Development and application of dual form conic formulation of multichannel active noise control filter design problem in frequency domain

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Study on the Cone Programming Reformulation of Active Noise Control Filter Design in the Frequency Domain

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Introduction

❑ Multichannel active noise control (ANC) systems

- Better performance when we need to create large-size quiet zone.
- Applications:

Interior of Vehicles

Range Hood

Infant Incubator

Air Conditioner

Introduction

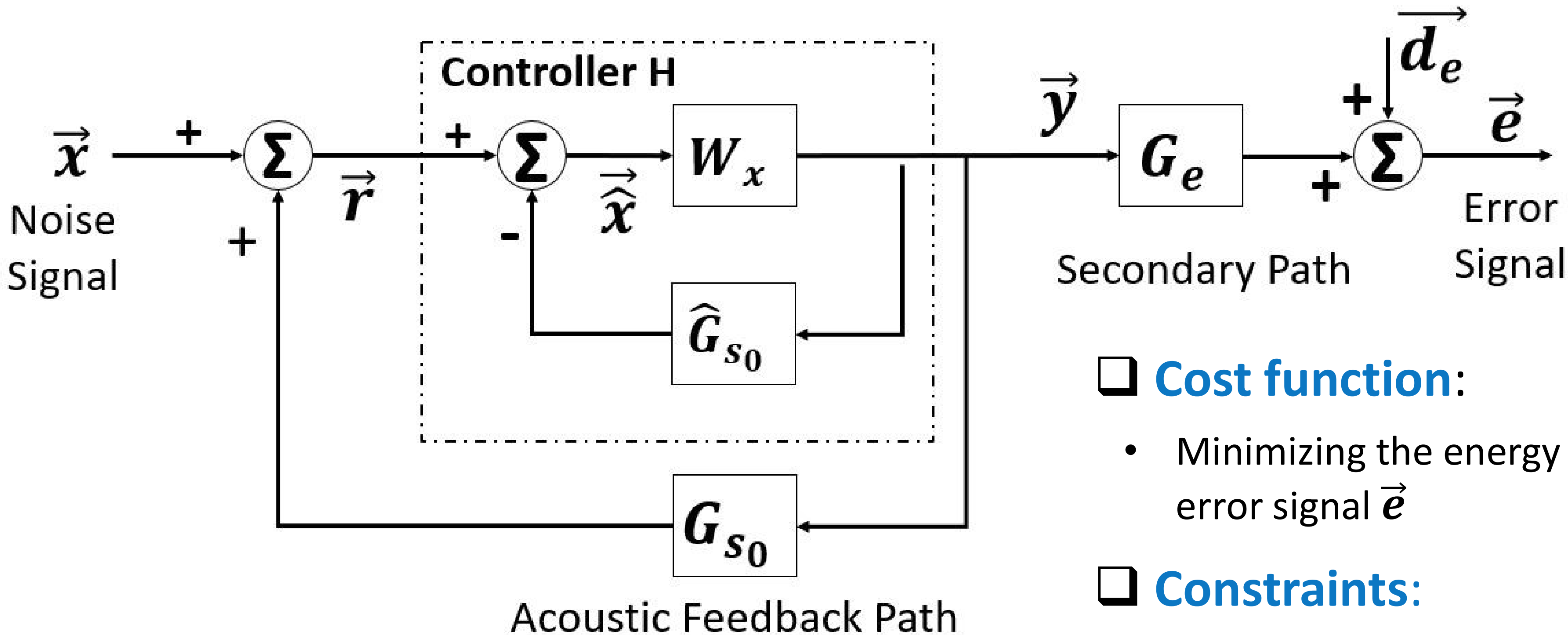
❑ Motivation of using frequency domain design

- Easier to specify frequency dependent constraints.
- Constraints in one frequency band will not affect performance of other bands.
- Usually, better ANC performance.

❑ Motivation of using improved cone programing form

- The computational complexity is usually significant for frequency-domain design method.
- It was demonstrated in previous study that by cone programming reformulation, the ANC design problem can be solved much more efficiently using the primal-dual interior-point algorithms.
- However, some numerical issues may occur when using the direct reformulated standard cone programming form. Thus, the effect on the numerical stability of different formulation approaches should be further investigated.

Active Noise Control System



- **Cost function**:
  - Minimizing the energy of error signal $\vec{e}$
- **Constraints**:
  - Disturbance enhancement
  - Stability
  - Robustness
  - Filter response

(Non-adaptive control is considered in the current work)

Review of Previous Work - Original Problem

Cost function:

$$\sum_{k=1}^{N_f} tr\left[\vec{\boldsymbol{E}}(f_k)\vec{\boldsymbol{E}}(f_k)^{\mathrm{H}}\right]$$

Total energy of e cross all frequencies

Constraints:

Enhancement: **Normalized energy of e:**

$$tr\left[\vec{E}(f_k)\vec{E}(f_k)^{\mathrm{H}}\right] \le A_e tr(\boldsymbol{S}_{d_e d_e}(f_k))$$

Stability: **Use Nyquist criterion:**

$$\min\left(\mathrm{Re}\left(\lambda\left(W_x(f_k)\hat{G}_{s0}(f_k)\right)\right)\right) > -1$$

Robustness: **$\boldsymbol{M}$- $\boldsymbol{\Delta}$ structure and small gain theory:**

$$\max\left(\sigma\left(W_x(f_k)\hat{G}_{s0}(f_k)\right)\right)B(f_k) \le 1$$

Filter response: **The magnitude of frequency response:**

$$\left|W_{x_{i,j}}(f_k)\right| \le C(f_k)$$

Review of Previous Work - Original Problem

Cost function: Total energy of e:

$$\sum_{k=1}^{N_f} tr\left[\vec{E}(f_k)\vec{E}(f_k)^{\mathrm{H}}\right],$$

Constraints:

Enhancement:

$$tr\left[\vec{E}(f_k)\vec{E}(f_k)^{\mathrm{H}}\right] \leq A_e tr(\boldsymbol{S}_{d_e d_e}(f_k))$$

Normalized energy of e at each frequency

Stability: **Use Nyquist criterion:**

$$\min\left(\mathrm{Re}\left(\lambda\left(W_x(f_k)\hat{G}_{s0}(f_k)\right)\right)\right) > -1$$

