An Asymptotically Exact Estimate of the Median Noise Eigenvalue of Sample Covariance Matrices

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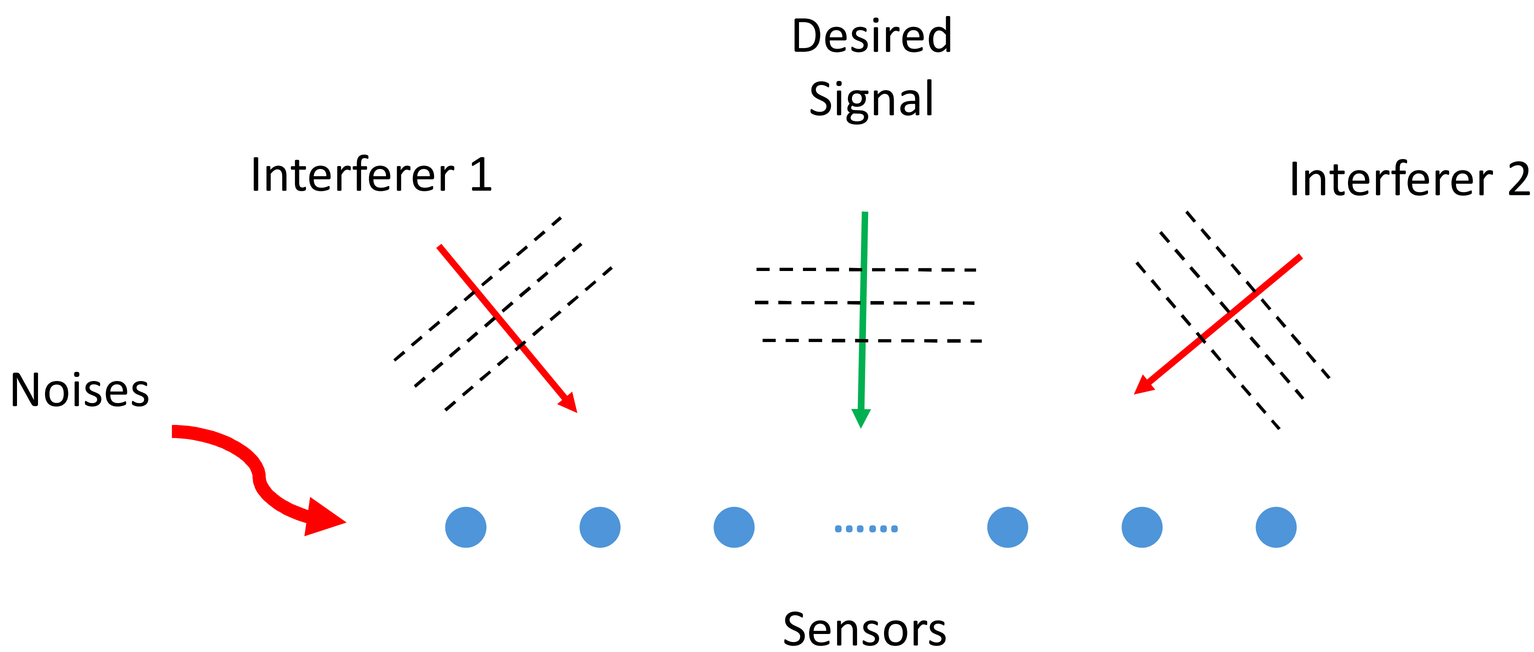
Adaptive beamforming for passive sonar is one of the original applications of data sciences in underwater acoustics

