# A NOTE ON 6-FUNCTOR FORMALISMS (UNDER CONSTRUCTION)

KEIMA AKASAKA

Abstract. We summarize key concepts and results on 6-functor formalisms, focusing on their foundational aspects.

Contents



## 1. Introduction

The aim of this note is to explain *what a 6-functor formalism is* and summarize the basic results and examples of 6-functor formalisms. A 6-functor formalism is a framework to formalize a cohomology theory, which provide a structured and systematic approach to handling cohomological operations.

The concept of six functors for sheaves originated in Grothendieck's work, and these are often referred to as Grothendieck's six operations. The first such formalism was developed to establish the étale cohomology of schemes, as formulated by Yifeng Liu and Weizhe Zheng [LZ17]. A subsequent formalism, intended for coherent cohomology of schemes, was provided by Dennis Gaitsgory and Nick Rozenblyum [GR17]. The former approach is based on heavy combinatorics of specific simplicial sets, while the latter uses the framework of $(\infty, 2)$-categories. More recently, Lucas Mann [Man22] has proposed a refined definition that combines favorable aspects of both methods. We will present Mann's definition after reviewing classical sheaf theory and Grothendieck's six operations.

**Notation 1.1.** We let

- Ab denote the category of abelian groups.
- $\mathrm{Pshv}(X)$ denote the category of presheaves on a topological space $X$.
- $\mathrm{Shv}(X)$ denote the category of sheaves on a topological space $X$.
- Cat denote the $\infty$-category of small $\infty$-categories.

## 2. Classical Sheaf Theory

In this section, we review the fundamental concepts and results of classical sheaf theory and Grothendieck's six functors. In section 2.1, we cover basic notions of the category of sheaves and the global section functor. In sections 2.2 and 2.3, we construct the six functors on the (derived) category of sheaves. In section 2.5, we highlight key results in sheaf theory: the proper base change theorem and projection formula.

### 2.1. **The Category of Sheaves.**

**Definition 2.1.** Let $X$ be a topological space. An (*abelian*) *presheaf* on $X$ is a functor

$$F : \mathrm{Open}(X)^{\mathrm{op}} \to \mathrm{Ab}.$$

A *morphism of presheaves* is a natural transformation between presheaves. We let $\mathrm{Pshv}(X)$ denote the category of presheaves on $X$ with morphisms of presheaves, and refer to it as the *category of presheaves* on $X$.

**Definition 2.2.** Let $X$ be a topological space, and let $F$ be a presheaf on $X$. We will say that $F$ is a *sheaf* if, for every complete full subcategory $\mathcal{U}$ of $\mathrm{Open}(X)$, $F$ carries every colimit which exists in $\mathcal{U}$ to a limit in Ab. We let $\mathrm{Shv}(X)$ denote the full subcategory of $\mathrm{Pshv}(X)$ spanned by sheaves on $X$, and refer to it as the *category of sheaves* on $X$.

**Proposition 2.3.** Let $X$ be a topological space. By the adjoint functor theorem, the inclusion $\mathrm{Shv}(X) \subseteq \mathrm{Pshv}(X)$ admits a left adjoint, called the *sheafification functor*

$$(-)^+ : \mathrm{Pshv}(X) \to \mathrm{Shv}(X).$$

**Definition 2.4.** Let $X$ be a topological space, and let $F$ be a sheaf on $X$. For an open subset $U$ of $X$, an element of $F(U)$ is called a *section* on $U$ of $F$. In particular, an element of $F(X)$ is called a *global section* of $F$. The construction $F \mapsto F(X)$ defines the *global section functor*

$$\Gamma(X; -) : \mathrm{Shv}(X) \to \mathrm{Ab}.$$

**Definition 2.5.** Let $X$ be a topological space, $F$ be a sheaf on $X$, $U$ be an open subset of $X$, and let $s$ be a section on $U$ of $F$. A *support* $\mathrm{supp}(s)$ of $s$ is the complement set $U \backslash V$, where $V$ is the largest open subset of $X$ on which $s|_V = 0$.

We can define the *kernel sheaf* $\mathrm{Ker}\,\varphi$ and the *image sheaf* $\mathrm{Im}\varphi$ for a morphism $\varphi$ of sheaves, and they endow an abelian category structure on the category of sheaves. Then we can consider exact sequences of sheaves.

**Proposition 2.6.** Let $X$ be a topological space. Then the category $\mathrm{Shv}(X)$ is an abelian category which has enough injectives.

**Definition 2.7.** A sequence of sheaves $F \xrightarrow{\varphi} G \xrightarrow{\psi} H$ is called *exact* if it satisfies $\mathrm{Im}\varphi = \mathrm{Ker}\,\psi$.

**Proposition 2.8.** Let $X$ be a topological space. Then the global section functor $\Gamma(X;-)$ : $\mathrm{Shv}(X) \to \mathrm{Ab}$ is left exact. That is, for every short exact sequence of sheaves $0 \to F \to G \to H \to 0$, $0 \to \Gamma(X;F) \to \Gamma(X;G) \to \Gamma(X;H)$ is exact in Ab.

2.2. **Grothendieck's Six Functors on the Category of Sheaves.** In this section, we recall Grothendieck's six functors. We first discuss how to construct a new sheaf from the given sheaves.

**Definition 2.9.** Let $X$ be a topological space, and let $F$ and $G$ be sheaves on $X$. For every open subset $U$ of $X$, we define

$$\underline{\mathrm{Hom}}(F,G)(U) := \mathrm{Hom}_{\mathrm{Shv}(X)}(F|_U, G|_U),$$

which induces a sheaf $\underline{\mathrm{Hom}}(F,G)$ on $X$. The construction $(F,G) \mapsto \underline{\mathrm{Hom}}(F,G)$ defines a bifunctor

$$\underline{\mathrm{Hom}} : \mathrm{Shv}(X)^{\mathrm{op}} \times \mathrm{Shv}(X) \to \mathrm{Shv}(X).$$

We will refer to it as the *sheaf hom functor*.

**Definition 2.10.** Let $X$ be a topological space, and let $F$ and $G$ be sheaves on $X$. For every open subset $U$ of $X$, we define

$$F \otimes' G(U) := F(U) \otimes_{\mathbb{Z}} G(U),$$

which induces a presheaf $F \otimes' G$ on $X$. We let $F \otimes G := (F \otimes' G)^+$ denote the sheafification of $F \otimes' G$. The construction $(F,G) \mapsto F \otimes G$ defines a bifunctor

$$\otimes : \mathrm{Shv}(X) \times \mathrm{Shv}(X) \to \mathrm{Shv}(X).$$

We will refer to it as the *tensor product functor*.

**Proposition 2.11.** Let $X$ be a topological space. The tensor product functor $\otimes : \mathrm{Shv}(X) \times \mathrm{Shv}(X) \to \mathrm{Shv}(X)$ defines a symmetric monoidal structure on $\mathrm{Shv}(X)$.

**Proposition 2.12.** Let $X$ be a topological space. Then there exists the following adjunction: For every objects $F, G$ and $H$ of $\mathrm{Shv}(X)$, we have an isomorphism

$$\mathrm{Hom}_{\mathrm{Shv}(X)}(F \otimes G, H) \simeq \mathrm{Hom}_{\mathrm{Shv}(X)}(F, \underline{\mathrm{Hom}}(G,H)).$$

**Proposition 2.13.** The tensor product functor $\otimes$ is right exact, and the sheaf hom functor $\underline{\mathrm{Hom}}$ is left exact.

We next study the operators which transform sheaves along a continuous map between topological spaces.

**Definition 2.14.** Let $f : X \to Y$ be a continuous map of topological spaces, and let $F$ be a sheaf on $X$. For every open subset $V$ of $Y$, we define

$$f_*F(V) := \Gamma(f^{-1}(V); F) = F(f^{-1}(V)),$$

which induces a sheaf $f_*F$ on $Y$. The construction $F \mapsto f_*F$ defines a functor

$$f_* : \mathrm{Shv}(X) \to \mathrm{Shv}(Y).$$

We will refer to it as the *direct image functor* or the *pushout functor*.

$$\begin{tikzcd}
\mathrm{Open}(X)^{\mathrm{op}} \arrow[dr, "F"] & \\
\mathrm{Open}(Y)^{\mathrm{op}} \arrow[u, "(f^{-1})^{\mathrm{op}}"] \arrow[r, "f_*F"'] & \mathrm{Ab}.
\end{tikzcd}$$

We can construct a left adjoint functor to the direct image functor using the left Kan extension.

**Definition 2.15.** Let $f : X \to Y$ be a continuous map of topological spaces, and let $G$ be a sheaf on $Y$. For every open subset $U$ of $X$, we define

$$f'G(U) := \operatorname*{colim}_{f(U)\subset V} G(V),$$

which induces a presheaf $f'G$ on $X$. We let $f^*G := (f'G)^+$ denote the sheafification of $f'G$. The construction $G \mapsto f^*G$ defines a functor

$$f^* : \mathrm{Shv}(Y) \to \mathrm{Shv}(X).$$

We will refer to it as the *inverse image functor* or the *pullback functor.*

$$\begin{tikzcd}
\mathrm{Open}(X)^{\mathrm{op}} \arrow[dr, "f^*G"] & \\
\mathrm{Open}(Y)^{\mathrm{op}} \arrow[u, "(f^{-1})^{\mathrm{op}}"] \arrow[r, "G"'] & \mathrm{Ab}.
\end{tikzcd}$$

**Proposition 2.16.** Let $f : X \to Y$ be a continuous map of topological spaces. Then there exists the following adjunction

$$f^* : \mathrm{Shv}(Y) \rightleftarrows \mathrm{Shv}(X) : f_*.$$

**Proposition 2.17.** Let $f : X \to Y$ be a continuous map of topological spaces. Then the inverse image functor $f^*$ is exact, and the direct image functor $f_*$ is left exact.

In the rest of this section, we will consider the fifth functor, the *proper direct image functor.* It does not admit a right adjoint functor in the setting of the category of sheaves. From here all topological spaces are assumed to be (finite-dimensional) locally compact Hausdorff.

**Definition 2.18.** Let $f : X \to Y$ be a continuous map of topological spaces. We will say that $f$ is *proper* if, for every compact subset $K$ of $Y$, the inverse image $f^{-1}(K)$ is a compact subset of $X$.

The properness of continuous maps can be verified locally; in other words, properness is a local property.

**Definition 2.19.** Let $f : X \to Y$ be a continuous map of topological spaces, and let $F$ be a sheaf on $X$. For every open subset $V$ of $Y$, we define

$$f_!F(V) := \{s \in f_*F(V) \mid f|_{\mathrm{supp}(s)} : \mathrm{supp}(s) \to V \text{ is proper}\},$$

which induces a sheaf $f_!F$ on $Y$. The construction $F \mapsto f_!F$ defines a functor

$$f_! : \mathrm{Shv}(X) \to \mathrm{Shv}(Y).$$

We will refer to it as the *proper direct image functor.*

**Remark 2.20.** Let $f : X \to Y$ be a continuous map of topological spaces. Is there a right adjoint functor $f^! : \mathrm{Shv}(Y) \to \mathrm{Shv}(X)$ to the proper direct image functor $f_!$? The answer is "No," except in the case where $f$ is an inclusion of closed subsets. This is because $f_!$ is not exact. However, this problem will be solved in the setting of the derived category of sheaves.

2.3. **Grothendieck's Six Functors on the Derived Category of Sheaves.** In this section, we consider Grothendieck's six functors in the setting of derived categories (see appendix A for notations). We first consider the derived functors of the sheaf hom functor $\underline{\mathrm{Hom}}$ and the tensor product functor $\otimes$.

