New Trends in Financial Mathematics

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Abstract

Recently, cryptocurrencies have been more and more popular and essential in financial market. And the global cryptocurrency market has surpassed US$3 trillion by the end of 2021, with Bitcoin accounting for 41.5% and Ethereum 18.7% of the total. Here, we want to restore the mean field game model, which was built by Zongxi Li, A. Max Reppen and Ronnie Sircar, in *A Mean Field Games Model for Cryptocurrency Mining,* to study the question of how centralization of reward and computational power occur in cryptocurrencies. Miners compete against each other for mining rewards by increasing their computational power. This leads to a novel mean field game of jump intensity control, which we solve explicitly for miners maximizing exponential utility, and handle numerically in the case of miners with power utilities. This paper will present generalizations of the mean field model of Li et al.'s paper and details that are not been shown in that paper such as implicit method process. Due to different number of miners and value of rewards, the result of equilibrium may show various characters which have some differences with their paper.

1. Introduction

In the Bitcoin network, miners follow proof-of-work protocol and solve math puzzles, which are designed as no better way of solving them than brute force calculation. In other words, the probability of solving one puzzle is proportional to the computational power the miner can provide. And once a miner gets a solution, the corresponding block is added on top of the blockchain and the miner receives the reward. Moreover, the difficulty of the puzzle varies to maintain a consistent solving time, for example 10 minutes. To be specific, if miners can solve the problem in 8 minutes, the system will make the problem harder so that the average time goes back to 10 minutes. In summary, the two important properties are that (1) the probability of obtaining the next reward is proportional to computational efforts and (2) the blocks and their rewards appear with a fixed average frequency (Li, Reppen & Sircar, 2019).

Bitcoin is a payment system maintained by a peer-to-peer network. Miners record transactions on the blockchain and achieve the decentralization of the payment system. Kondor, Pósfai, Csabai, and Vattay (2014) presented that the accumulation of bitcoins tends to occur among a small number of miners, which suggests centralization in the market.

The mean-field framework was developed to study systems with an infinite number of rational agents in competition, which arise naturally in many applications (Gomes & Saúde, 2013). Li et al.'s paper quantifies the competition between miners by adopting a tractable mean field games approach. The idea is that, with many participants, any particular player has little impact on any other player when they interact through the mean of their actions. As the number of players grows, one can first view an individual's decision-making problem as being against a mean field competition, knowing that their individual contribution to the mean field is infinitesimally small. The final step is a fixed-point condition in which the mean optimal control should coincide with the aforementioned mean field. This approach leads to a computationally tractable model, which is intuitively a good approximation to the finite player game, precisely because each player’s impact dissipates in the mean of many players.

In this paper, we will make a generalization of Li et al.’s paper and mainly focus on the power utility function of constant relative risk aversion (CRRA). Based on this function, we will give as much details as we can to restore their model. Especially, we will provide detailed implicit method processes that are needed in numerical method part. What’s more, we will discuss more about results with different parameters in the mining pool like different number of miners and various value of mining rewards.

2. Methods

To study the optimal strategy of miners in mining games, we formalize a mean field game model, where each miner is characterized by its wealth, and chooses its hash rate to maximize expected utility at a fixed time horizon. Miners' wealth changes because of the mining expenses and rewards. The instantaneous probability of receiving the reward is given by the probability of producing the next block:

$$\frac{\text{pl.}i'\text{s hash rate}}{\text{total hash rate}} = \frac{\text{pl.}i'\text{s hash rate}}{\text{\#players} \times \text{mean huash rate}} \approx \frac{\text{pl.}i'\text{s hash rate}}{\text{pl.}i'\text{s hash rat} \quad (\text{\#players}-1)\times\text{mean hash rate}} \quad ,$$

where the last expression is a good approximation when the number of players is large. Let $M = (\#\text{players} - 1)$, which is assumed to be large. In order to utilize computational advantages of mean field games technology, our model replaces the second term in the denominator by $(M \times \text{continuum mean hash rate})$. The purpose of using a continuum mean field games model instead of a finite player model is that it significantly reduces the mathematical complexity.

General structure of the mining problem

Here we illustrate the general structure of the mining problem. Consider a continuum of miners in Bitcoin mining over some finite time period $[t_0, T]$. Miners characterized by their wealth $x \in \mathbb{R}$, distributed at time t_0 according to an initial density function m_0. The representative miner provides hash rate α_t, incurring a linear cost per unit of time $c\alpha_t$, where $c > 0$ is interpreted as the cost of electricity, for simplicity, and is thus proportional to their hash rates. Since one of the important features of the Bitcoin protocol: the system always generates a reward on an almost fixed frequency that does not depend on the total hash rate, it is reasonable to model the total number of rewards in the system as a whole as a Poisson process with a constant intensity $K > 0$. The number of rewards each miner can receive is modelled by a counting process $N = (N_t)_{t \geq t_0}$ with a jump intensity $\lambda = (\lambda_t)_{t \geq t_0} > 0$. Let $M + 1$ be the total number of miners and $\bar{\alpha}_t \geq 0$ denote the mean hash rate across all miners. Hence, the reward intensity of an individual is

$$\lambda_t := \begin{cases} K \dfrac{\alpha_t}{(\alpha_t + M\bar{\alpha}_t)}, & \alpha_t > 0, \\ 0 & \alpha_t = 0. \end{cases}$$

Miners have initial wealth x distributed at time t_0 according to an initial density m_0. Then, an individual miner's wealth process $X = (X_t)_{t \geq t_0}$ follows

$$dX_t = -c\alpha_t \, dt + r \, dN_t,$$

where r is the value of the mining reward.

The miners' optimization problem

Afterwards, we discuss the miners' optimization problem. The setup of the continuum model is to consider the optimization problem of an individual miner, in response to any given action of the rest miners. Let $\alpha = (\alpha_t)_{t \geq t_0}$ be a Markovian control. The process α can then be associated with a function $(t, X_t) \mapsto \alpha(t, X_t; \bar{\alpha})$ of the current state. With such control, the wealth process X is a Markov process. The objective of the representative miner is to maximize the expected utility at fixed terminal time T. Assume the utility function U is strictly increasing and concave. The miner's value function is

$$v(t_0, x; \bar{\alpha}) = \sup_{\alpha \geq 0} \mathbb{E}\left[U(X_T) \middle| X_{t_0} = x\right].$$

In other words, given a fixed a mean hash rate $\bar{\alpha}$, a miner, with an initial wealth x at t_0, aims to maximize the expected value of his own utility at the ending time.

For a fixed mean hash rate $\bar{\alpha} > 0$, we first write down the HJB

$$\partial_t v + \sup_{\alpha \geq 0} \left(-c\alpha \partial_x v + \frac{K\alpha}{\alpha + M\bar{\alpha}_t} \Delta v \right) = 0,$$

with terminal condition $v(T, x) = U(x)$, and where

$$\Delta v = v(t, x + r; \bar{\alpha}) - v(t, x; \bar{\alpha}).$$

We assume that v is a classical solution of HJB. Then, for any time $t \in [t_0, T]$ and fixed $\bar{\alpha} > 0$, the value function $v(t, x; \bar{\alpha})$ is finite and strictly increasing in wealth x. $\Delta v > 0$ and $\partial_x v > 0$, and so the optimal hash rate is given by

$$\alpha^*(t, x; \bar{\alpha}) = \begin{cases} -M\bar{\alpha}_t + \sqrt{\dfrac{KM\bar{\alpha}_t \Delta v(t, x; \bar{\alpha})}{c\partial_x v(t, x; \bar{\alpha})}}, & \text{if } \bar{\alpha}_t < \dfrac{K\Delta v(t, x; \bar{\alpha})}{Mc\partial_x v(t, x; \bar{\alpha})}, \\ 0 & \text{otherwise.} \end{cases}$$

