Smoothing the Payoff for Efficient Computation of Option Pricing in Time-Stepping Setting

1 Motivation

To motivate our purposes, we consider the basket option under multi-dimensional GBM model where the process $\mathbf{X}$ is the discretized d-dimensional Black-Scholes model and the payoff function g is given by

$$g(\mathbf{X}(T)) = \max\left(\sum_{j=1}^{d} c_i X^{(j)}(T) - K, 0\right). \tag{1.1}$$

Precisely, we are interested in the d-dimensional lognormal example where the dynamics of the stock are given by

$$dX_t^{(j)} = \sigma^{(j)} X_t^{(j)} dB_t^{(i)}, \tag{1.2}$$

where $\{B^{(1)}, \dots, B^{(d)}\}$ are correlated Brownian motions with correlations ρ_{ij}.

We denote by $(z_1^{(j)}, \dots, z_N^{(j)})$ the N Gaussian independent rdvs that will be used to construct the path of the j-th asset $\overline{X}^{(j)}$, where $1 \leq j \leq d$ (d denotes the number of underlyings considered in the basket). We keep the same notations by denoting $\psi : (z_1^{(j)}, \dots, z_N^{(j)}) \to (B_1, \dots, B_N)$ the mapping of Brownian bridge construction and by $\Phi : (\Delta t, B_1^{(j)}, \dots, B_N^{(j)}) \to \overline{X}_T^{(j)}$, the mapping consisting of the time-stepping scheme. Then, we can express the option price as

$$\begin{aligned}
\mathrm{E}\left[g(\mathbf{X}(T))\right] &\approx \mathrm{E}\left[g\left(\Phi \circ \psi\right)\left(z_1^{(1)}, \dots, z_N^{(1)}, \dots, z_1^{(d)}, \dots, z_N^{(d)}\right)\right] \\
&= \int_{\mathbb{R}^{d\times N}} G(z_1^{(1)}, \dots, z_N^{(1)}, \dots, z_1^{(d)}, \dots, z_N^{(d)})) \rho_{d\times N}(\mathbf{z}) dz_1^{(1)} \dots dz_N^{(1)} \dots z_1^{(d)} \dots dz_N^{(d)},
\end{aligned} \tag{1.3}$$

where $G = g \circ \Phi \circ \psi$ and

$$\rho_{d\times N}(\mathbf{z}) = \frac{1}{(2\pi)^{d\times N/2}} e^{-\frac{1}{2}\mathbf{z}^T\mathbf{z}}.$$

In the discrete case, the numerical approximation of $X^{(j)}(T)$ satisfies

$$\begin{aligned}
\overline{X}_T^{(j)} &= \Phi(\Delta t, z_1^{(j)}, \Delta B_0^{(j)}, \dots, \Delta B_{N-1}^{(j)}), \quad 1 \leq j \leq d, \\
&= \Phi(\Delta t, \psi(z_1^{(j)}, \dots, z_N^{(j)})), \quad 1 \leq j \leq d,
\end{aligned} \tag{1.4}$$

and precisely, we have

$$
\begin{aligned}
\overline{X}^{(j)}(T) &= X_0^{(j)} \prod_{i=0}^{N-1} \left[1 + \frac{\sigma^{(j)}}{\sqrt{T}} z_1^{(j)} \Delta t + \sigma^{(j)} \Delta B_i^{(j)} \right], \quad 1 \leq j \leq d \\
&= \prod_{i=0}^{N-1} f_i^{(j)}(z_1^{(j)}), \quad 1 \leq j \leq d. \qquad (1.5)
\end{aligned}
$$

1.1 Step 1: Numerical smoothing

The first step of our idea is to smoothen the problem by solving the root finding problem in one dimension after using a sub-optimal linear mapping for the coarsest factors of the Brownian increments $\mathbf{z}_1 = (z_1^{(1)}, \dots, z_1^{(d)})$. In fact, let us define for a certain $\boldsymbol{\alpha} = (\alpha_1, \dots, \alpha_d)$, the linear transformation

$$
\begin{aligned}
\omega &= L(\mathbf{z}_1) \\
&= \sum_{i=1}^{d} \alpha_i z_1^{(i)} \qquad (1.6)
\end{aligned}
$$

