Study on the Cone Programming Reformulation of Active Noise Control Filter Design in the Frequency Domain

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PURDUE MECHANICAL ENGINEERING

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Introduction

- ❑ Multichannel active noise control (ANC) systems
  - Better performance when we need to create large-size quiet zone.
  - Applications:

Interior of Vehicles

Range Hood

Infant Incubator

Air Conditioner

Introduction

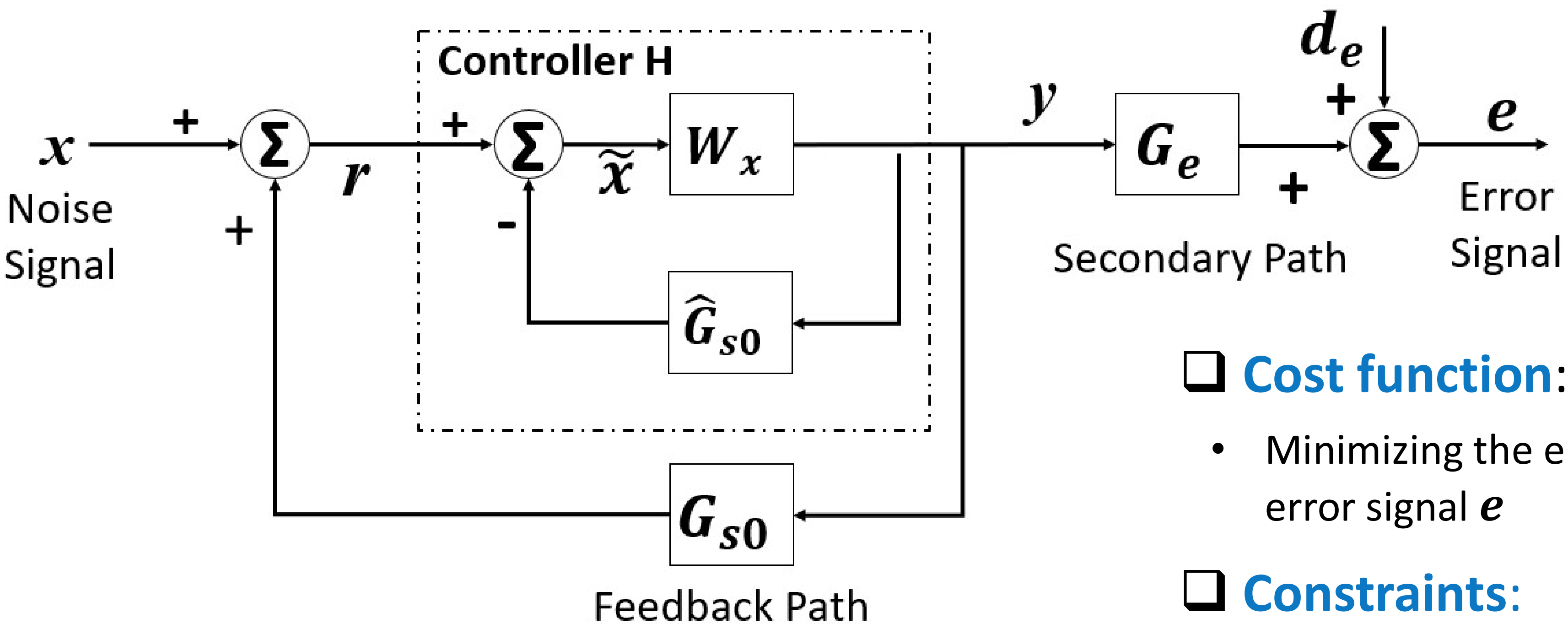
❑ Motivation of using frequency domain design

- Easier to specify frequency dependent constraints.
- Constraints in one frequency band will not affect performance of other bands.
- Usually, better ANC performance.
- Convenient to design sub-band filter structure

❑ Motivation of using cone programing

- The computational complexity is usually significant for frequency domain design method.
- In recent study of convex optimization, very efficient algorithms were developed for cone programing.
- Many optimization problems can be converted to cone programing.
- Potential to perform adaptive control in frequency domain with multiple constraints.