Robustness: **$\boldsymbol{M}$- $\boldsymbol{\Delta}$ structure and small gain theory:**

$$\max\left(\sigma\left(W_x(f_k)\hat{G}_{s0}(f_k)\right)\right)B(f_k) \leq 1$$

Filter response: **The magnitude of frequency response:**

$$\left|W_{x_{i,j}}(f_k)\right| \leq C(f_k)$$

Review of Previous Work - Original Problem

Cost function: **Total energy of e:**

$$\sum_{k=1}^{N_f} tr\left[\vec{E}(f_k)\vec{E}(f_k)^{\mathrm{H}}\right],$$

Constraints:

Enhancement: **Normalized energy of e:**

$$tr\left[\vec{E}(f_k)\vec{E}(f_k)^{\mathrm{H}}\right] \leq A_e tr(\boldsymbol{S}_{d_e d_e}(f_k))$$

Stability:

$$\min\left(\mathrm{Re}\left(\lambda\left(\boldsymbol{W}_x(f_k)\widehat{\boldsymbol{G}}_{s0}(f_k)\right)\right)\right) > -1$$

Nyquist criterion, on the right of -1 point

Robustness: ***M*- Δ structure and small gain theory:**

$$\max\left(\sigma\left(W_x(f_k)\hat{G}_{s0}(f_k)\right)\right)B(f_k) \leq 1$$

Filter response: **The magnitude of frequency response:**

$$\left|W_{x_{i,j}}(f_k)\right| \leq C(f_k)$$

Review of Previous Work - Original Problem

Cost function: **Total energy of e:**

$$\sum_{k=1}^{N_f} tr\left[\vec{E}(f_k)\vec{E}(f_k)^{\mathrm{H}}\right],$$

Constraints:

Enhancement: **Normalized energy of e:**

$$tr\left[\vec{E}(f_k)\vec{E}(f_k)^{\mathrm{H}}\right] \le A_e tr(\boldsymbol{S}_{d_e d_e}(f_k))$$

Stability:

$$\min\left(\mathrm{Re}\left(\lambda\left(\boldsymbol{W}_x(f_k)\widehat{\boldsymbol{G}}_{s0}(f_k)\right)\right)\right) > -1$$

It is convexified as:

$$\max\left(\lambda\left(\frac{-\boldsymbol{W}_x(f_k)\widehat{\boldsymbol{G}}_{s0}(f_k) + \left(-\boldsymbol{W}_x(f_k)\widehat{\boldsymbol{G}}_{s0}(f_k)\right)^{\mathrm{H}}}{2}\right)\right) - (1-\epsilon_s) \le 0$$

Robustness: $\boldsymbol{M}$- $\boldsymbol{\Delta}$ **structure and small gain theory:**

$$\max\left(\sigma\left(W_x(f_k)\hat{G}_{s0}(f_k)\right)\right)B(f_k) \le 1$$

Filter response: **The magnitude of frequency response:**

$$\left|W_{x_{i,j}}(f_k)\right| \le C(f_k)$$

Review of Previous Work - Original Problem

Cost function: **Total energy of e:**

$$\sum_{k=1}^{N_f} tr\left[\vec{E}(f_k)\vec{E}(f_k)^{\mathrm{H}}\right],$$

Constraints:

Enhancement: **Normalized energy of e:**

$$tr\left[\vec{E}(f_k)\vec{E}(f_k)^{\mathrm{H}}\right] \le A_e tr(\boldsymbol{S}_{d_e d_e}(f_k))$$

Stability: **Use Nyquist criterion:**

$$\min\left(\mathrm{Re}\left(\lambda\left(W_x(f_k)\hat{G}_{s0}(f_k)\right)\right)\right) > -1$$

Robustness:

$$\max\left(\sigma\left(\boldsymbol{W}_x(f_k)\widehat{\boldsymbol{G}}_{s0}(f_k)\right)\right)B(f_k) \le 1$$ ➡ ***M*- Δ structure and small gain theory**

Filter response: **The magnitude of frequency response:**

$$\left|W_{x_{i,j}}(f_k)\right| \le C(f_k)$$

Review of Previous Work - Original Problem

Cost function: **Total energy of e:**

$$\sum_{k=1}^{N_f} tr[\vec{E}(f_k)\vec{E}(f_k)^{\mathrm{H}}],$$

Constraints:

Enhancement: **Normalized energy of e:**

$$tr[\vec{E}(f_k)\vec{E}(f_k)^{\mathrm{H}}] \le A_e tr(\boldsymbol{S}_{d_e d_e}(f_k))$$

Stability: **Use Nyquist criterion:**

$$\min\left(\mathrm{Re}\left(\lambda\left(W_x(f_k)\hat{G}_{s0}(f_k)\right)\right)\right) > -1$$

Robustness: ***M*- Δ structure and small gain theory:**

$$\max\left(\sigma\left(W_x(f_k)\hat{G}_{s0}(f_k)\right)\right)B(f_k) \le 1$$

Filter response:

$$\left|\boldsymbol{W}_{x_{i,j}}(f_k)\right| \le C(f_k)$$

The magnitude of frequency response

Review of Previous Work - Conic Formulation

Original Problem

Cost function: **Total energy of e:**

$$\sum_{k=1}^{N_f} tr\left[\vec{\boldsymbol{E}}(f_k)\vec{\boldsymbol{E}}(f_k)^{\mathrm{H}}\right],$$

Constraints:

Enhancement: **Normalized energy of e:**

$$tr\left[\vec{\boldsymbol{E}}(f_k)\vec{\boldsymbol{E}}(f_k)^{\mathrm{H}}\right] \le A_e tr\left(\boldsymbol{S}_{d_e d_e}(f_k)\right)$$

Stability: **Use Nyquist criterion:**

$$\max\left(\lambda\left(\frac{-W_x(f_k)\hat{G}_{s0}(f_k) + \left(-W_x(f_k)\hat{G}_{s0}(f_k)\right)^{\mathrm{H}}}{2}\right)\right) - (1-\epsilon_s) \le 0$$