Equilibrium characterization

Now we look for a Markovian equilibrium of the mining game. Let $m(t, x; \bar{\alpha})$ be the resulting density of miners' wealth as a function of time and wealth. We say $\bar{\alpha}^*$ forms an equilibrium mean hush rate of mining game if

$$\bar{\alpha}_t^* = \int_{\mathbb{R}} \alpha^*(t, x; \bar{\alpha}^*) m(t, x; \bar{\alpha}^*)\, dx, \quad \forall t \in [t_0, T].$$

We assume that $\bar{\alpha}_t^* \neq 0$ for all t for the following reason. If $\bar{\alpha}_t^* = 0$, then each miner has an admissible control that dominates the choice of not mining. We assume the initial density $m_0(x)$ is continuously differentiable and satisfies

$$\int m_0(x)\, dx = 1.$$

The Fokker-Planck equation for m is given by

$$\partial_t m - \partial_x (c\alpha^*(t, x) m) - K \left(\frac{\alpha^*(t, x - r)\, m(t, x - r)}{\alpha^*(t, x - r) + M\bar{\alpha}_t^*} - \frac{\alpha^*(t, x)}{\alpha^*(t, x) + M\bar{\alpha}_t^*} m(t, x) \right) = 0,$$

with initial distribution $m(t_0, x) = m_0(x)$.

Numerical method

1. Initialize with a mean hash rate $t \mapsto \bar{\alpha}_t$, for instance as constant.

2. Solve for the value function v and the hash rate α^*:

   At time T, the value function is v known, so the optimal hash rate formula yields $\alpha^*(T, x; \bar{\alpha})$. This value is used as an approximation of $\alpha^*(T - dt, x\,; \bar{\alpha})$, which allows solving for v at $T - dt$, using the HJB equation:

$$\partial_t v + \left(-c\alpha^*(T, x; \bar{\alpha})\partial_x v + \frac{K\alpha^*(T, x; \bar{\alpha})}{\alpha^*(T, x; \bar{\alpha}) + M\bar{\alpha}_T} \Delta v \right) = 0.$$

   The Δv term is calculated explicitly using $v(T, x + r; \bar{\alpha}) - v(T, x; \bar{\alpha})$, while the other part is discretized by an implicit finite difference scheme. With the value function v at $T - dt$, we can get $\alpha^*(T - dt, x\,; \bar{\alpha})$. Repeat such time steps backwards until $t = 0$. This yields both functions v and α^*.

3. The next step is to solve the Fokker–Planck equation and get the mean field control. The $\alpha^*(t, x; \bar{\alpha})$ is obtained from the previous step allows solving Fokker–Planck equation for $m(t, x)$. In doing so, the following terms are discretized by an implicit finite difference scheme

$$\partial_t m - \partial_x(c\alpha^*(t, x)m) + \frac{K\alpha^*(t, x)}{\alpha^*(t, x) + M\bar{\alpha}_t} m\,,$$

   while m in

$$-\frac{K\alpha^*(t, x - r)}{\alpha^*(t, x - r) + M\bar{\alpha}_t} m(t, x - r)$$

   is evaluated in the previous time step.

4. Due to the large factor M scaling up the errors away from the equilibrium, so a direct

iteration will tend to overshoot. We therefore use an under-relaxation technique to reduce this effect. For some parameter $w \in [0,1]$ and for each time t, the relaxed iteration is given by

$$\bar{\alpha}_t^{new} = w\bar{\alpha}_t + (1-w)\int_{\mathbb{R}} \alpha^*(t,x;\bar{\alpha})m(t,x;\bar{\alpha})\,dx = w\bar{\alpha}_t + (1-w)\psi(\bar{\alpha})_t.$$

The choice of w has no impact on the equilibrium fixed point.

5. Finally, repeat from the first step with $\bar{\alpha} = \bar{\alpha}^{new}$, unless the difference is sufficiently small to terminate.

(The detailed implicit method processes are shown in Appendix)

3. Results

Here, we numerically solve for the equilibrium with liquidity constraints and for utility functions of constant relative risk aversion (CRRA), also known as power utility U, defined on $\mathbb{R} \geq 0$, namely

$$U(x) = \frac{1}{1-\gamma} x^{1-\gamma} \quad for\ \gamma \in (0,1), x > 0,$$

and admissible strategies α are such that the wealth process X remains positive. The HJB equation holds on $x > 0$ with boundary condition $V(t,0) = U(0) = 0.$ Then assume that the initial density m_0 has strictly positive support. This means all miners start with positive wealth and the problem is fully characterized on $\mathbb{R} \geq 0$.

In this problem, the value function cannot be found explicitly, so we need to solve it numerically. The Fokker–Planck equation has two parts to be considered to make sure that miners' wealth stays strictly positive. One is that the density m solves for $x \geq r$. On the other hand, if $0 < x < r$, there is no density at $x - r$ jumping to x, because there are no miners with negative wealth. So, for $0 < x < r$, the density m solves

$$\partial_t m - \partial_x (c\alpha^* m) + \frac{K\alpha^*}{\alpha^* + M\bar{\alpha}^*} m = 0\,.$$

The initial condition on all $x > 0$ is $m(t_0, x) = m_0(x)$.

In order to solve this problem, we need to set the parameters firstly and try to make all these parameters correctly and meaningfully be closed to the real life. In fact, the system will adjust the difficulty to make a reward available every 10 minutes on average. Li and his team (2019) find that, in reality, in the case of Bitcoin, this number is adjusted every 2016 blocks (about every two weeks). They make the simplifying assumption that this happens continuously in our model. Similarly, they also assume that also the miners' hash rates may change continuously. So, it is reasonable to model the total number of rewards in the system as a whole as a Poisson process with a constant intensity $K > 0$, where K^{-1} is approximately 10 minutes, which is $K^{-1} = 0.007$. Then they also assume the value of mining reward is $r = 3$. Even though the reward in reality has two part that, one is block reward set by the system as a fixed number per block and the second is transaction fees received from transactions included in the appended block, our focus is on the strategic decision of miners and the centralization in the competition so that they treat the reward as a constant. For simplification, we set the linear cost per hash rate $c = 2 \times 10^{-5}$, also the total number except one in the mining pool $M = 1000$ and the total time all miners competing in the mining pool $T = 90$. What's more, in the utility function, here we set $\gamma = 0.8$. In addition, we assume the initial distribution m_0 is normal distribution with mean 90 and standard deviation 5. Of course, all these parameters can be changed. And all the different results from changing parameters will be shown and illustrated later.

Figure 1 shows the distribution of the miners' wealth at $t = 30, 60, 90$ compared with the initial distribution m_0 (mean 90 and standard deviation 5). As time increases, the majority of the mass move to the left quickly and congregate near 0, most mass focusing on interval $x \in [0,40]$. At the terminal time, there is a small local peak at $x = 41$. This figure indicates that most miners lose their wealth rapidly, but only a small amount of miners may be lucky to keep some of their wealth around 41. Fatedly, most miners would have negative return in this situation. And also in Figure 2, we can directly see how fast the mass cumulated. In Matlab version, we could find out the cumulative density reaches to almost 1 at $x = 44$.

Most miners will lose their wealth to nearly zero in this competitive cryptocurrency mining game.