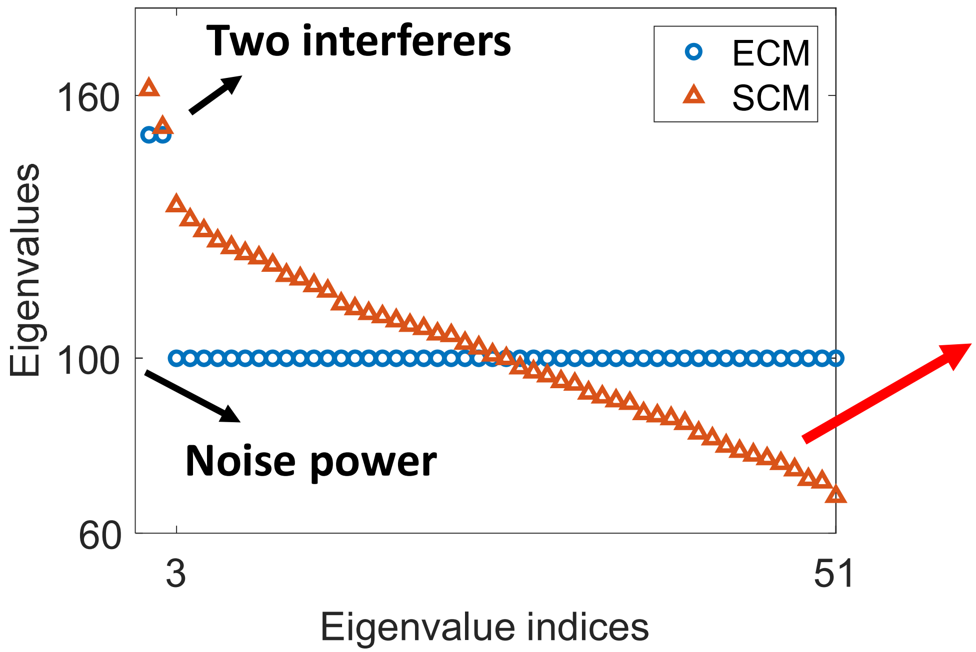


Most popular adaptive beamformer:
Minimum Variance Distortionless Response (**MVDR**) beamformer that tries to minimize the effect of interferers and noises while keeping the desired signal unchanged.

Using the SCM instead of ECM in MVDR degrades adaptive beamformer performance

- **Ideal case**: ensemble covariance matrix (ECM) → not available in real life
- **Practical case**: sample covariance matrix (SCM) → approaching ECM with more snapshots

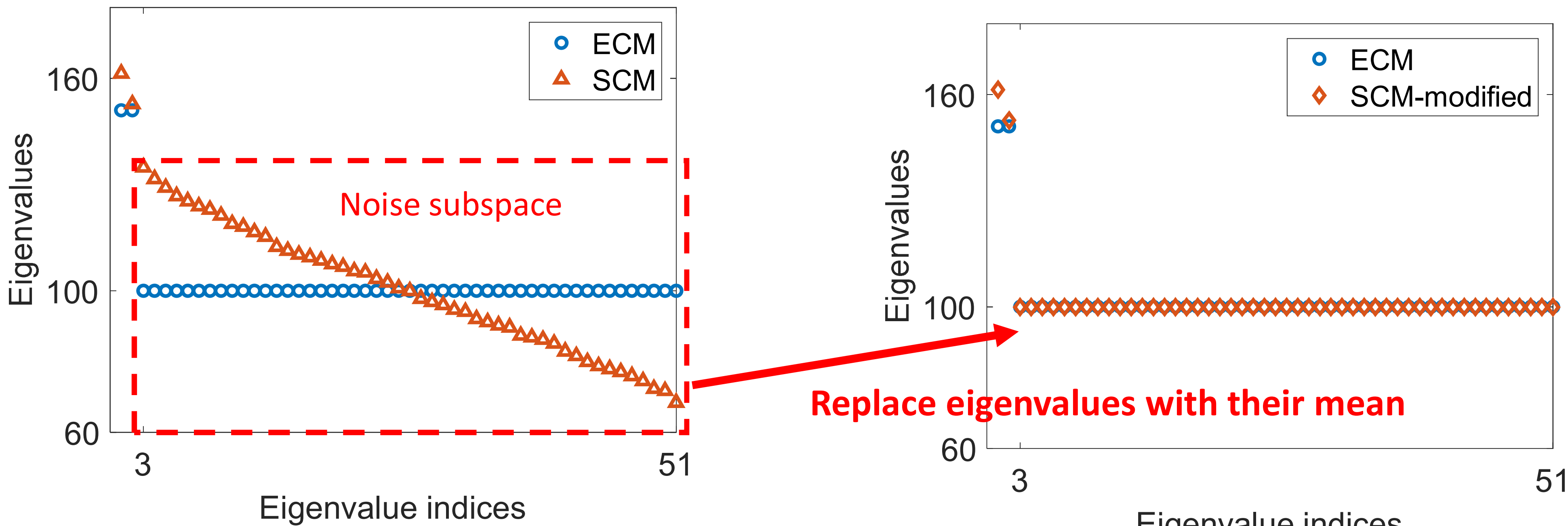
An example of 51 sensors, 2 interferers, SCM when $\frac{\text{\# of sensors}}{\text{\# of snapshots}} = \frac{1}{30}$



