Wk 3 - Lecture Note

Cross Product

★★★ Only Works in $\mathbb{R}^3$

$\underset{\sim}{u} \times \underset{\sim}{v} = [u_2v_3 - u_3v_2,\ u_3v_1 - u_1v_3,\ u_1v_2 - u_2v_1]$

* results in 1 vector

$u_2 \quad u_3 \quad u_1 \quad u_2$

$v_2 \quad v_3 \quad v_1 \quad v_2$

" blue lines - red lines "

ex) $\underset{\sim}{u} = [2,1,3]$ $\quad \underset{\sim}{v} = [5,0,4]$

i) $\underset{\sim}{u} \times \underset{\sim}{v} = [4,7,-5]$

$1 \quad 3 \quad 2 \quad 1$

$0 \quad 4 \quad 5 \quad 0$

$= [4-0,\ 15-8,\ 0-5]$

$= [4,\ 7,\ -5]$

ii) $\underset{\sim}{v} \times \underset{\sim}{u} = [-4,-7,5]$

Anti-commuted

$0 \quad 4 \quad 5 \quad 0$

$1 \quad 3 \quad 2 \quad 1$

$= [0-4,\ 8-15,\ 5-0]$

$= [-4,\ -7,\ 5]$

iii) $\underset{\sim}{u} \cdot (\underset{\sim}{u} \times \underset{\sim}{v})$

$= [2,1,3] \cdot [4,7,-5]$

$= 8 + 7 - 15 = 0$

$\underset{\sim}{u}$ and $\underset{\sim}{u} \times \underset{\sim}{v}$ are orthogonal

$\underset{\sim}{v} \cdot [\underset{\sim}{u} \times \underset{\sim}{v}]$

$= [5,0,4] \cdot [4,7,-5]$

$= 20 + 0 - 20 = 0$

$\underset{\sim}{v}$ and $\underset{\sim}{u} \times \underset{\sim}{v}$ are orthogonal

Q applies for all $\underset{\sim}{u}$ and $\underset{\sim}{v}$??

iv) $\underset{\sim}{u} \times \underset{\sim}{u} = [0,0,0]$ $\quad \left([\underbrace{u_2u_3 - u_3u_2}_{0},\ \underbrace{u_3u_1 - u_1u_3}_{0},\ \underbrace{u_1u_2 - u_2u_1}_{0}] \right)$

* Cross product of itself is always $\emptyset$.

Theorems (Prove myself !)

1) $\underset{\sim}{u} \times \underset{\sim}{v} = -\underset{\sim}{v} \times \underset{\sim}{u}$, $\quad \underset{\sim}{v} \times \underset{\sim}{u} = -\underset{\sim}{u} \times \underset{\sim}{v}$

2) $\underset{\sim}{u} \cdot (\underset{\sim}{u} \times \underset{\sim}{v}) = 0$, $\quad \underset{\sim}{v} \cdot (\underset{\sim}{u} \times \underset{\sim}{v}) = 0$

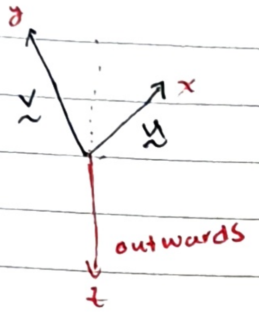
3) $\underset{\sim}{u} \times \underset{\sim}{u} = \underset{\sim}{0}$

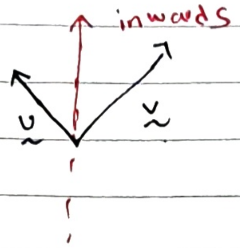
4) $\underset{\sim}{u} \times (\underset{\sim}{v} + \underset{\sim}{w}) = \underset{\sim}{u} \times \underset{\sim}{v} + \underset{\sim}{u} \times \underset{\sim}{w}$

5) $c(\underset{\sim}{u} \times \underset{\sim}{v}) = c\underset{\sim}{u} \times \underset{\sim}{v}$ or $\underset{\sim}{u} \times c\underset{\sim}{v}$

Direction of $\boxed{u \times v}$

has to be orthogonal to both u and v.





앞에 벡터가 오른쪽일 때, 바깥쪽
왼쪽일 때, 안쪽

$(u \times v) \times w \neq u \times (v \times w)$

(not always true)

(not associative)

Key Points

$u \times v$ is orthogonal to u and v

$\|u \times v\| =$ Area of ▱ (v, u)

$= \|v\| \sin\theta \, \|u\|$

$= \|u\|^2 \|v\|^2 (1 - \cancel{\sin^2}\theta)$

$\|u\|^2 \|v\|^2 - \|u\|^2 \|v\|^2 \cos^2\theta$

$\|u\|^2 \|v\|^2 - (u \cdot v)^2$

이걸 이래서야 되는데?