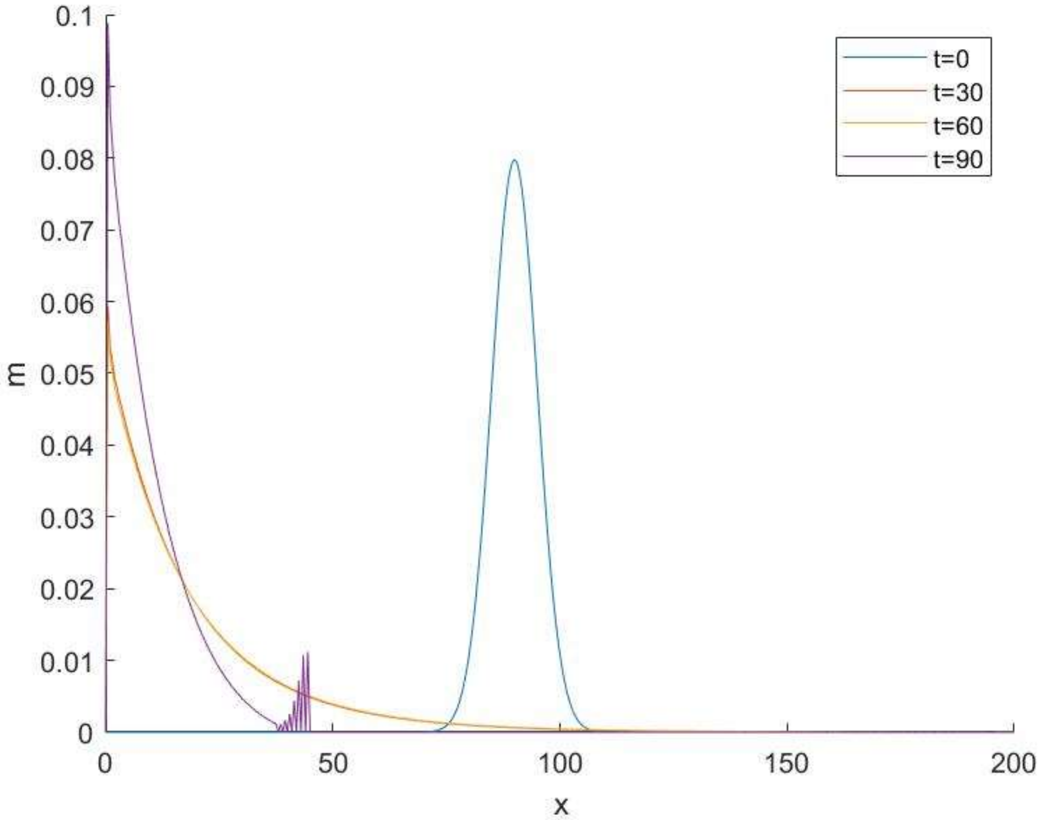

Figure 1: *The distribution of miners' wealth at different times. Parameters:* $K^{-1} = 0.007$, $r = 3$, $M = 1000$. *Note that as time pass by, the mass concentrate on 0. And at* $T = 90$, *small amount of mass congregates a small local peak at* $x = 44$.

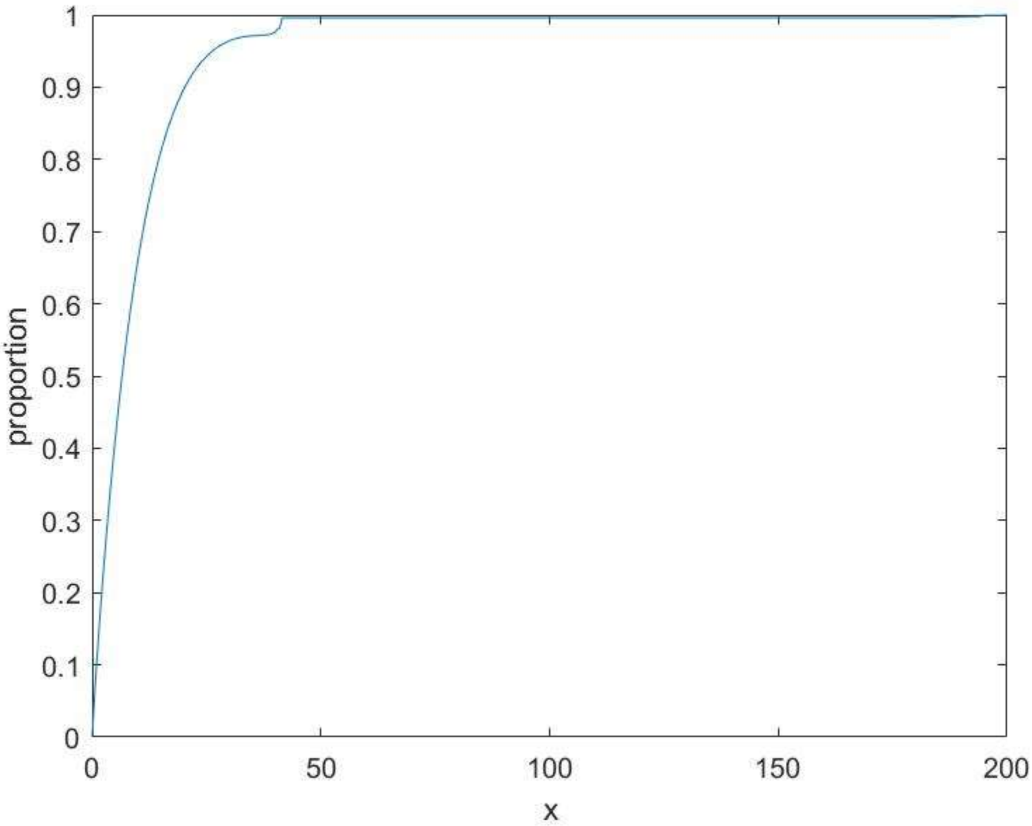

Figure 2: the cumulative proportion of miners at $t = 90$. Parameters are the same as Figure 1.

Then we will discuss more about different parameters situations to study the general phenomenon of mining competition. Different parameter r and M, which means various mining rewards and diverse number of participants in this game, may generate disparate results. And later, we will discuss situations about $r = 0.5,\ 1,\ 2$, $M = 1000,\ 2000,\ 3000, 10000$ and also $K = 1K, 2K, 3K, (10K)$.

Primarily, we discuss the details when $r = 0.5$. This is a more general case and it outputs more interesting results. As shown below, figure 3 presents that, when $r = 0.5$ and identical K, the mass density function has lower pick with larger number in the mining game. The means of these density functions may not change through miners' increase, but the variances of the density functions get larger and larger. The phenomenon indicates that, if more miners join the cryptocurrency mining game, the system will get bigger fluctuation, such that equal proportion of miners may get more wealth or lose wealth.

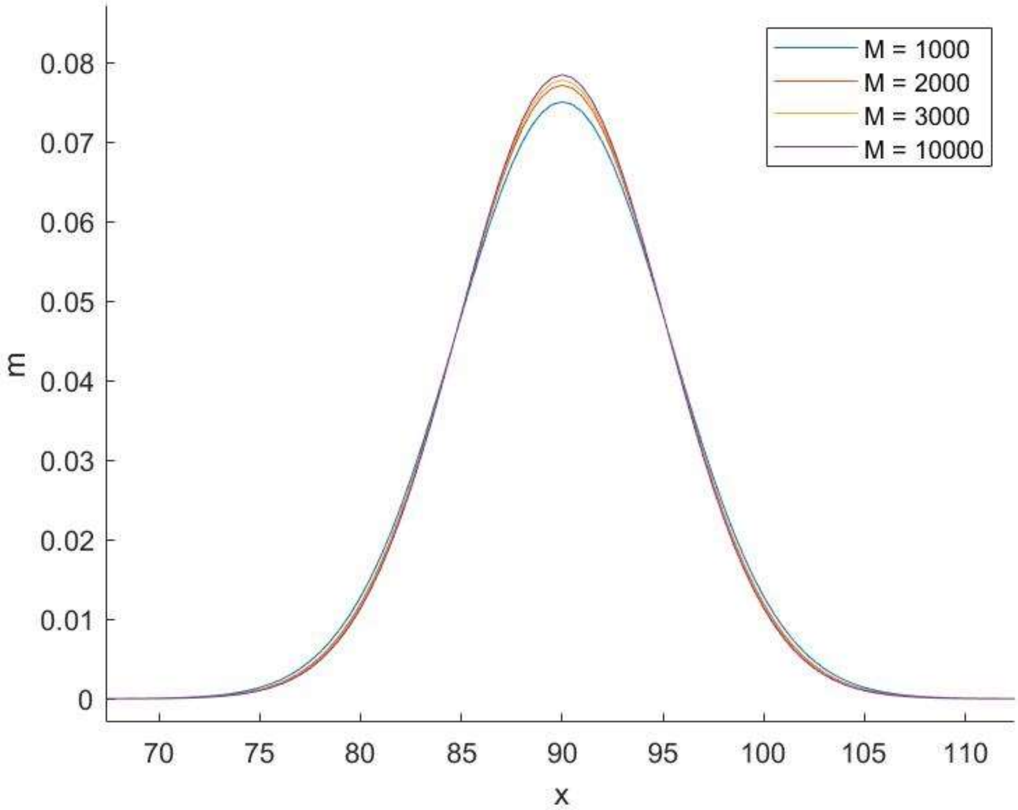

Figure 3: The distribution of miners' wealth at different miner number M *(*$r = 0.5, K^{-1} = 0.007$*).*

