THE ALGEBRAIC K-THEORY OF STABLE ∞-CATEGORIES

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Abstract. We summarize the algebraic K-theory of small stable ∞-categories.

Contents

1. Introduction

This paper is a summary of the workshop on the algebraic K-theory held in Kyoto in September 2024.

The aim of this note is to summarize fundamental results of the algebraic K-theory of stable ∞-categories: The additivity theorem, localization theorem, universality theorem and cofinality theorem. Moreover, we introduce both the connective and non-connective K-theory spectra. Detailed proofs are omitted, but references are provided for further study.

1.1. **Introduction.** The algebraic K-theory is a powerful tool for understanding algebraic and geometric structures, particularly in algebraic geometry and topology. It originated from attempts to study invariants of rings and schemes systematically.

Historically, K-theory was first introduced in the 1940s by Whitehead in the context of studying homotopy groups and higher algebraic structures. He defined the group $K_1(R)$ for a ring R. In the 1950s, Grothendieck extended these ideas in the context of algebraic geometry, defining the group $K_1(X)$ for a scheme X. It played a pivotal role in his proof of the Grothendieck-Riemann-Roch theorem. By the 1960s, mathematicians defined higher K-groups K_n for $n \geq 1$ using ad hoc constructions, which lacked a unified framework.

In the 1970s, Quillen revolutionized K-theory by introducing a systematic framework. He defined the connective K-theory spectrum $K(R)$, which satisfies $\pi_0 K(R) = K_0(R)$ and $\pi_1 K(R) = K_1(R)$. This construction is known as Quillen's Q-construction. In the 1980s, Waldhausen extended K-theory to categories with weak equivalences and cofibrations, now known as Waldhausen categories.

Blumberg-Gepner-Tabuada developed a universal approach to the algebraic K-theory for small stable ∞-categories. Their framework not only unified previous constructions but also allowed for defining non-connective K-theory, following earlier ideas of Bass and Thomason.

Today, the algebraic K-theory is a vibrant area of research, with connections to motivic homotopy theory, derived algebraic geometry, and topological cyclic homology. It plays a crucial role in understanding invariants of rings, schemes, and categories.

1.2. **Notations.** From here all categories are assumed to be (small) ∞-categories. We let

- An denote the category of small anima.
- Cat denote the category of small categories.
- $\mathrm{Cat}^{\mathrm{lex}}$ denote the category of small categories which admit finite limits, with left exact functors.
- $\mathrm{Cat}^{\mathrm{st}}$ denote the category of small stable categories with exact functors.
- $\mathrm{Cat}^{\mathrm{perf}}$ denote the category of small idempotent complete stable categories with exact functors.
- Sp denote the category of spectra.
- $\mathrm{Mon}_{\mathbb{E}_\infty}(\mathcal{E})$ (resp. $\mathrm{Grp}_{\mathbb{E}_\infty}(\mathcal{E})$) denote the category of commutative monoid (resp. group) objects in $\mathcal{E}$ for a category $\mathcal{E}$.

2. Preliminaries

Throughout this paper, some notions can be defined under less restrictive conditions. For instance, the algebraic K-theory can be defined for small categories with finite (co)limits. However, in this paper, we primarily work within the framework of small stable categories.

2.1. **The Grothendieck Group.** In this section, we review the definition of the Grothendieck group for stable categories.

Definition 2.1. Let $(\mathcal{C}, \oplus)$ be a stable category, and let X and Y be objects of $\mathcal{C}$. We let $[X]$ denote the connected component of X. The connected component set $\pi_0(\operatorname{core}\mathcal{C})$, together with the operation $+$ defined by

$$[X] + [Y] := [X \oplus Y]$$

form an ordinary monoid $(\pi_0(\operatorname{core}\mathcal{C}), +)$. We define the *Grothendieck group* $\mathcal{K}_0(\mathcal{C})$ of $\mathcal{C}$ as

$$\mathcal{K}_0(\mathcal{C}) := (\pi_0(\operatorname{core}\mathcal{C}), +)/\sim,$$

where $\sim$ is the equivalence relation generated by the following relation: $[X] = [X'] + [X'']$ whenever $X' \to X \to X''$ is a cofiber sequence in $\mathcal{C}$.

Remark 2.2. Let $(\mathcal{C}, \oplus)$ be a stable category. Then the connected component set $\pi_0(\operatorname{core}\mathcal{C})$ is the set of equivalence classes of objects of $\mathcal{C}$. Moreover, the Grothendieck group $\mathcal{K}_0(\mathcal{C})$ is actually abelian.

(1) The zero object 0 of $\mathcal{C}$ defines a unit element $[0]$ in $\mathcal{K}_0(\mathcal{C})$, since $X \to X \to 0$ is a cofiber sequence in $\mathcal{C}$ for every object X of $\mathcal{C}$.

(2) For every object X of $\mathcal{C}$, $[\Omega X]$ and $[\Sigma X]$ are inverse element of $[X]$ in $\mathcal{K}_0(\mathcal{C})$, since $\Omega X \to 0 \to X$ and $X \to 0 \to \Sigma X$ are cofiber sequences in $\mathcal{C}$.

(3) For every objects X and Y of $\mathcal{C}$, we have $[X] + [Y] = [Y] + [X]$, since $X \to X \oplus Y \to Y$ and $Y \to X \oplus Y \to X$ are cofiber sequences in $\mathcal{C}$.

Remark 2.3. We can also define the Grothendieck group as follows: Let $\mathcal{C}$ be a stable category, and let X be an object of $\mathcal{C}$. We let $\mathcal{K}_0(\mathcal{C})$ denote the free abelian group on generators $[X]$

modulo the relations given by $[X] = [X'] + [X'']$ whenever $X' \to X \to X''$ is a cofiber sequence in $\mathcal{C}$.

Example 2.4. We let FinTop_* denote the category of finite pointed spaces. Then the Grothendieck group $\mathcal{K}_0(\mathrm{FinTop}_*)$ is isomorphic to $\mathbb{Z}$. This isomorphism is given by the reduced Euler characteristic $[X] \mapsto \chi(X) - 1$.

Example 2.5. Let R be a ring. We let $\mathrm{Perf}(R)$ denote the category of perfect complexes over R. Then the Grothendieck group $\mathcal{K}_0(\mathrm{Perf}(R))$ is isomorphic to $K_0(R)$. The map $K_0(R) \to \mathcal{K}_0(\mathrm{Perf}(R))$ is defined as follows: It sends each finitely generated projective R-module P to the complex P concentrated in degree zero. The inverse $\mathcal{K}_0(\mathrm{Perf}(R)) \to K_0(R)$ sends each perfect complex $[P_*]$ to its alternating sum $\sum_n (-1)^n [P_n]$.

Remark 2.6 (Eilenberg Swindle)**.** Let $(\mathcal{C}, \oplus)$ be a stable category with countable coproducts. Then the Grothendieck group $\mathcal{K}_0(\mathcal{C})$ is trivial. Indeed, for every object X of $\mathcal{C}$, we have $[X] = 0$, since

$$\bigoplus_{n \geq 1} X \to \bigoplus_{n \geq 0} X \to X$$

is a cofiber sequence in $\mathcal{C}$ and the last two terms are equivalent. It can be generalized to the algebraic K-theory (see corollary 5.5).

Remark 2.7. Let $F : \mathcal{C} \to \mathcal{D}$ be an exact functor between stable categories. Then for every object X of $\mathcal{C}$, the construction $[X] \mapsto [F(X)]$ defines a group homomorphism $\mathcal{K}_0(F) : \mathcal{K}_0(\mathcal{C}) \to \mathcal{K}_0(\mathcal{D})$. The constructions $\mathcal{C} \mapsto \mathcal{K}_0(\mathcal{C})$ and $F \mapsto \mathcal{K}_0(F)$ determine a functor

$$\mathcal{K}_0 : \mathrm{Cat}^{\mathrm{st}} \to \mathrm{Ab}.$$

Remark 2.8. Let $\mathcal{C}$ be a stable category. Then the suspension $\Sigma : \mathcal{C} \to \mathcal{C}$ induces the map $\mathcal{K}_0(\Sigma) : \mathcal{K}_0(\mathcal{C}) \to \mathcal{K}_0(\mathcal{C})$ given by multiplication by -1 (remark 2.2).