Applications of Cross Product:

Length & Area

$\boxed{\text{if } u, v \in \mathbb{R}^3}$

$\|u \times v\|^2 = \|u\|^2 \times \|v\|^2 - (u \cdot v)^2$

$\|u \times v\| = \|u\| \|v\| \sin\theta$ — angle between u and v. (derived from above formula

$0 \leq \theta \leq \pi$

Assumed

Proof $\|u \times v\|^2$

$= \|u\|^2 \times \|v\|^2 - (u \cdot v)^2$

$= \|u\|^2 \times \|v\|^2 - (\|u\| \|v\| \cos\theta)^2$

$= \|u\|^2 \|v\|^2 - (\|u\|^2 \|v\|^2 \cos^2\theta)$

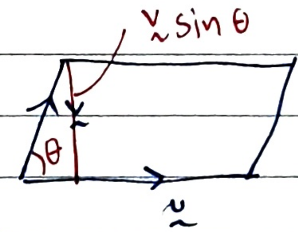
$= \|u\|^2 \|v\|^2 (1 - \cos^2\theta)$

$= \|u\|^2 \|v\|^2 \sin^2\theta$

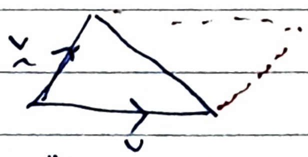
$\therefore \|u \times v\|^2 = \underbrace{\|u\|^2 \|v\|^2 \sin^2\theta}_{\text{positive}}$

$\underbrace{\|u \times v\|^2}_{\text{positive}}$

$\therefore \|u \times v\| = \|u\| \|v\| \sin\theta$ — assumed to be positive because θ between $0 \sim \pi$.



$A = \|u \times v\|$



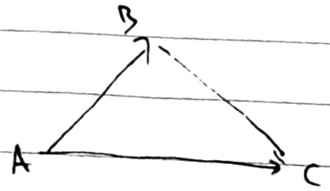
$A = \frac{1}{2} \|u \times v\|$

$A = (1,2,-1)$

$B = (2,4,1)$

$C = (0,2,-2)$

find area of $\triangle ABC$



$\overrightarrow{AB} = [1,2,2]$

$\overrightarrow{AC} = [-1,0,-1]$

$$\triangle ABC = \frac{1}{2}\|\overrightarrow{AB} \times \overrightarrow{AC}\|$$

$$= [-2-0, -2+1, 0+2]$$

$$= [-2, -1, 2]$$

$$\frac{1}{2}\|[-2,-1,2]\|$$

$$= \frac{1}{2}\sqrt{4+1+4} = \boxed{\frac{3}{2}}$$

Wk 3-2 lec note

~~Lines in $\mathbb{R}^2$, General form & Normal form~~

For any Dimension ⟨General, normal form⟩
try generalisation to $\mathbb{R}^3$ later

General form of line in plane ($\mathbb{R}^2$)

$ax + by = c$

(both a & b shouldn't be 0 at the same time)

if $b = 0$, $x = \frac{c}{a}$ = vertical line

if $a = 0$, $y = \frac{c}{b}$ = horizontal line

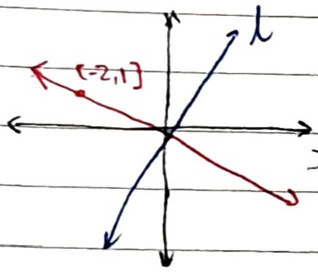
normal form

ℓ: $-2x + y = 0$ and ℓ: $y = 2x$

Normal form: $\underset{\sim}{n} \cdot \underset{\sim}{x} = 0$

normal vector, dot product, direction vector

$[a, b]$ ⟨orthogonal⟩



$y = 2x$

$-2x + y = 0 =$

$\cancel{-2x+y} = [-2,1] \cdot [x, y] = 0$

⟨Alias⟩

$\ell = \underset{\sim}{n} \cdot (\underset{\sim}{x} - \underset{\sim}{p}) = 0$

p is arbitary point on ℓ

$\underset{\sim}{n} = [-2,1]$ is orthognal to line ℓ.

