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Constrained optimal filter design for multi-channel active noise control via convex optimization

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ABSTRACT:
In many practical multi-channel active noise control (ANC) applications, various constraints need to be satisfied, such as stability, enhancement, etc. One way to enforce these constraints is to add a regularization term to the Wiener filter formulation, which, by tuning only a single parameter, can over satisfy many constraints and degrade the ANC performance. Another approach for non-adaptive ANC filter design that can produce better ANC performance is to directly solve the constrained optimization problem formulated based on the H_2/H_∞ control framework. However, such a formulation does not result in a convex optimization problem and its practicality can be limited by the significant computation time required in the solving process. In the presented work, the H_2/H_∞ formulation is convexified and a global minimum is guaranteed. It is then further reformulated into a cone programming problem which can be solved using specialized algorithms. Results show that the proposed method can produce better noise control performance than the regularization method. Compared with the traditional H_2/H_∞ formulation, the proposed method is more reliable and the computation time can be reduced by several orders, which, practically, provides a potential to extend its application to adaptive control.
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I. INTRODUCTION

In the past two decades, active noise control (ANC) techniques have been implemented in a wide range of applications, such as headrests,[1,2] automobiles,[3] aircraft,[4] motorcycle helmets,[5] ventilation windows,[6] etc. In many of these practical applications, various constraints, such as stability, robustness, controller output, disturbance enhancement, etc., need to be considered when designing the ANC filters.[7] The stability and robustness constraints are of particular importance when a feedback control strategy is used or a strong acoustic feedback path (i.e., the sound generated by secondary speakers can propagate to the reference microphones) is present.

One commonly used approach to ensure the satisfaction of those constraints is to introduce a regularization term in the filter coefficients optimization process.[7] Instead of minimizing the power of error signals only, it minimizes a weighted sum of error signal power and the power of the ANC filter response. This approach is also known as leaky LMS when implemented in an adaptive filter design process. Some methods were proposed in previous studies to choose appropriate leak factor for one particular constraint, such as output power constraint,[8,9] or disturbance enhancement constraint.[10] When multiple frequency dependent constraints need to be satisfied, the satisfaction of one constraint usually leads to an over satisfaction of others which, in turn, degrades the ANC performance.

A potentially better ANC filter design approach is to directly solve an optimization problem with multiple constraints formulated under the H_2/H_∞ control framework in the frequency domain.[1,11–14] However, one issue with this H_2/H_∞ based ANC filter design method applied in recent studies[3] is that the stability constraint expression, which involves obtaining the real part of a frequency response matrix's eigenvalues, is not a convex function of the filter coefficients. This, mathematically, makes the whole optimization problem non-convex, so that no global minimum solution is guaranteed (sometimes, the numerical method may not even converge to a local minimum solution). Practically, it means that this filter design method may not be reliable in a wide range of applications and the ANC performance may be sacrificed because a global minimum solution is not achieved in the numerical solving process. In addition to this non-convexity issue, the long computation time required in the solving process also leads to limitations in the practical implementation of this method. In principle, this H_2/H_∞ based filter design method can be made adaptive to slow time-varying system characteristics by continuously repeating the filter optimization process using the most recent measurements of reference and disturbance signals, and then updating the filter coefficients. However, since, in all previous applications of this H_2/H_∞ based ANC filter design method, the constrained optimization problem is solved by sequential quadratic programming (SQP) or similar algorithms,[3,15–17] it requires a computation time on the order of hours or tens of hours, depending on the filter order, channel count, as well as frequency range, which makes it impossible for this filter design method to be implemented in practical applications requiring filter

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adaptation. If the filter design computation time can be shortened to the order of seconds or minutes, a simple continuous repeating of this filter design calculation can extend this method to a wider ANC application such as variable-speed HVAC equipment which does not usually change speed within several minutes. Another practical advantage of shortening the computation time is that, even for a non-adaptive ANC system, it can accelerate the product development cycle in the commercialization of this ANC technique, which requires the ANC filter to be designed in each engineering iteration.

There are some preliminary studies by the authors that showed the benefits of using convex optimization in ANC filter design.[18,19] In this article, a convex optimization for multi-channel ANC system is presented with a more completed formulation process, practical experimental setup, and an extensive investigation on the comparison to other design approaches. The H_2/H_∞ framework in the previous work of Cheer and Elliott[3] was adopted as an initial formulation for constrained multi-channel ANC filter design. This filter design problem is formulated to a convex optimization problem by relaxing the non-convex stability constraint to its convex upper bound. The relaxed stability constraint was shown to be convex and, at the same time, still less conservative than other commonly used convex stability constraints. In order to reduce the computation time in the numerical solving process, it was then proposed to further convert this convex formulation into a cone programming formulation, which is solved by computationally efficient algorithms. The results demonstrate that, compared with the regularization parameter approaches, better noise control performance can be achieved by using the proposed method. Compared with the traditional H_2/H_∞ framework solved by the SQP method, the numerical solving process in the proposed work is more reliable and results in a significant reduction in the required computation time (reduce from the order of hours to seconds). This can significantly accelerate the development cycle of ANC products based on such a H_2/H_∞ framework and provide the potential to extend the application of this non-adaptive filter design method into adaptive active noise controls of slow time-varying systems.

II. THEORY

In this section, all the formulations are presented in the context of feedforward control. It is noted that they can be directly applied to feedback ANC as well, because, if an internal model structure is used, a feedback controller can be designed using an identical way as that for a feedforward controller.[3]

A. Review of regularization parameter and traditional H_2/H_∞ method

1. Control system description

The system block diagram of a typical multi-channel ANC feedforward controller is shown in Fig. 1(a). This ANC system includes N_r microphones as reference sensors, N_s loudspeakers as secondary sources, N_e microphones as error sensors. $\vec{\mathbf{x}}$ denotes the noise signals from the primary noise sources measured at reference sensor locations. $\vec{\mathbf{r}}$ denotes signal measured by the reference microphones when the ANC system is activated. $\vec{\mathbf{y}}$ is the output of the controller. $\vec{\mathbf{d}}_e$ denotes the disturbance signals. $\vec{\mathbf{e}}$ denotes the error signals. The total power of error signals is to be attenuated by the ANC system. $\mathbf{G}_{s_0}$ is the acoustic feedback path matrix that represents the acoustic responses of secondary sources at the reference sensors. The reference signals $\vec{\mathbf{r}}$ is the combination of primary noise signals, $\vec{\mathbf{x}}$, and the signal from the acoustic feedback path $\mathbf{G}_{s_0}$. To estimate the primary noise signals, $\vec{\mathbf{x}}$, from reference signals $\vec{\mathbf{r}}$, an internal model control (IMC) structure is applied to cancel the acoustic feedback path effect.[1,3,7] In the controller $\mathbf{H}$, the $\hat{\mathbf{G}}_{s_0}$ is a model of the physical feedback path, $\mathbf{G}_{s_0}$, which is assumed to be a perfect model, i.e., $\hat{\mathbf{G}}_{s_0} = \mathbf{G}_{s_0}$. Thus, this system becomes a standard feedforward system in Fig. 1(b). The possible feedback path modeling error in practice is considered as a model uncertainty in robustness constraints. $\mathbf{G}_e$ represents the acoustical responses matrix of the secondary sources at the error sensor positions. $\mathbf{W}_x$ is the frequency response matrix of the multi-channel ANC FIR filters that need to be designed. The FIR filter coefficients of the ANC filters, denoted as $\mathbf{w}_F$, are the variables that need to be calculated in the filter design problem. The optimal solution $\mathbf{w}_F^*$ obtained by solving the filter design problem can be directly implemented in the real-time signal processing controller *via* time-domain filtering.

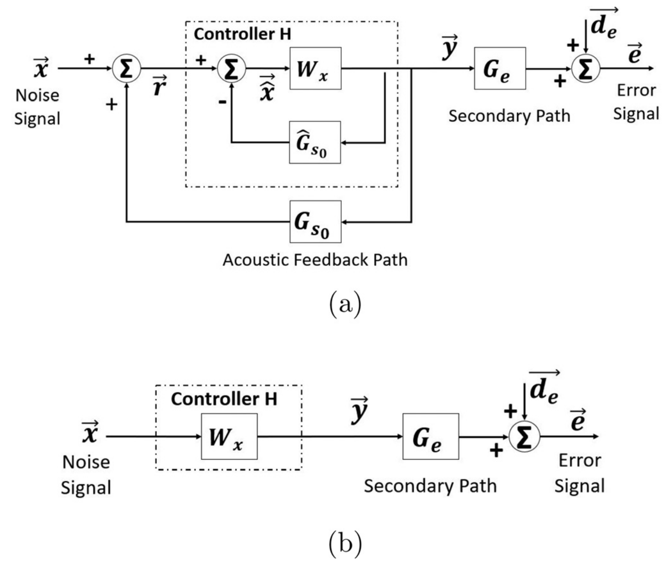

FIG. 1. Block diagram of the MIMO feedforward controllers (a) with the acoustic feedback path and internal model control structure, (b) in a standard feedforward form when assuming internal model control structure perfectly cancels the acoustic feedback path effect.