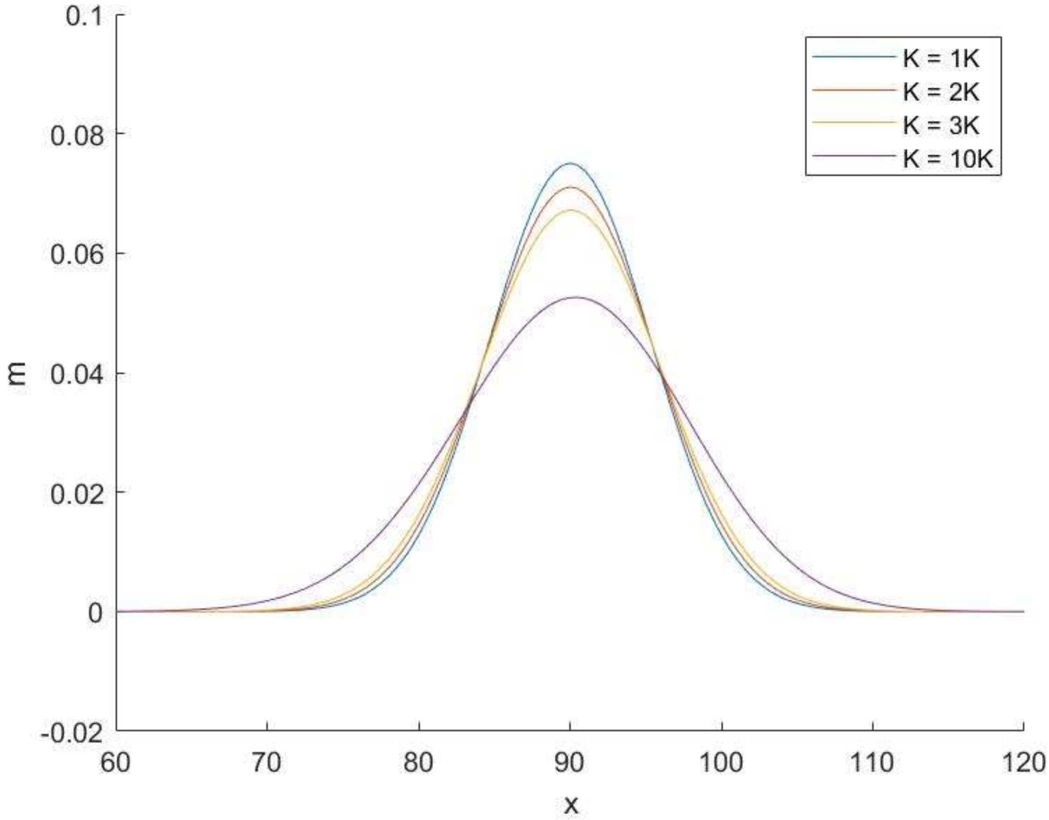

Figure 4: The distribution of miners' wealth at different Poisson process intensity K $(r = 0.5, M = 1000)$.

Then figure 4 shows the details of miners' density when K is various. Different K means different mean time of getting next block. The bigger K represents the system needs more time for miners to get next block. This phenomenon is more like that of different M. If K is bigger, the mining system would get larger variance, while the mean wealth will not change. And also, equal proportion of miners may get more wealth or lose wealth. However, the impact of changing K may be greater than that of changing M.

The reason why the miners' density functions are not congregate at zero wealth is that $r = 0.5$ we set initially is same as the wealth step size $dx = 0.5$, which makes the $\Delta v =$

$v(t, x+0.5; \bar{\alpha}) - v(t, x; \bar{\alpha})$. And because the $\partial_x v(t, x; \bar{\alpha}) = \big(v(t, x+dx; \bar{\alpha}) - v(t, x; \bar{\alpha})\big)/dx$, the optimal hash rate will become a constant

$$\alpha^*(t, x; \bar{\alpha}) = -M\bar{\alpha}_t + \sqrt{\frac{KM\bar{\alpha}_t dx}{c}}.$$

Since the optimal hash rate is a constant through the iterations, the density functions will not change.

When the reward value is not identical as the wealth step size, the result begins to have great distinctions. Here we discuss when $r = 1$ firstly. We could see from Figure 5 that there is a tremendous difference between situation $r = 0.5$ and $r = 1$. The density function rapidly congregate at zero wealth. And most of density function curves have small peaks around 8, which is an interesting phenomenon. In other words, if the reward value of mining cryptocurrencies is getting larger and different from the wealth increasing step size, there is a suggestion that do not join this game, since most of all miners will have a great possibility to lose their wealth terminally. Only a small number of miners may take the fortune to keep 10 dollars in their pockets.

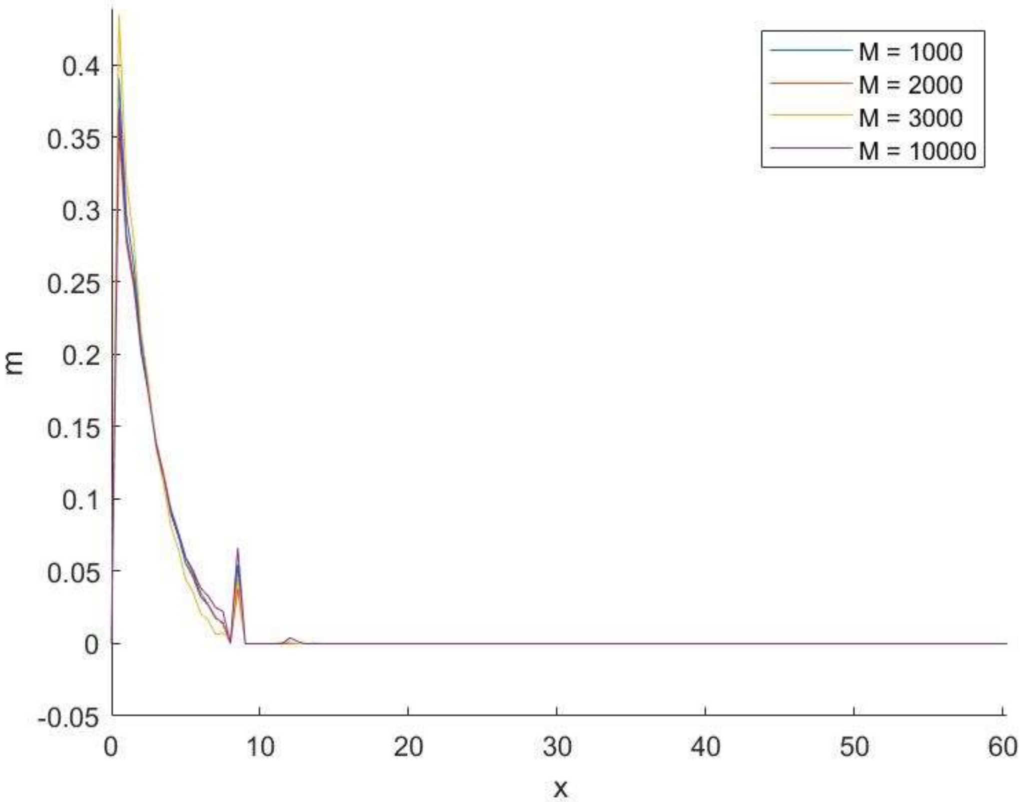

Figure 5: *The distribution of miners' wealth at different miner number* M *(*$r = 1, K^{-1} = 0.007$*)*