- Inverting the covariance matrix is a key step in MVDR beamformers
- The deviation from true noise power will harm the white noise gain

The DMR beamformer replaces smaller eigenvalues with their mean improves performance

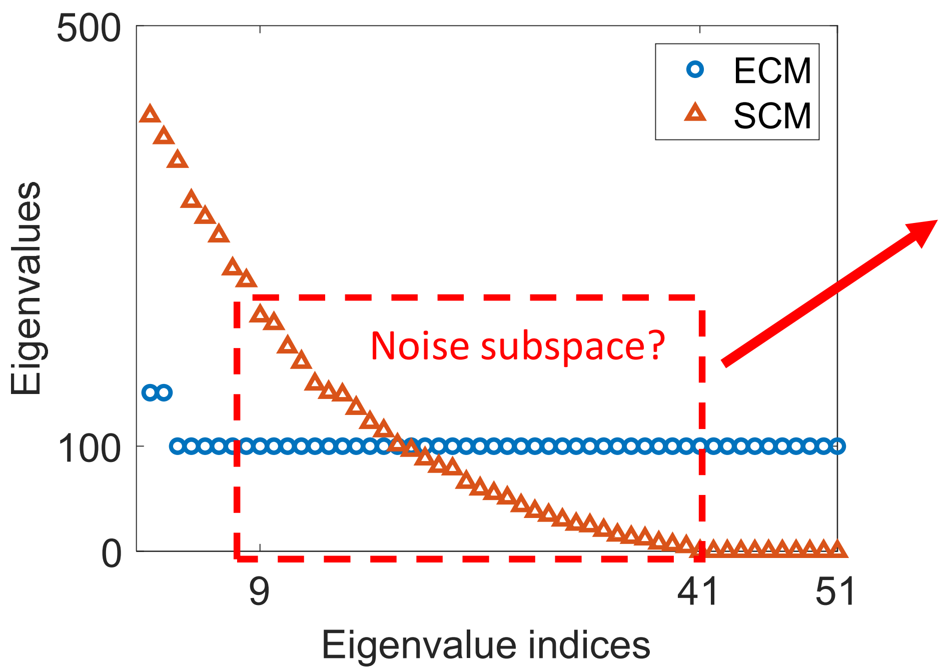
Dominant Mode Rejection (DMR) beamformer [Abraham & Owsley, 1990]:



Now the eigenvalues in the modified SCM look very similar to the ECM → better performance!

