HA1

SSY345 Sensor fusion and nonlinear filtering
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1a (i)

$$E[x] = \mu$$

$$E[x] = \int xp(x)dx$$

$$p(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-(\frac{1}{2}(x-\mu)\sigma^2(x-\mu))}$$

$$E[x] = \int x \frac{1}{\sigma\sqrt{2\pi}} e^{-(\frac{1}{2}(x-\mu)\sigma^2(x-\mu))} dx$$

Substitute with the following two equations:

$$t = \frac{x-\mu}{\sqrt{2}\sigma}$$

$$x = \sqrt{2}\sigma t + \mu$$

Multiplicate out all possible constants and solve:

$$E[x] = \frac{\sqrt{2}\sigma}{\sigma\sqrt{2\pi}} \int (\sqrt{2}\sigma t + \mu) e^{-t^2} dt$$

$$E[x] = \frac{1}{\sqrt{\pi}} \left(\int \sqrt{2}\sigma t e^{-t^2} dt + \int \mu e^{-t^2} dt \right)$$

$$\int_{-\infty}^{\infty} e^{-at} dt = \frac{\sqrt{t}}{\sqrt{a}}$$

$$E[x] = \frac{1}{\sqrt{\pi}} \left(\int \sqrt{2}\sigma t e^{-t^2} dt + \mu\sqrt{\pi} \right)$$

$$E[x] = \frac{1}{\sqrt{\pi}} \left(\sqrt{2}\sigma \left[e^{-t^2} - \frac{te^{-t^2}}{2t} \right]_{-\infty}^{\infty} + \mu\sqrt{\pi} \right)$$

$$E[x] = \frac{1}{\sqrt{\pi}} \left(\sqrt{2}\sigma \underbrace{\left[-\frac{1}{2} e^{-t^2} \right]_{-\infty}^{\infty}}_{=0} + \mu\sqrt{\pi} \right)$$

$$E[x] = \frac{\sqrt{\pi}\mu}{\sqrt{\pi}} = \mu$$

1a (ii)

$$E\left[(x-\mu)^2\right] = \sigma^2$$

$$E\left[(x-\mu)^2\right] = E\left[x^2\right] + E\left[-2x\mu\right] + E\left[\mu^2\right]$$

$$E\left[(x-\mu)^2\right] = E\left[x^2\right] - 2\mu E\left[x\right] + \mu^2$$

$$E\left[(x-\mu)^2\right] = E\left[x^2\right] - 2\mu^2 + \mu^2$$

$$E\left[(x-\mu)^2\right] = E\left[x^2\right] - \mu^2 = VAR\left[x\right]$$

1b (i)

$$E\left[x\right] = \int xp(x)$$

A constant can be taken out of a integral:

$$A\int xp(x) = \int Axp(x)$$

A constant won't change the probability because it just movesthe probability to Ax instead of x.

$$p(Ax) = p(x)$$

1b (ii)

$$Cov\left[x\right] = E\left[(x-E\left[x\right])(x-E\left[x\right])^\top\right]$$

$$Cov\left[Ax\right] = E\left[(Ax-E\left[Ax\right])(Ax-E\left[Ax\right])^\top\right]$$

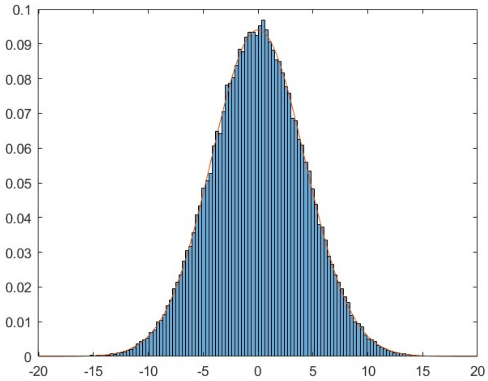
$$Cov\left[Ax\right] = E\left[(Ax-AE\left[x\right])(Ax-AE\left[x\right])^\top\right]$$

$$Cov\left[Ax\right] = A\underbrace{E\left[(x-E\left[x\right])(x-E\left[x\right])^\top\right]}_{Cov\left[x\right]}A^\top$$

1c

As seen in the equations from 1b it can be seen that A effects the mean by Ax and in this case A is not the Identity matrix and will therefore effect the mean. The same goes for the the covariance but it is effected by ACov(x)A' and as said before the covariance matrix will stay the same if A where a identity matrix. If A has zeroes on the off diagonal it won't change the shape of the graph becuase it won't effect the covariance. May only the variance.