For any 2 points x, p on line ℓ — $-2x + y = 1$ or $y = 2x + 1$

$\overrightarrow{PX} = \underset{\sim}{x} - \underset{\sim}{p}$

$\overrightarrow{PX}$ is parallel (same slope) to line ℓ. (in fact, it lies on ℓ)

$\therefore$ $\boxed{[-2,1] \text{ and } \overrightarrow{px} \text{ are orthogonal}}$

$-2x + y = 1$

$[-2, 1] \cdot [x, y] = 1$ $\quad$ $\underset{\sim}{n} \times \underset{\sim}{x} = 1$

when $x, y = (-1, -1)$

$[-2, 1] \cdot [-1, -1] = 2 - 1 = 1$

refer to geometric proof next page

⟨Algebraic Proof⟩

if $p = (-1, -1)$

let's note P's x-cord as P_x and y-cord as P_y

if $[-2,1][P_x, P_y] = 1$

P lies on line ℓ

and $\therefore$ P lies on the ℓ.

for any point $\vec{x}$ on line ℓ,

and point $p = [-1, -1]$ $\quad$ $\overrightarrow{px} = \underset{\sim}{p} - \underset{\sim}{x}$

~~$[-2,1] \cdot \underset{\sim}{x} = 0$ applies x~~

and $[-2,1]\underset{\sim}{x} = 0$ and $[-2,1][-1,-1] = 1$

$\therefore [-2,1] \cdot \underset{\sim}{p} = [-2,1] \cdot \underset{\sim}{x}$

$\therefore [-2,1] \cdot (\underset{\sim}{p} - \underset{\sim}{x}) = 0$ hence $\overrightarrow{px}$ is orthogonal to $[-2,1] = \underset{\sim}{n}$

Line ℓ with two points

$x = (x, y)$

$P = (P_1, P_2)$

ℓ direction vector $= \overrightarrow{PX} = \underset{\sim}{x} - \underset{\sim}{p}$ ← easier simplification

or $\overrightarrow{XP} = \underset{\sim}{p} - \underset{\sim}{x}$

$\boxed{\text{Normal Form: to general form}}$

$\ell = [a, b] \cdot [x - P_1, y - P_2] = 0$

~~a · [x − p1, y − p2]~~

$a(x - p_1) + b(y - p_2) = 0$

$ax - ap_1 + by - bp_2 = 0$

$ax + by = ap_1 + bp_2$

$ax + by = \boxed{[a, b] \cdot \underset{\sim}{p}}$

$ax + by = \boxed{\underset{\sim}{n} \cdot \underset{\sim}{p}}$

$ax + by = c \quad , \quad c = \underset{\sim}{n} \cdot \underset{\sim}{p}$

☆☆☆

수학적으로는,
아래서 위로 derive

linear algebra서 비슷
공식을 이용해서는
위에서 아래로 derive

$\boxed{\text{Geometric Proof}}$

$\ell = [a, b][\underset{\sim}{x} - \underset{\sim}{p}] = 0$

should be 0 cuz orthogona

vector on line ℓ

ex) line thru $(2, -1)$
perpendicular $[3, 4]$

$\boxed{\text{Q find general \& normal form}}$

$\ell : [3, 4] \cdot [x - 2, y + 1] = 0$

normal vector

$3(x - 2) + 4(y + 1) = 0$

$3x - 6 + 4y + 4 = 0$

$3x + 4y = \cancel{10}\ 2$

$[3, 4] \cdot [x, y] = \cancel{(3x + 4y)}\ \cancel{3}\ -(3p_1 + 4p_2)$

$\underset{\sim}{n} \cdot \underset{\sim}{p}$

$[3, 4][x, y] = \cancel{\underset{\sim}{n} \cdot \underset{\sim}{p}}\ 2$

$\underset{\sim}{n} \cdot \underset{\sim}{p} = [3, 4] \cdot [2, -1] = 2$

$-1 = -\frac{3}{4}(2) + b$

$1 = -\frac{3}{2} + b \quad b = \frac{1}{2}$

$6 + 4 = 10$

$y = -\frac{3}{4}x + \frac{1}{2}$

$4y = -3x + 2$

$y = \frac{-3}{4}x + \frac{1}{2}$

$3x + 4y \ -c = 0$

$3x + 4y = c$

~~$3(x-2) + 4(y+1) = 0$~~

$3(2) + 4(-1) = c$

$6 - 4 = c$

$c = 2$

normal:

Week 3-2 lec cont.

ex 1)

Line perpendicular to vector [3,4]
& goes thru (2,-1)

Q find normal and general form

Q Does loc of P Matter?