Active Noise Control System



- **Cost function**:
  - Minimizing the energy of error signal e
- **Constraints:**
  - Disturbance enhancement
  - Stability
  - Robustness
  - Filter response

(Non-adaptive control is considered in the current work)

Overview of Reformulation

Original Problem

Cost function:
Energy of ***e***
Constraints:
Enhancement
Stability
Robustness
Filter response

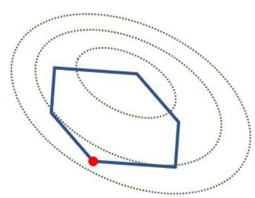
Apply → **Sequential Quadratic Programming Method**

(Commonly Used Method)

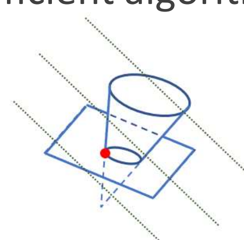
(Proposed Method)

Modified → **Convex Optimization** → Reformulate → **Cone Programming** → Apply → **Primal-dual Interior-Point**

Convex Optimization: Global minimum usually guaranteed



Cone Programming: Possible to apply efficient algorithms



Primal-dual Interior-Point:

- Efficient
- Upper-bound

Original Problem

Cost function:

$$\sum_{k=k_1}^{k_2} tr[E(f_k)E(f_k)^{\mathrm{H}}]$$

Total energy of e cross all frequencies

Constraints:

Enhancement: **Normalized energy of e:**

$$tr[E(f_k)E(f_k)^{\mathrm{H}}]D_e(f_k) \leq A_e$$

Stability: **Use Nyquist criterion:**

$$\min\left(\mathrm{Re}\left(\lambda\left(W_x(f_k)\hat{G}_{s0}(f_k)\right)\right)\right) > -1$$

Robustness: ***M*- Δ structure and small gain theory:**

$$\max\left(\sigma\left(W_x(f_k)\hat{G}_{s0}(f_k)\right)\right)B(f_k) \leq 1$$

Filter response: **The magnitude of frequency response:**

$$\left|W_{x_{i,j}}(f_k)\right| \leq C(f_k)$$

Original Problem

Cost function: **Total energy of e:**

$$\sum_{k=k_1}^{k_2} tr[E(f_k)E(f_k)^{\mathrm{H}}],$$

Constraints:

Enhancement:

$$tr[E(f_k)E(f_k)^{\mathrm{H}}]D_e(f_k) \leq A_e$$

Normalized energy of e at each frequency

Stability: **Use Nyquist criterion:**

$$\min\left(\mathrm{Re}\left(\lambda\left(W_x(f_k)\hat{G}_{s0}(f_k)\right)\right)\right) > -1$$

Robustness: ***M*- Δ structure and small gain theory:**

$$\max\left(\sigma\left(W_x(f_k)\hat{G}_{s0}(f_k)\right)\right)B(f_k) \leq 1$$

Filter response: **The magnitude of frequency response:**

$$\left|W_{x_{i,j}}(f_k)\right| \leq C(f_k)$$

Original Problem

Cost function: **Total energy of e:**

$$\sum_{k=k_1}^{k_2} tr[E(f_k)E(f_k)^{\mathrm{H}}],$$

Constraints:

Enhancement: **Normalized energy of e:**

$$tr[E(f_k)E(f_k)^{\mathrm{H}}]D_e(f_k) \le A_e$$

Stability:

$$\min\left(\mathrm{Re}\left(\lambda\left(W_x(f_k)\hat{G}_{s0}(f_k)\right)\right)\right) > -1 \Rightarrow$$ **Nyquist criterion, on the right of -1 point**

Robustness: ***M*- Δ structure and small gain theory:**

$$\max\left(\sigma\left(W_x(f_k)\hat{G}_{s0}(f_k)\right)\right)B(f_k) \le 1$$

Filter response: **The magnitude of frequency response:**

$$\left|W_{x_{i,j}}(f_k)\right| \le C(f_k)$$

Original Problem

Cost function: Total energy of e:

$$\sum_{k=k_1}^{k_2} tr[E(f_k)E(f_k)^{\mathrm{H}}]\,,$$

Constraints:

