A NOTE ON PRESENTABLE ∞-CATEGORIES

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Abstract. We summarize key concepts and results on presentable ∞-categories, focusing on their foundational aspects.

Contents

1. Introduction

We summarize key concepts and results on presentable ∞-categories, focusing on their foundational aspects. We primarily refer to [HTT, Chapter 5], but we also make use of [KNP24; kerodon; Lan21].

Many categories which arise naturally is *large*: They have a class of objects. However, large categories $\mathcal{C}$ can be determined by "small" categories $\mathcal{C}_0$ in some sense: That is, $\mathcal{C}$ is the equivalent to the category of Ind-objects of $\mathcal{C}_0$.

The aim of this note is to study these "good" large categories, called presentable categories in the setting of ∞-categories.

1.1. **Notations.** From here all categories are assumed to be ∞-categories. We let

- An denote the category of small anima.
- CAT denote the category of (not necessarily small) categories.
- Pr^{L} denote the category of presentable categories with left adjoint functors.

2. Yoneda's Lemma

2.1. **Small Simplicial Sets.** We recall the size conditions of simplicial sets. Let κ be a regular cardinal.

Definition 2.1 ([kerodon] Definition 03S2)**.** Let K be a simplicial set. We will say that K is *κ-small* if the collection of non-degenerate simplices of K is κ-small as a set. We will say that K is *small* if it is κ-small for some κ.

Definition 2.2 ([HTT] Definition 5.4.1.3)**.** Let $\mathcal{C}$ be a category. We will say that $\mathcal{C}$ is *essentially κ-small* if there exist a κ-small category $\mathcal{C}'$ and an equivalence of categories $\mathcal{C}' \to \mathcal{C}$. We will say that $\mathcal{C}$ is *essentially small* if it is essentially κ-small for some κ.

Definition 2.3 ([HTT] Section 5.4.1)**.** Let $\mathcal{C}$ be a category. We will say that $\mathcal{C}$ is *locally κ-small* if, for every pair of objects X and Y of $\mathcal{C}$, the mapping anima $\mathrm{Map}_{\mathcal{C}}(X, Y)$ is essentially κ-small. We will say that $\mathcal{C}$ is *locally small* if it is locally κ-small for some κ.

Definition 2.4 ([HTT] Definition 1.2.13.4)**.** Let $\mathcal{C}$ be a category, and let $f : K \to \mathcal{C}$ be a diagram of simplicial sets. We will refer to an initial object in the category $\mathcal{C}_{f/}$ as a *colimit* for f. Dually, we will refer to a final object in the category $\mathcal{C}_{/f}$ as a *limit* for f. If K is κ-small, then a colimit for f is called *κ-small*.

Definition 2.5 ([HTT] Definition 5.1.5.7)**.** Let $\mathcal{C}$ be a category, and let $\mathcal{C}'$ be a full subcategory of $\mathcal{C}$. We will say that $\mathcal{C}'$ is *stable under colimits* if, for every small diagram $f : K \to \mathcal{C}$ which admits a colimit $\overline{f} : K^{\triangleright} \to \mathcal{C}$, then the map $\overline{f}$ factors through $\mathcal{C}'$.

Let $\mathcal{C}$ be a category with small colimits, and let S be a collection of objects of $\mathcal{C}$. We will say that S *generates* $\mathcal{C}$ *under colimits* if the following condition is satisfied: For every full subcategory $\mathcal{C}'$ of $\mathcal{C}$ containing all elements of S, if $\mathcal{C}'$ is stable under colimits, then $\mathcal{C}'$ is equal to $\mathcal{C}$.

Let $f : S \to \mathcal{C}$ be a functor between categories. We will say that f *generates* $\mathcal{C}$ *under colimits* if its image $f(S)$ generates $\mathcal{C}$ under colimits.

2.2. **The Yoneda embedding.**

Definition 2.6 ([Lan21] Definition 4.2.3)**.** Let $\mathcal{C}$ be a category. The *twisted arrow category* $\mathrm{TwAr}(\mathcal{C})$ of $\mathcal{C}$ is the simplicial set defined by

$$\mathrm{TwAr}(\mathcal{C})_n := \mathrm{Hom}_{\mathrm{sSet}}([n] \star [n]^{\mathrm{op}}, \mathcal{C})$$

for every $n \geq 0$, where $\star$ denotes the join operator.