2.2. **Arrow Categories and Twisted Arrow Categories.** In this section, we recall the notions of (twisted) arrow categories and the category of sequences.

Definition 2.9. Let $\mathcal{C}$ be a category. We define the *arrow category* $\mathrm{Ar}(\mathcal{C})$ of $\mathcal{C}$ as

$$\mathrm{Ar}(\mathcal{C}) := \mathrm{Fun}([1], \mathcal{C}).$$

Definition 2.10 ([kerodon] Construction 03JG)**.** Let $\mathcal{C}$ be a category. The *twisted arrow category* $\mathrm{TwAr}(\mathcal{C})$ of $\mathcal{C}$ is the simplicial set defined by

$$\mathrm{TwAr}(\mathcal{C})_n := \mathrm{Hom}_{\mathrm{sSet}}([n] \star [n]^{\mathrm{op}}, \mathcal{C})$$

for every $n \geq 0$, where $\star$ denotes the join operator.

Remark 2.11. Let $\mathcal{C}$ be a category. We can describe objects and morphisms of $\mathrm{Ar}(\mathcal{C})$ and $\mathrm{TwAr}(\mathcal{C})$.

- The objects of both are morphisms in $\mathcal{C}$.
- A morphism from f to g in $\mathrm{Ar}(\mathcal{C})$ is a diagram, depicted as

$$\begin{tikzcd}
 & 1' & \\
 & & 1. \arrow[ul, "g"'] \\
0 \arrow[uur] \arrow[urr] \arrow[dr, "f"'] & & \\
 & 0' \arrow[uuu] \arrow[uur] &
\end{tikzcd}$$

- A morphism from f to g in $\mathrm{TwAr}(\mathcal{C})$ is a diagram, depicted as

$$\begin{tikzcd}
 & 1' \arrow[dd] & \\
0 \arrow[ur] \arrow[rr] \arrow[dr, "f"'] & & 1. \arrow[ul, "g"'] \\
 & 0' \arrow[ur] &
\end{tikzcd}$$

Remark 2.12. Let $\mathcal{C}$ be a category. Then there are two projections

$$s : \mathrm{TwAr}(\mathcal{C}) \to \mathcal{C} \quad \text{and} \quad t : \mathrm{TwAr}(\mathcal{C}) \to \mathcal{C}^{\mathrm{op}}$$

which are defined as follows: They send each n-simplex σ of $\mathrm{TwAr}(\mathcal{C})$ to the composition

$$[n] \hookrightarrow [n] \star [n]^{\mathrm{op}} \xrightarrow{\sigma} \mathcal{C} \quad \text{and} \quad [n]^{\mathrm{op}} \hookrightarrow [n] \star [n]^{\mathrm{op}} \xrightarrow{\sigma} \mathcal{C}.$$

respectively. Then these projections induce a right fibration

$$(s, t) : \mathrm{TwAr}(\mathcal{C}) \to \mathcal{C} \times \mathcal{C}^{\mathrm{op}}.$$

As a consequence, for a category $\mathcal{C}$, $\mathrm{TwAr}(\mathcal{C})$ is also a category.

Notation 2.13. Let $\mathcal{C}$ be a stable category. We let $\mathrm{Seq}(\mathcal{C})$ denote the full subcategory of $\mathrm{Fun}(\Delta^1 \times \Delta^1, \mathcal{C})$ spanned by the bifiber sequences in $\mathcal{C}$.

Remark 2.14. Let $\mathcal{C}$ be a stable category. Then we have an equivalence of categories $\mathrm{Seq}(\mathcal{C}) \simeq \mathrm{Ar}(\mathcal{C})$, which implies that the category $\mathrm{Seq}(\mathcal{C})$ is stable.

Notation 2.15. Let $\mathcal{C}$ be a stable category. We define functors from $\mathrm{Seq}(\mathcal{C})$ to $\mathcal{C}$ as follows:

$$\begin{aligned}
\mathrm{fib} : \mathrm{Seq}(\mathcal{C}) \to \mathcal{C} : (X \to Y \to Z) &\mapsto X, \\
\mathrm{mid} : \mathrm{Seq}(\mathcal{C}) \to \mathcal{C} : (X \to Y \to Z) &\mapsto Y, \\
\mathrm{cofib} : \mathrm{Seq}(\mathcal{C}) \to \mathcal{C} : (X \to Y \to Z) &\mapsto Z.
\end{aligned}$$

3. Localization Properties of Functors

In this section, we define various functors with localization properties: additive, Verdier-localizing, Karoubi-localizing, and grouplike functors.

We follow the terminology of [Cal+23]. In [Cal+23], these notions are defined for Poincaré-Verdier squares. We use the same terminology for Verdier squares.

3.1. **Verdier Sequences and Squares.**

In this section, we recall the notions of (split) Verdier sequences and Karoubi sequences and there relative versions: (split) Verdier squares and Karoubi squares.

Definition 3.1. Let $\mathcal{C} \xrightarrow{f} \mathcal{D} \xrightarrow{p} \mathcal{E}$ be a sequence in $\mathrm{Cat}^{\mathrm{st}}$. We will say that the sequence *has vanishing composition* if the composition pf is a zero object of $\mathrm{Cat}^{\mathrm{st}}$.

In this case, the composition pf is equivalent to the functor $\mathcal{C} \to 0 \to \mathcal{E}$, since the full subcategory of $\mathrm{Fun}^{\mathrm{ex}}(\mathcal{C}, \mathcal{E})$ spanned by the zero objects is contractible. That is, there exists the following commutative diagram:

$$\begin{tikzcd}
\mathcal{C} \arrow[r, "f"] \arrow[d] & \mathcal{D} \arrow[d, "p"] \\
0 \arrow[r] & \mathcal{E}.
\end{tikzcd}$$

We will say that the sequence is a *fiber* (resp. *cofiber*) *sequence* if the above diagram is cartesian (resp. cocartesian).

Definition 3.2 ([Cal+23] Definition A.1.1)**.** Let $\mathcal{C} \xrightarrow{f} \mathcal{D} \xrightarrow{p} \mathcal{E}$ be a sequence in $\mathrm{Cat}^{\mathrm{st}}$ with vanishing composition. We will say that it is *Verdier* if it is a bifiber sequence in $\mathrm{Cat}^{\mathrm{st}}$. In this case, we will refer to the functor f as the *Verdier inclusion* and to the functor p as the *Verdier projection.*

Definition 3.3 ([Cal+23] Definition A.2.4)**.** Let $\mathcal{C} \xrightarrow{f} \mathcal{D} \xrightarrow{p} \mathcal{E}$ be a Verdier sequence. We will say that it is *split* if the functor p admits both adjoint functors. In this case, we will refer to the functor f as the *split Verdier inclusion* and to the functor p as the *split Verdier projection.*

Definition 3.4 ([Cal+23] Definition A.3.5)**.** Let $\mathcal{C} \xrightarrow{f} \mathcal{D} \xrightarrow{p} \mathcal{E}$ be a sequence in $\mathrm{Cat}^{\mathrm{st}}$ with vanishing composition. We will say that it is *Karoubi* if its idempotent completion $\mathcal{C}^\natural \to \mathcal{D}^\natural \to \mathcal{E}^\natural$ is a bifiber sequence in $\mathrm{Cat}^{\mathrm{perf}}$. In this case, we will refer to the functor f as the *Karoubi inclusion* and to the functor p as the *Karoubi projection.*

We can characterize Verdier inclusions and projections (proposition 3.7). The fiber of an exact functor $f : \mathcal{C} \to \mathcal{D}$ can be computed by its kernel $\ker(f)$. On the other hand, its cofiber is described by the Verdier quotient.

Definition 3.5 ([Cal+23] Definition A.1.3)**.** Let $f : \mathcal{C} \to \mathcal{D}$ be an exact functor between stable categories. We will say that a morphism in $\mathcal{D}$ is an *equivalence modulo* $\mathcal{C}$ in $\mathcal{D}$ if its fiber (or equivalently, its cofiber) belongs in the essential image of f.
We define the category $\mathcal{D}/\mathcal{C}$ as the localization of $\mathcal{D}$ with respect to the set of equivalences modulo $\mathcal{C}$ in $\mathcal{D}$. We will refer to the category $\mathcal{D}/\mathcal{C}$ as the *Verdier quotient* of $\mathcal{D}$ by $\mathcal{C}$.

