Development and application of dual form conic formulation of multichannel active noise control filter design problem in frequency domain

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ABSTRACT

Active noise control filter design methods can be categorized as time-domain or frequency-domain methods. When multiple frequency-dependent constraints need to be specified, such as the enhancement constraint, stability constraint and robustness constraint, the optimal filter coefficients can be obtained more conveniently by solving a constrained optimization problem formulated using frequency-domain methods. However, the computational load for searching the global optimal solution is significantly high, if the number of channels, filter coefficients, or constraints is large. To improve computational efficiency, some previous work relaxed the traditional formulation to a convex problem, and then reformulated it to a cone programming problem. After this reformulation, efficient algorithms for cone programming problem, e.g., the primal-dual interior-point methods, can be applied to solve the filter design problem. However, some numerical issues may occur when solving the reformulated standard conic form directly. In this paper, the numerical instability issue for active noise control filter design problem is investigated. The original conic form was rearranged via dual formulation and different treatments of free variables are discussed. It is demonstrated that the proposed approach result is equivalent to the standard conic form but numerically more stable.

1. INTRODUCTION

Active noise control (ANC) technology was firstly proposed in Paul's patent in 1936 [1] and later in Olson and May's paper in 1953 [2]. In an ANC system, noise signal information is obtained by reference or error sensors, such as microphones and accelerometers. Based on the noise signal information, secondary sources, such as loudspeakers, are used to generate a secondary sound field (also referred to as anti-sound field) that attenuates the primary sound field at desired locations. In the past several decades, ANC has been implemented in a wide range of applications, such as headrest

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[3–6], automobiles [7,8], aircraft [9], ventilation windows [10], transformer [10], motorcycle helmet [11], MRI equipment [11], and infant incubator [11], etc.

The methods of designing ANC filters can be categorized as time-domain and frequency-domain methods depending on in which domain the ANC filter design problem is formulated [12]. Compared with the time-domain methods, frequency domain methods can usually achieve better noise control performance [13] by solving a constrained optimization problem, especially when multiple frequency-dependent constraints are required to be specified, such as disturbance enhancement constraint, stability constraint, robustness constraint, and filter response magnitude limit constraint [3,8,12,14]. However, the required computing power for searching the global optimal solution of formulated ANC filter design problem is significantly high, if a large-size quiet zone is needed, and the number of channels, filter coefficients, or constraints is large. Thus, the development of more efficient approaches to design control filters in frequency domain for multichannel ANC systems is of significant research and practical values.

One of the commonly used frequency-domain ANC filter design frameworks was developed by Rafaely and Elliott [3]. In their work, an H_2 performance criterion were used with H_2 and H_∞ constraints. A similar but less conservative formulation was then applied to the attenuation of interior road noise in vehicles by Cheer and Elliott [8]. Sequential quadratic programming (SQP) was used to solve the constrained optimization problems in both studies. However, the direct use of SQP is not computationally efficient, because the inherit mathematical structure of the ANC filter design problem is not exploited [15]. Recently, a cone programming formulation for frequency-domain multichannel ANC filter design was proposed by Zhuang and Liu [14]. In their work, the H_2/H_∞ formulation was relaxed to a convex programming and then reformulated into a cone programming problem, and a primal-dual interior-point algorithm designed for cone programming was applied to solve the optimization problem. It was demonstrated that, compared with the commonly used SQP, the computational time can be reduced significantly from the order of hours to seconds [14]. However, applying the primal-dual interior-point algorithms that designed specifically for cone programming requires some rearrangement to standard forms [16–18]. In some practical situations, numerical instability issue may occur if the direct rearranged form is used [14]. It was also mentioned in Zhuang and Liu's work [14] that using a different rearranging approach in CVX toolbox results in a standard form with better numerical stability. Thus, the effect on the numerical stability of different rearranging approaches should be further investigated for multichannel frequency-domain ANC filter design problems.

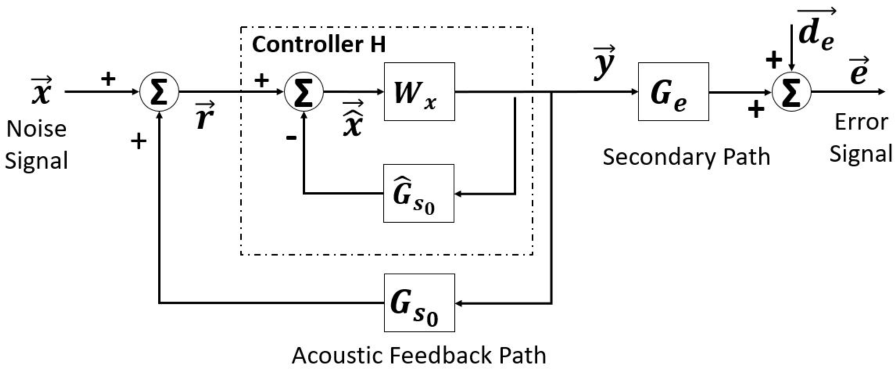
In this paper, the conic formulation of multichannel ANC filter design problem in the frequency domain proposed by Zhuang and Liu [14] was adopted. The formulation was first rearranged directly to standard semidefinite-quadratic-linear programming [19] by introducing a set of additional variables. The dual form of that conic formulation is formulated and can be solved without extra effort because the primal-dual interior-point algorithms can always obtain the optimal dual solution along with the optimal primal solution [16,17,20,21]. Two approaches to process the free variables were carried out for both the primal and the dual formulation, i.e., formulate the free variables into a second order cone, or split the free variables into two sets of non-negative variables. Thus, four different formulations were used in the current work. The simulation results show that, the numerical instability issue usually occurs when positive semidefinite cones exist, i.e., when stability or robustness constraints are applied. Compared with the other three reformulated forms, the standard form obtained from dual formulation and splitting free variables turned out to be numerically more stable.