**Definition 2.21.** Let $X$ be a topological space, and let $F$ and $G$ be objects of $C(\mathrm{Shv}(X))$. For every $n \geq 0$, we define

$$\underline{\mathrm{Hom}}^n(F, G) := \prod_{i \in \mathbb{Z}} \underline{\mathrm{Hom}}(F^i, G^{n+i}),$$

which induces a bifunctor $\underline{\mathrm{Hom}}^\bullet : K(\mathrm{Shv}(X))^{\mathrm{op}} \times K(\mathrm{Shv}(X)) \to K(\mathrm{Shv}(X))$. Moreover, it induces the right derived functor

$$\mathbb{R}\underline{\mathrm{Hom}} : D(X)^{\mathrm{op}} \times D(X) \to D(X)$$

of the sheaf hom functor $\underline{\mathrm{Hom}} : \mathrm{Shv}(X)^{\mathrm{op}} \times \mathrm{Shv}(X) \to \mathrm{Shv}(X)$. It can be calculated as follows: For every objects $F$ and $G$ of $D(X)$, $\mathbb{R}\underline{\mathrm{Hom}}(F, G) = Q \circ \underline{\mathrm{Hom}}^\bullet(F, G')$, where $G'$ is quasi-isomorphic to $G$ in $K(\mathrm{Shv}(X))$ consisting of flabby sheaves (definition A.9).

**Definition 2.22.** Let $X$ be a topological space, and let $F$ and $G$ be objects of $C(\mathrm{Shv}(X))$. For every $n \geq 0$, we define

$$F \otimes^n G := \bigoplus_{i+j=n} F^i \otimes G^j,$$

which induces a bifunctor $\otimes^\bullet : K(\mathrm{Shv}(X)) \times K(\mathrm{Shv}(X)) \to K(\mathrm{Shv}(X))$. Moreover, it induces the right derived functor

$$\overset{\mathrm{L}}{\otimes} : D(X) \times D(X) \to D(X)$$

of the tensor product functor $\otimes : \mathrm{Shv}(X) \times \mathrm{Shv}(X) \to \mathrm{Shv}(X)$. It can be calculated as follows: For every objects $F$ and $G$ of $D(X)$, $F \overset{\mathrm{L}}{\otimes} G = F' \otimes^\bullet G$, where $F'$ is quasi-isomorphic to $F$ in $K(\mathrm{Shv}(X))$ consisting of flat sheaves (definition A.14).

**Proposition 2.23.** Let $X$ be a topological space. The tensor product functor $\overset{\mathrm{L}}{\otimes} : D(X) \times D(X) \to D(X)$ defines a symmetric monoidal structure on $D(X)$.

**Proposition 2.24.** Let $X$ be a topological space. Then there exists the following adjunction: For every objects $F, G$ and $H$ of $D(X)$, we have an isomorphism

$$\mathbb{R}\,\mathrm{Hom}(F \overset{\mathrm{L}}{\otimes} G, H) \simeq \mathbb{R}\,\mathrm{Hom}(F, \mathbb{R}\underline{\mathrm{Hom}}(G, H)).$$

Recall that the direct image functor $f_*$ is left exact, and that the inverse image functor $f^*$ is exact.

**Definition 2.25.** Let $f : X \to Y$ be a continuous map of topological spaces. Then there exists the right derived functor

$$\mathbb{R}f_* : D(X) \to D(Y)$$

of the direct image functor $f_* : \mathrm{Shv}(X) \to \mathrm{Shv}(Y)$. We will refer to it as the *derived direct image functor*. It can be calculated as follows: For every object $F$ of $D(X)$, $\mathbb{R}f_*F = f_*F'$, where $F'$ is quasi-isomorphic to $F$ in $K(\mathrm{Shv}(X))$ consisting of flabby sheaves.

**Definition 2.26.** Let $f : X \to Y$ be a continuous map of topological spaces. Then there exists the right derived functor

$$\mathbb{R}f^* : D(Y) \to D(X)$$

of the inverse image functor $f^* : \mathrm{Shv}(Y) \to \mathrm{Shv}(X)$. We will refer to it as the *derived inverse image functor*. It can be calculated as follows: For every object $G$ of $D(Y)$, $\mathbb{R}f^*G = f^*G$,

**Proposition 2.27.** Let $f : X \to Y$ be a continuous map of topological spaces. Then there exists the following adjunction

$$\mathbb{R}f^* : D(Y) \rightleftarrows D(X) : \mathbb{R}f_*.$$

We next consider the derived functor the proper direct image functor $f_!$, and its right adjoint functor.

**Definition 2.28.** Let $f : X \to Y$ be a continuous map of topological spaces. Then there exists the right derived functor

$$\mathbb{R}f_! : D(X) \to D(Y)$$

of the proper direct image functor $f_! : \mathrm{Shv}(X) \to \mathrm{Shv}(Y)$. We will refer to it as the *derived proper direct image functor*. It can be calculated as follows: For every object $X$ of $D(X)$, $\mathbb{R}f_!F = f_!F'$, where $F'$ is a quasi-isomorphic object to $F$ in $K(\mathrm{Shv}(X))$ consisting of c-soft sheaves (definition A.12).

Let $f : X \to Y$ be a continuous map between topological spaces. From here we assume the following condition: *The proper direct image functor $f_! : \mathrm{Shv}(X) \to \mathrm{Shv}(Y)$ has finite cohomology dimension.*

**Theorem 2.29** (Verdier Duality)**.** *Under the above assumption, there exists a functor*

$$\mathbb{R}f^! : D(Y) \to D(X),$$

*which is right adjoint to the derived proper direct image functor $\mathbb{R}f_!$. That is, there exists the following adjunction*

$$\mathbb{R}f_! : D(X) \rightleftarrows D(Y) : \mathbb{R}f^!.$$

**Definition 2.30.** We will refer to the functor $\mathbb{R}f^! : D(Y) \to D(X)$ defined above as the *exceptional inverse image functor*.

2.4. **Sheaf Cohomology.** The goal of this section is to see some examples of six functors, and then define the sheaf cohomology and the cohomology of a topological space.

**Example 2.31.** Let $X$ be a topological space, and let $F$ be a sheaf on $X$. Then we have an equivalence

$$F \otimes \underline{\mathbb{Z}} \simeq F \simeq \underline{\mathbb{Z}} \otimes F.$$

That is, the constant sheaf $\underline{\mathbb{Z}}$ is a unit object of the symmetric monoidal structure on $\mathrm{Shv}(X)$. We denote this unit object by $\mathbf{1}$.

**Remark 2.32.** Let $*$ be a topological space consisting of one point. Then there exists an equivalence of categories $\mathrm{Shv}(*) \simeq \mathrm{Ab}$.

**Example 2.33.** The direct image of the projection to a point agrees with the global section: Let $X$ be a topological space, and let $p : X \to *$ be the projection. For a sheaf $F$ on $X$, we have

$$p_*F(*) \simeq \Gamma(p^{-1}(*); F) \simeq \Gamma(X; F).$$

Then we obtain an equivalence of functors $p_* \simeq \Gamma(X : -) : \mathrm{Shv}(X) \to \mathrm{Ab}$.

**Example 2.34.** The inverse image of the projection to a point agrees with the constant sheaf: Let $X$ be a topological space, and let $p : X \to *$ be the projection. For a sheaf $G$ on $*$, the value $G(*)$ can be identified with an abelian group $M$. We have

$$p' G(U) \simeq \operatorname*{colim}_{p(U) \subset *} G(*) \simeq M,$$

since the condition $p(U) \subset *$ is automatically satisfied. Thus its sheafification $p^* G$ is equivalent the constant sheaf $\underline{M}$. Then we obtain an equivalence of functors $p^* \simeq \underline{(-)} : \mathrm{Ab} \to \mathrm{Shv}(X)$.

**Remark 2.35.** Let $X$ be a topological space, and let $p : X \to *$ be the projection. By examples 2.33 and 2.34, there exists the following adjunction

$$p^* \simeq \underline{(-)} : \mathrm{Ab} \rightleftarrows \mathrm{Shv}(X) : \Gamma(X; -) \simeq p_*.$$

These derived functors define the following adjunction

$$p^* \simeq \underline{(-)} : D(\mathbb{Z}) \rightleftarrows D(X) : \mathbb{R}\Gamma(X; -) \simeq \mathbb{R}p_*.$$

**Definition 2.36.** Let $X$ be a topological space, and let $p : X \to *$ be the projection. We define the *cohomology* $\mathbb{R}\Gamma(X; \mathbb{Z})$ of $X$ by

$$\mathbb{R}\Gamma(X; \mathbb{Z}) := \mathbb{R}\Gamma(X; \mathbf{1}) \simeq \mathbb{R}p_* \mathbf{1}.$$

Similarly, we define the *cohomology with compact support* $\mathbb{R}\Gamma_c(X; \mathbb{Z})$ of $X$ by

$$\mathbb{R}\Gamma_c(X; \mathbb{Z}) := \mathbb{R}p_! \mathbf{1}.$$

2.5. **Some Remarkable Theorems.** In this section, we summarize notable results in sheaf theory, which are properties that 6-functor formalisms are expected to satisfy (see **??**). From here we omit the term "derived" and the derived symbol $\mathbb{R}$.

**Proposition 2.37** (Naturality)**.** Let $f : X \to Y$ be a continuous map of topological spaces. Then the functors $f_*, f_*, f_!$ and $f^!$ are compatible with composition.

**Proposition 2.38** (Symmetric Monoidality)**.** Let $f : X \to Y$ be a continuous map of topological spaces. Then the inverse direct image $f^* : D(Y) \to D(X)$ is symmetric monoidal.

**Theorem 2.39** (Proper Base Change)**.** *For every Cartesian diagram of topological spaces*

$$\begin{tikzcd} X' \arrow[r, "g'"] \arrow[d, "f'"'] \arrow[dr, phantom, "\lrcorner", very near start] & X \arrow[d, "f"] \\ Y' \arrow[r, "g"'] & Y, \end{tikzcd}$$

*there exists a natural isomorphism*

$$g^* f_! \simeq f'_! g'^*$$

*of functors $D(X) \to D(Y')$.*

**Theorem 2.40** (Projection Formula)**.** *Let $f : X \to Y$ be a proper map of topological spaces. Then there exist a natural isomorphism*

$$f_!(-) \otimes (-) \to f_!((-) \otimes f^*(-))$$

*of functors $D(X) \times D(Y) \to D(Y)$*

**Theorem 2.41** (Künneth Formula)**.** *Let $X$ and $Y$ be topological spaces. Then the canonical map*

$$\Gamma_c(X \times Y; \mathbb{Z}) \to \Gamma_c(X; \mathbb{Z}) \otimes \Gamma_c(Y; \mathbb{Z}).$$

*is an isomorphism in $D(\mathbb{Z})$.*

## 3. The Category of Correspondences

3.1. **Motivations.** Let us recall the setup of classical sheaf theory. We work with the category $\mathcal{C}$ of locally compact Hausdorff topological spaces. In this setting, there exists a class $E$ of morphisms in $\mathcal{C}$, where the exceptional functors $f_!$ and $f^!$ can be defined. Then we can

- For every object $X$ of $\mathcal{C}$, we obtain the category $\mathrm{D}(X)$ of sheaves on $X$.
- For every morphism $f : X \to Y$ in $\mathcal{C}$, we have functors

$$f^* : \mathrm{D}(Y) \to \mathrm{D}(X) \quad \text{and} \quad f_* : \mathrm{D}(X) \to \mathrm{D}(Y).$$

- For every morphism $f : X \to Y$ in $E$, we can define functors

$$f_! : \mathrm{D}(X) \to \mathrm{D}(Y) \quad \text{and} \quad f^! : \mathrm{D}(Y) \to \mathrm{D}(X).$$

- Additionally, there exist bifunctors

$$\otimes : \mathrm{D}(X) \times \mathrm{D}(X) \to \mathrm{D}(X) \quad \text{and} \quad \underline{\mathrm{Hom}} : \mathrm{D}(X)^{\mathrm{op}} \times \mathrm{D}(X) \to \mathrm{D}(X).$$

There is a natural map

$$\mathrm{D}(X) \otimes \mathrm{D}(Y) \to \mathrm{D}(X \times Y).$$

For $Y = X$, this induces a morphism

$$\mathrm{D}(X) \otimes \mathrm{D}(X) \to \mathrm{D}(X \times X) \xrightarrow{\Delta^*} \mathrm{D}(X),$$

where the second morphism is given by the diagonal $\Delta : X \to X \times X$.