The next proposition implies that the Verdier quotient is universal.

Proposition 3.6 ([NS18] Theorem 1.3.3)**.** Let $f : \mathcal{C} \to \mathcal{D}$ be an exact functor between stable categories. Then

(1) The Verdier quotient $\mathcal{D}/\mathcal{C}$ is stable, and the localization functor $\mathcal{D} \to \mathcal{D}/\mathcal{C}$ is exact.
(2) For every stable category $\mathcal{E}$, the restriction functor
$$\mathrm{Fun}^{\mathrm{ex}}(\mathcal{D}/\mathcal{C}, \mathcal{E}) \to \mathrm{Fun}^{\mathrm{ex}}(\mathcal{D}, \mathcal{E})$$
is fully faithful, and its essential image consists of the functors which vanish after composing with f.
(3) The sequence $\mathcal{C} \to \mathcal{D} \to \mathcal{D}/\mathcal{C}$ is a cofiber sequence in $\mathrm{Cat}^{\mathrm{st}}$.

Proposition 3.7 ([Cal+23] Corollary A.1.10)**.** Let $\mathcal{C} \xrightarrow{f} \mathcal{D} \xrightarrow{p} \mathcal{E}$ be a sequence in $\mathrm{Cat}^{\mathrm{st}}$ with vanishing composition. The following conditions are equivalent:

(1) The sequence is Verdier.
(2) The functor f is fully faithful and its essential image is closed under retracts in $\mathcal{D}$, and the functor p exhibits $\mathcal{E}$ as the Verdier quotient of $\mathcal{D}$ by $\mathcal{C}$.
(3) The functor f exhibits $\mathcal{C}$ as the kernel of p, and the functor p is a localization.

We can characterize split Verdier inclusions and projections.

Proposition 3.8 ([Cal+23] Corollary A.2.6)**.** Let $\mathcal{C} \xrightarrow{f} \mathcal{D} \xrightarrow{p} \mathcal{E}$ be a sequence in $\mathrm{Cat}^{\mathrm{st}}$ with vanishing composition. The following conditions are equivalent:

(1) The sequence is split Verdier.
(2) The functor p admits fully faithful both adjoint functors.
(3) The functor f is fully faithful and admits both adjoint functors.

We can characterize Karoubi inclusions and projections.

Proposition 3.9 ([Cal+23] Corollary A.3.8)**.** Let $\mathcal{C} \xrightarrow{f} \mathcal{D} \xrightarrow{p} \mathcal{E}$ be a sequence in $\mathrm{Cat}^{\mathrm{st}}$ with vanishing composition. The following conditions are equivalent:

(1) The sequence is Karoubi.

(2) The functor f is fully faithful and the functor p has the dense essential image $p(\mathcal{D}) \subseteq \mathcal{E}$, and the induced functor $\mathcal{D} \to p(\mathcal{D})$ is a Verdier projection.

We can describe Karoubi sequences using Ind-categories.

Theorem 3.10 (Thomason-Neeman's localization theorem)**.** *Let $\mathcal{C} \xrightarrow{f} \mathcal{D} \xrightarrow{p} \mathcal{E}$ be a sequence in* $\mathrm{Cat}^{\mathrm{st}}$ *with vanishing composition. The following conditions are equivalent:*

(1) The sequence is Karoubi.

(2) The sequence $\mathrm{Ind}(\mathcal{C}) \to \mathrm{Ind}(\mathcal{D}) \to \mathrm{Ind}(\mathcal{E})$ *is Verdier (of non-necessarily small categories).*

We next introduce the relative versions of these sequences.

Definition 3.11 ([Cal+23] Definition.1.5.1)**.** A square in $\mathrm{Cat}^{\mathrm{st}}$ is called

- *Verdier* if it is cartesian and its both vertical maps are Verdier projections.
- *split Verdier* if it is cartesian and its both vertical maps are split Verdier projections.
- *Karoubi* if it is cartesian after idempotent completion and its both vertical maps are Karoubi projections.

Remark 3.12. In definition 3.11, the condition that the square is cartesian can be replaced by the condition that it is cocartesian. (See proof A.1.)

3.2. **Additive and Grouplike Functors.** In this section, we define additive, Verdier-localizing, Karoubi-localizing, and grouplike functors.

Definition 3.13. Let $\mathcal{E}$ be a category with a terminal object, and let $F : \mathrm{Cat}^{\mathrm{st}} \to \mathcal{E}$ be a functor. We will say that F is *reduced* if $F(0)$ is equivalent to a terminal object of $\mathcal{E}$, where 0 is a zero object in $\mathrm{Cat}^{\mathrm{st}}$.

Definition 3.14 ([HLS23] Definition 2.1)**.** Let $\mathcal{E}$ be a category with finite limits, and let $F : \mathrm{Cat}^{\mathrm{st}} \to \mathcal{E}$ be a reduced functor. The functor F is called

- *Verdier-localizing* if it takes every Verdier square in $\mathrm{Cat}^{\mathrm{st}}$ to a cartesian square in $\mathcal{E}$.
- *additive* if it takes every split Verdier square in $\mathrm{Cat}^{\mathrm{st}}$ to a cartesian square in $\mathcal{E}$.
- *Karoubi-localizing* if it takes every Karoubi square in $\mathrm{Cat}^{\mathrm{st}}$ to a cartesian square in $\mathcal{E}$.

Every additive (resp. Verdier-localizing, Karoubi-localizing) functor $F : \mathrm{Cat}^{\mathrm{st}} \to \mathcal{E}$ sends split Verdier sequences (resp. Verdier sequences, Karoubi sequences) to fiber sequences in $\mathcal{E}$. If $\mathcal{E}$ is stable, the converse holds.

Proposition 3.15 ([Cal+23] Proposition 1.5.5)**.** Let $\mathcal{E}$ be a category with finite limits, and let $F : \mathrm{Cat}^{\mathrm{st}} \to \mathcal{E}$ be a reduced functor. If $\mathcal{E}$ is stable, then the following conditions are equivalent:

(1) The functor F is additive (resp. Verdier-localizing, Karoubi-localizing).

(2) The functor F takes every split Verdier sequence (resp. Verdier sequence, Karoubi sequence) in $\mathrm{Cat}^{\mathrm{st}}$ to a fiber sequence in $\mathcal{E}$. (See proof A.2.)

Definition 3.16 ([HLS23] Definition 2.1)**.** Let $\mathcal{E}$ be a category with finite limits, and let $F : \mathrm{Cat}^{\mathrm{st}} \to \mathcal{E}$ be an additive functor. We will say that F is *grouplike* if it lifts to the category $\mathrm{Mon}_{\mathbb{E}_\infty}(\mathcal{E})$ takes values in the full subcategory $\mathrm{Grp}_{\mathbb{E}_\infty}(\mathcal{E})$ of $\mathrm{Mon}_{\mathbb{E}_\infty}(\mathcal{E})$.

Example 3.17 ([Cal+23] Example 1.5.10)**.** We give some (counter)examples.

(1) The core functor core : $\mathrm{Cat}^{\mathrm{st}} \to \mathrm{An}$ is additive, but not grouplike.
(2) The algebraic K-theory $\mathcal{K} : \mathrm{Cat}^{\mathrm{st}} \to \mathrm{An}$ (definition 4.12) is Verdier-localizing (theorem 6.1) and grouplike (corollary 5.4), but not Karoubi-localizing.
(3) The connective K-theory spectrum $\mathcal{K} \circ (-)^\natural : \mathrm{Cat}^{\mathrm{st}} \to \mathrm{Sp}_{\geq 0}$ (definition 9.2) is Karoubi-localizing (example 8.9), thus is Verdier-localizing (proposition 3.27).
(4) The non-connective K-theory spectrum $\mathbb{K} : \mathrm{Cat}^{\mathrm{st}} \to \mathrm{Sp}$ (definition 9.8) is Karoubi-localizing.

