NOTES ON SIX-FUNCTOR FORMALISMS (UNDER CONSTRUCTION)

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Abstract. We summarize key concepts and results on six-functor formalisms, focusing on their foundational aspects.

Contents

1. Introduction

The aim of this note is to explain *what a six-functor formalism is* and summarize the basic results and examples of six-functor formalisms. A six-functor formalism is a framework to formalize a cohomology theory, which provide a structured and systematic approach to handling cohomological operations.

The concept of six functors for sheaves originated in Grothendieck's work, and these are often referred to as Grothendieck's six operations.

The first such formalism was developed to establish the étale cohomology of schemes, as formulated by Yifeng Liu and Weizhe Zheng [LZ17]. A subsequent formalism, intended for coherent cohomology of schemes, was provided by Dennis Gaitsgory and Nick Rozenblyum [GR17]. The former approach is based on heavy combinatorics of specific simplicial sets, while the latter uses the framework of $(\infty, 2)$-categories.

More recently, Lucas Mann [Man22] has proposed a refined definition that combines favorable aspects of both methods. We will present Mann's definition after reviewing classical sheaf theory and Grothendieck's six operations.

1.1. Notations.

We will identify 1-categories, quasi-categories, and complete Segal anima using the nerve and Rezk nerve constructions. From this point forward, all categories are assumed to be ∞-categories, specifically complete Segal anima.

In section 2, we let

- Cat denote the category of small categories.
- $\mathrm{Corr}(\mathcal{C}, \mathcal{C}_0)$ denote the category of correspondences for a geometric setup $(\mathcal{C}, \mathcal{C}_0)$.
- GS denote the category of geometric setups.

2. The Category of Correspondences

2.1. **Motivations.** Let us recall the setup of classical sheaf theory. We work with the category $\mathcal{C}$ of locally compact Hausdorff topological spaces. In this setting, there exists a class E of morphisms in $\mathcal{C}$, where the exceptional functors $f_!$ and $f^!$ can be defined. Then we can

- For every object X of $\mathcal{C}$, we obtain the category $\mathrm{D}(X)$ of sheaves on X.
- For every morphism $f : X \to Y$ in $\mathcal{C}$, we have functors

$$f^* : \mathrm{D}(Y) \to \mathrm{D}(X) \quad \text{and} \quad f_* : \mathrm{D}(X) \to \mathrm{D}(Y).$$

- For every morphism $f : X \to Y$ in E, we can define functors

$$f_! : \mathrm{D}(X) \to \mathrm{D}(Y) \quad \text{and} \quad f^! : \mathrm{D}(Y) \to \mathrm{D}(X).$$

- Additionally, there exist bifunctors

$$\otimes : \mathrm{D}(X) \times \mathrm{D}(X) \to \mathrm{D}(X) \quad \text{and} \quad \underline{\mathrm{Hom}} : \mathrm{D}(X)^{\mathrm{op}} \times \mathrm{D}(X) \to \mathrm{D}(X).$$

Moreover, there exists a natural map

$$\mathrm{D}(X) \otimes \mathrm{D}(Y) \to \mathrm{D}(X \times Y).$$

For $Y = X$, this induces a morphism

$$\mathrm{D}(X) \otimes \mathrm{D}(X) \to \mathrm{D}(X \times X) \xrightarrow{\Delta^*} \mathrm{D}(X),$$

where the second morphism is given by the diagonal $\Delta : X \to X \times X$.

In summary, the constructions $X \mapsto \mathrm{D}(X)$ and $f \mapsto f^*$ define a lax symmetric monoidal functor

$$\mathrm{D} : \mathcal{C}^{\mathrm{op}} \to \mathrm{Cat}.$$

Similarly, the constructions $X \mapsto \mathrm{D}(X)$ and $f \mapsto f_!$ define a functor

$$\mathrm{D}_E : \mathcal{C}_E \to \mathrm{Cat},$$

where $\mathcal{C}_E$ is the wide subcategory of $\mathcal{C}$ consisting of the morphisms in E.

To unify the different directions of morphisms f^* and $f_!$, we consider the category $\mathrm{Corr}(\mathcal{C}, E)$ of correspondences:

- The objects of $\mathrm{Corr}(\mathcal{C}, E)$ are the objects of $\mathcal{C}$.
- A morphism $X \to Y$ in $\mathrm{Corr}(\mathcal{C}, E)$ is a correspondence $X \leftarrow W \to Y$ in $\mathcal{C}$, where $W \to Y$ lies in E.

There are two inclusions

$$\mathcal{C}^{\mathrm{op}} \to \mathrm{Corr}(\mathcal{C}, E) : (Y \xrightarrow{f^{\mathrm{op}}} X) \mapsto (Y \xleftarrow{f} X \xrightarrow{\mathrm{id}_X} X),$$

$$E \to \mathrm{Corr}(\mathcal{C}, E) : (X \xrightarrow{f} Y) \mapsto (X \xleftarrow{\mathrm{id}_X} X \xrightarrow{f} Y).$$

Then the above discussion can be summarized as follows: A three-functor formalism is a lax symmetric monoidal functor

$$\mathrm{D} : \mathrm{Corr}(\mathcal{C}, E) \to \mathrm{Cat}.$$

To define a six-functor formalism, we don't need any extra data. That is, a six-functor formalism is a special case of a three-functor formalism, where three-functors $\otimes$, f^* and $f_!$ admit right adjoint functors.

2.2. Geometric Setups.

Definition 2.1 ([HM24] Definition 2.1.1)**.** A *geometric setup* $(\mathcal{C}, \mathcal{C}_0)$ consists of a category $\mathcal{C}$ and a wide subcategory $\mathcal{C}_0$ of $\mathcal{C}$ satisfying the following conditions:

(1) The subcategory $\mathcal{C}_0$ is closed under pullbacks along morphisms in $\mathcal{C}$.

(2) The subcategory $\mathcal{C}_0$ admits and the inclusion $\mathcal{C}_0 \subseteq \mathcal{C}$ preserves pullbacks.

Definition 2.2. Let $(\mathcal{C}, \mathcal{C}_0)$ and $(\mathcal{D}, \mathcal{D}_0)$ be geometric setups. A *morphism of geometric setups* from $(\mathcal{C}, \mathcal{C}_0)$ to $(\mathcal{D}, \mathcal{D}_0)$ is a functor $f : \mathcal{C} \to \mathcal{D}$ which preserves pullbacks of the form $X \xrightarrow{f} Y \xleftarrow{g} Z$ in $\mathcal{C}$ where f lies in $\mathcal{C}_0$. We let $\mathrm{Fun}^{\mathrm{GS}}(\mathcal{C}, \mathcal{D})$ denote the full subcategory of $\mathrm{Fun}(\mathcal{C}, \mathcal{D})$ spanned by morphisms of geometric setups.

Example 2.3. Let $\mathcal{C}$ be a category.

(1) If $\mathcal{C}$ admits pullbacks, then the pair $\mathcal{C}^\sharp := (\mathcal{C}, \mathcal{C})$ is a geometric setup.

(2) The pair $\mathcal{C}^\flat := (\mathcal{C}^{\mathrm{op}}, \mathrm{core}\mathcal{C})$ is a geometric setup.

(3) Let $(\mathcal{C}, \mathcal{C}_0)$ be a geometric setup. Then the inclusion $\mathcal{C}_0 \subseteq \mathcal{C}$ induces a morphism of geometric setups $\mathcal{C}_0^\sharp \to (\mathcal{C}, \mathcal{C}_0)$.

(4) Let $(\mathcal{C}, \mathcal{C}_0)$ be a geometric setup. Then the identity $\mathcal{C} \subseteq \mathcal{C}$ induces a morphism of geometric setups $\mathcal{C}^\flat \to (\mathcal{C}, \mathcal{C}_0)$.