2. Regularization parameter approach

The regularization parameter approach is briefly described here. More details can be found in Elliott's work on robust controller design.[7,20] Instead of the total signal power at error sensors, the objective function being minimized is

$$J = E\left[\vec{\mathbf{e}}^{\mathrm{T}}(n)\vec{\mathbf{e}}(n) + \beta\sum_{i=1}^{N_s}\sum_{j=1}^{N_r}\vec{\mathbf{w}}_{F_{i,j}}^{\mathrm{T}}\vec{\mathbf{w}}_{F_{i,j}}\right], \quad (1)$$

where i and j denote the i-th output channel and j-th input channel of the ANC filter. Each $\vec{\mathbf{w}}_{F_{i,j}}$ is a column vector that is composed of the ANC filter coefficients associated with the channel pair specified by i and j. β is the regularization parameter that needs to be tuned to satisfy all required constraints. By minimizing the objective function, the optimal filter can be written as

$$\vec{\mathbf{w}}_{opt} = -(\mathbf{R}_{rr} + \beta\mathbf{I}_{N_rN_sN_t})^{-1}\vec{\mathbf{r}}_{rd}, \quad (2)$$

where N_t denotes the order of FIR filters in each channel, $\mathbf{I}_{N_rN_sN_t}$ is a square identity matrix with a dimension of $N_rN_sN_t$ by $N_rN_sN_t$, $\mathbf{R}_{rr}$ is the auto-correlation matrix of filtered reference signals, and $\vec{\mathbf{r}}_{rd}$ is the cross correlation vector of filtered reference signal and disturbance signal.

When β increases, all the required constraints will be gradually satisfied but it can cause an over satisfaction of many constraints and thus results in a sacrifice of noise control performance.

3. Traditional H_2/H_∞ approach

The previous work of Cheer and Elliott[3] was presented here to introduce the traditional H_2/H_∞ approach. First, frequency response of an FIR ANC filter can be expressed as

$$\mathbf{W}_{x_{i,j}}(f) = \vec{\mathbf{F}}_z^{\mathrm{T}}(f)\vec{\mathbf{w}}_{F_{i,j}}, \quad (3)$$

where

$$\vec{\mathbf{F}}_z(f) = \begin{bmatrix} 1 & e^{-j2\pi f(1/f_s)} & e^{-j2\pi f(2/f_s)} & \cdots & e^{-j2\pi f[(N_t-1)/f_s]} \end{bmatrix}^{\mathrm{T}},$$

f denotes the frequency, and f_s denotes the sampling frequency. $\mathbf{W}_{x_{i,j}}$ means the i-th row and j-th column of frequency response matrix $\mathbf{W}_x$.

The objective function $J_0(\mathbf{w}_F)$ to be minimized is the total power of error signals across all desired frequencies

$$J_0(\mathbf{w}_F) = \sum_{k=1}^{N_f} J(f_k), \quad (4)$$

where the $J(f_k)$ is the power of error signal at k-th frequency f_k. N_f denotes the total number of frequency points in the desired noise attenuation band. Following the work of Cheer and Elliott,[3] $J(f_k)$ is expressed as

$$\begin{aligned} J(f_k) &= \mathrm{tr}\left[E(\vec{\mathbf{e}}(f_k)\vec{\mathbf{e}}^{\mathrm{H}}(f_k))\right] \\ &= \mathrm{tr}\Big[E((\vec{\mathbf{d}}_e(f_k) + \mathbf{G}_e(f_k)\mathbf{W}_x(f_k)\vec{\mathbf{x}}(f_k)) \\ &\quad \times (\vec{\mathbf{d}}_e(f_k) + \mathbf{G}_e(f_k)\mathbf{W}_x(f_k)\vec{\mathbf{x}}(f_k))^{\mathrm{H}})\Big] \\ &= \mathrm{tr}\Big[\mathbf{G}_e(f_k)\mathbf{W}_x(f_k)\mathbf{S}_{xx}(f_k)\mathbf{W}_x^{\mathrm{H}}(f_k)\mathbf{G}_e^{\mathrm{H}}(f_k) \\ &\quad + \mathbf{G}_e(f_k)\mathbf{W}_x(f_k)\mathbf{S}_{xd_e}(f_k) \\ &\quad + \mathbf{S}_{xd_e}^{\mathrm{H}}(f_k)\mathbf{W}_x^{\mathrm{H}}(f_k)\mathbf{G}_e^{\mathrm{H}}(f_k) + \mathbf{S}_{d_ed_e}(f_k)\Big], \end{aligned} \quad (5)$$

where

$$\mathbf{W}_x(f_k) = \begin{bmatrix} \mathbf{W}_{x_{1,1}}(f_k) & \cdots & \mathbf{W}_{x_{1,N_r}}(f_k) \\ \vdots & \ddots & \vdots \\ \mathbf{W}_{x_{N_s,1}}(f_k) & \cdots & \mathbf{W}_{x_{N_s,N_r}}(f_k) \end{bmatrix},$$

and E is the expectation operator, H denotes the conjugate transpose of a matrix, $\mathbf{S}_{xx}(f_k)$ and $\mathbf{S}_{d_ed_e}(f_k)$ are the cross-spectral density matrices of $\vec{\mathbf{x}}$ and $\vec{\mathbf{d}}_e$, respectively, at frequency f_k, and $\mathbf{S}_{xd_e}(f_k)$ is the cross-spectral density matrix between the primary noise signals $\vec{\mathbf{x}}$ and the disturbance signals $\vec{\mathbf{d}}_e$ at frequency f_k.

Disturbance enhancement constraints are added to prevent large enhancements. A simplified version of Cheer and Elliott's work[3,7] is used

$$J(f_k)D_e(f_k) \le A_e(f_k), \quad (6)$$

where

$$D_e(f_k) = \frac{1}{\mathrm{tr}\left(\mathbf{S}_{d_ed_e}(f_k)\right)},$$

and $D_e(f_k)$ is the reciprocal of total disturbance energy at k-th frequency; $A_e(f_k)$ is specified as the upper bound of the enhancement ratio of error signals at frequency f_k.

Even for feedforward control, the stability and robustness constraints also need to be considered because of the use of feedback cancellation path $\hat{\mathbf{G}}_{s_0}$ in the controller $\mathbf{H}$. In the current work, the controller stability is ensured by limiting the open-loop response trajectory of controller $\mathbf{H}$ (i.e., the eigenvalues of the open-loop frequency response matrix) to be at the right-hand side of the Nyquist point in the Laplace domain,[3,7] which can be expressed as

$$\min\left(\Re\left(\lambda\left(\mathbf{W}_x(f_k)\hat{\mathbf{G}}_{s_0}(f_k)\right)\right)\right) > -1, \quad (7)$$

where $\lambda()$ denotes the eigenvalues of a matrix and $\Re()$ denotes the real part of a complex number. Note that, compared with Cheer and Elliott's work,[3] $\hat{\mathbf{G}}_{s_0}(f_k)\mathbf{W}_x(f_k)$ is changed to $\mathbf{W}_x(f_k)\hat{\mathbf{G}}_{s_0}(f_k)$ since this will not change the value of non-zero eigenvalues. The size of $\hat{\mathbf{G}}_{s_0}(f_k)\mathbf{W}_x(f_k)$ is smaller when the number of secondary sources N_s is smaller than the number of reference sensors N_r.

For robustness constraints, with the $\mathbf{M}$-$\boldsymbol{\Delta}$ structure and the small-gain theory applied,[3,7] they can be expressed as

$$\max\left(\sigma\left(\mathbf{W}_x(f_k)\hat{\mathbf{G}}_{s_0}(f_k)\right)\right)B(f_k) \le 1, \quad (8)$$

where $\sigma()$ denotes the singular values of a matrix and $B(f_k)$ is the upper bound on the output multiplicative plant uncertainty at frequency f_k. To reduce the problem size, as explained in the stability constraint Eq. (7), $\mathbf{W}_x(f_k)\hat{\mathbf{G}}_{s_0}(f_k)$ is used instead of $\hat{\mathbf{G}}_{s_0}(f_k)\mathbf{W}_x(f_k)$.

