Week 4-1 Lecture Note

Planes in $\mathbb{R}^3$

General form

$ax + by + cz = d$ ← ~~at least~~ one of a, b, c should be non-zero

Normal form

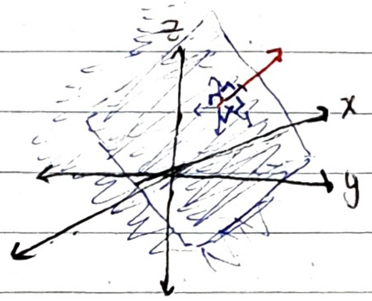
$\underset{\sim}{n} = [a, b, c]$ $\quad \underset{\sim}{x} = [x, y, z]$ — can't be $\underset{\sim}{0}$

$\underset{\sim}{n} \cdot \underset{\sim}{x} = d \iff [a, b, c][x, y, z] = d$

if $d = 0$, $(n \cdot \underset{\sim}{x} = 0)$

$\underset{\sim}{n} \cdot \underset{\sim}{x}$ are orthogonal

$\therefore$ infinitely many points (x, y, z) $\underset{\sim}{x}$ is orthogonal to $\underset{\sim}{n}$.



★ Given a point P

$\underset{\sim}{n}(\underset{\sim}{x} - P) = 0$

ex) $2x - y + z = 5$

$\underset{\sim}{n}: [2, -1, 1][x, y, z] = 5$

$(0, 0, 5)$ satisfies above eq.

$\therefore$ $P = [0, 0, 5]$

$[2, -1, 1][\underset{\sim}{x} - [0, 0, 5]] = 0$

Double check (expand)

$2(x) - 1(y) + (z - 5) = 0$

$2x - y + z - 5 = 0$

$2x - y + z = 5$

gf obtained again

ex2) $P = (1, 0, 6)$, $\perp$ to $[1, 2, 3]$

$[1, 2, 3][[x, y, z] - [1, 0, 6]] = 0$

(expand ~~for 계산~~) ~ to get general form

$[1, 2, 3][(x-1), y, (z-6)] = 0$

$1(x-1) + 2(y) + 3(z-6) = 0$

$x - 1 + 2y + 3z - 18 = 0$

$x + 2y + 3z - 19 = 0$

$x + 2y + 3z = 19$

Vector form for planes

$\underset{\sim}{x} = p + \underset{\sim}{d} r, \; r \in \mathbb{R}$ (for line)

$\underset{\sim}{x} = p + s\underset{\sim}{u} + t\underset{\sim}{v}, \; s, v \in \mathbb{R}$

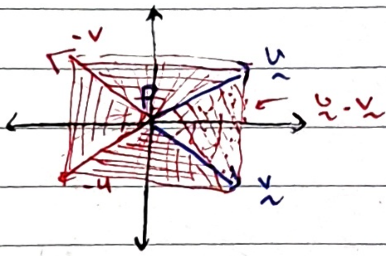
(for plane)

Parametric Eqs

$x = p_1 + su_1 + tv_1$

$y = p_2 + su_2 + tv_2$

$z = p_3 + su_3 + tv_3$

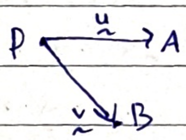


G.F $= x + 2y + 3z = 19$

$P = (19, 0, 0)$ → find 2 more points.

$A = (1, 9, 0)$,

$B = (1, 0, 6)$



$\vec{PA} = \underset{\sim}{u} = (1, 9, 0) - (19, 0, 0)$

$= [-18, 9, 0]$

$= 2[-2, 1, 0]$

$\underset{\sim}{v} = [1, 0, 6] - [19, 0, 0]$

$= [-18, 0, 6]$

$= 3[-6, 0, 1]$

$\underset{\sim}{x} = [19, 0, 0] + s2[-2, 1, 0] + t3[-6, 0, 1]$

V.F

$\boxed{\underset{\sim}{x} = [19, 0, 0] + s[-2, 1, 0] + t[-6, 0, 1]}$

$\left.\begin{array}{l} x = 19 - 2s - 6t \\ y = s \\ z = t \end{array}\right\} s, t \in \mathbb{R}$