Proposition 3.18 ([HLS23] Observation 2.2)**.** The additive, Verdier-localizing, Karoubi-localizing functors preserve finite products. (See proof A.3.)

3.3. **Additive Grouplike vs. Extension-splitting.** We can characterize additive grouplike functors by extension-splitting functors. We will use lemma 3.20 and proposition 3.21 in the proof of the additivity theorem (corollary 5.4).

Definition 3.19. Let $\mathcal{E}$ be a category with finite limits, and let $F : \mathrm{Cat}^{\mathrm{st}} \to \mathcal{E}$ be a reduced functor. We will say that F is *extension-splitting* if, for every stable category $\mathcal{C}$, the fiber-cofiber map

$$(\mathrm{fib}, \mathrm{cofib}) : \mathrm{Seq}(\mathcal{C}) \to \mathcal{C}^2$$

induces an equivalence $F(\mathrm{Seq}(\mathcal{C})) \to F(\mathcal{C})^2$.

We show that additive grouplike functors and extension-splitting functors are equivalent (proposition 3.21).

Lemma 3.20 ([HLS23] Lemma 2.5)**.** Let $\mathcal{E}$ be a category with finite limits, and let $F : \mathrm{Cat}^{\mathrm{st}} \to \mathcal{E}$ be a reduced and product-preserving functor. The following conditions are equivalent:

(1) The functor F is extension-splitting.
(2) The functor F sends the source-target projection $(s, t) : \mathrm{Ar}(\mathcal{C}) \to \mathcal{C}^2$ for every object stable category $\mathcal{C}$ to an equivalence in $\mathcal{E}$.

Proposition 3.21 ([HLS23] Proposition 2.4)**.** Let $\mathcal{E}$ be a category with finite limits, and let $F : \mathrm{Cat}^{\mathrm{st}} \to \mathcal{E}$ be a reduced functor. The following conditions are equivalent:

(1) The functor F is additive grouplike.
(2) The functor F is extension-splitting.

3.4. **Additive vs. Verdier-localizing.** In this section, we recall Waldhausen's fibration theorem. We will use it in the proof of the localization theorem (theorem 6.1).

Notation 3.22. Let $\mathcal{D}$ be a stable category, let $\mathcal{C}$ be a stable full subcategory of $\mathcal{D}$, and let $\mathcal{I}$ be a category. We let $\mathrm{Fun}^{\mathcal{C}}(\mathcal{I}, \mathcal{D})$ denote the full subcategory of $\mathrm{Fun}(\mathcal{I}, \mathcal{D})$ spanned by the functors which take every map in $\mathcal{I}$ to an equivalence modulo $\mathcal{C}$.

Theorem 3.23 (Waldhausen's fibration theorem)**.** *Let $\mathcal{C} \to \mathcal{D} \to \mathcal{E}$ be a Verdier sequence, and let $F : \mathrm{Cat}^{\mathrm{st}} \to \mathrm{An}$ be an additive grouplike functor. Then, for every $n \geq 0$, the constant map*

$$\mathrm{const} : \mathcal{D} \to \mathrm{Fun}^{\mathcal{C}}([n], \mathcal{D}) : X \mapsto (X \to \cdots \to X)$$

induces a bifiber sequence of $\mathbb{E}_\infty$-groups

$$F(\mathcal{C}) \to F(\mathcal{D}) \to |F\, \mathrm{Fun}^{\mathcal{C}}([-], \mathcal{D})|.$$

We can deduce when an additive functor becomes a Verdier-localizing functor.

Corollary 3.24 ([HLS23] Corollary 2.10)**.** Let $F : \mathrm{Cat}^{\mathrm{st}} \to \mathrm{An}$ be an additive functor. These conditions are equivalent:

(1) The functor F is Verdier-localizing.

(2) For every Verdier sequence $\mathcal{C} \to \mathcal{D} \to \mathcal{E}$, the canonical map $|F\,\mathrm{Fun}^{\mathcal{C}}([-], \mathcal{D})| \to F(\mathcal{E})$ is an equivalence of anima.

3.5. **Verdier-localizing vs. Karoubi-localizing.** The relationship between Verdier-localizing and Karoubi-localizing functors is as follows.

Definition 3.25. Let $f : \mathcal{C} \to \mathcal{D}$ be a functor between categories. We will say that f *has the dense image* if, for every object X of $\mathcal{D}$, there exists an object Y in the essential image of $\mathcal{C}$ such that Y is a retract of X.

Definition 3.26. Let $f : \mathcal{C} \to \mathcal{D}$ be an exact functor between stable categories. We will say that f is a *Karoubi equivalence* if it is fully faithful and has the dense image.

Proposition 3.27 ([HLS23] Observation 2.12)**.** Let $\mathcal{E}$ be a category with finite limits, and let $F : \mathrm{Cat}^{\mathrm{st}} \to \mathcal{E}$ be a reduced functor. The following conditions are equivalent:

(1) The functor F is Karoubi-localizing.

(2) The functor F is Verdier-localizing and inverts Karoubi equivalences.

We can construct Karoubi-localization functors from Verdier-localizing functors using the idempotent completion.

Proposition 3.28 ([HLS23] Lemma 2.13)**.** Let $\mathcal{E}$ be a category with finite limits, and let $F : \mathrm{Cat}^{\mathrm{st}} \to \mathcal{E}$ be a Verdier-localizing functor. Suppose that F takes every cartesian square in $\mathrm{Cat}^{\mathrm{st}}$ whose vertical maps are dense inclusions, to a cartesian square in $\mathcal{E}$. Then the functor $F \circ (-)^{\natural} : \mathrm{Cat}^{\mathrm{st}} \to \mathcal{E}$ is Karoubi-localizing.

4. The Algebraic K-Theory of Stable ∞-Categories

In this section, we recall the Q-construction and define the algebraic K-theory. Furthermore, we introduce the S-construction and demonstrate that the definitions of the algebraic K-theory given by these two constructions are equivalent.

4.1. **Simplicial Objects.** In this section, we recall the basic notions of simplicial objects.

Definition 4.1. The inclusion $\mathrm{N}(\mathbb{\Delta}) \subseteq \mathrm{Cat}$ induces an adjunction

$$\mathrm{asscat} : \mathrm{Fun}(\mathrm{N}(\mathbb{\Delta})^{\mathrm{op}}, \mathrm{An}) \rightleftarrows \mathrm{Cat} : \mathrm{N}^r .$$

We will refer to the left adjoint as the *associated category functor*, and to the right adjoint as the *Rezk nerve*.

Definition 4.2. Let $\mathcal{C}$ be a category. We will refer to a functor

$$X : \mathrm{N}(\mathbb{\Delta})^{\mathrm{op}} \to \mathcal{C}$$

as a *simplicial object* of $\mathcal{C}$. We will say that X is a *simplicial anima* if $\mathcal{C}$ is An.

Remark 4.3. Let $\mathcal{C}$ be a category. For every $n \geq 0$, we have an equivalence of anima

$$\mathrm{N}^r_n(\mathcal{C}) \simeq \mathrm{Map}_{\mathrm{Cat}}([n], \mathcal{C}) \simeq \mathrm{core}\,\mathrm{Fun}([n], \mathcal{C}).$$

Notation 4.4. We let $[n]$ denote the category the ordinary nerve $\mathrm{N}([n])$, instead of Δ^n. On the other hand, we let Δ^n denote the functor

$$\Delta^n := \mathrm{Map}_{\mathrm{Cat}}(-, [n]) : \mathrm{N}(\mathbb{\Delta})^{\mathrm{op}} \to \mathrm{An}.$$

Then we have an equivalence of simplicial anima $\mathrm{N}^r([n]) \simeq \Delta^n$.

We define the Segal condition and completeness specifically for simplicial anima, although these concepts are applicable to every category.

Definition 4.5. Let $X : \mathrm{N}(\mathbb{\Delta})^{\mathrm{op}} \to \mathrm{An}$ be a simplicial anima. We will say that X is *Segal* if the n-spine inclusion $\mathrm{sp}^n \subseteq \Delta^n$ induces an equivalence of anima

$$X_n \simeq \mathrm{Map}_{\mathrm{Fun}(\mathrm{N}(\mathbb{\Delta})^{\mathrm{op}},\mathrm{An})}(\Delta^n, X) \to \mathrm{Map}_{\mathrm{Fun}(\mathrm{N}(\mathbb{\Delta})^{\mathrm{op}},\mathrm{An})}(\mathrm{sp}^n, X) \simeq X_1 \times_{X_0} \cdots \times_{X_0} X_1$$

for every $n \geq 0$.