In practical applications, a constraint on the amplitude of the ANC filter response, $\mathbf{W}_{x_{i,j}}(f_k)$, is usually needed[1] to ensure that the loudspeakers always operates in their linear response range, which can be written as

$$|\mathbf{W}_{x_{i,j}}(f_k)| \le C_{i,j}(f_k), \tag{9}$$

where $C_{i,j}(f_k)$ is the required upper bound on the amplitude of filter $\mathbf{w}_{F_{i,j}}$ at frequency f_k. Note that if $C_{i,j}(f_k)$ is small enough and Eq. (9) is satisfied at some frequencies, Eqs. (6), (7), (8) are also automatically satisfied at those frequencies. So, for those frequency bands where $C_{i,j}(f_k)$ is small, Eq. (9) can be used to replace Eqs. (6), (7), (8) to reduce the calculation effort of solving the optimization problem. Also, it is found that when the energy of the disturbance signal is small at some frequency bands, e.g., near Nyquist frequency due to anti-aliasing filtering, the use of response limits, Eq. (9), to replace Eqs. (6), (7), (8) on these frequency bands can help improve the numerical behavior during solving the optimization problem.

As a summary of the H_2/H_∞ formulation of ANC applications, an optimization problem for the multi-channel ANC filter design in the frequency domain can be constructed by using Eq. (4) as the objective function, and Eqs. (6), (7), (8), (9) as constraints,

$$\begin{aligned}
\min_{\mathbf{w}_F}.\quad & \sum_{k=1}^{N_f} J(f_k) \\
\text{s.t.}\quad & J(f_k)D_e(f_k) \le A_e(f_k),\ \text{for all}\ f_k, \\
& \min\left(\Re\left(\lambda\left(\mathbf{W}_x(f_k)\hat{\mathbf{G}}_{s0}(f_k)\right)\right)\right) > -1,\ \text{for all}\ f_k, \\
& \max\left(\sigma\left(\mathbf{W}_x(f_k)\hat{\mathbf{G}}_{s0}(f_k)\right)\right)B(f_k) \le 1,\ \text{for all}\ f_k, \\
& |\mathbf{W}_{x_{i,j}}(f_k)| \le C_{i,j}(f_k),\ \text{for all}\ f_k,\ i, \text{and}\ j.
\end{aligned} \tag{10}$$

Cheer and Elliott proposed to solve this optimization problem Eq. (10) by sequential quadratic programming (SQP). But the problem solving process can be quite time consuming since it involves complicated non-smooth constraints, especially for high order multi-channel filter $\mathbf{w}_F$ and a large number of frequency components. Also, the SQP cannot guarantee to converge to the global optimal solution for such a general problem. To overcome these difficulties, modifications and reformulations of Eq. (10) are proposed and discussed in the following.

B. Proposed convex formulation

In this section, the original optimization problem, Eq. (10), is modified and reformulated into a convex optimization problem. Then, the global optimal solution can usually be found.[21] Computationally efficient algorithms can also be applied to this convex problem.

A general convex optimization problem requires the objective function and all inequality constraint functions to be convex.[21] Equation (10) does not meet this requirement since the function in the inequalities for stability constraint is not convex (which can easily be shown by checking the definition of convexity). Some modifications and reformulations thus need to be applied. However, all the other constraints and the objective function listed in Eq. (10) are already convex. For the benefit of analyzing the convexity of the optimization problem and applying algorithms, some simplifications to Eq. (10) are also introduced in Sec. II B 2.

1. Convexification of stability constraint

Since the stability constraint is the only constraint that is not convex, only this constraint needs to be relaxed to accomplish the problem convexification. Equaction (7) can first be reformulated as

$$\max\left(\Re\left(\lambda\left(-\mathbf{W}_x(f_k)\hat{\mathbf{G}}_{s0}(f_k)\right)\right)\right) < 1. \tag{11}$$

It is noted that the operation of taking real part of eigenvalues is not a convex function. For brevity, $\mathbf{A}_s$ is used to represent $-\mathbf{W}_x(f_k)\hat{\mathbf{G}}_{s0}(f_k)$ in the following modification process. And $\lambda_1(\mathbf{A}_s)$ denotes the eigenvalue of $-\mathbf{W}_x(f_k)\hat{\mathbf{G}}_{s0}(f_k)$ with the largest real part. Then, Eq. (11) can be expressed as

$$\Re(\lambda_1(\mathbf{A}_s)) = \frac{\lambda_1(\mathbf{A}_s) + \lambda_1(\mathbf{A}_s)^*}{2} < 1, \tag{12}$$

where * denotes the complex conjugate of a complex number. Now suppose $\boldsymbol{v}_1$ is an eigenvector associated with $\lambda_1(\mathbf{A}_s)$, so $\lambda_1(\mathbf{A}_s) = \boldsymbol{v}_1^{\mathrm{H}}\mathbf{A}_s\boldsymbol{v}_1/||\boldsymbol{v}_1||_2^2$. Then Eq. (12) becomes

$$\Re(\lambda_1(\mathbf{A}_s)) = \frac{\boldsymbol{v}_1^{\mathrm{H}}\left(\dfrac{\mathbf{A}_s + \mathbf{A}_s^{\mathrm{H}}}{2}\right)\boldsymbol{v}_1}{||\boldsymbol{v}_1||_2^2} < 1. \tag{13}$$

In order to achieve convexity, Eq. (13) can be relaxed by enforcing it to be satisfied not only for $\boldsymbol{v}_1$, but for any arbitrary vectors, which is equivalent to

$$\max\left(\lambda\left(\frac{\mathbf{A}_s + \mathbf{A}_s^{\mathrm{H}}}{2}\right)\right) < 1. \tag{14}$$

Since Eq. (14) is obtained by relaxing Eq. (13), it serves as an upper bound for Eq. (11), i.e., a more restricted stability constraint. This constraint function is convex because it takes the largest eigenvalue of a Hermitian matrix[21] and this matrix is obtained by linear transform of unknown variables, $\vec{\mathbf{w}}_F$. The use of Eq. (14) is still a less conservative stability constraint compared with some other constraints used in previous studies. For example, sometimes controller stability is ensured by limiting the eigenvalues of $\mathbf{W}_x(f_k)\hat{\mathbf{G}}_{s0}(f_k)$ to be inside the unit circle,[7] which is not convex either. There are some other stability constraints used in some studies that are indeed convex, e.g., limiting the multiplication of the largest singular value of $\mathbf{W}_x(f_k)$ and the largest singular value of $\hat{\mathbf{G}}_{s0}(f_k)$ to be less than 1,[7]

$$||\mathbf{W}_x(f_k)||_2||\hat{\mathbf{G}}_{s0}(f_k)||_2 < 1, \tag{15}$$

or sometimes a slightly less conservative but still convex constraint was used in previous studies,

$$||\mathbf{W}_x(f_k)\hat{\mathbf{G}}_{s_0}(f_k)||_2 = ||\mathbf{A}_s||_2 < 1, \tag{16}$$

where $||\ \ ||_2$ denotes the spectral norm of a matrix. To show that Eq. (14) is less conservative, it can be first shown that

$$\begin{aligned} \max\left(\lambda\left(\frac{\mathbf{A}_s + \mathbf{A}_s^{\mathrm{H}}}{2}\right)\right) &\leq \left|\left|\frac{\mathbf{A}_s + \mathbf{A}_s^{\mathrm{H}}}{2}\right|\right|_2 \\ &\leq \frac{||\mathbf{A}_s||_2 + ||\mathbf{A}_s^{\mathrm{H}}||_2}{2} \\ &= ||\mathbf{A}_s||_2 \\ &\leq ||\mathbf{W}_x(f_k)||_2||\hat{\mathbf{G}}_{s_0}(f_k)||_2, \end{aligned} \tag{17}$$

where the first inequality comes from the fact that the maximum eigenvalue of a matrix is always smaller or equal to its spectral norm; the second and fourth inequalities come from the triangle property and sub-multiplicativity properties of the spectral norm. This proves that Eq. (14) is less conservative compared with Eq. (15) or Eq. (16). Thus, the associated ANC performance resulting from the proposed method can, in principle, be better.

