Linear control Assignment2

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1 Pre-info

From the assigment1 subquestion(c the matrices A B are:

$$
A = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & \frac{D_2}{B_f} & -\frac{D_2}{B_f} & 0 & 0 \\ -\frac{D_1}{J_1} & \frac{D_1}{J_1} & 0 & -\frac{K_T \cdot K_E}{J_1 \cdot R} & 0 \\ \frac{D_1}{J_2} & -\frac{(D_1 + D_2)}{J_2} & \frac{D_2}{J_2} & 0 & 0 \end{bmatrix}
$$
$$
B = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & \frac{1}{B_f} \\ \frac{K_T}{J_1 \cdot R} & 0 \\ 0 & 0 \end{bmatrix} \tag{1}
$$

We call matrices of $c)(1)$ $C1, D1$, matrices of $c)(2)$ $C2, D2$, the parameter D_1 D_2 is differ from matrix D1 D2:

$$
C1 = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}
$$
$$
C2 = \begin{bmatrix} 0 & 0 & 0 & -\frac{K_E}{R} & 0 \\ 0 & \frac{D_2}{B_f} & -\frac{D_2}{B_f} & 0 & 0 \end{bmatrix}
$$
$$
D1 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}
$$
$$
D2 = \begin{bmatrix} \frac{1}{R} & 0 \\ 0 & \frac{1}{B_f} \end{bmatrix} \tag{2}
$$

There are some parameters:

$$
KE = 1e-1 \; KT = 1e-1 \; J1 = 1e-5 \; J2 = 4e-5 \; Bf = 2e-3 \; D_2 = 2
$$
$$
R > 0, D \in R
$$

2 Question a

The dimension of the system n is 5, hence we can get the controllability matrix M_{con} and observability matrices M_{ob1} and M_{ob2} of row-reduced form:

$$M_{con} = \begin{bmatrix} B \\ AB \\ \vdots \\ AB^4 \end{bmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 & -100000\,D_1 & 0 & \frac{100000000\,D_1}{R} & 250000000\,D_1\,R & -\frac{12500000000\left(8\,D_1 - D_1{}^2\,R^2\right)}{R^2} & -250000000000\,D_1\,R \\ 0 & 1 & 0 & 0 & \frac{10000\,D_1}{R} & 0 & -\frac{10000000\,D_1}{R^2} & -25000000\,D_1 & -\frac{1250000000\,D_1\left(D_1\,R^2-8\right)}{R^3} & 25000000000\,D_1 \\ 0 & 0 & 1 & 0 & -\frac{1000}{R} & 0 & -\frac{100000\left(D_1\,R^2-10\right)}{R^2} & 0 & \frac{200000000\left(D_1\,R^2-5\right)}{R^3} & 250000000\,D_1\,R \\ 0 & 0 & 0 & 1 & \frac{10\,D_1}{R} & 0 & \frac{10000\,D_1\,(R-1)}{R^2} & -25000\,D_1 - 50000 & -\frac{250000\,D_1\left(40\,R+5\,D_1\,R^2+2\,R^2-40\right)}{R^3} & 50000000 \\ 0 & 0 & 0 & 0 & 0 & 1 & \frac{10\,D_1}{R} & -1000 & -\frac{10000\,D_1}{R^2} & 950000 - 25000\,D_1 \end{pmatrix} \tag{3}$$

$$M_{ob1} = \begin{bmatrix} C_1 \\ C_1A \\ \vdots \\ C_1A^4 \end{bmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \tag{4}$$

$$M_{ob2} = \begin{bmatrix} C_2 \\ C_2A \\ \vdots \\ C_2A^4 \end{bmatrix} = \begin{pmatrix} 1 & 0 & -1 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \tag{5}$$