Enhancement: **Normalized energy of e:**

$$tr[E(f_k)E(f_k)^{\mathrm{H}}]D_e(f_k) \leq A_e$$

Stability: **Use Nyquist criterion:**

$$\min\left(\mathrm{Re}\left(\lambda\left(W_x(f_k)\hat{G}_{s0}(f_k)\right)\right)\right) > -1$$

Robustness:

$$\max\left(\sigma\left(W_x(f_k)\hat{G}_{s0}(f_k)\right)\right)B(f_k) \leq 1$$

***M*- Δ structure and small gain theory**

Filter response: **The magnitude of frequency response:**

$$\left|W_{x_{i,j}}(f_k)\right| \leq C(f_k)$$

Original Problem

Cost function: Total energy of e:

$$\sum_{k=k_1}^{k_2} tr[E(f_k)E(f_k)^{\mathrm{H}}],$$

Constraints:

Enhancement: **Normalized energy of e:**

$$tr[E(f_k)E(f_k)^{\mathrm{H}}]D_e(f_k) \leq A_e$$

Stability: **Use Nyquist criterion:**

$$\min\left(\mathrm{Re}\left(\lambda\left(W_x(f_k)\hat{G}_{s0}(f_k)\right)\right)\right) > -1$$

Robustness: ***M*- Δ structure and small gain theory:**

$$\max\left(\sigma\left(W_x(f_k)\hat{G}_{s0}(f_k)\right)\right)B(f_k) \leq 1$$

Filter response:

$$\left|W_{x_{i,j}}(f_k)\right| \leq C(f_k)$$

The magnitude of frequency response

Modification

Original Problem

Cost function: **Total energy of e:**

$$\sum_{k=k_1}^{k_2} tr[E(f_k)E(f_k)^{\mathrm{H}}],$$

Constraints:

Enhancement: **Normalized energy of e:**

$$tr[E(f_k)E(f_k)^{\mathrm{H}}]D_e(f_k) \leq A_e$$

Stability: **Use Nyquist criterion:**

$$\min\left(\mathrm{Re}\left(\lambda\left(W_x(f_k)\hat{G}_{s0}(f_k)\right)\right)\right) > -1$$

Robustness: ***M*- Δ structure and small gain theory:**

$$\max\left(\sigma\left(W_x(f_k)\hat{G}_{s0}(f_k)\right)\right)B(f_k) \leq 1$$

Filter response: **The magnitude of frequency response:**

$$\left|W_{x_{i,j}}(f_k)\right| \leq C(f_k)$$

Standard General Convex Problem

Cost function: $f_0(x)$

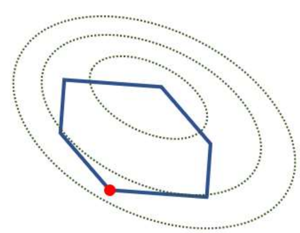
Constraints: $f_i(x) \leq 0, \qquad i = 1, 2, 3 \ \ldots$

$$Ax = b$$

$f_0(x)$ **to be a convex function**

$f_i(x)$ **to be a convex function**

A, b **to be a constant matrix and vector**



Modification

Original Problem → **General Convex Problem**

Cost function: $w^{\mathrm{T}}\left(\sum_{k=k_1}^{k_2} A_J(f_k)\right)w + 2\mathrm{Re}\left(\sum_{k=k_1}^{k_2} b_J^{\mathrm{T}}(f_k)\right)w + \sum_{k=k_1}^{k_2} c_J(f_k)$ →
- Quadratic
- Hessian $A_J(f_k)$ positive semidefinite

Convex ✓

Constraints:

Enhancement: $w^{\mathrm{T}} A_J(f_k)\, w + 2\mathrm{Re}\left(b_J^{\mathrm{T}}(f_k)\right)w + c_J(f_k) - \dfrac{A_e}{D_e(f_k)} \leq 0$ →
- Quadratic
- Hessian $A_J(f_k)$ positive semidefinite

