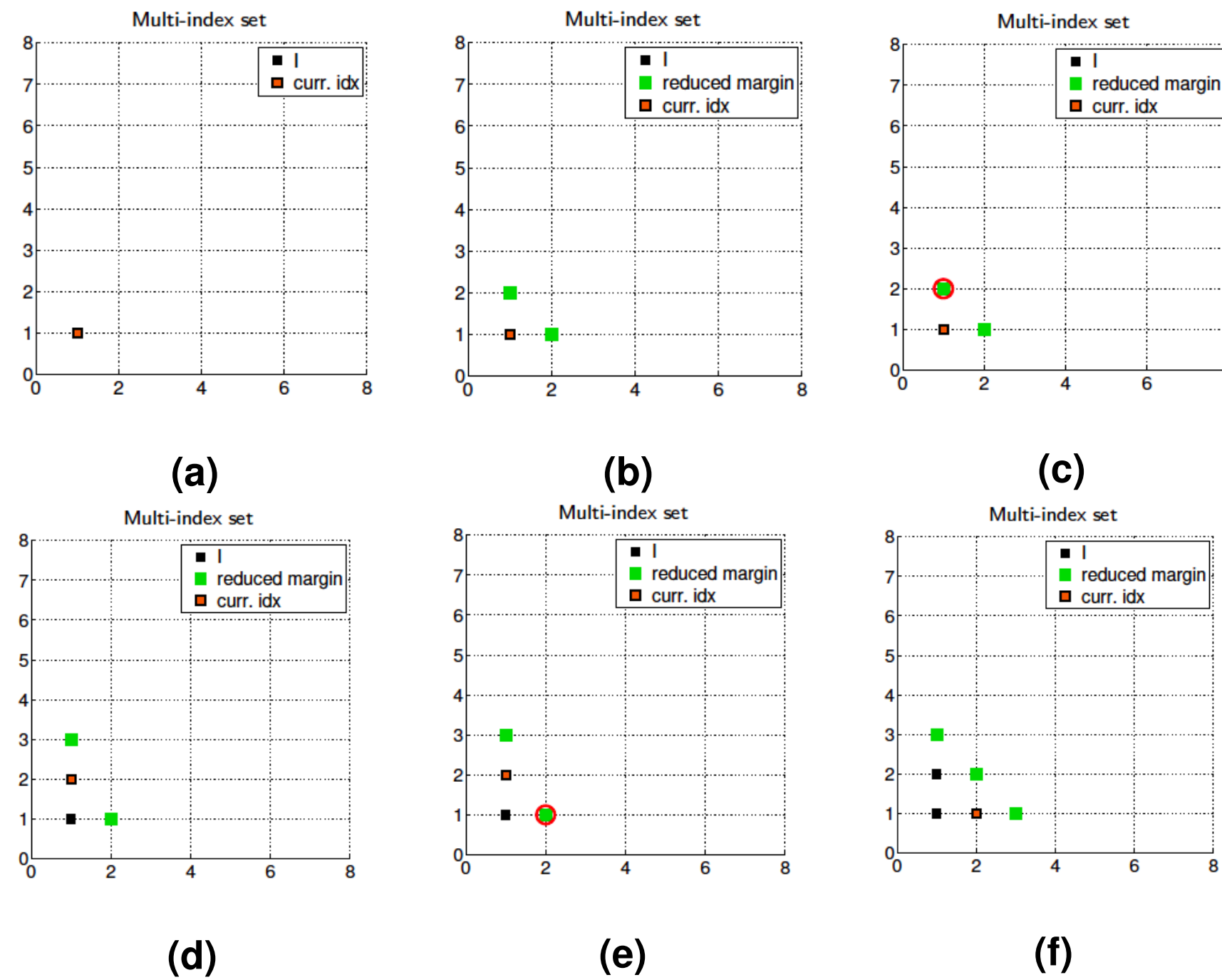


Figure 3: Construction of the index set for ASGQ method. **A posteriori, adaptive construction:** Given an index set \mathcal{I}_{k_i} , compute the profits of the neighbor indices and select the most profitable one.