Robustness: ***M*- Δ structure and small gain theory:**

$$\max\left(\sigma\left(\boldsymbol{W}_x(f_k)\hat{\boldsymbol{G}}_{s0}(f_k)\right)\right) B(f_k) \le 1$$

Filter response: **The magnitude of frequency response:**

$$\left|\boldsymbol{W}_{x_{i,j}}(f_k)\right| \le C(f_k)$$

Standard Cone programming

$$\begin{aligned} \min. \quad & (\vec{\mathbf{c}}^{l})^{\mathrm{T}}\vec{\mathbf{x}}^{l} + (\vec{\mathbf{c}}^{q})^{\mathrm{T}}\vec{\mathbf{x}}^{q} + (\vec{\mathbf{c}}^{s})^{\mathrm{T}}\vec{\mathbf{x}}^{s}, \\ \text{s.t.} \quad & \mathbf{A}^{l}\vec{\mathbf{x}}^{l} + \mathbf{A}^{q}\vec{\mathbf{x}}^{q} + \mathbf{A}^{s}\vec{\mathbf{x}}^{s} = \vec{\mathbf{b}}, \\ & \vec{\mathbf{x}}^{l} \in \Re_{+}^{k_l}, \vec{\mathbf{x}}^{q} \in K^{q}, \vec{\mathbf{x}}^{s} \in K^{s} \end{aligned}$$

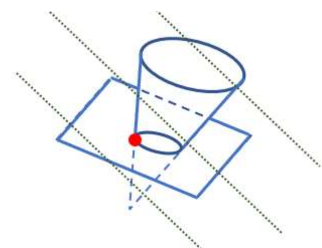
Where,

$$K^q = K_1^q \times \ldots \times K_{k_q}^q$$ **Second order cones**

$$K_i^q = \left\{(y, \vec{\mathbf{x}}) \in \Re \times \Re^{n_i - 1} : y \ge \|\vec{\mathbf{x}}\|_2\right\}$$

$$K^s = K_1^s \times \ldots \times K_{k_s}^s$$ **Positive semidefinite cones**

$$K_i^s = \left\{\mathrm{vec}(X) \in \Re^{n_i^2} : X \in \Re^{n_i \times n_i} \text{ is positive semidefinite}\right\}$$



Review of Previous Work - Conic Formulation

Original Problem

Cost function: Total energy of e:

$$\sum_{k=1}^{N_f} tr\left[\vec{\boldsymbol{E}}(f_k)\vec{\boldsymbol{E}}(f_k)^{\mathrm{H}}\right],$$

Constraints:

Enhancement: **Normalized energy of e:**

$$tr\left[\vec{\boldsymbol{E}}(f_k)\vec{\boldsymbol{E}}(f_k)^{\mathrm{H}}\right] \leq A_e tr(\boldsymbol{S}_{d_e d_e}(f_k))$$

Stability: **Use Nyquist criterion:**

$$\max\left(\lambda\left(\frac{-W_x(f_k)\hat{G}_{s0}(f_k) + \left(-W_x(f_k)\hat{G}_{s0}(f_k)\right)^{\mathrm{H}}}{2}\right)\right) - (1-\epsilon_s) \leq 0$$

Robustness: ***M*- Δ structure and small gain theory:**

$$\max\left(\sigma\left(\boldsymbol{W}_x(f_k)\widehat{\boldsymbol{G}}_{s0}(f_k)\right)\right) B(f_k) \leq 1$$

Filter response: **The magnitude of frequency response:**

$$\left|\boldsymbol{W}_{x_{i,j}}(f_k)\right| \leq C(f_k)$$

Conic Formulation

Cost function: $t_0 + \sum_{k=1}^{N_f} \vec{\boldsymbol{b}}_J^{\mathrm{T}}(f_k)\,\vec{\boldsymbol{w}}$

Constraints:

$$\|\boldsymbol{M}_0\vec{\boldsymbol{w}}\|_2 \leq \sqrt{t_0\,\tilde{t}_0}\;, \qquad \tilde{t}_0 = 1$$

$$t_{1,k} + \vec{\boldsymbol{b}}_J^{\mathrm{T}}(f_k)\vec{\boldsymbol{w}} + tr(\boldsymbol{S}_{d_e d_e}(f_k))(1 - A_e(f_k)) = 0$$

$$\|\boldsymbol{M}_{1,\mathrm{k}}\vec{\boldsymbol{w}}\|_2 \leq \sqrt{t_{1,k}\;\tilde{t}_{1,k}}\;, \qquad \tilde{t}_{1,k} = 1$$

$$\|\boldsymbol{F}_z(f_k)\,\vec{\boldsymbol{w}}_{F_{i,j}}\|_2 \leq t_{2,\mathrm{i,j},k}\;, \qquad t_{2,\mathrm{i,j},k} = C(f_k)$$

$$\boldsymbol{W}_x(f_k)\widehat{\boldsymbol{G}}_{s_0}(f_k) + \left(\boldsymbol{W}_x(f_k)\widehat{\boldsymbol{G}}_{s_0}(f_k)\right)^{\mathrm{H}} + 2(1-\epsilon_s)\boldsymbol{I}_{N_s} \succcurlyeq 0$$

$$\begin{bmatrix} \frac{1}{B(f_k)}\boldsymbol{I}_{N_s} & \boldsymbol{W}_x(k)\widehat{\boldsymbol{G}}_{s_0}(f_k) \\ \left(\boldsymbol{W}_x(f_k)\widehat{\boldsymbol{G}}_{s_0}(f_k)\right)^{H} & \frac{1}{B(f_k)}\boldsymbol{I}_{N_s} \end{bmatrix} \succcurlyeq 0$$

Review of Previous Work - Summary

- Previous work showed that this conic formulation can be solved much more efficiently.
- Numerical issues may occur sometimes, i.e., the solver may fail to obtain a searching direction when the current solution is close to optimal solution.
- It is found that different treatments of free variables in conic formulation have different numerical behaviors.