"Mean" works poorly in snapshot deficient case

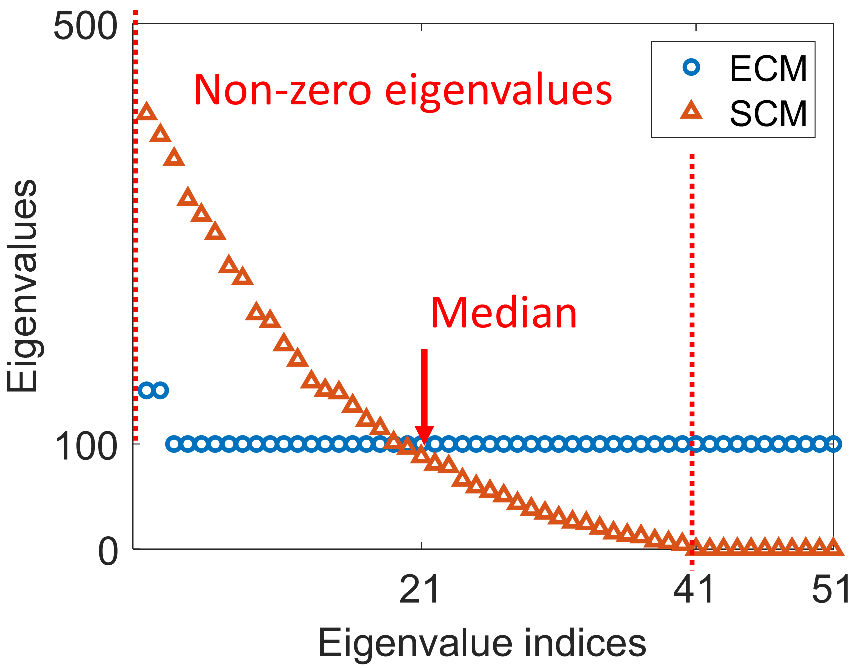
Snapshot deficient case: $\frac{\text{\# of sensors}}{\text{\# of snapshots}} = \frac{5}{4}$



- Mean of guessed noise eigenvalues = 81
  → much smaller than 100
- The noise power is usually negatively biased

Median can be a more accurate estimate than Mean

$$\frac{\#\text{ of sensors}}{\#\text{ of snapshots}} = \frac{5}{4}$$



Advantages of using median value:

- No need to guess noise subspace size
- Robust to outliers

A remaining challenge:

Noise power $\neq$ Median eigenvalue = 88

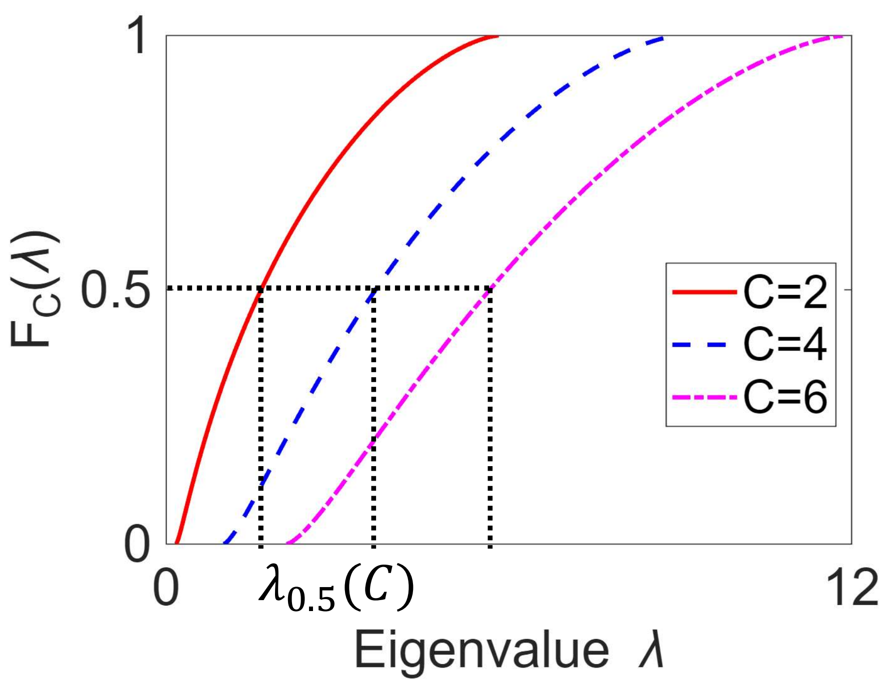
One goal of this talk:

Noise power **?** Median eigenvalue

Key to the expression: Marchenko-Pastur (MP) distribution

$C = \frac{\text{\# of sensors}}{\text{\# of snapshots}}$

The MP distribution $F_C(\lambda)$ from random matrix theory is the CDF of eigenvalues of SCM



$$\text{noise power} = \frac{\text{median of eigs(SCM)}}{\lambda_{0.5}(C)}$$

→ is an unbiased estimator of the noise power for a given C

Numerical fitting of MP distribution to get $\lambda_{0.5}(C)$

[Anchieta & Buck, 2022] used numerical regression fitting to show:

- A simple 1st order approximation:

  $\lambda_{0.5}(C) = C - 0.345$

  approximates the solution of $F_C(\lambda) = 0.5$

- This approximation improved the performance of Dominant Mode Rejection beamformers.

But why?

Trust me, simply plugging it back does not work

Why do we need theoretical analysis?

- Numerical results depends on the fitting range:
  $C \in (1,2)$ → $\lambda_{0.5}(C) = C - 0.353$
  $C \in (1,5)$ → $\lambda_{0.5}(C) = C - 0.345$
  $C \in (1,10)$ → $\lambda_{0.5}(C) = C - 0.340$
  $C \in (1,100)$ → $\lambda_{0.5}(C) = C - 0.335$
  ……
  Will it fail at some point when C is large? Will it converge to a constant?

- The lack of theoretical understanding of this approximation may thwart further exploration of using other order statistics (k-th largest eigenvalue).
  We may derive other order statistics using the insight from median point derivation.

→ **In this talk, we present a theoretical derivation of the 1st order approximation.**

Proposed asymptotically exact estimate

There are several challenges in solving $F_C(\lambda) - 0.5 = 0$:

A mixture of several $\boldsymbol{sin^{-1}}()$**, square roots,** and **ratios of polynomials**.

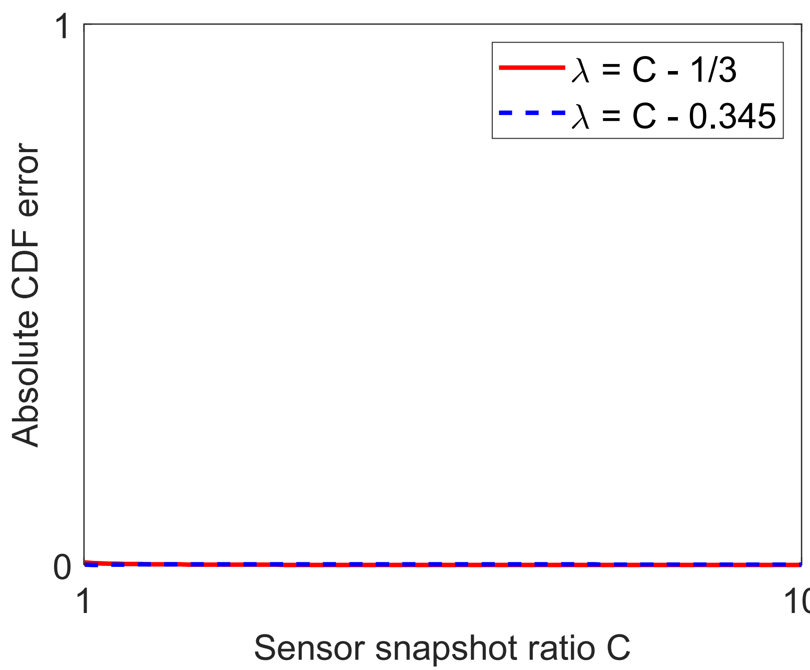
Four main steps:

1. Replacing $\mathbf{sin^{-1}}()$ with its power series expansion
2. Apply exponents after power series expansion → **square roots** disappear
3. Rotate the $C - \lambda$ coordinate by 45° and rearranging it to **series of polynomials**.
4. Solve the dominant term: $F_C(\lambda) - 0.5 \quad \rightarrow \quad \infty$ unless $\lambda_{0.5}(C) = C - 1/3$