$y = 2x + 1$

$-2y + y = 1$

NOPE as long as P lies on ℓ.

$P_1 = (2,5)$ $P_2 = (-2,-3)$

$\ell = [-2,1]\cdot[x,y] = \underset{\sim}{N} P_1 = -4+5 = 1$

$[-2,1]\cdot[x,y] = \underset{\sim}{N} P_2 = 4-3 = 1$

Normal form

$\underset{\sim}{n} = [3,4]$ (normal vector)

$[3,4]\cdot[\underset{\sim}{x} - \underset{\sim}{p}] = 0$, $\underset{\sim}{p} = [2,-1]$

$$\boxed{\ell : [3,4]\cdot([x,y] - [2,-1]) = 0}$$

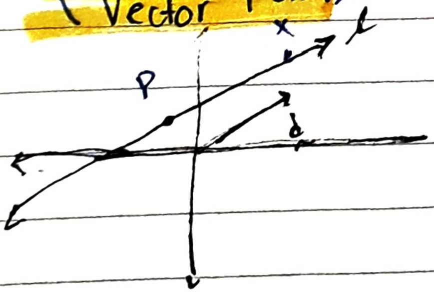
General form

$[3,4][x,y] = 3\times 2 + 4(-1)$

$$\boxed{[3,4][x,y] = 2}$$

Vector Form & Parametric Form (only applies to lines in $\mathbb{R}^2$)

⟨Vector Form⟩



let P be a point on ℓ

$\overrightarrow{PX}$ lies on ℓ $\therefore$ $\overrightarrow{PX} \parallel \ell$

$t \in \mathbb{R}$

$\therefore \overrightarrow{PX} = t \times \underset{\sim}{d}$ (t is scalar)

direction vector

$\ell \parallel \underset{\sim}{d}$

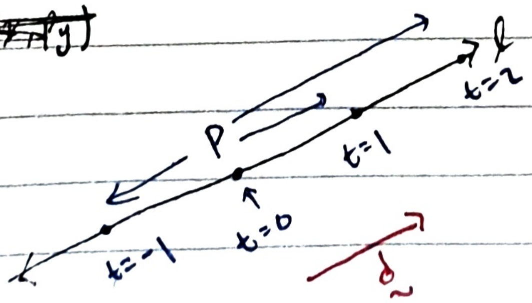
$\ell \parallel \overrightarrow{PX}$

$\therefore \underset{\sim}{x} - \underset{\sim}{p} = t \times \underset{\sim}{d}$

or

$$\boxed{\underset{\sim}{x} = t \times \underset{\sim}{d} + \underset{\sim}{p}}$$

~~When p = (px, py)~~



Parametric Equations

when $\underset{\sim}{p} = [p_1, p_2]$

and $\underset{\sim}{d} = [d_1, d_2]$

$$(x,y) \in \begin{cases} x = p_1 + t d_1 \\ y = p_2 + t d_2 \end{cases}$$

ex 2)

ℓ thru (1,2) and (5,7)

a vector form & parametric equations

direction vector $\underset{\sim}{d} = [4,5]$

vector form $\boxed{\underset{\sim}{x} = [1,2] + t[4,5]}$ $t \in \mathbb{R}$

parametric eqs $\boxed{\begin{aligned} x &= 1 + 4t \\ y &= 2 + 5t \end{aligned} \quad t \in \mathbb{R}}$

Generalisation to $\mathbb{R}^3$

Vector form

$$\underset{\sim}{x} = \underset{\sim}{p} + t\underset{\sim}{d} \quad (\underset{\sim}{x}, \underset{\sim}{p}, \underset{\sim}{d} \in \mathbb{R}^3)$$

$$(t \in \mathbb{R})$$

point on line

$$\left\{\begin{aligned} x &= p_1 + d_1 \\ y &= p_2 + d_2 \\ z &= p_3 + d_3 \end{aligned}\right) \text{ parametric equations}$$

general form

$$ax + by + cz = k$$

$$\underbrace{[a,b,c]}_{\underset{\sim}{n}} \cdot \underbrace{[x,y,z]}_{\underset{\sim}{x}} = k$$

normal form

$$k = \underset{\sim}{n} \times \underset{\sim}{d}$$

$$= [x,y,z] \cdot [a,b,c]$$

$$[a,b,c] \cdot [x - p_1, y - p_2, z - p_3] = 0$$

$$\underset{\sim}{n} \cdot (\underset{\sim}{x} - \underset{\sim}{p}) = 0$$