Proof. (1) and (3) are clear. (2) and (4) follow from the fact that the pullback of isomorphisms is also an isomorphism. □

Definition 2.4. We define the simplicial category GS^Δ as follows:

- The objects of GS^Δ are geometric setups.
- For every pair of objects $(\mathcal{C}, \mathcal{C}_0)$ and $(\mathcal{D}, \mathcal{D}_0)$ of GS^Δ, the mapping space

$$\mathrm{Map}_{\mathrm{GS}^\Delta}((\mathcal{C}, \mathcal{C}_0), (\mathcal{D}, \mathcal{D}_0)) := \mathrm{core}\,\mathrm{Fun}^{\mathrm{GS}}(\mathcal{C}, \mathcal{D}).$$

Then the simplicial nerve of GS^Δ is a category by [HTT, Proposition 1.1.5.10]. We let GS denote the simplicial nerve $\mathrm{N}_\Delta(\mathrm{GS}^\Delta)$. We will refer to GS as the *category of geometric setups.*

Lemma 2.5 ([HM24] Lemma 2.1.6)**.** The category GS admits small limits.

Proof. Let $I \to \mathrm{GS} : i \mapsto (\mathcal{C}_i, (\mathcal{C}_0)_i)$ be a small diagram. We show that a pair $(\lim_i \mathcal{C}_i, \lim_i(\mathcal{C}_0)_i)$ is a limit in GS. □

2.3. The Category of Correspondences.

Definition 2.6 ([HM24] Definition 2.2.1)**.** For every $n \geq 0$, we let

(1) Σ_n denote the poset of pairs (i, j) with $0 \leq i \leq j \leq n$, where $(i, j) \leq (i', j')$ if and only if $i \leq i'$ and $j' \leq j$.

(2) $\Sigma_n^!$ denote the subposet of Σ_n, where $(i, j) \leq (i', j')$ if and only if $i \leq i'$ and $j' = j$.

Remark 2.7. For every $n \geq 0$, the pair $(\Sigma_n, \Sigma_n^!)$ is a geometric setup by definition. Thus the construction $[n] \mapsto (\Sigma_n, \Sigma_n^!)$ defines a cosimplicial object

$$(\Sigma, \Sigma^!) : \mathbb{\Delta} \to \mathrm{GS}.$$

Definition 2.8 ([HM24] Definition 2.2.10)**.** Let $(\mathcal{C}, \mathcal{C}_0)$ be a geometric setup. We define a functor

$$\mathrm{Corr}(\mathcal{C}, \mathcal{C}_0) : \mathbb{\Delta}^{\mathrm{op}} \to \mathrm{An}$$

by

$$\mathrm{Corr}(\mathcal{C}, \mathcal{C}_0)_n := \mathrm{Map}_{\mathrm{GS}}((\Sigma_n, \Sigma_n^!), (\mathcal{C}, \mathcal{C}_0))$$

for every $n \geq 0$. We will refer to it as the *category of correspondences*.

Remark 2.9. The construction $(\mathcal{C}, \mathcal{C}_0) \mapsto \mathrm{Corr}(\mathcal{C}, \mathcal{C}_0)$ induces a functor

$$\mathrm{Corr} : \mathrm{GS} \to \mathrm{Fun}(\Delta^{\mathrm{op}}, \mathrm{An}).$$

$$\begin{tikzcd}
\mathrm{Fun}(\Delta^{\mathrm{op}}, \mathrm{An}) & \\
\Delta \arrow[u, "よ"] \arrow[r, "{(\Sigma, \Sigma^!)}"'] & \mathrm{GS}. \arrow[ul, "\mathrm{Corr}"']
\end{tikzcd}$$

Proposition 2.10 ([HM24] Proposition 2.2.9)**.** Let $(\mathcal{C}, \mathcal{C}_0)$ be a geometric setup. Then the category of correspondences $\mathrm{Corr}(\mathcal{C}, \mathcal{C}_0) : \Delta^{\mathrm{op}} \to \mathrm{An}$ is a complete Segal anima. Thus the functor $\mathrm{Corr} : \mathrm{GS} \to \mathrm{Fun}(\Delta^{\mathrm{op}}, \mathrm{An})$ factors through the full subcategory of complete Segal anima.

Lemma 2.11 ([HM24] Lemma 2.2.11)**.** The functor $\mathrm{Corr} : \mathrm{GS} \to \mathrm{Fun}(\Delta^{\mathrm{op}}, \mathrm{An})$ preserves small limits.

Proof. Let $I \to \mathrm{GS} : i \mapsto (\mathcal{C}_i, (\mathcal{C}_0)_i)$ be a small diagram. We show that a natural transformation

$$\mathrm{Corr}(\lim_i \mathcal{C}_i, \lim_i (\mathcal{C}_0)_i) \to \lim_i \mathrm{Corr}(\mathcal{C}_i, (\mathcal{C}_0)_i)$$

is an isomorphism. Since the limits in $\mathrm{Fun}(\Delta^{\mathrm{op}}, \mathrm{An})$ can be calculated pointwise, it suffices to show that a morphism of anima

$$\mathrm{Corr}(\lim_i \mathcal{C}_i, \lim_i (\mathcal{C}_0)_i)_n \to \lim_i \mathrm{Corr}(\mathcal{C}_i, (\mathcal{C}_0)_i)_n$$

is an isomorphism for every $n \geq 0$. It follows from the fact the functor $\mathrm{Map}_{\mathrm{GS}}((\Sigma_n, \Sigma_n^!), -)$ commutes with limits for every $n \geq 0$. $\square$

2.4. **Two Inclusions to $\mathrm{Corr}(\mathcal{C}, \mathcal{C}_0)$.** In this section, we construct two inclusions $\mathcal{C}_0 \to \mathrm{Corr}(\mathcal{C}, \mathcal{C}_0)$ and $\mathcal{C}^{\mathrm{op}} \to \mathrm{Corr}(\mathcal{C}, \mathcal{C}_0)$.

Lemma 2.12. (1) For every $n \geq 0$, the first-component projection $(\Sigma_n, \Sigma_n^!) \to [n]^\sharp$ is a morphism of geometric setups.

(2) For every $n \geq 0$, the second-component projection $(\Sigma_n, \Sigma_n^!) \to [n]^{\mathrm{op},\flat}$ is a morphism of geometric setups.

Moreover, these constructions define morphisms of cosimplicial objects $(\Sigma, \Sigma^!) \to [-]^\sharp$ and $(\Sigma, \Sigma^!) \to [-]^{\mathrm{op},\flat}$.

Lemma 2.13. not sure. For every $n \geq 0$, the second-component projection $(\Sigma_n, \Sigma_n^!) \to [n]^{\mathrm{op},\flat}$ is an isomorphism of geometric setups.

Remark 2.14. Let $(\mathcal{C}, \mathcal{C}_0)$ be a geometric setup. Then the composition

$$\mathrm{Map}_{\mathrm{Cat}}([n], \mathcal{C}_0) \simeq \mathrm{Map}_{\mathrm{GS}}([n]^\sharp, \mathcal{C}_0^\sharp) \to \mathrm{Map}_{\mathrm{GS}}((\Sigma_n, \Sigma_n^!), \mathcal{C}_0^\sharp) \to \mathrm{Map}_{\mathrm{GS}}((\Sigma_n, \Sigma_n^!), (\mathcal{C}, \mathcal{C}_0))$$

defines a morphism of anima

$$(\mathcal{C}_0)_n \to \mathrm{Corr}(\mathcal{C}, \mathcal{C}_0)_n.$$

Since all constructions are functorial, it determines a morphism of cosimplicial objects

$$\mathcal{C}_0 \to \mathrm{Corr}(\mathcal{C}, \mathcal{C}_0).$$

Remark 2.15. Let $(\mathcal{C}, \mathcal{C}_0)$ be a geometric setup. Then the composition

$$\mathrm{Map}_{\mathrm{Cat}}([n]^{\mathrm{op}}, \mathcal{C}) \simeq \mathrm{Map}_{\mathrm{GS}}([n]^{\mathrm{op},\flat}, \mathcal{C}^\flat) \to \mathrm{Map}_{\mathrm{GS}}((\Sigma_n, \Sigma_n^!), C^\flat) \to \mathrm{Map}_{\mathrm{GS}}((\Sigma_n, \Sigma_n^!), (\mathcal{C}, \mathcal{C}_0))$$

defines a morphism of anima

$$(\mathcal{C}^{\mathrm{op}})_n \to \mathrm{Corr}(\mathcal{C}, \mathcal{C}_0)_n.$$

Since all constructions are functorial, it determines a morphism of cosimplicial objects

$$\mathcal{C}^{\mathrm{op}} \to \mathrm{Corr}(\mathcal{C}, \mathcal{C}_0).$$

2.5. The Operad Structure on $\mathrm{Corr}(\mathcal{C}, \mathcal{C}_0)$.

Lemma 2.16 ([HM24] Lemma 2.3.1)**.** Let $(\mathcal{C}, \mathcal{C}_0)$ be a geometric setup. We denote

$$\mathcal{C}_0^- := (\mathcal{C}_0^{\mathrm{op}})^{\amalg,\mathrm{op}} \times_{\mathrm{Fin}_*^{\mathrm{op}}} \mathrm{coreFin}_*.$$

Then the pair $((\mathcal{C}_0^{\mathrm{op}})^{\amalg,\mathrm{op}}, \mathcal{C}_0^-)$ is a geometric setup.

