THE HIGHER ALGEBRAIC K-THEORY OF STABLE ∞-CATEGORIES

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Abstract. We summarize the higher algebraic K-theory of stable ∞-categories.

Contents

1. Introduction

This paper is a summary of the workshop on the higher algebraic K-theory held in Kyoto in September 2024. To make it easier to read, all proofs are included in Appendix.

1.1. **Notation.** From here all categories are assumed to be ∞-categories. We let

- An denote the category of small anima.
- Cat denote the category of small categories.
- $\mathrm{Cat}^{\mathrm{lex}}$ denote the category of small categories which admit finite limits, with left exact functors.
- $\mathrm{Cat}^{\mathrm{st}}$ denote the category of small stable categories with exact functors.
- $\mathrm{Cat}^{\mathrm{perf}}$ denote the category of small idempotent complete stable categories with exact functors.
- Sp denote the category of spectra.

2. Preliminaries

In this section, we recall some basic notions of ∞-categories.

2.1. **The Grothendieck Group.** In this section, we review the definition of the Grothendieck group for stable categories.

Definition 2.1. Let $(\mathcal{C}, \oplus)$ be a stable category, and let X and Y be objects of $\mathcal{C}$. We let $[X]$ denote the connected component of X. The connected component set $\pi_0(\mathrm{core}\,\mathcal{C})$, together with the operation $+$ defined by

$$[X] + [Y] := [X \oplus Y]$$

forms an ordinary monoid $(\pi_0(\operatorname{core}\mathcal{C}), +)$. We define the *Grothendieck group* $\mathcal{K}_0(\mathcal{C})$ of $\mathcal{C}$ as

$$\mathcal{K}_0(\mathcal{C}) := (\pi_0(\operatorname{core}\mathcal{C}), +)/\sim$$

where $\sim$ is the equivalence relation generated by the following relation: $[X] = [X'] + [X'']$ whenever $X' \to X \to X''$ is a cofiber sequence in $\mathcal{C}$.

Remark 2.2. Let $(\mathcal{C}, \oplus)$ be a stable category. Then the connected component set $\pi_0(\operatorname{core}\mathcal{C})$ is the set of equivalence classes of objects of $\mathcal{C}$. Moreover, the Grothendieck group $\mathcal{K}_0(\mathcal{C})$ is actually abelian.

(1) The zero object 0 of $\mathcal{C}$ is a unit object $[0]$ of $\mathcal{K}_0(\mathcal{C})$, since $X \to X \to 0$ is a cofiber sequence in $\mathcal{C}$ for every object X of $\mathcal{C}$.
(2) For every object X of $\mathcal{C}$, $[\Omega X]$ and $[\Sigma X]$ are inverse objects of $[X]$ in $\mathcal{K}_0(\mathcal{C})$, since $\Omega X \to 0 \to X$ and $X \to 0 \to X$ are cofiber sequences in $\mathcal{C}$.
(3) For every objects X and Y of $\mathcal{C}$, we have $[X]+[Y] = [Y]+[X]$, since $X \to X \oplus Y \to Y$ and $Y \to X \oplus Y \to X$ are cofiber sequences in $\mathcal{C}$.

Remark 2.3 (Eilenberg swindle)**.** Let $\mathcal{C}$ be a stable category with countable coproducts. Then the Grothendieck group $\mathcal{K}_0(\mathcal{C})$ is trivial. Indeed, for every object X of $\mathcal{C}$,

$$X \to \oplus_{n \geq 0} X \to \oplus_{n \geq 1} X$$

is a cofiber sequence in $\mathcal{C}$, and that the last two terms are equivalent. It can be generalized to the algebraic K-theory (see corollary 5.5).

Remark 2.4. The construction $\mathcal{C} \mapsto \mathcal{K}_0(\mathcal{C})$ determine a functor $\mathcal{K}_0 : \mathrm{h}\,\mathrm{Cat}^{\mathrm{st}} \to \mathrm{Ab}$.

2.2. **Arrow Categories and Twisted Arrow Categories.** In this section, we recall the notions of (twisted) arrow categories.

Definition 2.5. Let $\mathcal{C}$ be a category. We define the *arrow category* $\mathrm{Ar}(\mathcal{C})$ of $\mathcal{C}$ as

$$\mathrm{Ar}(\mathcal{C}) := \mathrm{Fun}([1], \mathcal{C}).$$

Definition 2.6. Let $\mathcal{C}$ be a category. The *twisted arrow category* $\mathrm{TwAr}(\mathcal{C})$ of $\mathcal{C}$ is the simplicial set which is defined by

$$\mathrm{TwAr}(\mathcal{C})_n := \mathrm{Hom}_{\mathrm{sSet}}([n] \star [n]^{\mathrm{op}}, \mathcal{C})$$

for every $n \geq 0$, where $\star$ is the join operator. By [HA, Proposition 5.2.1.3], $\mathrm{TwAr}(\mathcal{C})$ is a category.

Remark 2.7. Let $\mathcal{C}$ be a category. Let see the objects and morphisms of $\mathrm{Ar}(\mathcal{C})$ and $\mathrm{TwAr}(\mathcal{C})$.

- The objects of both are morphisms in $\mathcal{C}$.
- A morphism from f to g in $\mathrm{Ar}(\mathcal{C})$ is a diagram, depicted as

$$\begin{tikzcd}
 & 1' & \\
 & & 1. \arrow[ul, "g"'] \\
0 \arrow[uur] \arrow[urr] \arrow[r, "f"'] & 0' \arrow[uu] \arrow[ur] &
\end{tikzcd}$$

- A morphism from f to g in $\mathrm{TwAr}(\mathcal{C})$ is a diagram, depicted as

$$\begin{tikzcd}
 & 1' \arrow[dd] & \\
0 \arrow[ur] \arrow[rr] \arrow[dr, "f"'] & & 1. \arrow[ul, "g"'] \\
 & 0' \arrow[ur] &
\end{tikzcd}$$

Notation 2.8. Let $\mathcal{C}$ be a stable category. We let $\mathrm{Seq}(\mathcal{C})$ denote the full subcategory of $\mathrm{Fun}(\Delta^1 \times \Delta^1, \mathcal{C})$ spanned by the bifiber sequences in $\mathcal{C}$.

Remark 2.9. Let $\mathcal{C}$ be a stable category. Then we have an equivalence of categories $\mathrm{Seq}(\mathcal{C}) \simeq \mathrm{Ar}(\mathcal{C})$, which implies that the category $\mathrm{Seq}(\mathcal{C})$ is stable.

Notation 2.10. Let $\mathcal{C}$ be a stable category. We define functors from $\mathrm{Seq}(\mathcal{C})$ to $\mathcal{C}$ as follows:

$$\begin{aligned}
\mathrm{fib} : \mathrm{Seq}(\mathcal{C}) \to \mathcal{C} : (X \to Y \to Z) \mapsto X,\\
\mathrm{mid} : \mathrm{Seq}(\mathcal{C}) \to \mathcal{C} : (X \to Y \to Z) \mapsto Y,\\
\mathrm{cofib} : \mathrm{Seq}(\mathcal{C}) \to \mathcal{C} : (X \to Y \to Z) \mapsto Z.
\end{aligned}$$

3. Localization Properties of Functors

In this section, we define various functors with localizing properties and recall their relations: additive, Verdier-localizing, Karoubi-localizing, grouplike functors.

We follow the terminology of [Cal+23]. In [Cal+23], these notions are defined for Poincaré-Verdier squares. We use the same terminology for Verdier squares.

3.1. **Verdier Sequences and Squares.**

In this section, we recall the notions of (split) Verdier sequences and Karoubi sequences and there relative versions: (split) Verdier squares and Karoubi squares.

Definition 3.1. Let $\mathcal{C} \xrightarrow{f} \mathcal{D} \xrightarrow{p} \mathcal{E}$ be a sequence in $\mathrm{Cat}^{\mathrm{st}}$. We will say that the sequence has *vanishing composition* if the composition pf is a zero object of $\mathrm{Cat}^{\mathrm{st}}$.