2a



Figur 1: Distribution of z

As can be seen in the figure they are almost the same. Red is analytically and the histogram is the approx. When trying different values it can be seen that the histogram follows better the larger the N is. This make sense because the larger the N is the more it will converge to the statistical representation.

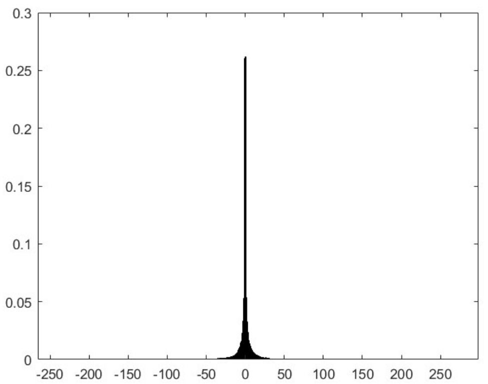
$mu_y = 0$

$var_y = 18$

$mu_{yapprox} = -0.0270$

$var_{yapprox} = 18.0399$

2b



Figur 2: Distribution of z

It cannot be calculated analytically and therefore it cannot be compared.

2c

It can be seen that when doing a linear transformation it can be calculated analytically and it will have the same shape however that is not the case for the non-linear.

3a

No, it cannot be done because the distribution of x effects the distribution of y and the distribution of both are unknown.

3b

Yes, it can be calculated because x is a constant and H is deterministic this means that H(x) can be seen as a constant. Which means that y becomes normally distributed around the mean H(x) + mean of r (=0) with a standard deviation of σ_r

3c

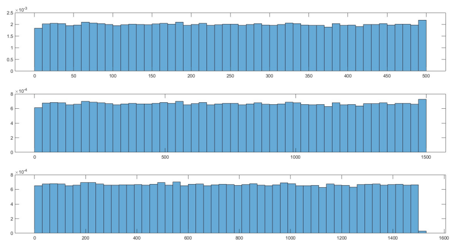
It is the same as the two above. The linearity does not effect them because it is still an deterministic function. Which means that P(y) is not possible and $P(y|x)$ is possible.

3d

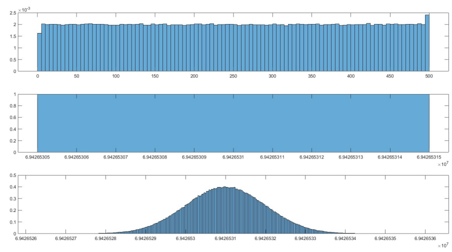
Now both are possible because there are information about x and Hx. Thereby the distribution of H(x) can always be calculated which means that the distribution y always can be calculated.

3e

a



(a) Linear

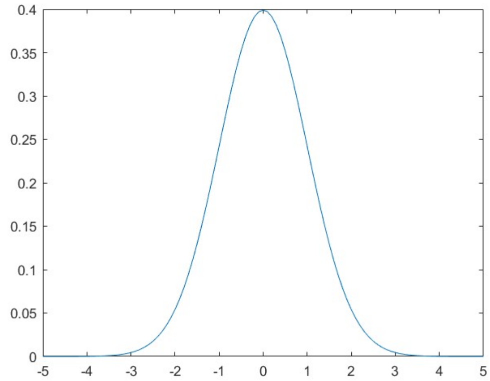


(b) Non-linear

Figur 3: Figure 1 = P(x), Figure 2 = P(hx), Figure 3 = P(y)

