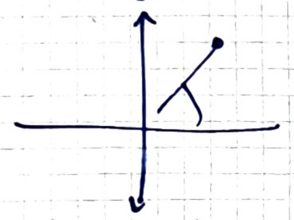
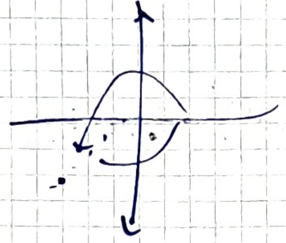
# Week 2-1 lecture

~~Arg~~ arg$(z)$ = argument of $z = a+bi$

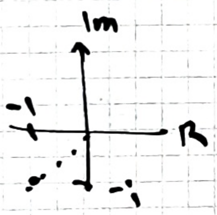
is the "angle" made with the positive axis.

$\arg(1+i) = \frac{\pi}{4}$ $\qquad$ $\arg(-1-i) = -\frac{3}{4}\pi = \frac{5}{4}\pi$





ex 1) Write 4 arguments for $z = -1-i$




$\theta = -\frac{3}{4}\pi$ or $\frac{5\pi}{4}$ or $\frac{13\pi}{4}$ or $-\frac{11\pi}{4}$

capital A for principal arg

**Principal argument** $\quad$ Arg$(z)$

$-\pi < \theta \leq \pi \quad = \quad \text{Arg}(z) \in (-\pi, \pi]$

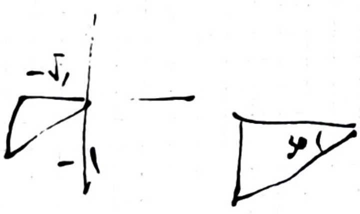
Arg$(z)$

if $y \geq 0 \Rightarrow$ ~~Arg~~ $\theta \in [0, \pi]$

if $y < 0 \Rightarrow \theta \in (-\pi, 0)$

$z = -\sqrt{3} - i$ $\quad$ find $|z|$ and Arg$(z)$

$|z| = \sqrt{3+1} = 2$



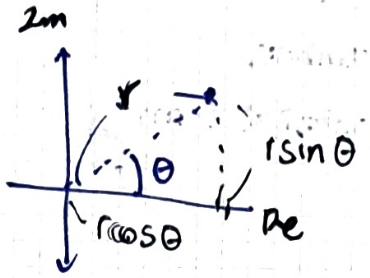

Arg$(z) = -150° = -\frac{5}{6}\pi$

# Polar form

$$z = r(\cos\theta + i\sin\theta)$$

~~$(z = a+bi)$~~

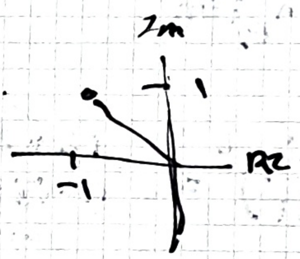
$$\begin{aligned} z &= a + bi \\ &= r\cos\theta + r\sin\theta\, i \\ &= r(\cos\theta + i\sin\theta) \end{aligned}$$




(a) polar form of $-1+i$

$r = |z| = \sqrt{2}$

$$\text{p.f} = \sqrt{2}\left(\cos\tfrac{3\pi}{4} + i\sin\tfrac{3\pi}{4}\right)$$




(b) $z = 2\left(\cos\left(\frac{\pi}{3}\right) + i\sin\frac{\pi}{3}\right)$

$$= 2\left(\frac{1}{2} + i\frac{\sqrt{3}}{2}\right) = 1 + \sqrt{3}i$$

# Euler's formula

$$\underline{e^{i\theta} = \cos\theta + i\sin\theta}$$

(will be proved in wk 8)

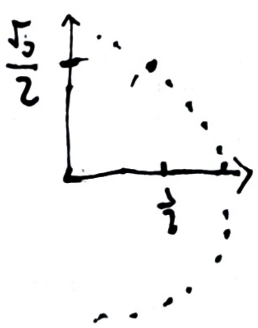
All lies on the unit circle when

if $\theta = \pi$

$$e^{i\pi} = \overset{-1}{\cancel{\cos\pi}} + \overset{0}{\cancel{i\sin\pi}}$$

$e^{i\pi} + 1 = 0$ "Euler's Identity"

