Mathematical model of viscoelasticity

Dynamic vicoelastic problem of generalised Maxwell solid is represented as a second kind of Volterra integral equation with expontially decaying kernel. Our aim is to solve the hyperbolic PDE with memory terms by spatially continuous Galerkin finite element method (CGFEM) and Crank-Nicolson finite difference scheme for time discretisation.

We breifly introduce the model problem and give a numerical scheme. *Stability and error analysis as well as more details are seen in my research parper and PhD thesis (the link will appear here soon).*

Let $\Omega \in \mathbb{R}^d$ be our open bounded for $d = 2, 3$ and $[0, T]$ be a time domain for $T > 0$. The model problem is given by

$$\rho \ddot{u}(t) - \nabla \cdot D\nabla \left(u(t) - \sum_{q=1}^{N_\varphi} \psi_q(t) \right) = f(t)$$

where $D > 0$, ρ is a density, u is a displacement, f is an external force and ${\psi_q}_{q=1}^{N_\varphi}$ is a set of internal variables of displacement form defined by

$$\psi_q(t) := \frac{\varphi_q}{\tau_q} \int_0^t e^{-(t-s)/\tau_q} u(s)\, ds, \qquad \text{for } q = 1, \ldots, N_\varphi,$$

for sets of positive constants $\{\varphi_q\}_{q=0}^{N_\varphi}$ and $\{\tau_q\}_{q=1}^{N_\varphi}$ with $\sum_{q=0}^{N_\varphi} \varphi_q = 1$.

For the stability, we impose pure Dirichlet boundary or mixed boundary with a positive measure of Dirichlet boundary Γ_D. Let us assume a homogeneous Dirichlet boundary condition. Then we have the following boundary condition such that

$$\begin{aligned} u(t,x) &= 0, \qquad \text{for } x \in \Gamma_D,\ \forall t \text{ (Dirichlet boundary)}, \\ D\nabla \left(u(t) - \sum_{q=1}^{N_\varphi} \psi_q(t) \right) \cdot \boldsymbol{n} &= g_N, \qquad \text{for } x \in \Gamma_N,\ \forall t \text{ (Neumann boundary)}, \end{aligned}$$

where $\boldsymbol{n}$ is an outward normal vector.

In addition, use of integration by parts leads us to obtain internal variables of velocity form as following. $\psi_q(t) = \varphi_q u(t) - \varphi_q e^{-t/\tau_q} u_0 - \zeta_q(t)$ where

$$\zeta_q(t) = \int_0^t \varphi_q e^{-(t-s)/\tau_q}\,\dot{u}(s)\,ds.$$

Define a finite element space $V^h \subset V = \{v \in H^1(\Omega) | v = 0 \text{ for } x \in \Gamma_D\}$ of Lagrange finite element. Then we can derive variational problem with respect to internal variables.

Displacement form (P1)

$$(\rho\ddot{u}(t), v)_{L_2(\Omega)} + a(u(t), v) - \sum_{q=1}^{N_\varphi} a(\psi_q(t), v) = F_d(t; v) \qquad \forall v \in V, \tag{1}$$

$$\tau_q a(\dot{\psi}_q(t), v) + a(\psi_q(t), v) = \varphi_q a(u(t), v) \qquad \forall v \in V,\ q = 1, \ldots, N_\varphi \tag{2}$$

where $a(w, v) = (D\nabla w, \nabla v)_{L_2(\Omega)}$ and $F_d(t; v) = (f(t), v)_{L_2(\Omega)} + (g_N(t), v)_{L_2(\Gamma_N)}$ with $u(0) = u_0$, $\dot{u}(0) = w_0$ and $\psi_q(0) = 0$, $\forall q \in \{1, \ldots, N_\varphi\}$.

In a similar way, we can also obtain a weak formulation of velocity form.

Velocity form (P2)

$$(\rho\ddot{u}(t), v)_{L_2(\Omega)} + \varphi_0 a(u(t), v) + \sum_{q=1}^{N_\varphi} a(\zeta_q(t), v) = F_v(t; v) \qquad \forall v \in V, \tag{3}$$

$$\tau_q a(\dot{\zeta}_q(t), v) + a(\zeta_q(t), v) = \tau_q \varphi_q a(\dot{u}(t), v) \qquad \forall v \in V,\ q = 1, \ldots, N_\varphi \tag{4}$$

where $F_v(t; v) = F_d(t; v) - \sum_{q=1}^{N_\varphi} \varphi_q e^{-t/\tau_q} a(u_0, v)$ with $u(0) = u_0$, $\dot{u}(0) = w_0$ and $\zeta_q(0) = 0$, $\forall q \in \{1, \ldots, N_\varphi\}$.