In this case, the composition pf is equivalent to the functor $\mathcal{C} \to 0 \to \mathcal{E}$, since the full subcategory of $\mathrm{Fun}^{\mathrm{ex}}(\mathcal{C}, \mathcal{E})$ spanned by the zero objects is contractible. That is, there exists the following commutative diagram:

$$\begin{tikzcd}
\mathcal{C} \arrow[r, "f"] \arrow[d] & \mathcal{D} \arrow[d, "p"] \\
0 \arrow[r] & \mathcal{E}.
\end{tikzcd}$$

We will say that the sequence is a fiber (resp. cofiber) sequence if the above diagram is a Cartesian (resp. coCartesian) diagram.

Definition 3.2 ([Cal+23] Definition A.1.1)**.** Let $\mathcal{C} \xrightarrow{f} \mathcal{D} \xrightarrow{p} \mathcal{E}$ be a sequence in $\mathrm{Cat}^{\mathrm{st}}$ with vanishing composition. We will say that this sequence is *Verdier* if it is a bifiber sequence in $\mathrm{Cat}^{\mathrm{st}}$. In this case, we will refer to the functor f as the *Verdier inclusion* and to the functor p as the *Verdier projection*.

Definition 3.3 ([Cal+23] Definition A.2.4)**.** Let $\mathcal{C} \xrightarrow{f} \mathcal{D} \xrightarrow{p} \mathcal{E}$ be a Verdier sequence. We will say that this sequence is *split* if the functor p admits both adjoint functors. In this case, we will refer to the functor f as the *split Verdier inclusion* and to the functor p as the *split Verdier projection*.

Definition 3.4 ([Cal+23] Definition A.3.5)**.** Let $\mathcal{C} \xrightarrow{f} \mathcal{D} \xrightarrow{p} \mathcal{E}$ be a sequence in $\mathrm{Cat}^{\mathrm{st}}$ with vanishing composition. We will say that this sequence is *Karoubi* if its idempotent completion $\mathcal{C}^{\natural} \to \mathcal{D}^{\natural} \to \mathcal{E}^{\natural}$ is a bifiber sequence in $\mathrm{Cat}^{\mathrm{perf}}$. In this case, we will refer to the functor f as the *Karoubi inclusion* and to the functor p as the *Karoubi projection.*

We can characterize Verdier inclusions and projections (**??**). The fiber of exact functor $f : \mathcal{C} \to \mathcal{D}$ can be computed by its kernel category $\ker(f)$. On the other hand, its cofiber is described by the Verdier quotient.

Definition 3.5 ([Cal+23] Definition A.1.3)**.** Let $f : \mathcal{C} \to \mathcal{D}$ be an exact functor between stable categories. We will say that a morphism in $\mathcal{D}$ is an *equivalence modulo* $\mathcal{C}$ in $\mathcal{D}$ if its fiber (or equivalently, its cofiber) belongs in the essential image of f.

We define the category $\mathcal{D}/\mathcal{C}$ as the localization of $\mathcal{D}$ with respect to the set of equivalences modulo $\mathcal{C}$ in $\mathcal{D}$. We will refer to the category $\mathcal{D}/\mathcal{C}$ as the *Verdier quotient* of $\mathcal{D}$ by $\mathcal{C}$.

The next proposition implies that the Verdier quotient is universal.

Proposition 3.6 ([NS18] Theorem.1.3.3)**.** Let $f : \mathcal{C} \to \mathcal{D}$ be an exact functor between stable categories. Then

(1) The Verdier quotient $\mathcal{D}/\mathcal{C}$ is stable, and the localization functor $\mathcal{D} \to \mathcal{D}/\mathcal{C}$ is exact.

(2) For every stable category $\mathcal{E}$, the restriction functor
$$\mathrm{Fun}^{\mathrm{ex}}(\mathcal{D}/\mathcal{C}, \mathcal{E}) \to \mathrm{Fun}^{\mathrm{ex}}(\mathcal{D}, \mathcal{E})$$
is fully faithful, and its essential image consists of the functors which vanish after composing with f.

(3) The sequence $\mathcal{C} \to \mathcal{D} \to \mathcal{D}/\mathcal{C}$ is a cofiber sequence in $\mathrm{Cat}^{\mathrm{st}}$.

Proposition 3.7 ([Cal+23] Corollary A.1.10)**.** Let $\mathcal{C} \xrightarrow{f} \mathcal{D} \xrightarrow{p} \mathcal{E}$ be a sequence in $\mathrm{Cat}^{\mathrm{st}}$ with vanishing composition. The following conditions are equivalent:

(1) The sequence is Verdier.

(2) The functor f is fully faithful and its essential image is closed under retracts in $\mathcal{D}$, and the functor p exhibits $\mathcal{E}$ as the Verdier quotient of $\mathcal{D}$ by $\mathcal{C}$.

(3) The functor f exhibits $\mathcal{C}$ as the kernel of p, and the functor p is a localization.

We can characterize split Verdier inclusions and projections.

Proposition 3.8 ([Cal+23] Corollary A.2.6)**.** Let $\mathcal{C} \xrightarrow{f} \mathcal{D} \xrightarrow{p} \mathcal{E}$ be a sequence in $\mathrm{Cat}^{\mathrm{st}}$ with vanishing composition. The following conditions are equivalent:

(1) The sequence is split Verdier.

(2) The functor p admits fully faithful both adjoint functors.

(3) The functor f is fully faithful and admits both adjoint functors.

We can characterize Karoubi inclusions and projections.

Proposition 3.9 ([Cal+23] Corollary A.3.8)**.** Let $\mathcal{C} \xrightarrow{f} \mathcal{D} \xrightarrow{p} \mathcal{E}$ be a sequence in $\mathrm{Cat}^{\mathrm{st}}$ with vanishing composition. The following conditions are equivalent:

(1) The sequence is Karoubi.

(2) The functor f is fully faithful and the functor p has the dense essential image $p(\mathcal{D}) \subseteq \mathcal{E}$, and the induced functor $\mathcal{D} \to p(\mathcal{D})$ is a Verdier projection.

We can describe Karoubi sequences using Ind-categories.

Theorem 3.10 (Thomason-Neeman's localization theorem). *Let $\mathcal{C} \xrightarrow{f} \mathcal{D} \xrightarrow{p} \mathcal{E}$ be a sequence in* $\mathrm{Cat}^{\mathrm{st}}$ *with vanishing composition. The following conditions are equivalent:*

(1) The sequence is Karoubi.

(2) The sequence $\mathrm{Ind}(\mathcal{C}) \to \mathrm{Ind}(\mathcal{D}) \to \mathrm{Ind}(\mathcal{E})$ *is Verdier (of non-necessarily small categories).*

We next introduce the relative versions of these sequences.

Definition 3.11 ([Cal+23] Definition.1.5.1)**.** A square in $\mathrm{Cat}^{\mathrm{st}}$ is called

- *Verdier* if it is Cartesian and its both vertical maps are Verdier projections.
- *split Verdier* if it is Cartesian and its both vertical maps are split Verdier projections.
- *Karoubi* if it is Cartesian after idempotent completion and its both vertical maps are Karoubi projections.

Remark 3.12. In definition 3.11, the condition that the square is Cartesian can be replaced by the condition that it is coCartesian. (See proof A.1.)

3.2. **Additive and Grouplike Functors.** In this section, we define additive, Verdier-localizing, Karoubi-localizing, and grouplike functors.

Definition 3.13. Let $\mathcal{E}$ be a category with a terminal object, and let $F : \mathrm{Cat}^{\mathrm{st}} \to \mathcal{E}$ be a functor. We will say that F is *reduced* if $F(0)$ is equivalent to a terminal object of $\mathcal{E}$, where 0 is a zero object in $\mathrm{Cat}^{\mathrm{st}}$.