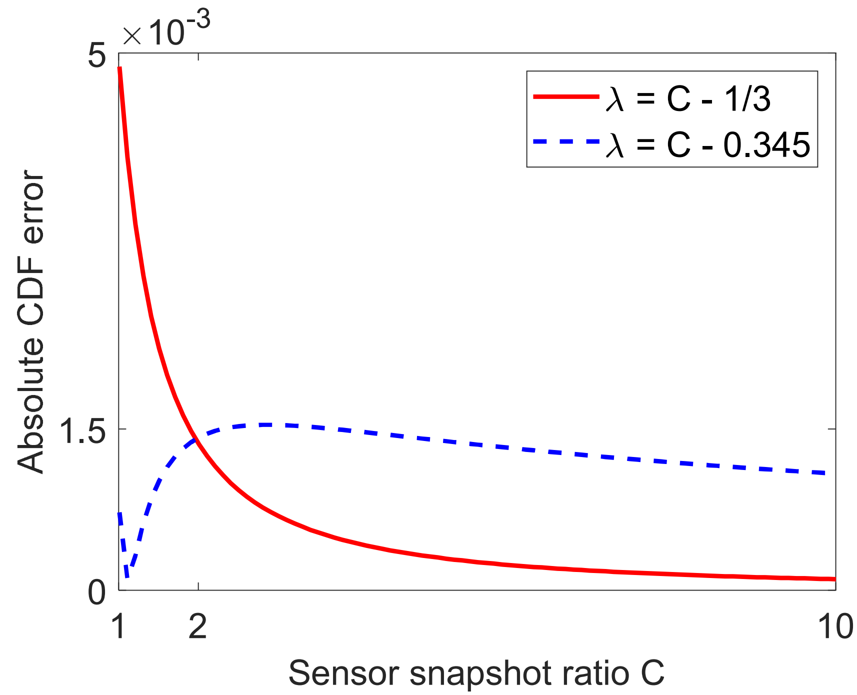
→ Thus, it is an **asymptotically exact estimate**

Analytically derived formulation is more accurate at higher C → asymptotic

Check if $|F_C(\lambda) - 0.5| = 0$?



Zoom-in



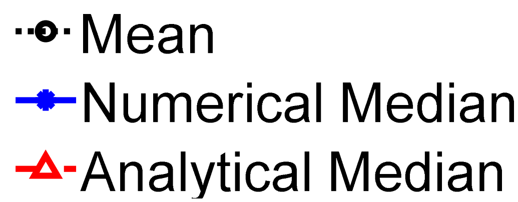
Both of them work well compared with CDF range ($0 \leq \mathrm{CDF} \leq 1$)