Proof. □

Proposition 2.17 ([HM24] Proposition 2.3.3)**.** Let $(\mathcal{C}, \mathcal{C}_0)$ be a geometric setup. We denote

$$\mathrm{Corr}(\mathcal{C}, \mathcal{C}_0)^\otimes := \mathrm{Corr}((\mathcal{C}_0^{\mathrm{op}})^{\amalg,\mathrm{op}}, \mathcal{C}_0^-).$$

Then $\mathrm{Corr}(\mathcal{C}, \mathcal{C}_0)^\otimes$ is an operad whose underlying category is $\mathrm{Corr}(\mathcal{C}, \mathcal{C}_0)$.

Proposition 2.18 ([HM24] Proposition 2.3.5)**.** The functor $(\mathcal{C}, \mathcal{C}_0) \to \mathrm{Corr}(\mathcal{C}, \mathcal{C}_0)^\otimes$ enhances to a functor

$$\mathrm{Corr}^\otimes : \mathrm{GS} \to \mathrm{Op}.$$

Proposition 2.19 ([HM24] Proposition 2.3.7)**.** Let $(\mathcal{C}, \mathcal{C}_0)$ be a geometric setup. The functor $\mathcal{C}^{\mathrm{op}} \to \mathrm{Corr}(\mathcal{C}, \mathcal{C}_0)$ enhances to a functor

$$\mathcal{C}^{\mathrm{op},\amalg} \to \mathrm{Corr}(\mathcal{C}, \mathcal{C}_0)^\otimes.$$

2.6. The Symmetric Monoidal Structure on $\mathrm{Corr}(\mathcal{C}, \mathcal{C}_0)$.

Proposition 2.20. Let $(\mathcal{C}, \mathcal{C}_0)$ be a geometric setup. If $\mathcal{C}$ admits finite products, then $\mathrm{Corr}(\mathcal{C}, \mathcal{C}_0)^\otimes$ is a symmetric monoidal category.

Proposition 2.21. Let $(\mathcal{C}, \mathcal{C}_0)$ be a geometric setup. The functor $(\mathcal{C}, \mathcal{C}_0) \to \mathrm{Corr}(\mathcal{C}, \mathcal{C}_0)^\otimes$ enhances to a functor

$$\mathrm{Corr}^\otimes : \mathrm{GS} \to \mathrm{CMon}.$$

If $\mathcal{C} \to \mathcal{D}$ preserves finite products, $\mathrm{Corr}(\mathcal{C}, \mathcal{C}_0)^\otimes \to \mathrm{Corr}(\mathcal{D}, \mathcal{D}_0)^\otimes$ is a symmetric monoidal functor.

Proposition 2.22.

3. Six-functor formalisms

3.1. Three-functor Formalisms.

Definition 3.1 ([HM24] Definition 3.1.1)**.** Let $(\mathcal{C}, E)$ be a geometric setup. A *three-functor formalism* on $(\mathcal{C}, E)$ is a map of operads

$$\mathrm{D} : \mathrm{Corr}(\mathcal{C}, E)^{\otimes} \to \mathrm{Cat}^{\times}.$$

Remark 3.2. Let $(\mathcal{C}, E)$ be a geometric setup, and let $\mathrm{D} : \mathrm{Corr}(\mathcal{C}, E)^{\otimes} \to \mathrm{Cat}^{\times}$ be a three-functor formalism on $(\mathcal{C}, E)$.

(1) By precomposing with the map of operads $(\mathcal{C}^{\mathrm{op}})^{\amalg} \to \mathrm{Corr}(\mathcal{C}, E)^{\otimes}$, we obtain a map of operads $\mathrm{D} : (\mathcal{C}^{\mathrm{op}})^{\amalg} \to \mathrm{Cat}^{\times}$. By [HA, Theorem 2.4.3.18], it corresponds to a functor

$$\mathrm{D}^* : \mathcal{C}^{\mathrm{op}} \to \mathrm{CMon}.$$

(2) By precomposing with the map of operads $\mathcal{C}_E \to \mathrm{Corr}(\mathcal{C}, E)^{\otimes}$, we obtain a map of operads (?)

$$\mathrm{D}_! : \mathcal{C}_E \to \mathrm{Cat}^{\times}.$$

Notation 3.3. Let $(\mathcal{C}, E)$ be a geometric setup, and let $\mathrm{D} : \mathrm{Corr}(\mathcal{C}, E)^{\otimes} \to \mathrm{Cat}^{\times}$ be a three-functor formalism on $(\mathcal{C}, E)$. We will use the following notations.

(1) For every object X of $\mathcal{C}$, the category $\mathrm{D}(X) = \mathrm{D}^*(X)$ admits a symmetric monoidal structure, which we denote by

$$\otimes : \mathrm{D}(X) \times \mathrm{D}(X) \to \mathrm{D}(X).$$

(2) For every morphism $f : X \to Y$ in $\mathcal{C}$, we obtain a symmetric monoidal functor

$$f^* := \mathrm{D}^*(f) : \mathrm{D}(Y) \to \mathrm{D}(X).$$

(3) For every morphism $f : X \to Y$ in E, we obtain a functor

$$f_! := \mathrm{D}_!(f) : \mathrm{D}(X) \to \mathrm{D}(Y).$$

Lemma 3.4. Let $(\mathcal{C}, E)$ be a geometric setup, and let $\mathrm{D} : \mathrm{Corr}(\mathcal{C}, E)^{\otimes} \to \mathrm{Cat}^{\times}$ be a three-functor formalism on $(\mathcal{C}, E)$. For every pair of morphisms $g : X' \to X$ in $\mathcal{C}$ and $f : X' \to Y$ in E, we obtain a natural morphism

$$f_! g^* : \mathrm{D}(X) \to \mathrm{D}(Y).$$

$$\begin{tikzcd} X & X' \arrow[l, "g"'] \arrow[d, "f"] \\ & Y \end{tikzcd} \quad \overset{\mathrm{D}}{\longmapsto} \quad \begin{tikzcd} \mathrm{D}(X) \arrow[r, "g^*"] & \mathrm{D}(X') \arrow[d, "f_!"] \\ & \mathrm{D}(Y). \end{tikzcd}$$

Proof. The morphisms $g : X' \to X$ in $\mathcal{C}$ and $f : X' \to Y$ in E determines correspondences h_1 and h_2 respectively.

$$\begin{tikzcd} X & X' \arrow[l, "g"'] \arrow[d, equal] \\ & X' \end{tikzcd} \qquad \begin{tikzcd} X' \arrow[r, equal] & X' \arrow[d, "f"] \\ & Y. \end{tikzcd}$$

These morphisms define functors

$$g^* : \mathrm{D}(X) \to \mathrm{D}(X') \quad \text{and} \quad f_! : \mathrm{D}(X') \to \mathrm{D}(Y)$$

respectively. On the other hands, the composition h_2h_1 is given by the correspondence

$$\begin{tikzcd} X & X' \arrow[l, "g"'] \arrow[r, equal] \arrow[d, equal] \arrow[dr, phantom, "\llcorner"] & X' \arrow[d, equal] \\ & X' \arrow[r, equal] & X' \arrow[d, "f"] \\ & & Y. \end{tikzcd}$$