(a) $e^{\frac{\pi}{3}i}$ on complex plane




(b) $2+3i$ in complex plane

~~$\sqrt{13}$~~

$r = \sqrt{13}$

$\theta = \sin^{-1}\left(\frac{3}{\sqrt{13}}\right)$ or $\tan^{-1}\left(\frac{3}{2}\right)$

$$z = \sqrt{13}\, e^{i(\tan^{-1}(\frac{3}{2}))}$$

$\times$ , $\div$ Much easier for complex numbers in Polar exponential form

$z = re^{i\theta}$

$w = se^{i\theta}$

$z \times w = r \cdot s \cdot e^{i(\theta + \theta)}$

$\frac{z}{w} = \frac{r}{s} e^{i(\theta - \theta)}$

$z^n = r^n e^{i\theta n}$

ex) $\left(\frac{2+2i}{1-\sqrt{3}i}\right)^6 = \frac{\overbrace{(2+2i)}^{z}{}^6}{\underbrace{(1-\sqrt{3}i)}_{w}{}^6}$

$z = 2+2i = 2\sqrt{2}\left(\sqrt{2} + \sqrt{2}i\right)$

$= 2\sqrt{2}\, e^{i\frac{\pi}{4}}$

$w = 1 - \sqrt{3}i$

$r = 2$ $\quad = 2e^{i\frac{-\pi}{6}}$

$\theta = -\frac{\pi}{3}$

$\sqrt{2}^6 \, e^{i\left(\frac{6\pi}{4} + \pi\right)} = 8e^{i\frac{1}{2}\pi} = 8\left(\cos\tfrac{1}{2}\pi + i\sin\tfrac{1}{2}\pi\right)$

$= \boxed{8i}$

Roots of Complex Numbers

$4^{th}$ roots of $-1+i$

$z^4 = -1+i$ $\quad r = \sqrt{2}$ $\quad \theta = \tan^{-1}(-1) = \frac{3\pi}{4}$

$z^4 = \sqrt{2}\, e^{i\frac{3\pi}{4}}$

$2^{\frac{1}{8}} e^{i\frac{3}{16}\pi}$

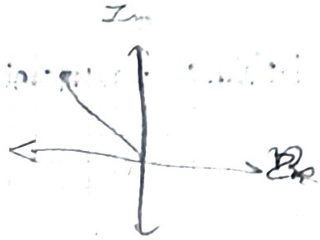
$A^2 = \sqrt{2}\, e^{i\frac{3\pi}{4}}$

$A = \pm$

$$z^4 = \frac{-1+i}{w}$$

$$w = \sqrt{2}\left(\cos\frac{-3\pi}{4} + i\sin\left(-\frac{3\pi}{4}\right)\right)$$

$$A^2 = \sqrt{2}\, e^{i\left(-\frac{3\pi}{4}\right)}$$




$$A = \left(2^{\frac{1}{4}}\right)^2$$

or

$$\cancel{\left(-2^{\frac{1}{4}}\right)^2} \quad \boxed{r \text{ can't be negative}}$$

**lec 2-2 note**

$$z^n = re^{i\theta}$$

$$z = r^{\frac{1}{n}} e^{\frac{i\theta}{n}}$$

$$= r^{\frac{1}{n}} e^{\frac{i(\theta+2\pi k)}{n}} \quad k = 0, 1, 2 \ldots n-1$$

$$z^4 = -1+i$$

$$2^{\frac{1}{8}} = \sqrt[8]{2}$$

$$z = \sqrt{2}^{\frac{1}{4}} e^{i\frac{(\theta+2\pi k)}{4}} \quad k = 0 \sim 3$$

$$\cancel{z = \sqrt[8]{2}}$$ if $k = n$,

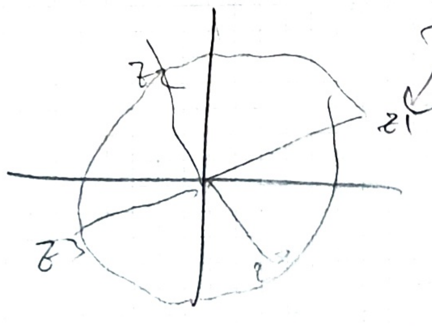
$$r = \sqrt[8]{2}$$

$z =$

$$\theta = \frac{+\frac{3\pi}{4}}{4}, \frac{\frac{11\pi}{4}}{4}, \frac{\frac{19\pi}{4}}{4}, \frac{\frac{27\pi}{4}}{4}$$

$$\theta = -\frac{3\pi}{16}, \frac{11\pi}{16}, \frac{19\pi}{16}, \frac{27\pi}{16}$$

$$z = 2^{\frac{1}{8}}\left(e^{i\theta}\right) \quad \theta \in \left\{\frac{3\pi}{16}, \frac{11\pi}{16}, \frac{19\pi}{16}, \frac{27\pi}{16}\right\}$$




roots of z.