Definition 3.14 ([HLS23] Definition 2.1)**.** Let $\mathcal{E}$ be a category with finite limits, and let $F : \mathrm{Cat}^{\mathrm{st}} \to \mathcal{E}$ be a reduced functor. The functor F is called

- *Verdier-localizing* if it takes every Verdier square in $\mathrm{Cat}^{\mathrm{st}}$ to a Cartesian square in $\mathcal{E}$.
- *additive* if it takes every split Verdier square in $\mathrm{Cat}^{\mathrm{st}}$ to a Cartesian square in $\mathcal{E}$.
- *Karoubi-localizing* if it takes every Karoubi square in $\mathrm{Cat}^{\mathrm{st}}$ to a Cartesian square in $\mathcal{E}$.

Every additive (resp. Verdier-localizing, Karoubi-localizing) functor $F : \mathrm{Cat}^{\mathrm{st}} \to \mathcal{E}$ sends split Verdier sequences (resp. Verdier sequences, Karoubi sequences) to fiber sequences. If $\mathcal{E}$ is stable, the converse holds.

Proposition 3.15 ([Cal+23] Proposition 1.5.5)**.** Let $\mathcal{E}$ be a category with finite limits, and let $F : \mathrm{Cat}^{\mathrm{st}} \to \mathcal{E}$ be a reduced functor. If $\mathcal{E}$ is stable, then the following conditions are equivalent:

(1) The functor F is additive (resp. Verdier-localizing, Karoubi-localizing).

(2) The functor F takes every split Verdier sequence (resp. Verdier sequence, Karoubi sequence) in $\mathrm{Cat}^{\mathrm{st}}$ to a fiber sequence in $\mathcal{E}$. (See proof A.2.)

Definition 3.16 ([HLS23] Definition 2.1)**.** Let $\mathcal{E}$ be a category with finite limits, and let $F : \mathrm{Cat}^{\mathrm{st}} \to \mathcal{E}$ be an additive functor. We will say that F is *grouplike* if it lifts to the category $\mathrm{Mon}_{\mathbb{E}_\infty}(\mathcal{E})$ takes values in the full subcategory$\mathrm{Grp}_{\mathbb{E}_\infty}(\mathcal{E})$ of $\mathrm{Mon}_{\mathbb{E}_\infty}(\mathcal{E})$.

Example 3.17 ([Cal+23] Example 1.5.10)**.** We give some (counter)examples.

(1) The core functor $\mathrm{core} : \mathrm{Cat}^{\mathrm{st}} \to \mathrm{An}$ is additive, but not grouplike.

(2) The algebraic K-theory $\mathcal{K} : \mathrm{Cat}^{\mathrm{st}} \to \mathrm{An}$ and the algebraic K-theory spectrum $\mathcal{K}_{\geq 0} : \mathrm{Cat}^{\mathrm{st}} \to \mathrm{Sp}$ are Verdier-localizing (theorem 6.1) and grouplike (corollary 5.4), but not Karoubi-localizing.

(3) The non-connective K-theory spectrum $\mathbb{K} : \mathrm{Cat}^{\mathrm{st}} \to \mathrm{Sp}$ is Karoubi-localizing. (TBA)

(4) The functor $\mathcal{K} \circ (-)^\natural : \mathrm{Cat}^{\mathrm{st}} \to \mathrm{An}$ is Karoubi-localizing (corollary 8.8), thus is Verdier-localizing (**??**).

(5) The functor $\mathcal{K}_{\geq 0} \circ (-)^\natural : \mathrm{Cat}^{\mathrm{st}} \to \mathrm{Sp}$ is additive, but not Verdier-localizing.

Proposition 3.18 ([HLS23] Observation 2.2)**.** The additive, Verdier-localizing, Karoubi-localizing functors preserve finite products. (See proof A.3.)

3.3. **Additive Grouplike vs. Extension-splitting.** We can characterize additive grouplike functors by extension-splitting functors. We will use lemma 3.20 and proposition 3.21 in the proof of corollary 5.4.

Definition 3.19. Let $\mathcal{E}$ be a category with finite limits, and let $F : \mathrm{Cat}^{\mathrm{st}} \to \mathcal{E}$ be a reduced functor. We will say that F is *extension-splitting* if, for every stable category $\mathcal{C}$, the fiber-cofiber map

$$(\mathrm{fib}, \mathrm{cofib}) : \mathrm{Seq}(\mathcal{C}) \to \mathcal{C}^2$$

induces an equivalence $F(\mathrm{Seq}(\mathcal{C})) \to F(\mathcal{C})^2$.

We show that additive grouplike functors and extension-splitting functors are equivalent (proposition 3.21).

Lemma 3.20 ([HLS23] Lemma 2.5)**.** Let $\mathcal{E}$ be a category with finite limits, and let $F : \mathrm{Cat}^{\mathrm{st}} \to \mathcal{E}$ be a reduced and product-preserving functor. The following conditions are equivalent:

(1) The functor F is extension-splitting.

(2) The functor F sends the source-target projection $(s, t) : \mathrm{Ar}(\mathcal{C}) \to \mathcal{C}^2$ for every object $\mathcal{C}$ of $\mathrm{Cat}^{\mathrm{st}}$ to an equivalence in $\mathcal{E}$.

Proposition 3.21 ([HLS23] Proposition 2.4)**.** Let $\mathcal{E}$ be a category with finite limits, and let $F : \mathrm{Cat}^{\mathrm{st}} \to \mathcal{E}$ be a reduced functor. The following conditions are equivalent:

(1) The functor F is additive grouplike.

(2) The functor F is extension-splitting.

3.4. **Additive vs. Verdier-localizing.** In this section, we recall Waldhausen's fibration theorem. We will use this theorem in the proof of the localization theorem (theorem 6.1).

Notation 3.22. Let $\mathcal{D}$ be a stable category, let $\mathcal{C}$ be a stable full subcategory of $\mathcal{D}$, and let $\mathcal{I}$ be a category. We let $\mathrm{Fun}^{\mathcal{C}}(\mathcal{I}, \mathcal{D})$ denote the full subcategory of $\mathrm{Fun}(\mathcal{I}, \mathcal{D})$ spanned by the functors which take every maps in $\mathcal{I}$ to equivalences modulo $\mathcal{C}$.

Theorem 3.23 (Waldhausen's fibration theorem)**.** *Let $\mathcal{C} \to \mathcal{D} \to \mathcal{E}$ be a Verdier sequence, and let $F : \mathrm{Cat}^{\mathrm{st}} \to \mathrm{An}$ be an additive grouplike functor. Then, for every $n \geq 0$, the constant map*

$$\mathrm{const} : \mathcal{D} \to \mathrm{Fun}^{\mathcal{C}}([n], \mathcal{D}) : X \mapsto (X \to \cdots \to X)$$

induces a bifiber sequence of $\mathbb{E}_\infty$-groups

$$F(\mathcal{C}) \to F(\mathcal{D}) \to |F\,\mathrm{Fun}^{\mathcal{C}}([-], \mathcal{D})|.$$

We can deduce when an additive functor becomes a Verdier-localizing functor.

Corollary 3.24 ([HLS23] Corollary 2.10)**.** Let $F : \mathrm{Cat}^{\mathrm{st}} \to \mathrm{An}$ be an additive functor. These conditions are equivalent:

(1) The functor F is Verdier-localizing.

(2) For every Verdier sequence $\mathcal{C} \to \mathcal{D} \to \mathcal{E}$, the canonical map $|F\,\mathrm{Fun}^{\mathcal{C}}([-], \mathcal{D})| \to F(\mathcal{E})$ is an equivalence of anima.

3.5. **Verdier-localizing vs. Karoubi-localizing.** The relationship between Verdier-localizing and Karoubi-localizing functors is as follows.

Definition 3.25. Let $f : \mathcal{C} \to \mathcal{D}$ be a functor between categories. We will say that f *has the dense image* if, for every object X of $\mathcal{D}$, there exists an object Y in the essential image of $\mathcal{C}$ such that Y is a retract of X.

Definition 3.26. Let $f : \mathcal{C} \to \mathcal{D}$ be an exact functor between stable categories. We will say that f is a *Karoubi equivalence* if it is fully faithful and has the dense image.