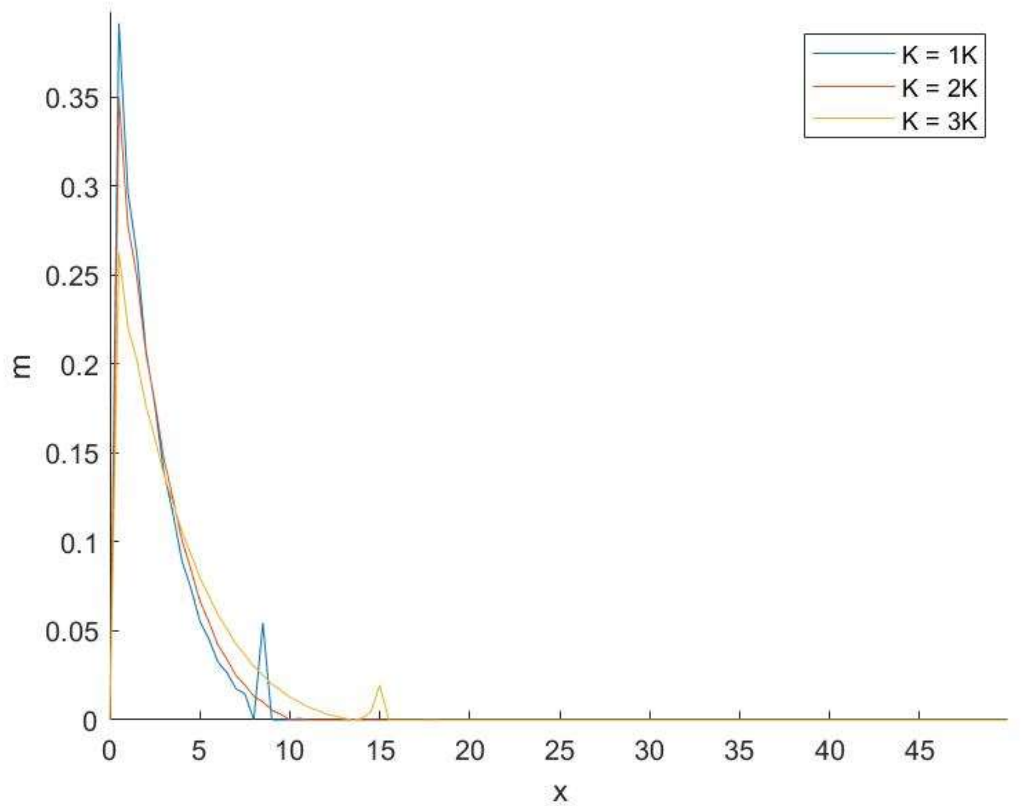

Figure 6: *The distribution of miners' wealth at different Poisson process intensity* K $(r = 1, M =$

1000).

Then Figure 6 shows the changes clearly as the Poisson process intensity K gets bigger. We zoom in the figure to focus on wealth $x < 50$ and find that with Poisson intensity getting bigger, the mass density function m congregates less severely. And also, the small peaks around $x = 10$ would move to the right-hand side as K getting bigger. The results illustrate that if the longer mean time for all miners to get the next block in the mining system, the fewer miners will lose their wealth in the mining competitions.

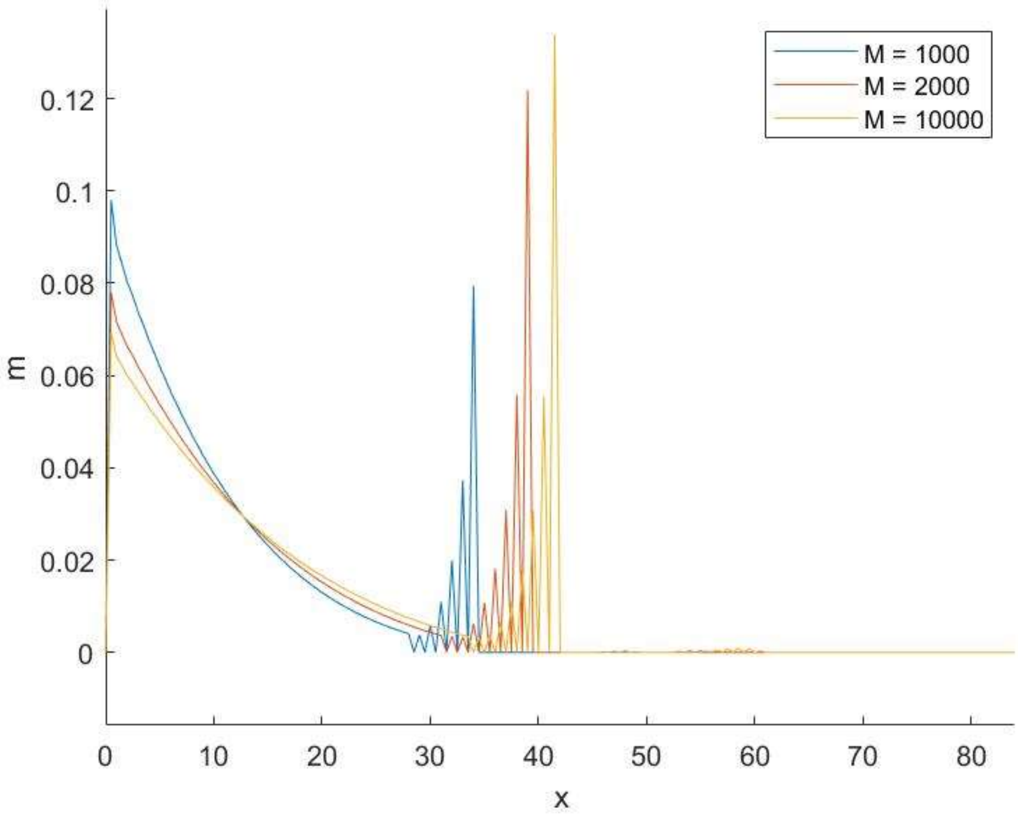

Figure 7: *The distribution of miners' wealth at different miner number* M $(r = 2, K^{-1} = 0.007)$

