Wk 3-1 Lecture Note

Functions

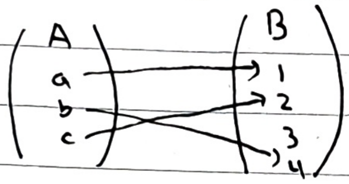
Image : ~~f~~ (range)

f(a) is called the image of a under f.

Preimage : ~~(domain)~~

$f^{-1}(b)$ is called pre-image of b under f.

$f: A \to B$



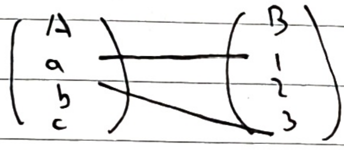
Domain : A $\{a, b, c\}$

Codomain : B $\{1, 2, 3, 4\}$

range : f(A) $\{1, 2, 4\}$

Is it a function?

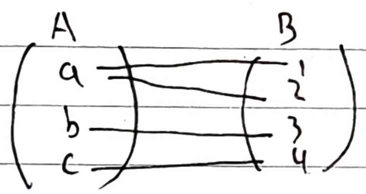
① $f: A \to B$



Not a function because f(b) is undefined.

every value in domain should be defined !

② $f: A \to B$



Not a function because f(a) gets mapped to 2 values ! $\{1, 2\}$

③ $f: \mathbb{R} \to \mathbb{R}$ $f(x) = \frac{1}{x}$

Not a function because f(0) is undefined

④ $f: \mathbb{N} \to \mathbb{Z}$ $f(x) = \sqrt{x}$

Not a function because for some natural number N such as 2 $\sqrt{2}$ is not defined under any integer.

Natural Domain

↳ set of real numbers which the defining formula gives a well-defined real value.

ex) Natural domain for $f(\frac{1}{x})$

$\mathbb{D} = \mathbb{R} \setminus \{0\}$

Extension : to set of complex numbers

$f: \mathbb{C} \to \mathbb{C}$

ex) $f(z) = e^z$

domain : $\mathbb{C}$

range : $\mathbb{C} \setminus \{0\}$

↳ No matter how big/small x is $R(z)$ does not map to 0.

$e^x \cdot e^{yi} = e^x [\cos y + i \sin y]$

$Re(z) = -e^x \sim e^x$ $\quad x \in \mathbb{R}$

$Im(z) = ie^x \sim ie^x$ $\quad x \in \mathbb{R}$

$e^{-\infty}$

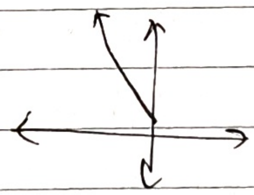
$e^x \mapsto (\frac{1}{e^{\infty}}, e^{\infty})$

Surjective, Injective & bijective.

① Surjective : (onto) Range = Codomain ~~(f(a) = B)~~

② Injective : (one-to-one) for any $\{a_1, a_2 \mid a_1 \neq a_2\}, a_1, a_2 \in D\} \to f(a_1) \neq f(a_2)$

③ bijective : subjective & injective



$f: (-\infty, 0] \to [0, \infty)$ $\boxed{\text{bijective}}$

$x \to x^2$

CoDomain = Range

& ~~one-to-one~~

$f: (-\infty, 0] \to \mathbb{R}$, $x \to x^2$

Injective, not surjective

To prove surjectivity,

- find range

or go the other way around...

for $y = [0, \infty)$ there exists $-\sqrt{y} \in \mathbb{R}$ $(-\infty, 0]$

$f(-\sqrt{y}) \to y$

Surjective

range$(f) = \{x^2 \mid x \in (a, -\infty, 0]\}$

$= [0, \infty)$

Injective

if $x_1 = x_2$

$f(x) = f(x_2)$

$x_1^2 = x_2^2$

$x_1 \pm \sqrt{x_2^2} = \pm x_2$, $x_1, x_2 \leq 0$

$\therefore x_1 = x_2$ (only one possible mapping for each x)

Combining functions

for function f, g

- $f+g$
- $f-g$
- $f \times g$

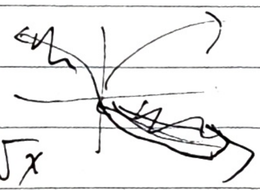
} $x \in$ ~~Domain~~ $\mathrm{Domain}(f) \cap \mathrm{Domain}(g)$

- $\frac{f}{g}$ } $x \in \mathrm{Domain}(A) \cap \mathrm{Domain}(g)$ and $g(x) \neq 0$.

Composing functions

if $\boxed{\begin{array}{l} f: A \to B \\ g: B \to C \end{array}}$

$g \circ f : A \to C$



if $f: x \to \sqrt{x}$

$g: x \to -\sqrt{x}$

then $f: [0,\infty) \to [0,\infty)$

$g: [0,\infty) \to (-\infty, 0]$

$g \circ f : [0,\infty) \to [-\infty, 0]$

ex) $g \circ f(4) = g(f(4)) = g(2) = -\sqrt{2}$

$\boxed{\text{ex2}}$

$f: \mathbb{R} \to \mathbb{R}$, $x \to x^3+1$

$g: \mathbb{R} \to [0,\infty)$, $x \to e^x$

$g \circ f : \mathbb{R} \to (0,\infty)$

$x \to e^{x^3+1}$

$f \circ g : \mathbb{R} \to [1,\infty)$

$x \to (e^x)^3+1$

$x^2-1 \leq 0$

$x^2-1 > 0$ $x^2 \leq 1$

$[-1,1] \to [0,1]$

$\boxed{y = \sqrt{-x^2+1}}$

$y = \pm\sqrt{-x^2+1}$

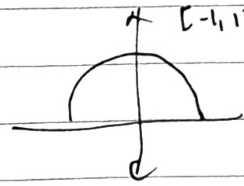
$y^2 = -x^2+1$

$x^2+y^2 = 1$

Vertical line Test

function's graph must pass vertical line test

ex) ~~sem~~ upper semi-circle



$[-1,1] \to [0,1]$

Horizontal line Test

one-to-one