The rest of this paper is organized as follows. In section 2, the framework of formulating the constrained optimization problem for ANC filter design in frequency domain is described, then different rearranging approaches are presented. In section 3, simulation results based on experimental data are used to investigate the numerical stability behavior of different rearranging approaches. The conclusions are drawn in section 4.

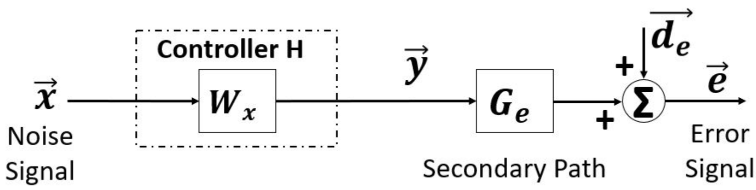
2. THEORY

In this section, formulations on ANC filter design are presented in the context of feedforward control. But it is noted that the same approach can be applied to design feedback controller as well when an internal model structure is used [8].

2.1. Review on the cone programming formulation for ANC filter design reformulated from H_2/H_∞ framework



(a)



(b)

Figure 1: Block diagram of the MIMO feedforward controllers (a) with the acoustic feedback path and internal model structure, (b) in a standard feedforward form when assuming internal model structure perfectly cancel the acoustic feedback path effect.

Previous works on conic formulation of frequency-domain multichannel ANC filter design by Zhuang and Liu [14] is reviewed first, which serves as the background for the investigation of different further reformulations and related numerical properties presented in the following subsections. The active noise control system block diagram is shown in Figure 1a. N_r, N_s, and N_e are used to denote the number of reference sensors, secondary sources, and error sensors respectively. In the control diagram, $\vec{\mathbf{x}}$ denotes the primary noise signals at reference sensor locations. $\vec{\mathbf{y}}$ denotes the output signals of the controller $\mathbf{H}$. $\mathbf{G}_{s_0}$ denotes the acoustic feedback path matrix. $\vec{\mathbf{r}}$ denotes the reference signals measured by the reference sensors. $\mathbf{G}_e$ denotes the acoustical responses matrix of the secondary sources at the error sensor locations. $\vec{\mathbf{d}}_e$ denotes the disturbance signals, i.e., the primary noise at error sensor locations. $\vec{\mathbf{e}}$ denotes the error signals which is the sum of primary and secondary sound fields at error sensor locations. The internal model control (IMC) structure is used to cancel the acoustic feedback path effect [3,8,22]. In the current work, $\hat{\mathbf{G}}_{s_0}$ is assume to be a perfect model (i.e., $\hat{\mathbf{G}}_{s_0} = \mathbf{G}_{s_0}$) in current work. With this model structure, the system can be described in a standard feedforward configuration (shown in Figure 1b), even if the original system involves feedback control. $\mathbf{W}_x$ denotes the frequency response matrix of the multichannel ANC FIR filters. The filter coefficients, $\mathbf{w}_F$, are to be designed. In the current work, FIR filters are used and each element of matrix $\mathbf{W}_x(f_k)$ is a linear function of the coefficients $\mathbf{w}_F$ [14]:

$$\begin{aligned} \mathbf{W}_{x_{i,j}}(f_k) &= \mathbf{F}_z(f_k)\vec{\mathbf{w}}_{F_{i,j}}, \\ \mathbf{F}_z(f_k) &= \begin{bmatrix} 1 & e^{-j2\pi f_k \frac{1}{f_s}} & e^{-j2\pi f_k \frac{2}{f_s}} & \dots & e^{-j2\pi f_k \frac{N_t-1}{f_s}} \end{bmatrix}, \end{aligned} \tag{1}$$

where f_k denotes the k-th frequency component, f_s denotes the sampling frequency, N_t denotes the length of FIR filters. i and j denote the i-th output and j-th input channel. The elements in each column vector $\vec{\mathbf{w}}_{F_{i,j}}$ are the coefficients of ANC filter associated with the channel specified by i and j. $\mathbf{W}_{x_{i,j}}(f_k)$ denotes the i-th row and j-th column of the frequency response matrix $\mathbf{W}_x(f_k)$.

By using the H_2/H_∞ framework, multichannel ANC filter design problem can be formulated as a constrained optimization problem as [14]:

$$\min_{\mathbf{w}_F}. \quad \sum_{k=1}^{N_f} J(f_k), \tag{2a}$$

$$\text{s.t.} \quad J(f_k) \leq A_e(f_k)\, \mathrm{tr}\left(\mathbf{S}_{d_e d_e}(f_k)\right), \tag{2b}$$

$$\min\left(\Re\left(\lambda\left(\mathbf{W}_x(f_k)\hat{\mathbf{G}}_{s_0}(f_k)\right)\right)\right) > -1, \tag{2c}$$

$$\max\left(\sigma\left(\mathbf{W}_x(f_k)\hat{\mathbf{G}}_{s_0}(f_k)\right)\right) B(f_k) \leq 1, \tag{2d}$$

$$|\mathbf{W}_{x_{i,j}}(f_k)| \leq C_{i,j}(f_k), \tag{2e}$$

for all f_k, i, and j. Eq. (2a) is the total power of error signals across all desired frequencies that is to be minimized. $J(f_k)$ is the power of the error signals at k-th frequency [8], which is expressed as:

$$J(f_k) = \mathrm{tr}[\mathbf{G}_e(f_k)\mathbf{W}_x(f_k)\mathbf{S}_{xx}(f_k)\mathbf{W}_x^{\mathrm{H}}(f_k)\mathbf{G}_e^{\mathrm{H}}(f_k) + \mathbf{G}_e(f_k)\mathbf{W}_x(f_k)\mathbf{S}_{xd_e}(f_k) + \mathbf{S}_{xd_e}^{\mathrm{H}}(f_k)\mathbf{W}_x^{\mathrm{H}}(f_k)\mathbf{G}_e^{\mathrm{H}}(f_k) + \mathbf{S}_{d_e d_e}(f_k)], \tag{3}$$

where N_f denotes the total number of frequency points in the desired noise attenuation band, H denotes the conjugate transpose of a matrix, $\mathbf{S}_{xx}(f_k)$ and $\mathbf{S}_{d_e d_e}(f_k)$ are the cross spectral density matrices of $\vec{\mathbf{x}}$ and $\vec{\mathbf{d}}_e$ respectively at frequency f_k, $\mathbf{S}_{xd_e}(f_k)$ is the cross spectral density matrix between $\vec{\mathbf{x}}$ and $\vec{\mathbf{d}}_e$ at frequency f_k. Eq. (2b) is enhancement constraint that prevents high enhancement in some frequency bands, and $A_e(f_k)$ is a frequency-dependent value that is specified at frequency f_k to limit the enhancement. Due to the use of internal model control structure, Eq. (2c) and Eq. (2d) are required to ensure that the controller $\mathbf{H}$ is stable and robust [8], where $\lambda()$ and $\sigma()$ denote the eigenvalues and singular values of a matrix, $\Re()$ denotes the real part of a complex number, min() and max() denote the minimum and maximum value of a set of real numbers, and $B(f_k)$ is the upper bound on the output multiplicative plant uncertainty at frequency f_k. Eq. (2e) is frequency response amplitude constraint to ensure that the loudspeakers operate in its linear response range and satisfy power consumption requirement, where $C_{i,j}(f_k)$ is the required upper bound on the amplitude of filter $\mathbf{w}_{F_{i,j}}$ at frequency f_k.

To solve the frequency-domain ANC filter design problem more efficiently, Zhuang and Liu [14] reformulated Eq. (2) to be an equivalent cone programming problem, which is expressed as:

$$\min. \quad t_0 + \sum_{k=1}^{N_f} \vec{\mathbf{b}}_J^{\mathrm{T}}(f_k)\,\vec{\mathbf{w}}, \tag{4a}$$

$$\text{s.t.} \quad \|\mathbf{M}_0\vec{\mathbf{w}}\|_2 \leq \sqrt{t_0\tilde{t}_0}, \tag{4b}$$

$$\tilde{t}_0 = 1, \tag{4c}$$

$$t_{1,k} + \vec{\mathbf{b}}_J^{\mathrm{T}}(f_k)\,\vec{\mathbf{w}} + \mathrm{tr}\left(\mathbf{S}_{d_e d_e}(f_k)\right)(1 - A_e(f_k)) = 0, \tag{4d}$$

$$\|\mathbf{M}_{1,k}\,\vec{\mathbf{w}}\|_2 \leq \sqrt{t_{1,k}\,\tilde{t}_{1,k}}, \tag{4e}$$

$$\tilde{t}_{1,k} = 1, \tag{4f}$$

$$\|\mathbf{F}_z(f_k)\vec{\mathbf{w}}_{F_{i,j}}\|_2 \leq t_{2,i,j,k}, \tag{4g}$$

$$t_{2,i,j,k} = C_{i,j}(f_k), \tag{4h}$$

$$\mathbf{W}_x(f_k)\hat{\mathbf{G}}_{s_0}(f_k) + \left(\mathbf{W}_x(f_k)\hat{\mathbf{G}}_{s_0}(f_k)\right)^{\mathrm{H}} + 2(1-\epsilon_s)\mathbf{I}_{N_s} \geq 0, \tag{4i}$$

$$\begin{bmatrix} \frac{1}{B(f_k)}\mathbf{I}_{N_s} & \mathbf{W}_x(f_k)\hat{\mathbf{G}}_{s_0}(f_k) \\ \left(\mathbf{W}_x(f_k)\hat{\mathbf{G}}_{s_0}(f_k)\right)^{\mathrm{H}} & \frac{1}{B(f_k)}\mathbf{I}_{N_s} \end{bmatrix} \geq 0. \tag{4j}$$

The problem is minimized with respect to $\vec{\mathbf{w}}$, t_0, $\tilde{t}_0$, $t_{1,k}$, $\tilde{t}_{1,k}$, and $t_{2,i,j,k}$ for all i, j, and k. In Eqs. (4a), (4b), (4c), the matrix $\mathbf{M}_0$ is a matrix such that $\mathbf{M}_0^{\mathrm{T}}\mathbf{M}_0 = \Re\left(\sum_{k=1}^{N_f} \mathbf{A}_J(f_k)\right)$ [14] and

$$\begin{aligned}
\mathbf{A}_J(f_k) &= \left(\mathbf{G}_e^{\mathrm{H}}(f_k)\mathbf{G}_e(f_k)\right) \otimes \mathbf{S}_{xx}^{\mathrm{T}}(f_k) \otimes \left(\mathbf{F}_z^{\mathrm{H}}(f_k)\mathbf{F}_z(f_k)\right),\\
\vec{\mathbf{b}}_J(f_k) &= 2\Re\left(\mathrm{vec}\left((\mathbf{S}_{xd_e}(f_k)\mathbf{G}_e(f_k)) \otimes \mathbf{F}_z^{\mathrm{T}}(f_k)\right)\right),\\
\vec{\mathbf{w}} &= \begin{bmatrix}\vec{\mathbf{w}}_{F_{1,1}}^{\mathrm{T}} & \cdots & \vec{\mathbf{w}}_{F_{1,N_r}}^{\mathrm{T}} & \vec{\mathbf{w}}_{F_{2,1}}^{\mathrm{T}} & \cdots & \vec{\mathbf{w}}_{F_{N_s,N_r}}^{\mathrm{T}}\end{bmatrix}^{\mathrm{T}},
\end{aligned} \tag{5}$$