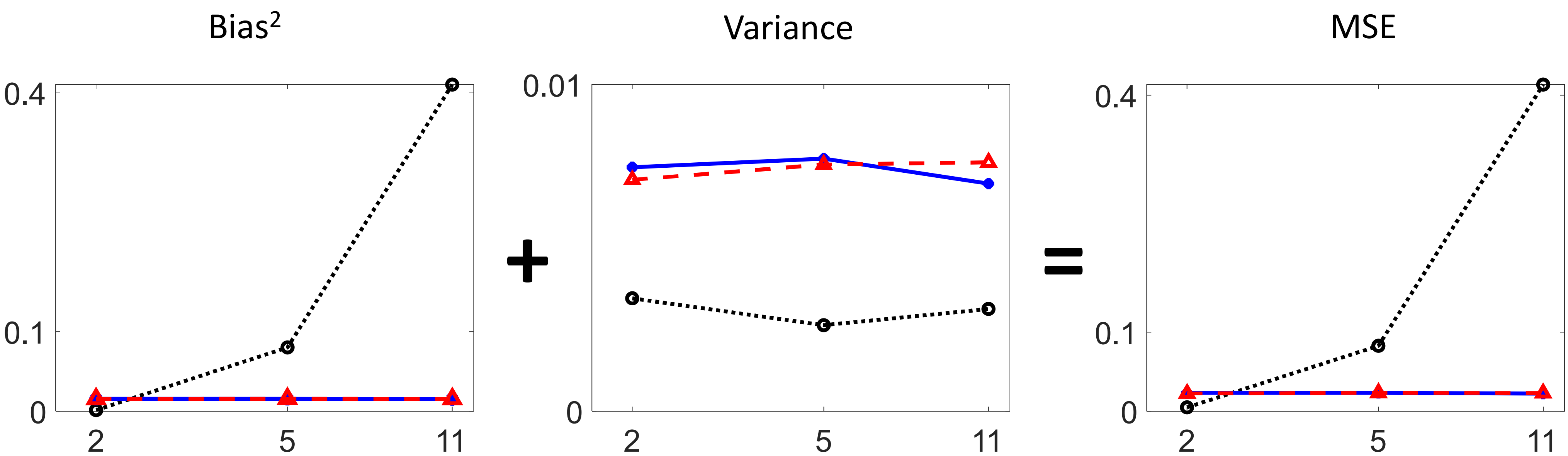
Analytically derived formulation is more accurate when $C > 2$ (snapshot deficient environments)

Median estimator reduces bias by increasing the variance slightly

Estimating the noise power:

- 2 interferers
- Sensor number to snapshot ratio C = 4.25
- 1000 Monte Carlo Trials



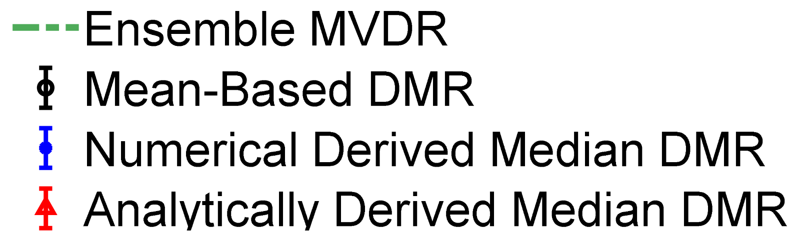


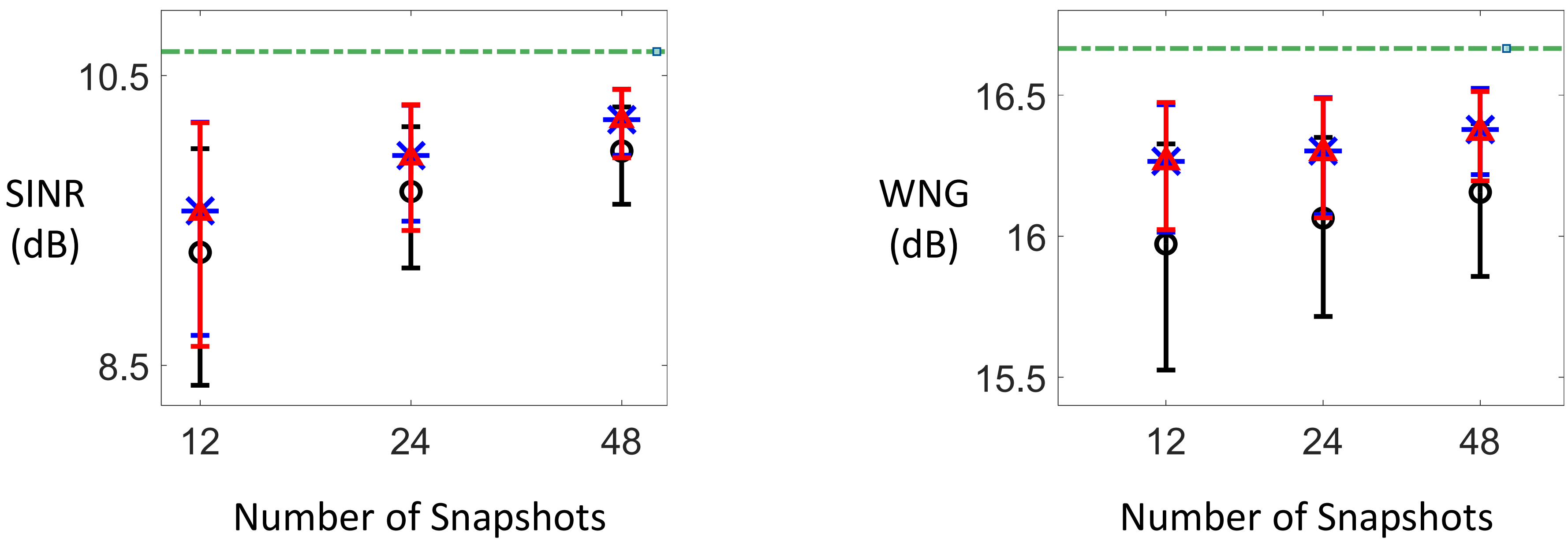
x axis: guessed interferer subspace size