In a general convex optimization problem, the inequality constraints need to be non-strict. So a very small positive constant ϵ_s is introduced to ensure the strict stability when expressed as non-strict constraint. Equation (14) can now become:

$$\max\left(\lambda\left(\frac{\mathbf{A}_s + \mathbf{A}_s^{\mathrm{H}}}{2}\right)\right) - (1-\epsilon_s) \leq 0, \tag{18}$$

where

$$\mathbf{A}_s = -\mathbf{W}_x(f_k)\hat{\mathbf{G}}_{s_0}(f_k).$$

2. Convex formulation of objective function and other constraint

The objective function, Eq. (4), can be simplified into a standard quadratic form. First, Eq. (5) can be simplified to a standard quadratic form as (see the Appendix for detail),

$$J(f_k) = \vec{\mathbf{w}}^{\mathrm{T}}\mathbf{A}_J(f_k)\vec{\mathbf{w}} + \vec{\mathbf{b}}_J^{\mathrm{T}}(f_k)\vec{\mathbf{w}} + c_J(f_k), \tag{19}$$

where

$$\begin{aligned} &\mathbf{A}_J(f_k) = \Re\Big(\big(\mathbf{G}_e^{\mathrm{H}}(f_k)\mathbf{G}_e(f_k)\big)\otimes\mathbf{S}_{xx}^{\mathrm{T}}(f_k)\otimes\Big(\vec{\mathbf{F}}_z^{*}(f_k)\vec{\mathbf{F}}_z^{\mathrm{T}}(f_k)\Big)\Big), \\ &\vec{\mathbf{b}}_J(f_k) = 2\Re\Big(\mathrm{vec}\Big(\big(\mathbf{S}_{xd_e}(f_k)\mathbf{G}_e(f_k)\big)\otimes\vec{\mathbf{F}}_z(f_k)\Big)\Big), \\ &c_J(f_k) = \mathrm{tr}\big(\mathbf{S}_{d_e d_e}(f_k)\big), \\ &\vec{\mathbf{w}} = \begin{bmatrix}\vec{\mathbf{w}}_{F_{1,1}}^{\mathrm{T}} & \cdots & \vec{\mathbf{w}}_{F_{1,N_r}}^{\mathrm{T}} & \vec{\mathbf{w}}_{F_{2,1}}^{\mathrm{T}} & \cdots & \vec{\mathbf{w}}_{F_{N_s,N_r}}^{\mathrm{T}}\end{bmatrix}^{\mathrm{T}}, \end{aligned}$$

vec() converts a matrix to a vector by stacking the columns[22] and $\otimes$ denotes Kronecker product. $\mathbf{G}_e^{\mathrm{H}}(f_k)\mathbf{G}_e(f_k)$, $\mathbf{F}_z^{*}(f_k)\vec{\mathbf{F}}^{\mathrm{T}}(f_k)$ and $\mathbf{S}_{xx}^{\mathrm{T}}(f_k)$ are always positive semidefinite by their definitions. So $\mathbf{A}_J(f_k)$ is positive semidefinite because of the Kronecker product properties. After dropping the constant term, the original objective function Eq. (4) can then be expressed as

$$J_0 = \vec{\mathbf{w}}^{\mathrm{T}}\left(\sum_{k=1}^{N_f}\mathbf{A}_J(f_k)\right)\vec{\mathbf{w}} + \sum_{k=1}^{N_f}\vec{\mathbf{b}}_J^{\mathrm{T}}(f_k)\,\vec{\mathbf{w}}, \tag{20}$$

which is convex because the Hessian matrix of J_0, i.e., $2\sum_{k=1}^{N_f}\mathbf{A}_J(f_k)$, is positive semidefinite since it is a summation of positive semidefinite matrices. Also, the dimension of the objective function is independent of the number of frequencies after this simplification. Thus, even if a fine frequency resolution is chosen in the objective function, it will not increase the computational complexity.

The disturbance enhancement constraint, Eq. (6), can be simplified similarly,

$$\vec{\mathbf{w}}^{\mathrm{T}}\mathbf{A}_J(f_k)\vec{\mathbf{w}} + \vec{\mathbf{b}}_J^{\mathrm{T}}(f_k)\,\vec{\mathbf{w}} + \tilde{c}_J(f_k) \leq 0, \tag{21}$$

where

$$\tilde{c}_J(f_k) = c_J(f_k)(1 - A_e(f_k)).$$

For robustness constraint, Eq. (8), it can be simply expressed as

$$\max\Big(\sigma\Big(\mathbf{W}_x(f_k)\hat{\mathbf{G}}_{s_0}(f_k)\Big)\Big)B(f_k) - 1 \leq 0, \tag{22}$$

which is already in a convex form because the maximum singular value is a matrix norm, and each matrix element is obtained by linear transform of variables $\vec{\mathbf{w}}_F$.[21]

For filter response magnitude constraint, Eq. (9), by using Eq. (3), it can be expressed as

$$||\vec{\mathbf{F}}_z^{\mathrm{T}}(f_k)\vec{\mathbf{w}}_{F_{i,j}}||_2 - C_{i,j}(f_k) \leq 0, \tag{23}$$

which is convex because it is the $\mathbf{L}_2$ norm of a vector obtained by linear transforms of the variables.[21]

3. Summary of convexified problem

The convex formulation of designing constrained optimal multi-channel ANC filter can be constructed by using Eq. (20) as objective function and Eqs. (18), (21), (22), (23) as constraints,

$$\begin{aligned} &\min_{\mathbf{w}_F}.\ \vec{\mathbf{w}}^{\mathrm{T}}\left(\sum_{k=1}^{N_f}\mathbf{A}_J(f_k)\right)\vec{\mathbf{w}} + \sum_{k=1}^{N_f}\vec{\mathbf{b}}_J^{\mathrm{T}}(f_k)\,\vec{\mathbf{w}}, \\ &\text{s.t.}\ \max\left(\lambda\left(\frac{\mathbf{A}_s + \mathbf{A}_s^{\mathrm{H}}}{2}\right)\right) - (1-\epsilon_s) \leq 0, \quad \text{for all } f_k, \\ &\quad \vec{\mathbf{w}}^{\mathrm{T}}\mathbf{A}_J(f_k)\vec{\mathbf{w}} + \vec{\mathbf{b}}_J^{\mathrm{T}}(f_k)\,\vec{\mathbf{w}} + \tilde{c}_J(f_k) \leq 0, \quad \text{for all } f_k, \\ &\quad \max(\sigma(\mathbf{A}_s))B(f_k) - 1 \leq 0, \quad \text{for all } f_k, \\ &\quad ||\vec{\mathbf{F}}_z^{\mathrm{T}}(f_k)\vec{\mathbf{w}}_{F_{i,j}}||_2 - C_{i,j}(f_k) \leq 0, \quad \text{for all } f_k,\ i, \text{and } j. \end{aligned} \tag{24}$$

Because of the convexity, a global optimal solution of this optimization problem Eq. (24) can always be found by using a gradient-based algorithm.[21] This simplified convex formulation Eq. (24) can also provide useful insights into ANC system design. For example, if no constraints are applied, it is usually preferred that the Hessian of objective function is strictly positive-definite, so that the global minimum is unique and the convergence rate of optimization algorithm is fast.[21] If the Hessian matrix of the objective is singular, the resulting optimal filter coefficients may not be unique and tend to be unnecessarily large without proper further constraints. By observing Hessian matrix of J_0 [i.e., $2\sum_{k=1}^{N_f}\mathbf{A}_J(f_k)$], the strict positive definiteness preference implies that it is preferred to have (1) the number of frequencies in the objective function larger than the number of filter coefficients (since $\mathbf{A}_J(f_k)$ is rank-one matrix), (2) different channels of reference signals being uncorrelated and having similar power (i.e., $\mathbf{S}_{xx}$ is close to identity), (3) the number of secondary sources being no larger than the number of error sensors [otherwise $\mathbf{G}_e^{\mathrm{H}}(f_k)\mathbf{G}_e(f_k)$ will be singular]. However, even if the above-mentioned conditions are not satisfied, with the help of constraints in Eqs. (18), (21), (22), and (23), appropriate control filters can still be found by the proposed method.