Conic Formulation

Cost function: $t_0 + \sum_{k=1}^{N_f} \vec{\boldsymbol{b}}_J^{\mathrm{T}}(f_k)\,\vec{\boldsymbol{w}}$

Constraints: $\|\boldsymbol{M}_0\vec{\boldsymbol{w}}\|_2 \le \sqrt{t_0\,\tilde{t}_0}\ , \qquad \tilde{t}_0 = 1$

$$t_{1,k} + \vec{\boldsymbol{b}}_J^{\mathrm{T}}(f_k)\vec{\boldsymbol{w}} + tr(\boldsymbol{S}_{d_e d_e}(f_k))(1 - A_e(f_k)) = 0$$

$$\|\boldsymbol{M}_{1,\mathrm{k}}\vec{\boldsymbol{w}}\|_2 \le \sqrt{t_{1,k}\ \tilde{t}_{1,k}}\ , \qquad \tilde{t}_{1,k} = 1$$

$$\|\boldsymbol{F}_z(f_k)\,\vec{\boldsymbol{w}}_{F_{i,j}}\|_2 \le t_{2,\mathrm{i,j},k}\ , \qquad t_{2,\mathrm{i,j},k} = C(f_k)$$

$$\boldsymbol{W}_x(f_k)\widehat{\boldsymbol{G}}_{s_0}(f_k) + \left(\boldsymbol{W}_x(f_k)\widehat{\boldsymbol{G}}_{s_0}(f_k)\right)^{\mathrm{H}} + 2(1-\epsilon_s)\boldsymbol{I}_{N_s} \succcurlyeq 0$$

$$\begin{bmatrix} \frac{1}{B(f_k)}\boldsymbol{I}_{N_s} & \boldsymbol{W}_x(k)\widehat{\boldsymbol{G}}_{s_0}(f_k) \\ \left(\boldsymbol{W}_x(f_k)\widehat{\boldsymbol{G}}_{s_0}(f_k)\right)^{H} & \frac{1}{B(f_k)}\boldsymbol{I}_{N_s} \end{bmatrix} \succcurlyeq 0$$

Conic Formulation

For simplification, denote the conic form as:

$$\begin{aligned} \min . \quad & \begin{bmatrix} (\vec{\mathbf{c}}_w)^{\mathrm{T}} & (\vec{\mathbf{c}}_x)^{\mathrm{T}} \end{bmatrix} \begin{bmatrix} \vec{\mathbf{w}} \\ \vec{\mathbf{x}} \end{bmatrix}, \\ \text{s.t.} \quad & \mathbf{A}\vec{\mathbf{w}} + \mathbf{B}\vec{\mathbf{x}} = \vec{\mathbf{b}}, \\ & \vec{\mathbf{w}} \in \Re^{N_r N_s N_t}, \\ & \vec{\mathbf{x}} \in K, \end{aligned}$$

Where,

$\vec{\mathbf{x}}$ is introduced to represent each constraints

K Represents the Cartesian product of cones for constraints

New variables are required to represent these conic constraints

Conic Formulation

Cost function: $t_0 + \sum_{k=1}^{N_f} \vec{\boldsymbol{b}}_J^{\mathrm{T}}(f_k)\,\vec{\boldsymbol{w}}$

Constraints:

$$\|\boldsymbol{M}_0\vec{\boldsymbol{w}}\|_2 \le \sqrt{t_0\ \tilde{t}_0}\ , \qquad \tilde{t}_0 = 1$$

$$t_{1,k} + \vec{\boldsymbol{b}}_J^{\mathrm{T}}(f_k)\vec{\boldsymbol{w}} + tr(\boldsymbol{S}_{d_e d_e}(f_k))(1 - A_e(f_k)) = 0$$

$$\|\boldsymbol{M}_{1,\mathrm{k}}\vec{\boldsymbol{w}}\|_2 \le \sqrt{t_{1,k}\ \tilde{t}_{1,k}}\ , \qquad \tilde{t}_{1,k} = 1$$

$$\|\boldsymbol{F}_z(f_k)\,\vec{\boldsymbol{w}}_{F_{i,j}}\|_2 \le t_{2,\mathrm{i,j},k}\ , \qquad t_{2,\mathrm{i,j},k} = C(f_k)$$

$$\boldsymbol{W}_x(f_k)\widehat{\boldsymbol{G}}_{s_0}(f_k) + \left(\boldsymbol{W}_x(f_k)\widehat{\boldsymbol{G}}_{s_0}(f_k)\right)^{\mathrm{H}} + 2(1-\epsilon_s)\boldsymbol{I}_{N_s} \succcurlyeq 0$$

$$\begin{bmatrix} \dfrac{1}{B(f_k)}\boldsymbol{I}_{N_s} & \boldsymbol{W}_x(k)\widehat{\boldsymbol{G}}_{s_0}(f_k) \\ \left(\boldsymbol{W}_x(f_k)\widehat{\boldsymbol{G}}_{s_0}(f_k)\right)^{H} & \dfrac{1}{B(f_k)}\boldsymbol{I}_{N_s} \end{bmatrix} \succcurlyeq 0$$

Conic Formulation - The Direct Reformulation

$$\begin{aligned} \min.\quad & \begin{bmatrix}(\vec{\mathbf{c}}_w)^{\mathrm{T}} & (\vec{\mathbf{c}}_x)^{\mathrm{T}}\end{bmatrix}\begin{bmatrix}\vec{\mathbf{w}}\\ \vec{\mathbf{x}}\end{bmatrix}, \\ \text{s.t.}\quad & \mathbf{A}\vec{\mathbf{w}} + \mathbf{B}\vec{\mathbf{x}} = \vec{\mathbf{b}}, \\ & \vec{\mathbf{w}} \in \mathfrak{R}^{N_r N_s N_t}, \\ & \vec{\mathbf{x}} \in K, \end{aligned}$$

Free variables

Convert into a second order cone

$$\begin{aligned} \min.\quad & \begin{bmatrix}0 & (\vec{\mathbf{c}}_w)^{\mathrm{T}} & (\vec{\mathbf{c}}_x)^{\mathrm{T}}\end{bmatrix}\begin{bmatrix}w_0\\ \vec{\mathbf{w}}\\ \vec{\mathbf{x}}\end{bmatrix}, \\ \text{s.t.}\quad & \mathbf{A}\vec{\mathbf{w}} + \mathbf{B}\vec{\mathbf{x}} = \vec{\mathbf{b}}, \\ & \underline{w_0 \geq \|\vec{\mathbf{w}}\|_2} \\ & \vec{\mathbf{x}} \in K, \end{aligned}$$