Remark 2.7. Let $\mathcal{C}$ be a category. Then there are two projections

$$s : \mathrm{TwAr}(\mathcal{C}) \to \mathcal{C} \quad \text{and} \quad t : \mathrm{TwAr}(\mathcal{C}) \to \mathcal{C}^{\mathrm{op}}$$

which are defined as follows: They send each n-simplex σ of $\mathrm{TwAr}(\mathcal{C})$ to the composition

$$[n] \hookrightarrow [n] \star [n]^{\mathrm{op}} \xrightarrow{\sigma} \mathcal{C} \quad \text{and} \quad [n]^{\mathrm{op}} \hookrightarrow [n] \star [n]^{\mathrm{op}} \xrightarrow{\sigma} \mathcal{C}.$$

respectively. Then these projections induce a right fibration

$$(s, t) : \mathrm{TwAr}(\mathcal{C}) \to \mathcal{C} \times \mathcal{C}^{\mathrm{op}}.$$

As a consequence, for a category $\mathcal{C}$, $\mathrm{TwAr}(\mathcal{C})$ is also a category.

Definition 2.8 ([Lan21] Definition 4.2.5)**.** Let $\mathcal{C}$ be a category. We let

$$\mathrm{Map}_{\mathcal{C}}(-, -) : \mathcal{C}^{\mathrm{op}} \times \mathcal{C} \to \mathrm{An}$$

denote the functor obtained by the straightening of the right fibration $(s, t) : \mathrm{TwAr}(\mathcal{C}) \to \mathcal{C} \times \mathcal{C}^{\mathrm{op}}$.

Definition 2.9 ([Lan21] Definition 4.2.9)**.** Let $\mathcal{C}$ be a category. We let

$$よ : \mathcal{C} \to \mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathrm{An})$$

denote the right adjoint functor to $\mathrm{Map}_{\mathcal{C}}(-,-) : \mathcal{C}^{\mathrm{op}} \times \mathcal{C} \to \mathrm{An}$. We will refer to it as the (*contravariant*) *Yoneda embedding*. Similarly, we can define the *covariant Yoneda embedding*

$$\text{よ} : \mathcal{C}^{\mathrm{op}} \to \mathrm{Fun}(\mathcal{C}, \mathrm{An})$$

as the right adjoint functor to $\mathrm{Map}_{\mathcal{C}}(-,-) : \mathcal{C}^{\mathrm{op}} \times \mathcal{C} \to \mathrm{An}$.

There are other constructions of the Yoneda embedding. These are at least objectwise equivalent to each other.

Remark 2.10. Recall that there exists the adjunction between the 1-category sSet of simplicial sets and the 1-category Cat_Δ of sSet-enriched 1-categories:

$$\mathfrak{C} : \mathrm{sSet} \rightleftarrows \mathrm{Cat}_\Delta : \mathrm{N}_\Delta$$

where $\mathfrak{C}$ is the *rigidification functor* and N_Δ is the *simplicial nerve* or *(homotopy) coherent nerve*.

Construction 2.11 ([HTT] Section 5.1.3)**.** Let K be a simplicial set. We have a simplicial functor

$$\mathfrak{C}[K]^{\mathrm{op}} \times \mathfrak{C}[K] \to \mathrm{Kan} : (X, Y) \mapsto \mathrm{Sing}|\mathrm{Map}_{\mathfrak{C}[K]}(X, Y)|$$

where Kan is the 1-category of anima. The functor $\mathfrak{C}$, in general, does not commute with products, but there is a natural map

$$\mathfrak{C}[K^{\mathrm{op}} \times K] \to \mathfrak{C}[K]^{\mathrm{op}} \times \mathfrak{C}[K].$$

Thus we can obtain a simplicial functor

$$\mathfrak{C}[K^{\mathrm{op}} \times K] \to \mathfrak{C}[K]^{\mathrm{op}} \times \mathfrak{C}[K] \to \mathrm{Kan}.$$

Using the adjunction $\mathfrak{C} \dashv \mathrm{N}_\Delta$ and the fact that $\mathrm{An} \simeq \mathrm{N}_\Delta(\mathrm{Kan})$, we get a map of simplicial sets

$$K^{\mathrm{op}} \times K \to \mathrm{An}.$$

By further using the adjunction $(K^{\mathrm{op}} \times -) \dashv \mathrm{Fun}(K^{\mathrm{op}}, -)$, we have a map

$$\text{よ} : K \to \mathrm{Fun}(K^{\mathrm{op}}, \mathrm{An}).$$

We will refer to the functor よ constructed above (or more generally, to every functor equivalent to j) as the *(contravariant) Yoneda embedding*. Similarly, we can define the *(covariant) Yoneda functor* $\text{よ} : K^{\mathrm{op}} \to \mathrm{Fun}(K, \mathrm{An})$.