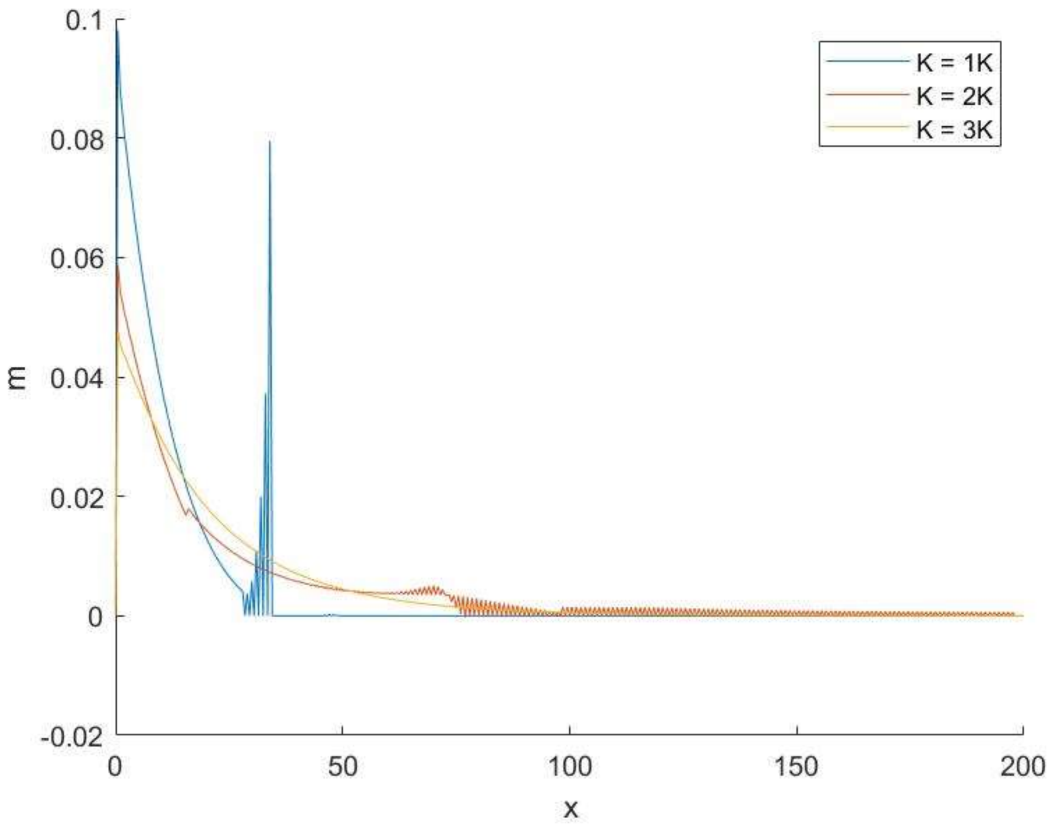

Figure 8: *The distribution of miners' wealth at different Poisson process intensity* K $(r = 2, M = 1000)$.

Figure 7 and figure 8 state the phenomenon when $r = 2$. We zoom in the picture on x-axis to have a better view of density functions. And there are actually some distinctions between $r = 1$ and $r = 2$. When M gets larger, the intercept distance on y-axis become smaller. Then these density function have an intersection around $x = 12$ and the peaks of density functions move to right hand side and become higher. The graph presents an interesting phenomenon of the mining game when the mining reward $r = 2$ similar with the previous one. Most miner will lose their money at terminal in this game and their wealth is likely to end up in the interval $x \in [0, 30]$, while only a small number of miners may have chances to save approximately 30 dollars. Identically, more miners join the game, miners may save more money in the end. However, with increase of the Poisson intensity K, the density functions generate a little difference. The density functions are not well-converged when x is large and the local peaks disappear when $K = 2K$ and $K = 3K$, but the trends of density functions are still the same, which are that, increase on Poisson intensity K takes a positive impact on miners' wealth which makes miners have more opportunity to save money from the game. By the way, the result from $K = 10K$ is not shown in above figure 8 because when K get larger, it's really difficult to find the initial mean hash rate basin so as to output the correct figures.

The results of density function m when $r = 3$ are shown below. The two figures show the different phenomena at various the total number in the mining game M and Poisson intensity K. When M increases, the mass density around 0 wealth drops dramatically at first and then slow down. The local peak of mass density function continues moving to the left approximately from $x = 45$ to $x = 20$ with increase of M from 1000 to 10000. Identically, with the increase of Poisson intensity K, the density function presents the same response that it drops quickly at beginning and then slows down. But surprisingly, the local peak of density function disappears as soon as the K increased, similar with the previous reward value case.

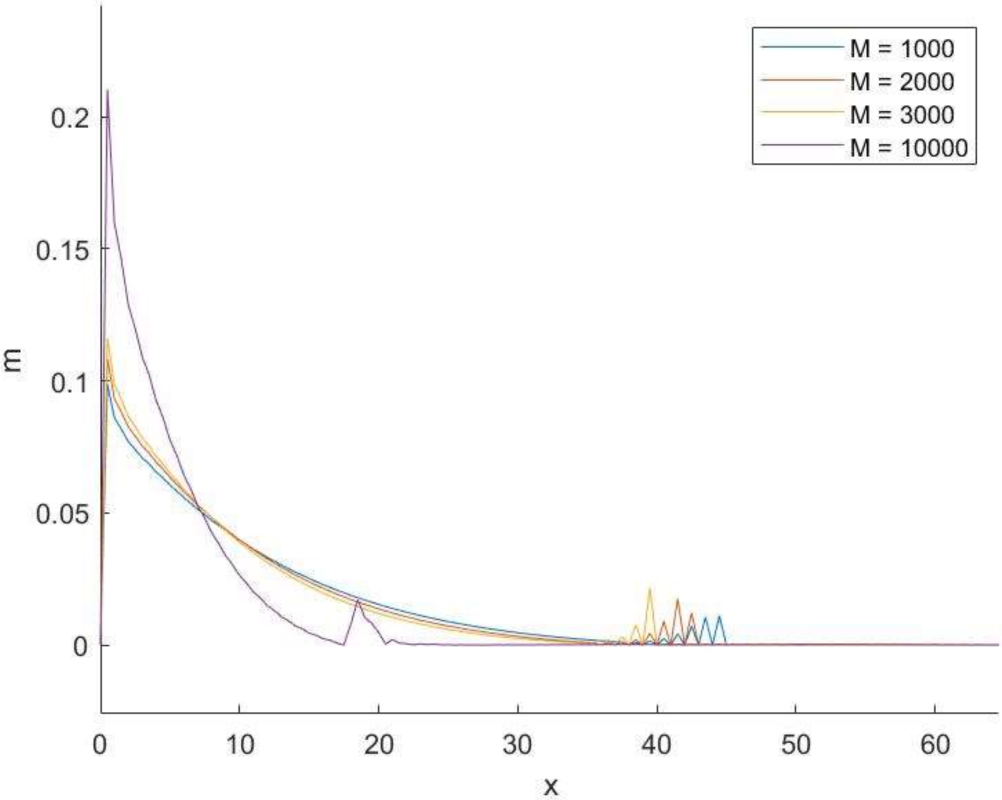

Figure 9: *The distribution of miners' wealth at different miner number* M *(*$r = 3, K^{-1} = 0.007$*)*

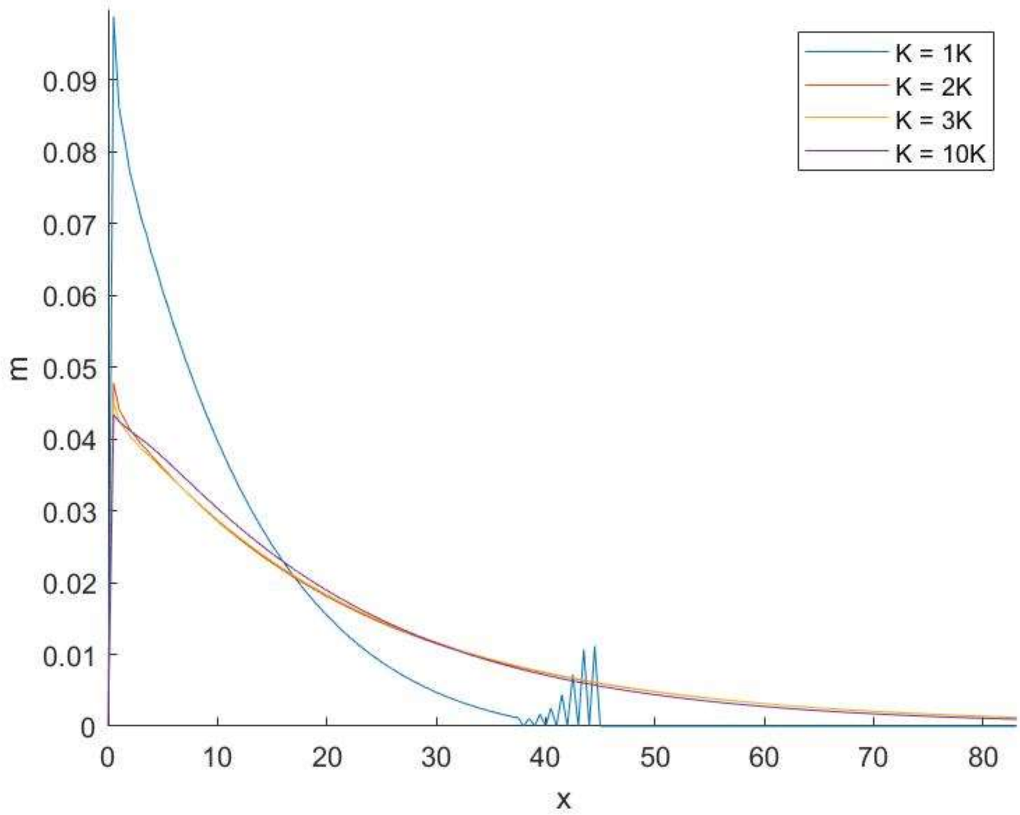

Figure 10: *The distribution of miners' wealth at different Poisson process intensity* K $(r = 3, M = 1000)$

