SSY281 Model Predictive Control
Assignment 6 – MPC stability

Instructions

The assignments comprise an important part of the examination in this course. Hence, it is important to comply with the following rules and instructions:

- The assignment is pursued and reported individually.
- The findings from each assignment are described in a short report, written by each student independently.
- The report should provide clear and concise answers to the questions, including your motivations, explanations, observations from simulations, etc. Conclusions should be supported by relevant results if applicable; e.g., the system is stable since the eigenvalues, $[0.5\ , 0.2+0.5j\ , 0.2-0.5j]$, are inside the unit circle. Figures included in the report should have legends, should be readable, should have proper scaling to illustrate the relevant information, and axes should be labeled. Try to verify your solutions if possible; e.g., plot the inputs and outputs and see whether they respect the constraints.
- Since the assignments are part of the examination in the course, plagiarism is of course not allowed. If we observe that this happens anyway, it will be reported.
- The report should be uploaded to Canvas *before the deadline*. A report uploaded a second or a day after the deadline are penalized equally. Name the report as A6.pdf.
- A MATLAB code should be uploaded which reproduces all numbers and figures in your report. Make sure that one can run your code and see your results without any error. Name the MATLAB script as A6.m.

Table 1: Points per question

Question:	1	2	3	4	Total
Points:	5	6	1	3	15

1. **Lyapunov functions**
Consider the discrete-time system

$$x(k+1) = Ax(k) + Bu(k), \tag{1}$$

where

$$A = \begin{bmatrix} 1.0041 & 0.0100 & 0 & 0 \\ 0.8281 & 1.0041 & 0 & -0.0093 \\ 0.0002 & 0.0000 & 1 & 0.0098 \\ 0.0491 & 0.0002 & 0 & 0.9629 \end{bmatrix}, \quad B = \begin{bmatrix} 0.0007 \\ 0.1398 \\ 0.0028 \\ 0.5605 \end{bmatrix}.$$

(a) [2p] Assume $u(k) = 0$ and consider the Lyapunov function $V(x) = x^\top Sx$ and the Lyapunov equation $A^\top SA - S = -Q$. Assume $S = I_4$ and find Q; what can you say about stability of system (1) given these S and Q matrices? Find the eigenvalues of A and argue whether the system is stable or not.

(b) [2p] Consider $u(k) = -Kx(k)$ where

$$K = \begin{bmatrix} 114.3879 & 12.7189 & -1.2779 & -1.5952 \end{bmatrix},$$

and $S = I_4$. Find Q for the closed-loop system; what can you say about the stability given these S and Q matrices?

(c) [1p] Consider $u(k) = -Kx(k)$ as in the previous part and choose $Q = I_4$. Find S and use it to argue whether the closed-loop system is stable.

Note: one may use `eig(A)` and `dlyap(A',Q)` (note the transpose!) in MATLAB.

2. **Stability with receding horizon control**
Consider an RHC applied to the system (1), based on minimizing the cost function

$$V_N(x) = x(N)^\top P_f x(N) + \sum_{i=0}^{N-1} \big(x(i)^\top Qx(i) + u(i)^\top Ru(i)\big), \tag{2}$$

with the parameters

$$Q = I_4, \quad R = 1. \tag{3}$$

(a) [1p] What is the effect of Q on the stability when $N = 1$?

(b) [1p] With $P_f = Q$, find the shortest N and the corresponding feedback gain K, such that the RH controller stabilizes the system.

(c) [2p] Verify whether $V_N(x)$ is a Lyapunov function for the closed-loop system with the feedback gain in b) and interpret the result.

(d) [2p] Find a P_f for which the system is stable for any $N \geq 1$. Consider this Pf and $N = 1$, $Q = I_4$, $R = 1$ and find the corresponding feedback gain K that minimizes the cost function $V_N(x)$. Verify whether $V_N(x)$ is a Lyapunov function for the new closed-loop system.

3. [1p] **Receding horizon controller example**
Consider a system described as (1) with

$$A = \begin{bmatrix} \bar{A}_{(n-1)\times n} \\ 0_{1\times n} \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix},$$

where $\bar{A}$ is an arbitrary matrix. Assume that the open-loop system is *unstable* but *controllable.*

Show that the receding horizon controller obtained by minimizing the cost function (2) with $N = 1$ and

$$P_f = \operatorname{diag}(\begin{bmatrix} p_1 & p_2 & \dots & p_n \end{bmatrix}), \quad p_1, \dots, p_n > 0,$$

cannot stabilize the system, regardless of Q and R.

4. [3p] **Stability for constrained systems**

Consider the system
$$x(k+1) = Ax(k) + Bu(k),$$
with
$$A = \begin{bmatrix} 2 & 0.1 \\ 0 & 1.5 \end{bmatrix}, \ B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \ Q = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \ R = 100, \ N = 2.$$
Furthermore, consider $|u(t)| \leq 1$ as the input constraint and
$$\begin{bmatrix} -0.5 \\ -0.5 \end{bmatrix} \leq x(N) \leq \begin{bmatrix} 0.5 \\ 0.5 \end{bmatrix}$$
as the terminal constraint. Find the largest control invariant set as the system's terminal set, $\mathcal{X}_f$, and it's N-step backward reachable set, $\mathcal{X}_0$. Consider $x(0)^\top = \begin{bmatrix} -0.1 & 1.3 \end{bmatrix}$ and find the closed-loop trajectories using MPC when the terminal penalty is

- $P_f = I_2$
- P_f as the Riccati solution

Plot X_0, X_f, and the closed-loop trajectories for the above cases in the same plot and discuss the feasibility and the stability of the closed-loop system in each case.

Note: In your plot, sketch transparent sets so the state trajectories could be observed clearly. You may select 'alpha' in your plot to tune the transparency level, `plot(X,'alpha',0.1)` for instance.