SPANS

if $S = \{v_1, v_2 \dots v_n\}$ is a set of vectors in $\mathbb{R}^3$,

$\boxed{\text{span} = \text{set of all linear combinations of } \{v_1, v_2 \dots v_n\}}$

for $\{v_1, v_2 \dots v_n\}$ in $\mathbb{R}^3$

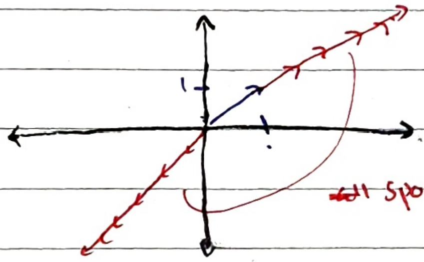
$\text{span} = \{ \underset{\sim}{v} \in \mathbb{R}^3 \mid \underset{\sim}{v} = c_1 \underset{\sim}{v}_1 + c_2 \underset{\sim}{v}_2 \dots c_n \underset{\sim}{v}_n \text{, for some } c \in \mathbb{R} \}$

ex) $\underset{\sim}{v} \in \mathbb{R}^2$

$\text{span}(\underset{\sim}{v})$ ~~span~~ $= \{ \underset{\sim}{v} \in \mathbb{R}^2 \mid \underset{\sim}{v} = c_1 \underset{\sim}{v}_1 , c \in \mathbb{R} \}$

if $\underset{\sim}{v}_1 = [1,1]$

$\text{span}(\underset{\sim}{v}_1) = \{ \underset{\sim}{v} \in \mathbb{R}^2 \mid \underset{\sim}{v} = c[1,1], c \in \mathbb{R} \}$



all spans of v_1

ooo

In $\mathbb{R}^2$, $\text{span}(\underset{\sim}{v})$ is the line through $\boxed{\text{origin}}$ & $\boxed{\underset{\sim}{d} = \underset{\sim}{v}}$

$\underset{\sim}{v} \neq \underset{\sim}{0}$

$\text{span}(\underset{\sim}{v})$ only if a line goes through origin with $\underset{\sim}{d} = \underset{\sim}{v}$

ex2)

$\underset{\sim}{v}_1 = [1,2,0]$ and $v_2 = [-2,3,5]$

$\text{span}(\underset{\sim}{v}_1, \underset{\sim}{v}_2) = \{ \underset{\sim}{v} \in \mathbb{R}^3 \mid \underset{\sim}{v} = c_1 \underset{\sim}{v}_1 + c_2 \underset{\sim}{v}_2 \text{ with } c_1, c_2 \in \mathbb{R} \}$

= for $\text{span}(\underset{\sim}{v}_1, v_2 \dots v_n)$, it must pass thru origin!

Q is $[8,-5,-15] \in \text{span}([1,2,0],[-2,3,5])$?

$[8,-5,-15] = c_1[1,2,0] + c_2[-2,3,5]$

$= \cancel{c_1 + 2c_2 + 2c_2 + 3c_2 + 5c_2}$

$8 = c_1 - 2c_2$

$-5 = 2c_1 + 3c_2$

$-15 = 5c_2$

Spanning set

if $S = \mathbb{R}^2$, S is spanning set

eg. $v_1 = [1, 0]$, $v_2 = [0, 1]$

$[x, y] = c_1[1, 0] + c_2[0, 1]$

$= [c_1, c_2]$ ~~an~~ with $c_1 c_2 \in \mathbb{R}$

$\therefore$ span(v_1, v_2) is spanning set

Week 4-2 lecture

(spanning set cont...)

In $\mathbb{R}^3$, spanning set $= [c_1, c_2, c_3]$

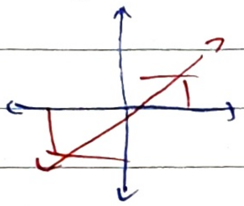
In $\mathbb{R}^n$, spanning set $= [c_1, c_2 \cdots c_n]$

ex) In $\mathbb{R}^2$, is ~~span~~ $S = \{[1,1], [-2,-2]\}$ spanning set ?

$[x, y] = c_1[1,1] + c_2[-2,-2]$

$$\left.\begin{aligned} x &= c_1 - 2c_2 \\ y &= c_1 - 2c_2 \end{aligned}\right\} = A$$

$[x, y] = [A, A]$ not a spanning set



Never goes outside the line

* [1, 1] and [-2, -2] are scalar multiples of each other.