Proposition 3.27 ([HLS23] Observation 2.12)**.** Let $\mathcal{E}$ be a category with finite limits, and let $F : \mathrm{Cat}^{\mathrm{st}} \to \mathcal{E}$ be a reduced functor. The following conditions are equivalent:

(1) The functor F is Karoubi-localizing.
(2) The functor F is Verdier-localizing and inverts Karoubi equivalences.

We can construct Karoubi-localization functors from Verdier-localizing functors using the idempotent completion.

Proposition 3.28 ([HLS23] Lemma 2.13)**.** Let $\mathcal{E}$ be a category with finite limits, and let $F : \mathrm{Cat}^{\mathrm{st}} \to \mathcal{E}$ be a Verdier-localizing functor. Suppose that F takes every Cartesian square in $\mathrm{Cat}^{\mathrm{st}}$ whose vertical maps are dense inclusions, to a Cartesian square in $\mathcal{E}$. Then the functor $F \circ (-)^{\natural} : \mathrm{Cat}^{\mathrm{st}} \to \mathcal{E}$ is Karoubi-localizing.

4. The Higher K-Theory of Stable ∞-Categories

4.1. Simplicial Objects.

Definition 4.1. The inclusion $\mathrm{N}(\Delta) \subseteq \mathrm{Cat}$ induces an adjunction

$$\mathrm{asscat} : \mathrm{Fun}(\mathrm{N}(\Delta)^{\mathrm{op}}, \mathrm{An}) \rightleftarrows \mathrm{Cat} : \mathrm{N}^r\,.$$

We will refer to the left adjoint as the *associated category functor*, and to the right adjoint as the *Rezk nerve*.

Definition 4.2. Let $\mathcal{C}$ be a category. We will refer to a functor

$$X : \mathrm{N}(\Delta)^{\mathrm{op}} \to \mathcal{C}$$

as a *simplicial object* of $\mathcal{C}$. We will say that X is a *simplicial anima* if $\mathcal{C}$ is An.

Remark 4.3. Let $\mathcal{C}$ be a category. For every $n \geq 0$, we have an equivalence of anima

$$\mathrm{N}^r_n(\mathcal{C}) \simeq \mathrm{Map}_{\mathrm{Cat}}([n], \mathcal{C}) \simeq \mathrm{core}\,\mathrm{Fun}([n], \mathcal{C}).$$

Notation 4.4. We let $[n]$ denote the category the ordinary nerve $\mathrm{N}([n])$ of $[n]$, instead of Δ^n. On the other hand, we let Δ^n denote the functor

$$\Delta^n := \mathrm{Map}_{\mathrm{Cat}}(-, [n]) : \mathrm{N}(\Delta)^{\mathrm{op}} \to \mathrm{An}.$$

Then we have an equivalence of functors $\mathrm{N}^r([n]) \simeq \Delta^n$.

We define the Segal condition and completeness specifically for simplicial anima, although these concepts are applicable to every category.

Definition 4.5. Let $X : \mathrm{N}(\Delta)^{\mathrm{op}} \to \mathrm{An}$ be a simplicial anima. We will say that X is *Segal* if the n-spine inclusion $\mathrm{sp}^n \subseteq \Delta^n$ induces an equivalence of anima

$$X_n \simeq \mathrm{Map}_{\mathrm{Fun}(\mathrm{N}(\Delta)^{\mathrm{op}},\mathrm{An})}(\Delta^n, X) \to \mathrm{Map}_{\mathrm{Fun}(\mathrm{N}(\Delta)^{\mathrm{op}},\mathrm{An})}(\mathrm{sp}^n, X) \simeq X_1 \times_{X_0} \cdots \times_{X_0} X_1$$

for every $n \geq 0$.

The Segal condition can be interpreted as stating that a Segal simplicial anima has a unique spine lifting up to a choice of contractible spaces.

Definition 4.6. Let $X : \mathrm{N}(\Delta)^{\mathrm{op}} \to \mathrm{An}$ be a Segal simplicial anima. We will say that X is *complete* if the following diagram is a Cartesian diagram in An.

$$\begin{tikzcd} X_0 \arrow[r, "\mathrm{diag}"] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & X_0 \times X_0 \arrow[d, "{(s,s)}"] \\ X_3 \arrow[r, "{(d^{\{0,2\}}, d^{\{1,3\}})}"'] & X_1 \times X_1 \end{tikzcd}$$

The completeness condition can be understood as indicating that the higher simplices of a complete Segal simplicial anima correspond to equivalences related to its degenerate edges.

Proposition 4.7. The Rezk nerve $\mathrm{N}^r : \mathrm{Cat} \to \mathrm{Fun}(\mathrm{N}(\Delta)^{\mathrm{op}}, \mathrm{An})$ is fully faithful. Moreover, its essential image precisely consists of complete Segal simplicial anima.

4.2. **The algebraic K-Theory.**

Definition 4.8. Let $\mathcal{C}$ be a category with finite limits. For every $n \geq 0$, we let $Q_n(\mathcal{C})$ denote the full subcategory of $\mathrm{Fun}(\mathrm{TwAr}[n], \mathcal{C})$ spanned by the diagrams which take every square in $\mathrm{TwAr}[n]$ to a Cartesian square in $\mathcal{C}$.

The construction $n \mapsto Q_n(\mathcal{C})$ determines a functor

$$Q(\mathcal{C}) : \mathrm{N}(\Delta)^{\mathrm{op}} \to \mathrm{Cat}^{\mathrm{lex}}$$

and furthermore, the construction $\mathcal{C} \mapsto Q(\mathcal{C})$ defines a functor

$$Q : \mathrm{Cat}^{\mathrm{lex}} \to \mathrm{Fun}(\mathrm{N}(\Delta)^{\mathrm{op}}, \mathrm{Cat}^{\mathrm{lex}}).$$

We will refer to this functor as the *(Quillen's) Q-construction*.

Notation 4.9. For every $n \geq 0$, we let $\mathcal{J}_n$ denote the full subcategory of $\mathrm{TwAr}[n]$ spanned by the images of objects $(i \leq j)$ in $[n]$ satisfying $j \leq i + 1$.

Lemma 4.10. Let $\mathcal{C}$ be a category with finite limits, and let $F : \mathrm{TwAr}[n] \to \mathcal{E}$ be a functor. The following conditions are equivalent:

(1) The functor F belongs to $Q_n(\mathcal{C})$.
(2) The functor F is the right Kan extension of its restriction to $\mathcal{J}_n$ along the inclusion $\mathcal{J}_n \subseteq \mathrm{TwAr}[n]$.

(See proof B.1.)

The next corollary follows from lemma 4.10 immediately.

Corollary 4.11. Let $\mathcal{C}$ be a category with finite limits. Then the restriction of $\mathrm{Fun}(\mathrm{TwAr}[n], \mathcal{C})$ along the inclusion $\mathcal{J}_n \subseteq \mathrm{TwAr}[n]$ induces an equivalence of categories

$$Q_n(\mathcal{C}) \to \mathrm{Fun}(\mathcal{J}_n, \mathcal{C}).$$

Proposition 4.12. Let $\mathcal{C}$ be a category with finite limits. Then the simplicial object in $\mathrm{Cat}^{\mathrm{lex}}$

$$Q(\mathcal{C}) : \mathrm{N}(\Delta)^{\mathrm{op}} \to \mathrm{Cat}^{\mathrm{lex}}$$

is complete Segal. In particular, the simplicial anima

$$\mathrm{core}\, Q(\mathcal{C}) : \mathrm{N}(\Delta)^{\mathrm{op}} \to \mathrm{An}$$

is complete Segal. (See proof B.2.)

Remark 4.13. Corollary 4.11 implies that, if $\mathcal{C}$ is stable, so is $Q_n(\mathcal{C})$. Therefore we obtain functors

$$Q(\mathcal{C}) : \mathrm{N}(\Delta)^{\mathrm{op}} \to \mathrm{Cat}^{\mathrm{st}} \quad \text{and} \quad Q : \mathrm{Cat}^{\mathrm{st}} \to \mathrm{Fun}(\mathrm{N}(\Delta)^{\mathrm{op}}, \mathrm{Cat}^{\mathrm{st}}).$$

Moreover, for every stable category $\mathcal{C}$, the category $Q_n(\mathcal{C})$ is a complete Segal simplicial anima, since $\mathrm{Cat}^{\mathrm{st}}$ is stable under finite limits in Cat.