where T denotes the transpose of a matrix, $\otimes$ denotes Kronecker product, and vec() denotes the vectorization operator that converts a matrix to a column vector by stacking the columns vertically [23]. In Eqs. (4d), (4e), (4f), the matrix $\mathbf{M}_{1,k}$ is a matrix such that $\mathbf{M}_{1,k}^{\mathrm{H}}\mathbf{M}_{1,k} = \mathbf{A}_J(f_k)$ [14]. In Eqs. (4i), (4j), $\mathbf{I}_{N_s}$ is an identity matrix with dimension N_s by N_s, ϵ_s is a small positive constant to ensure strict stability, and $\succeq 0$ means the matrix is positive semidefinite.

2.2. The direct reformulation of ANC filter design into standard cone programming form

Eq. (4) in a cone programming with linear objective function and constraints expressed as affine equalities, (rotated) Lorentz cones, and positive semidefinite cones. To solve this optimization problem using primal-dual interior-point algorithms that designed specifically for cone programming, e.g., using solvers such as SeDuMi [16, 20], SDPT3 [17, 19], or MOSEK [18], it is necessary to rearrange it to a standard semidefinite-quadratic-linear programming (SQLP) [19] (sometimes also referred to as mixed semidefinite and second order cone optimization problem [20]). In general, the standard SQLP problem can be expressed as:

$$\begin{aligned}
\min.\quad & (\vec{\mathbf{c}}^l)^{\mathrm{T}}\vec{\mathbf{x}}^l + (\vec{\mathbf{c}}^q)^{\mathrm{T}}\vec{\mathbf{x}}^q + (\vec{\mathbf{c}}^s)^{\mathrm{T}}\vec{\mathbf{x}}^s,\\
\text{s.t.}\quad & \mathbf{A}^l\vec{\mathbf{x}}^l + \mathbf{A}^q\vec{\mathbf{x}}^q + \mathbf{A}^s\vec{\mathbf{x}}^s = \vec{\mathbf{b}},\\
& \vec{\mathbf{x}}^l \in \Re_+^{k_l}, \vec{\mathbf{x}}^q \in K^q, \vec{\mathbf{x}}^s \in K^s
\end{aligned} \tag{6}$$

where, $\Re_+^{k_l}$ is the nonnegative orthant with dimension k_l, $K^q = K_1^q \times ... \times K_{k_q}^q$ is a Cartesian product of k_q second order cones (sometimes also referred to as Lorentz cones):

$$K_i^q = \left\{(y, \vec{\mathbf{x}}) \in \Re \times \Re^{n_i - 1} : y \geq \|\vec{\mathbf{x}}\|_2\right\}, \tag{7}$$

and $K^s = K_1^s \times ... \times K_{k_s}^s$ is a Cartesian product of k_s vectorizations of positive semidefinite cones:

$$K_i^s = \left\{\mathrm{vec}(X) \in \Re^{n_i^2} : X \in \Re^{n_i \times n_i} \text{ is positive semidefinite}\right\}. \tag{8}$$

By comparing the two formulations in Eq. (4) and Eq. (6), a direct and simple approach to reformulate Eq. (4) into the form of Eq. (6) is to introduce a set of new variables $\vec{\mathbf{x}} = [\vec{\mathbf{x}}_1\ \vec{\mathbf{x}}_2\ ...\ \vec{\mathbf{x}}_M]^{\mathrm{T}}$ where each $\vec{\mathbf{x}}_i$ represents one of the constraints in Eq. (4), i.e., $\vec{\mathbf{x}}_i \in K_i$ where K_i is a second order cone or a vectorized positive semidefinite cone. The relationship between $\vec{\mathbf{w}}$ and $\vec{\mathbf{x}}_i$ is applied by adding equality constraint $\mathbf{A}_i\vec{\mathbf{w}} + \mathbf{B}_i\vec{\mathbf{x}} = \vec{\mathbf{b}}_i$ with appropriate constants $\mathbf{A}_i$, $\mathbf{B}_i$, and $\vec{\mathbf{b}}_i$ [24]. Then, Eq. (4) can be equivalently rearranged as:

$$\begin{aligned}
\min.\quad & \begin{bmatrix}(\vec{\mathbf{c}}_w)^{\mathrm{T}} & (\vec{\mathbf{c}}_x)^{\mathrm{T}}\end{bmatrix}\begin{bmatrix}\vec{\mathbf{w}}\\ \vec{\mathbf{x}}\end{bmatrix},\\
\text{s.t.}\quad & \mathbf{A}\vec{\mathbf{w}} + \mathbf{B}\vec{\mathbf{x}} = \vec{\mathbf{b}},\\
& \vec{\mathbf{w}} \in \Re^{N_r N_s N_t},\\
& \vec{\mathbf{x}} \in K,
\end{aligned} \tag{9}$$

where $\mathbf{A}$ is obtained by vertically concatenating all $\mathbf{A}_i$, $\mathbf{B}$ is obtained by diagonally concatenating all $\mathbf{B}_i$, $\vec{\mathbf{b}}$ is obtained by vertically concatenating all $\vec{\mathbf{b}}_i$, and $K = K_1 \times K_2 \times ... \times K_M$. It is noted that elements in $\vec{\mathbf{w}}$ are free variables, i.e., there are no cone constraints associated with $\vec{\mathbf{w}}$. There are two standard approaches to formulate free variables $\vec{\mathbf{w}}$ into a cone [25]. First approach is to formulate $\vec{\mathbf{w}}$ into a second order cone with the help of an additional variable w_0:

$$
\begin{aligned}
\min. \quad & \begin{bmatrix} 0 & (\vec{\mathbf{c}}_w)^{\mathrm{T}} & (\vec{\mathbf{c}}_x)^{\mathrm{T}} \end{bmatrix} \begin{bmatrix} w_0 \\ \vec{\mathbf{w}} \\ \vec{\mathbf{x}} \end{bmatrix}, \\
\text{s.t.} \quad & \mathbf{A}\vec{\mathbf{w}} + \mathbf{B}\vec{\mathbf{x}} = \vec{\mathbf{b}}, \\
& \begin{bmatrix} w_0 \\ \vec{\mathbf{w}} \end{bmatrix} \in K_0^q, \\
& \vec{\mathbf{x}} \in K,
\end{aligned}
\tag{10}
$$

where K_0^q denotes a second order cone, i.e., $w_0 \geq \|\vec{\mathbf{w}}\|_2$. Another approach is to split $\vec{\mathbf{w}}$ as the difference of two sets of nonnegative variables, i.e., $\vec{\mathbf{w}} = \vec{\mathbf{w}}_1 - \vec{\mathbf{w}}_2$, which gives:

$$
\begin{aligned}
\min. \quad & \begin{bmatrix} (\vec{\mathbf{c}}_w)^{\mathrm{T}} & -(\vec{\mathbf{c}}_w)^{\mathrm{T}} & (\vec{\mathbf{c}}_x)^{\mathrm{T}} \end{bmatrix} \begin{bmatrix} \vec{\mathbf{w}}_1 \\ \vec{\mathbf{w}}_2 \\ \vec{\mathbf{x}} \end{bmatrix}, \\
\text{s.t.} \quad & \mathbf{A}\vec{\mathbf{w}}_1 - \mathbf{A}\vec{\mathbf{w}}_2 + \mathbf{B}\vec{\mathbf{x}} = \vec{\mathbf{b}}, \\
& \vec{\mathbf{w}}_1 \in \Re_+^{N_r N_s N_t}, \\
& \vec{\mathbf{w}}_2 \in \Re_+^{N_r N_s N_t}, \\
& \vec{\mathbf{x}} \in K.
\end{aligned}
\tag{11}
$$

It is known that the existence of free variables may compromise the numerical behavior [16,17,20,21], thus, Eq. (10) and Eq. (11) are considered as two different forms after SQLP reformulation in the current work. The numerical stability of using these two different forms is compared in the result section.

2.3. Reformulation of ANC filter design into standard cone programming using dual formulation

When the primal-dual interior-point algorithms are used to solve the cone programming problem, the optimal dual solution will be obtained along with the optimal primal solution. Thus, instead of solving the direct SQLP formulations (Eq. (10) or Eq. (11)), it is possible to solve the dual problem of Eq. (10) or Eq. (11), i.e., treat the dual form of the direct standard SQLP formulation as a primal problem that is solved by standard solvers. The optimal filter coefficients can then be obtained from the optimal dual solution given by the primal-dual interior-point algorithm. These types of reformulation may result in different numerical stability characteristics when standard algorithms are used.

For the direct formulation expressed in Eq. (10), since all the cones are self-dual, its dual form can then be expressed as:

$$
\begin{aligned}
\min. \quad & -\vec{\mathbf{b}}^{\mathrm{T}}\vec{\mathbf{y}}, \\
\text{s.t.} \quad & s_0 = 0, \\
& \mathbf{A}^{\mathrm{T}}\vec{\mathbf{y}} + \vec{\mathbf{s}}_w = \vec{\mathbf{c}}_w, \\
& \mathbf{B}^{\mathrm{T}}\vec{\mathbf{y}} + \vec{\mathbf{s}}_x = \vec{\mathbf{c}}_x, \\
& \vec{\mathbf{y}} \in \Re^{N_b}, \\
& \begin{bmatrix} s_0 \\ \vec{\mathbf{s}}_w \end{bmatrix} \in K_0^q, \\
& \vec{\mathbf{s}}_x \in K,
\end{aligned}
\tag{12}
$$

where, $\vec{\mathbf{y}}$, s_0, $\vec{\mathbf{s}}_w$, and $\vec{\mathbf{s}}_x$ are dual variables and N_b is the dimension of $\vec{\mathbf{b}}$. It is noted that there are no cone constraints related to $\vec{\mathbf{y}}$, i.e., elements in $\vec{\mathbf{y}}$ are free variables. Since $s_0 = 0$ and $\begin{bmatrix} s_0 & \vec{\mathbf{s}}_w \end{bmatrix}^{\mathrm{T}} \in K_0^q$, it is obvious that $\vec{\mathbf{s}}_w = \vec{\mathbf{0}}$. Thus, Eq. (12) can be further simplified as:

$$
\begin{aligned}
\min. \quad & -\vec{\mathbf{b}}^{\mathrm{T}}\vec{\mathbf{y}}, \\
\text{s.t.} \quad & \mathbf{A}^{\mathrm{T}}\vec{\mathbf{y}} = \vec{\mathbf{c}}_w, \\
& \mathbf{B}^{\mathrm{T}}\vec{\mathbf{y}} + \vec{\mathbf{s}}_x = \vec{\mathbf{c}}_x, \\
& \vec{\mathbf{y}} \in \Re^{N_b}, \\
& \vec{\mathbf{s}}_x \in K,
\end{aligned}
\tag{13}
$$