Since D is a functor, we obtain

$$\mathrm{D}(h_2h_1) \simeq \mathrm{D}(h_2)\,\mathrm{D}(h_1) \simeq f_!g^* : \mathrm{D}(X) \to \mathrm{D}(Y).$$

□

Proposition 3.5 ([HM24] Proposition 3.1.8)**.** Let $(\mathcal{C}, E)$ be a geometric setup, and let $\mathrm{D} : \mathrm{Corr}(\mathcal{C}, E)^{\otimes} \to \mathrm{Cat}^{\times}$ be a three-functor formalism on $(\mathcal{C}, E)$. Then the associated three functors $\otimes$, f^* and $f_!$ satisfy the following conditions:

(1) The functors f^* and $f_!$ are natural.
(2) For every morphism $f : X \to Y$ in $\mathcal{C}$, the functor $f^* : \mathrm{D}(Y) \to \mathrm{D}(X)$ is symmetric monoidal.
(3) (Proper base change) For every cartesian diagram in $\mathcal{C}$

$$\begin{tikzcd} X' \arrow[r, "g'"] \arrow[d, "f'"'] \arrow[dr, phantom, "\lrcorner", very near start] & X \arrow[d, "f"] \\ Y' \arrow[r, "g"'] & Y, \end{tikzcd}$$

where f in E (thus also f' in E), there exists a natural equivalence

$$g^*f_! \simeq f'_!g'^*$$

of functors $\mathrm{D}(X) \to \mathrm{D}(Y')$.

$$\begin{tikzcd} \mathrm{D}(X') \arrow[d, "f'_!"'] & \mathrm{D}(X) \arrow[l, "g'^*"'] \arrow[d, "f_!"] \\ \mathrm{D}(Y') & \mathrm{D}(Y). \arrow[l, "g^*"] \end{tikzcd}$$

(4) (The projection formula) For every morphism $f : X \to Y$ in E and every pair of objects F of $\mathrm{D}(X)$ and G of $\mathrm{D}(Y)$, there exists a natural equivalence

$$f_!F \otimes G \to f_!(F \otimes f^*G)$$

of functors $\mathrm{D}(X) \times \mathrm{D}(Y) \to \mathrm{D}(Y)$.

$$\begin{tikzcd}
\mathrm{D}(Y) \times \mathrm{D}(Y) \arrow[dd, "\otimes"'] & \mathrm{D}(X) \times \mathrm{D}(Y) \arrow[l, "(f_!, \mathrm{id}_{\mathrm{D}(Y)})"'] \arrow[d, "(\mathrm{id}_{\mathrm{D}(X)}, f^*)"] \\
 & \mathrm{D}(X) \times \mathrm{D}(X) \arrow[d, "\otimes"] \\
\mathrm{D}(Y) & \mathrm{D}(X). \arrow[l, "f_!"]
\end{tikzcd}$$

Proof. (1) and (2) follow from the definitions.

We next show (3). The morphisms $f : X \to Y$ in E and $g : Y' \to Y$ in $\mathcal{C}$ determine correspondences h_1 and h_2 respectively.

$$\begin{tikzcd}
X \arrow[r, equal] & X \arrow[d, "f"] \\
 & Y
\end{tikzcd}
\qquad
\begin{tikzcd}
Y & Y' \arrow[l, "g"'] \arrow[d, equal] \\
 & Y'.
\end{tikzcd}$$

The composition $h_2 h_1$ is given by the correspondence

$$\begin{tikzcd}
X \arrow[r, equal] & X \arrow[d, "f"] \arrow[dr, phantom, "\llcorner"] & X' \arrow[l, "g'"'] \arrow[d, "f'"] \\
 & Y & Y' \arrow[l, "g"'] \arrow[d, equal] \\
 & & Y'.
\end{tikzcd}$$

The first map $h_1 : X \xleftarrow{\mathrm{id}_X} X \xrightarrow{f} Y$ defines a functor

$$f_! : \mathrm{D}(X) \to \mathrm{D}(Y).$$

The second map $h_2 : Y \xleftarrow{g'} Y' \xrightarrow{\mathrm{id}_{Y'}} Y'$ defines a functor

$$g^* : \mathrm{D}(Y) \to \mathrm{D}(Y').$$

The composition $h_2 h_1 : X \xleftarrow{g'} X' \xrightarrow{f'} Y'$ defines a functor

$$f'_! g'^* : \mathrm{D}(X) \to \mathrm{D}(Y').$$

Thus we obtain a natural equivalence $g^* f_! \simeq f'_! g'^*$, since D is a functor.

We finally show (4). Let $f : X \to Y$ be a morphism in E. Consider the morphism $X \times Y \to Y$ given by the correspondence

$$\begin{tikzcd}
X \times Y & X \arrow[l, "(\mathrm{id}_X, f)"'] \arrow[d, "f"] \\
 & Y.
\end{tikzcd}$$

We obtain two factorizations of $X \times Y \to Y$ as follows: The morphism $X \times Y \to Y \times Y \to Y$ which is given by the correspondence

$$\begin{tikzcd}
X \times Y \arrow[r, equal] & X \times Y \arrow[d, "{(f, \mathrm{id}_X)}"] & X \arrow[l, "{(\mathrm{id}_X, f)}"'] \arrow[d, "f"] \arrow[dl, phantom, "\llcorner", very near start] \\
& Y \times Y & Y \arrow[l, "\Delta_Y"'] \arrow[d, equal] \\
& & Y.
\end{tikzcd}$$

The morphism $X \times Y \to X \times X \to X \to Y$ which is given by the correspondence

$$\begin{tikzcd}
X \times Y & X \times X \arrow[l, "{(\mathrm{id}_X, f)}"'] \arrow[d, equal] & X \arrow[l, "\Delta_X"'] \arrow[r, equal] \arrow[d, equal] \arrow[dl, phantom, "\llcorner", very near start] & X \arrow[d, equal] \arrow[dl, phantom, "\llcorner", very near start] \\
& X \times X & X \arrow[l, "\Delta_X"'] \arrow[r, equal] \arrow[d, equal] & X \arrow[d, equal] \arrow[dl, phantom, "\llcorner", very near start] \\
& & X \arrow[r, equal] & X \arrow[d, "f"] \\
& & & Y.
\end{tikzcd}$$

The considering morphism $X \times Y \xleftarrow{(\mathrm{id}_X, f)} X \xrightarrow{f} Y$ defines a functor

$$D(X) \times D(Y) \to D(Y).$$

The first factorization $X \times Y \xleftarrow{(\mathrm{id}_X, f)} X \xrightarrow{f} Y$ defines a functor

$$\begin{aligned}
D(X) \times D(Y) &\to D(Y) \times D(Y) \to D(Y) \\
(F, G) &\mapsto (f_!F, G) \mapsto f_!F \otimes G.
\end{aligned}$$

The second factorization $X \times Y \xleftarrow{(\mathrm{id}_X, f)} X \times X \xleftarrow{\Delta_X} X \xrightarrow{f} Y$ defines a functor

$$\begin{aligned}
D(X) \times D(Y) &\to D(X) \times D(X) \to D(X) \to D(Y) \\
(F, G) &\mapsto (F, f^*G) \mapsto F \otimes f^*G \mapsto f_!(F \otimes f^*G).
\end{aligned}$$

This implies that a morphism $f_!F \otimes G \to f_!(F \otimes f^*G)$ is a natural equivalence in $D(Y)$, since D is a functor. □

3.2. **Six-functor formalisms.**

Definition 3.6 ([HM24] Definition 3.2.1)**.** Let $(\mathcal{C}, E)$ be a geometric setup. A *six-functor formalism* on $(\mathcal{C}, E)$ is a three-functor formalism $D : \mathrm{Corr}(\mathcal{C}, E)^{\otimes} \to \mathrm{Cat}^{\times}$ satisfying the following conditions:

(1) For every object X of $\mathcal{C}$, the symmetric monoidal structure on $D(X)$ is closed.

(2) The functors $f^* : D(Y) \to D(X)$ and $f_! : D(X) \to D(Y)$ admit right adjoint functors respectively.