Some convex toolboxes can be used to solve this problem Eq. (24), e.g., the CVX toolbox.[23,24] This toolbox reformulates the general disciplined convex optimization problem into a cone programming problem and use cone programming solvers to solve it. To save the time on reformulation in CVX toolbox, this reformulation to cone programming can also be done analytically and use cone programming solvers directly, e.g., SDPT3,[25–27] SeDuMi,[28,29] or MOSEK,[30] where primal-dual interior-point methods designed specifically for cone programming is implemented. Those algorithms can process non-differentiable constraints in Eq. (24) efficiently. In the current work, the cone programming reformulation is carried out analytically using in a standard procedure[18,19,21,28] and implemented by a direct coding of the analytically reformulated expressions into the proper interface format to call a cone programming solver. No automatic reformulation tools, such as the CVX toolbox, are used in this work.

A flow chart showing the procedure of implementing the traditional H_2/H_∞ method and the proposed method is shown in Fig. 2. Compared with the traditional H_2/H_∞ method, the proposed method does not significantly complicate the implementation procedure.

III. EXPERIMENTAL RESULTS

A. Description of experimental condition

An experiment was carried out to investigate the proposed method, which is an ANC system installed on the wind channel of a central air handling system. There are two speakers (served as secondary sources), four reference microphones and four error microphones in this system. The experimental setup as well as individual components are shown in Fig. 3. The key dimensions of the components are shown in Fig. 4. A non-adaptive controller is designed and implemented in the experiment using measured system characteristics under a nominal operating condition. Different number of channels will be selected to investigate the ANC performance dependence on channel counts. For convenience, the terminology "$xN_ryN_szN_e$ case" is used in this paper to denote a study case with x reference microphones, y control speakers, and z error microphones. In this section, "regularization parameters method" refers to the method using Eq. (2), "traditional H_2/H_∞ method" refers to the method using Eq. (10), and "proposed method" refers to the method using convex formulation Eq. (24).

When acquiring measurement data, the sampling rate of data acquisition system is 48 kHz. Two million sampling points for each channel were acquired for calculating the correlation matrices and frequency responses, where a hamming window of 48 000 points is used for averaging with 50% overlapping (83 times averages).

B. Comparison of regularization parameters method and proposed method

In this subsection, noise control performance of constrained control filters designed by regularization parameters method and proposed method is compared. Disturbance enhancement and robust stability constraint are particularly focused on. For regularization parameters, the parameter β

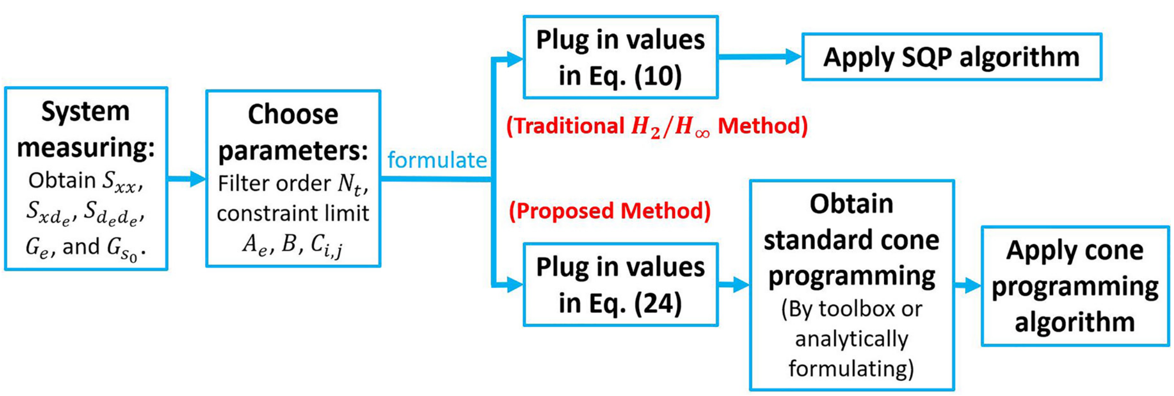

FIG. 2. (Color online) A flow chart showing the procedure of implementing the traditional H_2/H_∞ method and the proposed method.

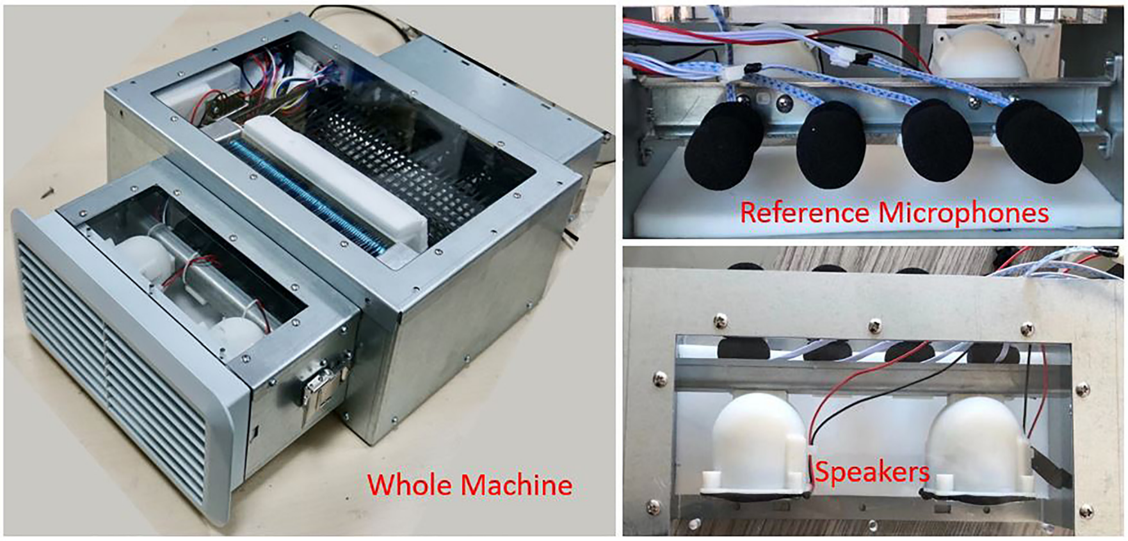

FIG. 3. (Color online) The picture of the experimental setup and individual components.

in Eq. (2) is gradually increased from zero to an appropriate value such that all constraints are satisfied. For the proposed method, constraints are directly specified in the convex optimization formulation Eq. (24) and no parameter tuning is needed. This also indicates the better robustness in implementing the proposed method, the same problem formulation can be applied to a wide range of applications without additional tuning effort from users. In Figs. 5(a) and 5(c), "ANC OFF" denotes the sound pressure power spectral density function (PSD) of original primary sound disturbance signals averaged among all the error microphones. "Normalized PSD" is the ratio of the sensor averaged PSD when ANC system is in operation to the primary noise PSD first averaged among error sensors and then averaged over the whole frequency band when ANC system is turned off. System stability behavior shown in Figs. 5(b) and 5(d) is investigated by checking the maximum real part of eigenvalue of open loop responses matrix $-\mathbf{W}_x(f_k)\hat{\mathbf{G}}_{s_0}(f_k)$. Thus, when they are all smaller than 1 across all frequencies, the control filters are considered to be internally stable. The stability constraint limit (black dot line) is set to be 0.9 to ensure strict stability.

In Fig. 5, the $4N_r2N_s4N_e$ case is considered. From Fig. 5(b) where a sampling rate of 3000 Hz is used, when β is higher than or equal to 0.02, the stability constraint is satisfied (i.e., the open-loop frequency response is below the stability constraint limit). The control filter design by the proposed method is also stable. From Fig. 5(d), a case with 8000 Hz sampling rate, system stability requirement is satisfied when β is higher than or equal to 0.3. Similarly, the