The Segal condition can be interpreted as stating that a Segal simplicial anima has a unique spine lifting up to a choice of contractible spaces.

Definition 4.6. Let $X : \mathrm{N}(\mathbb{\Delta})^{\mathrm{op}} \to \mathrm{An}$ be a Segal simplicial anima. We will say that X is *complete* if the following diagram is a cartesian diagram in An.

$$\begin{tikzcd} X_0 \arrow[r, "\mathrm{diag}"] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & X_0 \times X_0 \arrow[d, "{(s,s)}"] \\ X_3 \arrow[r, "{(d^{\{0,2\}}, d^{\{1,3\}})}"'] & X_1 \times X_1 \end{tikzcd}$$

The completeness condition can be understood as indicating that the higher simplices of a complete Segal simplicial anima correspond to equivalences related to its degenerate edges.

Proposition 4.7. The Rezk nerve $\mathrm{N}^r : \mathrm{Cat} \to \mathrm{Fun}(\mathrm{N}(\mathbb{\Delta})^{\mathrm{op}}, \mathrm{An})$ is fully faithful. Moreover, its essential image precisely consists of complete Segal simplicial anima.

4.2. **The Algebraic K-Theory.**

Definition 4.8 ([HLS23] Definition 3.1)**.** Let $\mathcal{C}$ be a category with finite limits. For every $n \geq 0$, we let $Q_n(\mathcal{C})$ denote the full subcategory of $\mathrm{Fun}(\mathrm{TwAr}[n], \mathcal{C})$ spanned by the diagrams which take every square in $\mathrm{TwAr}[n]$ to a cartesian square in $\mathcal{C}$.

The construction $n \mapsto Q_n(\mathcal{C})$ determines a functor

$$Q(\mathcal{C}) : \mathrm{N}(\mathbb{\Delta})^{\mathrm{op}} \to \mathrm{Cat}^{\mathrm{lex}}$$

and furthermore, the construction $\mathcal{C} \mapsto Q(\mathcal{C})$ defines a functor

$$Q : \mathrm{Cat}^{\mathrm{lex}} \to \mathrm{Fun}(\mathrm{N}(\mathbb{\Delta})^{\mathrm{op}}, \mathrm{Cat}^{\mathrm{lex}}).$$

We will refer to this functor as the *(Quillen's) Q-construction.*

Proposition 4.9 ([HLS23] Proposition 3.2)**.** Let $\mathcal{C}$ be a category with finite limits. Then the simplicial object in $\mathrm{Cat}^{\mathrm{lex}}$

$$Q(\mathcal{C}) : \mathrm{N}(\mathbb{\Delta})^{\mathrm{op}} \to \mathrm{Cat}^{\mathrm{lex}}$$

is complete Segal. In particular, the simplicial anima

$$\mathrm{core}\, Q(\mathcal{C}) : \mathrm{N}(\mathbb{\Delta})^{\mathrm{op}} \to \mathrm{An}$$

is complete Segal. (See proof A.7.)

Remark 4.10. Corollary A.6 implies that, if $\mathcal{C}$ is stable, so is $Q_n(\mathcal{C})$. Therefore we obtain functors

$$Q(\mathcal{C}) : \mathrm{N}(\mathbb{\Delta})^{\mathrm{op}} \to \mathrm{Cat}^{\mathrm{st}} \quad \text{and} \quad Q : \mathrm{Cat}^{\mathrm{st}} \to \mathrm{Fun}(\mathrm{N}(\mathbb{\Delta})^{\mathrm{op}}, \mathrm{Cat}^{\mathrm{st}}).$$

Moreover, for every stable category $\mathcal{C}$, the category $Q(\mathcal{C})$ is a complete Segal simplicial object in $\mathrm{Cat}^{\mathrm{st}}$, since $\mathrm{Cat}^{\mathrm{st}}$ is stable under finite limits in Cat.

Definition 4.11 ([HLS23] Definition 3.3)**.** Let $\mathcal{C}$ be a category with finite limits. Then we define the *category of spans* $\mathrm{Span}(\mathcal{C})$ in $\mathcal{C}$ as

$$\mathrm{Span}(\mathcal{C}) := \mathrm{asscat}\,\mathrm{core}\,Q(\mathcal{C}).$$

The construction $\mathcal{C} \mapsto \mathrm{Span}(\mathcal{C})$ determines a functor

$$\mathrm{Span} : \mathrm{Cat}^{\mathrm{lex}} \to \mathrm{Cat}.$$

Definition 4.12 ([HLS23] Definition 3.4)**.** Let $\mathcal{C}$ be a stable category. Then we define the *algebraic K-anima* (or *algebraic K-theory anima*, or *projective class anima*) $\mathcal{K}(\mathcal{C})$ of $\mathcal{C}$ as

$$\mathcal{K}(\mathcal{C}) := \Omega|\,\mathrm{Span}(\mathcal{C})| \simeq \Omega|\,\mathrm{core}\,Q(\mathcal{C})|$$

where the base object of the loop space is given by the zero object of $\mathrm{Span}(\mathcal{C})$.

The construction $\mathcal{C} \mapsto \mathcal{K}(\mathcal{C})$ determines a functor

$$\mathcal{K} : \mathrm{Cat}^{\mathrm{st}} \to \mathrm{An}.$$

We will refer to this functor as the *algebraic K-theory* (or *algebraic K-functor*).

Definition 4.13. Let $\mathcal{C}$ be a stable category. For every $n \geq 0$, we define the *n-th K-group* $\mathcal{K}_n(\mathcal{C})$ of $\mathcal{C}$ as

$$\mathcal{K}_n(\mathcal{C}) := \pi_n\mathcal{K}(\mathcal{C}).$$

Proposition 4.14. Let $\mathcal{C}$ be a stable category. Then we have an isomorphism of groups

$$\pi_0\mathcal{K}(\mathcal{C}) = \mathcal{K}_0(\mathcal{C}),$$

where $\mathcal{K}_0(\mathcal{C})$ is the Grothendieck group of $\mathcal{C}$ (definition 2.1).

4.3. **Waldhausen's S-Construction.** In this section, we construct the algebraic K-theory using Waldhausen's S-construction.

Definition 4.15. Let $\mathcal{C}$ be a stable category. An *$[n]$-gapped object* of $\mathcal{C}$ is a functor $F : \mathrm{Ar}[n] \to \mathcal{C}$ which satisfies the following conditions:

(1) For every $0 \leq i \leq n$, $F(i,i)$ is a zero object of $\mathcal{C}$.
(2) For every $i \leq j \leq k$, the following diagram is a (co)cartesian diagram in $\mathcal{C}$.

$$\begin{tikzcd} F(i,j) \arrow[r] \arrow[d] & F(i,k) \arrow[d] \\ 0 \simeq F(j,j) \arrow[r] \arrow[ur, phantom, "\ulcorner"] & F(j,k) \end{tikzcd}$$

We let $S_n(\mathcal{C})$ denote the full subcategory of $\mathrm{Fun}(\mathrm{Ar}[n], \mathcal{C})$ spanned by the $[n]$-gapped objects of $\mathcal{C}$.

Remark 4.16. Let $\mathcal{C}$ be a stable category. We can describe the low-dimensional simplices of $S_n(\mathcal{C})$.

- The category $S_0(\mathcal{C})$ is the full subcategory of $\mathcal{C}$ spanned by the zero objects of $\mathcal{C}$. Thus $S_0(\mathcal{C})$ is contractible.
- The category $S_1(\mathcal{C})$ is equivalent to $\mathcal{C}$, since every object of $S_1(\mathcal{C})$ is of the form $0 \to X \to 0$, where X is an object of $\mathcal{C}$.
- The category $S_2(\mathcal{C})$ is equivalent to the arrow category $\mathrm{Ar}(\mathcal{C})$ of $\mathcal{C}$, since every object of $S_2(\mathcal{C})$ is of the form $0 \to X' \to X \to X'' \to 0$, where $X' \to X \to X''$ is a cofiber sequence in $\mathcal{C}$.