In summary, the constructions $X \mapsto \mathrm{D}(X)$ and $f \mapsto f^*$ define a lax symmetric monoidal functor

$$\mathrm{D}_{\mathcal{C}} : \mathcal{C}^{\mathrm{op}} \to \mathrm{Cat}.$$

Similarly, the constructions $X \mapsto \mathrm{D}(X)$ and $f \mapsto f_!$ define a functor

$$\mathrm{D}_E : \mathcal{C}_E \to \mathrm{Cat},$$

where $\mathcal{C}_E$ is the wide subcategory of $\mathcal{C}$ consisting of the morphisms in $E$.

To unify the different directions of morphisms $f^*$ and $f_!$, we consider the category $\mathrm{Corr}(\mathcal{C}, E)$ of correspondences:

- The objects of $\mathrm{Corr}(\mathcal{C}, E)$ are the objects of $\mathcal{C}$.
- A morphism $X \to Y$ in $\mathrm{Corr}(\mathcal{C}, E)$ is a correspondence $X \leftarrow W \to Y$ in $\mathcal{C}$, where $W \to Y$ lies in $E$.

There are two inclusions

$$\mathcal{C}^{\mathrm{op}} \to \mathrm{Corr}(\mathcal{C}, E) : (f : X \to Y) \mapsto (Y \xleftarrow{f} X \xrightarrow{\mathrm{id}_X} X),$$

$$E \to \mathrm{Corr}(\mathcal{C}, E) : (f : X \to Y) \mapsto (X \xleftarrow{\mathrm{id}_X} X \xrightarrow{f} Y).$$

Then the above discussion can be summarized as follows: A 3-functor formalism is a lax symmetric monoidal functor

$$\mathrm{D} : \mathrm{Corr}(\mathcal{C}, E) \to \mathrm{Cat}.$$

To define a 6-functor formalism, we don't need any extra data. That is, a 6-functor formalism is a special case of a 3-functor formalism, where 3-functors $\otimes$, $f^*$ and $f_!$ admit right adjoint functors.

In this section, we define the category of correspondences, which is a central ingredient in constructing 6-functor formalisms From here all categories are assumed to be $\infty$-categories.

### 3.2. Geometric Setups.

**Definition 3.1** ([HM24] Definition 2.1.1)**.** A *geometric setup* $(\mathcal{C}, \mathcal{C}_0)$ consists of a category $\mathcal{C}$ and a wide subcategory $\mathcal{C}_0$ of $\mathcal{C}$ satisfying the following conditions:

(0) The category $\mathcal{C}$ admits pullbacks.

(1) The subcategory $\mathcal{C}_0$ is stable under pullbacks along morphisms in $\mathcal{C}$.

(2) The subcategory $\mathcal{C}_0$ admits and the inclusion $\mathcal{C}_0 \subseteq \mathcal{C}$ preserves pullbacks.

**Definition 3.2.** Let $(\mathcal{C}, \mathcal{C}_0)$ and $(\mathcal{C}', \mathcal{C}'_0)$ be geometric setups. A *morphism of geometric setups* from $(\mathcal{C}, \mathcal{C}_0)$ to $(\mathcal{C}', \mathcal{C}'_0)$ is a functor $f : \mathcal{C} \to \mathcal{C}'$ satisfying $f(\mathcal{C}_0) \subseteq \mathcal{C}'_0$. We let $\mathrm{Fun}^{\mathrm{GS}}(\mathcal{C}, \mathcal{C}')$ denote the full subcategory of $\mathrm{Fun}(\mathcal{C}, \mathcal{C}')$ spanned by morphisms of geometric setups.

**Notation 3.3.** We define the simplicial category $\mathrm{GS}^{\Delta}$ as follows:

- The objects of $\mathrm{GS}^{\Delta}$ are geometric setups.
- For every pair of objects $(\mathcal{C}, \mathcal{C}_0)$ and $(\mathcal{C}', \mathcal{C}'_0)$ of $\mathrm{GS}^{\Delta}$, the mapping space

$$\mathrm{Map}_{\mathrm{GS}^{\Delta}}((\mathcal{C}, \mathcal{C}_0), (\mathcal{C}', \mathcal{C}'_0)) := \mathrm{core}\,\mathrm{Fun}^{\mathrm{GS}}(\mathcal{C}, \mathcal{C}').$$

By [**HTT**], the simplicial nerve of $\mathrm{GS}^{\Delta}$ is a category.

**Definition 3.4.** We let GS denote the simplicial nerve $\mathrm{N}_{\Delta}(\mathrm{GS}^{\Delta})$. We will refer to GS as the *category of geometric setups.*

**Remark 3.5.** We can identity a geometric setup $(\mathcal{C}, \mathcal{C}_0)$ with a pair $(\mathcal{C}, E)$ where $\mathcal{C}$ is a category and $E$ is a class of morphisms in $\mathcal{C}$ satisfying the following conditions:

(1) The class $E$ contains all equivalences, is stable under composition and pullbacks along morphisms in $\mathcal{C}$.

(2) For every morphism $f : X \to Y$ in $E$, the diagonal $\Delta : X \to X \times_Y X$ lies in $E$ (see lemma 3.6).

Similarly, We can identity a morphism $f : (\mathcal{C}, \mathcal{C}_0) \to (\mathcal{C}', \mathcal{C}'_0)$ of geometric setups with a functor $f : (\mathcal{C}, E) \to (\mathcal{C}', E)$ which preserves pullbacks of the form $X \leftarrow W \to Y$ where $W \to Y$ lies in $E$.

Conversely, given a pair $(\mathcal{C}, E)$ which satisfies the above conditions, we can obtain a geometric setup $(\mathcal{C}, \mathcal{C}_E)$ where $\mathcal{C}_E$ denote the wide subcategory of $\mathcal{C}$ consisting of the morphisms in $E$. Similarly, we can get a morphism of geometric setups.

**Lemma 3.6** ([HM24] Lemma 2.1.5)**.** Let $\mathcal{C}$ be a category, and let $E$ be a class of morphisms in $E$ satisfying the condition (1) of remark 3.5. The following conditions are equivalent:

(1) The subcategory $\mathcal{C}_E$ satisfies the condition (2) of definition 3.1.

(2) The class $E$ satisfies the condition (2) of remark 3.5.

(3) The class $E$ is right cancellative.

### 3.3. The Category of Correspondences.

**Notation 3.7** ([HM24] Definition 2.2.1)**.** Let $[n]$ be an object of $\mathbb{\Delta}$.

(1) We define the simplicial set $\Sigma^n$ by $\Sigma^n_k := \mathrm{Hom}_{\mathrm{sSet}}([k] \star [k]^{\mathrm{op}}, [n])$ for every $k \geq 0$.

(2) We let $\Sigma^n_{\mathrm{v}}$ denote the subsimplicial set of $\Sigma^n$ satisfying the following condition: Its projection to the second component is a degenerate edge.

(3) We let $\mathbb{A}^n$ denote the subsimplicial set of $\Sigma^n$ spanned by the image of objects $(i, j)$ of $[-] \star [-]^{\mathrm{op}}$ satisfying $j \leq i + 1$.

**Remark 3.8.** The simplicial set $\Sigma^n$ is an example of twisted arrow categories (see [HA] Construction 5.2.1.1). By [HA, Proposition 5.2.1.3], $\Sigma^n$ is a category. Then $\Sigma^n_{\mathrm{v}}$ and $\mathbb{A}^n$ are full subcategories of $\Sigma^n$.

**Remark 3.9.** Let $[n]$ be an object of $\mathbb{\Delta}$. The pair $(\Sigma^n, \Sigma^n_{\mathrm{v}})$ is a geometric setup by definition. Then the construction $[n] \mapsto (\Sigma^n, \Sigma^n_{\mathrm{v}})$ defines cosimplicial object

$$(\Sigma^-, \Sigma^-_{\mathrm{v}}) : \mathbb{\Delta} \to \mathrm{GS}.$$

**Notation 3.10.** Let $(\mathcal{C}, \mathcal{C}_0)$ be a geometric setup. We define the functor $\mathrm{Corr}(\mathcal{C}, \mathcal{C}_0) : \mathbb{\Delta}^{\mathrm{op}} \to \mathrm{An}$ by

$$\mathrm{Corr}(\mathcal{C}, \mathcal{C}_0)_n := \mathrm{Map}_{\mathrm{GS}}((\Sigma^n, \Sigma^n_{\mathrm{v}}), (\mathcal{C}, \mathcal{C}_0))$$

for every $n \geq 0$.

**Remark 3.11.** The construction $[n] \mapsto \mathrm{Corr}(\mathcal{C}, \mathcal{C}_0)_n$ defines a cosimplicial anima

$$\mathrm{Corr}(\mathcal{C}, \mathcal{C}_0) : \mathbb{\Delta} \to \mathrm{An}.$$

The construction $(\mathcal{C}, \mathcal{C}_0) \mapsto \mathrm{Corr}(\mathcal{C}, \mathcal{C}_0)$ determines a functor

$$\mathrm{Corr} : \mathrm{GS} \to \mathrm{Fun}(\mathbb{\Delta}^{\mathrm{op}}, \mathrm{An}).$$

$$\begin{tikzcd} \mathrm{Fun}(\mathbb{\Delta}^{\mathrm{op}}, \mathrm{An}) & \\ \mathbb{\Delta} \arrow[u, "よ"] \arrow[r, "(\Sigma^-, \Sigma^-_{\mathrm{v}})"'] & \mathrm{GS}. \arrow[ul, "\mathrm{Corr}"'] \end{tikzcd}$$

**Lemma 3.12** ([HM24] Lemma 2.1.6, 2.2.11)**.** The category GS is complete, and the functor $\mathrm{Corr} : \mathrm{GS} \to \mathrm{Fun}(\mathbb{\Delta}^{\mathrm{op}}, \mathrm{An})$ preserves limits.

The following proposition follows from the similar argument that Quillen's Q-construction defines a complete Segal anima in algebraic K-theory.

**Proposition 3.13** ([HM24] Proposition 2.2.9)**.** The functor $\mathrm{Corr} : \mathrm{GS} \to \mathrm{Fun}(\mathbb{\Delta}^{\mathrm{op}}, \mathrm{An})$ factors through the full subcategory of complete Segal anima. That is, for every geometric setup $(\mathcal{C}, \mathcal{C}_0)$, the functor $\mathrm{Corr}(\mathcal{C}, \mathcal{C}_0) : \mathbb{\Delta}^{\mathrm{op}} \to \mathrm{An}$ is a complete Segal anima.

**Definition 3.14** ([HM24] Definition 2.2.10)**.** Let $(\mathcal{C}, \mathcal{C}_0)$ be a geometric setup. We will refer to the complete Segal anima $\mathrm{Corr}(\mathcal{C}, \mathcal{C}_0) : \mathbb{\Delta}^{\mathrm{op}} \to \mathrm{An}$ as the *category of correspondences*.

**Remark 3.15.** Let $(\mathcal{C}, \mathcal{C}_0)$ be a geometric setup. Then the lower simplices of $\mathrm{Corr}(\mathcal{C}, \mathcal{C}_0)$ can be described as follows:

- The objects of $\mathrm{Corr}(\mathcal{C}, \mathcal{C}_0)$ are the objects of $\mathcal{C}$.
- A morphism $X \to Y$ in $\mathrm{Corr}(\mathcal{C}, \mathcal{C}_0)$ is a correspondence

$$\begin{tikzcd} X & X' \arrow[l, "h"'] \arrow[d, "v"] \\ & Y \end{tikzcd}$$

in $\mathcal{C}$, where $X'$ is an object of $\mathcal{C}$ and $v$ is a morphism in $E$.