Notation 3.7. Let $(\mathcal{C}, E)$ be a geometric setup, and let $D : \mathrm{Corr}(\mathcal{C}, E)^{\otimes} \to \mathrm{Cat}^{\times}$ be a six-functor formalism on $(\mathcal{C}, E)$. We will use the following notations:

(1) We denote the internal hom in $\mathrm{D}(X)$ (i.e. the right adjoint functor to $\otimes$) by
$$\underline{\mathrm{Hom}} : \mathrm{D}(X)^{\mathrm{op}} \times \mathrm{D}(X) \to \mathrm{D}(X).$$
(2) For every morphism $f : X \to Y$ in $\mathcal{C}$, we denote the right adjoint functor to f^* by
$$f_* : \mathrm{D}(X) \to \mathrm{D}(Y).$$
(3) For every morphism $f : X \to Y$ in E, we denote the right adjoint functor to $f_!$ by
$$f^! : \mathrm{D}(Y) \to \mathrm{D}(X).$$

Proposition 3.8 ([HM24] Proposition 3.2.2)**.** Let $(\mathcal{C}, E)$ be a geometric setup, and let $D : \mathrm{Corr}(\mathcal{C}, E)^{\otimes} \to \mathrm{Cat}^{\times}$ be a six-functor formalism on $(\mathcal{C}, E)$. Then the associated six functors $\otimes \dashv \underline{\mathrm{Hom}}$, $f^* \dashv f_*$ and $f_! \dashv f^!$ satisfy the following conditions:

(1) The functors f_* and $f^!$ are natural.
(2) For every cartesian diagram in $\mathcal{C}$
$$\begin{tikzcd} X' \arrow[r, "g'"] \arrow[d, "f'"'] \arrow[dr, phantom, "\lrcorner", very near start] & X \arrow[d, "f"] \\ Y' \arrow[r, "g"'] & Y, \end{tikzcd}$$
where f in E (thus also f' in E), there exists a natural equivalence
$$g^! f_* \simeq f'_* g'^!$$
of functors $\mathrm{D}(X) \to \mathrm{D}(Y')$.
$$\begin{tikzcd} \mathrm{D}(X') \arrow[d, "f'_*"'] & \mathrm{D}(X) \arrow[l, "g'^!"'] \arrow[d, "f_*"] \\ \mathrm{D}(Y') & \mathrm{D}(Y). \arrow[l, "g^!"] \end{tikzcd}$$

Proof. (1) follows from proposition 3.5 and the fact that the adjoint is also natural. (2) follows from proposition 3.5 by passing to right adjoint functors. □

3.3. **Constructions of six-functor formalisms.**

Definition 3.9 ([HM24] Definition 3.3.2)**.** Let $(\mathcal{C}, E)$ be a geometric setup. We will say that E *has a suitable decomposition* (I, P) if there exists a pair of subclasses of morphisms (I, P) of E satisfying the following conditions:

(1) The classes I and P contain all equivalences, and are stable under composition and pullbacks.
(2) Every morphism f in E admits a factorization $f = gj$, where g in P and j in I.
(3) For every morphism $f : X \to Y$ in I (resp. P), the diagonal $\Delta : X \to X \times_Y X$ lies in I (resp. P).
(4) Every morphism f in $I \cap P$ is n-truncated for some $n \geq -2$ (which may depend on f).

Remark 3.10. By **??**, the condition (2) of definition 3.9 is equivalent to the following condition: The classes I and P are right cancellative in $\mathcal{C}$.

Definition 3.11. Let $(\mathcal{C}, E)$ be a geometric setup, and let $\mathrm{D} : \mathcal{C}^{\mathrm{op}} \to \mathrm{CMon}$ be a functor. Suppose that E has a suitable decomposition (I, P). For every morphism $f : X \to Y$ in $\mathcal{C}$, we denote $f^* := \mathrm{D}(f) : \mathrm{D}(Y) \to \mathrm{D}(X)$. We will say that D *satisfies a suitable decomposition condition* for (I, P) if it satisfies the following conditions:

(1) For every morphism j in I, the functor j^* admits a left adjoint functor $j_! : \mathrm{D}(X) \to \mathrm{D}(Y)$, which satisfies proper base change and the projection formula.

(2) For every morphism g in P, the functor g^* admits a right adjoint functor $g_* : \mathrm{D}(X) \to \mathrm{D}(Y)$, which satisfies the $*$-version base change and the $*$-version projection formula.

(3) For every cartesian diagram in $\mathcal{C}$

$$\begin{tikzcd} X' \arrow[r, "j'"] \arrow[d, "g'"'] \arrow[dr, phantom, "\lrcorner", very near start] & X \arrow[d, "g"] \\ Y' \arrow[r, "j"'] & Y, \end{tikzcd}$$

where j in I (thus also j' in I), and g in P (thus also g' in P), the morphism

$$j_! g'_* \to g_* j'_!$$

of functors $\mathrm{D}(X') \to \mathrm{D}(Y)$ is an equivalence.

$$\begin{tikzcd} \mathrm{D}(X') \arrow[r, "j'_!"] \arrow[d, "g'_*"'] & \mathrm{D}(X) \arrow[d, "g_*"] \\ \mathrm{D}(Y') \arrow[r, "j_!"'] & \mathrm{D}(Y). \end{tikzcd}$$

Remark 3.12. We impose the condition (3) in definition 3.11 only when the diagram is cartesian. This restriction arises because the morphism $j_! g'_* \to g_* j'_!$ is defined using the adjunction:

$$\begin{aligned} \mathrm{Hom}_{\mathrm{D}(Y)}(j_! g'_*(-), g_* j'_!(-)) &\simeq \mathrm{Hom}_{\mathrm{D}(X)}(g^* j_! g'_*(-), j'_!(-)) \\ &\simeq \mathrm{Hom}_{\mathrm{D}(X)}(j'_! g'^* g'_*(-), j'_!(-)). \end{aligned}$$

Here the second equivalence uses proper base change which is defined only for cartesian diagrams. Thus, we cannot *a priori* define a natural equivalence. However, we can still deduce the same equivalence as a consequence of the axioms, even when the diagram is not cartesian (lemmas 3.13 and 3.14). The following lemmata do not needed to prove proposition 3.16.

Lemma 3.13 ([Sch22] Construction 4.3)**.** Let $(\mathcal{C}, E)$ be a geometric setup. Suppose that E has a suitable decomposition (I, P). Then, for every morphism f in $I \cap P$, there exists a natural equivalence $f_! \simeq f_*$ between the left and right adjoint functors of f^*.

Proof. Let $f : X \to Y$ be a morphism in $I \cap P$. By (4) of definition 3.9, f is n-truncated for some $n \geq -2$. Consider the following diagram.

$$\begin{tikzcd} X \arrow[dr, dashed, "\Delta"] \arrow[drr, bend left, "\mathrm{id}_X"] \arrow[ddr, bend right, "\mathrm{id}_X"'] & & \\ & X \times_Y X \arrow[r, "f'"'] \arrow[d, "f'"] \arrow[dr, phantom, "\lrcorner", very near start] & X \arrow[d, "f"] \\ & X \arrow[r, "f"'] & Y. \end{tikzcd}$$

By (3) of definition 3.9, the diagonal $\Delta : X \to X \times_Y X$ lies in $I \cap P$. By (4) of definition 3.9, Δ is n-truncated for some $n \geq -2$. It is well known that, if f is n-truncated, then Δ is $(n-1)$-truncated.

We prove the statement by induction on n.

Base case ($n = -2$): Assume that f is -2-truncated, which means f is an equivalence. In this case, the left and right adjoint functors of f^* coincide. Then, we obtain an equivalence $f_! \simeq f_*$. This establishes the base case.