Convex ✓

Robustness: $\max\left(\sigma\left(W_x(f_k)\hat{G}_{s0}(f_k)\right)\right)B(f_k) - 1 \leq 0$ → Matrix norm Convex ✓

Filter response: $\|F_z(f_k)\, w_{F_{i,j}}\|_2 - C(f_k) \leq 0$ → Vector norm Convex ✓

Stability: $\min\left(\mathrm{Re}\left(\lambda\left(W_x(f_k)\hat{G}_{s0}(f_k)\right)\right)\right) > -1$ → NOT Convex ✗

Replace by its upper-bound:

$$\max\left(\lambda\left(\frac{-W_x(f_k)\hat{G}_{s0}(f_k) + \left(-W_x(f_k)\hat{G}_{s0}(f_k)\right)^{\mathrm{H}}}{2}\right)\right) - (1-\epsilon_s) \leq 0$$

→ Easy to prove convexity by $\max(\lambda(A)) = \sup_{\|x\|_2=1} x^T A x$ Convex ✓

Reformulation

Convex Problem

Cost function:

$$w^{\mathrm{T}}\left(\sum_{k=k_1}^{k_2} A_J(f_k)\right) w + 2\mathrm{Re}\left(\sum_{k=k_1}^{k_2} b_J^{\mathrm{T}}(f_k)\right) w + \sum_{k=k_1}^{k_2} c_J(f_k)$$ **Quadratic**

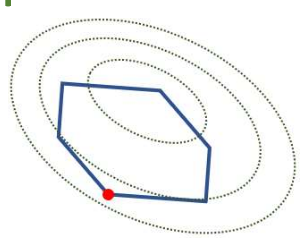
Constraints:

$$w^{\mathrm{T}} A_J(f_k)\, w + 2\mathrm{Re}\big(b_J^{\mathrm{T}}(f_k)\big) w \; + \; c_J(f_k) - \frac{A_e}{D_e(f_k)} \leq 0$$ **Quadratic**

$$\max\left(\lambda\left(\frac{-W_x(f_k)\hat{G}_{s0}(f_k) + \left(-W_x(f_k)\hat{G}_{s0}(f_k)\right)^{\mathrm{H}}}{2}\right)\right) - (1-\epsilon_s) \leq 0$$ **Max Eigenvalue**

$$\max\Big(\sigma\big(W_x(f_k)\hat{G}_{s0}(f_k)\big)\Big) B(f_k) - 1 \leq 0$$ **Max Singular Value**

$$\|F_z(f_k)\, w_{F_{i,j}}\|_2 - C(f_k) \leq 0$$ **Vector Norm**



Standard Cone Programming

Cost function: $c^{\mathrm{T}} x$

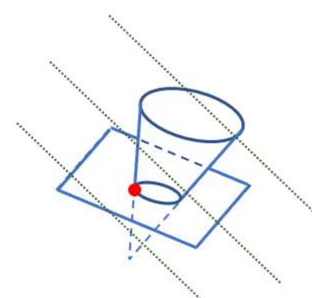
Constraints: $x \in K_i, \qquad i = 1, 2, 3\ \ldots$

$$Ax = b$$

c **to be a constant vector**

K_i **to be a convex cone**

A, b **to be a constant matrix and vector**



Reformulation

Convex Problem ➡ **Cone Programming**

- **Reformulate quadratic cost function**

  Cost function: $x^{\mathrm{T}} A\, x + b^{\mathrm{T}} x + c$

  $\Updownarrow$

  Cost function: $t_0 + b^{\mathrm{T}} x$ ➡ Linear cost function

  Constraints: $\|\sqrt{A}\, x\|_2 \leq \sqrt{t_0\, \tilde{t}_0}$ ➡ Rotated second-order cone