Then from (1.5), we have

$$
\overline{X}^{(j)}(T) = \prod_{i=0}^{N-1} g_i^{(j)}(w), \quad 1 \leq j \leq d, \qquad (1.7)
$$

where

$$
\begin{aligned}
g_i^{(j)}(w) &= X_0^{(j)} \left[1 + \frac{\sigma^{(j)}}{\sqrt{T}} \left(\frac{w - \sum_{l=1, l \neq j} \alpha_l z_1^{(l)}}{\alpha_j} \right) \Delta t + \sigma^{(j)} \Delta B_i^{(j)} \right] \\
&= X_0^{(j)} \left[1 + \frac{\sigma^{(j)} \Delta t}{\alpha_j \sqrt{T}} w - \frac{\sigma^{(j)}}{\sqrt{T}} \left(\frac{\sum_{l=1, l \neq j} \alpha_l z_1^{(l)}}{\alpha_j} \right) \Delta t + \sigma^{(j)} \Delta B_i^{(j)} \right] \qquad (1.8)
\end{aligned}
$$

Therefore, in order to determine w^*, we need to solve

$$
x = \sum_{j=1}^{d} c_j \prod_{i=0}^{N-1} g_i^{(j)}(w^*(x)), \qquad (1.9)
$$

which implies that the location of the kink point for the approximate problem is equivalent to finding the roots of the polynomial $P(w_*(K))$, given by

$$
P(w^*(K)) = \left(\sum_{j=1}^{d} c_j \prod_{i=0}^{N-1} g_i^{(j)}(w^*) \right) - K. \qquad (1.10)
$$

Using **Newton iteration method**, we use the expression $P' = \frac{dP}{dw^*}$, and we can easily show that

$$P'(w) = \sum_{j=1}^{d} c_j \frac{\sigma^{(j)}\Delta}{\alpha_j\sqrt{T}} \left(\prod_{i=0}^{N-1} g_i^{(j)}(w) \right) \left[\sum_{i=0}^{N-1} \frac{1}{g_i^{(j)}(w)} \right]. \tag{1.11}$$

Question 2: One question that arises here: Do we have to optimize over $\boldsymbol{\alpha}$ to get the optimal linear transformation? If yes what will be the metric to be used to optimize with respect to it? If no, how do we check that choices of $\boldsymbol{\alpha}$ are good choices, at least not bad chosen directions?

1.2 Step 2: Integration

At this stage, we want to perform the pre-integrating step with respect to w^*. In fact, we have from (1.3)

$$\begin{aligned}
\mathrm{E}\left[g(\mathbf{X}(T))\right] &= \int_{\mathbb{R}^{d\times N}} G(z_1^{(1)}, \dots, z_N^{(1)}, \dots, z_1^{(d)}, \dots, z_N^{(d)}))\rho_{d\times N}(\mathbf{z}) dz_1^{(1)} \dots dz_N^{(1)} \dots z_1^{(d)} \dots dz_N^{(d)} \\
&= \int_{\mathbb{R}^{d\times(N-1)}} \left(\int_{\mathbb{R}} G(w, \mathbf{z}_{-1}^{(1)}, \dots, \mathbf{z}_{-1}^{(d)}) \rho_w(w) dw \right) \rho_{d\times(N-1)}(\mathbf{z}_{-1}^{(1)}, \dots, \mathbf{z}_{-1}^{(d)}) d\mathbf{z}_{-1}^{(1)} \dots d\mathbf{z}_{-1}^{(d)} \\
&= \int_{\mathbb{R}^{d\times(N-1)}} h(\mathbf{z}_{-1}^{(1)}, \dots, \mathbf{z}_{-1}^{(d)}) \rho_{d\times(N-1)}(\mathbf{z}_{-1}^{(1)}, \dots, \mathbf{z}_{-1}^{(d)}) d\mathbf{z}_{-1}^{(1)} \dots d\mathbf{z}_{-1}^{(d)}, \\
&= \mathrm{E}\left[h(\mathbf{z}_{-1}^{(1)}, \dots, \mathbf{z}_{-1}^{(d)}) \right],
\end{aligned} \tag{1.12}$$

where $\rho_w \sim \mathcal{N}(0, \sum_{j=1}^{d} \alpha_j^2)$ and

$$\begin{aligned}
h(\mathbf{z}_{-1}^{(1)}, \dots, \mathbf{z}_{-1}^{(d)}) &= \int_{\mathbb{R}} G(w, \mathbf{z}_{-1}^{(1)}, \dots, \mathbf{z}_{-1}^{(d)}) \rho_w(w) dw \\
&= \int_{-\infty}^{w^*} G(w, \mathbf{z}_{-1}^{(1)}, \dots, \mathbf{z}_{-1}^{(d)}) \rho_w(w) dw + \int_{w^*}^{+\infty} G(w, \mathbf{z}_{-1}^{(1)}, \dots, \mathbf{z}_{-1}^{(d)}) \rho_w(w) dw
\end{aligned} \tag{1.13}$$