Corollary 2.12 ([Lan21] Proposition 4.2.11)**.** Let $\mathcal{C}$ be a category. Then the Yoneda embedding よ is fully faithful.

Proposition 2.13 ([HTT] Proposition 5.1.3.2)**.** Let $\mathcal{C}$ be a small category. Then the Yoneda embedding よ preserves small limits which exist in $\mathcal{C}$.

For a category $\mathcal{C}$, $\mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathrm{An})$ is freely generated by the Yoneda embedding よ under small colimits.

Theorem 2.14 ([HTT] Theorem 5.1.5.6)**.** *Let $\mathcal{C}$ be a small category, and let $\mathcal{D}$ with small colimits. Then the functor よ induces an equivalence of categories*

$$\mathrm{Fun}^{\mathrm{colim}}(\mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathrm{An}), \mathcal{D}) \to \mathrm{Fun}(\mathcal{C}, \mathcal{D}).$$

The inverse is given by a left Kan extension along よ.

$$\begin{tikzcd} \mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathrm{An}) \arrow[dr, "\mathrm{Lan}_{\text{よ}} f"] & \\ \mathcal{C} \arrow[u, "\text{よ}"] \arrow[r, "f"'] & \mathcal{D} \end{tikzcd}$$

3. The ∞-Category of Ind-objects

3.1. Filtered ∞-Categories.

Definition 3.1 ([HTT] Definition.5.3.1.7)**.** Let $\mathcal{I}$ be a category. We will say that $\mathcal{I}$ is *κ-filtered* if, for every κ-small simplicial set K and every diagram $f : K \to \mathcal{I}$, there exists a map $\overline{f} : K^{\triangleright} \to \mathcal{I}$ extending f.

$$\begin{tikzcd} K \arrow[r, "f"] \arrow[d, hook, "i"'] & \mathcal{I} \\ K^{\triangleright} \arrow[ur, dashed, "\overline{f}"'] & \end{tikzcd}$$

We will say that $\mathcal{C}$ is *filtered* if it is ω-filtered. If a category $\mathcal{I}$ is κ-filtered, then a diagram $\mathcal{I} \to \mathcal{C}$ is called *κ-filtered.* Similarly, in this case, a colimit for $\mathcal{I} \to \mathcal{C}$ is called *κ-filtered.*

Remark 3.2 ([HTT] Remark.5.3.1.9)**.** Let $\mathcal{C}$ be a category. The following conditions are equivalent:

(1) The category $\mathcal{C}$ is κ-filtered.
(2) For every diagram $f : K \to \mathcal{C}$ where K is a κ-small simplicial set, the category $\mathcal{C}_{f/}$ is not empty.

Let $q : \mathcal{C} \to \mathcal{C}'$ be a categorical equivalence. It is obvious that $\mathcal{C}_{p/}$ is not empty if and only if $\mathcal{C}_{qp/}$ is not empty. Consequently, $\mathcal{C}$ is κ-filtered if and only if $\mathcal{C}'$ is κ-filtered.

We provide a characterization of κ-filtered categories using colimit diagrams.

Definition 3.3 ([HTT] Definition.5.3.3.1)**.** Let $\mathcal{C}$ be a category. We will say that $\mathcal{C}$ is *κ-closed* if every diagram $p : K \to \mathcal{C}$ where K is a κ-small simplicial set, admits a colimit $\overline{p} : K^{\triangleright} \to \mathcal{C}$.

If a category $\mathcal{C}$ is κ-closed, we can construct κ-small colimits functionally.

Construction 3.4. Let $\mathcal{C}$ be a category, and let K be a simplicial set. Suppose that every diagram $p : K \to \mathcal{C}$ admits a colimit in $\mathcal{C}$. We let $\mathcal{D}$ denote the full subcategory of $\mathrm{Fun}(K^{\triangleright}, \mathcal{C})$ spanned by the colimit diagrams. [HTT] Proposition.4.3.2.15 implies that the restriction $\mathcal{D} \to \mathrm{Fun}(K, \mathcal{C})$ is a trivial fibration. Thus it has a section $s : \mathrm{Fun}(K, \mathcal{C}) \to \mathcal{D}$. Let $\mathrm{ev}_\infty : \mathrm{Fun}(K^{\triangleright}, \mathcal{C}) \to \mathcal{C}$ be a functor defined by evaluation at the cone point of $K^{\triangleright}$. We will refer to the composition

$$\operatorname*{colim}_K : \mathrm{Fun}(K, \mathcal{C}) \xrightarrow{s} \mathcal{D} \subseteq \mathrm{Fun}(K^{\triangleright}, \mathcal{C}) \xrightarrow{\mathrm{ev}_\infty} \mathcal{C}$$

as a *colimit functor* for p.

