wk 2-1 Lecture

linear combination

v is a lineal combination of vectors $v_1, v_2 \cdots v_q$

If there are scalars such that

$$v = \{c_1 v_1 + c_2 v_2 \cdots c_k v_k\}$$

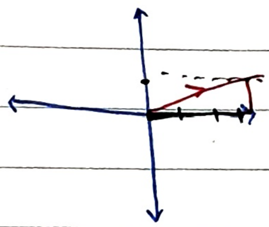
coefficients

ex)

$$\begin{bmatrix} -1 \\ 2 \\ -11 \end{bmatrix} \text{ is linear comb of } \begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix} \text{ and } \begin{bmatrix} 3 \\ 0 \\ 7 \end{bmatrix}$$

because $2\begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} - \begin{bmatrix} 3 \\ 0 \\ 7 \end{bmatrix} = \begin{bmatrix} 2-3 \\ 2-0 \\ -4-7 \end{bmatrix} = \begin{bmatrix} -1 \\ 2 \\ -11 \end{bmatrix}$

$$\begin{bmatrix} 3 \\ 1 \end{bmatrix} = 3\begin{bmatrix} 1 \\ 0 \end{bmatrix} + 1\begin{bmatrix} 0 \\ 1 \end{bmatrix}$$



Show that when $v_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $v_2 = \begin{bmatrix} -2 \\ -2 \end{bmatrix}$

$v = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$ v is NOT a linear comb of v_1 and v_2

Assume $\begin{bmatrix} 1 \\ 3 \end{bmatrix} = a\begin{bmatrix} 1 \\ 1 \end{bmatrix} + b\begin{bmatrix} -2 \\ -2 \end{bmatrix}$

$$= \begin{bmatrix} a-2b \\ a-2b \end{bmatrix}$$

$1 = -2b + a$

$3 = -2b + a$

$-2 = 0$

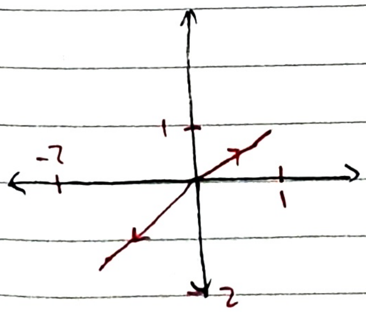
~~$a = 2$ $b = -\frac{1}{2}$~~

Based on assumption, there exists no constant a, b.

$\therefore$ contradiction

$\therefore$ Assumption is wrong

$\therefore$ not a linear combination.



Geometrically, because v_1 and v_2 are on the same line, linear comb of v_1 and v_2 will always be on the same line.

$\therefore$ v is not on the line $y=x$.

Dot Product

$\underset{\sim}{v} = [v_1, v_2 \cdots v_n]$

$\underset{\sim}{w} = [w_1, w_2 \cdots w_n]$

$v \cdot w = \sum v_1 w_1 + v_2 w_2 + v_3 w_3 \cdots v_n w_n$

$\underset{\sim}{u} = [1, 2]$, $\underset{\sim}{v} = [2, -3]$

$u \cdot v = 2 - 6 = -4$

$\underset{\sim}{u} \cdot \underset{\sim}{v} = \underset{\sim}{v} \cdot \underset{\sim}{u}$ commutative

$\underset{\sim}{u} \cdot (\underset{\sim}{v} + \underset{\sim}{w}) = \underset{\sim}{u} \cdot \underset{\sim}{v} + \underset{\sim}{u} \cdot \underset{\sim}{w}$ distributive

$(c\underset{\sim}{u}) \cdot \underset{\sim}{v} = c(\underset{\sim}{u} \cdot \underset{\sim}{v})$ constant can be pulled out

* $u \cdot u \geq 0$

for $\underset{\sim}{u} = [u_1, u_2 \cdots u_n]$

$\underset{\sim}{u} \cdot \underset{\sim}{u} = \sum u_1^2 + u_2^2 \cdots u_n^2$

= sum of ~~post~~ non-negative integers

≥ 0

More formal approach

let $\underset{\sim}{u} = [u_1, u_2 \cdots u_n]$

Want to show $u \cdot \underset{\sim}{u} \geq 0$

we know $\underset{\sim}{u} \cdot \underset{\sim}{u} = \sum u_1^2 + u_2^2 + u_3^2 \cdots u_n^2$

Since $u_1, u_2 \cdots u_n \in \mathbb{R}$, $u_1^2, u_2^2 \cdots u_n^2 \in$ ~~NNI~~ N (~~NNI = {x}~~)

~~Sums~~ sum of natural numbers are bounded inside natural numbers.

Therefore, $\underset{\sim}{u} \cdot \underset{\sim}{u} \geq 0$

$\underset{\sim}{u} \cdot \underset{\sim}{u} = 0$
$\longleftrightarrow$
$\underset{\sim}{u} = \underset{\sim}{0}$

Assume $\boxed{\underset{\sim}{u} \cdot \underset{\sim}{u} = 0}$

$\underset{\sim}{u} \cdot \underset{\sim}{u} = u_1^2 + u_2^2 \cdots u_n^2$

Since $u_1^2, u_2^2 \cdots u_n^2$ are all positive, squared.

for their sum to be zero, all elements of $\underset{\sim}{u}$ must be 0.