where $\vec{\mathbf{s}}_w$ has been removed from the problem completely, and the problem size can be greatly reduced. Similarly, for direct formulation expressed in Eq. (11), its dual form can be expressed as:

$$
\begin{aligned}
\min. \quad & -\vec{\mathbf{b}}^{\mathrm{T}}\vec{\mathbf{y}}, \\
\text{s.t.} \quad & \mathbf{A}^{\mathrm{T}}\vec{\mathbf{y}} + \vec{\mathbf{s}}_{w_1} = \vec{\mathbf{c}}_w, \\
& -\mathbf{A}^{\mathrm{T}}\vec{\mathbf{y}} + \vec{\mathbf{s}}_{w_2} = -\vec{\mathbf{c}}_w, \\
& \mathbf{B}^{\mathrm{T}}\vec{\mathbf{y}} + \vec{\mathbf{s}}_x = \vec{\mathbf{c}}_x, \\
& \vec{\mathbf{y}} \in \Re^{N_b}, \\
& \vec{\mathbf{s}}_{w_1} \in \Re_+^{N_r N_s N_t}, \\
& \vec{\mathbf{s}}_{w_2} \in \Re_+^{N_r N_s N_t}, \\
& \vec{\mathbf{s}}_x \in K,
\end{aligned}
\tag{14}
$$

where, $\vec{\mathbf{y}}$, $\vec{\mathbf{s}}_{w_1}$, $\vec{\mathbf{s}}_{w_2}$, and $\vec{\mathbf{s}}_x$ are dual variables. By comparing the first two equality constraint in Eq. (14), it is obvious that $\vec{\mathbf{s}}_{w_1} = -\vec{\mathbf{s}}_{w_2}$. Also, since both $\vec{\mathbf{s}}_{w_1}$ and $\vec{\mathbf{s}}_{w_2}$ are non-negative, $\vec{\mathbf{s}}_{w_1} = \vec{\mathbf{s}}_{w_2} = \vec{\mathbf{0}}$. Thus, Eq. (12) can be simplified to the exact same form in Eq. (13). When using primal-dual interior-point algorithms to solve Eq. (13) (treat it as a primal problem), two different approaches to process free variables $\vec{\mathbf{y}}$ are similar as mentioned above, i.e., formulate free variables in a second order cone:

$$
\begin{aligned}
\min. \quad & \begin{bmatrix} 0 & -\vec{\mathbf{b}}^{\mathrm{T}} \end{bmatrix} \begin{bmatrix} y_0 \\ \vec{\mathbf{y}} \end{bmatrix}, \\
\text{s.t.} \quad & \mathbf{A}^{\mathrm{T}}\vec{\mathbf{y}} = \vec{\mathbf{c}}_w, \\
& \mathbf{B}^{\mathrm{T}}\vec{\mathbf{y}} + \vec{\mathbf{s}}_x = \vec{\mathbf{c}}_x, \\
& \begin{bmatrix} y_0 \\ \vec{\mathbf{y}} \end{bmatrix} \in K_0^q, \\
& \vec{\mathbf{s}}_x \in K.
\end{aligned}
\tag{15}
$$

or split free variables into two sets of non-negative variables:

$$
\begin{aligned}
\min. \quad & \begin{bmatrix} -\vec{\mathbf{b}}^{\mathrm{T}} & \vec{\mathbf{b}}^{\mathrm{T}} \end{bmatrix} \begin{bmatrix} \vec{\mathbf{y}}_1 \\ \vec{\mathbf{y}}_2 \end{bmatrix}, \\
\text{s.t.} \quad & \mathbf{A}^{\mathrm{T}}\vec{\mathbf{y}}_1 - \mathbf{A}^{\mathrm{T}}\vec{\mathbf{y}}_2 = \vec{\mathbf{c}}_w, \\
& \mathbf{B}^{\mathrm{T}}\vec{\mathbf{y}}_1 - \mathbf{B}^{\mathrm{T}}\vec{\mathbf{y}}_2 + \vec{\mathbf{s}}_x = \vec{\mathbf{c}}_x, \\
& \vec{\mathbf{y}}_1 \in \Re_+^{N_b}, \\
& \vec{\mathbf{y}}_2 \in \Re_+^{N_b}, \\
& \vec{\mathbf{s}}_x \in K.
\end{aligned}
\tag{16}
$$

Similarly, Eq. (15) and Eq. (16) are also considered as two different forms after dual formulation in the current work. The numerical stability characteristics of the primal-dual interior-point algorithm when applied to these two different formulations and to the formulations presented in Section 2.2 are compared in the next section.

3. SIMULATION RESULTS

In this section, the numerical stability behaviors of solving different formulations of the ANC filter design problem are compared. All four different but equivalent formulations derived in the last section are studied, i.e., Eq. (10) (denoted as "Form 1"), Eq. (11) (denoted as "Form 2"), Eq. (15) (denoted as "Form 3"), and Eq. (16) (denoted as "Form 4"). Numerical stability issue may occur when the solution in one iteration is close to the optimal solution or the boundary of feasible set for typical primal-dual interior-point algorithms [17, 20, 26]. The duality gap is usually used as one of the practical stopping criteria of the algorithm, which is the difference between primal and dual objective function values and it converges to zero when the solution in the iteration is approaching the optimal solution [16, 21]. Thus, in the current work, the value of duality gap of the iteration before numerical issue occurs is used to represent the numerical stability characteristics of each formulation. In the present work, a smaller duality gap result is considered to be a more stable formulation, since it implies the last iteration gives a solution closer to the global optimal solution.