Proposition 3.5 ([HTT] Proposition.5.3.3.3)**.** Let $\mathcal{I}$ be a category. The following conditions are equivalent:

(1) The category $\mathcal{I}$ is κ-filtered.
(2) The colimit functor $\operatorname*{colim}_{\mathcal{I}} : \mathrm{Fun}(\mathcal{I}, \mathrm{An}) \to \mathrm{An}$ preserves κ-small limits.

3.2. Compact Objects.

Definition 3.6 ([HTT] Definition.5.3.4.5)**.** Let $\mathcal{C}$ with κ-filtered small colimits, and let $f : \mathcal{C} \to \mathcal{D}$ be a functor of categories. We will say that f is *κ-continuous* if it preserves κ-filtered colimits.

Let $\mathcal{C}$ be a category with κ-filtered colimits, and let X be an object of $\mathcal{C}$. We will say that X is *κ-compact* if the functor $\text{よ}_X : \mathcal{C} \to \mathrm{An}$ is κ-continuous. We will say that X is *compact* if it is ω-compact.

Remark 3.7. In [KNP24], they define a κ-compact object X as follows: We will say that X is *κ-compact* if the canonical map

$$\operatorname*{colim}_{i \in \mathcal{I}} \mathrm{Map}_{\mathcal{C}}(X, Y_i) \to \mathrm{Map}_{\mathcal{C}}(X, \operatorname*{colim}_{i \in \mathcal{I}} Y_i)$$

is an equivalence for every κ-filtered small diagram $Y : \mathcal{I} \to \mathcal{C}$.

Notation 3.8 ([HTT] Notation.5.3.4.6)**.** Let $\mathcal{C}$ be a category with κ-filtered colimits. We let $\mathcal{C}^{\kappa}$ denote the full subcategory of $\mathcal{C}$ spanned by the κ-compact objects of $\mathcal{C}$.

Proposition 3.9 ([HTT] Corollary.5.3.4.15)**.** Let $\mathcal{C}$ be a category with small κ-filtered colimits. Then $\mathcal{C}^{\kappa}$ is stable under the κ-small colimits which exist in $\mathcal{C}$. That is, a κ-small colimit of the κ-compact objects is κ-compact.

Proof. Let $Y : \mathcal{I} \to \mathcal{C}$ be a κ-filtered small diagram, and let $X : \mathcal{J} \to \mathcal{C}$ be a κ-small diagram of κ-compact objects. We want to show that a map

$$\operatorname*{colim}_{i \in \mathcal{I}} \mathrm{Map}_{\mathcal{C}}(\operatorname*{colim}_{j \in \mathcal{J}} X_j, Y_i) \to \mathrm{Map}_{\mathcal{C}}(\operatorname*{colim}_{j \in \mathcal{J}} X_j, \operatorname*{colim}_{i \in \mathcal{I}} Y_i)$$

is an equivalence. We may write

$$\begin{aligned}
\operatorname*{colim}_{i \in \mathcal{I}} \mathrm{Map}_{\mathcal{C}}(\operatorname*{colim}_{j \in \mathcal{J}} X_j, Y_i) &\simeq \operatorname*{colim}_{i \in \mathcal{I}} \lim_{j \in \mathcal{J}} \mathrm{Map}_{\mathcal{C}}(X_j, Y_i) \\
&\simeq \lim_{j \in \mathcal{J}} \operatorname*{colim}_{i \in \mathcal{I}} \mathrm{Map}_{\mathcal{C}}(X_j, Y_i) \\
&\simeq \lim_{j \in \mathcal{J}} \mathrm{Map}_{\mathcal{C}}(X_j, \operatorname*{colim}_{i \in \mathcal{I}} Y_i) \\
&\simeq \mathrm{Map}_{\mathcal{C}}(\operatorname*{colim}_{j \in \mathcal{J}} X_j, \operatorname*{colim}_{i \in \mathcal{I}} Y_i).
\end{aligned}$$

$\square$

3.3. **The ∞-Category of Ind-objects.** We showed that the category $\mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathrm{An})$ is freely generated by the functor よ under small colimits (theorem 2.14). We next consider the analogue situation only under κ-filtered small colimits.

