The Delicate General Stoke's Formula

a short introduction to differential forms and its unification

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For the Well-Known Newton Leibniz's theorem, irrotational field and its protential,there's obviously sth like similarity:

$$\int_a^b \frac{dF}{dx}dx = F(b) - F(a) \tag{1}$$

$$\int_a^b \nabla\Phi \cdot d\mathbf{r} = \Phi(b) - \Phi(a) \tag{2}$$

And then for this short journey we must introduce a tiny formalism: we are kind of using ∂D to stand for the directed(or strictly,orienting) boundary of the regin D.

With this formalation,and a littel Vector Analysis, the well-know *Gauss Theorem* and *Stoke's Theorem* can be written as follow:

$$\oiint_{\partial D} \mathbf{E} \cdot d\mathbf{S} = \iiint_D \nabla \cdot \mathbf{E} dV \tag{3}$$

$$\oint_{\partial D} \mathbf{E} \cdot d\mathbf{r} = \iint_D \nabla \times \mathbf{E} \cdot d\mathbf{S} \tag{4}$$

For the Stoke's theorem(4), choose $\mathbf{E}(P(x,y), Q(x,y), 0)$,and D a region on the x-y plane,we have:

$$\iint_D \nabla \times \mathbf{E} \cdot d\mathbf{S} = \iint_D \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & 0 \end{vmatrix} \cdot d\mathbf{S} = \iint_D (\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y})\mathbf{k} \cdot d\mathbf{S}$$

$$= \iint_D (\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}) dxdy$$

which precisely gives the Green’ theorm.

$$\iint_D (\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}) dxdy = \oint_{\partial D} Pdx + Qdy \tag{5}$$

What’s the relation between them? And why we write the line and surface integration as follow:

$$\oint_{\partial D} \mathbf{E} \cdot d\mathbf{r} = \int_{\partial D} Pdx + Qdy + Rdz$$

$$\iint_{\partial D} \mathbf{E} \cdot d\mathbf{S} = \iint_{\partial D} Pdydz + Qdzdx + Rdxdy$$

Are they indeed imply somthing?

Without so much more sophisticated mathematical proof, I can’t wait to share with you the compact,meticulous *Stoke’s Formula*:

$$\int_{\partial D} \omega = \int_D d\omega$$

To feel the power inside this equation, we need to know that $\omega, d\omega$ is something call *differential forms*,and that "d" is an differential operator call *exterior differentiation*, which is quite similar to "d" in the common calculus, but not just the same,and which can be apply to a differential forms. Hiden in this equation there’s a knid of new product operation between two d("something") call *wedge product*(or exterior product),as "$\wedge$",which is antisymmetry and defined as follow:

$$d(d\omega) = 0 \tag{6}$$

$$dx \wedge dx = 0 \tag{7}$$

$$dx \wedge dy = -dy \wedge dx \tag{8}$$

$$dF(x, y, z) = \frac{\partial F}{\partial x} dx + \frac{\partial F}{\partial y} dy + \frac{\partial F}{\partial z} dz \tag{9}$$

Quite annoying? Let’s take a try. Take Green Theorem as an example. Suppose $\mathbf{E}$ is a vector field (or mathematically: $\mathbf{E} : \mathbb{R}^2 \to \mathbb{R}^2$) define on region D within x-y plane, whith has two component $\mathbf{E}(P, Q)$,a line integrate running arround the edge of D is:

$$\int_{\partial D} \mathbf{E} \cdot d\mathbf{r} = \int_{\partial D} Pdx + Qdy$$

Now glance at the left hand side of the Stoke's Fomula, we just denote the integrand as $\omega = Pdx + Qdy$, now it's time to do "exterior differention" with the notation of wedge product:

$$\begin{aligned}
d(d\omega) &= d(Pdx + Qdy) \\
&= dPdx + Pd(dx) + dQdy + Qd(dy) \qquad //\text{still use the product rule} \\
&= (\frac{\partial P}{\partial x}dx + \frac{\partial P}{\partial y}dy) \wedge dx + (\frac{\partial Q}{\partial x}dx + \frac{\partial Q}{\partial y}dy) \wedge dy \quad \text{(1)gives d(dx)=0} \\
&= \frac{\partial P}{\partial x}dx \wedge dx + \frac{\partial P}{\partial y}dy \wedge dx + \frac{\partial Q}{\partial x}dx \wedge dy + \frac{\partial Q}{\partial y}dy \wedge dy \\
&= \frac{\partial P}{\partial y}dy \wedge dx + \frac{\partial Q}{\partial x}dx \wedge dy \qquad //\text{(2)gives} \quad dx \wedge dx = 0 \\
&= (\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y})dx \wedge dy \qquad //\text{use the (3) antisymmetry law}
\end{aligned}$$

which precisly give the *Green Theorem*.

Now here comes the unification of Fundamental theorem of Calculus, take $\omega = F(x)$ and "D" a interval of $\mathbb{R}$, $d\omega$ gives $\frac{dF(x)}{dx}dx$, and $\int_D d\omega = \int_D \frac{dF(x)}{dx}dx$, the general stoke's formula gives the former equals to $\int_{\partial D} \omega = \int_{\partial D} F(x)$, now we come to a strange integration without "d", the edge of an line interval is two end points, naturally it's $\int_{\partial D} F(x) = F(b) - F(a)$

And if we choose the differential forms $\omega = F(x, y, z)$, and "D" a curve in space. $d\omega$ gives, $\frac{\partial F}{\partial x}dx + \frac{\partial F}{\partial y}dy + \frac{\partial F}{\partial z}dz$. With this formula we have:

$$\int_D \frac{\partial F}{\partial x}dx + \frac{\partial F}{\partial y}dy + \frac{\partial F}{\partial z}dz = \int_{\partial D} F(x, y, z)$$

which return the equation (2).

And the same way, given $\omega = Pdx + Qdy + Rdz$,(which we call differential 1-forms) gives back

$$d\omega = (\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y})dx \wedge dy + (\frac{\partial P}{\partial x} - \frac{\partial P}{\partial y})dx \wedge dy + (\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y})dx \wedge dy$$

Now take a look at the Gauss' theorem.

Takes $\omega = Pdy \wedge dz + Qdz \wedge dx + Rdx \wedge dy$ (2-forms)gives back:

$$\begin{aligned}
d\omega &= d(Pdy \wedge dz + Qdz \wedge dx + Rdx \wedge dy) \\
&= (\frac{\partial P}{\partial x}dx + \frac{\partial P}{\partial y}dy + \frac{\partial P}{\partial z}dz)dy \wedge dz \\
&\quad + (\frac{\partial Q}{\partial x}dx + \frac{\partial Q}{\partial y}dy + \frac{\partial Q}{\partial z}dz)dz \wedge dx \\
&\quad + (\frac{\partial R}{\partial x}dx + \frac{\partial R}{\partial y}dy + \frac{\partial R}{\partial z}dz)dx \wedge dy \\
&= \frac{\partial P}{\partial x}dx \wedge dy \wedge dz + \frac{\partial Q}{\partial y}dy \wedge dz \wedge dx + \frac{\partial R}{\partial z}dz \wedge dx \wedge dy \\
&= (\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z})dx \wedge dy \wedge dz \qquad (10)
\end{aligned}$$

Look back to the Thomas Calculus, a great book with almost 1200 pages, talks form the notation of function and map, to the integration on vector field. Such a grand book finally end up with such a grand summation:

> **The integral of a differential operator acting on a field over a region equals the sum of the field components appropriate to the operator over the boundary of the region.**