The ANC system used for in the current simulation work has two reference microphones ($N_r = 2$), two loudspeakers ($N_s = 2$), and two error microphones ($N_e = 2$). So there are 2×2 (a total of four) FIR filters need to be designed in the optimization problem. Sampling frequency f_s is 3000 Hz and the desired noise attenuation band is from 100 Hz to 1400 Hz. In the attenuation band, frequency interval used in expressing the total power of error signals (Eq. (2a)) is 1 Hz. Below 100 Hz and above 1400 Hz, frequency interval used in constraints limiting the filter response magnitude (Eq. (2e)) is 3 Hz. The length of FIR filter for each channel and the number of other three constraints vary in different simulation cases. SeDuMi [16, 20] was used to implement primal-dual interior-point algorithms. In this section, the processor used in the current work is Intel(R) Core(TM) i7-7700 CPU @ 3.60 GHz, and programming platform is Matlab installed on a Windows 10 64-bit operating system. The experimental data were scaled via dividing by the mean power of reference signals across all frequencies before using algorithms to solve.

The duality gaps before numerical issues occur for different simulation cases are shown in Figure 2. Overall, the duality gaps presented in Figure 2a, 2b, and 2c are usually very small for all four formulations when only disturbance enhancement constraints are involved. However, when either stability or robustness constraints are involved, some of the cases may have severe numerical issues, i.e., the final duality gap is relatively large. It is noted that stability and robustness constraints are eventually reformulated to positive semidefinite cones, and disturbance enhancement, cost function, and filter response magnitude constraints are reformulated to second order cones. This suggests that the numerical issue in ANC filter design problems occurs when positive semidefinite constraints are involved.

Comparing the duality gap of all four formulations at different cases in Figure 2, overall, Form 4 (i.e., the dual reformulation with a split treatment of free variables) has relatively small duality gaps compared with the other three formulations. Although in Figure 2d, "Form 4" also has a relatively large duality gap when FIR order is high (larger than or equal to 160 for each channel), it is still not worse than the other three forms. These demonstrate that, compared with the other three forms, "Form 4" is considered as a relatively more numerically stable formulation when solving the multichannel frequency-domain ANC filter design problems.

To confirm that a small duality gap is indeed necessary, The comparison of ANC noise control performance for different forms using 128-point FIR filters when 66 evenly spaced frequencies are used for the enhancement, stability, and robustness constraints are shown in Figure 3. The

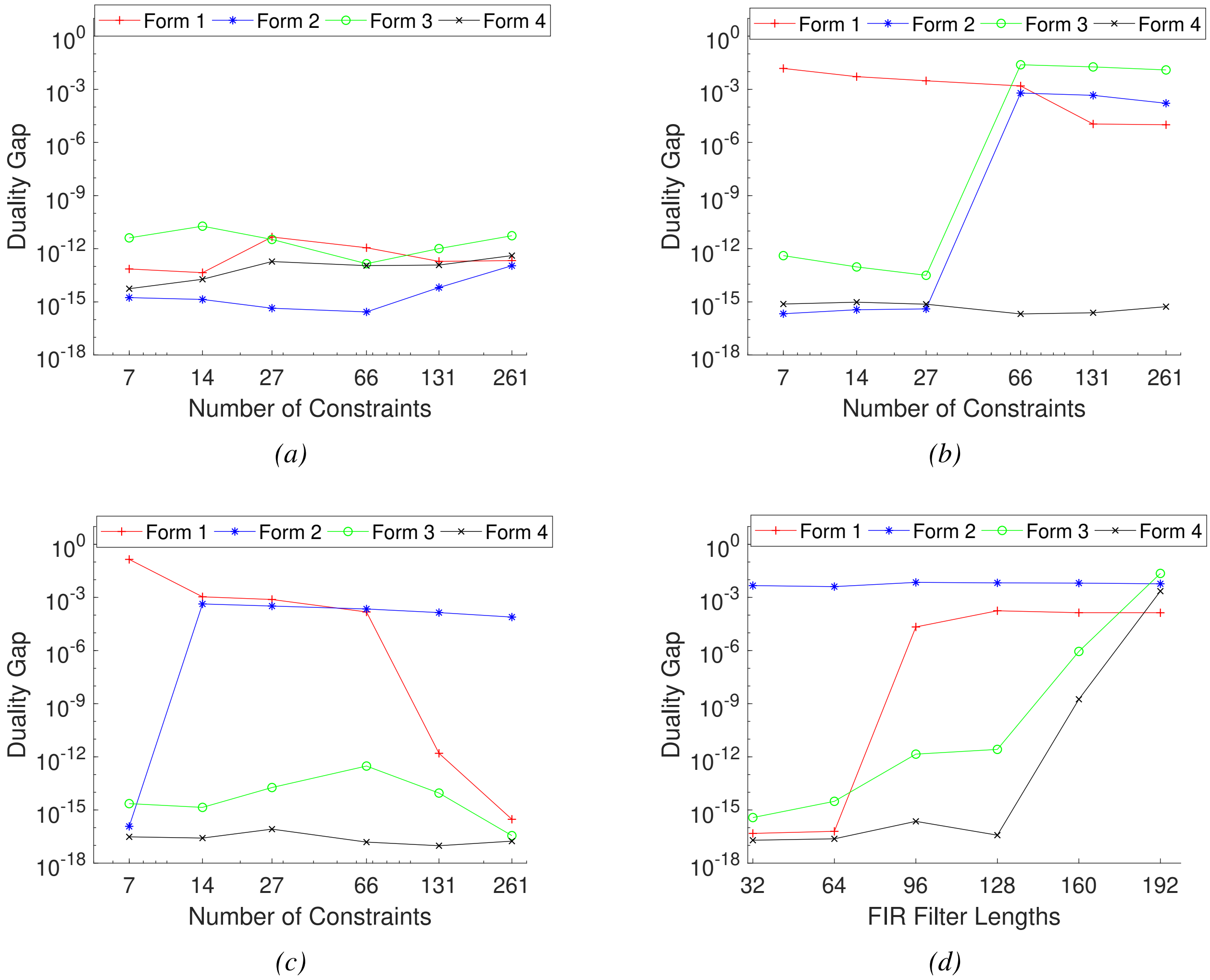


Figure 2: The duality gap before numerical issues occur for different forms when using (a) 128-point FIR filters and different number of enhancement constraints (frequency components are evenly spaced in the attenuation band), (b) 128-point FIR filters and different number of stability constraints evenly spaced in the attenuation band, (c) 128-point FIR filters and different number of robustness constraints (frequency components are evenly spaced in the attenuation band), and (d) 66 evenly spaced frequencies are used for all three constraints (frequency components are evenly spaced in the attenuation band).