Obviously, the observability matrices are rank-deficient. As M_{con} is the reduced row echelon form of the controllability matrix, we can judge from the elements in the first 6 columns in equation3 that M_{con} is full rank. There is no set of values for which the matrices lose rank:

$$\begin{pmatrix} 1 & 0 & 0 & 0 & -100000\,D_1 & 0 \\ 0 & 1 & 0 & 0 & \frac{10000\,D_1}{R} & 0 \\ 0 & 0 & 1 & 0 & -\frac{1000}{R} & 0 \\ 0 & 0 & 0 & 1 & \frac{10\,D_1}{R} & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

Both R and D_1 are assumed to be nonzero, positive and real parameter values. To calculate the determinant, we convert the controllability matrix (observability matrix) to square matrix by multiplying its transform:

```
% question a try to find numericall solution
```

```
cona = [B A*B A^2*B A^3*B A^4*B];
ob1a = [C1; C1*A; C1*A^2; C1*A^3; C1*A^4];
ob2a = [C2; C2*A; C2*A^2; C2*A^3; C2*A^4];

%checking if there is any set of values for which the matrix loose rank
assume(R>0);
assume(D_1>0);

%Both controllability matrix and observability matrix are not square
%matrix.
con2 = cona'*cona;
[conR,conD1,conParams,conConds] = solve(det(con2)==0,[R,D_1],"ReturnConditions",true)

pob1 = ob1a'*ob1a;
[ob1aR,ob1aD1,ob1aParams,ob1aConds] = ...
    solve(det(pob1)==0,[R,D_1],"ReturnConditions",true)
```

The result says:

$$
\begin{aligned}
conR &= x \\
conD_1 &= y \\
conParams &= [x, y] \\
conConds &= 0 < x \ \ \& \ \ x < sym(\infty) \ \ \& \ \ 0 < y \ \ \& \ \ y < sym(\infty) \\
ob1aR &= 10/(z+10) \\
ob1aD_1 &= z \\
ob1aParams &= z \\
ob1aConds &= 0 < z
\end{aligned}
$$

From the output,In certain cases, when satisfy the condition, the controllability matrix can be rank-deficient. If it satisfy $ob1aR = 10/(z+10)$, the observability matrix in case 1 is rank-deficient.
This result is contradictory to row-reduced matrix. The reasons cause the contradiction are probably:
1.) The numerical properties being changed due to row reduction;
2.)Pole-zero cancellation due to assignment.

3 Question b

As what we discussed in Question a, the controllability is always full rank, hence the system is controllable and stabilizable. The rank of M_{ob1} is 5, which is equal to the dimensions of the system, hence this system is observable and detectable. While the rank of the other observability matrix M_{ob2} is 4, we need to verify whether the unobservable states(modes) are stable.The eigenvalues of the system are:

$$\begin{pmatrix} 0 & -\sigma_4-\sigma_1-\sigma_3 & \sigma_3-\sigma_1-\sigma_4 & \sigma_4-\sigma_1-\sigma_2 & \sigma_4-\sigma_1+\sigma_2 \end{pmatrix}$$

where

$$\sigma_1 = \frac{1000\,\overline{R}+1000}{4\,\overline{R}}$$

$$\sigma_2 = \frac{\overline{\sqrt{-12\,\sigma_{12}\,\sigma_9-\sigma_{13}{}^2\,\sigma_9-\sigma_8-\sigma_6-\sigma_7}}}{\sigma_5}$$

$$\sigma_3 = \frac{\overline{\sqrt{\sigma_6-\sigma_{13}{}^2\,\sigma_9-\sigma_8-12\,\sigma_{12}\,\sigma_9-\sigma_7}}}{\sigma_5}$$

$$\sigma_4 = \frac{\overline{\sigma_9}}{6\,\overline{\sigma_{10}{}^{1/6}}}$$

$$\sigma_5 = 6\,\overline{\left(\frac{12\,\sigma_{15}}{R}+\sigma_{13}{}^2-\frac{9\,(1000\,R+1000)^4}{64\,R^4}+9\,\sigma_{10}{}^{2/3}-6\,\sigma_{13}\,\sigma_{10}{}^{1/3}-\frac{3\,(1000\,R+1000)\,\sigma_{18}}{R^2}+\frac{3\,\sigma_{16}\,\sigma_{17}}{4\,R^3}\right)^{1/4}\,\overline{\sigma_{10}{}^{1/6}}}$$