Definition 3.10 ([HTT] Section.5.3.5)**.** Let $\mathcal{C}$ be a small category. We define $\mathrm{Ind}_{\kappa}(\mathcal{C})$ as the smallest full subcategory of $\mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathrm{An})$ which contains the image of よ and is stable under κ-filtered colimits. When $\kappa = \omega$, we will write $\mathrm{Ind}(\mathcal{C})$ for $\mathrm{Ind}_{\kappa}(\mathcal{C})$. We will refer to $\mathrm{Ind}(\mathcal{C})$ as the category of *Ind-objects* of $\mathcal{C}$.

If a category $\mathcal{C}$ admits κ-small colimits, we can easily characterize the category $\mathrm{Ind}_{\kappa}(\mathcal{C})$.

Proposition 3.11 ([HTT] Corollary.5.3.5.4)**.** Let $\mathcal{C}$ be a small category with κ-small colimits, and let $F : \mathcal{C}^{\mathrm{op}} \to \mathrm{An}$ be a functor. The following conditions are equivalent:

(1) The functor F belongs to $\mathrm{Ind}_{\kappa}(\mathcal{C})$.
(2) The functor F preserves κ-small limits.

In particular, if $\mathcal{C}$ admits κ-small colimits, $\mathrm{Ind}_{\kappa}(\mathcal{C})$ admits small limits.

Proposition 3.12 ([HTT] Proposition.5.3.5.5)**.** Let $\mathcal{C}$ be a small category, and let よ $: \mathcal{C} \to \mathrm{Ind}_{\kappa}(\mathcal{C})$ be the Yoneda embedding. Then for every object X of $\mathcal{C}$, よX is κ-compact of $\mathrm{Ind}_{\kappa}(\mathcal{C})$.

Proof. Let $Y : \mathcal{I} \to \mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathrm{An})$ be a κ-filtered small diagram. We may write

$$\begin{aligned}
\operatorname*{colim}_{i\in\mathcal{I}} \mathrm{Map}_{\mathrm{Ind}_\kappa(\mathcal{C})}(よX, Y_i) &\simeq \operatorname*{colim}_{i\in\mathcal{I}} \mathrm{Map}_{\mathrm{Fun}(\mathcal{C}^{\mathrm{op}},\mathrm{An})}(よX, Y_i) \\
&\simeq \operatorname*{colim}_{i\in\mathcal{I}}(Y_i(X)) \\
&\simeq (\operatorname*{colim}_{i\in\mathcal{I}} Y_i)(X) \\
&\simeq \mathrm{Map}_{\mathrm{Fun}(\mathcal{C}^{\mathrm{op}},\mathrm{An})}(よX, \operatorname*{colim}_{i\in\mathcal{I}} Y_i) \\
&\simeq \mathrm{Map}_{\mathrm{Ind}_\kappa(\mathcal{C})}(よX, \operatorname*{colim}_{i\in\mathcal{I}} Y_i).
\end{aligned}$$

□

We show that the category $\mathrm{Ind}_\kappa(\mathcal{C})$ is freely generated by $\mathcal{C}$ under κ-filtered colimits.

Proposition 3.13 ([HTT] Proposition.5.3.5.10)**.** Let $\mathcal{C}$ be a small category, and let $\mathcal{D}$ be a category with small κ-filtered colimits. Then the functor $よ : \mathcal{C} \to \mathrm{Ind}_\kappa(\mathcal{C})$ induces an equivalence of categories

$$\mathrm{Fun}^{\mathrm{colim}_{\kappa\text{-filt}}}(\mathrm{Ind}_\kappa(\mathcal{C}), \mathcal{D}) \to \mathrm{Fun}(\mathcal{C}, \mathcal{D}).$$

The inverse is given by a left Kan extension ([HTT] Lemma.5.3.5.8).

$$\begin{tikzcd}
\mathrm{Ind}_\kappa(\mathcal{C}) \arrow[dr, "F"] & \\
\mathcal{C} \arrow[u, "よ"] \arrow[r, "f"'] & \mathcal{D}
\end{tikzcd}$$

We will refer to this inverse as the Ind_κ*-extension* $F : \mathrm{Ind}_\kappa(\mathcal{C}) \to \mathcal{D}$ of the functor $f : \mathcal{C} \to \mathcal{D}$.