Inductive step: Assume that the statement holds for every n-truncated morphisms. i.e. for every n-truncated morphism g, we have $g_! \simeq g_*$. Let f be a $(n+1)$-truncated morphism. Then the diagonal Δ is n-truncated. By the induction hypothesis, we have an equivalence $\Delta_! \simeq \Delta_*$. Then we have

$$f_! \simeq f_! \mathrm{id}_{X!} \simeq f_! \mathrm{id}_{X*} \simeq f_! g_* \Delta_* \simeq f_! g_* \Delta_! \simeq f_* h_! \Delta_! \simeq f_* \mathrm{id}_{X!} \simeq f_*.$$

□

Lemma 3.14 ([Sch22] Construction 4.4)**.** Let $(\mathcal{C}, E)$ be a geometric setup. Suppose that E has a suitable decomposition (I, P). Then, for every commutative diagram in $\mathcal{C}$

$$\begin{tikzcd}
X' \arrow[r, "j'"] \arrow[d, "g'"'] \arrow[dr, phantom, "\lrcorner", very near start] & X \arrow[d, "g"] \\
Y' \arrow[r, "j"'] & Y,
\end{tikzcd}$$

where j in I (thus also j' in I), and g in P (thus also g' in P), we have an equivalence

$$j_! g'_* \to g_* j'_!$$

of functors $\mathrm{D}(X') \to \mathrm{D}(Y)$.

Proof. Consider the following diagram.

$$\begin{tikzcd}
X' \arrow[dr, dashed, "h"] \arrow[drr, bend left, "j'"] \arrow[ddr, bend right, "g'"'] & & \\
& X \times_Y Y' \arrow[r, "j''"'] \arrow[d, "g''"] \arrow[dr, phantom, "\lrcorner", very near start] & X \arrow[d, "g"] \\
& Y' \arrow[r, "j"'] & Y.
\end{tikzcd}$$

By remark 3.10, h lies in $I \cap P$. By lemma 3.13, we obtain an equivalence $h_! \simeq h_*$. Then we have

$$j_! g'_* \simeq j_! g''_* h_* \simeq j_! g''_* h_! \simeq g_* j''_! h_! \simeq g_* j'_!.$$

□

We can also show that if a morphism f in E has two factorizations $f = gj = g'j'$, then the two induced factorizations of $f_!$ are equivalent. It is, of course, the corollary of proposition 3.16. To prove this statement (proposition 3.16), we need the condition (2) of definition 2.1 and **??**.

Proposition 3.15 ([Sch22] Construction 4.5)**.** Let $(\mathcal{C}, E)$ be a geometric setup. Suppose that E has a suitable decomposition (I, P). Let $f : X \to Y$ be a morphism in E such that there exist two factorizations $f = gj = g'j'$, where g and g' in P, and j and j' in I. Then there exists a natural equivalence

$$g_* j_! \simeq g'_* j_!$$

of functors $\mathrm{D}(X) \to \mathrm{D}(Y)$.

Proof. Consider the following diagram.

$$\begin{tikzcd}
X \arrow[dr, dashed, "\Delta"] \arrow[drrr, bend left, "\mathrm{id}_X"] \arrow[dddr, bend right, "\mathrm{id}_X"'] & & & \\
& X \times_Y X \arrow[rr, "f'"] \arrow[dd, "f'"] \arrow[dr, dashed, "h'"] \arrow[dr, phantom, "\lrcorner", very near start] & & X \arrow[d, "j"] \\
& & \overline{X} \times_Y \overline{X}' \arrow[r, "\gamma"'] \arrow[d, "\varepsilon"] \arrow[dr, phantom, "\lrcorner", very near start] & \overline{X} \arrow[d, "g"] \\
& X \arrow[r, "j'"'] & \overline{X}' \arrow[r, "g'"'] & Y.
\end{tikzcd}$$

By (2) of **??**, the diagonal $\Delta : X \to X \times_Y X$ and the induced morphism h' lie in E. Then the composition $h = h'\Delta$ lies in E. By (2) of definition 3.9, h admits a factorization $h = \beta\alpha$, where β in P and α in I. Then we obtain the following diagram.

$$\begin{tikzcd}
X \arrow[rrr, "\mathrm{id}_X"] \arrow[dr, "\alpha"] \arrow[ddd, "\mathrm{id}_X"'] & & & X \arrow[dd, "j"] \\
& \overline{X}'' \arrow[dr, "\beta"] \arrow[r, phantom, "(1)"] & & \\
& \arrow[d, phantom, "(2)"] & \overline{X} \times_Y \overline{X}' \arrow[r, "\gamma"] \arrow[d, "\varepsilon"] \arrow[dr, phantom, "\lrcorner", very near start] & \overline{X} \arrow[d, "g"] \\
X \arrow[rr, "j'"'] & & \overline{X}' \arrow[r, "g'"'] & Y,
\end{tikzcd}$$

where α in I, β, γ and ε in P. From the diagrams (1) and (2), we get equivalences

$$j_! \simeq (\gamma\beta)_*\alpha_! \quad \text{and} \quad j'_! \simeq (\varepsilon\beta)_*\alpha_!.$$

Then we have

$$g_* j_! \simeq g_*(\gamma\beta)_*\alpha_! \simeq (g\gamma)_*\beta_*\alpha_! \simeq (g'\varepsilon)_*\beta_*\alpha_! \simeq g'_*(\varepsilon\beta)_*\alpha_! \simeq g'_* j'_!.$$

□

Proposition 3.16 ([HM24] Proposition 3.3.3)**.** Let $(\mathcal{C}, E)$ be a geometric setup, and let $\mathrm{D} : \mathcal{C}^{\mathrm{op}} \to \mathrm{CMon}$ be a functor. Suppose that E has a suitable decomposition (I, P) and D satisfies a suitable decomposition condition for (I, P). Then D can be extended to a three-functor formalism

$$\mathrm{D} : \mathrm{Corr}(\mathcal{C}, E)^{\otimes} \to \mathrm{Cat}^{\times},$$

on $(\mathcal{C}, E)$ such that for every morphism j in I, $j_!$ is left adjoint to j^*, and for every morphism g in P, $g_!$ is right adjoint to g^*.

Corollary 3.17. Let $(\mathcal{C}, E)$ be a geometric setup, and let $\mathrm{D} : \mathcal{C}^{\mathrm{op}} \to \mathrm{CMon}$ be a functor. In addition to proposition 3.16, suppose that D satisfies the following conditions:

(1) For every object X of $\mathcal{C}$, the symmetric monoidal structure on $\mathrm{D}(X)$ is closed.

(2) For every morphism $f : X \to Y$ in $\mathcal{C}$, the functor $f^* : \mathrm{D}(Y) \to \mathrm{D}(X)$ admit a right adjoint functor $f_* : \mathrm{D}(X) \to \mathrm{D}(Y)$.

(3) For every morphism $g : X \to Y$ in P, the functor $g_* = g_! : \mathrm{D}(X) \to \mathrm{D}(Y)$ admit a right adjoint functor $g^! : \mathrm{D}(Y) \to \mathrm{D}(X)$.

Then the three-functor formalism obtained by proposition 3.16 is a six-functor formalism on $(\mathcal{C}, E)$.

3.4. **Extensions of six-functor formalisms.**

Definition 3.18. Let $(\mathcal{C}, E)$ be a geometric setup. We will say that $(\mathcal{C}, E)$ is a (*subcanonical*) *site* setup if $\mathcal{C}$ is a (subcanonical) site.

Definition 3.19 ([HM24] Definition 3.4.1)**.** Let $(\mathcal{C}, E)$ be a site setup, and let $\mathrm{D} : \mathrm{Corr}(\mathcal{C}, E)^{\otimes} \to \mathrm{Cat}^{\times}$ be a three-functor formalism on $(\mathcal{C}, E)$. We will say that D is *sheafy* if the induced functor $\mathrm{D}^* : \mathcal{C}^{\mathrm{op}} \to \mathrm{Cat}$ (remark 3.2) is a sheaf.

Proposition 3.20 ([HM24] Proposition 3.4.2)**.** Let $(\mathcal{C}, E)$ be a subcanonical site setup, and let $\mathrm{D} : \mathrm{Corr}(\mathcal{C}, E)^{\otimes} \to \mathrm{Cat}^{\times}$ be a sheafy three-functor formalism. We let $\mathcal{X} := \mathrm{Shv}(\mathcal{C})$ denote the topos of sheaves on $\mathcal{C}$. We let E' be the collection of morphisms $f' : X \to Y$ in $\mathcal{X}$ such that, for every morphism $g : よ(Y) \to Y'$ from an object Y of $\mathcal{C}$, the pullback of f' along g lies in E. Then

(1) The inclusion $よ : \mathcal{C} \subseteq \mathcal{X}$ induces a morphism of geometric setups $(\mathcal{C}, E) \to (\mathcal{X}, E')$.