Error Comparison

\mathcal{E}_{tot} : the total error of approximating the expectation in (2)

- When using ASGQ estimator, Q_N

$$\mathcal{E}_{\text{tot}} \leq |C_{RB} - C_{RB}^N| + |C_{RB}^N - Q_N| \leq \mathcal{E}_B(N) + \mathcal{E}_Q(\text{TOL}_{\text{ASGQ}}, N), \quad (4)$$
 where \mathcal{E}_Q is the quadrature error, \mathcal{E}_B is the bias, TOL_{ASGQ} is a user selected tolerance for ASGQ method.

- When using randomized QMC or MC estimator, $Q_N^{\text{MC (QMC)}}$

$$\mathcal{E}_{\text{tot}} \leq |C_{RB} - C_{RB}^N| + |C_{RB}^N - Q_N^{\text{MC (QMC)}}| \leq \mathcal{E}_B(N) + \mathcal{E}_S(M, N), \quad (5)$$
 where \mathcal{E}_S is the statistical error, M is the number of samples used for MC or randomized QMC method.

- The number of samples, M^{QMC} and M^{MC} , are chosen so that the statistical errors of QMC, $\mathcal{E}_{S, \text{QMC}}(M^{\text{QMC}})$, and MC, $\mathcal{E}_{S, \text{MC}}(M^{\text{MC}})$, satisfy

$$\mathcal{E}_{S, \text{QMC}}(M^{\text{QMC}}) = \mathcal{E}_{S, \text{MC}}(M^{\text{MC}}) = \mathcal{E}_B(N) = \frac{\mathcal{E}_{\text{tot}}}{2}, \quad (6)$$

Numerical Experiments

Table 1: Reference solution, using MC with 500 time steps and number of samples, $M = 8 \times 10^6$, of call option price under the rough Bergomi model, for different parameter constellations. The numbers between parentheses correspond to the statistical errors estimates.

Parameters	Reference solution
Set 1: $H = 0.07, K = 1, S_0 = 1, T = 1, \rho = -0.9, \eta = 1.9, \xi_0 = 0.235^2$	0.0791 (5.6e-05)
Set 2: $H = 0.02, K = 1, S_0 = 1, T = 1, \rho = -0.7, \eta = 0.4, \xi_0 = 0.1$	0.1246 (9.0e-05)
Set 3: $H = 0.02, K = 0.8, S_0 = 1, T = 1, \rho = -0.7, \eta = 0.4, \xi_0 = 0.1$	0.2412 (5.4e-05)
Set 4: $H = 0.02, K = 1.2, S_0 = 1, T = 1, \rho = -0.7, \eta = 0.4, \xi_0 = 0.1$	0.0570 (8.0e-05)

- The first set is the **closest to the empirical findings** [5, 3], suggesting that $H \approx 0.1$. The choice of values $\nu = 1.9$ and $\rho = -0.9$ is justified by [1].
- For the remaining three sets, we wanted to test the potential of our method for a **very rough case**, where variance reduction methods are inefficient.

Relative Errors and Computational Gains of the Different Methods.

Table 2: In this table, we highlight the computational gains achieved by ASGQ and QMC over MC method to meet a certain error tolerance. We note that the ratios are computed **for the best configuration with Richardson extrapolation for each method**.

Parameter set	Total relative error	CPU time ratio (MC/ASGQ)	CPU time ratio (MC/QMC)
Set 1	1%	15	10
Set 2	0.2%	21.5	73.2
Set 3	0.4%	26.7	21.3
Set 4	2%	5	10