The following proposition will be useful throughout this paper.

Proposition 3.14 ([HTT] Proposition.5.3.5.11)**.** Let $f : \mathcal{C} \to \mathcal{D}$ be a functor between categories. Suppose that $\mathcal{D}$ admits small κ-filtered colimits. Let $F : \mathrm{Ind}_\kappa(\mathcal{C}) \to \mathcal{D}$ be the Ind_κ-extension of f. Then

(1) If the functor f is fully faithful and its essential image consists of κ-compact objects of $\mathcal{D}$, then F is fully faithful.

(2) If additionally to (1), the image of f generate $\mathcal{D}$ under κ-filtered colimits, then F is an equivalence.

Proof. (1): Let X and Y be objects of $\mathrm{Ind}_\kappa(\mathcal{C})$. From the definition of $\mathrm{Ind}_\kappa(\mathcal{C})$, X and Y are of the form

$$X \simeq \operatorname*{colim}_{i\in\mathcal{I}} よX_i, \quad \text{and } Y \simeq \operatorname*{colim}_{j\in\mathcal{J}} よY_j$$

for some filtered diagrams $\mathcal{I} \to \mathcal{C}$ and $\mathcal{J} \to \mathcal{C}$. We want to show that a map

$$\mathrm{Map}_{\mathrm{Ind}_\kappa(\mathcal{C})}(X, Y) \to \mathrm{Map}_{\mathcal{D}}(F(X), F(Y))$$

is an equivalence. We may write

$$\begin{aligned}
\operatorname{Map}_{\operatorname{Ind}_\kappa(\mathcal{C})}(X, Y) &\simeq \operatorname{Map}_{\operatorname{Ind}_\kappa(\mathcal{C})}(\operatorname*{colim}_{i \in \mathcal{I}} \text{よ} X_i, \operatorname*{colim}_{j \in \mathcal{J}} \text{よ} Y_j) \\
&\simeq \lim_{i \in \mathcal{I}} \operatorname*{colim}_{j \in \mathcal{J}} \operatorname{Map}_{\operatorname{Ind}_\kappa(\mathcal{C})}(\text{よ} X_i, \text{よ} Y_j) \\
&\simeq \lim_{i \in \mathcal{I}} \operatorname*{colim}_{j \in \mathcal{J}} \operatorname{Map}_{\mathcal{C}}(X_i, Y_j) \\
&\simeq \lim_{i \in \mathcal{I}} \operatorname*{colim}_{j \in \mathcal{J}} \operatorname{Map}_{\mathcal{D}}(f(X_i), f(Y_j)) \\
&\simeq \operatorname{Map}_{\mathcal{D}}(\operatorname*{colim}_{i \in \mathcal{I}} f(X_i), \operatorname*{colim}_{j \in \mathcal{J}} f(Y_j)) \\
&\simeq \operatorname{Map}_{\mathcal{D}}(\operatorname*{colim}_{i \in \mathcal{I}} F(\text{よ} X_i), \operatorname*{colim}_{j \in \mathcal{J}} F(\text{よ} Y_j)) \\
&\simeq \operatorname{Map}_{\mathcal{D}}(F(\operatorname*{colim}_{i \in \mathcal{I}} \text{よ} X_i), F(\operatorname*{colim}_{j \in \mathcal{J}} \text{よ} Y_j)) \\
&\simeq \operatorname{Map}_{\mathcal{D}}(F(X), F(Y)).
\end{aligned}$$

(2): The essential image of F contains the image of f and is stable under small κ-filtered colimits. Thus F is essentially surjective. □

Proposition 3.15 ([HTT] Proposition.5.3.5.14)**.** Let $\mathcal{C}$ be a small category with κ-small colimits. Then the functor $\text{よ} : \mathcal{C} \to \operatorname{Ind}_\kappa(\mathcal{C})$ preserves κ-small colimits which exist in $\mathcal{C}$.