- A composition of morphisms $X \to Y$ and $Y \to Z$ in $\mathrm{Corr}(\mathcal{C}, E)$ represented by diagrams $X \xleftarrow{h_1} X' \xrightarrow{v_1} Y$ and $Y \xleftarrow{h_2} Y' \xrightarrow{v_2} Z$ in $\mathcal{C}$ respectively, is a correspondence

$$\begin{tikzcd} X & X' \arrow[l, "h_1"'] \arrow[d, "v_1"] & X \times_Y Y' \arrow[l] \arrow[d] \arrow[dl, phantom, "\lrcorner", very near start] \\ & Y & Y' \arrow[l, "h_2"'] \arrow[d, "v_2"] \\ & & Z \end{tikzcd}$$

in $\mathcal{C}$. Since $v_1$ and $v_2$ lie in $E$, the right vertical also lies in $E$.

**Remark 3.16.** Let $[n]$ be an object of $\Delta$.

(1) The pair $([n], [n])$ is a geometric setup. The construction $[n] \mapsto ([n], [n])$ defines a cosimplicial object

$$([-], [-]) : \Delta \to \mathrm{GS}.$$

The construction $(\Sigma^n, \Sigma^n_{\mathrm{v}}) \mapsto ([n], [n])$ determines a map of cosimplicial objects

$$\Sigma^-, \Sigma^-_{\mathrm{v}} \to ([-], [-]).$$

Precomposing it induces a natural transformation

$$\mathcal{C}_0 \to \mathrm{Corr}(\mathcal{C}, \mathcal{C}_0),$$

since we have an equivalence

$$\mathrm{Hom}_{\mathrm{GS}}(([-], [-]), (\mathcal{C}, \mathcal{C}_0)) \simeq \mathrm{Hom}_{\mathrm{Cat}}([-], \mathcal{C}_0).$$

(2) The pair $([n]^{\mathrm{op}}, \mathrm{core}[n])$ is a geometric setup. The construction $[n] \mapsto ([n]^{\mathrm{op}}, \mathrm{core}[n])$ defines a cosimplicial object

$$([-]^{\mathrm{op}}, \mathrm{core}[-]) : \Delta \to \mathrm{GS}.$$

The construction $(\Sigma^n, \Sigma^n_{\mathrm{v}}) \mapsto ([n]^{\mathrm{op}}, \mathrm{core}[n])$ determines a map of cosimplicial objects

$$\Sigma^-, \Sigma^-_{\mathrm{v}} \to ([-]^{\mathrm{op}}, \mathrm{core}[-]).$$

Precomposing it induces a natural transformation

$$\mathcal{C}^{\mathrm{op}} \to \mathrm{Corr}(\mathcal{C}, \mathcal{C}_0),$$

since we have an equivalence

$$\mathrm{Hom}_{\mathrm{GS}}(([-]^{\mathrm{op}}, \mathrm{core}[-]), (\mathcal{C}, \mathcal{C}_0)) \simeq \mathrm{Hom}_{\mathrm{Cat}}([-], \mathcal{C}^{\mathrm{op}}).$$

### 3.4. The Operad Structure on $\mathrm{Corr}(\mathcal{C}, \mathcal{C}_0)$.

**Lemma 3.17** ([HM24] Lemma 2.3.1)**.** Let $(\mathcal{C}, \mathcal{C}_0)$ be a geometric setup. We denote

$$\mathcal{C}_0^- := (\mathcal{C}_0^{\mathrm{op}})^{\amalg,\mathrm{op}} \times_{\mathrm{Fin}_*^{\mathrm{op}}} \mathrm{coreFin}_*.$$

Then the pair $((\mathcal{C}_0^{\mathrm{op}})^{\amalg,\mathrm{op}}, \mathcal{C}_0^-)$ is a geometric setup.

**Proposition 3.18** ([HM24] Proposition 2.3.1)**.** Let $(\mathcal{C}, \mathcal{C}_0)$ be a geometric setup. We denote

$$\mathrm{Corr}(\mathcal{C}, \mathcal{C}_0)^{\otimes} := \mathrm{Corr}((\mathcal{C}_0^{\mathrm{op}})^{\amalg,\mathrm{op}}, \mathcal{C}_0^-).$$

Then $\mathrm{Corr}(\mathcal{C}, \mathcal{C}_0)^{\otimes}$ is an operad whose underlying category is $\mathrm{Corr}(\mathcal{C}, \mathcal{C}_0)$.

# 4. Six-functor formalisms

## 4.1. Three-functor Formalisms.

**Definition 4.1** ([HM24] Definition 3.1.1)**.** Let $(\mathcal{C}, E)$ be a geometric setup. A *3-functor formalism* on $(\mathcal{C}, E)$ is a map of operads

$$\mathrm{D} : \mathrm{Corr}(\mathcal{C}, E)^{\otimes} \to \mathrm{Cat}^{\times}.$$

**Remark 4.2.** Let $(\mathcal{C}, E)$ be a geometric setup, and let $\mathrm{D} : \mathrm{Corr}(\mathcal{C}, E)^{\otimes} \to \mathrm{Cat}^{\times}$ be a 3-functor formalism on $(\mathcal{C}, E)$.

(1) By precomposing D with the map of operads $(\mathcal{C}^{\mathrm{op}})^{\amalg} \to \mathrm{Corr}(\mathcal{C}, E)^{\otimes}$, we obtain a map of operads $\mathrm{D} : (\mathcal{C}^{\mathrm{op}})^{\amalg} \to \mathrm{Cat}^{\times}$. By [HA, Theorem 2.4.3.18], it corresponds to a functor

$$\mathrm{D}^* : \mathcal{C}^{\mathrm{op}} \to \mathrm{CMon}.$$

(2) By precomposing D with the map of operads $\mathcal{C}_E \to \mathrm{Corr}(\mathcal{C}, E)^{\otimes}$, we obtain a map of operads (?)

$$\mathrm{D}_! : \mathcal{C}_E \to \mathrm{Cat}^{\times}.$$

**Notation 4.3.** Let $(\mathcal{C}, E)$ be a geometric setup, and let $\mathrm{D} : \mathrm{Corr}(\mathcal{C}, E)^{\otimes} \to \mathrm{Cat}^{\times}$ be a 3-functor formalism on $(\mathcal{C}, E)$. We will use the following notations.

(1) For every object $X$ of $\mathcal{C}$, the category $\mathrm{D}(X) = \mathrm{D}^*(X)$ admits a symmetric monoidal structure, which we denote by

$$\otimes : \mathrm{D}(X) \times \mathrm{D}(X) \to \mathrm{D}(X).$$

(2) For every morphism $f : X \to Y$ in $\mathcal{C}$, we obtain a symmetric monoidal functor

$$f^* := \mathrm{D}^*(f) : \mathrm{D}(Y) \to \mathrm{D}(X).$$

(3) For every morphism $f : X \to Y$ in $E$, we obtain a functor

$$f_! := \mathrm{D}_!(f) : \mathrm{D}(X) \to \mathrm{D}(Y).$$

**Lemma 4.4.** Let $(\mathcal{C}, E)$ be a geometric setup, and let $\mathrm{D} : \mathrm{Corr}(\mathcal{C}, E)^{\otimes} \to \mathrm{Cat}^{\times}$ be a 3-functor formalism on $(\mathcal{C}, E)$. For every pair of morphisms $g : X' \to X$ in $\mathcal{C}$ and $f : X' \to Y$ in $E$, we obtain a natural morphism

$$f_! g^* : \mathrm{D}(X) \to \mathrm{D}(Y).$$

$$\begin{tikzcd} X & X' \arrow[l, "g"'] \arrow[d, "f"] & & \mathrm{D}(X) \arrow[r, "g^*"] & \mathrm{D}(X') \arrow[d, "f_!"] \\ & Y & {} \arrow[r, mapsto, "\mathrm{D}"] & {} & \mathrm{D}(Y). \end{tikzcd}$$

*Proof.* The morphisms $g : X' \to X$ in $\mathcal{C}$ and $f : X' \to Y$ in $E$ determines correspondences $h_1$ and $h_2$ respectively.

$$\begin{tikzcd} X & X' \arrow[l, "g"'] \arrow[d, equal] & X' \arrow[r, equal] & X' \arrow[d, "f"] \\ & X' & & Y. \end{tikzcd}$$

These morphisms define functors

$$g^* : \mathrm{D}(X) \to \mathrm{D}(X') \quad \text{and} \quad f_! : \mathrm{D}(X') \to \mathrm{D}(Y)$$

respectively. On the other hands, the composition $h_2h_1$ is given by the correspondence

$$\begin{tikzcd} X & X' \arrow[l, "g"'] \arrow[r, equal] \arrow[d, equal] \arrow[dr, phantom, "\llcorner"] & X' \arrow[d, equal] \\ & X' \arrow[r, equal] & X' \arrow[d, "f"] \\ & & Y. \end{tikzcd}$$

Since D is a functor, we obtain

$$\mathrm{D}(h_2h_1) \simeq \mathrm{D}(h_2)\,\mathrm{D}(h_1) \simeq f_!g^* : \mathrm{D}(X) \to \mathrm{D}(Y).$$

□

**Proposition 4.5** ([HM24] Proposition 3.1.8)**.** Let $(\mathcal{C}, E)$ be a geometric setup, and let $\mathrm{D} : \mathrm{Corr}(\mathcal{C}, E)^{\otimes} \to \mathrm{Cat}^{\times}$ be a 3-functor formalism on $(\mathcal{C}, E)$. Then the associated three functors $\otimes, f^*$ and $f_!$ satisfy the following conditions:

(1) The functors $f^*$ and $f_!$ are natural.

(2) For every morphism $f : X \to Y$ in $\mathcal{C}$, the functor $f^* : \mathrm{D}(Y) \to \mathrm{D}(X)$ is symmetric monoidal.

(3) (Proper base change) For every cartesian diagram in $\mathcal{C}$

$$\begin{tikzcd} X' \arrow[r, "g'"] \arrow[d, "f'"'] \arrow[dr, phantom, "\lrcorner", very near start] & X \arrow[d, "f"] \\ Y' \arrow[r, "g"'] & Y, \end{tikzcd}$$

where $f$ in $E$ (thus also $f'$ in $E$), there exists a natural equivalence

$$g^*f_! \simeq f'_!g'^*$$

of functors $\mathrm{D}(X) \to \mathrm{D}(Y')$.

$$\begin{tikzcd} \mathrm{D}(X') \arrow[d, "f'_!"'] & \mathrm{D}(X) \arrow[l, "g'^*"'] \arrow[d, "f_!"] \\ \mathrm{D}(Y') & \mathrm{D}(Y). \arrow[l, "g^*"] \end{tikzcd}$$

(4) (The projection formula) For every morphism $f : X \to Y$ in $E$ and every pair of objects $F$ of $\mathrm{D}(X)$ and $G$ of $\mathrm{D}(Y)$, there exists a natural equivalence

$$f_!F \otimes G \to f_!(F \otimes f^*G)$$

of functors $\mathrm{D}(X) \times \mathrm{D}(Y) \to \mathrm{D}(Y)$.

$$\begin{tikzcd}
\mathrm{D}(Y) \times \mathrm{D}(Y) \arrow[dd, "\otimes"'] & \mathrm{D}(X) \times \mathrm{D}(Y) \arrow[l, "(f_!, \mathrm{id}_{\mathrm{D}(Y)})"'] \arrow[d, "(\mathrm{id}_{\mathrm{D}(X)}, f^*)"] \\
& \mathrm{D}(X) \times \mathrm{D}(X) \arrow[d, "\otimes"] \\
\mathrm{D}(Y) & \mathrm{D}(X). \arrow[l, "f_!"]
\end{tikzcd}$$

*Proof.* (1) and (2) follow from the definitions.