Numerical Complexity of the Different Methods with the Different Configurations

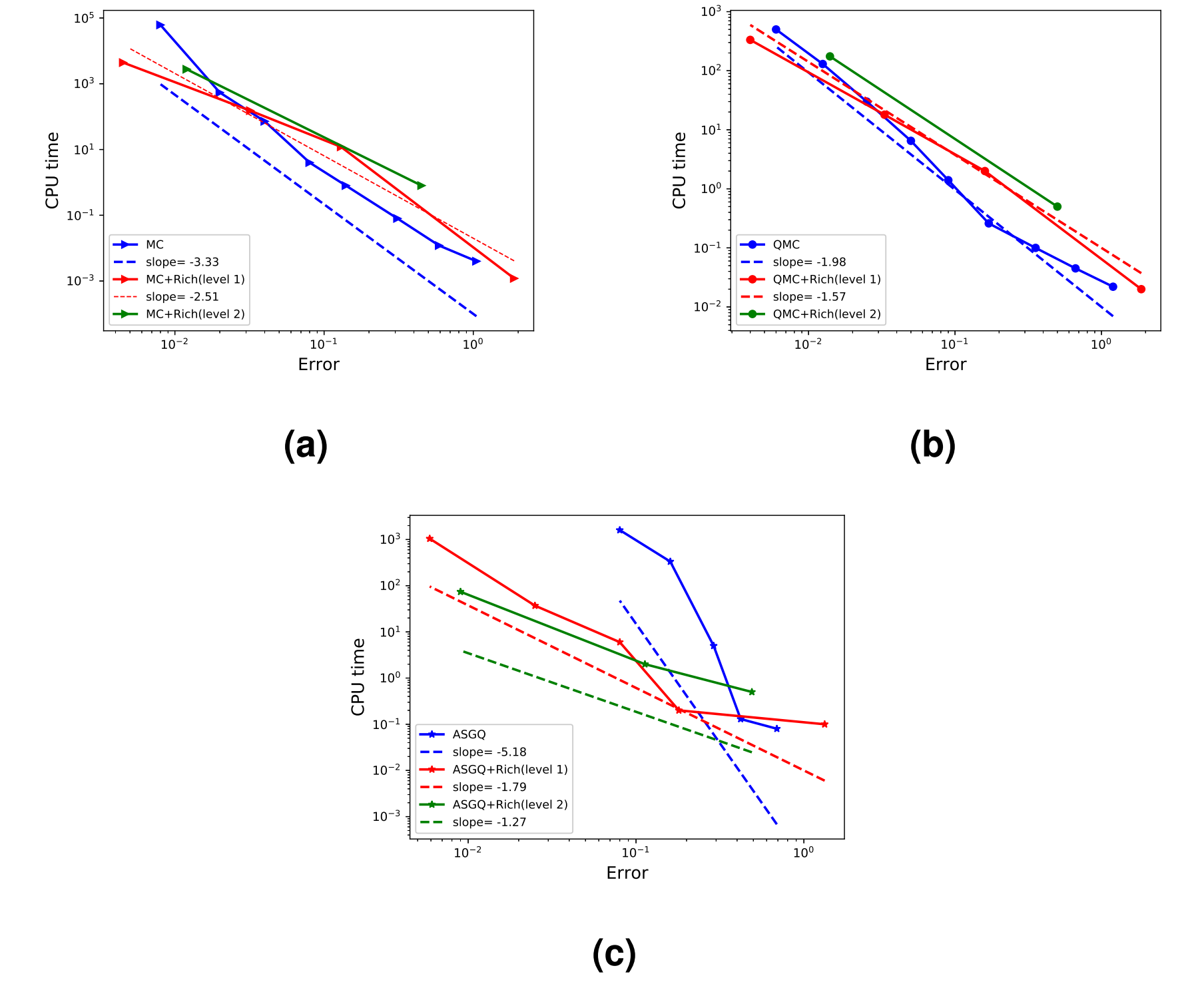


Figure 4: Comparing the numerical complexity of the different methods with the different configurations in terms of the level of Richardson extrapolation, for the case of **parameter set 1 in Table 1**. a) **MC methods**. b) **QMC methods**. d) **ASGQ methods**.