  $\tilde{t}_0 = 1$ ➡ Linear constraint

- **Reformulate quadratic constraints**

  Constraints : $x^{\mathrm{T}} A\, x + b^{\mathrm{T}} x + c \leq 0$

  $\Updownarrow$

  Constraints: $t_1 + b^{\mathrm{T}} x + c = 0$ ➡ Linear constraint

  $\|\sqrt{A}\, x\|_2 \leq \sqrt{t_1\, \tilde{t}_1}$ ➡ Rotated second-order cone

  $\tilde{t}_1 = 1$ ➡ Linear constraint

- **The vector norm itself meets second-order cone requirement**

Reformulation

Convex Problem ➡ **Cone Programming**

- **Reformulate eigenvalue constraints**

Constraints : $\max\left(\lambda\left(\frac{A(x)+A(x)^{\mathrm{H}}}{2}\right)\right)-\epsilon \le 0$ ➡ Each $A_{i,j}$ is the linearly related to x

⇕

Constraints: $-A(x)-A(x)^{\mathrm{H}}+2\epsilon I \succcurlyeq 0$ ➡ Positive semidefinite cone
Each $A_{i,j}$ is the linearly related to x

- **Reformulate singular value constraints**

Constraints : $\max(\sigma(A(x)))-\epsilon \le 0$ ➡ Each $A_{i,j}$ is the linearly related to x

⇕

Constraints: $\begin{bmatrix} \epsilon I & A(x) \\ A(x)^H & \epsilon I \end{bmatrix} \succcurlyeq 0$ ➡ Positive semidefinite cone
Each $A_{i,j}$ is the linearly related to x

Reformulation

Convex Problem

Cost function:

$$w^{\mathrm{T}}\left(\sum_{k=k_1}^{k_2} A_J(f_k)\right)w + 2\mathrm{Re}\left(\sum_{k=k_1}^{k_2} b_J^{\mathrm{T}}(f_k)\right)w + \sum_{k=k_1}^{k_2} c_J(f_k)$$

Constraints:

$$w^{\mathrm{T}} A_J(f_k)\, w + 2\mathrm{Re}\left(b_J^{\mathrm{T}}(f_k)\right)w + c_J(f_k) - \frac{A_e}{D_e(f_k)} \leq 0$$

$$\max\left(\lambda\left(\frac{-W_x(f_k)\hat{G}_{s0}(f_k) + \left(-W_x(f_k)\hat{G}_{s0}(f_k)\right)^{\mathrm{H}}}{2}\right)\right) - (1-\epsilon_s) \leq 0$$

$$\max\left(\sigma\left(W_x(f_k)\hat{G}_{s0}(f_k)\right)\right)B(f_k) - 1 \leq 0$$

$$\|F_z(f_k)\, w_{F_{i,j}}\|_2 - C(f_k) \leq 0$$

Cone Programming

Cost function: $t_0 + 2\mathrm{Re}\left(\sum_{k=k_1}^{k_2} b_J^{\mathrm{T}}(f_k)\right)w$

Constraints: $\|M_0\, w\|_2 \leq \sqrt{t_0\, \tilde{t}_0}$, $\tilde{t}_0 = 1$

$$t_{1,k} + 2\mathrm{Re}\left(b_J^{\mathrm{T}}(f_k)\right)w + c_J(f_k) - \frac{A_e}{D_e(k)} = 0$$

$$\|M_{1,k}\, w\|_2 \leq \sqrt{t_{1,k}\, \tilde{t}_{1,k}} \ , \qquad \tilde{t}_{1,k} = 1$$

$$W_x(f_k)\hat{G}_{s0}(f_k) + \left(W_x(f_k)\hat{G}_{s0}(f_k)\right)^{\mathrm{H}} + 2(1-\epsilon_s) \succcurlyeq 0$$

$$\begin{bmatrix} \frac{1}{B(k)} I_{N_s} & W_x(k)\hat{G}_{s0}(k) \\ \left(W_x(k)\hat{G}_{s0}(k)\right)^H & \frac{1}{B(k)} I_{N_s} \end{bmatrix} \succcurlyeq 0$$