We next show (3). The morphisms $f : X \to Y$ in $E$ and $g : Y' \to Y$ in $\mathcal{C}$ determine correspondences $h_1$ and $h_2$ respectively.

$$\begin{tikzcd}
X \arrow[r, equal] & X \arrow[d, "f"] \\
& Y
\end{tikzcd}
\qquad
\begin{tikzcd}
Y & Y' \arrow[l, "g"'] \arrow[d, equal] \\
& Y'.
\end{tikzcd}$$

The composition $h_2 h_1$ is given by the correspondence

$$\begin{tikzcd}
X \arrow[r, equal] & X \arrow[d, "f"] & X' \arrow[l, "g'"'] \arrow[d, "f'"] \arrow[dl, phantom, "\llcorner"] \\
& Y & Y' \arrow[l, "g"'] \arrow[d, equal] \\
& & Y'.
\end{tikzcd}$$

The first map $h_1 : X \xleftarrow{\mathrm{id}_X} X \xrightarrow{f} Y$ defines a functor

$$f_! : \mathrm{D}(X) \to \mathrm{D}(Y).$$

The second map $h_2 : Y \xleftarrow{g'} Y' \xrightarrow{\mathrm{id}_{Y'}} Y'$ defines a functor

$$g^* : \mathrm{D}(Y) \to \mathrm{D}(Y').$$

The composition $h_2 h_1 : X \xleftarrow{g'} X' \xrightarrow{f'} Y'$ defines a functor

$$f'_! g'^* : \mathrm{D}(X) \to \mathrm{D}(Y').$$

Thus we obtain a natural equivalence $g^* f_! \simeq f'_! g'^*$, since $\mathrm{D}$ is a functor.

We finally show (4). Let $f : X \to Y$ be a morphism in $E$. Consider the morphism $X \times Y \to Y$ given by the correspondence

$$\begin{tikzcd}
X \times Y & X \arrow[l, "(\mathrm{id}_X, f)"'] \arrow[d, "f"] \\
& Y.
\end{tikzcd}$$

We obtain two factorizations of $X \times Y \to Y$ as follows: The morphism $X \times Y \to Y \times Y \to Y$ which is given by the correspondence

$$\begin{tikzcd}
X \times Y \arrow[r, equal] & X \times Y \arrow[d, "{(f, \mathrm{id}_X)}"] & X \arrow[l, "{(\mathrm{id}_X, f)}"'] \arrow[d, "f"] \arrow[dl, phantom, very near start, "\llcorner"] \\
& Y \times Y & Y \arrow[l, "\Delta_Y"'] \arrow[d, equal] \\
& & Y.
\end{tikzcd}$$

The morphism $X \times Y \to X \times X \to X \to Y$ which is given by the correspondence

$$\begin{tikzcd}
X \times Y & X \times X \arrow[l, "{(\mathrm{id}_X, f)}"'] \arrow[d, equal] & X \arrow[l, "\Delta_X"'] \arrow[r, equal] \arrow[d, equal] \arrow[dl, phantom, very near start, "\llcorner"] & X \arrow[d, equal] \arrow[dl, phantom, very near start, "\llcorner"] \\
& X \times X & X \arrow[l, "\Delta_X"'] \arrow[r, equal] \arrow[d, equal] & X \arrow[d, equal] \arrow[dl, phantom, very near start, "\llcorner"] \\
& & X \arrow[r, equal] & X \arrow[d, "f"] \\
& & & Y.
\end{tikzcd}$$

The considering morphism $X \times Y \xleftarrow{(\mathrm{id}_X, f)} X \xrightarrow{f} Y$ defines a functor

$$\mathrm{D}(X) \times \mathrm{D}(Y) \to \mathrm{D}(Y).$$

The first factorization $X \times Y \xleftarrow{(\mathrm{id}_X, f)} X \xrightarrow{f} Y$ defines a functor

$$\begin{aligned}
\mathrm{D}(X) \times \mathrm{D}(Y) &\to \mathrm{D}(Y) \times \mathrm{D}(Y) \to \mathrm{D}(Y) \\
(F, G) &\mapsto (f_! F, G) \mapsto f_! F \otimes G.
\end{aligned}$$

The second factorization $X \times Y \xleftarrow{(\mathrm{id}_X, f)} X \times X \xleftarrow{\Delta_X} X \xrightarrow{f} Y$ defines a functor

$$\begin{aligned}
\mathrm{D}(X) \times \mathrm{D}(Y) &\to \mathrm{D}(X) \times \mathrm{D}(X) \to \mathrm{D}(X) \to \mathrm{D}(Y) \\
(F, G) &\mapsto (F, f^* G) \mapsto F \otimes f^* G \mapsto f_!(F \otimes f^* G).
\end{aligned}$$

This implies that a morphism $f_! F \otimes G \to f_!(F \otimes f^* G)$ is a natural equivalence in $\mathrm{D}(Y)$, since D is a functor. $\square$

### 4.2. **Six-functor formalisms.**

**Definition 4.6** ([HM24] Definition 3.2.1)**.** Let $(\mathcal{C}, E)$ be a geometric setup. A *6-functor formalism* on $(\mathcal{C}, E)$ is a 3-functor formalism $\mathrm{D} : \mathrm{Corr}(\mathcal{C}, E)^{\otimes} \to \mathrm{Cat}^{\times}$ satisfying the following conditions:

(1) For every object $X$ of $\mathcal{C}$, the symmetric monoidal structure on $\mathrm{D}(X)$ is closed.
(2) The functors $f^* : \mathrm{D}(Y) \to \mathrm{D}(X)$ and $f_! : \mathrm{D}(X) \to \mathrm{D}(Y)$ admit right adjoint functors respectively.

**Notation 4.7.** Let $(\mathcal{C}, E)$ be a geometric setup, and let $D : \mathrm{Corr}(\mathcal{C}, E)^{\otimes} \to \mathrm{Cat}^{\times}$ be a 6-functor formalism on $(\mathcal{C}, E)$. We will use the following notations:

(1) We denote the internal hom in $\mathrm{D}(X)$ (i.e. the right adjoint functor to $\otimes$) by

$$\underline{\mathrm{Hom}} : \mathrm{D}(X)^{\mathrm{op}} \times \mathrm{D}(X) \to \mathrm{D}(X).$$

(2) For every morphism $f : X \to Y$ in $\mathcal{C}$, we denote the right adjoint functor to $f^*$ by

$$f_* : \mathrm{D}(X) \to \mathrm{D}(Y).$$

(3) For every morphism $f : X \to Y$ in $E$, we denote the right adjoint functor to $f_!$ by

$$f^! : \mathrm{D}(Y) \to \mathrm{D}(X).$$

**Proposition 4.8** ([HM24] Proposition 3.2.2)**.** Let $(\mathcal{C}, E)$ be a geometric setup, and let $D : \mathrm{Corr}(\mathcal{C}, E)^{\otimes} \to \mathrm{Cat}^{\times}$ be a 6-functor formalism on $(\mathcal{C}, E)$. Then the associated six functors $\otimes \dashv \underline{\mathrm{Hom}}$, $f^* \dashv f_*$ and $f_! \dashv f^!$ satisfy the following conditions:

(1) The functors $f_*$ and $f^!$ are natural.
(2) For every cartesian diagram in $\mathcal{C}$

$$\begin{tikzcd} X' \arrow[r, "g'"] \arrow[d, "f'"'] \arrow[dr, phantom, "\lrcorner", very near start] & X \arrow[d, "f"] \\ Y' \arrow[r, "g"'] & Y, \end{tikzcd}$$

where $f$ in $E$ (thus also $f'$ in $E$), there exists a natural equivalence

$$g^! f_* \simeq f'_* g'^!$$

of functors $\mathrm{D}(X) \to \mathrm{D}(Y')$.

$$\begin{tikzcd} \mathrm{D}(X') \arrow[d, "f'_*"'] & \mathrm{D}(X) \arrow[l, "g'^!"'] \arrow[d, "f_*"] \\ \mathrm{D}(Y') & \mathrm{D}(Y). \arrow[l, "g^!"] \end{tikzcd}$$

*Proof.* (1) follows from proposition 4.5 and the fact that the adjoint is also natural. (2) follows from proposition 4.5 by passing to right adjoint functors. □

### 4.3. Constructions of 6-functor formalisms.

**Definition 4.9** ([HM24] Definition 3.3.2)**.** Let $(\mathcal{C}, E)$ be a geometric setup. We will say that $E$ *has a suitable decomposition* $(I, P)$ if there exists a pair of subclasses of morphisms $(I, P)$ of $E$ satisfying the following conditions:

(1) The classes $I$ and $P$ contain all equivalences, and are stable under composition and pullbacks.
(2) Every morphism $f$ in $E$ admits a factorization $f = gj$, where $g$ in $P$ and $j$ in $I$.
(3) For every morphism $f : X \to Y$ in $I$ (resp. $P$), the diagonal $\Delta : X \to X \times_Y X$ lies in $I$ (resp. $P$).
(4) Every morphism $f$ in $I \cap P$ is $n$-truncated for some $n \geq -2$ (which may depend on $f$).

**Remark 4.10.** By lemma 3.6, the condition (2) of definition 4.9 is equivalent to the following condition: The classes $I$ and $P$ are right cancellative in $\mathcal{C}$.

**Definition 4.11.** Let $(\mathcal{C}, E)$ be a geometric setup, and let $\mathrm{D} : \mathcal{C}^{\mathrm{op}} \to \mathrm{CMon}$ be a functor. Suppose that $E$ has a suitable decomposition $(I, P)$. For every morphism $f : X \to Y$ in $\mathcal{C}$, we denote $f^* := \mathrm{D}(f) : \mathrm{D}(Y) \to \mathrm{D}(X)$. We will say that D *satisfies a suitable decomposition condition* for $(I, P)$ if it satisfies the following conditions:

(1) For every morphism $j$ in $I$, the functor $j^*$ admits a left adjoint functor $j_! : \mathrm{D}(X) \to \mathrm{D}(Y)$, which satisfies proper base change and the projection formula.

(2) For every morphism $g$ in $P$, the functor $g^*$ admits a right adjoint functor $g_* : \mathrm{D}(X) \to \mathrm{D}(Y)$, which satisfies the $*$-version base change and the $*$-version projection formula.

(3) For every cartesian diagram in $\mathcal{C}$

$$\begin{tikzcd} X' \arrow[r, "j'"] \arrow[d, "g'"'] \arrow[dr, phantom, "\lrcorner", very near start] & X \arrow[d, "g"] \\ Y' \arrow[r, "j"'] & Y, \end{tikzcd}$$

where $j$ in $I$ (thus also $j'$ in $I$), and $g$ in $P$ (thus also $g'$ in $P$), the morphism

$$j_! g'_* \to g_* j'_!$$

of functors $\mathrm{D}(X') \to \mathrm{D}(Y)$ is an equivalence.

$$\begin{tikzcd} \mathrm{D}(X') \arrow[r, "j'_!"] \arrow[d, "g'_*"'] & \mathrm{D}(X) \arrow[d, "g_*"] \\ \mathrm{D}(Y') \arrow[r, "j_!"'] & \mathrm{D}(Y). \end{tikzcd}$$

**Remark 4.12.** We impose the condition (3) in definition 4.11 only when the diagram is cartesian. This restriction arises because the morphism $j_! g'_* \to g_* j'_!$ is defined using the adjunction:

$$\begin{aligned} \mathrm{Hom}_{\mathrm{D}(Y)}(j_! g'_*(-), g_* j'_!(-)) &\simeq \mathrm{Hom}_{\mathrm{D}(X)}(g^* j_! g'_*(-), j'_!(-)) \\ &\simeq \mathrm{Hom}_{\mathrm{D}(X)}(j'_! g'^* g'_*(-), j'_!(-)). \end{aligned}$$

Here the second equivalence uses proper base change which is defined only for cartesian diagrams. Thus, we cannot *a priori* define a natural equivalence. However, we can still deduce the same equivalence as a consequence of the axioms, even when the diagram is not cartesian (lemmas 4.13 and 4.14). The following lemmata do not needed to prove proposition 4.16.