$$\sigma_6 = 3\,\sqrt{6}\,\sigma_{14}\,\sqrt{3\,\sqrt{3}\,\sigma_{11}+2\,\sigma_{13}{}^3+27\,\sigma_{14}{}^2-72\,\sigma_{13}\,\sigma_{12}}$$

$$\sigma_7 = 12\,\sigma_{13}\,\sigma_{10}{}^{1/3}\,\sigma_9$$

$$\sigma_8 = 9\,\sigma_{10}{}^{2/3}\,\sigma_9$$

$$\sigma_9 = \sqrt{\frac{12\,\sigma_{15}}{R}+\sigma_{13}{}^2-\frac{9\,(1000\,R+1000)^4}{64\,R^4}+9\,\sigma_{10}{}^{2/3}-6\,\sigma_{13}\,\sigma_{10}{}^{1/3}-\frac{3\,(1000\,R+1000)\,\sigma_{18}}{R^2}+\frac{3\,\sigma_{16}\,\sigma_{17}}{4\,R^3}}$$

$$\sigma_{10} = \frac{\sqrt{3}\,\sigma_{11}}{18}+\frac{\sigma_{13}{}^3}{27}+\frac{\sigma_{14}{}^2}{2}-\frac{4\,\sigma_{13}\,\sigma_{12}}{3}$$

$$\sigma_{11} = \sqrt{4\,\sigma_{13}{}^3\,\sigma_{14}{}^2-16\,\sigma_{13}{}^4\,\sigma_{12}+27\,\sigma_{14}{}^4+128\,\sigma_{13}{}^2\,\sigma_{12}{}^2-256\,\sigma_{12}{}^3-144\,\sigma_{13}\,\sigma_{14}{}^2\,\sigma_{12}}$$

$$\sigma_{12} = \frac{\sigma_{15}}{R}-\frac{3\,(1000\,R+1000)^4}{256\,R^4}-\frac{(1000\,R+1000)\,\sigma_{18}}{4\,R^2}+\frac{\sigma_{16}\,\sigma_{17}}{16\,R^3}$$

$$\sigma_{13} = \frac{\sigma_{17}}{R}-\frac{3\,\sigma_{16}}{8\,R^2}$$

$$\sigma_{14} = \frac{\sigma_{18}}{R}+\frac{(1000\,R+1000)^3}{8\,R^3}-\frac{(1000\,R+1000)\,\sigma_{17}}{2\,R^2}$$

$$\sigma_{15} = 25000000000\,D_1+5000000000\,D_1\,R$$

$$\sigma_{16} = (1000\,R+1000)^2$$

$$\sigma_{17} = 50000\,R+125000\,D_1\,R+1000000$$

$$\sigma_{18} = 25000000\,D_1+125000000\,D_1\,R+50000000$$

"0" is not in LHP, which means that mode is not asymptotically stable. Through PBH-Test, the rank of the matrix $\begin{pmatrix} A-\lambda I \\ C_2 \end{pmatrix}$ is 4, this system (for case2) is not observable either, thus this system is not detectable.