If we compare the three cases shown before, $r = 1, 2, 3$ with different M and K, we could find an interesting phenomenon that the density functions are all congregate around 0 and, with reward value r getting larger and larger, the density function m at 0 wealth drops more and more drastically at first, relatively, and then drops slowly, no matter change parameters M or K. And in the various-K cases, when the Poisson process intensity K increase, the local peak would disappear with higher K, especially r is larger.

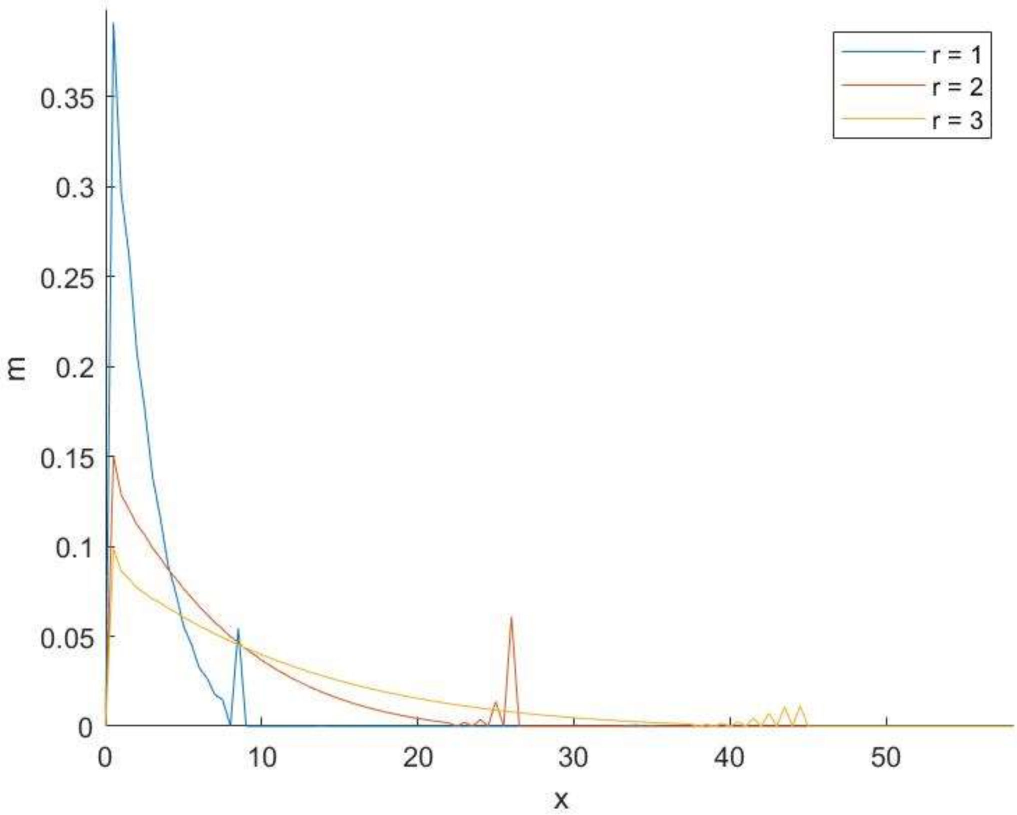

Figure 11: The mass density function m with different reward value r $(M = 1000, K^{-1} = 0.007)$

In the end, from the horizontal comparison of $r = 1, 2, 3$, presented in Figure 11 and Figure 12, we could conclude that, if reward value r increases, then the mass density value of low wealth will get lower. And the local peak of mass density function will move to the right-hand side and will probably decrease. The phenomenon illustrates that higher mining reward could make fewer miners lose their money to empty, but no matter how, most miners in the game would lose their money in the end.

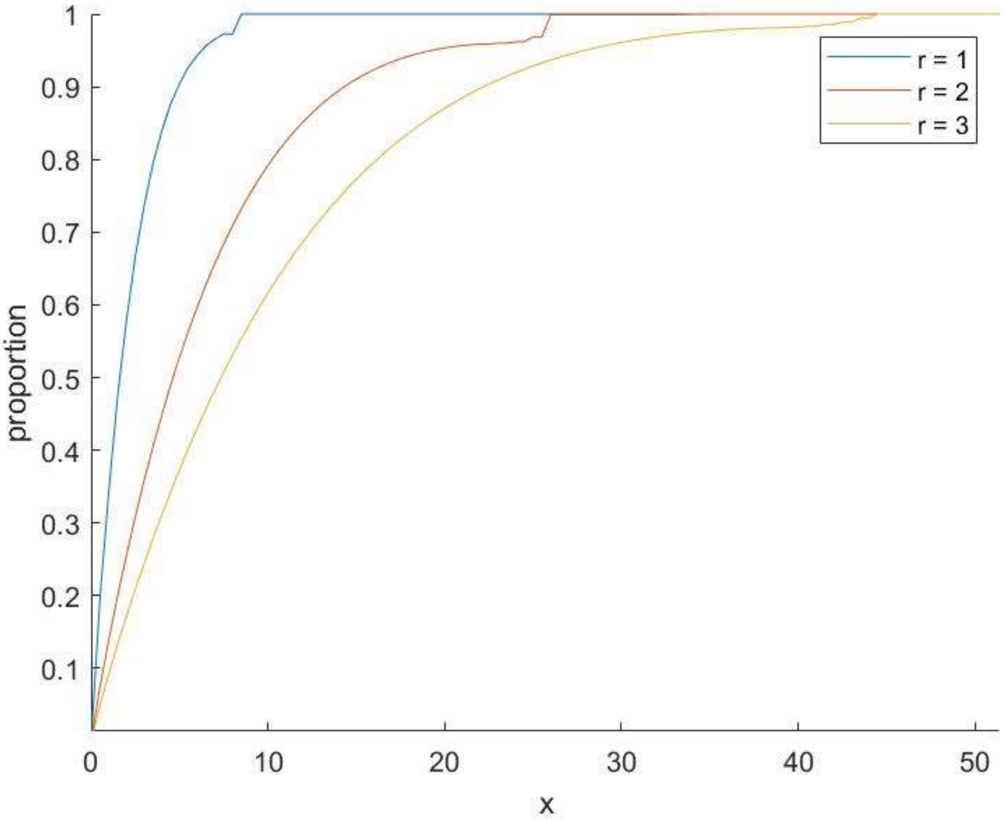

Figure 12: the cumulative proportion of miners with different reward value r ($M = 1000, K^{-1} = 0.007$)

4. Conclusion

This paper develops stochastic models for the mining of cryptocurrencies that used a novel mean field game of intensity control. Miners compete through their computation efforts, their hash rates, to gain wealth in the mining game. Remarkably, the equilibrium is found implicitly and efficiently for the power utility functions in this paper. In other cases, the equilibrium can be found explicitly and easily but not described in this paper.