Remark 4.17. Let $\mathcal{C}$ be a stable category. We have an equivalence of categories

$$S_n(\mathcal{C}) \simeq \mathrm{Fun}([n-1], \mathcal{C})$$

for every $n \geq 0$. Thus, if $\mathcal{C}$ is stable, so is $S_n(\mathcal{C})$.

Definition 4.18. The construction $n \mapsto S_n(\mathcal{C})$ determines a functor

$$S(\mathcal{C}) : \mathrm{N}(\mathbb{\Delta})^{\mathrm{op}} \to \mathrm{Cat}^{\mathrm{st}}$$

and furthermore, the construction $\mathcal{C} \mapsto S(\mathcal{C})$ defines a functor

$$S : \mathrm{Cat}^{\mathrm{st}} \to \mathrm{Fun}(\mathrm{N}(\mathbb{\Delta})^{\mathrm{op}}, \mathrm{Cat}^{\mathrm{st}}).$$

We will refer to this functor as *(Waldhausen's) S-construction.*

Definition 4.19. Let $\mathcal{C}$ be a stable category. Then we define the *algebraic K-anima* $\mathcal{K}_S(\mathcal{C})$ of $\mathcal{C}$ as

$$\mathcal{K}_S(\mathcal{C}) := \Omega |\operatorname{core} S(\mathcal{C})|.$$

The construction $\mathcal{C} \mapsto \mathcal{K}_S(\mathcal{C})$ determines a functor

$$\mathcal{K}_S : \mathrm{Cat}^{\mathrm{st}} \to \mathrm{An}.$$

We will refer to this functor as the *algebraic K-theory.*

Remark 4.20. Let $\mathcal{C}$ be a stable category. Then the anima $|\operatorname{core} S(\mathcal{C})|$ admits the canonical base point given by the map

$$0 \simeq \operatorname{core} S_0(\mathcal{C}) \to |\operatorname{core} S(\mathcal{C})|.$$

Moreover, $|\operatorname{core} S(\mathcal{C})|$ is connected, since the canonical map

$$0 \simeq \pi_0 \operatorname{core} S_0(\mathcal{C}) \to \pi_0 |\operatorname{core} S(\mathcal{C})|$$

is surjective.

Proposition 4.21. Let $\mathcal{C}$ be a stable category. Then two definitions of the algebraic K-anima (definitions 4.12 and 4.19) induce an equivalence of anima

$$\mathcal{K}(\mathcal{C}) \simeq \mathcal{K}_S(\mathcal{C}).$$

5. The Additivity Theorem

The goal of this section is to prove the additivity theorem.

Theorem 5.1 ([HLS23] Theorem 4.1: The Additivity Theorem)**.** *Let $\mathcal{C}$ be a stable category. Then the source-target projection induces an equivalence of anima*

$$|\operatorname{Span}(s,t)| : |\operatorname{Span}(\operatorname{Ar}(\mathcal{C}))| \to |\operatorname{Span}(\mathcal{C})|^2.$$

Before proving theorem 5.1, we show some corollaries. We need some lemmata.

Lemma 5.2. Let $\mathcal{C}$ and $\mathcal{D}$ be stable categories, and let $F' \to F \to F''$ be a cofiber sequence of exact functors from $\mathcal{C}$ to $\mathcal{D}$. Then we have

$$\mathcal{K}(F) = \mathcal{K}(F') + \mathcal{K}(F'').$$

Proof. Consider the following functors

$$\mathrm{mid}, \mathrm{fib} + \mathrm{cofib} : \operatorname{Seq}(\operatorname{Fun}^{\mathrm{ex}}(\mathcal{C}, \mathcal{D})) \to \operatorname{Fun}^{\mathrm{ex}}(\mathcal{C}, \mathcal{D}).$$

Since $\mathcal{K}$ is extension-splitting (see proposition 3.21), by Waldhausen's additivity theorem (TBA), we have an equivalence

$$\mathcal{K}(\mathrm{mid}) \simeq \mathcal{K}(\mathrm{fib}) + \mathcal{K}(\mathrm{cofib}).$$

We obtain the assertion by applying it to the cofiber sequence $F' \to F \to F''$. □

Lemma 5.3. The algebraic K-theory $\mathcal{K} : \mathrm{Cat}^{\mathrm{st}} \to \mathrm{An}$ preserves finite products.

Proof. Let $\mathcal{C}$ and $\mathcal{D}$ be two stable categories. Two projections $\mathcal{C} \leftarrow \mathcal{C} \times \mathcal{D} \to \mathcal{D}$ induce a natural morphism

$$\mathcal{K}(\mathcal{C} \times \mathcal{D}) \to \mathcal{K}(\mathcal{C}) \times \mathcal{K}(\mathcal{D}).$$

Unwinding the definition, it suffices to show that $Q : \mathrm{Cat}^{\mathrm{st}} \to \operatorname{Fun}(\mathrm{N}(\mathbb{\Delta}), \mathrm{Cat}^{\mathrm{st}})$ preserves finite products. It follows from that we have an equivalence

$$\operatorname{Fun}(\operatorname{TwAr}[n], \mathcal{C} \times \mathcal{D}) \times \operatorname{Fun}(\operatorname{TwAr}[n], \mathcal{C}) \times \operatorname{Fun}(\operatorname{TwAr}[n], \mathcal{D})$$

and it restricts an equivalence

$$Q(\mathcal{C} \times \mathcal{D}) \simeq Q(\mathcal{C}) \times Q(\mathcal{D}).$$

□

Corollary 5.4 ([HLS23] Corollary 4.2)**.** The algebraic K-theory $\mathcal{K} : \mathrm{Cat}^{\mathrm{st}} \to \mathrm{An}$ is additive grouplike.

Proof. By proposition 3.21, it suffices to show that $\mathcal{K}$ is a reduced functor and it is extension splitting. We have

$$\mathcal{K}(0) \simeq \Omega|\operatorname{Span}(0)| \simeq \Omega|\operatorname{core} Q(0)| \simeq 0.$$

Then by lemmas 3.20 and 5.3, it is enough to show that $\mathcal{K}$ sends the source-target projection $(s,t) : \operatorname{Ar}(\mathcal{C}) \to \mathcal{C}^2$ to an equivalence of anima. That is, it suffices to show that there is an equivalence $\mathcal{K}(s,t) : \mathcal{K}(\operatorname{Ar}(\mathcal{C})) \to \mathcal{K}(\mathcal{C})^2$.

By theorem 5.1, we have an equivalence $|\operatorname{Span}(\operatorname{Ar}(\mathcal{C}))| \to |\operatorname{Span}(\mathcal{C})|^2$. Since the loop functor Ω preserves limits, we obtain an equivalence

$$\mathcal{K}(s,t) : \mathcal{K}(\operatorname{Ar}(\mathcal{C})) \to \mathcal{K}(\mathcal{C})^2.$$

□

Corollary 5.5 (Eilenberg swindle)**.** Let $\mathcal{C}$ be a stable category with countable coproducts. Then the algebraic K-theory anima vanishes. That is, we have an equivalence of anima

$$\mathcal{K}(\mathcal{C}) \simeq 0.$$

Proof. Consider the following exact functor

$$F : \mathcal{C} \to \mathcal{C} : X \mapsto \bigoplus_{n \in \mathbb{N}} X_n.$$

Then there exists a cofiber sequence

$$\mathrm{id}_{\mathcal{C}} \to F \to F$$

of exact functors on $\mathcal{C}$. By lemma 5.2, we have an equivalence

$$\mathcal{K}(F) \simeq \mathcal{K}(\mathrm{id}_{\mathcal{C}}) + \mathcal{K}(\mathcal{F}).$$

Thus we have $\mathcal{K}(\mathrm{id}_{\mathcal{C}}) \simeq 0$ and, in particular $\mathcal{K}(\mathcal{C}) \simeq 0$. □

Corollary 5.6. Let $\mathcal{C}$ be a stable category. Then the suspension $\Sigma : \mathcal{C} \to \mathcal{C}$ induces the map

$$\mathcal{K}(\Sigma) : \mathcal{K}(\mathcal{C}) \to \mathcal{K}(\mathcal{C})$$

given by multiplication by -1.