Form 1

Split as two sets of nonnegative variables

$$\begin{aligned} \min.\quad & \begin{bmatrix}(\vec{\mathbf{c}}_w)^{\mathrm{T}} & -(\vec{\mathbf{c}}_w)^{\mathrm{T}} & (\vec{\mathbf{c}}_x)^{\mathrm{T}}\end{bmatrix}\begin{bmatrix}\vec{\mathbf{w}}_1\\ \vec{\mathbf{w}}_2\\ \vec{\mathbf{x}}\end{bmatrix}, \\ \text{s.t.}\quad & \mathbf{A}\vec{\mathbf{w}}_1 - \mathbf{A}\vec{\mathbf{w}}_2 + \mathbf{B}\vec{\mathbf{x}} = \vec{\mathbf{b}}, \\ & \underline{\vec{\mathbf{w}}_1 \in \mathfrak{R}_+^{N_r N_s N_t}}, \\ & \underline{\vec{\mathbf{w}}_2 \in \mathfrak{R}_+^{N_r N_s N_t}}, \\ & \vec{\mathbf{x}} \in K. \end{aligned}$$

Form 2

Where,

$\vec{\mathbf{x}}$ is introduced to represent each constraints

K Represents the Cartesian product of cones for constraints

Conic Formulation - The Dual Reformulation

Form 1

$$
\begin{aligned}
\min.\quad & \begin{bmatrix} 0 & (\vec{\mathbf{c}}_w)^{\mathrm{T}} & (\vec{\mathbf{c}}_x)^{\mathrm{T}} \end{bmatrix} \begin{bmatrix} w_0 \\ \vec{\mathbf{w}} \\ \vec{\mathbf{x}} \end{bmatrix}, \\
\text{s.t.}\quad & \mathbf{A}\vec{\mathbf{w}} + \mathbf{B}\vec{\mathbf{x}} = \vec{\mathbf{b}}, \\
& \begin{bmatrix} w_0 \\ \vec{\mathbf{w}} \end{bmatrix} \in K_0^q, \\
& \vec{\mathbf{x}} \in K,
\end{aligned}
$$

Form 2

$$
\begin{aligned}
\min.\quad & \begin{bmatrix} (\vec{\mathbf{c}}_w)^{\mathrm{T}} & -(\vec{\mathbf{c}}_w)^{\mathrm{T}} & (\vec{\mathbf{c}}_x)^{\mathrm{T}} \end{bmatrix} \begin{bmatrix} \vec{\mathbf{w}}_1 \\ \vec{\mathbf{w}}_2 \\ \vec{\mathbf{x}} \end{bmatrix}, \\
\text{s.t.}\quad & \mathbf{A}\vec{\mathbf{w}}_1 - \mathbf{A}\vec{\mathbf{w}}_2 + \mathbf{B}\vec{\mathbf{x}} = \vec{\mathbf{b}}, \\
& \vec{\mathbf{w}}_1 \in \mathfrak{R}_+^{N_r N_s N_t}, \\
& \vec{\mathbf{w}}_2 \in \mathfrak{R}_+^{N_r N_s N_t}, \\
& \vec{\mathbf{x}} \in K.
\end{aligned}
$$

Dual formulation

Both forms have the same simplified dual formulation:

$$
\begin{aligned}
\min.\quad & -\vec{\mathbf{b}}^{\mathrm{T}}\vec{\mathbf{y}}, \\
\text{s.t.}\quad & \mathbf{A}^{\mathrm{T}}\vec{\mathbf{y}} = \vec{\mathbf{c}}_w, \\
& \mathbf{B}^{\mathrm{T}}\vec{\mathbf{y}} + \vec{\mathbf{s}}_x = \vec{\mathbf{c}}_x, \\
& \vec{\mathbf{y}} \in \mathfrak{R}^{N_b}, \\
& \vec{\mathbf{s}}_x \in K,
\end{aligned}
$$

Where,

$\vec{\mathbf{y}}$ is the dual variable associated with equality constraints

$\vec{\mathbf{s}}_x$ is the dual variable associated with conic constraints

The Dual Reformulation

The dual formulation

$$
\begin{aligned}
\min. \quad & -\vec{\mathbf{b}}^{\mathrm{T}}\vec{\mathbf{y}}, \\
\text{s.t.} \quad & \mathbf{A}^{\mathrm{T}}\vec{\mathbf{y}} = \vec{\mathbf{c}}_w, \\
& \mathbf{B}^{\mathrm{T}}\vec{\mathbf{y}} + \vec{\mathbf{s}}_x = \vec{\mathbf{c}}_x, \\
& \vec{\mathbf{y}} \in \Re^{N_b}, \\
& \vec{\mathbf{s}}_x \in K,
\end{aligned}
$$

Free variables

Convert into a second order cone

$$
\begin{aligned}
\min. \quad & \begin{bmatrix} 0 & -\vec{\mathbf{b}}^{\mathrm{T}} \end{bmatrix} \begin{bmatrix} y_0 \\ \vec{\mathbf{y}} \end{bmatrix}, \\
\text{s.t.} \quad & \mathbf{A}^{\mathrm{T}}\vec{\mathbf{y}} = \vec{\mathbf{c}}_w, \\
& \mathbf{B}^{\mathrm{T}}\vec{\mathbf{y}} + \vec{\mathbf{s}}_x = \vec{\mathbf{c}}_x, \\
& y_0 \geq \|\vec{\mathbf{y}}\|_2 \\
& \vec{\mathbf{s}}_x \in K.
\end{aligned}
$$