4 Question c

In this question, R and D_1 are set to 1 and 20 respectively. Using the standard Matlab routines the new form of controllability matrix con and observability matrices $ob1$ and $ob2$ are:

```
con = 1.0e+17 *

0       0       0.0000   0        -0.0000   0        -0.0000    0          0.0003 ...
       0.0005
0       0       0        0         0        0.0000    0.0000   -0.0000   -0.0001 ...
       0.0001
0       0.0000  0       -0.0000    0        0.0000    0        -0.0000    0.0001 ...
       0.0045
0.0000  0      -0.0000   0        -0.0000   0         0.0003    0.0005    0        ...
    -1.0000
0       0       0        0.0000    0.0000  -0.0000   -0.0001    0.0001   -0.0775   ...
    -0.1000

ob1 = 1.0e+15 *

0         0.0000    0         0         0
0         0         0         0         0.0000
0         0         0         0         0.0000
0.0000   -0.0000    0.0000    0         0
0.0000   -0.0000    0.0000    0         0
0         0.0000   -0.0000    0.0000   -0.0000
0         0.0000   -0.0000    0.0000   -0.0000
-0.0013   0.0013    0.0000   -0.0000    0.0000
-0.0013   0.0013    0.0000   -0.0000    0.0000
1.0250   -1.0050   -0.0200   -0.0008    0.0013

ob2 = 1.0e+15 *

0         0         0        -0.0000    0
0         0.0000   -0.0000    0         0
0.0000   -0.0000    0         0.0000    0
0        -0.0000    0.0000    0         0.0000
-0.0000   0.0000    0         0.0000   -0.0000
0.0000    0.0000   -0.0000    0        -0.0000
-0.0003   0.0003   -0.0000   -0.0000    0.0000
-0.0005  -0.0004    0.0009    0.0000    0.0000
0.7000   -0.7200    0.0200    0         0.0003
-0.7750   1.6525   -0.8775   -0.0010   -0.0004
```

And all rank of these three matrices are 4, therefore these two systems are neither controllable nor observable. The condition number of the matrix can be expressed as:

$$\kappa(a) = \frac{\sigma_{max}(A)}{\sigma_{min}(A)} \tag{6}$$

where $\sigma_{max(A)}$ and $\sigma_{min}(A)$ are maximal and minimal singular values of matrix A respectively. Then the condition number can be computed:

```
cond_con = 1.019769e+15
cond_ob1 = 3.064639e+15
cond_ob2 = 1.367984e+16
```

These large values also indicate that it is extremely difficult to control or observe both of these two systems in some mode. The eigenvalues of the system are:

$$\begin{pmatrix} 0 & -\sigma_3 - \sigma_2 - 500 & \sigma_2 - \sigma_3 - 500 & \sigma_3 + \sigma_1 - 500 & \sigma_3 - \sigma_1 - 500 \end{pmatrix}$$

where

$$\begin{aligned}
\sigma_1 &= \frac{\sqrt{-1502500000000\,\sqrt{\sigma_4} - 24600000\,\sigma_5{}^{1/3}\,\sqrt{\sigma_4} - 9\,\sigma_5{}^{2/3}\,\sqrt{\sigma_4} - 7500000000000000\,\sqrt{6}\,\sqrt{10}\,\sqrt{6\,\sqrt{3}\,\sqrt{482541790} + 228761}}}{6\,\sigma_5{}^{1/6}\,\sigma_4{}^{1/4}} \\
\sigma_2 &= \frac{\sqrt{7500000000000000\,\sqrt{6}\,\sqrt{10}\,\sqrt{6\,\sqrt{3}\,\sqrt{482541790} + 228761} - 24600000\,\sigma_5{}^{1/3}\,\sqrt{\sigma_4} - 9\,\sigma_5{}^{2/3}\,\sqrt{\sigma_4} - 1502500000000\,\sqrt{\sigma_4}}}{6\,\sigma_5{}^{1/6}\,\sigma_4{}^{1/4}} \\
\sigma_3 &= \frac{\sqrt{\sigma_4}}{6\,\sigma_5{}^{1/6}} \\
\sigma_4 &= 9\,\sigma_5{}^{2/3} - 12300000\,\sigma_5{}^{1/3} + 1502500000000 \\
\sigma_5 &= \frac{250000000000000\,\sqrt{3}\,\sqrt{482541790}}{9} + \frac{2859512500000000000000}{27}
\end{aligned} \tag{7}$$