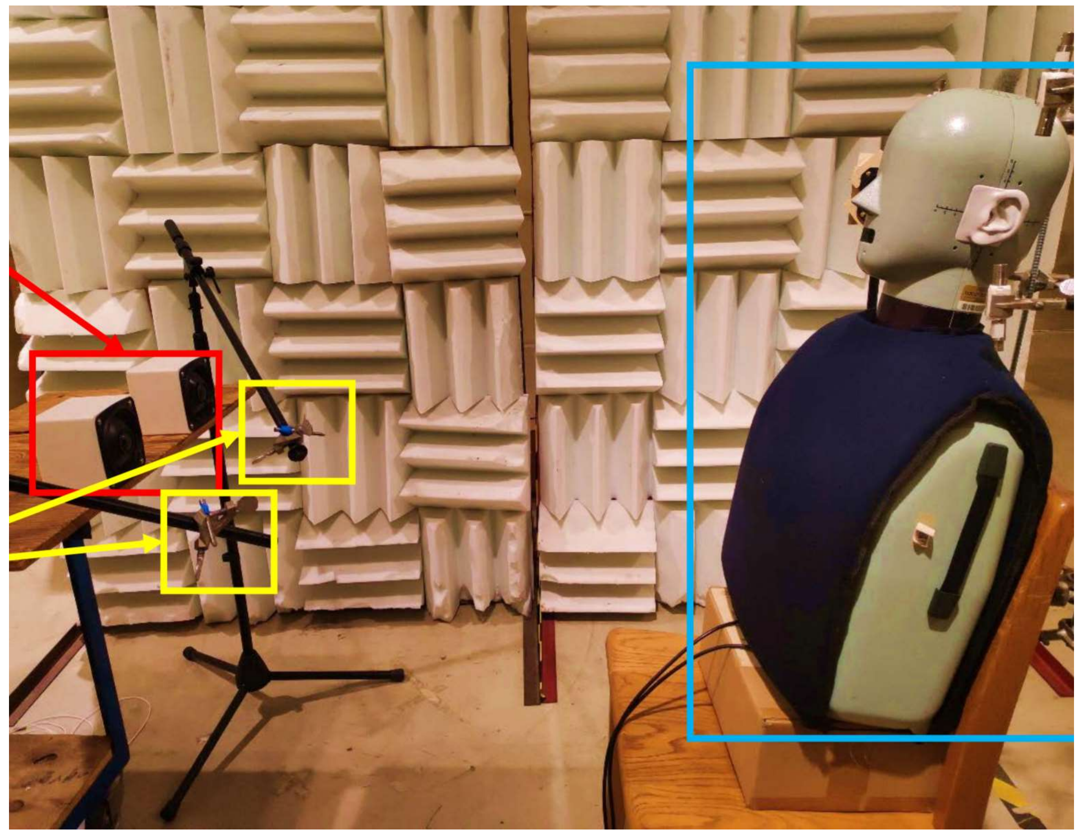
$$\|F_z(f_k)\, w_{F_{i,j}}\|_2 \leq t_{3,k} \ , \qquad t_{3,k} = C(f_k)$$

Results

Off-line Simulation based on experimental data

Experiment description:

- 2 reference microphones
- 2 control loudspeakers
- 2 error microphones
- sampling frequency is 8000 Hz