Proof. Let $X : \mathcal{I} \to \mathcal{C}$ be a κ-small diagram. We want to show that a map

$$\operatorname*{colim}_{i \in \mathcal{I}} \text{よ} X_i \to \text{よ} \operatorname*{colim}_{i \in \mathcal{I}} X_i$$

is an equivalence. By Yoneda's lemma, it is enough to show that a map

$$\operatorname{Map}_{\operatorname{Ind}_\kappa(\mathcal{C})}(\text{よ} \operatorname*{colim}_{i \in \mathcal{I}} X_i, F) \to \operatorname{Map}_{\operatorname{Ind}_\kappa(\mathcal{C})}(\operatorname*{colim}_{i \in \mathcal{I}} \text{よ} X_i, F)$$

is an equivalence for every functor $F : \mathcal{C}^{\mathrm{op}} \to \mathrm{An}$. We have equivalences

$$\operatorname{Map}_{\operatorname{Ind}_\kappa(\mathcal{C})}(\text{よ} \operatorname*{colim}_{i \in \mathcal{I}} X_i, F) \simeq F(\operatorname*{colim}_{i \in \mathcal{I}} X_i)$$
$$\operatorname{Map}_{\operatorname{Ind}_\kappa(\mathcal{C})}(\operatorname*{colim}_{i \in \mathcal{I}} \text{よ} X_i, F) \simeq \lim_{i \in \mathcal{I}} \operatorname{Map}_{\operatorname{Ind}_\kappa(\mathcal{C})}(\text{よ} X_i, F) \simeq \lim_{i \in \mathcal{I}} F(X_i).$$

Since F preserves κ-small limit from proposition 3.11, these are equivalent. □

Corollary 3.16 ([HTT] Example.5.3.6.8)**.** Let $\mathcal{C}$ be a small category with κ-small colimits. Then $\operatorname{Ind}_\kappa(\mathcal{C})$ admits small colimits. Moreover, for every category $\mathcal{D}$ with small colimits, the restriction along よ induces an equivalence of categories

$$\operatorname{Fun}^{\mathrm{colim}}(\operatorname{Ind}_\kappa(\mathcal{C}), \mathcal{D}) \to \operatorname{Fun}^{\mathrm{colim}_{\kappa\text{-filt}}}(\mathcal{C}, \mathcal{D}).$$

Proof. Every small colimit can be written as a κ-filtered colimit of κ-small colimits. It follows from the definition of $\operatorname{Ind}_\kappa(\mathcal{C})$ and proposition 3.15. □

4. Presentable ∞-Categories

4.1. Accessible ∞-Categories.

Definition 4.1 ([HTT] Definition.5.4.2.1)**.** Let $\mathcal{C}$ be a category. We will say that $\mathcal{C}$ is *κ-accessible* if there exist a small category $\mathcal{C}^0$ and an equivalence of categories

$$\operatorname{Ind}_\kappa(\mathcal{C}^0) \to \mathcal{C}.$$

We will say that $\mathcal{C}$ is *accessible* if it is κ-accessible for some κ.

Definition 4.2 ([HTT] Definition.5.4.2.5)**.** Let $f : \mathcal{C} \to \mathcal{D}$ be a functor between categories. We will say that f is *accessible* if it is κ-continuous for some κ.

We can characterize accessible categories as follows:

Proposition 4.3 ([HTT] Proposition.5.4.2.2)**.** Let $\mathcal{C}$ be a category. Then $\mathcal{C}$ is accessible if and only if the following conditions are satisfied:

(1) The category $\mathcal{C}$ is locally small, and the category $\mathcal{C}^{\kappa}$ is essentially small.

(2) The category $\mathcal{C}$ admits κ-filtered small colimits.

(3) The category $\mathcal{C}^{\kappa}$ generates $\mathcal{C}$ under κ-filtered small colimits.

4.2. **Presentable ∞-Categories.**

Definition 4.4 ([HTT] Definition.5.5.0.1)**.** Let $\mathcal{C}$ be a category. We will say that $\mathcal{C}$ is *presentable* if $\mathcal{C}$ is accessible and admits small colimits.

Theorem 4.5 ([HTT] Theorem.5.5.1.1)**.** *Let $\mathcal{C}$ be a category. The following conditions are equivalent:*

(1) The category $\mathcal{C}$ is presentable.

(2) The category $\mathcal{C}$ is accessible, and the full subcategory $\mathcal{C}^{\kappa}$ admits κ-small colimits for every regular cardinal κ.

(3) There exists a regular cardinal κ such that $\mathcal{C}$ is κ-accessible, and $\mathcal{C}^{\kappa}$ admits κ-small colimits.

(4) There exist a regular cardinal κ, a small category $\mathcal{D}$ which admits κ-small colimits, and an equivalence of categories $\mathrm{Ind}_{\kappa}(\mathcal{D}) \to \mathcal{C}$.