Form 3

Split as two sets of nonnegative variables

$$
\begin{aligned}
\min. \quad & \begin{bmatrix} -\vec{\mathbf{b}}^{\mathrm{T}} & \vec{\mathbf{b}}^{\mathrm{T}} \end{bmatrix} \begin{bmatrix} \vec{\mathbf{y}}_1 \\ \vec{\mathbf{y}}_2 \end{bmatrix}, \\
\text{s.t.} \quad & \mathbf{A}^{\mathrm{T}}\vec{\mathbf{y}}_1 - \mathbf{A}^{\mathrm{T}}\vec{\mathbf{y}}_2 = \vec{\mathbf{c}}_w, \\
& \mathbf{B}^{\mathrm{T}}\vec{\mathbf{y}}_1 - \mathbf{B}^{\mathrm{T}}\vec{\mathbf{y}}_2 + \vec{\mathbf{s}}_x = \vec{\mathbf{c}}_x, \\
& \vec{\mathbf{y}}_1 \in \Re_+^{N_b}, \\
& \vec{\mathbf{y}}_2 \in \Re_+^{N_b}, \\
& \vec{\mathbf{s}}_x \in K.
\end{aligned}
$$

Form 4

The Direct and Dual Reformulation - Summary

Form 1

$$
\begin{aligned}
\min.\quad & \begin{bmatrix} 0 & (\vec{\mathbf{c}}_w)^{\mathrm{T}} & (\vec{\mathbf{c}}_x)^{\mathrm{T}} \end{bmatrix} \begin{bmatrix} w_0 \\ \vec{\mathbf{w}} \\ \vec{\mathbf{x}} \end{bmatrix}, \\
\text{s.t.}\quad & \mathbf{A}\vec{\mathbf{w}} + \mathbf{B}\vec{\mathbf{x}} = \vec{\mathbf{b}}, \\
& w_0 \geq ||\vec{\mathbf{w}}||_2 \\
& \vec{\mathbf{x}} \in K,
\end{aligned}
$$

Form 3

$$
\begin{aligned}
\min.\quad & \begin{bmatrix} 0 & -\vec{\mathbf{b}}^{\mathrm{T}} \end{bmatrix} \begin{bmatrix} y_0 \\ \vec{\mathbf{y}} \end{bmatrix}, \\
\text{s.t.}\quad & \mathbf{A}^{\mathrm{T}}\vec{\mathbf{y}} = \vec{\mathbf{c}}_w, \\
& \mathbf{B}^{\mathrm{T}}\vec{\mathbf{y}} + \vec{\mathbf{s}}_x = \vec{\mathbf{c}}_x, \\
& y_0 \geq ||\vec{\mathbf{y}}||_2 \\
& \vec{\mathbf{s}}_x \in K.
\end{aligned}
$$

Form 2

$$
\begin{aligned}
\min.\quad & \begin{bmatrix} (\vec{\mathbf{c}}_w)^{\mathrm{T}} & -(\vec{\mathbf{c}}_w)^{\mathrm{T}} & (\vec{\mathbf{c}}_x)^{\mathrm{T}} \end{bmatrix} \begin{bmatrix} \vec{\mathbf{w}}_1 \\ \vec{\mathbf{w}}_2 \\ \vec{\mathbf{x}} \end{bmatrix}, \\
\text{s.t.}\quad & \mathbf{A}\vec{\mathbf{w}}_1 - \mathbf{A}\vec{\mathbf{w}}_2 + \mathbf{B}\vec{\mathbf{x}} = \vec{\mathbf{b}}, \\
& \vec{\mathbf{w}}_1 \in \mathfrak{R}_+^{N_r N_s N_t}, \\
& \vec{\mathbf{w}}_2 \in \mathfrak{R}_+^{N_r N_s N_t}, \\
& \vec{\mathbf{x}} \in K.
\end{aligned}
$$

Form 4

$$
\begin{aligned}
\min.\quad & \begin{bmatrix} -\vec{\mathbf{b}}^{\mathrm{T}} & \vec{\mathbf{b}}^{\mathrm{T}} \end{bmatrix} \begin{bmatrix} \vec{\mathbf{y}}_1 \\ \vec{\mathbf{y}}_2 \end{bmatrix}, \\
\text{s.t.}\quad & \mathbf{A}^{\mathrm{T}}\vec{\mathbf{y}}_1 - \mathbf{A}^{\mathrm{T}}\vec{\mathbf{y}}_2 = \vec{\mathbf{c}}_w, \\
& \mathbf{B}^{\mathrm{T}}\vec{\mathbf{y}}_1 - \mathbf{B}^{\mathrm{T}}\vec{\mathbf{y}}_2 + \vec{\mathbf{s}}_x = \vec{\mathbf{c}}_x, \\
& \vec{\mathbf{y}}_1 \in \mathfrak{R}_+^{N_b}, \\
& \vec{\mathbf{y}}_2 \in \mathfrak{R}_+^{N_b}, \\
& \vec{\mathbf{s}}_x \in K.
\end{aligned}
$$

Results

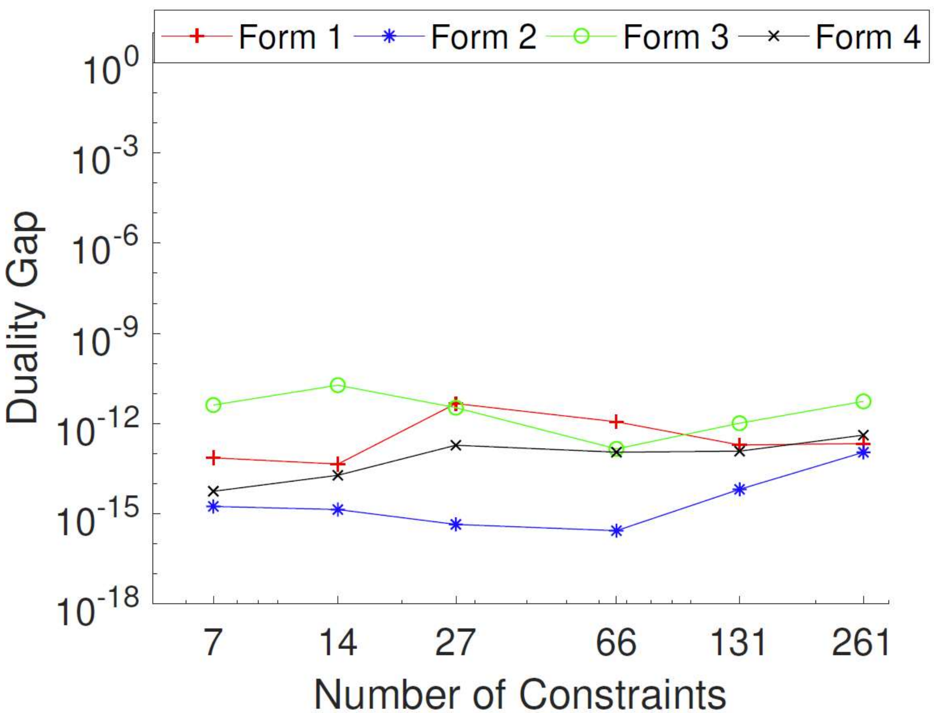
Off-line Simulation based on experimental data