Red: Noise source

Yellow: Reference Microphones

Blue: Dummy, place for error microphone

Results

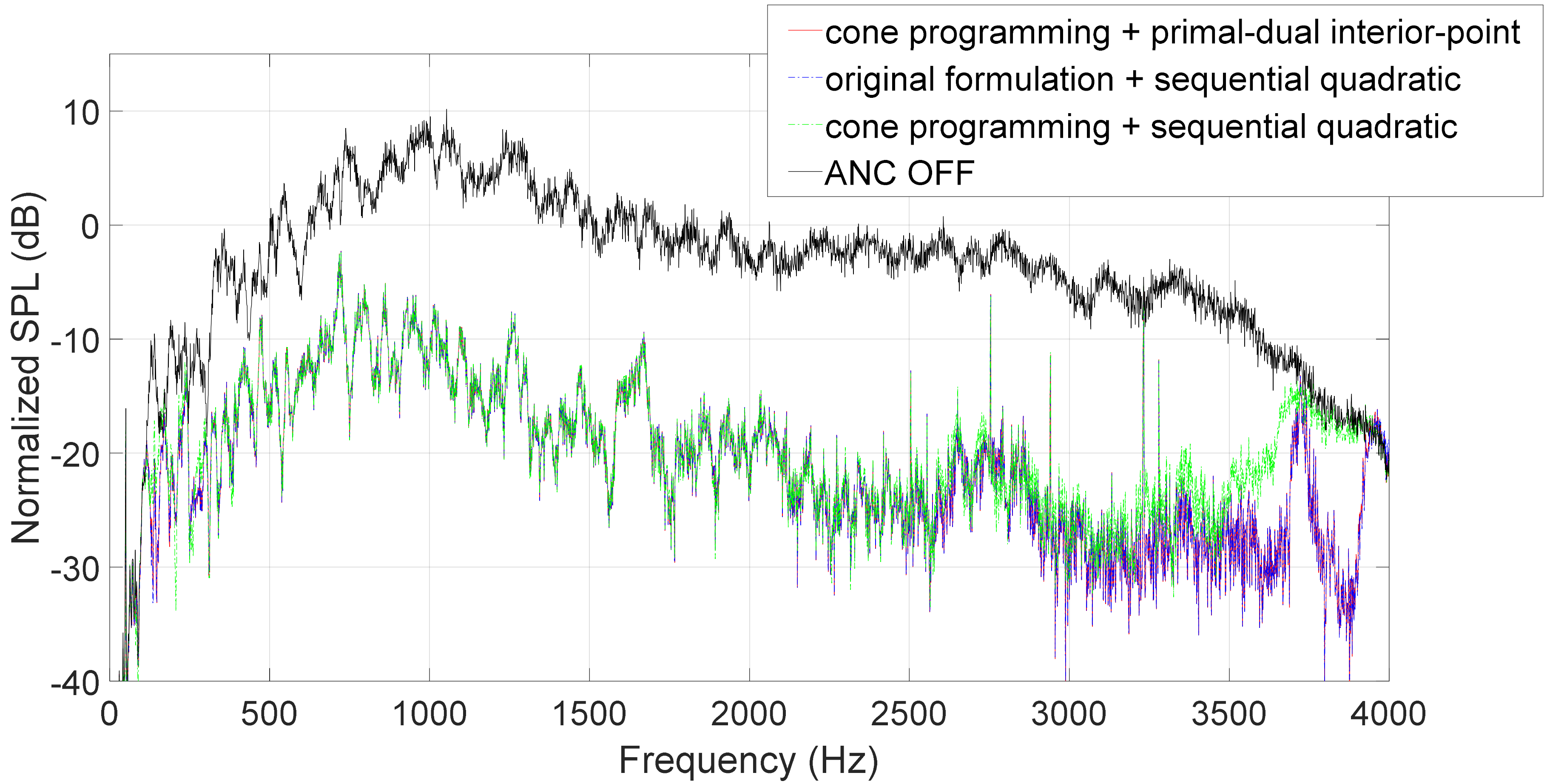
Table: Computation time for two problem sizes using different formulation-algorithm combinations

FIR length	cone programming + primal-dual interior-point	original formulation + sequential quadratic	cone programming + sequential quadratic
64	8.0 s	1790.4 s	1943.2 s
128	28.4 s	7504.9 s	5980.7 s

It is more **efficient**, if the filter design problem is reformulated to **cone programming** and solved by the **primal-dual interior-point method** (although the scale of the problem is much larger).

Results

Simulation of the attenuation performance for FIR length 128 using experimental data



Conclusions

- The ANC filter design problem can be modified and reformulated to a cone programming problem.

- The calculation using the primal-dual interior-point method for cone programming can be faster, compared with that using the commonly used sequential quadratic programming method.

- In the future, if the efficiency of this method can be further improved, it is possible to consider making this filter design problem adaptive.

Thank you !