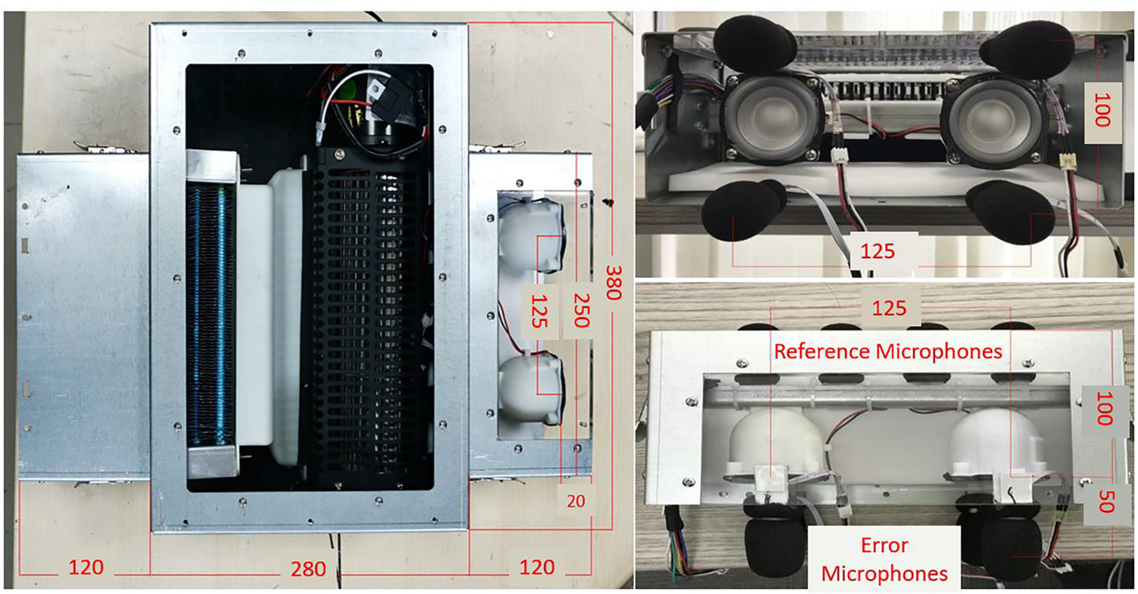

FIG. 4. (Color online) The dimensions in the experiment setup (unit: mm).

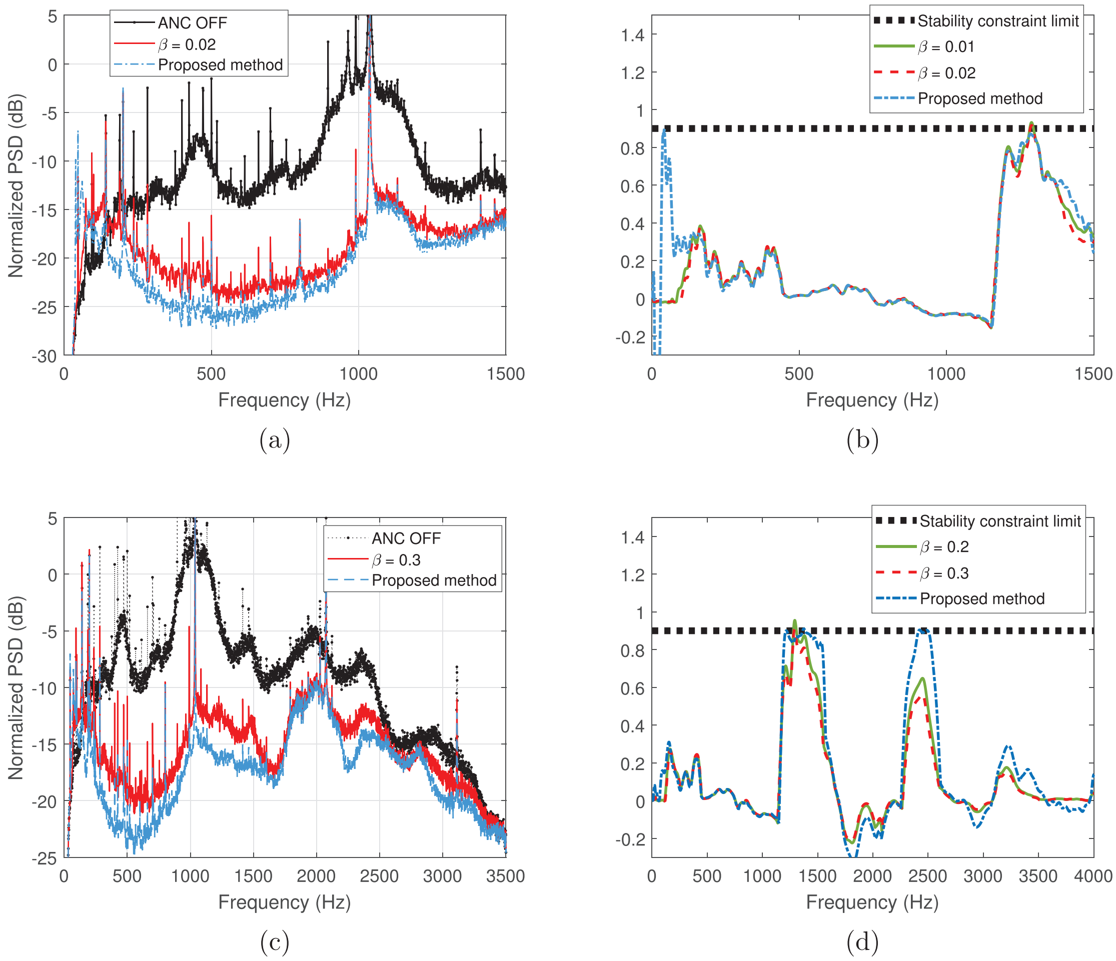

FIG. 5. (Color online) The comparison of regularization parameters method and proposed method using FIR order of 128 in each channel of $4N_r2N_s4N_e$ case for (a) noise control performance for 3000 Hz sampling rate, (b) stability behavior for 3000 Hz sampling rate, (c) noise control performance for 8000 Hz sampling rate, (d) stability behavior for 8000 Hz sampling rate.

control filter design by the proposed method is stable. By comparing the open-loop frequency response of filters designed by the regularization method and the proposed method, there are some frequency bands where filter designed by the regularization method is noticeably lower than (farther away from the constraint limit) that designed by the proposed method. This suggests an over satisfaction of stability constraint when the regularization method is used, which can lead to a degradation of ANC performance. This negative influence on ANC performance can be seen in the results shown in Figs. 5(a) and 5(c). Overall, it is clear that, if a strong acoustic feedback path exists and stability constraints needs to be satisfied, the proposed method can achieve better noise control performance than traditional regularization parameter method.

C. Comparison of traditional H_2/H_∞ method and proposed method

1. Comparison of computational efficiency

The computational efficiency of traditional H_2/H_∞ formulation and proposed convex formulation are compared with the same set of frequencies chosen for objective function and constraints. In this comparison study, the sampling rate is set to be 8000 Hz, the desired noise deduction frequency range is specified to be from 100 to 3400 Hz. In this range, 826 equally spaced frequency points are used in calculating the total energy of error signals, i.e., $N_f = 826$ in the objective functions in Eqs. (10) and (24). Also, 166 equally spaced frequency points are used in the disturbance enhancement constraints (limited to a maximum enhancement of 3 dB); 111 equally spaced frequency points are used in the stability constraints; 67 equally spaced frequency points are used in robustness constraints. There are 81 frequency points in filter response magnitude constraints that covers the frequency band of below 100 Hz and above 3400 Hz to improve the numerical performance and to satisfy the speaker linear response requirement as mentioned in Sec. II A. In the filter design process, the computing platform used in the current work is personal computer with an Intel(R) Core(TM) i7–7700 CPU @ 3.60 GHz, and the numerical algorithms are implemented in a MATLAB platform installed on Windows 10 64-bit operating system. It is noted that when using the proposed method [Eq. (24)], it is analytically formulated in a cone programming format[18,19] and SeDuMi[28] solver is used.

The computational time required to solve the optimization problems is listed in Table. I. In the $4N_r2N_s4N_e$ case, no computational time is presented for the traditional H_2/H_∞ formulation when the order of the FIR ANC filter in each channel is larger than or equal to 32 (the required computational time for the traditional method is over several days) and there is no clear indication of achieving a convergence even for a local minimum. It is shown in Table I that, compared with traditional method, the proposed convex formulation has a significantly higher computational efficiency. Even for relatively low filter orders (e.g., an order of 16), the computational time is reduced from tens of hours to several seconds. If a high filter order is used to obtain a reasonable ANC performance in practice, e.g., an order 64 or 128, where solving the traditional H_2/H_∞ formulation is impractical, the computation for the proposed formulation only takes less than half a minute for the filter order of 64 and less than two minutes for the filter order of 128.

The fact that, even for a long FIR length ($N_t = 128$), the computation of designing a control filter with eight channels for this air-handling system can still be updated within two minutes suggests that the proposed H_2/H_∞ framework has the potential to be applied to a much wider practical applications than the non-adaptive situations studied in related previous literature. Because for variable speed HVAC equipment, equipment speed is usually not changed within several minutes. However, if the traditional formulation is employed, even for a short length ($N_t = 16$), it takes more than one day to update all control filters once which cannot follow the time varying changing of the equipment operating condition. Another practical advantage brought by the improvement of computational efficiency is that, even for a non-adaptive ANC system, it can accelerate the product development cycle in the commercialization of this ANC technique, since many product prototypes involve a large number of engineering iterations and the design problem will be solved multiple times.