(2) There exists a minimal choice of E'.

(3) The sheafy three-functor formalism $\mathrm{D} : \mathrm{Corr}(\mathcal{C}, E)^{\otimes} \to \mathrm{Cat}^{\times}$ can be uniquely extended to a sheafy three-functor formalism $\mathrm{D}' : \mathrm{Corr}(\mathcal{X}, E')^{\otimes} \to \mathrm{Cat}^{\times}$.

Moreover, if D is a presentable sheafy six-functor formalism, so is D'.

Theorem 3.21 ([HM24] Theorem 3.4.11)**.** *Let $(\mathcal{C}, E)$ be a subcanonical site setup, and let $\mathrm{D} : \mathrm{Corr}(\mathcal{C}, E)^{\otimes} \to \mathrm{Cat}^{\times}$ be a sheafy presentable six-functor formalism. We let $\mathcal{X} := \mathrm{Shv}(\mathcal{C})$ denote the topos of sheaves on $\mathcal{C}$. Then there exists a collection of morphisms E' in $\mathcal{X}$ satisfying the following conditions:*

(1) The inclusion $よ : \mathcal{C} \subseteq \mathcal{X}$ induces a morphism of geometric setups $(\mathcal{C}, E) \to (\mathcal{X}, E')$.

(2) There exists a minimal choice of E'.

(3) The class E' is $*$-local on the target, $!$-local, and tame.

(4) The sheafy presentable six-functor formalism $\mathrm{D} : \mathrm{Corr}(\mathcal{C}, E)^{\otimes} \to \mathrm{Cat}^{\times}$ can be uniquely extended to a sheafy six-functor formalism $\mathrm{D}' : \mathrm{Corr}(\mathcal{X}, E')^{\otimes} \to \mathrm{Cat}^{\times}$.

4. Other Results

4.1. **Corollaries of Six-functor Formalisms.** In this section, we show some corollaries of proposition 3.8. We fix a geometric setup $(\mathcal{C}, E)$ and a six-functor formalism $\mathrm{D} : \mathrm{Corr}(\mathcal{C}, E)^{\otimes} \to \mathrm{Cat}^{\times}$.

We have the local representation of "Verdier duality".

Corollary 4.1. For every morphism $f : X \to Y$ in E, and every pair of objects F of $\mathrm{D}(X)$ and G of $\mathrm{D}(Y)$, there exists a natural equivalence

$$\underline{\mathrm{Hom}}_{\mathrm{D}(Y)}(f_! F, G) \simeq f_* \underline{\mathrm{Hom}}_{\mathrm{D}(X)}(F, f^! G)$$

of functors $\mathrm{D}(X) \times \mathrm{D}(Y) \to \mathrm{D}(Y)$.

Proof. For every object H of $\mathrm{D}(X)$, we have

$$\begin{aligned}\mathrm{Hom}_{\mathrm{D}(X)}(H, \underline{\mathrm{Hom}}_{\mathrm{D}(Y)}(f_!F, G)) &\simeq \mathrm{Hom}_{\mathrm{D}(X)}(H \otimes f_!F, G) \\ &\simeq \mathrm{Hom}_{\mathrm{D}(X)}(f_!(f^*H \otimes F), G) \\ &\simeq \mathrm{Hom}_{\mathrm{D}(Y)}(f^*H \otimes F, f^!G) \\ &\simeq \mathrm{Hom}_{\mathrm{D}(Y)}(f^*H, \underline{\mathrm{Hom}}_{\mathrm{D}(X)}(F, f^!G)) \\ &\simeq \mathrm{Hom}_{\mathrm{D}(X)}(H, f_*\underline{\mathrm{Hom}}_{\mathrm{D}(X)}(F, f^!G)).\end{aligned}$$

Here, the second equivalence uses the projection formula. The desired assertion follows from Yoneda's lemma. $\square$

Corollary 4.2. For every morphism $f : X \to Y$ in E, and every pair of objects F and G of $\mathrm{D}(Y)$, there exists a natural equivalence

$$f^!\underline{\mathrm{Hom}}_{\mathrm{D}(Y)}(F, G) \simeq \underline{\mathrm{Hom}}_{\mathrm{D}(X)}(f^*F, f^!G)$$

of functors $\mathrm{D}(Y) \times \mathrm{D}(Y) \to \mathrm{D}(X)$.

Proof. For every object H of $\mathrm{D}(X)$, we have

$$\begin{aligned}\mathrm{Hom}_{\mathrm{D}(X)}(H, f^!\underline{\mathrm{Hom}}_{\mathrm{D}(Y)}(F, G)) &\simeq \mathrm{Hom}_{\mathrm{D}(Y)}(f_!H, \underline{\mathrm{Hom}}_{\mathrm{D}(Y)}(F, G)) \\ &\simeq \mathrm{Hom}_{\mathrm{D}(Y)}(f_!H \otimes F, G) \\ &\simeq \mathrm{Hom}_{\mathrm{D}(Y)}(f_!(H \otimes f^*F), G) \\ &\simeq \mathrm{Hom}_{\mathrm{D}(X)}(H \otimes f^*F, f^!G) \\ &\simeq \mathrm{Hom}_{\mathrm{D}(X)}(H, \underline{\mathrm{Hom}}_{\mathrm{D}(X)}(f^*F, f^!G)).\end{aligned}$$

Here, the third equivalence uses the projection formula. The desired assertion follows from Yoneda's lemma. $\square$

Corollary 4.3. For every morphism $f : X \to Y$ in E, there exists a natural morphism

$$f^!(-) \otimes f^*(-) \to f^!(- \otimes -)$$

from $\mathrm{D}(Y) \times \mathrm{D}(Y)$ to $\mathrm{D}(X)$.

Proof. For every pair of objects F and G of $\mathrm{D}(Y)$, we have

$$\begin{aligned}\mathrm{Hom}_{\mathrm{D}(X)}(f^!(F) \otimes f^*(G), f^!(F \otimes G)) &\simeq \mathrm{Hom}_{\mathrm{D}(Y)}(f_!(f^!(F) \otimes f^*(G)), F \otimes G) \\ &\simeq \mathrm{Hom}_{\mathrm{D}(Y)}(f_!f^!F \otimes G, F \otimes G).\end{aligned}$$

Here, the second equivalence uses the projection formula. From the unit $f_!f^!F \to F$ of the adjunction $f_! \dashv f^!$, we obtain the canonical map

$$f_!f^!F \otimes G \to F \otimes G.$$

Then we get a natural morphism

$$f^!(F) \otimes f^*(G) \to f^!(F \otimes G).$$

$\square$

Definition 4.4. Suppose that $\mathcal{C}$ admits finite products. Let X, Y and Z be objects of $\mathcal{C}$ such that projections q_{XY}, q_{XZ} and q_{YZ} from $X \times Y \times Z$ to $X \times Y$, $X \times Z$ and $Y \times Z$ respectively lie in E

$$\begin{tikzcd} & X \times Y \times Z \arrow[dl, "q_{XY}"'] \arrow[d, "q_{XZ}"] \arrow[dr, "q_{YZ}"] & \\ X \times Y & X \times Z & Y \times Z. \end{tikzcd}$$

For every pair of objects F_{XY} of $\mathrm{D}(X \times Y)$ and F_{YZ} of $\mathrm{D}(Y \times Z)$, we define the *composition* $F_{XY} \circ F_{YZ}$ of F_{XY} and F_{YZ} on Y by

$$F_{XY} \circ F_{YZ} := q_{XZ!}(q_{XY}^* F_{XY} \otimes q_{YZ}^* F_{YZ}).$$

$$\begin{tikzcd} & \mathrm{D}(X \times Y \times Z) \arrow[d, "q_{XZ!}"] & \\ \mathrm{D}(X \times Y) \arrow[ur, "q_{XY}^*"] & \mathrm{D}(X \times Z) & \mathrm{D}(Y \times Z). \arrow[ul, "q_{YZ}^*"'] \end{tikzcd}$$

Additionally suppose that $\mathcal{C}$ admits a terminal object $*$. Consider the case $Z \simeq *$. For every pair of objects K of $\mathrm{D}(X \times Y)$ and F of $\mathrm{D}(Y)$, we will refer to the composition

$$K \circ F \simeq q_{X!}(K \otimes q_Y^* F)$$

as the *integral transformation* of F by K.