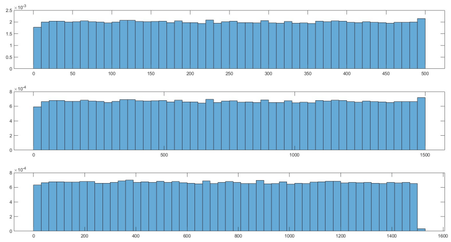
The non linear is Gaussian just because I choose x^3 as H(x). It will still not make it possible to calculate p(y)

b

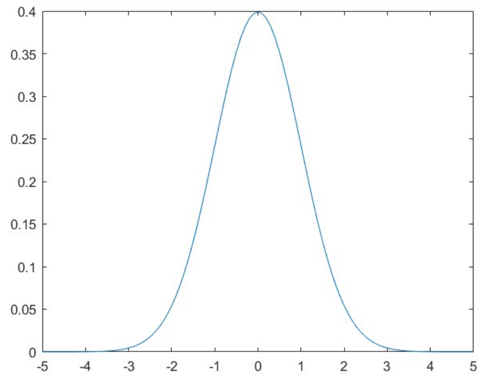


Figur 4: Distribution of y when x = 0 ($P(y|x = 0)$)

c

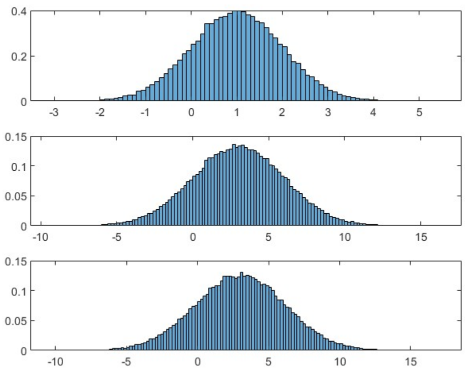


(a) Figure 1 = P(x), Figure 2 = P(hx), Figure 3 = P(y)

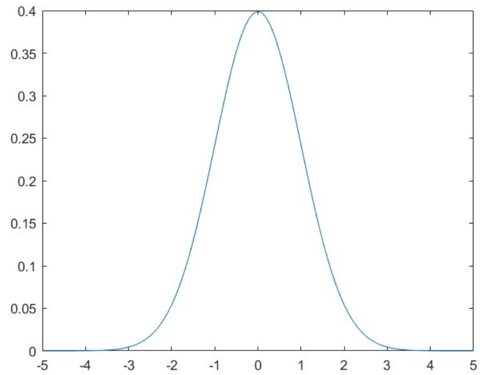


(b) Distribution of y when x = 0 ($P(y|x = 0)$)

d

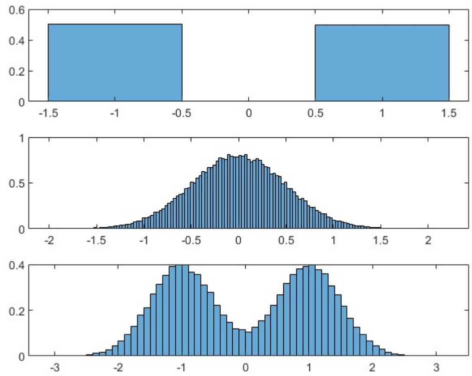


(a) Figure 1 = P(x), Figure 2 = P(hx), Figure 3 = P(y)



(b) Distribution of y when x = 0 ($P(y|x = 0)$)

4a



Figur 7: Figure 1 = P(x), Figure 2 = P(w), Figure 3 = P(y)

It is two Gaussian distributions with two different means. The reason why it looks like that is because x can only be two different values and w is Gaussian distributed.

4b

I would have guessed $\theta = 1$ because w is Gaussian distributed and it is more likely that w = -0.3 than w = 1.7. This can be confirmed by looking at the graphs from 4a.