(5) There exists a small category $\mathcal{D}$ such that $\mathcal{C}$ is an accessible localization of $\mathrm{Fun}(\mathcal{D}^{\mathrm{op}}, \mathrm{An})$.

Remark 4.6. Let $\mathcal{C}$ be a presentable category. It follows from proposition 3.11 and theorem 4.5 that $\mathcal{C}$ admits small limits.

The following theorem is the *adjoint functor theorem* in the setting of ∞-categories.

Theorem 4.7 ([HTT] Corollary.5.5.2.9)**.** *Let $F : \mathcal{C} \to \mathcal{D}$ be a functor between presentable categories. Then*

(1) The functor F has a right adjoint if and only if F preserves small colimits.

(2) The functor F has a left adjoint if and only if F is accessible and preserves small limits.

Theorem 4.7 suggests that an appropriate concept of morphisms between presentable categories are described by pairs of adjoint functors.

Definition 4.8 ([HTT] Definition.5.5.3.1)**.** Let $\mathrm{Pr}^{\mathrm{L}} \subseteq \mathrm{CAT}$ denote the (very big) category whose objects are presentable categories and whose morphisms are left adjoint (or colimit-preserving) functors.

The next results imply that the category Pr^{L} is stable under various categorical constructions.

Example 4.9. The category An is presentable.

If $\mathcal{C}$ is a small category, then the category $\mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathrm{An})$ is presentable ([HTT] Proposition.5.5.3.6).

If $\mathcal{C}$ is a small category, then the categories $\mathcal{C}_{/f}$ and $\mathcal{C}_{f/}$ are presentable for every diagram $f : K \to \mathcal{C}$, where K is a small simplicial set. ([HTT] Proposition.5.5.3.10, 5.5.3.11).

Proposition 4.10 ([HTT] Proposition.5.5.3.6)**.** Let $\mathcal{C}$ be a small category, and let $\mathcal{D}$ be a presentable category. Then the category $\mathrm{Fun}(\mathcal{C}, \mathcal{D})$ is presentable.

Proposition 4.11 ([HTT] Proposition.5.5.3.8)**.** Let $\mathcal{C}$ and $\mathcal{D}$ be presentable categories. Then the category $\mathrm{Fun}^{\mathrm{colim}}(\mathcal{C}, \mathcal{D})$ is presentable.

Proposition 4.11 implies that the category $\mathrm{Fun}^{\mathrm{colim}}(\mathcal{C}, \mathcal{D})$ can be regarded as an internal mapping object in Pr^{L}. We can show that there exists a *tensor product* $\otimes$ left adjoint to this functor. The operation $\otimes$ endows a symmetric monoidal structure on Pr^{L}. Proposition 4.11 shows that this symmetric monoidal structure is closed.

Proposition 4.12 ([HTT] Proposition.5.5.3.13)**.** The category Pr^{L} admits small colimits, and the inclusion $\mathrm{Pr}^{\mathrm{L}} \subseteq \mathrm{CAT}$ preserves small limits.

4.3. Compactly Generated ∞-Categories.

Definition 4.13 ([HTT] Definition.5.5.7.1)**.** Let $\mathcal{C}$ be a category. We will say that $\mathcal{C}$ is *κ-compactly generated* if $\mathcal{C}$ is presentable and κ-accessible. We will say that $\mathcal{C}$ is *compactly generated* if it is ω-compactly generated.

Proposition 4.14 ([HTT] Section.5.5.7)**.** Let $\mathcal{C}$ be a category. The following conditions are equivalent:

(1) The category $\mathcal{C}$ is κ-compactly generated.

(2) There exist a small category $\mathcal{D}$ which admits κ-small colimits and an equivalence $\mathrm{Ind}_\kappa(\mathcal{D}) \to \mathcal{C}$. Moreover, we can choose $\mathcal{D}$ to be the full subcategory $\mathcal{C}^\kappa$ of κ-compact objects of $\mathcal{C}$.

Proposition 4.15 ([HTT] Proposition.5.5.7.2)**.** Let $\mathcal{C}$ and $\mathcal{D}$ be categories with κ-filtered colimit, and $L : \mathcal{C} \rightleftarrows \mathcal{D} : R$ be an adjunction. Then

(1) If the functor R is κ-continuous, then the functor L preserves κ-compact objects.

(2) If $\mathcal{C}$ is κ-accessible and the functor L preserves κ-compact objects, then the functor R is κ-continuous.

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