Definition 4.14. Let $\mathcal{C}$ be a category with finite limits. Then we define the *category of spans* in $\mathcal{C}$ as

$$\mathrm{Span}(\mathcal{C}) := \mathrm{asscat}\,\mathrm{core}\,Q(\mathcal{C}).$$

The construction $\mathcal{C} \mapsto \mathrm{Span}(\mathcal{C})$ determines a functor

$$\mathrm{Span} : \mathrm{Cat}^{\mathrm{lex}} \to \mathrm{Cat}.$$

Definition 4.15. Let $\mathcal{C}$ be a stable category. Then we define the *algebraic K-anima* (or *algebraic K-theory anima*, or *projective class anima*) as

$$\mathcal{K}(\mathcal{C}) := \Omega|\,\mathrm{Span}(\mathcal{C})| \simeq \Omega|\,\mathrm{core}\,Q(\mathcal{C})|$$

where the base object of the loop space is given by the zero object of $\mathrm{Span}(\mathcal{C})$.

The construction $\mathcal{C} \mapsto \mathcal{K}(\mathcal{C})$ determines a functor

$$\mathcal{K} : \mathrm{Cat}^{\mathrm{st}} \to \mathrm{An}.$$

We will refer to this functor as the *algebraic K-theory* (or *algebraic K-functor*).

Definition 4.16. Let $\mathcal{C}$ be a stable category. For every $n \geq 1$, we define the *n-th K-group* of $\mathcal{C}$ as the abelian group

$$\mathcal{K}_n(\mathcal{C}) := \pi_n\mathcal{K}(\mathcal{C}).$$

Proposition 4.17. Let $\mathcal{C}$ be a stable category. Then we have an equivalence

$$\pi_0\mathcal{K}(\mathcal{C}) \simeq \mathcal{K}_0(\mathcal{C})$$

where $\mathcal{K}_0(\mathcal{C})$ is the Grothendieck group of $\mathcal{C}$.

4.3. **Waldhausen's S-Construction.**

In this section, we construct the algebraic K-theory using Waldhausen's S-construction.

Definition 4.18. Let $\mathcal{C}$ be a stable category. An *$[n]$-gapped object* of $\mathcal{C}$ is a functor $F : \mathrm{Ar}[n] \to \mathcal{C}$ which satisfies the following properties:

(1) For every $0 \leq i \leq n$, $F(i,i)$ is a zero object of $\mathcal{C}$.
(2) For every $i \leq j \leq k$, the following diagram is a (co)Cartesian diagram in $\mathcal{C}$.

$$\begin{tikzcd} F(i,j) \arrow[r] \arrow[d] & F(i,k) \arrow[d] \\ 0 \simeq F(j,j) \arrow[r] \arrow[ur, phantom, "\ulcorner"] & F(j,k) \end{tikzcd}$$

We let $S_n(\mathcal{C})$ denote the full subcategory of $\mathrm{Fun}(\mathrm{Ar}[n], \mathcal{C})$ spanned by the $[n]$-gapped objects of $\mathcal{C}$.

Remark 4.19. Let $\mathcal{C}$ be a stable category. We can describe the low-dimensional simplices of $S_n(\mathcal{C})$.

- The category $S_0(\mathcal{C})$ is the full subcategory of $\mathcal{C}$ spanned by the zero objects of $\mathcal{C}$. Thus $S_0(\mathcal{C})$ is contractible.

- The category $S_1(\mathcal{C})$ is equivalent to $\mathcal{C}$, since every object of $S_1(\mathcal{C})$ is of the form $0 \to X \to 0$, where X is an object of $\mathcal{C}$.
- The category $S_2(\mathcal{C})$ is equivalent to the arrow category $\mathrm{Ar}(\mathcal{C})$ of $\mathcal{C}$, since every object of $S_2(\mathcal{C})$ is of the form $0 \to X' \to X \to X'' \to 0$, where $X' \to X \to X''$ is a cofiber sequence in $\mathcal{C}$.

Remark 4.20. Let $\mathcal{C}$ be a stable category. We have an equivalence of categories

$$S_n(\mathcal{C}) \simeq \mathrm{Fun}([n-1], \mathcal{C})$$

for every $n \geq 0$. Thus, if $\mathcal{C}$ is stable, then $S_n(\mathcal{C})$ is stable.

Definition 4.21. The construction $n \mapsto S_n(\mathcal{C})$ determines a functor

$$S(\mathcal{C}) : \mathrm{N}(\Delta)^{\mathrm{op}} \to \mathrm{Cat}^{\mathrm{st}}$$

and furthermore, the construction $\mathcal{C} \mapsto S(\mathcal{C})$ determines a functor

$$S : \mathrm{Cat}^{\mathrm{st}} \to \mathrm{Fun}(\mathrm{N}(\Delta)^{\mathrm{op}}, \mathrm{Cat}^{\mathrm{st}}).$$

We will refer to this functor as *(Waldhausen's) S-construction.*

Definition 4.22. Let $\mathcal{C}$ be a stable category. Then we define the *algebraic K-anima* as

$$\mathcal{K}_S(\mathcal{C}) := \Omega|\operatorname{core} S(\mathcal{C})|.$$

The construction $\mathcal{C} \mapsto \mathcal{K}_S(\mathcal{C})$ determines a functor

$$\mathcal{K}_S : \mathrm{Cat}^{\mathrm{st}} \to \mathrm{An}.$$

We will refer to this functor as the *algebraic K-theory.*

Remark 4.23. Let $\mathcal{C}$ be a stable category. Then the anima $|\operatorname{core} S(\mathcal{C})|$ admits a canonical base point given by a map

$$0 \simeq \operatorname{core} S_0(\mathcal{C}) \to |\operatorname{core} S(\mathcal{C})|.$$

Moreover, $|\operatorname{core} S(\mathcal{C})|$ is connected, since the canonical map

$$0 \simeq \pi_0 \operatorname{core} S_0(\mathcal{C}) \to \pi_0|\operatorname{core} S(\mathcal{C})|$$

is surjective.

Proposition 4.24. The two definitions of algebraic K-anima (definitions 4.15 and 4.22) induce an equivalence of anima

$$\mathcal{K}(\mathcal{C}) \simeq \mathcal{K}_S(\mathcal{C})$$

for every stable category $\mathcal{C}$.

5. The Additivity Theorem

5.1. **The Additivity Theorem.** The goal of this section is to prove the additivity theorem.

Theorem 5.1 ([HLS23] Theorem.4.1 (The Additivity Theorem))**.** *Let $\mathcal{C}$ be a stable category. Then the source-target projection induces an equivalence of anima*

$$|\mathrm{Span}(s,t)| : |\mathrm{Span}(\mathrm{Ar}(\mathcal{C}))| \to |\mathrm{Span}(\mathcal{C})|^2.$$

The proof of theorem 5.1 follows from the next two propositions.

Proposition 5.2. Let $\mathcal{C}$ be a stable category. Then there are canonical equivalences of categories

$$\mathrm{Span}(\mathcal{C}) \to \mathrm{Span}(\mathcal{C}^{\mathrm{op}}) \quad \text{and} \quad \mathrm{Span}(\mathrm{Ar}(\mathcal{C})) \simeq \mathrm{Span}(\mathrm{TwAr}(\mathcal{C})).$$

Moreover, they fit together into a natural commutative diagram

$$\begin{tikzcd} \mathrm{Span}(\mathrm{Ar}(\mathcal{C})) \arrow[r, "\simeq"] \arrow[d, "{(s,t)}"'] & \mathrm{Span}(\mathrm{TwAr}(\mathcal{C})) \arrow[d, "{(s,t)}"] \\ \mathrm{Span}(\mathcal{C}) \times \mathrm{Span}(\mathcal{C}) \arrow[r, "\simeq"'] & \mathrm{Span}(\mathcal{C}) \times \mathrm{Span}(\mathcal{C}^{\mathrm{op}}). \end{tikzcd}$$

Proposition 5.3. Let $\mathcal{C}$ be a stable category. Then the source-target projection

$$(s,t) : \mathrm{Span}(\mathrm{TwAr}(\mathcal{C})) \to \mathrm{Span}(\mathcal{C}) \times \mathrm{Span}(\mathcal{C}^{\mathrm{op}})$$

is cofinal.