Take "0" to PHB-Test, the rank of the matrix $\begin{pmatrix} A - \lambda I & B \end{pmatrix}$ is 5 (full rank), meaning that the only not stable mode is controllable, hence it is stabalizable; The rank of the matrix $\begin{pmatrix} A - \lambda I \\ C_1 \end{pmatrix}$ is 5 (full rank), meaning that the only not stable mode is observable, hence it is detectable; The rank of the matrix $\begin{pmatrix} A - \lambda I \\ C_2 \end{pmatrix}$ is 4 (rank deficient), meaning that it is undetectable.

5 d

Set sampling time $T_s = 0.001$, using Matlab function *c2d* to generate discrete time system function:

```
Ts = 1e-03;
A = double(expm(A * Ts))
```

The matrix A_d is what question d asks:

$$A_d = \begin{bmatrix} 0.4010 & 0.5964 & 0.0026 & 0.0004 & 0.0002 \\ 0.2076 & 0.7749 & 0.0175 & 0.0001 & 0.0009 \\ 0.0585 & 0.5686 & 0.3729 & 0.0000 & 0.0004 \\ -782.8328 & 773.7475 & 9.0853 & -0.0489 & 0.5964 \\ 342.5299 & -370.9589 & 28.4290 & 0.1491 & 0.7749 \end{bmatrix} \tag{8}$$

6 e

```
syms s t
matrix = inv(eye(size(A))*s-A)
exp_A = vpa(ilaplace(matrix))
Bd1 = vpa(int(exp_A,t,0,1e-3)*B,5)
```

$$B_{d1} = \begin{bmatrix} 0.0031544 & 0.00028574 \\ 0.00015945 & 0.0032115 \\ 0.000028574 & 0.31676 \\ 4.4993 & 1.3088 \\ 0.58514 & 8.7634 \end{bmatrix} \tag{9}$$

Verifying the result using c2d function:

```
sys2 = ss(A,B,C2,D2);
case2_d = c2d(sys2,Ts);
[Ad2,Bd2,Cd2,Dd2] = ssdata(case2_d);
```

$$B_{d2} = \begin{bmatrix} 0.0032 & 0.0003 \\ 0.0002 & 0.0032 \\ 0.0000 & 0.3168 \\ 4.4993 & 1.3088 \\ 0.5851 & 8.7634 \end{bmatrix} \tag{10}$$

7 f

The minimal systems require to be both controllable and observable. According to part a, for both system c_1 and c_2, they are always controllable.So by judging the observable matrix, we can know if the system is minimal.

```
sys1 = ss(A,B,C1,D1);
case1_d = c2d(sys1,Ts);
[Ad,Bd,Cd1,Dd1]= ssdata(case1_d)

sys2 = ss(A,B,C2,D2);
case2_d = c2d(sys2,Ts);
[Ad2,Bd2,Cd2,Dd2] = ssdata(case2_d);

M_dob1 = rref(obsv(case1_d))
rank(M_dob1)
M_dob2 = rref(obsv(case2_d))
rank(M_dob2)
```

From the output, we know C_1 is a full rank(rank=5) observability matrix, which is observable, while C_2 is not a full rank observability matrix(rank=4), which is unobservable. This leads to the conclusion: C_1 is a minimal system, and C_2 is not.
Then analyzing stability:

```
% check if its eigenvalues are in the region of stability
eig_Ad = eig(Ad)
```

The result is :

$$eig_{Ad} = \begin{bmatrix} 0.0619 + 0.6733i \\ 0.0619 - 0.6733i \\ 1.0000 + 0.0000i \\ 0.7627 + 0.0000i \\ 0.3881 + 0.0000i \end{bmatrix} \tag{11}$$

If all eigenvalues are inside the unit circle, the system is stable. In this case, there is an eigenvalue $1.0000 + 0.0000i$ that numerically lies on the unit circle. So the system is margin stable.