**Lemma 4.13** ([Sch22] Construction 4.3)**.** Let $(\mathcal{C}, E)$ be a geometric setup. Suppose that $E$ has a suitable decomposition $(I, P)$. Then, for every morphism $f$ in $I \cap P$, there exists a natural equivalence $f_! \simeq f_*$ between the left and right adjoint functors of $f^*$.

*Proof.* Let $f : X \to Y$ be a morphism in $I \cap P$. By (4) of definition 4.9, $f$ is $n$-truncated for some $n \geq -2$. Consider the following diagram.

$$\begin{tikzcd} X \arrow[dr, dashed, "\Delta"] \arrow[drr, bend left, "\mathrm{id}_X"] \arrow[ddr, bend right, "\mathrm{id}_X"'] & & \\ & X \times_Y X \arrow[r, "f'"] \arrow[d, "f'"] \arrow[dr, phantom, "\lrcorner", very near start] & X \arrow[d, "f"] \\ & X \arrow[r, "f"'] & Y. \end{tikzcd}$$

By (3) of definition 4.9, the diagonal $\Delta : X \to X \times_Y X$ lies in $I \cap P$. By (4) of definition 4.9, $\Delta$ is $n$-truncated for some $n \geq -2$. It is well known that, if $f$ is $n$-truncated, then $\Delta$ is $(n-1)$-truncated.

We prove the statement by induction on $n$.

Base case ($n = -2$): Assume that $f$ is $-2$-truncated, which means $f$ is an equivalence. In this case, the left and right adjoint functors of $f^*$ coincide. Then, we obtain an equivalence $f_! \simeq f_*$. This establishes the base case.

Inductive step: Assume that the statement holds for every $n$-truncated morphisms. i.e. for every $n$-truncated morphism $g$, we have $g_! \simeq g_*$. Let $f$ be a $(n+1)$-truncated morphism. Then the diagonal $\Delta$ is $n$-truncated. By the induction hypothesis, we have an equivalence $\Delta_! \simeq \Delta_*$. Then we have

$$f_! \simeq f_! \mathrm{id}_{X!} \simeq f_! \mathrm{id}_{X*} \simeq f_! g_* \Delta_* \simeq f_! g_* \Delta_! \simeq f_* h_! \Delta_! \simeq f_* \mathrm{id}_{X!} \simeq f_*.$$

□

**Lemma 4.14** ([Sch22] Construction 4.4)**.** Let $(\mathcal{C}, E)$ be a geometric setup. Suppose that $E$ has a suitable decomposition $(I, P)$. Then, for every commutative diagram in $\mathcal{C}$

$$\begin{tikzcd} X' \arrow[r, "j'"] \arrow[d, "g'"'] \arrow[dr, phantom, "\lrcorner", very near start] & X \arrow[d, "g"] \\ Y' \arrow[r, "j"'] & Y, \end{tikzcd}$$

where $j$ in $I$ (thus also $j'$ in $I$), and $g$ in $P$ (thus also $g'$ in $P$), we have an equivalence

$$j_! g'_* \to g_* j'_!$$

of functors $\mathrm{D}(X') \to \mathrm{D}(Y)$.

*Proof.* Consider the following diagram.

$$\begin{tikzcd} X' \arrow[dr, dashed, "h"] \arrow[drr, bend left, "j'"] \arrow[ddr, bend right, "g'"'] & & \\ & X \times_Y Y' \arrow[r, "j''"'] \arrow[d, "g''"] \arrow[dr, phantom, "\lrcorner", very near start] & X \arrow[d, "g"] \\ & Y' \arrow[r, "j"'] & Y. \end{tikzcd}$$

By remark 4.10, $h$ lies in $I \cap P$. By lemma 4.13, we obtain an equivalence $h_! \simeq h_*$. Then we have

$$j_! g'_* \simeq j_! g''_* h_* \simeq j_! g''_* h_! \simeq g_* j''_! h_! \simeq g_* j'_!.$$

□

We can also show that if a morphism $f$ in $E$ has two factorizations $f = gj = g'j'$, then the two induced factorizations of $f_!$ are equivalent. It is, of course, the corollary of proposition 4.16. To prove this statement (proposition 4.16), we need the condition (2) of definition 3.1 and remark 3.5.

**Proposition 4.15** ([Sch22] Construction 4.5)**.** Let $(\mathcal{C}, E)$ be a geometric setup. Suppose that $E$ has a suitable decomposition $(I, P)$. Let $f : X \to Y$ be a morphism in $E$ such that there exist two factorizations $f = gj = g'j'$, where $g$ and $g'$ in $P$, and $j$ and $j'$ in $I$. Then there exists a natural equivalence

$$g_* j_! \simeq g'_* j_!$$

of functors $\mathrm{D}(X) \to \mathrm{D}(Y)$.

*Proof.* Consider the following diagram.

$$\begin{tikzcd}
X \arrow[dr, dashed, "\Delta"] \arrow[drrr, bend left, "\mathrm{id}_X"] \arrow[dddr, bend right, "\mathrm{id}_X"'] & & & \\
& X \times_Y X \arrow[rr, "f'"] \arrow[dr, dashed, "h'"] \arrow[dd, "f'"] & & X \arrow[d, "j"] \\
& & \overline{X} \times_Y \overline{X}' \arrow[r, "\gamma"] \arrow[d, "\varepsilon"] & \overline{X} \arrow[d, "g"] \\
& X \arrow[r, "j'"'] & \overline{X}' \arrow[r, "g'"'] & Y.
\end{tikzcd}$$

By (2) of lemma 3.6, the diagonal $\Delta : X \to X \times_Y X$ and the induced morphism $h'$ lie in $E$. Then the composition $h = h'\Delta$ lies in $E$. By (2) of definition 4.9, $h$ admits a factorization $h = \beta\alpha$, where $\beta$ in $P$ and $\alpha$ in $I$. Then we obtain the following diagram.

$$\begin{tikzcd}
X \arrow[rrr, "\mathrm{id}_X"] \arrow[dr, "\alpha"] \arrow[ddd, "\mathrm{id}_X"'] & & & X \arrow[dd, "j"] \\
& \overline{X}'' \arrow[dr, "\beta"] \arrow[r, phantom, "(1)"] & & \\
& & \overline{X} \times_Y \overline{X}' \arrow[r, "\gamma"] \arrow[d, "\varepsilon"] & \overline{X} \arrow[d, "g"] \\
X \arrow[rr, "j'"'] \arrow[ur, phantom, "(2)"] & & \overline{X}' \arrow[r, "g'"'] & Y,
\end{tikzcd}$$

where $\alpha$ in $I$, $\beta$, $\gamma$ and $\varepsilon$ in $P$. From the diagrams (1) and (2), we get equivalences

$$j_! \simeq (\gamma\beta)_*\alpha_! \quad \text{and} \quad j'_! \simeq (\varepsilon\beta)_*\alpha_!.$$

Then we have

$$g_* j_! \simeq g_*(\gamma\beta)_*\alpha_! \simeq (g\gamma)_*\beta_*\alpha_! \simeq (g'\varepsilon)_*\beta_*\alpha_! \simeq g'_*(\varepsilon\beta)_*\alpha_! \simeq g'_* j'_!.$$

□

**Proposition 4.16** ([HM24] Proposition 3.3.3)**.** Let $(\mathcal{C}, E)$ be a geometric setup, and let $\mathrm{D} : \mathcal{C}^{\mathrm{op}} \to \mathrm{CMon}$ be a functor. Suppose that $E$ has a suitable decomposition $(I, P)$ and $\mathrm{D}$ satisfies a suitable decomposition condition for $(I, P)$. Then $\mathrm{D}$ can be extended to a 3-functor formalism

$$\mathrm{D} : \mathrm{Corr}(\mathcal{C}, E)^{\otimes} \to \mathrm{Cat}^{\times},$$

on $(\mathcal{C}, E)$ such that for every morphism $j$ in $I$, $j_!$ is left adjoint to $j^*$, and for every morphism $g$ in $P$, $g_!$ is right adjoint to $g^*$.

**Corollary 4.17.** Let $(\mathcal{C}, E)$ be a geometric setup, and let $\mathrm{D} : \mathcal{C}^{\mathrm{op}} \to \mathrm{CMon}$ be a functor. In addition to proposition 4.16, suppose that $\mathrm{D}$ satisfies the following conditions:

(1) For every object $X$ of $\mathcal{C}$, the symmetric monoidal structure on $\mathrm{D}(X)$ is closed.

(2) For every morphism $f : X \to Y$ in $\mathcal{C}$, the functor $f^* : \mathrm{D}(Y) \to \mathrm{D}(X)$ admit a right adjoint functor $f_* : \mathrm{D}(X) \to \mathrm{D}(Y)$.
(3) For every morphism $g : X \to Y$ in $P$, the functor $g_* = g_! : \mathrm{D}(X) \to \mathrm{D}(Y)$ admit a right adjoint functor $g^! : \mathrm{D}(Y) \to \mathrm{D}(X)$.

Then the 3-functor formalism obtained by proposition 4.16 is a 6-functor formalism on $(\mathcal{C}, E)$.

### 4.4. **Extensions of 6-functor formalisms.**

**Definition 4.18.** Let $(\mathcal{C}, E)$ be a geometric setup. We will say that $(\mathcal{C}, E)$ is a (*subcanonical*) *site* setup if $\mathcal{C}$ is a (subcanonical) site.

**Definition 4.19** ([HM24] Definition 3.4.1)**.** Let $(\mathcal{C}, E)$ be a site setup, and let $\mathrm{D} : \mathrm{Corr}(\mathcal{C}, E)^{\otimes} \to \mathrm{Cat}^{\times}$ be a 3-functor formalism on $(\mathcal{C}, E)$. We will say that D is *sheafy* if the induced functor $\mathrm{D}^* : \mathcal{C}^{\mathrm{op}} \to \mathrm{Cat}$ (remark 4.2) is a sheaf.

**Proposition 4.20** ([HM24] Proposition 3.4.2)**.** Let $(\mathcal{C}, E)$ be a subcanonical site setup, and let $\mathrm{D} : \mathrm{Corr}(\mathcal{C}, E)^{\otimes} \to \mathrm{Cat}^{\times}$ be a sheafy 3-functor formalism. We let $\mathcal{X} := \mathrm{Shv}(\mathcal{C})$ denote the topos of sheaves on $\mathcal{C}$. We let $E'$ be the collection of morphisms $f' : X \to Y$ in $\mathcal{X}$ such that, for every morphism $g : よ(Y) \to Y'$ from an object $Y$ of $\mathcal{C}$, the pullback of $f'$ along $g$ lies in $E$. Then

(1) The inclusion $よ : \mathcal{C} \subseteq \mathcal{X}$ induces a morphism of geometric setups $(\mathcal{C}, E) \to (\mathcal{X}, E')$.
(2) There exists a minimal choice of $E'$.
(3) The sheafy 3-functor formalism $\mathrm{D} : \mathrm{Corr}(\mathcal{C}, E)^{\otimes} \to \mathrm{Cat}^{\times}$ can be uniquely extended to a sheafy 3-functor formalism $\mathrm{D}' : \mathrm{Corr}(\mathcal{X}, E')^{\otimes} \to \mathrm{Cat}^{\times}$.

Moreover, if D is a presentable sheafy 6-functor formalism, so is D′.