Comparing the Numerical Complexity of the Best Configurations

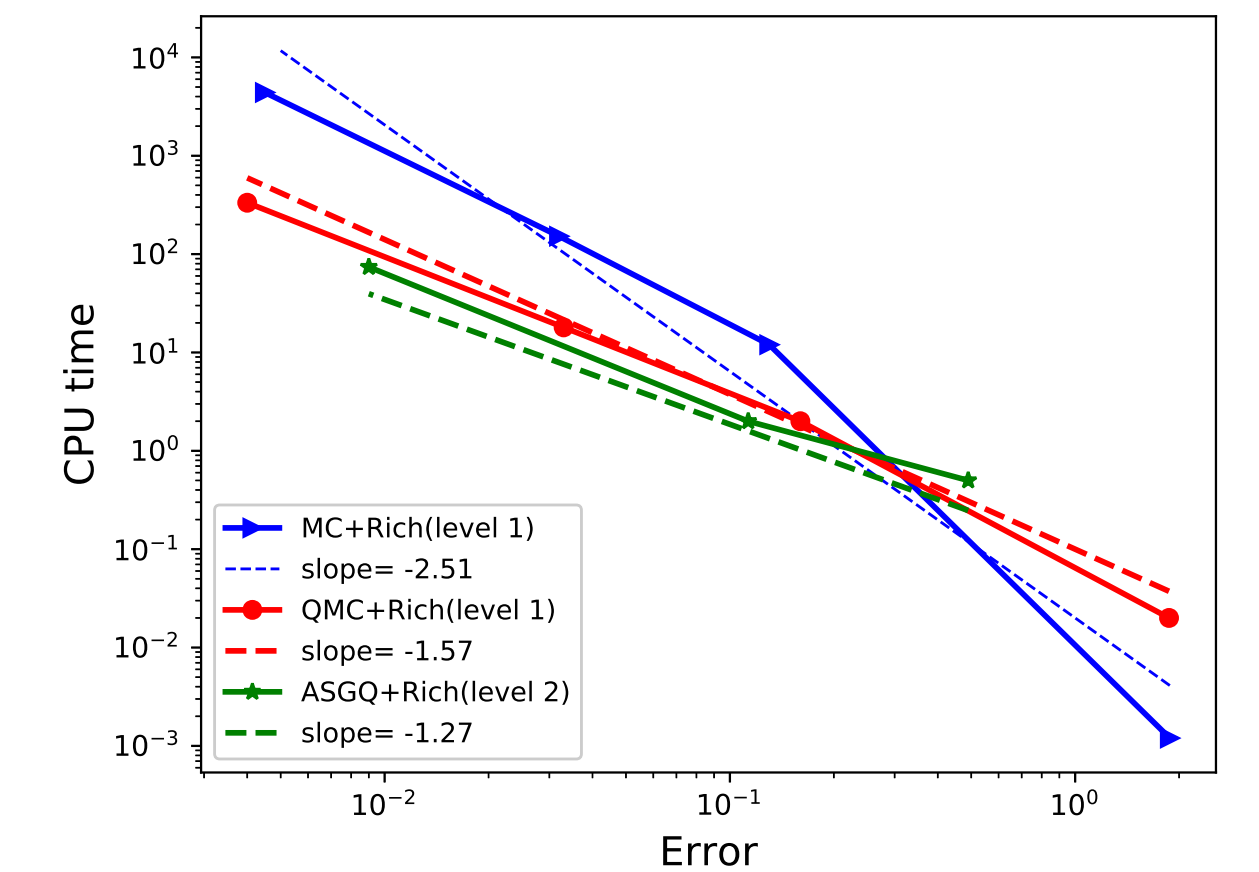


Figure 5: Computational work comparison for the different methods **with the best configurations concluded from Figure 4**, for the case of **parameter set 1 in Table 1**.

Acknowledgements

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References

- [1] C. Bayer, P. Friz, and J. Gatheral. Pricing under rough volatility. *Quantitative Finance*, 16(6):887–904, 2016.
- [2] C. Bayer, P. K. Friz, P. Gassiat, J. Martin, and B. Stempert. A regularity structure for rough volatility. *arXiv preprint arXiv:1710.07481*, 2017.
- [3] M. Bennesden, A. Lunde, and M. S. Pakkanen. Decoupling the short-and long-term behavior of stochastic volatility. *arXiv preprint arXiv:1610.00332*, 2016.
- [4] M. Bennesden, A. Lunde, and M. S. Pakkanen. Hybrid scheme for Brownian semistationary processes. *Finance and Stochastics*, 21(4):931–965, 2017.
- [5] J. Gatheral, T. Jaisson, and M. Rosenbaum. Volatility is rough. *Quantitative Finance*, 18(6):933–949, 2018.
- [6] A.-L. Haji-Ali, F. Nobile, L. Tamellini, and R. Tempone. Multi-index stochastic collocation for random pdes. *Computer Methods in Applied Mechanics and Engineering*, 306:95–122, 2016.
- [7] R. McCrickard and M. S. Pakkanen. Turbocharging Monte Carlo pricing for the rough Bergomi model. *Quantitative Finance*, pages 1–10, 2018.
- [8] A. Neuenkirch and T. Shalako. The order barrier for strong approximation of rough volatility models. *arXiv preprint arXiv:1606.03854*, 2016.