Proof. Applying lemma 5.2 to the cofiber sequence $\mathrm{id} \to 0 \to \Sigma$, we obtain

$$\mathcal{K}(0) \simeq \mathcal{K}(\mathrm{id}) + \mathcal{K}(\Sigma).$$

Then we have $\mathcal{K}(\mathrm{id}) \simeq \mathcal{K}(\Sigma)$. It implies that $\mathcal{K}(\Sigma) : \mathcal{K}(\mathcal{C}) \to \mathcal{K}(\mathcal{C})$ is given by multiplication by -1. □

The proof of theorem 5.1 follows from the next two propositions.

Proposition 5.7 ([HLS23] Proposition 4.3)**.** Let $\mathcal{C}$ be a stable category. Then there are canonical equivalences of categories

$$\mathrm{Span}(\mathcal{C}) \to \mathrm{Span}(\mathcal{C}^{\mathrm{op}}) \quad \text{and} \quad \mathrm{Span}(\mathrm{Ar}(\mathcal{C})) \simeq \mathrm{Span}(\mathrm{TwAr}(\mathcal{C})).$$

Moreover, they fit together into a natural commutative diagram

$$\begin{tikzcd}
\mathrm{Span}(\mathrm{Ar}(\mathcal{C})) \arrow[r, "\simeq"] \arrow[d, "\mathrm{Span}(s,t)"'] & \mathrm{Span}(\mathrm{TwAr}(\mathcal{C})) \arrow[d, "\mathrm{Span}(s,t)"] \\
\mathrm{Span}(\mathcal{C}) \times \mathrm{Span}(\mathcal{C}) \arrow[r, "\simeq"'] & \mathrm{Span}(\mathcal{C}) \times \mathrm{Span}(\mathcal{C}^{\mathrm{op}}).
\end{tikzcd}$$

Proposition 5.8 ([HLS23] Proposition 4.4)**.** Let $\mathcal{C}$ be a stable category. Then the source-target projection

$$\mathrm{Span}(s,t) : \mathrm{Span}(\mathrm{TwAr}(\mathcal{C})) \to \mathrm{Span}(\mathcal{C}) \times \mathrm{Span}(\mathcal{C}^{\mathrm{op}})$$

is cofinal.

6. The Localization Theorem

The goal of this section is to prove the localization theorem.

Theorem 6.1 ([HLS23] Theorem 6.1: The Localization Theorem)**.** *The algebraic K-theory* $\mathcal{K} : \mathrm{Cat}^{\mathrm{st}} \to \mathrm{An}$ *is Verdier-localizing.*

Remark 6.2. By corollary 3.24, an additive grouplike functor $F : \mathrm{Cat}^{\mathrm{st}} \to \mathrm{An}$ is Verdier-localizing if and only if it satisfies the following condition:

(∗) For every Verdier sequence $\mathcal{C} \to \mathcal{D} \to \mathcal{E}$, the canonical map $|F\,\mathrm{Fun}^{\mathcal{C}}([-], \mathcal{D})| \to F(\mathcal{E})$ is an equivalence of anima.

The next proposition implies that it is enough to prove that the core functor satisfies (∗).

Proposition 6.3 ([HLS23] Proposition 6.2)**.** Let $F : \mathrm{Cat}^{\mathrm{st}} \to \mathrm{An}$ be an additive functor. If F satisfies (∗), then $|FQ(-)|$ and $\Omega|FQ(-)|$ also satisfy (∗).

Remark 6.4. If the core functor satisfies (∗), then the algebraic K-theory also satisfies (∗) since $\mathcal{K}(-) \simeq \Omega|\operatorname{core} Q(-)|$.

Notation 6.5. Let $\mathcal{D}$ be a stable category, and let $\mathcal{C}$ be a stable subcategory of $\mathcal{D}$. We let $\mathcal{D}_{\mathcal{C}}$ denote the full subcategory of $\mathcal{D}$ spanned by the equivalences modulo $\mathcal{C}$ in $\mathcal{D}$.

Remark 6.6. Let $\mathcal{D}$ be a stable category, and let $\mathcal{C}$ be a stable subcategory of $\mathcal{D}$. We obtain an equivalence of anima

$$\operatorname{core} \mathrm{Fun}^{\mathcal{C}}([-], \mathcal{D}) \simeq \operatorname{core} \mathrm{Fun}([-], \mathcal{D}_{\mathcal{C}}) \simeq \mathrm{Map}_{\mathrm{Cat}}([-], \mathcal{D}_{\mathcal{C}}) \simeq \mathrm{N}^r(\mathcal{D}_{\mathcal{C}}).$$

Then we have an equivalence

$$|\operatorname{core} \mathrm{Fun}^{\mathcal{C}}([-], \mathcal{D})| \simeq |\mathcal{D}_{\mathcal{C}}|,$$

since the canonical map $|\,\mathrm{N}^r(\mathcal{D}_{\mathcal{C}})| \to |\mathcal{D}_{\mathcal{C}}|$ is an equivalence

Thus it suffices to show the following proposition.

Proposition 6.7 ([HLS23] Proposition 6.6)**.** Let $\mathcal{D}$ be a stable category, and let $\mathcal{C}$ be a stable subcategory of $\mathcal{D}$. Then the map

$$|\mathcal{D}_{\mathcal{C}}| \to \operatorname{core}(\mathcal{D}/\mathcal{C})$$

is faithful. If the inclusion $\mathcal{C} \subseteq \mathcal{D}$ is a Verdier inclusion, then the above map is an equivalence.

Proposition 6.7 is a special case of the following proposition.

Proposition 6.8 ([HLS23] Proposition 6.8)**.** Let $\mathcal{C}$ be a category, and let S be a subcategory of $\mathcal{C}$. If S is closed under 2-out-of-3 and pushouts in $\mathcal{C}$, then the map

$$|S| = S[S^{-1}] \to \mathcal{C}$$

is faithful. Moreover, the following conditions are equivalent:

(1) The inclusion $|S| \subseteq \operatorname{core} \mathcal{C}[S^{-1}]$ is fully faithful.

(2) The category S is closed under 2-out-of-6 in $\mathcal{C}$.

(3) A morphism in $\mathcal{D}$ belongs to S if and only if its source and target are in S and it is invertible in $\mathcal{C}[S^{-1}]$.

7. The Universality Theorem

The goal of this section is to prove the universality theorem.

Theorem 7.1 ([HLS23] Theorem 5.1: The Universality Theorem)**.** *The algebraic K-theory* $\mathcal{K} : \mathrm{Cat}^{\mathrm{st}} \to \mathrm{An}$ *is an initial additive grouplike functor under the core functor* $\mathrm{core} : \mathrm{Cat}^{\mathrm{st}} \to \mathrm{An}$*. That is, the natural map* $\tau : \mathrm{core} \Rightarrow \mathcal{K}$ *is an initial object in* $\mathrm{Fun}(\mathrm{Cat}^{\mathrm{st}}, \mathrm{An})^{\mathrm{add,grp}}_{\mathrm{core}\,/}$.

Notation 7.2. Let $F : \mathrm{Cat}^{\mathrm{st}} \to \mathrm{An}$ be a reduced functor. We denote a functor

$$GF(-) := \Omega|FQ(-)| : \mathrm{Cat}^{\mathrm{st}} \to \mathrm{An}.$$

Example 7.3. the functor $G\,\mathrm{core}$ is equivalent to the algebraic K-theory $\mathcal{K} : \mathrm{Cat}^{\mathrm{st}} \to \mathrm{An}$.

Proof of theorem 7.1. We want to show that the natural transformation $\tau : \mathrm{core} \Rightarrow \mathcal{K}$ induces an equivalence

$$\tau^* : \mathrm{Nat}(\mathcal{K}, F) \to \mathrm{Nat}(\mathrm{core}, F)$$

for every additive grouplike functor $F : \mathrm{Cat}^{\mathrm{st}} \to \mathrm{An}$.