$\therefore$ all elements of $\underset{\sim}{u}$ must be 0 (because $\sqrt{0} = 0$)

$\therefore \boxed{\underset{\sim}{u} = [0, 0, \cdots 0] = \underset{\sim}{0}}$

Assume $\underset{\sim}{u} = \underset{\sim}{0}$, $\underset{\sim}{u} \cdot \underset{\sim}{u} = \sum u_1^2 + u_2^2 + u_3^2 \cdots u_n^2$

$= 0 + 0 + \cdots 0 = 0$ $\therefore \boxed{\underset{\sim}{u} \cdot \underset{\sim}{u} = 0}$

Length of vectors

$\underset{\sim}{v} = [a, b]$

$\text{len}(\underset{\sim}{v}) = \sqrt{a^2+b^2}$

$\underset{\sim}{v} = [a, b, c]$

$\text{len}(\underset{\sim}{v}) = \sqrt{\left(\sqrt{a^2+b^2}\right)^2 + c^2} = \sqrt{a^2+b^2+c^2}$

$\|v\|$

General formula $\boxed{\text{len}(\underset{\sim}{v}) = \sqrt{\underset{\sim}{v} \cdot \underset{\sim}{v}}}$

$\|\underset{\sim}{v}\| = \sqrt{\underset{\sim}{v} \cdot \underset{\sim}{v}}$

$\underset{\sim}{v} \cdot \underset{\sim}{v} = 0 \iff \|\underset{\sim}{v}\| = 0$

$\therefore \|\underset{\sim}{v}\| = 0 \iff \underset{\sim}{v} = \underset{\sim}{0}$

Week 2-2 Lecture

$||v||$ = length of vector $\underset{\sim}{v}$

$|c|$ = absolute val of scalar c

$||c \cdot v|| = ||v|| \cdot |c|$

Given a vector $\underset{\sim}{v}$ and a scalar c,

I want to show that $||c \cdot v|| = ||v|| \cdot |c|$

let $\underset{\sim}{v} = [v_1, v_2 \cdots v_n]$

then $c \cdot \underset{\sim}{v} = [cv_1, cv_2 \cdots cv_n]$

~~By definition,~~ We know that $||c \cdot v|| = \sqrt{c\underset{\sim}{v} \cdot c\underset{\sim}{v}}$

$$\begin{aligned} & cv \cdot cv \\ &= \Sigma\, c^2v_1^2 + c^2v_2^2 \cdots c^2v_n^2 \\ &= c^2(v_1^2 + v_2^2 \cdots v_n^2) \\ &= c^2\, v \cdot v \end{aligned}$$

by definition

$$\therefore ||c \cdot v|| = \sqrt{c^2(v \cdot v)} = |c|\sqrt{v \cdot v} = |c| \times ||v||$$

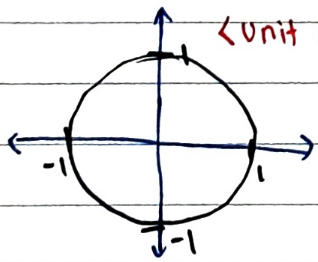
Vector with length 1 = UNIT VECTOR

show $\begin{bmatrix}1\\0\end{bmatrix}$ is a unit vector

$||\begin{bmatrix}1\\0\end{bmatrix}|| = \sqrt{1^2+0^2} = \sqrt{1^2} = 1$

Because $\begin{bmatrix}1\\0\end{bmatrix}$'s len is 1, it is unit vector

also



(Unit circle)

↑ Geometrically, unit vectors (of dimenension 2) lie on unit vector

In $\mathbb{R}^3$, unit vectors lie on unit sphere

v = unit vector × c

$\underset{\sim}{v} = [1, 1]$ $\quad ||v|| = \sqrt{2}$

$\underset{\sim}{v} = \sqrt{2}\left[\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right]$

unit vector

$|u \cdot v| \leq \|u\| \times \|v\|$ (proof - not needed fact) Cauchy-Schwahn

$\|u+v\| \leq \|u\| + \|v\|$ ★ Triangle Inequality

Triangle Inequality

$\|u+v\| \leq \|u\| + \|v\|$ $\quad \|u+v\|^2 \leq \|u\|^2 + \|v\|^2 + 2\|u\|\|v\|$

$\Rightarrow \sqrt{(u+v)^2} \leq \sqrt{u^2} + \sqrt{v^2}$

$\Rightarrow (u+v)\cdot(u+v) \leq (u \cdot u) + (v \cdot v) + 2\sqrt{u^2 v^2}$

$u \cdot (u+v) + v \cdot (u+v) \leq (u \cdot u) + (v \cdot v) + 2\sqrt{u^2 \cdot v^2}$