# Solving polynomial equations

$p(z) = a_n z^n + a_{n-1} z^{n-1} \cdots a_1 z + a_0$

$\downarrow$

roots

$p(a) = 0$ , $a \in \mathbb{C}$

If coefficients are real numbers...

- complex roots occur in conjugate pairs.

$\therefore$ If $a_n a_{n-1} \cdots a_1 a_0 \in \mathbb{R}$ and

$z = a$ is a root of $p(z) = a_n z^n + \cdots a_1 z + a_0$

then $z = \bar{a}$ is also the root of $p(z)$.

(proof in lec ~~note~~ slides using

property $\overline{zw} = \bar{z} \cdot \bar{w}$, $\overline{z+w} = \bar{z} + \bar{w}$

$p(z) = z^4 - 2z^3 + 6z^2 - 2z + 5$

Show $i$ is a root

$p(i) = (+1) - 2(-i) + (-6) - 2i + 5$

$= 1 + 2i - 6 - 2i + 5 = 0$

Hence, $-i$ is also a root.

$\therefore (z-i)(z+i)$ is a factor of $p(z)$.

$p(z) = (z^2+1)q(x)$

$$\begin{array}{r} z^2 - 2z + 5 \\ z^2+1 \,\big)\, z^4 - 2z^3 + 6z^2 - 2z + 5 \\ -\, z^4 \quad\quad\quad - z^2 \quad\quad\quad\quad \\ \hline -2z^3 + 5z^2 - \cancel{2z} + 5 \\ +2z^3 \quad\quad\quad + 2z \quad\quad \end{array}$$

$z^2 - 2z + 5 = 0$

$(z-1)^2 = -4$

$A = 2i$ or $-2i$

$z - 1 = 2i$

$\boxed{z = 1 + 2i \text{ or } 1 - 2i}$

## Complex exponential function

$e^z = e^{x+iy} = \underline{e^x e^{iy}}$   when $z = x + yi$

### Properties of $e^z$

(1) $|e^z| = \cancel{|e^x||e|}$

$|e^x e^{iy}| = |e^x|\,|e^{iy}| = |e^x| \times 1 = e^x$

(2) $\arg(e^z) = y \left\{ e^z = \underbrace{(e^x)}_{r}\underbrace{(e^{yi})}^{\theta} \right.$

(3) $e^{z+w} = e^z \times e^w$

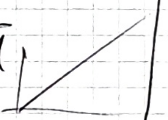
(4) $e^{-z} = \dfrac{1}{e^z}$

$z = re^{i\theta} = r\cos\theta + ir\sin\theta$

(5) $\overline{e^z} = e^{\bar{z}}$

modulus

$e^z = \underset{\sqrt{2}}{\underline{e^x}} \times e^{iy} = \boxed{1+i}$   $\theta = \frac{\pi}{4}$



$e^x = \sqrt{2}$   $e^{iy} = e^{i\frac{\pi}{4}}$

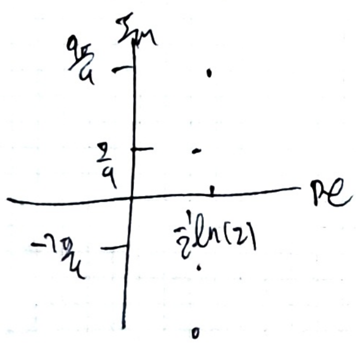
$x = \ln(\sqrt{2})$
$= \ln(2^{\frac{1}{2}}) = \frac{1}{2}\ln(2)$

$y = \frac{\pi}{4} + 2k\pi$

periodic function
period: $2\pi k i$

~~$z = \frac{1}{2}\ln(2) +$~~

$z = \frac{1}{2}\ln(2) + (\frac{\pi}{4} + 2\pi k)$   $k \in \mathbb{Z}$



$$\cos\theta = \frac{e^{i\theta} + e^{-i\theta}}{2}$$

$$\sin\theta = \frac{e^{i\theta} - e^{-i\theta}}{2}$$

$$e^{i\theta} = \cos\theta + i\sin\theta \qquad e^{-i\theta} = \frac{1}{e^{i\theta}}$$

$$\cos\theta = e^{i\theta} - i\sin\theta$$

veng? 4

$$\sin^3\theta = \frac{1}{4}(3\sin\theta - \sin(3\theta))$$