Note that the composition (especially the integral transformation) can be defined in the setting of three-functor formalisms D. However, there exists the right adjoint functor to it if D is a six-functor formalism.

Corollary 4.5. Suppose that $\mathcal{C}$ admits finite products and a terminal object. Let K be an object of $\mathrm{D}(X \times Y)$. Then the integral transformation $K \circ - : \mathrm{D}(Y) \to \mathrm{D}(X)$ admits a right adjoint functor.

Proof. Let F be an object of $\mathrm{D}(Y)$ and let G be an object of $\mathrm{D}(X)$, we have

$$\begin{aligned} \mathrm{Hom}_{\mathrm{D}(X)}(q_{X!}(K \otimes q_Y^* F), G) &\simeq \mathrm{Hom}_{\mathrm{D}(X \times Y)}(K \otimes q_Y^* F, q_X^! G) \\ &\simeq \mathrm{Hom}_{\mathrm{D}(X \times Y)}(q_Y^* F, \underline{\mathrm{Hom}}_{\mathrm{D}(X \times Y)}(K, q_X^! G)) \\ &\simeq \mathrm{Hom}_{\mathrm{D}(Y)}(F, q_{Y*}\underline{\mathrm{Hom}}_{\mathrm{D}(X \times Y)}(K, q_X^! G)). \end{aligned}$$

Then the functor

$$q_{Y*}\underline{\mathrm{Hom}}_{\mathrm{D}(X \times Y)}(K, q_X^!(-)) : \mathrm{D}(X) \to \mathrm{D}(Y)$$

is right adjoint to the integral transformation. □

4.2. **Sheaf Cohomology and Künneth Formula.** We fix a geometric setup $(\mathcal{C}, E)$ and a three-functor formalism $\mathrm{D} : \mathrm{Corr}(\mathcal{C}, E)^{\otimes} \to \mathrm{Cat}^{\times}$. Suppose that $\mathcal{C}$ admits a terminal object $*$.

Definition 4.6. Let X be an object of $\mathcal{C}$, and let $p_X : X \to *$ be the projection. For an object F of $\mathrm{D}(X)$, we define the *cohomology* $\Gamma(X; F)$ *of* X *with coefficient in* F by

$$\Gamma(X; F) := p_{X*}F.$$

Suppose that the projection p_X lies in E. Similarly, we define the *cohomology with compact support* $\Gamma_c(X; *)$ *of* X *with coefficient in* F by

$$\Gamma_c(X; F) := p_{X!}F.$$

We prove the important result on cohomology: Künneth Formula.

Proposition 4.7 (Künneth Formula)**.** Let X and Y be objects of $\mathcal{C}$ such that projections p_X and p_Y lie in E. For every pair of objects F of $\mathrm{D}(X)$ and G of $\mathrm{D}(Y)$, there exists an equivalence in $D(*)$

$$\Gamma_c(X \times Y; q_X^* F \otimes q_Y^* G) \simeq \Gamma_c(X; F) \otimes \Gamma_c(Y; G).$$

Proof. Consider the following diagram.

$$\begin{tikzcd}
X \times Y \arrow[r, "q_Y"] \arrow[d, "q_X"'] \arrow[dr, "p_{X\times Y}"] & Y \arrow[d, "p_Y"] \\
X \arrow[r, "p_X"'] & *.
\end{tikzcd}$$

Then p_X, p_Y, q_X, q_Y and $p_{X\times Y}$ lie in E, since E is closed under pullbacks and composition. We have

$$\begin{aligned}
\Gamma_c(X \times Y; q_X^* F \otimes q_Y^* G) &\simeq p_{X\times Y!}(q_X^* F \otimes q_Y^* G) \\
&\simeq p_{X!} q_{X!}(q_X^* F \otimes q_Y^* G) \\
&\simeq p_{Y!}(F \otimes q_{X!} q_Y^* G) \\
&\simeq p_{Y!}(F \otimes p_X^* p_{Y!} G) \\
&\simeq p_{X!} F \otimes p_{Y!} G \\
&\simeq \Gamma_c(X; F) \otimes \Gamma_c(Y; G).
\end{aligned}$$

Here, the third and fifth equivalences use the projection formula, and the third uses proper base change. □

We can prove it as the proof of proposition 3.5.

Proof. Consider the following diagram.

$$\begin{tikzcd}
X \times Y \arrow[r, "p_{X\times Y}"] \arrow[d, "\Delta_{X\times Y}"'] & * \\
(X \times Y)^2 \arrow[r, "(q_X, q_Y)"'] & X \times Y. \arrow[u, "p_{X\times Y}"']
\end{tikzcd}$$

Then p_X, p_Y, q_X, q_Y and $p_{X\times Y}$ lie in E.

The morphism $X \times Y \xrightarrow{p_{X\times Y}} *$ in E determines a correspondence

$$\begin{tikzcd}
X \times Y \arrow[r, equal] & X \times Y \arrow[r, equal] \arrow[d, "(p_X, p_Y)"] \arrow[dr, phantom, "\lrcorner", very near start] & X \times Y \arrow[d, "p_{X\times Y}"] \\
& * \times * & * \arrow[l, "\Delta_*"'] \arrow[d, equal] \\
& & *.
\end{tikzcd}$$

The morphism $X \xrightarrow{\Delta_{X\times Y}} (X\times Y)^2 \xrightarrow{(q_X,q_Y)} X\times Y \xrightarrow{(p_X,p_Y)} *\times* \simeq *$ determines a correspondence

$$
\begin{tikzcd}
X \times Y & (X \times Y)^2 \arrow[l, "{(q_X, q_Y)}"'] \arrow[d, equal] & X \times Y \arrow[l, "\Delta_{X\times Y}"'] \arrow[r, equal] \arrow[d, equal] \arrow[dl, phantom, "\llcorner", very near start] & X \times Y \arrow[d, equal] \arrow[dl, phantom, "\llcorner", very near start] \\
& (X \times Y)^2 & X \times Y \arrow[l, "\Delta_{X\times Y}"'] \arrow[r, equal] \arrow[d, equal] & X \times Y \arrow[d, equal] \arrow[dl, phantom, "\llcorner", very near start] \\
& & X \times Y \arrow[r, equal] & X \times Y \arrow[d, "p_{X\times Y}"] \\
& & & *.
\end{tikzcd}
$$

The first correspondence defines a functor

$$
\begin{aligned}
&\mathrm{D}(X) \times \mathrm{D}(Y) \to * \times * \to * \\
&(F, G) \mapsto (p_{X!}F, p_{Y!}G) \mapsto p_{X!}F \otimes p_{Y!}G \simeq \Gamma_c(X; F) \otimes \Gamma_c(Y; G).
\end{aligned}
$$

The second correspondence defines a functor

$$
\begin{aligned}
&\mathrm{D}(X) \times \mathrm{D}(Y) \to (\mathrm{D}(X) \times \mathrm{D}(Y))^2 \to \mathrm{D}(X) \times \mathrm{D}(Y) \to D(*) \\
&(F, G) \mapsto (q_X^*F, q_Y^*G) \mapsto q_X^*F \otimes q_Y^*G \mapsto p_{X\times Y!}q_X^*F \otimes q_Y^*G \simeq \Gamma_c(X \times Y; q_X^*F \otimes q_Y^*G).
\end{aligned}
$$

This implies that a morphism $\Gamma_c(X \times Y; q_X^*F \otimes q_Y^*G) \to \Gamma_c(X; F) \otimes \Gamma_c(Y; G)$ is a natural equivalence in $\mathrm{D}(Y)$, since D is a functor. $\square$