performance is shown at a 1 Hz interval for the whole frequency range. "ANC OFF" denotes the sound pressure power spectral density function (PSD) of original disturbance signals averaged among all the error microphones. "Normalized SPL" is the ratio of the sensor averaged PSD when ANC system is in operation to the PSD first averaged among error sensors and then averaged over the whole frequency band when "ANC OFF". In Fig. 3, it is observed that when "Form 2" is used, which is also implied by a large duality gap (6.61×10^{-3}), there is little noise control performance. The performance when using "Form 1" (duality gap : 1.76×10^{-4}) is slightly worse than "Form 3" (duality gap : 2.63×10^{-12}) and "Form 4" (duality gap : 3.78×10^{-17}). This confirms that a small duality gap is indication of a accurate solution in ANC filter design problems.

4. CONCLUSIONS

In this article, different approaches to obtain a standard SQLP form of multichannel ANC filter design in the frequency domain is formulated. First, the previously formulated conic form of ANC

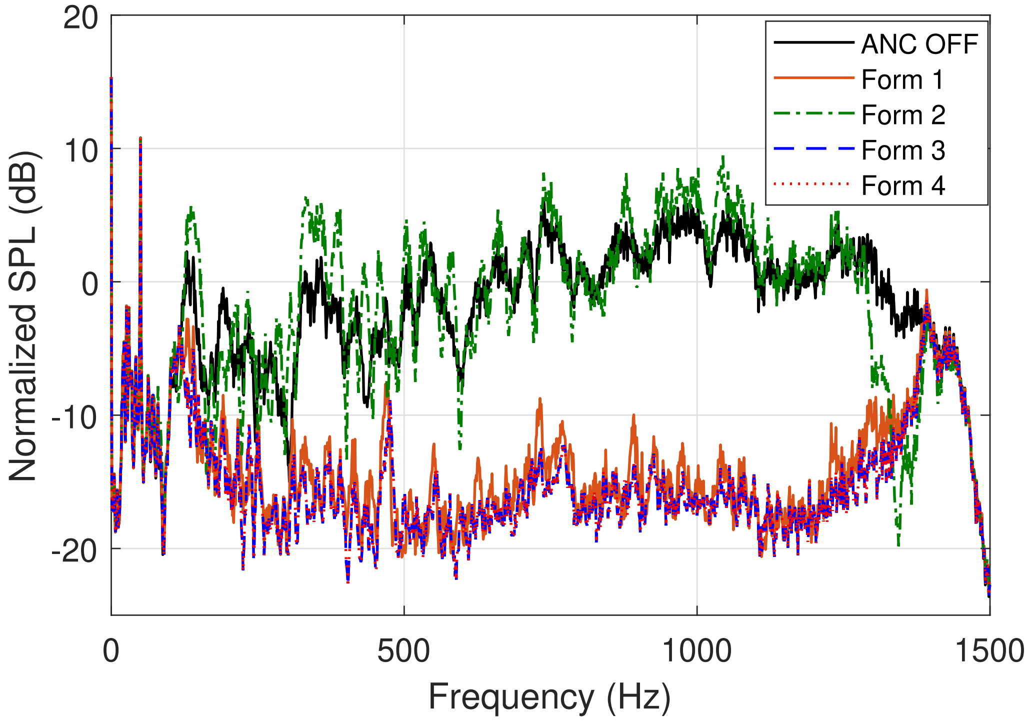


Figure 3: The comparison of ANC noise control performance for different forms using 128-point FIR filters when 66 frequencies are used for each of the enhancement, stability, and robustness constraints. The frequency interval for plotting is 1 Hz.

filter design is directly reformulated into a standard SQLP form by introducing a set of additional variables. Two different approaches to process free variables in this direct SQLP formulation, i.e., formulate free variables into a second order cone, or split free variables into two sets of non-negative variables, which results in two different SQLP formulations. Then it is shown that the dual problem of these two direct formulations will be the same after some simplification. By solving this dual problem (treating it as a primal problem), the optimal filter coefficients can be obtained from the optimal dual solution without extra computational effort because the primal-dual interior-point algorithms always solves the primal problem together with its dual problem. The above mentioned two approaches to process free variables can also be applied to this dual formulation. Thus, four equivalent but different SQLP formulations are derived in the current work.

Simulation results demonstrated that, numerical issues may occur when positive semidefinite cones are involved, i.e., when stability or robustness constraints are applied. Compared with the other three SQLP formulations, the one obtained by dual formulations and then splitting free variables into two sets of non-negative variables, has a better numerical stability behavior.

In the future, further studies can be conducted to explain the numerical characteristics of the dual formulation of ANC filter design problem. For example, the simulation result shows that when filter order is high enough, there will still be some numerical instability for all the four forms. Other reformulation approaches may be used to further improve the numerical stability by exploiting the problem structure of the ANC filter design problem. Besides the numerical stability, the computational efficiency of different forms can also be investigated.

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