2. Comparison of noise control performance

Besides significant improvement in computational time, the proposed method is also more reliable compared with the traditional method, i.e., it is less likely to encounter numerical issues. First, because the proposed method uses a convex formulation, a global optimal solution can always be found. The algorithm for convex problem can handle non-differentiable ANC constraint functions in a rigorous way, while the gradient can only be estimated by finite difference method if algorithms similar to SQP are used, which may cause numerical instability. Moreover, sometimes, SQP does not converge even to a local minimum solution before it terminates at a sufficiently large maximum iterations, or a sufficiently small step length, which could require further user intervention in actual implementation.

The case where both traditional and proposed method converge to satisfactory result is shown in Figs. 6(a) and 6(b). It is shown that the proposed method can achieve a noise control performance similar to the traditional method, which suggests that the replacement of the non-convex stability constraint in traditional formulation with a more restrictive convex stability constraint does not produce any noticeable influence on the final noise control performance. This confirms that the use of the stability constraint relaxation approach in the convexification process is appropriate.

The advantage of the proposed method is significant when the case becomes more complicated, e.g., when the order of the FIR filter in each channel becomes higher. In Figs. 6(c) and 6(d), a case where the traditional formulation solved by SQP is not satisfactory is presented. In this case, the final step size of SQP is smaller than 10^{-6} which is usually considered as an achievement of a converged solution, however, when compared with the resulting performance of the proposed method, the traditional method does not have satisfactory noise control performance and the stability constraint is not satisfied. In contrast, the proposed method still achieves satisfactory noise control performance and satisfies required constraints. In fact, the proposed method can still achieve satisfactory noise control performance while satisfying constraints for even higher filter orders, which is shown in Fig. 7.

TABLE I. Computation time of different problem sizes using different formulation methods.

System	N_t	Traditional method	Proposed method
$2N_r2N_s2N_e$	4	1.78×10^2 s	1.22 s
$2N_r2N_s2N_e$	8	4.79×10^3 s	1.42 s
$2N_r2N_s2N_e$	16	1.75×10^4 s	1.38 s
$2N_r2N_s2N_e$	32	3.37×10^4 s	2.43 s
$4N_r2N_s4N_e$	4	1.37×10^3 s	1.58 s
$4N_r2N_s4N_e$	8	3.24×10^4 s	2.08 s
$4N_r2N_s4N_e$	16	9.52×10^4 s	2.26 s
$4N_r2N_s4N_e$	32		6.17 s
$4N_r2N_s4N_e$	64		22.58 s
$4N_r2N_s4N_e$	128		94.44 s

D. Other characteristics of the proposed method

It is pointed out in Sec. II B 2 that the proposed formulation will result in a problem complexity that is independent of the number of frequency points used in the objective function. To confirm this, the set of data for a $4N_r2N_s4N_e$ configuration was used to test the required time for the proposed method when the frequency resolution in the objective function varies. In this test, the FIR filter order is chosen to be 128. The results are listed in Table II. Instead of showing the total time required for solving the optimization problem, Table II shows two parts of the total time separately. The column "Solving time" denotes the time spent in solving the cone programming problem reformulated from the convex optimization formulation described in Eq. (24). It is demonstrated that as the number of frequency points used in the objective function increases, the time required for solving the optimization problem remains relatively the same. "Constructing time" is the preprocessing

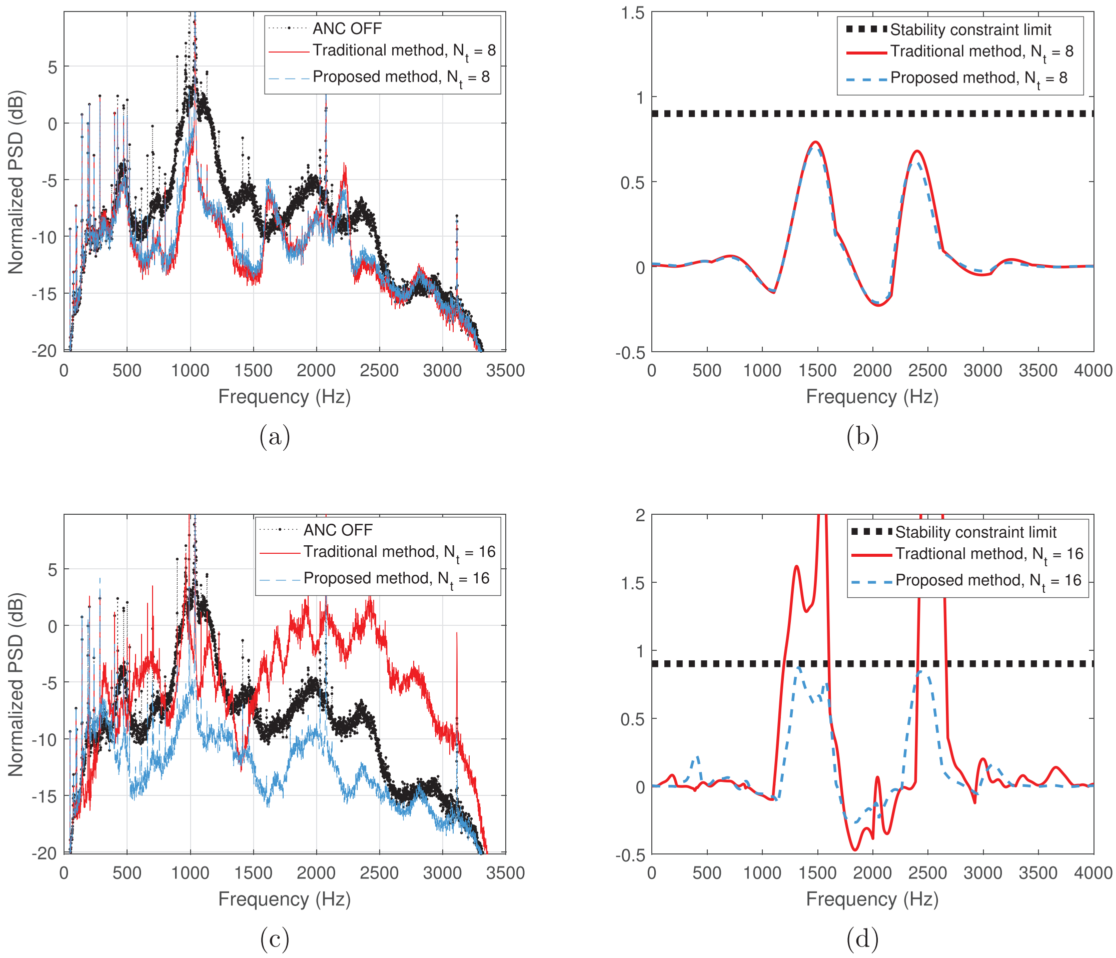

FIG. 6. (Color online) The comparison of traditional H_2/H_∞ method and proposed convex method of $4N_r2N_s4N_e$ case in 8000 Hz sampling rate for (a) noise control performance of FIR order of 8 in each channel, (b) stability behavior of FIR order of 8 in each channel, (c) noise control performance of FIR order of 16 in each channel, (d) stability behavior of FIR order of 16 in each channel.

time required for processing measured data to be in the SeDuMi interface format. It is reasonable that the constructing time increases as the increase in frequency resolution used in the objective function (i.e., more data is used).

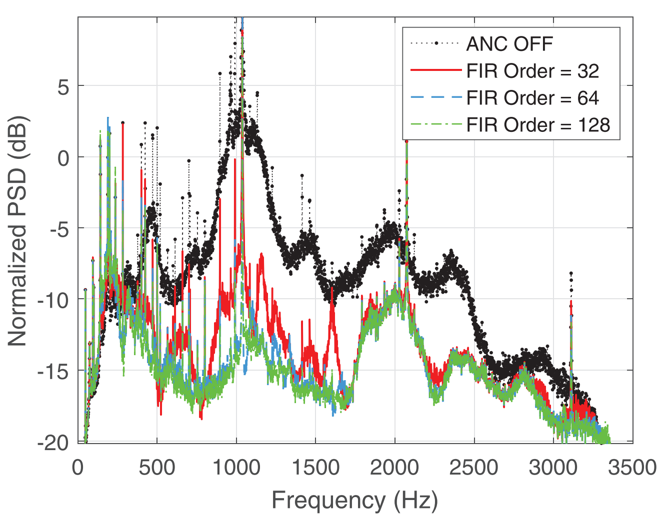

FIG. 7. (Color online) Comparison of ANC performance of the "$4N_r2N_s4N_e$" system with different choices of filter order N_t designed by proposed method.