$2 \times u \cdot v$

$\boxed{(\cancel{u \cdot u}) + (u \cdot v) + (v \cdot u) + (\cancel{v \cdot v})} \leq (\cancel{u \cdot u}) + (\cancel{v \cdot v}) + 2\cancel{\sqrt{u^2 \cdot v^2}}$

$(u \cdot v) + (v \cdot u) \leq 0$

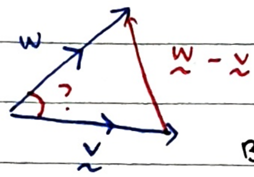
$(u \cdot v) + (u \cdot v) \leq 0$

$2(u \cdot v) \leq 0$

$\|u\| + \|v\| + 2(u \cdot v)$ } $|\text{scalar}| \geq \text{scalar}$

$\leq \|u\| + \|v\| + 2|u \cdot v|$ } cauchy-schwahn

$\leq \|u\| + \|v\| + 2\|u\|\|v\|$



By cosine rule,

$|w - v|^2 = |w|^2 + |v|^2 - 2|w||v| \cos\theta$

$= \sqrt{(w - v) \cdot (w - v)}$

$= (w \cdot w) - (v \cdot w) - (w \cdot v) + (v \cdot v)$

$= \cancel{|w|^2} + \cancel{|v|^2} - 2(v \cdot w)$

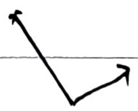
$\cancel{2}(v \cdot w) = \cancel{2}|v||w| \cos\theta$

$\cos\theta = \dfrac{(v \cdot w)}{|v| \times |w|}$

Angles between vectors

$\cos\theta = \dfrac{u \cdot v}{\|u\|\,\|v\|}$

ex angle between is $\pi/2$ (orthognal) if $u \cdot x = 0$



$u = [1, 1]$ $\quad u \cdot v = -1 + 1 = \emptyset$

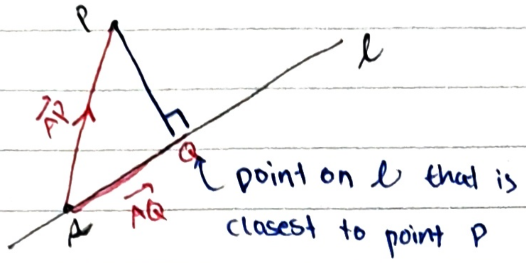
$v = [-1, 1]$

Projection | covered next w/c

$\text{Proj}_u V = \dfrac{u \cdot v}{\|v\|^2} v$

Wk 2-2 lect cont.

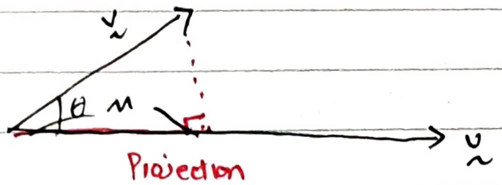
Projection



↑ point on ℓ that is closest to point P

"Project" $\vec{AP}$ on line ℓ

to $\boxed{\text{find } \vec{AQ}}$



$$\text{Proj}_{\underset{\sim}{u}}(\underset{\sim}{v}) = \left(\frac{(\underset{\sim}{u}\cdot\underset{\sim}{v})}{(\underset{\sim}{u}\cdot\underset{\sim}{u})}\right)\underset{\sim}{u}$$

$M = |v| \times \cos\theta$

proj $= M \cdot \underset{\sim}{u}$

proj $= (|v| \times \cos\theta) \times \underset{\sim}{u}$

proj $= \dfrac{(|u| \times |v| \times \cos\theta) \times \underset{\sim}{u}}{|u|}$

$= \dfrac{(\underset{\sim}{u}\cdot\underset{\sim}{v})}{|\underset{\sim}{u}|}\underset{\sim}{u}$

$|u| M = |u| \times |v| \times \cos\theta$

$|u| M = (\underset{\sim}{u}\cdot\underset{\sim}{v})$

$M =$

$\text{Proj}_{\underset{\sim}{u}}(\underset{\sim}{v})$'s length $=$

$|\tilde{v}| \times \cos\theta = \dfrac{|\underset{\sim}{u}| \times |\underset{\sim}{v}| \times \cos\theta}{|\underset{\sim}{u}|}$

$= \dfrac{(\underset{\sim}{u}\cdot\underset{\sim}{v})}{|\underset{\sim}{u}|}$

's Direction $= \dfrac{\underset{\sim}{u}}{|\underset{\sim}{u}|}$ (unit vector)

$$\frac{(u\cdot v)}{|\underset{\sim}{u}|} \times \frac{\underset{\sim}{u}}{|\underset{\sim}{u}|} = \frac{(u\cdot v)}{|\underset{\sim}{u}|^2}\underset{\sim}{u}$$

$$= \frac{(\underset{\sim}{u}\cdot\underset{\sim}{v})}{(\underset{\sim}{u}\cdot\underset{\sim}{u})}\underset{\sim}{u}$$