Experiment description:

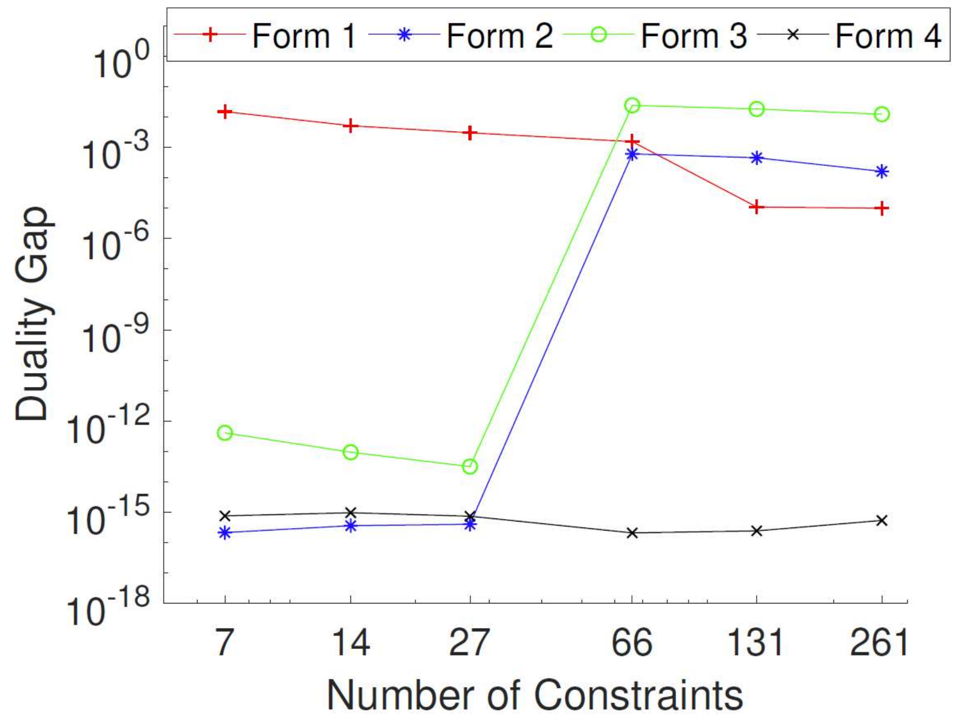
- 2 reference microphones, 2 control loudspeakers, 2 error microphones
- sampling frequency is 3000 Hz
- Filter length for each channel is 128
- SeDuMi is used to implement primal-dual interior-point algorithm for cone programming
- Duality gap is used to represent numerical stability characteristics (Smaller duality gap means more numerically stable)