We show some corollaries of theorem 5.1.

Corollary 5.4. The algebraic K-theory $\mathcal{K} : \mathrm{Cat}^{\mathrm{st}} \to \mathrm{An}$ is additive grouplike.

Proof. By **??**, it suffices to show that $\mathcal{K}$ is a reduced functor and it is extension splitting. We have

$$\mathcal{K}(0) \simeq \Omega|\,\mathrm{Span}(0)| \simeq \Omega|\,\mathrm{core}\, Q(0)| \simeq 0.$$

By **??**, the algebraic K-theory $\mathcal{K}$ preserves finite products. Then by **??**, it is enough to show that $\mathcal{K}$ sends the source-target projection $(s,t) : \mathrm{Ar}(\mathcal{C}) \to \mathcal{C}$ to an equivalence of anima. That is, there is an equivalence $\mathcal{K}(s,t) : \mathcal{K}(\mathrm{Ar}(\mathcal{C})) \to \mathcal{K}(\mathcal{C})^2$.

By theorem 5.1, we have an equivalence $|\,\mathrm{Span}(\mathrm{Ar}(\mathcal{C}))| \to |\,\mathrm{Span}(\mathcal{C})|^2$. Since the loop functor Ω preserves limits, we obtain an equivalence

$$\mathcal{K}(s,t) : \mathcal{K}(\mathrm{Ar}(\mathcal{C})) \to \mathcal{K}(\mathcal{C})^2.$$

□

Corollary 5.5 (Eilenberg swindle)**.** Let $\mathcal{C}$ be a stable category with countable coproducts. Then the algebraic K-theory anima vanishes.

$$\mathcal{K}(\mathcal{C}) \simeq 0.$$

Proof. We first prove the following proposition: Let $\mathcal{C}$ and $\mathcal{D}$ be stable categories, and let $F' \to F \to F''$ be a cofiber sequence of exact functors from $\mathcal{C}$ to $\mathcal{D}$. Then we have

$$\mathcal{K}(F) = \mathcal{K}(F') + \mathcal{K}(F'').$$

Consider the following functors

$$\mathrm{mid}, \mathrm{fib} + \mathrm{cofib} : \mathrm{Seq}(\mathrm{Fun}^{\mathrm{ex}}(\mathcal{C}, \mathcal{D})) \to \mathrm{Fun}^{\mathrm{ex}}(\mathcal{C}, \mathcal{D}).$$

Since $\mathcal{K}$ is extension splitting (see **??**), by Waldhausen's Additivity Theorem (reference to be added), we have an equivalence

$$\mathcal{K}(\mathrm{mid}) \simeq \mathcal{K}(\mathrm{fib}) + \mathcal{K}(\mathrm{cofib}).$$

We obtain the assertion by applying to this proposition to the cofiber sequence $F' \to F \to F''$.

Consider the functor

$$F : \mathcal{C} \to \mathcal{C} : X \mapsto \bigoplus_{n \in \mathbb{N}} X_n.$$

Then there exists a cofiber sequence

$$\mathrm{id}_{\mathcal{C}} \to F \to F$$

of exact functors on $\mathcal{C}$. By the above proposition, we have an equivalence

$$\mathcal{K}(F) \simeq \mathcal{K}(\mathrm{id}_{\mathcal{C}}) + \mathcal{K}(\mathcal{F}).$$

Thus we have $\mathcal{K}(\mathrm{id}_{\mathcal{C}}) \simeq 0$ and $\mathcal{K}(\mathcal{C}) \simeq 0$. $\square$

5.2. **The algebraic K-Theory Spectrum.** We can define an algebraic K-theory spectrum $\mathcal{K}_{\geq 0} : \mathrm{Cat}^{\mathrm{st}} \to \mathrm{Sp}$.

Definition 5.6. Corollary 5.4 implies that the K-theory functor lifts to a functor

$$\mathcal{K} : \mathrm{Cat}^{\mathrm{st}} \to \mathrm{Grp}_{\mathbb{E}_\infty}(\mathrm{An}).$$

Since we have the equivalence $\mathrm{Grp}_{\mathbb{E}_\infty}(\mathrm{An}) \simeq \mathrm{Sp}_{\geq 0}$, we obtain a functor

$$\mathcal{K}_{\geq 0} : \mathrm{Cat}^{\mathrm{st}} \to \mathrm{Sp}_{\geq 0} \subseteq \mathrm{Sp}.$$

We will refer to this functor as the *algebraic K-theory spectrum*.

Remark 5.7. There is the equivalence $\Omega^\infty : \mathrm{Sp}_{\geq 0} \rightleftarrows \mathrm{Grp}_{\mathbb{E}_\infty}(\mathrm{An}) : \Sigma^\infty$. We can recover the algebraic K-functor from algebraic K-theory spectrum as

$$\mathcal{K} \simeq \Omega^\infty \mathcal{K}_{\geq 0} : \mathrm{Cat}^{\mathrm{st}} \to \mathrm{Sp}_{\geq 0} \simeq \mathrm{Grp}_{\mathbb{E}_\infty}(\mathrm{An})$$

since Σ^∞ is fully faithful.

6. The Localization Theorem

The goal of this section is to prove the localization theorem.

Theorem 6.1 ([HLS23] Theorem.6.1 (The Localization Theorem))**.** *The algebraic K-theory $\mathcal{K} : \mathrm{Cat}^{\mathrm{st}} \to \mathrm{An}$ and the algebraic K-theory spectrum $\mathcal{K}_{\geq 0} : \mathrm{Cat}^{\mathrm{st}} \to \mathrm{Sp}$ are Verdier-localizing.*

By the corollary of theorem 3.23, an additive grouplike functor $F : \mathrm{Cat}^{\mathrm{st}} \to \mathrm{An}$ is Verdier-localizing if and only if it satisfies the following condition:

($*$) For every Verdier sequence $\mathcal{C} \to \mathcal{D} \to \mathcal{E}$, the canonical map $|F\,\mathrm{Fun}^{\mathcal{C}}([-], \mathcal{D})| \to F(\mathcal{E})$ is an equivalence of anima.

The next proposition implies that it is enough to prove that the core functor satisfies ($*$).

Proposition 6.2. Let $F : \mathrm{Cat}^{\mathrm{st}} \to \mathrm{An}$ be an additive functor. If F satisfies ($*$), then $|FQ(-)|$ and $\Omega|FQ(-)|$ also satisfy ($*$).

Remark 6.3. If the core functor satisfies ($*$), then the K-theory functor also satisfies ($*$). Indeed, we can write

$$\mathcal{K}(-) \simeq \Omega|\,\mathrm{Span}(-)| \simeq \Omega|\,\mathrm{asscat}\,\mathrm{core}\, Q(-)| \simeq \Omega|\,\mathrm{core}\, Q(-)|.$$

To prove the proposition, we need the following lemma.

Lemma 6.4. Verdier sequences are stable under applying the functor

$$\mathrm{Fun}(\mathcal{I}, -) : \mathrm{Cat}^{\mathrm{st}} \to \mathrm{Cat}^{\mathrm{st}}$$

for every finite poset $\mathcal{I}$.