Better DMR beamformer performance

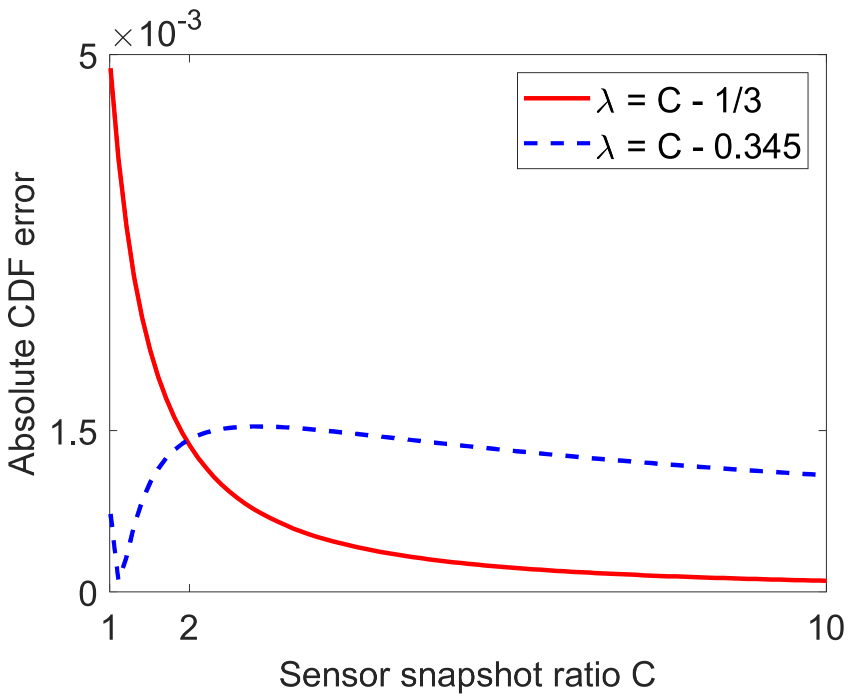
Applied to DMR beamformer:

- 2 interferers
- Guessed interferers number: 11
- 1000 Monte Carlo Trials
- Vertical line: 90% confidence intervals





Conclusion



- $\lambda_{0.5}(C) = C - 1/3$ is analytically derived by an asymptotically exact random matrix theory analysis
- It is more accurate at higher C value
  → asymptotic
- Median estimator for noise power reduces bias by increasing the variance slightly
- Median-based Dominant Mode Rejection beamformer has higher SINR and WNG
- In the future, this technique can be applied to order statistics other than median.

References

- Abraham, Douglas A., and Norm L. Owsley. "Beamforming with dominant mode rejection." *Conference Proceedings on Engineering in the Ocean Environment*. IEEE, 1990.
- Anchieta, David Campos, and John R. Buck. "Improving the robustness of the dominant mode rejection beamformer with median filtering." *IEEE Access* 10 (2022): 120146-120154.