Results

Comparison of duality gap for different forms in different cases



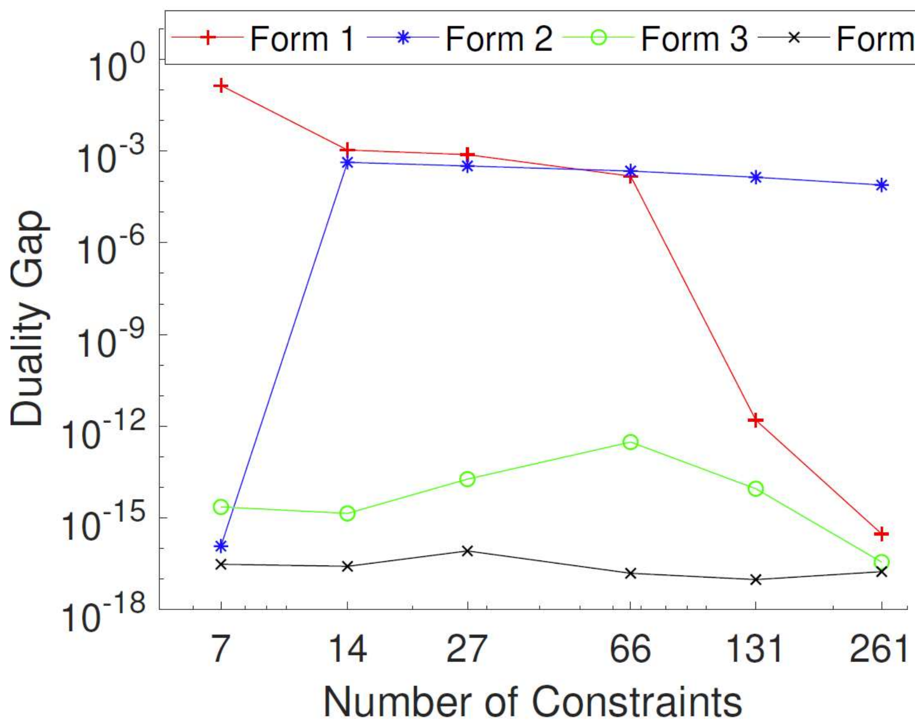
Use only enhancement constraint



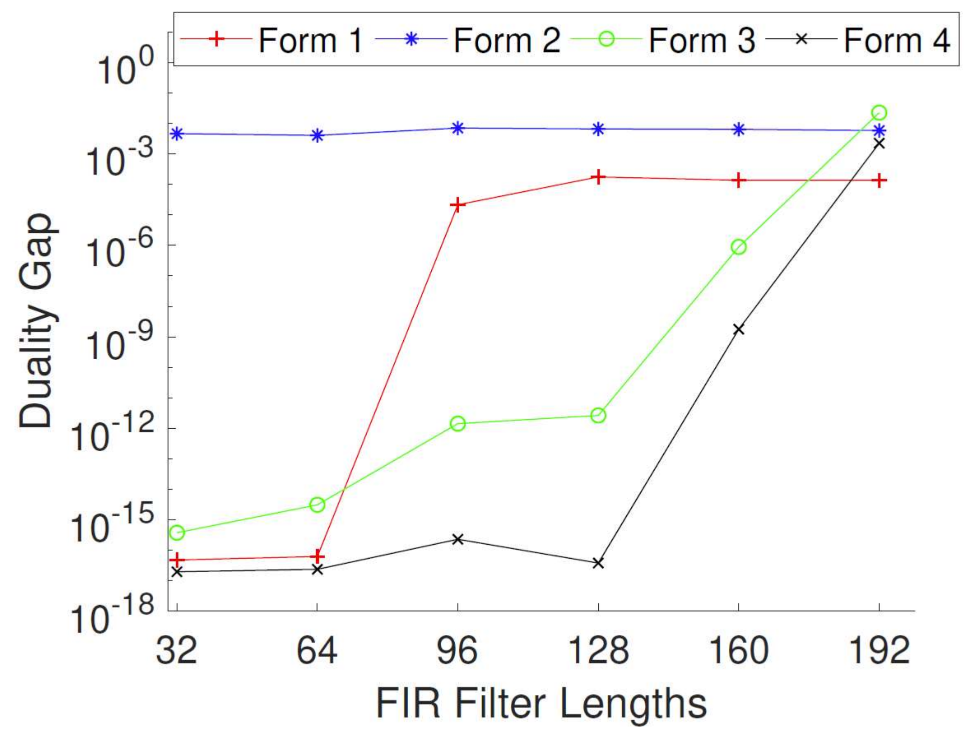
Use only stability constraint

Results

Comparison of duality gap for different forms in different cases



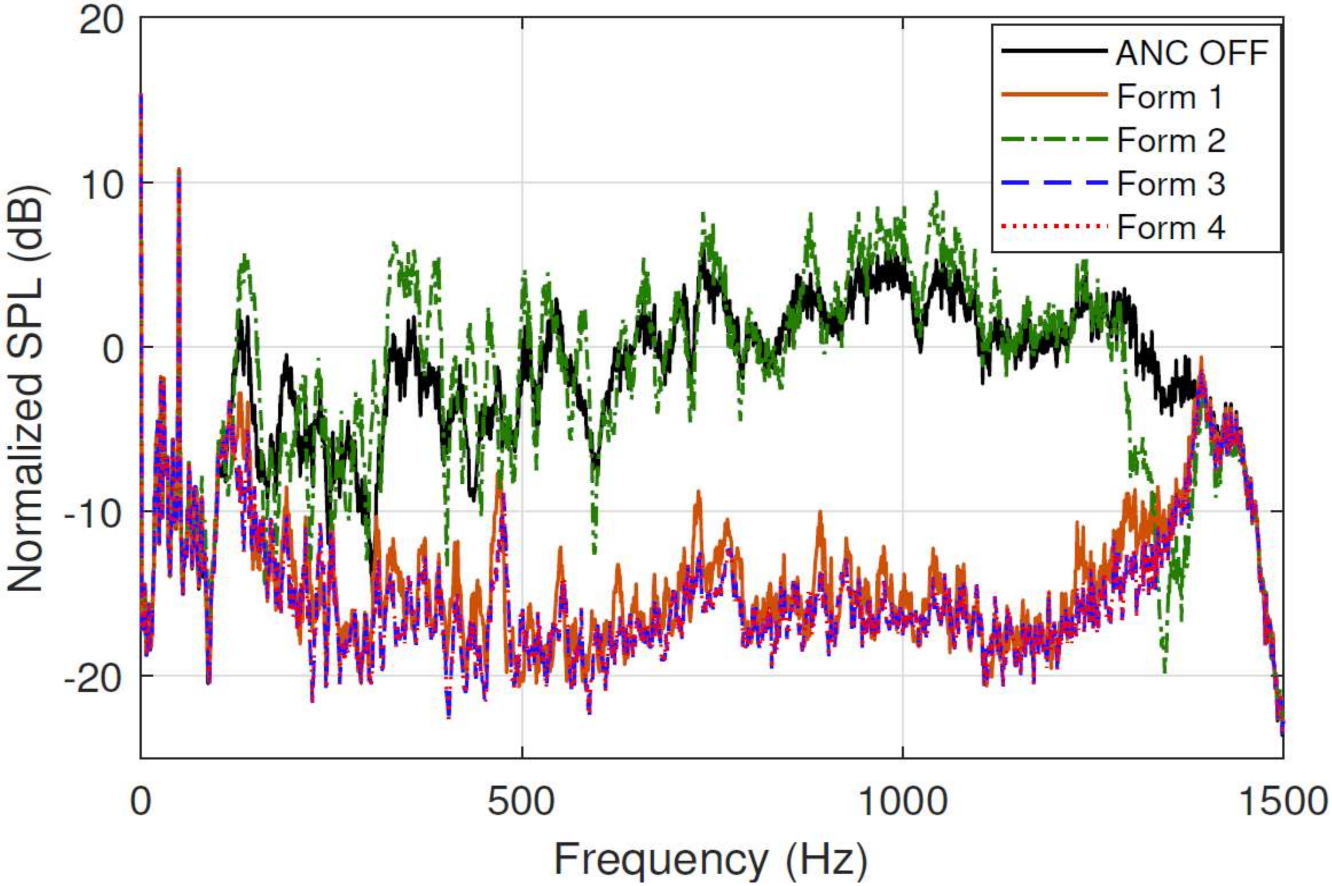
Use only robustness constraint



Use all constraints

Results

Comparison of ANC performance for different forms



Form	Duality Gap
1	1.76×10^{-4}
2	6.61×10^{-3}
3	2.63×10^{-12}
4	3.78×10^{-1}

The performance of using form 1 and 2 are worse than using form 3 and 4.

This demonstrates that a small duality gap is required.

Conclusions

- Numerical issues may occur when positive semidefinite cones are involved, i.e., when stability and robustness constraints are applied.

- Form 4, using the dual formulation and then splitting free variables into two sets of non-negative variables, has a better numerical stability behavior.

- In the future, other reformulation approaches may be used to further improve the numerical stability by exploiting the problem structure of the ANC filter design problem.

Thank you !