From the above discussion, we need to show that the core functor satisfies ($*$). Let $\mathcal{D}$ be a stable category, and let $\mathcal{C}$ be a stable subcategory of $\mathcal{D}$. We let $\mathcal{D}_{\mathcal{C}}$ denote the full subcategory of $\mathcal{D}$ spanned by the equivalences modulo $\mathcal{C}$ in $\mathcal{D}$. Then we obtain

$$\mathrm{core}\,\mathrm{Fun}^{\mathcal{C}}([-], \mathcal{D}) \simeq \mathrm{core}\,\mathrm{Fun}([-], \mathcal{D}_{\mathcal{C}}) \simeq \mathrm{Map}_{\mathrm{Cat}}([-], \mathcal{D}_{\mathcal{C}}) \simeq \mathrm{N}^r(\mathcal{D}_{\mathcal{C}}).$$

Since the canonical map $|\,\mathrm{N}^r(\mathcal{D}_{\mathcal{C}})| \to |\mathcal{D}_{\mathcal{C}}|$ is an equivalence of anima, we have an equivalence

$$|\,\mathrm{core}\,\mathrm{Fun}^{\mathcal{C}}([-], \mathcal{D})| \simeq |\mathcal{D}_{\mathcal{C}}|.$$

Thus it suffices to show the following proposition.

Proposition 6.5. Let $\mathcal{D}$ be a stable category, and let $\mathcal{C}$ be a stable subcategory of $\mathcal{D}$. We let $\mathcal{D}_{\mathcal{C}}$ denote the full subcategory of $\mathcal{D}$ spanned by the equivalences modulo $\mathcal{C}$ in $\mathcal{D}$. Then the map

$$|\mathcal{D}_{\mathcal{C}}| \to \mathrm{core}(\mathcal{D}/\mathcal{C})$$

is faithful. If the inclusion $\mathcal{C} \subseteq \mathcal{D}$ is a Verdier inclusion, then this map is an equivalence.

This proposition is a special case of the following proposition.

Proposition 6.6. Let $\mathcal{C}$ be a category, and let S be a subcategory of $\mathcal{C}$. If S is closed under 2-out-of-3 and pushouts in $\mathcal{C}$, then a map

$$|S| = S[S^{-1}] \to \mathcal{C}$$

is faithful. Moreover, the following conditions are equivalent:

(1) The inclusion $|S| \subseteq \mathrm{core}\,\mathcal{C}[S^{-1}]$ is fully faithful.

(2) The category S is closed under 2-out-of-6 in $\mathcal{C}$.

(3) A morphism in $\mathcal{D}$ belongs to S if and only if its source and target are in S and it is invertible in $\mathcal{C}[S^{-1}]$.

7. The Universality Theorem

The goal of this section is to prove the universality theorem.

Theorem 7.1 ([HLS23] Theorem.5.1 (The Universality Theorem))**.** *The algebraic K-theory* $\mathcal{K} : \mathrm{Cat}^{\mathrm{st}} \to \mathrm{An}$ *is an initial additive grouplike functor under the core functor* $\mathrm{core} : \mathrm{Cat}^{\mathrm{st}} \to \mathrm{An}$. *That is, the natural map* $\tau : \mathrm{core} \Rightarrow \mathcal{K}$ *is an initial object in the category* $\mathrm{Fun}(\mathrm{Cat}^{\mathrm{st}}, \mathrm{An})^{\mathrm{add,grp}}_{\mathrm{core}/}$.

Notation 7.2. Let $F : \mathrm{Cat}^{\mathrm{st}} \to \mathrm{An}$ be a reduced functor. We denote a functor

$$GF(-) := \Omega|FQ(-)| : \mathrm{Cat}^{\mathrm{st}} \to \mathrm{An}.$$

For example, the functor $G\,\mathrm{core}$ is equivalent to the algebraic K-theory $\mathcal{K}$.

Proof. We want to show that the natural transformation $\tau : \mathrm{core} \Rightarrow \mathcal{K}$ induces an equivalence

$$\tau^* : \mathrm{Nat}(\mathcal{K}, F) \to \mathrm{Nat}(\mathrm{core}, F)$$

for every additive grouplike functor $F : \mathrm{Cat}^{\mathrm{st}} \to \mathrm{An}$.

Now consider the following diagram

$$\begin{tikzcd}
\mathrm{Nat}(G\,\mathrm{core}, F) \arrow[r, "(\eta_{\mathrm{core}})^*"] \arrow[d, "G"] \arrow[dd, bend right=60, "(\eta_F)^*"'] & \mathrm{Nat}(\mathrm{core}, F) \arrow[d, "G"'] \arrow[dd, bend left=60, "(\eta_F)^*"] \\
\mathrm{Nat}(GG\,\mathrm{core}, GF) \arrow[r, "(G\eta_{\mathrm{core}})^*"] \arrow[d, "(\eta_{G\,\mathrm{core}})^*"] & \mathrm{Nat}(G\,\mathrm{core}, GF) \arrow[d, "(\eta_{\mathrm{core}})^*"'] \\
\mathrm{Nat}(G\,\mathrm{core}, GF) \arrow[r, "(\eta_{\mathrm{core}})^*"] & \mathrm{Nat}(\mathrm{core}, GF)
\end{tikzcd}$$

where the upper square commutes since G is a functor, and the other there parts commute since η is natural. Suppose the red-colored maps are equivalent, then we can show that the upper horizontal map $(\eta_{\mathrm{core}})^*$ is an equivalence. If we apply this to the case $F \simeq \mathrm{core}$, then we obtain the desired result. This assumption follows from the next two propositions. $\square$

The next proposition implies that $(\eta_F)_*$ and $(G\eta_{\mathrm{core}})^*$ are equivalences.

Proposition 7.3. Let $F : \mathrm{Cat}^{\mathrm{st}} \to \mathrm{An}$ be an additive grouplike functor. Then the natural transformation

$$\eta_F : F \Rightarrow GF$$

is an equivalence.

The next proposition implies that $(\eta_{G\,\mathrm{core}})$ is an equivalence.

Proposition 7.4. Let $F : \mathrm{Cat}^{\mathrm{st}} \to \mathrm{An}$ be an additive functor. Then the two natural transformations

$$\eta_{GF}, G\eta_F : GF \Rightarrow GGF$$

differ by an automorphism of the target. That is, the following diagram commutes

$$\begin{tikzcd} GF \arrow[r, "G\eta_F"] \arrow[dr, "\eta_{GF}"'] & GGF \arrow[d, "\simeq"] \\ & GGF. \end{tikzcd}$$

8. The Cofinality Theorem

The goal of this section is to prove the cofinality theorem.

Theorem 8.1 ([HLS23] Theorem.7.1 (The Cofinality Theorem))**.** *Let $f : \mathcal{C} \to \mathcal{D}$ be a functor between stable categories. If f is a dense inclusion, then it induces a fiber sequence*

$$\mathcal{K}(\mathcal{C}) \to \mathcal{K}(\mathcal{D}) \to \mathcal{K}_0(\mathcal{D})/\mathcal{K}_0(\mathcal{C}).$$

In particular, maps of abelian groups

$$\mathcal{K}_i(\mathcal{C}) \to \mathcal{K}_i(\mathcal{D})$$

are isomorphisms for every $i \geq 1$, and there exists a short exact sequence

$$0 \to \mathcal{K}_0(\mathcal{C}) \to \mathcal{K}_0(\mathcal{D}) \to \mathcal{K}_0(\mathcal{D})/\mathcal{K}_0(\mathcal{C}) \to 0.$$

Definition 8.2. Let $f : X \to Y$ be a map of $\mathbb{E}_\infty$-monoids in An. We will say that f is *cofinal* if it satisfies the following conditions:

(1) The map $\pi_0(f) : \pi_0(X) \to \pi_0(Y)$ is an inclusion.

(2) For every object y in $\pi_0(Y)$, there exists an object y' in $\pi_0(Y)$ such that $y + y'$ in $\pi_0(X)$.

We will say that a cofinal map is *dense* if it satisfies the following conditions:

(3) An object y in $\pi_0(Y)$ belongs to $\pi_0(X)$ if there exists an object x in $\pi_0(X)$ such that $x + y$ in $\pi_0(X)$.