The practical advantage of this property is that the choice of frequency interval for objective function can be sufficiently small to ensure the overall performance without increasing the solving time significantly.

The number of iterations of more choices of filter lengths with the same setting of constraints are listed in Table III. In practice, the number of iterations usually does not change significantly with the increase in problem size and is usually within 100 iterations,[21] which can be demonstrated from results in Table III. The increase in required solving time with the increase in problem size is mainly from the increase in the calculation time in each iteration. This relatively fixed number of iterations in the numerical solving process is also one important reason why the

TABLE II. Computation time of different number of frequency points used in objective function by using proposed method when FIR filter order is 128.

Number of frequencies	Solving time	Constructing time
195	84.61 s	3.37 s
413	84.10 s	4.76 s
826	87.07 s	7.38 s
1651	84.08 s	12.76 s
3302	84.56 s	23.55 s

TABLE III. Required iterations of different filter order of the "$4N_r2N_s4N_e$" system when using proposed method.

Filter order N_t	Iterations
4	28
8	28
16	18
32	20
64	22
128	22

proposed method is more reliable. The small number of iterations makes the convergence time more predictable, which, from a practical point of view, there is a fixed adaptation rate if this method is used in adaptive control via continuous repeating the filter design process. Thus, it is easy to judge the practical applicability of the proposed method for a specific situation where the time-varying rate of signal characteristics can be estimated.

IV. CONCLUSION

In this article, a convex formulation for designing constrained optimal multi-channel ANC filter is proposed. The traditional H_2/H_∞ framework is relaxed to a convex problem by replacing the non-convex stability constraint function with its convex upper bound, which is still less conservative compared with other convex stability constraints used in previous studies. The objective function and other constraint functions are greatly simplified to commonly used convex functions. Thus, efficient algorithms can be applied to solve this problem through a further reformulation to an equivalent cone programming problem.

Compared with commonly used regularization parameters method to ensure the satisfaction of constraints, the proposed method can achieve better noise control performance, especially when the system is complicated (e.g., systems with multiple channels, strong acoustic feedback path, high sampling rate, etc.).

Compared with the traditional H_2/H_∞ method, the proposed method is more reliable because the global minimum can always be found and the required number of iterations in the numerical solving process is relatively small and independent of the filter order. Finer frequency resolution in the cost function can be chosen without significantly increasing the required solving time. The computational time can be reduced from the order of hours to seconds, which can significantly accelerate the commercialized product development cycle. With all these benefits, the proposed method allows the use of H_2/H_∞ control framework to a wider range of practical applications.

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APPENDIX

The reformulation of simplifying Eq. (5) is presented here. For brevity, f_k is ignored from the formulation. First, considered the trace of $\mathbf{G}_e\mathbf{W}_x\mathbf{S}_{xx}\mathbf{W}_x^{\mathrm{H}}\mathbf{G}_e^{\mathrm{H}}$. It can be expressed as a Frobenius inner product by using vectorization operator, vec(), which converts a matrix to a vector by stacking the columns,[22]

$$\mathrm{tr}\left[\mathbf{G}_e\mathbf{W}_x\mathbf{S}_{xx}\mathbf{W}_x^{\mathrm{H}}\mathbf{G}_e^{\mathrm{H}}\right] = \mathrm{vec}(\mathbf{W}_x^{\mathrm{H}}\mathbf{G}_e^{\mathrm{H}})^{\mathrm{H}}\mathrm{vec}(\mathbf{S}_{xx}\mathbf{W}_x^{\mathrm{H}}\mathbf{G}_e^{\mathrm{H}}). \tag{A1}$$

Then, use the relation between vec() and the Kronecker product: $\mathrm{vec}(\mathbf{A}B) = (\mathbf{I}\otimes\mathbf{A})\mathrm{vec}(B) = (B^{\mathrm{T}}\otimes\mathbf{I})\mathrm{vec}(A)$, where $\otimes$ denotes the Kronecker product of two matrices. Equation (A1) can be reformulated as

$$\mathrm{vec}(\mathbf{W}_x^{\mathrm{T}})^{\mathrm{T}}(\mathbf{G}_e^{\mathrm{T}}\otimes\mathbf{I}_{N_r})(\mathbf{I}_{N_e}\otimes\mathbf{S}_{xx})(\mathbf{G}_e^{*}\otimes\mathbf{I}_{N_r})\mathrm{vec}(\mathbf{W}_x^{\mathrm{H}}). \tag{A2}$$

Note that Eq. (A2) can be further simplified using mixed-product property of Kronecker product: $(\mathbf{A}\otimes\mathbf{B})(\mathbf{C}\otimes\mathbf{D}) = (\mathbf{AC})\otimes(BD)$,

$$\mathrm{vec}(\mathbf{W}_x^{\mathrm{T}})^{\mathrm{T}}((\mathbf{G}_e^{\mathrm{H}}\mathbf{G}_e)^{\mathrm{T}}\otimes\mathbf{S}_{xx})\mathrm{vec}(\mathbf{W}_x^{\mathrm{H}}). \tag{A3}$$

Since Eq. (A3) is a scalar, taking a transpose of Eq. (A3) will result in the same value. Thus, Eq. (A3) equals

$$\mathrm{vec}(\mathbf{W}_x^{\mathrm{T}})^{\mathrm{H}}(\mathbf{G}_e^{\mathrm{H}}\mathbf{G}_e\otimes\mathbf{S}_{xx}^{\mathrm{T}})\mathrm{vec}(\mathbf{W}_x^{\mathrm{T}}). \tag{A4}$$

If coefficients of control filters $\vec{\mathbf{w}}_F$ is rearranged as the order in $\vec{\mathbf{w}}$ in Eq. (19), then we have

$$\mathrm{vec}(\mathbf{W}_x^{\mathrm{T}}) = (\mathbf{I}_{N_s}\otimes\mathbf{I}_{N_r}\otimes\vec{\mathbf{F}}_z^{\mathrm{T}})\vec{\mathbf{w}}. \tag{A5}$$

By using Eq. (A5) and those Kronecker product properties mentioned above, Eq. (A4) can be finally simplified as

$$\vec{\mathbf{w}}^{\mathrm{T}}(\mathbf{G}_e^{\mathrm{H}}\mathbf{G}_e)\otimes\mathbf{S}_{xx}^{\mathrm{T}}\otimes\left(\mathbf{F}_z^{*}\vec{\mathbf{F}}_z^{\mathrm{T}}\right)\vec{\mathbf{w}}. \tag{A6}$$

Since $\vec{\mathbf{w}}$ is a real-valued vector, Eq. (A6) can be simplified as

$$\vec{\mathbf{w}}^{\mathrm{T}}\Re\left((\mathbf{G}_e^{\mathrm{H}}\mathbf{G}_e)\otimes\mathbf{S}_{xx}^{\mathrm{T}}\otimes\left(\mathbf{F}_z^{*}\vec{\mathbf{F}}_z^{\mathrm{T}}\right)\right)\vec{\mathbf{w}}. \tag{A7}$$

In a similar way of rearranging Eqs. (A1)–(A6), we have

$$\mathrm{tr}[\mathbf{G}_e\mathbf{W}_x\mathbf{S}_{xd_e}] = \mathrm{vec}\left((\mathbf{S}_{xd_e}\mathbf{G}_e)\otimes\vec{\mathbf{F}}_z\right)^{\mathrm{T}}\vec{\mathbf{w}}. \tag{A8}$$

Considering that Eq. (A8) is a scalar, we have

$$\mathrm{tr}\left[\mathbf{G}_e\mathbf{W}_x\mathbf{S}_{xd_e} + \mathbf{S}_{xd_e}^{\mathrm{H}}\mathbf{W}_x^{\mathrm{H}}\mathbf{G}_e^{\mathrm{H}}\right] = 2\Re\left(\mathrm{vec}\left((\mathbf{S}_{xd_e}\mathbf{G}_e)\otimes\vec{\mathbf{F}}_z\right)\right)^{\mathrm{T}}\vec{\mathbf{w}}. \tag{A9}$$

Then, Eq. (5) is simplified to the standard quadratic form in Eq. (19).

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