**Theorem 4.21** ([HM24] Theorem 3.4.11)**.** *Let $(\mathcal{C}, E)$ be a subcanonical site setup, and let $\mathrm{D} : \mathrm{Corr}(\mathcal{C}, E)^{\otimes} \to \mathrm{Cat}^{\times}$ be a sheafy presentable 6-functor formalism. We let $\mathcal{X} := \mathrm{Shv}(\mathcal{C})$ denote the topos of sheaves on $\mathcal{C}$. Then there exists a collection of morphisms $E'$ in $\mathcal{X}$ satisfying the following conditions:*

*(1) The inclusion $よ : \mathcal{C} \subseteq \mathcal{X}$ induces a morphism of geometric setups $(\mathcal{C}, E) \to (\mathcal{X}, E')$.*
*(2) There exists a minimal choice of $E'$.*
*(3) The class $E'$ is $*$-local on the target, !-local, and tame.*
*(4) The sheafy presentable 6-functor formalism $\mathrm{D} : \mathrm{Corr}(\mathcal{C}, E)^{\otimes} \to \mathrm{Cat}^{\times}$ can be uniquely extended to a sheafy 6-functor formalism $\mathrm{D}' : \mathrm{Corr}(\mathcal{X}, E')^{\otimes} \to \mathrm{Cat}^{\times}$.*

## 5. Other Results

### 5.1. **Corollaries of Six-functor Formalisms.**
In this section, we show some corollaries of proposition 4.8. We fix a geometric setup $(\mathcal{C}, E)$ and a 6-functor formalism $\mathrm{D} : \mathrm{Corr}(\mathcal{C}, E)^{\otimes} \to \mathrm{Cat}^{\times}$.

We have the local representation of "Verdier duality".

**Corollary 5.1.** For every morphism $f : X \to Y$ in $E$, and every pair of objects $F$ of $\mathrm{D}(X)$ and $G$ of $\mathrm{D}(Y)$, there exists a natural equivalence

$$\underline{\mathrm{Hom}}_{\mathrm{D}(Y)}(f_! F, G) \simeq f_* \underline{\mathrm{Hom}}_{\mathrm{D}(X)}(F, f^! G)$$

of functors $\mathrm{D}(X) \times \mathrm{D}(Y) \to \mathrm{D}(Y)$.

*Proof.* For every object $H$ of $\mathrm{D}(X)$, we have

$$\begin{aligned}\mathrm{Hom}_{\mathrm{D}(X)}(H, \underline{\mathrm{Hom}}_{\mathrm{D}(Y)}(f_!F, G)) &\simeq \mathrm{Hom}_{\mathrm{D}(X)}(H \otimes f_!F, G)\\ &\simeq \mathrm{Hom}_{\mathrm{D}(X)}(f_!(f^*H \otimes F), G)\\ &\simeq \mathrm{Hom}_{\mathrm{D}(Y)}(f^*H \otimes F, f^!G)\\ &\simeq \mathrm{Hom}_{\mathrm{D}(Y)}(f^*H, \underline{\mathrm{Hom}}_{\mathrm{D}(X)}(F, f^!G))\\ &\simeq \mathrm{Hom}_{\mathrm{D}(X)}(H, f_*\underline{\mathrm{Hom}}_{\mathrm{D}(X)}(F, f^!G)).\end{aligned}$$

Here, the second equivalence uses the projection formula. The desired assertion follows from Yoneda's lemma. ☐

**Corollary 5.2.** For every morphism $f : X \to Y$ in $E$, and every pair of objects $F$ and $G$ of $\mathrm{D}(Y)$, there exists a natural equivalence

$$f^!\underline{\mathrm{Hom}}_{\mathrm{D}(Y)}(F, G) \simeq \underline{\mathrm{Hom}}_{\mathrm{D}(X)}(f^*F, f^!G)$$

of functors $\mathrm{D}(Y) \times \mathrm{D}(Y) \to \mathrm{D}(X)$.

*Proof.* For every object $H$ of $\mathrm{D}(X)$, we have

$$\begin{aligned}\mathrm{Hom}_{\mathrm{D}(X)}(H, f^!\underline{\mathrm{Hom}}_{\mathrm{D}(Y)}(F, G)) &\simeq \mathrm{Hom}_{\mathrm{D}(Y)}(f_!H, \underline{\mathrm{Hom}}_{\mathrm{D}(Y)}(F, G))\\ &\simeq \mathrm{Hom}_{\mathrm{D}(Y)}(f_!H \otimes F, G)\\ &\simeq \mathrm{Hom}_{\mathrm{D}(Y)}(f_!(H \otimes f^*F), G)\\ &\simeq \mathrm{Hom}_{\mathrm{D}(X)}(H \otimes f^*F, f^!G)\\ &\simeq \mathrm{Hom}_{\mathrm{D}(X)}(H, \underline{\mathrm{Hom}}_{\mathrm{D}(X)}(f^*F, f^!G)).\end{aligned}$$

Here, the third equivalence uses the projection formula. The desired assertion follows from Yoneda's lemma. ☐

**Corollary 5.3.** For every morphism $f : X \to Y$ in $E$, there exists a natural morphism

$$f^!(-) \otimes f^*(-) \to f^!(- \otimes -)$$

from $\mathrm{D}(Y) \times \mathrm{D}(Y)$ to $\mathrm{D}(X)$.

*Proof.* For every pair of objects $F$ and $G$ of $\mathrm{D}(Y)$, we have

$$\begin{aligned}\mathrm{Hom}_{\mathrm{D}(X)}(f^!(F) \otimes f^*(G), f^!(F \otimes G)) &\simeq \mathrm{Hom}_{\mathrm{D}(Y)}(f_!(f^!(F) \otimes f^*(G)), F \otimes G)\\ &\simeq \mathrm{Hom}_{\mathrm{D}(Y)}(f_!f^!F \otimes G, F \otimes G).\end{aligned}$$

Here, the second equivalence uses the projection formula. From the unit $f_!f^!F \to F$ of the adjunction $f_! \dashv f^!$, we obtain the canonical map

$$f_!f^!F \otimes G \to F \otimes G.$$

Then we get a natural morphism

$$f^!(F) \otimes f^*(G) \to f^!(F \otimes G).$$

☐

**Definition 5.4.** Suppose that $\mathcal{C}$ admits finite products. Let $X$, $Y$ and $Z$ be objects of $\mathcal{C}$ such that projections $q_{XY}$, $q_{XZ}$ and $q_{YZ}$ from $X \times Y \times Z$ to $X \times Y$, $X \times Z$ and $Y \times Z$ respectively lie in $E$

$$\begin{tikzcd} & X \times Y \times Z \arrow[dl, "q_{XY}"'] \arrow[d, "q_{XZ}"] \arrow[dr, "q_{YZ}"] & \\ X \times Y & X \times Z & Y \times Z. \end{tikzcd}$$

For every pair of objects $F_{XY}$ of $\mathrm{D}(X \times Y)$ and $F_{YZ}$ of $\mathrm{D}(Y \times Z)$, we define the *composition* $F_{XY} \circ F_{YZ}$ of $F_{XY}$ and $F_{YZ}$ on $Y$ by

$$F_{XY} \circ F_{YZ} := q_{XZ!}(q_{XY}^* F_{XY} \otimes q_{YZ}^* F_{YZ}).$$

$$\begin{tikzcd} & \mathrm{D}(X \times Y \times Z) \arrow[d, "q_{XZ!}"] & \\ \mathrm{D}(X \times Y) \arrow[ur, "q_{XY}^*"] & \mathrm{D}(X \times Z) & \mathrm{D}(Y \times Z). \arrow[ul, "q_{YZ}^*"'] \end{tikzcd}$$

Additionally suppose that $\mathcal{C}$ admits a terminal object $*$. Consider the case $Z \simeq *$. For every pair of objects $K$ of $\mathrm{D}(X \times Y)$ and $F$ of $\mathrm{D}(Y)$, we will refer to the composition

$$K \circ F \simeq q_{X!}(K \otimes q_Y^* F)$$

as the *integral transformation* of $F$ by $K$.

Note that the composition (especially the integral transformation) can be defined in the setting of 3-functor formalisms D. However, there exists the right adjoint functor to it if D is a 6-functor formalism.

**Corollary 5.5.** Suppose that $\mathcal{C}$ admits finite products and a terminal object. Let $K$ be an object of $\mathrm{D}(X \times Y)$. Then the integral transformation $K \circ - : \mathrm{D}(Y) \to \mathrm{D}(X)$ admits a right adjoint functor.

*Proof.* Let $F$ be an object of $\mathrm{D}(Y)$ and let $G$ be an object of $\mathrm{D}(X)$, we have

$$\begin{aligned} \mathrm{Hom}_{\mathrm{D}(X)}(q_{X!}(K \otimes q_Y^* F), G) &\simeq \mathrm{Hom}_{\mathrm{D}(X\times Y)}(K \otimes q_Y^* F, q_X^! G) \\ &\simeq \mathrm{Hom}_{\mathrm{D}(X\times Y)}(q_Y^* F, \underline{\mathrm{Hom}}_{\mathrm{D}(X\times Y)}(K, q_X^! G)) \\ &\simeq \mathrm{Hom}_{\mathrm{D}(Y)}(F, q_{Y*}\underline{\mathrm{Hom}}_{\mathrm{D}(X\times Y)}(K, q_X^! G)). \end{aligned}$$

Then the functor

$$q_{Y*}\underline{\mathrm{Hom}}_{\mathrm{D}(X\times Y)}(K, q_X^!(-)) : \mathrm{D}(X) \to \mathrm{D}(Y)$$

is right adjoint to the integral transformation. □

5.2. **Sheaf Cohomology and Künneth Formula.** We fix a geometric setup $(\mathcal{C}, E)$ and a 3-functor formalism $\mathrm{D} : \mathrm{Corr}(\mathcal{C}, E)^{\otimes} \to \mathrm{Cat}^{\times}$. Suppose that $\mathcal{C}$ admits a terminal object $*$.

**Definition 5.6.** Let $X$ be an object of $\mathcal{C}$, and let $p_X : X \to *$ be the projection. For an object $F$ of $\mathrm{D}(X)$, we define the *cohomology* $\Gamma(X; F)$ *of* $X$ *with coefficient in* $F$ by

$$\Gamma(X; F) := p_{X*}F.$$

Suppose that the projection $p_X$ lies in $E$. Similarly, we define the *cohomology with compact support* $\Gamma_c(X; *)$ *of* $X$ *with coefficient in* $F$ by

$$\Gamma_c(X; F) := p_{X!}F.$$

We prove the important result on cohomology: Künneth Formula.

**Proposition 5.7** (Künneth Formula)**.** Let $X$ and $Y$ be objects of $\mathcal{C}$ such that projections $p_X$ and $p_Y$ lie in $E$. For every pair of objects $F$ of $\mathrm{D}(X)$ and $G$ of $\mathrm{D}(Y)$, there exists an equivalence in $D(*)$

$$\Gamma_c(X \times Y; q_X^* F \otimes q_Y^* G) \simeq \Gamma_c(X; F) \otimes \Gamma_c(Y; G).$$

*Proof.* Consider the following diagram.

$$\begin{tikzcd}
X \times Y \arrow[r, "q_Y"] \arrow[d, "q_X"'] \arrow[dr, "p_{X\times Y}"] & Y \arrow[d, "p_Y"] \\
X \arrow[r, "p_X"'] & *.
\end{tikzcd}$$

Then $p_X, p_Y, q_X, q_Y$ and $p_{X\times Y}$ lie in $E$, since $E$ is closed under pullbacks and composition. We have

$$\begin{aligned}
\Gamma_c(X \times Y; q_X^* F \otimes q_Y^* G) &\simeq p_{X\times Y!}(q_X^* F \otimes q_Y^* G) \\
&\simeq p_{X!} q_{X!}(q_X^* F \otimes q_Y^* G) \\
&\simeq p_{Y!}(F \otimes q_{X!} q_Y^* G) \\
&\simeq p_{Y!}(F \otimes p_X^* p_{Y!} G) \\
&\simeq p_{X!} F \otimes p_{Y!} G \\
&\simeq \Gamma_c(X; F) \otimes \Gamma_c(Y; G).
\end{aligned}$$

Here, the third and fifth equivalences use the projection formula, and the third uses proper base change. □

We can prove it as the proof of proposition 4.5.