Definition 8.3. Let $F : \mathrm{Cat}^{\mathrm{st}} \to \mathrm{An}$ be an additive functor. We will say that F is *Karoubian* if it satisfies the following conditions:

(1) The functor F takes every dense inclusion between stable categories to a dense map of $\mathbb{E}_\infty$-monoids.

(2) The functor F preserves every Cartesian square in $\mathrm{Cat}^{\mathrm{st}}$ whose vertical maps are dense.

Example 8.4. The core functor $\mathrm{core} : \mathrm{Cat}^{\mathrm{st}} \to \mathrm{An}$ is Karoubian.

Theorem 8.1 holds for a broader class of additive Karoubian functors.

Notation 8.5. Let $F : \mathrm{Cat}^{\mathrm{st}} \to \mathrm{An}$ be an additive Karoubian functor. We will refer to the functor

$$F^{\mathrm{grp}} := \Omega|FQ-|$$

as the *group completion* of F. For example, the functor $(\mathrm{core})^{\mathrm{grp}}$ is equivalent to the algebraic K-theory $\mathcal{K}$.

Theorem 8.6. *Let $F : \mathrm{Cat}^{\mathrm{st}} \to \mathrm{An}$ be an additive Karoubian functor. For every dense inclusion $\mathcal{C} \subseteq \mathcal{D}$ between stable categories, the canonical map of $\mathbb{E}_\infty$-monoids*

$$F(\mathcal{D})/F(\mathcal{C}) \to F^{\mathrm{grp}}(\mathcal{D})/F^{\mathrm{grp}}(\mathcal{C})$$

is an equivalence. Hence maps of abelian groups

$$\pi_i F^{\mathrm{grp}}(\mathcal{C}) \to \pi_i F^{\mathrm{grp}}(\mathcal{D})$$

are isomorphisms for every $i \geq 1$, and there exists a short exact sequence

$$0 \to \pi_0 F^{\mathrm{grp}}(\mathcal{C}) \to \pi_0 F^{\mathrm{grp}}(\mathcal{D}) \to \pi_0 F^{\mathrm{grp}}(\mathcal{D})/\pi_0 F^{\mathrm{grp}}(\mathcal{C}) \to 0.$$

Corollary 8.7. If a functor $F : \mathrm{Cat}^{\mathrm{st}} \to \mathrm{An}$ is additive Karoubian, then so is the group completion F^{grp} of F.

Corollary 8.8. Let $F : \mathrm{Cat}^{\mathrm{st}} \to \mathrm{An}$ be an additive Karoubian functor. If the functor F^{grp} is Verdier-localizing, then the functor

$$F^{\mathrm{grp}} \circ (-)^{\natural} : \mathrm{Cat}^{\mathrm{st}} \to \mathrm{An}$$

is Karoubi-localizing. In particular, the functor $\mathcal{K} \circ (-)^{\natural} : \mathrm{Cat}^{\mathrm{st}} \to \mathrm{An}$ is Karoubi-localizing.

Corollary 8.9. Let $\mathcal{D}$ be a stable category, and let $\mathcal{C}$ be a dense stable subcategory of $\mathcal{D}$. Then the canonical maps of abelian groups

$$\mathcal{K}_i(\mathcal{C}) \to \mathcal{K}_i(\mathcal{D})$$

are isomorphisms for every $i \geq 1$.

Appendix A. Proofs in Section 3

Proof A.1 (Remark 3.12)**.** We show that every Verdier sequence square is a coCartesian diagram in $\mathrm{Cat}^{\mathrm{st}}$. Consider the following Verdier sequence.

$$\begin{tikzcd}
\mathcal{D} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathcal{D}' \arrow[d] \\
\mathcal{E} \arrow[r] & \mathcal{E}',
\end{tikzcd}$$

where the vertical maps are Verdier projections. Then we can extend to the following diagram.

$$\begin{tikzcd}
\mathcal{C} \arrow[r] \arrow[d] & \mathcal{D} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathcal{D}' \arrow[d] \\
0 \arrow[r] & \mathcal{E} \arrow[r] & \mathcal{E}'
\end{tikzcd}$$

By definition, the left and outer squares are biCartesian squares. Then the right square is also a biCartesian square.

Proof A.2 (Proposition 3.15)**.** (1) ⇒ (2) is obvious. (2) ⇒ (1): Consider the following Verdier sequence.

$$\begin{tikzcd} \mathcal{D} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathcal{D}' \arrow[d] \\ \mathcal{E} \arrow[r] & \mathcal{E}', \end{tikzcd}$$

Then we can extend to the following diagram.

$$\begin{tikzcd} \mathcal{C} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathcal{D} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathcal{D}' \arrow[d] \\ 0 \arrow[r] & \mathcal{E} \arrow[r] & \mathcal{E}' \end{tikzcd}$$

By definition, the left, right, outer squares are Cartesian squares. Thus the sequences $\mathcal{C} \to \mathcal{D} \to \mathcal{E}$ and $\mathcal{C} \to \mathcal{D}' \to E'$ are Verdier sequences in $\mathrm{Cat}^{\mathrm{st}}$. By assumption, the sequences $F(\mathcal{C}) \to F(\mathcal{D}) \to F(\mathcal{E})$ and $F(\mathcal{C}) \to F(\mathcal{D}') \to F(\mathcal{E}')$ are fiber sequences in $\mathcal{E}$. Then the left and outer squares in the following diagram are Cartesian squares.

$$\begin{tikzcd} F(\mathcal{C}) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & F(\mathcal{D}) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & F(\mathcal{D}') \arrow[d] \\ * \arrow[r] & F(\mathcal{E}) \arrow[r] & F(\mathcal{E}') \end{tikzcd}$$

Then the right square is also a Cartesian square.

Proof A.3 (Proposition 3.18)**.** We show that every Verdier-localizing functor preserves finite products. Let $\mathcal{E}$ be a category with finite limits, and let $F : \mathrm{Cat}^{\mathrm{st}} \to \mathcal{E}$ be a Verdier-localizing functor. The following diagram is a Cartesian square in $\mathrm{Cat}^{\mathrm{st}}$.

$$\begin{tikzcd} \mathcal{C} \times \mathcal{D} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathcal{D} \arrow[d] \\ \mathcal{C} \arrow[r] & 0. \end{tikzcd}$$

Applying the functor F, we obtain the following Cartesian square in $\mathcal{E}$.

$$\begin{tikzcd} F(\mathcal{C} \times \mathcal{D}) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & F(\mathcal{D}) \arrow[d] \\ F(\mathcal{C}) \arrow[r] & *. \end{tikzcd}$$

This implies that

$$F(\mathcal{C} \times \mathcal{D}) \simeq F(\mathcal{C}) \times F(\mathcal{D}).$$

Appendix B. Proofs in Section 4

Proof B.1 (Lemma 4.10)**.** The map

$$e_i : [1] \to [n] : 0 \mapsto i \text{ and } 1 \mapsto i+1$$

in Δ induces an equivalence of categories

$$\mathcal{J}_n \simeq \mathcal{J}_1 \coprod_{\mathcal{J}_0} \mathcal{J}_1 \cdots \coprod_{\mathcal{J}_0} \mathcal{J}_1 \simeq \mathrm{TwAr}[1] \coprod_{\mathrm{TwAr}[0]} \mathrm{TwAr}[1] \cdots \coprod_{\mathrm{TwAr}[0]} \mathrm{TwAr}[1]$$

in $\mathrm{Cat}^{\mathrm{lex}}$. Then the right Kan extension along the inclusion $\mathcal{J}_n \subseteq \mathrm{TwAr}[n]$ factors through $n(n-1)/2$-times the right Kan extension along the inclusion $\mathcal{J}_i \subseteq \mathrm{TwAr}[i]$ for $2 \leq n$. The right Kan extension along the inclusion $\mathcal{J}_2 \subseteq \mathrm{TwAr}[2]$ correspondences the operation of taking the pullback. Then the equivalence of conditions follows from that a functor F belong to $Q_n(\mathcal{C})$ if and only if each square is a Cartesian square.

Proof B.2 (Proposition 4.12)**.**

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