More ~~systematic~~ approaches

To say that [1,1], [-2,-2] ~~are~~ 's linear combination

~~not a~~ is not a spanning set :

find counterexample,

$[1, 0] = c_1[1,1] + c_2[-2,-2]$

$$\left.\begin{aligned} 1 &= c_1 - 2c_2 \\ 0 &= c_1 - 2c_2 \end{aligned}\right\} \text{no solution (for any } [x, y])$$

$\Rightarrow$ not a spanning set

Linear Independence (vectors only k(sum) ~~intersect~~ up to $\underset{\sim}{0}$ at the origin (not co-linear)

To show $\underset{\sim}{v}$ and $\underset{\sim}{u}$ are linearly Dependent,

suppose $c_1 \underset{\sim}{v} + c_2 \underset{\sim}{u} = \underset{\sim}{0}$

if only possible set of value for

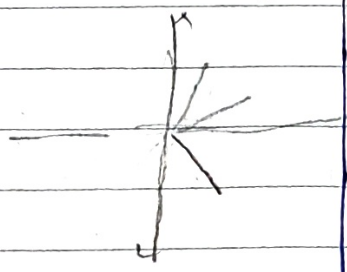
$\{c_1, c_2\} = \{0, 0\} \Rightarrow$ INDEPENDENT

Else $\Rightarrow$ Dependent

ex) $\underset{\sim}{u} = [2,1]$, $\underset{\sim}{v} = [1,-1]$, $w = [1,2]$

$6 + 3 - 3 = 6$
$3 - 3 + 3 = 6$

$x = 2c_1 + c_2 + c_3 = 0$
$y = c_1 - c_2 + 2c_3 = 0$
$x + y = 3c_1 + 3c_3 = 0$
$c_1 + c_3 = 0$



$c_1 = 1$, $c_3 = -1$, $c_2 = -1$
$\therefore (c_1, c_2, c_3) \neq \underset{\sim}{0}$
$\therefore \underset{\sim}{u}, \underset{\sim}{v}, \underset{\sim}{w}$ are linearly dependent!

$\underline{\underset{\sim}{u} - \underset{\sim}{w} = \underset{\sim}{v}}$

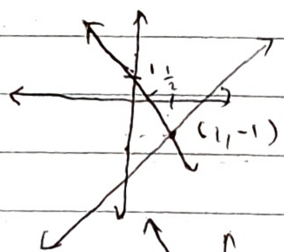
$\underset{\sim}{u} - \underset{\sim}{v} - \underset{\sim}{w} = 0$
$[c_1, c_2, c_3] = [1, -1, -1] \neq \underset{\sim}{0}$
$\therefore$ dependent

$\underset{\sim}{u}, \underset{\sim}{v}$ are ($\mathbb{R}^2$) LINEARLY DEPENDENT $\iff$ $\underset{\sim}{u}, \underset{\sim}{v}$ are scalar multiples of each other

$\underset{\sim}{v_1}, v_2 \dots v_n$ are linearly dependent $\iff$ one of the vector can be expressed as linear combination of other vectors

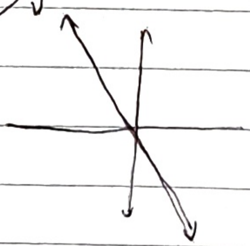
Systems of linear equations (In $\mathbb{R}^2$)

$2x + y = 1$
$x - y = 2$
$3x = 3$
$x = 1, y = -1$



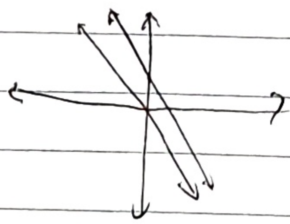
unique solution

$2x + y = 1$
$4x + 2y = 2$
$0 = 0$



Infinitely many solutions.
~~$y = -2x + 1$~~
$\{(x,y) \mid x + 2y = 1 \text{ with } x, y \in \mathbb{R}\}$

$2x + y = 1$
$2x + y = 0$



No Solution.
(parallel)

Even in $\mathbb{R}^n$, $\left\{\begin{array}{l} \text{one unique} \\ \text{infinite} \\ \text{none } \cancel{\text{set}} \end{array}\right\}$ & only 3 outcomes for system of linear eqs.

consistent $\left[\begin{array}{l} \text{unique} \\ \text{infinite} \end{array}\right.$ * technical terms

inconsistent - none

$$\left[\begin{array}{ccccc|c} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} & b_1 \\ a_{21} & \vdots & \vdots & \ddots & a_{2n} & b_2 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{n1} & a_{n2} & a_{n3} & \cdots & a_{nn} & b_n \end{array}\right]$$

Q is $[8,-5,15]$ in span$([1,2,0], [-2,3,5]$?

$$\left[\begin{array}{cc|c} c_1 & -2c_2 & 8 \\ 2c_1 & 3c_2 & -5 \\ 0 & 5c_2 & 15 \end{array}\right]$$

$x = \cancel{\ \ } \cdot c_1 + -2c_2 = 8$

$y = 2c_1 + 3c_2 = -5$

$z = 5c_2 = 15$

$$= \left[\begin{array}{cc|c} 1 & -2 & 8 \\ 2 & 3 & -5 \\ 0 & 5 & 15 \end{array}\right]$$