*Proof.* Consider the following diagram.

$$\begin{tikzcd}
X \times Y \arrow[r, "p_{X\times Y}"] \arrow[d, "\Delta_{X\times Y}"'] & * \\
(X \times Y)^2 \arrow[r, "(q_X, q_Y)"'] & X \times Y. \arrow[u, "p_{X\times Y}"']
\end{tikzcd}$$

Then $p_X, p_Y, q_X, q_Y$ and $p_{X\times Y}$ lie in $E$.

The morphism $X \times Y \xrightarrow{p_{X\times Y}} *$ in $E$ determines a correspondence

$$\begin{tikzcd}
X \times Y \arrow[r, equal] & X \times Y \arrow[r, equal] \arrow[d, "{(p_X, p_Y)}"] \arrow[dr, phantom, "\lrcorner"] & X \times Y \arrow[d, "p_{X\times Y}"] \\
& * \times * & * \arrow[l, "\Delta_*"'] \arrow[d, equal] \\
& & *.
\end{tikzcd}$$

The morphism $X \xrightarrow{\Delta_{X\times Y}} (X\times Y)^2 \xrightarrow{(q_X,q_Y)} X\times Y \xrightarrow{(p_X,p_Y)} *\times * \simeq *$ determines a correspondence

$$\begin{tikzcd}
X \times Y & (X \times Y)^2 \arrow[l, "{(q_X, q_Y)}"'] \arrow[d, equal] & X \times Y \arrow[l, "\Delta_{X\times Y}"'] \arrow[r, equal] \arrow[d, equal] \arrow[dr, phantom, "\llcorner", very near start] & X \times Y \arrow[d, equal] \\
& (X \times Y)^2 & X \times Y \arrow[l, "\Delta_{X\times Y}"'] \arrow[r, equal] \arrow[d, equal] \arrow[dr, phantom, "\llcorner", very near start] & X \times Y \arrow[d, equal] \\
& & X \times Y \arrow[r, equal] & X \times Y \arrow[d, "p_{X\times Y}"] \\
& & & *.
\end{tikzcd}$$

The first correspondence defines a functor

$$\begin{aligned}
&\mathrm{D}(X) \times \mathrm{D}(Y) \to * \times * \to * \\
&(F, G) \mapsto (p_{X!}F, p_{Y!}G) \mapsto p_{X!}F \otimes p_{Y!}G \simeq \Gamma_c(X; F) \otimes \Gamma_c(Y; G).
\end{aligned}$$

The second correspondence defines a functor

$$\begin{aligned}
&\mathrm{D}(X) \times \mathrm{D}(Y) \to (\mathrm{D}(X) \times \mathrm{D}(Y))^2 \to \mathrm{D}(X) \times \mathrm{D}(Y) \to D(*) \\
&(F, G) \mapsto (q_X^* F, q_Y^* G) \mapsto q_X^* F \otimes q_Y^* G \mapsto p_{X\times Y!} q_X^* F \otimes q_Y^* G \simeq \Gamma_c(X \times Y; q_X^* F \otimes q_Y^* G).
\end{aligned}$$

This implies that a morphism $\Gamma_c(X \times Y; q_X^* F \otimes q_Y^* G) \to \Gamma_c(X; F) \otimes \Gamma_c(Y; G)$ is a natural equivalence in $\mathrm{D}(Y)$, since D is a functor. $\square$

## Appendix A. Derived Categories and Derived Functors

A.1. **Derived Categories and Derived Functors.** We recall the classical notions of derived categories and (right) derived functors.

**Notation A.1.** Let $\mathcal{A}$ be an abelian category. We let

- $C(\mathcal{A})$ denote the category of complexes of $\mathcal{A}$.
- $K(\mathcal{A})$ denote the homotopy category of complexes of $\mathcal{A}$.
- $D(\mathcal{A})$ denote the derived category of complexes of $\mathcal{A}$.

Let $F : \mathcal{A} \to \mathcal{B}$ be a functor between abelian categories. We also denote by $F : K(\mathcal{A}) \to K(\mathcal{B})$ the induced functor between homotopy categories.

Let $\mathcal{A}$ and $\mathcal{B}$ be abelian categories. If a functor $F : K(\mathcal{A}) \to K(\mathcal{B})$ between homotopy categories is exact, then it preserves quasi-isomorphisms. In this case, it induces a functor $F : D(\mathcal{A}) \to D(\mathcal{B})$ between derived categories. However, in general, it does not necessarily preserves quasi-isomorphisms. That is, it does not send objects which are isomorphic in $D(\mathcal{A})$ to objects which are isomorphic in $D(\mathcal{B})$. To avoid this issue, we consider derived functors.

**Definition A.2.** Let $F : \mathcal{A} \to \mathcal{B}$ be a left exact functor between abelian categories. A *right derived functor* $(\mathbb{R}F, \tau)$ is the left Kan extension of $Q_{\mathcal{B}}F$ along $Q_{\mathcal{A}}$.

$$\begin{tikzcd}
D(\mathcal{A}) \arrow[rrd, "\mathbb{R}F"] & & \\
K(\mathcal{A}) \arrow[u, "Q_{\mathcal{A}}"] \arrow[r, "F"'] \arrow[ur, phantom, "\tau \nearrow"] & K(\mathcal{B}) \arrow[r, "Q_{\mathcal{B}}"'] & D(\mathcal{B})
\end{tikzcd}$$

We consider the conditions under which derived functors exist.

**Definition A.3.** Let $F : \mathcal{A} \to \mathcal{B}$ be a left exact functor between abelian categories, and let $\mathcal{I}$ be an additive full subcategory of $\mathcal{A}$. We will say that $\mathcal{I}$ is *$F$-injective* if it satisfies the following conditions:

(1) For every object $A$ of $\mathcal{A}$, there exist an object $I$ of $\mathcal{I}$ and a monomorphism $A \to I$.

(2) Let $0 \to A \to B \to C \to 0$ is a short exact sequence in $\mathcal{A}$. If $A$ and $B$ are in $\mathcal{I}$, then $C$ is also in $\mathcal{I}$.

(3) Let $0 \to A \to B \to C \to 0$ is a short exact sequence in $\mathcal{A}$. If $A$ is in $\mathcal{I}$, then $0 \to F(A) \to F(B) \to F(C) \to 0$ is a short exact sequence in $\mathcal{B}$.

**Proposition A.4.** Let $F : \mathcal{A} \to \mathcal{B}$ be a left exact functor between abelian categories. Suppose that there exists an $F$-injective additive full subcategory $\mathcal{I}$ of $\mathcal{A}$. Then there exists the right derived functor $\mathbb{R}F : D(\mathcal{A}) \to D(\mathcal{B})$ of $F$. Moreover, $\mathbb{R}F$ preserves quasi-isomorphisms.

**Remark A.5.** Let $F : \mathcal{A} \to \mathcal{B}$ be a left exact functor between abelian categories. If $\mathcal{A}$ has enough injectives, then the additive full subcategory spanned by injective objects of $\mathcal{A}$ is $F$-injective. Moreover, there exists the right derived functor $\mathbb{R}F : D(\mathcal{A}) \to D(\mathcal{B})$ of $F$.

We consider when the derived functor of a composition of two functors is itself a derived functor.

**Proposition A.6.** Let $F : \mathcal{A} \to \mathcal{B}$ and $G : \mathcal{B} \to \mathcal{C}$ be left exact functors between abelian categories. Suppose that there exist an $F$-injective additive full subcategory $\mathcal{I}_{\mathcal{A}}$ of $\mathcal{A}$ and a $G$-injective additive full subcategory $\mathcal{I}_{\mathcal{B}}$ of $\mathcal{B}$ which satisfy $F(\mathcal{I}_{\mathcal{A}}) \subseteq \mathcal{I}_{\mathcal{B}}$. Then $\mathcal{I}_{\mathcal{A}}$ is $GF$-injective and there exists a natural isomorphism $\mathbb{R}(G \circ F) \simeq \mathbb{R}G \circ \mathbb{R}F$.

We consider the derived category of sheaves and the right derived functor of the global section functor.

**Notation A.7.** Let $X$ be a topological space. We let $D(X)$ denote the derived category of sheaves on $X$.

**Example A.8.** Let $X$ be a topological space, and let $\Gamma(X;-) : \mathrm{Shv}(X) \to \mathrm{Ab}$ be the global section functor. Then an additive full subcategory of $\mathrm{Shv}(X)$ spanned by flabby sheaves is $\Gamma(X;-)$-injective. Thus there exists the right derived functor

$$\mathbb{R}\Gamma(X;-) : D(X) \to D(\mathrm{Ab}).$$

of the global section functor $\Gamma(X;-) : \mathrm{Shv}(X) \to \mathrm{Ab}$.

A.2. **Flabby, C-soft, and Flat Sheaves.** In this section, we recall the notions of flabby, c-soft, and flat sheaves.

**Definition A.9.** Let $X$ be a topological space. A sheaf $F$ on $X$ is called *flabby* (or *flasque*) if, for every open subset $U$ of $X$, a morphism $F(X) \to F(U)$ is surjective.

**Proposition A.10.** Let $X$ be a topological space. Then for every sheaf $F$ on $X$, there exist a flabby sheaf $[F]$ and a mono morphism $F \to [F]$.

**Proposition A.11.** Let $0 \to F \to G \to H \to 0$ be an exact sequence of sheaves. If $F$ is flabby, then the global section functor $\Gamma(X;F) : \mathrm{Shv}(X) \to \mathrm{Ab}$ is exact. Moreover, if $F$ and $G$ are flabby, then so is $H$.

**Definition A.12.** Let $X$ be a topological space. A sheaf $F$ on $X$ is called *c-soft* if, for every compact subset $K$ of $X$, a morphism $\Gamma(X;F) \to \Gamma(K, F_K)$ is surjective.

**Proposition A.13.** Let $f : X \to Y$ be a continuous map of locally compact Hausdorff spaces. Then the full subcategory of $\mathrm{Shv}(X)$ spanned by c-soft sheaves is $f_!$-injective.

**Definition A.14.** Let $X$ be a topological space. A sheaf $F$ on $X$ is called *flat* if the tensor product functor $F \otimes - : \mathrm{Shv}(X) \to \mathrm{Shv}(X)$ is exact.

**Proposition A.15.** Let $X$ be a topological space. Then for every sheaf $F$ on $X$, there exist a flat sheaf $P$ and an epimorphism $P \to F$.

## References

[GR17] Dennis Gaitsgory and Nick Rozenblyum. *A study in derived algebraic geometry Volume I: Correspondences and duality*. 2017. URL: https://people.math.rochester.edu/faculty/doug/otherpapers/gaitsgory2.pdf.

[HA] Jacob Lurie. *Higher Algebra*. 2017. URL: https://people.math.harvard.edu/~lurie/papers/HA.pdf.

[HM24] Claudius Heyer and Lucas Mann. *6-Functor Formalisms and Smooth Representations*. 2024. arXiv: 2410.13038 [math.CT]. URL: https://arxiv.org/abs/2410.13038.

[LZ17] Yifeng Liu and Weizhe Zheng. *Enhanced six operations and base change theorem for higher Artin stacks*. 2017. arXiv: 1211.5948 [math.AG]. URL: https://arxiv.org/abs/1211.5948.

[Man22] Lucas Mann. *A p-Adic 6-Functor Formalism in Rigid-Analytic Geometry*. 2022. arXiv: 2206.02022 [math.AG]. URL: https://arxiv.org/abs/2206.02022.

[Sch22] Peter Scholze. *Six-Functor Formalisms*. 2022. URL: https://people.mpim-bonn.mpg.de/scholze/SixFunctors.pdf.