Eqs (2) and (4) are governed by ODEs from differentiating internal variables.

Fully discrete formulation

Let us discretise time space. Define $\Delta t = T/N$ for $N \in \mathbb{N}$ and $t_n = n\Delta t$ for $n = 0, \ldots, N$. We approximate the solution by

$$u(t_n, x) \approx Z_h^n(x) \in V^h, \qquad \dot{u}(t_n, x) \approx W_h^n(x) \in V^h \text{ for } n = 0, \ldots, N,$$

based on Crank-Nicolson method with

$$\frac{W_h^{n+1}(x)+W_h^n(x)}{2}=\frac{Z_h^{n+1}(x)-Z_h^n(x)}{\Delta t}\ \text{for}\ n=0,\dots,N-1.$$

In this manner, we can also consider discrete solutions of internal variables such as Ψ_{hq} and $\mathcal{S}_{hq}$ for each q.

In the end, we can formulate fully discrete schemes;

Displacement form (P1)

$$\left(\rho\frac{W_h^{n+1}-W_h^n}{\Delta t},v\right)_{L_2(\Omega)}+a\left(\frac{Z_h^{n+1}+Z_h^n}{2},v\right)-\sum_{q=1}^{N_\varphi}a\left(\frac{\Psi_{hq}^{n+1}+\Psi_{hq}^n}{2},v\right)=\frac{F_d(t_{n+1};v)+F_d(t_n;v)}{2}, \tag{6}$$

$$\tau_q a\left(\frac{\Psi_{hq}^{n+1}-\Psi_{hq}^n}{\Delta t},v\right)+a\left(\frac{\Psi_{hq}^{n+1}+\Psi_{hq}^n}{2},v\right)=\varphi_q a\left(\frac{Z_h^{n+1}+Z_h^n}{2},v\right)\ \text{for each}\ q, \tag{7}$$

$$a(Z_h^0,v)=a(u_0,v), \tag{8}$$

$$(W_h^0,v)_{L_2(\Omega)}=(w_0,v)_{L_2(\Omega)}, \tag{9}$$

$$\Psi_{hq}^0=0\qquad\text{for each}\ q, \tag{10}$$

for all $v\in V^h$.

Velocity form (P2)

$$\left(\rho\frac{W_h^{n+1}-W_h^n}{\Delta t},v\right)_{L_2(\Omega)}+\varphi_0 a\left(\frac{Z_h^{n+1}+Z_h^n}{2},v\right)+\sum_{q=1}^{N_\varphi}a\left(\frac{\mathcal{S}_{hq}^{n+1}+\mathcal{S}_{hq}^n}{2},v\right)=\frac{F_v(t_{n+1};v)+F_v(t_n;v)}{2}, \tag{11}$$

$$\tau_q a\left(\frac{\mathcal{S}_{hq}^{n+1}-\mathcal{S}_{hq}^n}{\Delta t},v\right)+a\left(\frac{\mathcal{S}_{hq}^{n+1}+\mathcal{S}_{hq}^n}{2},v\right)=\tau_q\varphi_q a\left(\frac{W_h^{n+1}+W_h^n}{2},v\right)\ \text{for each}\ q, \tag{12}$$

$$a(Z_h^0,v)=a(u_0,v), \tag{13}$$

$$(W_h^0,v)_{L_2(\Omega)}=(w_0,v)_{L_2(\Omega)}, \tag{14}$$

$$\mathcal{S}_{hq}^0=0\qquad\text{for each}\ q, \tag{15}$$

for all $v\in V^h$.

Numerical Experiments

Let $u = e^{-t}\sin(xy)$ be the exact solution on the unit square. While we set $\varphi_0 = 0.5$, $\varphi_1 = 0.1$, $\varphi_2 = 0.4$, $\tau_1 = 0.5$, $\tau_2 = 1.5$, $\rho = 1$ and $D = 1$, we can obatain internal variables for $q = 1, 2$, the source term f and the Neumann boundary condition g_N.

According to the fully discrete formulations of dispalcement form Eqs (6)-(10) and velocity form Eqs (11)-(15), code implemetation has been carried out in **CG_P1.py** (displacement form) and **CG_P2.py** (velocity form). Here we can observe and compare numerical errors and convergent rates with respect to a degree of polynomial and the form of internal variables.

- Run **main_Figure1.sh** to illustrate graphs of error convergence rates for linear and quadratic polynomial bases when $h \approx \Delta t$.
- Run **main_Table.sh** to describe tables of numerical error for fixed either h or Δt where a degree of polynomial is 2.