Our main finding is that, most of miners tend to lose their wealth in the mining game, no matter the initial wealth heterogeneity among the miners, while miners with less initial wealth may have bigger chance to collect some rewards in this game. This leads to more concentrate mining, which against the very fundamental principle of mining being decentralized, and a gaming phenomenon that fewer miners bring money in this game.

More detailly, there are some ways to reduce the concentration of mining, which are, first, increase the total miners in the mining pool; second, increase the difficulty of the puzzles and thus lengthen the time for miners to reach the next blockchain; third, increase the value of the mining reward, like rise the price of Bitcoin. By the way, the second and third ways are more effective than the first way, and the first, weakest, way sometimes may generate negative impact in some cases, like case $r = 0.5$ discussed previously in this paper. Back to the point, as shown in previous figures, we see that these ways are efficient to reduce the concentration in this mining game but not efficient enough, since most of miners' wealth is still congregate in area less than 50 dollars.

5. Acknowledgments

Throughout the writing of this dissertation, I have received a great deal of support and assistance. I would first like to thank my supervisor, Dr. MOU Chenchen, whose expertise was invaluable in formulating the research questions and methodology. Your insightful feedback pushed me to sharpen my thinking and brought my work to a higher level. Then I would also like to thank the two authors of the paper *A Mean Field Games Model for Cryptocurrency Mining*, Dr. LI Zongxi and Dr. Anders Max Reppen, for their valuable guidance and data provided throughout my studies. You provided me with the tools that I needed to choose the right direction and successfully complete my dissertation.

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7. Appendices

Implicit method process

$$\partial_t V + \left(-c\alpha^* \partial_x V + \frac{K\alpha^* \Delta V}{\alpha^* + M\bar{\alpha}}\right) = 0$$

$$\Rightarrow \frac{V_x^t - V_x^{t-1}}{dt} - c\alpha_x^* \frac{V_{x+1}^{t-1} - V_{x-1}^{t-1}}{2dx} + \frac{K\alpha_x^* \Delta V}{\alpha_x^* + M\bar{\alpha}} = 0$$

$$\xRightarrow{F_x = \frac{K\alpha_x^* \Delta V}{\alpha_x^* + M\bar{\alpha}}} V_x^t - V_x^{t-1} - c\alpha_x^* \frac{dt}{2dx}\left(V_{x+1}^{t-1} - V_{x-1}^{t-1}\right) + dtF_x = 0$$

$$\xRightarrow{h = \frac{dt}{2dx}} V_x^t + dtF_x = -c\alpha_x^* h\, V_{x-1}^{t-1} + V_x^{t-1} + c\alpha_x^* h\, V_{x+1}^{t-1}$$

$\Rightarrow$

$$\begin{pmatrix} 1 & c\alpha_2^* h & 0 & & 0 & 0 & 0 \\ -c\alpha_3^* h & 1 & c\alpha_3^* h & \dots & 0 & 0 & 0 \\ 0 & -c\alpha_4^* h & 1 & & 0 & 0 & 0 \\ & \vdots & & \ddots & & \vdots & \\ 0 & 0 & 0 & & 1 & c\alpha_{end-3}^* h & 0 \\ 0 & 0 & 0 & \dots & -c\alpha_{end-2}^* h & 1 & c\alpha_{end-2}^* h \\ 0 & 0 & 0 & & 0 & -c\alpha_{end-1}^* h & 1 \end{pmatrix} \cdot \begin{pmatrix} V_2^{t-1} \\ V_3^{t-1} \\ V_4^{t-1} \\ \vdots \\ V_{end-3}^{t-1} \\ V_{end-2}^{t-1} \\ V_{end-1}^{t-1} \end{pmatrix}$$

$$= \begin{pmatrix} V_2^t + dtF_2 + hc\alpha_2^* V_1^{t-1} \\ V_3^t + dtF_3 \\ V_4^t + dtF_4 \\ \vdots \\ V_{end-3}^t + dtF_{end-3} \\ V_{end-2}^t + dtF_{end-2} \\ V_{end-1}^t + dtF_{end-1} - hc\alpha_{end-1}^* V_{end}^{t-1} \end{pmatrix}$$

$$\partial_t m - \partial_x(c\alpha^* m) - K\left(\frac{\alpha^*(x-r)\, m(x-r)}{\alpha^*(x-r)+M\bar{\alpha}^*} - \frac{\alpha^* m}{\alpha^* + M\bar{\alpha}^*}\right) = 0$$

$$\Longrightarrow \partial_t m - \partial_x(c\alpha^* m) + K\frac{\alpha^* m}{\alpha^* + M\bar{\alpha}^*} = K\frac{\alpha^*(x-r)\, m(x-r)}{\alpha^*(x-r)+M\bar{\alpha}^*}$$

$$\xrightarrow{H=K\frac{\alpha^*(x-r)\, m(x-r)}{\alpha^*(x-r)+M\bar{\alpha}^*}} \frac{m_x^{t+1}-m_x^t}{dt} - \frac{c{\alpha^*}_{x+1}^{t+1} m_{x+1}^{t+1} - c{\alpha^*}_{x-1}^{t+1} m_{x-1}^{t+1}}{2dx} + \frac{K{\alpha^*}_x^t}{{\alpha^*}_x^t + M{\bar{\alpha}^*}^t} m_x^t = H$$

$$\xrightarrow{G_x=\frac{K{\alpha^*}_x^t}{{\alpha^*}_x^t+M{\bar{\alpha}^*}^t}} m_x^{t+1} - m_x^t - \frac{dt}{2dx}\left(c{\alpha^*}_{x+1}^{t+1} m_{x+1}^{t+1} - c{\alpha^*}_{x-1}^{t+1} m_{x-1}^{t+1}\right) + dt\, G_x\, m_x^t = H\, dt$$

$$\xrightarrow{h=\frac{dt}{2dx}} (1-dtG_x)m_x^t + Hdt = hc{\alpha^*}_{x-1}^{t+1} m_{x-1}^{t+1} + m_x^{t+1} - hc{\alpha^*}_{x+1}^{t+1} m_{x+1}^{t+1}$$

$$\Longrightarrow$$

$$\begin{pmatrix} 1 & -hc{\alpha^*}_3^{t+1} & 0 & & 0 & 0 & 0 \\ hc{\alpha^*}_2^{t+1} & 1 & -hc{\alpha^*}_4^{t+1} & \cdots & 0 & 0 & 0 \\ 0 & hc{\alpha^*}_3^{t+1} & 1 & & 0 & 0 & 0 \\ & \vdots & & \ddots & & \vdots & \\ 0 & 0 & 0 & & 1 & -hc{\alpha^*}_{end-2}^{t+1} & 0 \\ 0 & 0 & 0 & \cdots & hc{\alpha^*}_{end-3}^{t+1} & 1 & -hc{\alpha^*}_{end-1}^{t+1} \\ 0 & 0 & 0 & & 0 & hc{\alpha^*}_{end-2}^{t+1} & 1 \end{pmatrix}$$

$$\cdot \begin{pmatrix} m_2^{t+1} \\ m_3^{t+1} \\ m_4^{t+1} \\ \vdots \\ m_{end-3}^{t+1} \\ m_{end-2}^{t+1} \\ m_{end-1}^{t+1} \end{pmatrix} = \begin{pmatrix} (1-dtG_2)m_2^t - hc{\alpha^*}_1^{t+1} m_1^{t+1} \\ (1-dtG_3)m_3^t \\ (1-dtG_4)m_4^t \\ \vdots \\ (1-dtG_{end-3})m_{end-3}^t \\ (1-dtG_{end-2})m_{end-2}^t \\ (1-dtG_{end-1})m_{end-1}^t + hc{\alpha^*}_{end}^{t+1} m_{end}^{t+1} \end{pmatrix} + Hdt$$