4c

$$p(y) = \int p(y|\theta)p(\theta)$$

$$p(\theta) = 0.5(\delta(\theta - 1) + \delta(\theta + 1))$$

$$p(y|\theta) = \frac{e^{\frac{-(y-\theta)^2}{2\sigma^2}}}{\sigma\sqrt{2\pi}}$$

$$p(y) = \int \frac{e^{\frac{-(y-\theta)^2}{2\sigma^2}}}{\sigma\sqrt{2\pi}} 0.5(\delta(\theta - 1) + \delta(\theta + 1))$$

$$p(y) = 0.5\frac{e^{\frac{-(y-1)^2}{2\sigma^2}}}{\sigma\sqrt{2\pi}} + 0.5\frac{e^{\frac{-(y+1)^2}{2\sigma^2}}}{\sigma\sqrt{2\pi}}$$

4d

$$p(\theta|y) = \frac{p(y|\theta)p(\theta)}{p(y)}$$

For $\theta = 1$

$$p(\theta|y) = \frac{0.5\frac{e^{\frac{-(y-1)^2}{2\sigma^2}}}{\sigma\sqrt{2\pi}}}{0.5\frac{e^{\frac{-(y-1)^2}{2\sigma^2}}}{\sigma\sqrt{2\pi}} + 0.5\frac{e^{\frac{-(y+1)^2}{2\sigma^2}}}{\sigma\sqrt{2\pi}}}$$

For $\theta = -1$

$$p(\theta|y) = \frac{0.5\frac{e^{\frac{-(y+1)^2}{2\sigma^2}}}{\sigma\sqrt{2\pi}}}{0.5\frac{e^{\frac{-(y-1)^2}{2\sigma^2}}}{\sigma\sqrt{2\pi}} + 0.5\frac{e^{\frac{-(y+1)^2}{2\sigma^2}}}{\sigma\sqrt{2\pi}}}$$

4e

$$\hat{\theta}_{MMSE} = \sum \theta p(\theta|y)$$

For $\theta = 1$

$$p(\theta|y) = \frac{0.5\frac{e^{\frac{-(y-1)^2}{2\sigma^2}}}{\sigma\sqrt{2\pi}}}{0.5\frac{e^{\frac{-(y-1)^2}{2\sigma^2}}}{\sigma\sqrt{2\pi}} + 0.5\frac{e^{\frac{-(y+1)^2}{2\sigma^2}}}{\sigma\sqrt{2\pi}}}$$

For $\theta = -1$

$$p(\theta|y) = \frac{0.5\frac{e^{\frac{-(y+1)^2}{2\sigma^2}}}{\sigma\sqrt{2\pi}}}{0.5\frac{e^{\frac{-(y-1)^2}{2\sigma^2}}}{\sigma\sqrt{2\pi}} + 0.5\frac{e^{\frac{-(y+1)^2}{2\sigma^2}}}{\sigma\sqrt{2\pi}}}$$

$$\hat{\theta}_{MMSE} = \frac{0.5\frac{e^{\frac{-(y-1)^2}{2\sigma^2}}}{\sigma\sqrt{2\pi}}}{0.5\frac{e^{\frac{-(y-1)^2}{2\sigma^2}}}{\sigma\sqrt{2\pi}} + 0.5\frac{e^{\frac{-(y+1)^2}{2\sigma^2}}}{\sigma\sqrt{2\pi}}} - \frac{0.5\frac{e^{\frac{-(y+1)^2}{2\sigma^2}}}{\sigma\sqrt{2\pi}}}{0.5\frac{e^{\frac{-(y-1)^2}{2\sigma^2}}}{\sigma\sqrt{2\pi}} + 0.5\frac{e^{\frac{-(y+1)^2}{2\sigma^2}}}{\sigma\sqrt{2\pi}}}$$

$$\hat{\theta}_{MMSE} = \frac{e^{\frac{-(y-1)^2}{2\sigma^2}} - e^{\frac{-(y+1)^2}{2\sigma^2}}}{e^{\frac{-(y-1)^2}{2\sigma^2}} + e^{\frac{-(y+1)^2}{2\sigma^2}}}$$