Now consider the following diagram

$$\begin{tikzcd}
\mathrm{Nat}(G\,\mathrm{core}, F) \arrow[r, "(\eta_{\mathrm{core}})^*"] \arrow[d, "G"] \arrow[dd, bend right=60, "(\eta_F)^*"'] & \mathrm{Nat}(\mathrm{core}, F) \arrow[d, "G"'] \arrow[dd, bend left=60, "(\eta_F)^*"] \\
\mathrm{Nat}(GG\,\mathrm{core}, GF) \arrow[r, "(G\eta_{\mathrm{core}})^*"] \arrow[d, "(\eta_{G\,\mathrm{core}})^*"] & \mathrm{Nat}(G\,\mathrm{core}, GF) \arrow[d, "(\eta_{\mathrm{core}})^*"'] \\
\mathrm{Nat}(G\,\mathrm{core}, GF) \arrow[r, "(\eta_{\mathrm{core}})^*"] & \mathrm{Nat}(\mathrm{core}, GF).
\end{tikzcd}$$

The upper square commutes since G is a functor, and the other three parts commute since η is natural. If the red-colored maps are equivalent, then we can show that the upper horizontal map $(\eta_{\mathrm{core}})^*$ is an equivalence. If we apply this to the case F is core, then we obtain the desired result. This assumption follows from the next two propositions. $\square$

The next proposition implies that $(\eta_F)_*$ and $(G\eta_{\mathrm{core}})^*$ are equivalences.

Proposition 7.4 ([HLS23] Proposition 5.2)**.** Let $F : \mathrm{Cat}^{\mathrm{st}} \to \mathrm{An}$ be an additive grouplike functor. Then the natural transformation

$$\eta_F : F \Rightarrow GF$$

is an equivalence.

The next proposition implies that $(\eta_{G\,\mathrm{core}})$ is an equivalence.

Proposition 7.5 ([HLS23] Proposition 5.3)**.** Let $F : \mathrm{Cat}^{\mathrm{st}} \to \mathrm{An}$ be an additive functor. Then the two natural transformations

$$\eta_{GF}, G\eta_F : GF \Rightarrow GGF$$

differ by an automorphism of the target. That is, the following diagram commutes

$$\begin{tikzcd}
GF \arrow[r, "G\eta_F"] \arrow[dr, "\eta_{GF}"'] & GGF \arrow[d, "\simeq"] \\
& GGF.
\end{tikzcd}$$

8. The Cofinality Theorem

The goal of this section is to prove the cofinality theorem.

Theorem 8.1 ([HLS23] Theorem 7.1: The Cofinality Theorem)**.** *Let $f : \mathcal{C} \subseteq \mathcal{D}$ be a dense inclusion of stable categories. Then it induces a fiber sequence*

$$\mathcal{K}(\mathcal{C}) \to \mathcal{K}(\mathcal{D}) \to \mathcal{K}_0(\mathcal{D})/\mathcal{K}_0(\mathcal{C}).$$

In particular, maps of abelian groups

$$\mathcal{K}_i(\mathcal{C}) \to \mathcal{K}_i(\mathcal{D})$$

are isomorphisms for every $i \geq 1$, and there exists a short exact sequence

$$0 \to \mathcal{K}_0(\mathcal{C}) \to \mathcal{K}_0(\mathcal{D}) \to \mathcal{K}_0(\mathcal{D})/\mathcal{K}_0(\mathcal{C}) \to 0.$$

Theorem 8.1 holds for a broader class of additive Karoubian functors (theorem 8.5).

Definition 8.2 ([HLS23] Definition 7.4)**.** Let $f : X \to Y$ be a map of $\mathbb{E}_\infty$-monoids. We will say that f is *cofinal* if it satisfies the following conditions:

(1) The map $\pi_0(f) : \pi_0(X) \to \pi_0(Y)$ is an inclusion.

(2) For every element x in $\pi_0(X)$, there exists an element x' in $\pi_0(X)$ such that $x + x'$ is in $\pi_0(Y)$.

We will say that a cofinal map is *dense* if it satisfies the following condition:

(3) The sequence of $\mathbb{E}_\infty$-monoids $0 \to \pi_0(X) \to \pi_0(Y) \to \pi_0(Y)/\pi_0(X) \to 0$ is exact.

Or equivalently,

(3') An element y in $\pi_0(Y)$ belongs to $\pi_0(X)$ if there exists an element x in $\pi_0(X)$ such that $x + y$ is in $\pi_0(X)$.

Definition 8.3 ([HLS23] Definition 7.6)**.** Let $F : \mathrm{Cat}^{\mathrm{st}} \to \mathrm{An}$ be an additive functor. We will say that F is *Karoubian* if it satisfies the following conditions:

(1) The functor F takes every dense inclusion between stable categories to a dense map of $\mathbb{E}_\infty$-monoids.

(2) The functor F preserves every cartesian square in $\mathrm{Cat}^{\mathrm{st}}$ whose vertical maps are dense.

Definition 8.4. Let $F : \mathrm{Cat}^{\mathrm{st}} \to \mathrm{An}$ be an additive Karoubian functor. We will refer to the functor

$$F^{\mathrm{grp}}(-) := \Omega|FQ(-)| : \mathrm{Cat}^{\mathrm{st}} \to \mathrm{An}$$

as the *group completion* of F.

Theorem 8.5 ([HLS23] Theorem 7.7)**.** *Let $F : \mathrm{Cat}^{\mathrm{st}} \to \mathrm{An}$ be an additive Karoubian functor. For every dense inclusion $\mathcal{C} \subseteq \mathcal{D}$ of stable categories, the canonical map of $\mathbb{E}_\infty$-monoids*

$$F(\mathcal{D})/F(\mathcal{C}) \to F^{\mathrm{grp}}(\mathcal{D})/F^{\mathrm{grp}}(\mathcal{C})$$

is an equivalence. Hence maps of abelian groups

$$\pi_i F^{\mathrm{grp}}(\mathcal{C}) \to \pi_i F^{\mathrm{grp}}(\mathcal{D})$$

are isomorphisms for every $i \geq 1$, and there exists a short exact sequence

$$0 \to \pi_0 F^{\mathrm{grp}}(\mathcal{C}) \to \pi_0 F^{\mathrm{grp}}(\mathcal{D}) \to \pi_0 F^{\mathrm{grp}}(\mathcal{D})/\pi_0 F^{\mathrm{grp}}(\mathcal{C}) \to 0.$$

Corollary 8.6 ([HLS23] Corollary 7.8)**.** Let $F : \mathrm{Cat}^{\mathrm{st}} \to \mathrm{An}$ be an additive Karoubian functor. Then the group completion

$$F^{\mathrm{grp}} : \mathrm{Cat}^{\mathrm{st}} \to \mathrm{An}$$

of F is also additive Karoubian. Moreover if F^{grp} is Verdier-localizing, then the functor

$$F^{\mathrm{grp}} \circ (-)^{\natural} : \mathrm{Cat}^{\mathrm{st}} \to \mathrm{An}$$

is Karoubi-localizing.

Example 8.7. The core functor $\mathrm{core} : \mathrm{Cat}^{\mathrm{st}} \to \mathrm{An}$ is (additive) Karoubian.

(1): Let $\mathcal{C} \hookrightarrow \mathcal{D}$ be a dense inclusion of stable categories. Then the map $\mathrm{core}\, f : \mathrm{core}\, \mathcal{C} \to \mathrm{core}\, \mathcal{D}$ is cofinal. Let d be an element in $\pi_0(\mathrm{core}\, \mathcal{D})$, and let c be an element in $\pi_0(\mathrm{core}\, \mathcal{C})$ such that $c + d$ is in $\pi_0(\mathrm{core}\, \mathcal{C})$. Then we have an equivalence $d \simeq \mathrm{fib}(c \oplus d \to c)$. The element d is in $\pi_0(\mathrm{core}\, \mathcal{C})$ since $\mathcal{C}$ is closed under fibers. That is, $\mathrm{core}\, f$ is dense.

(2): The core functor preserves limits, since it is a right adjoint functor.

Example 8.8. The group completion

$$\mathrm{core}^{\mathrm{grp}} : \mathrm{Cat}^{\mathrm{st}} \to \mathrm{An}$$

of $\mathrm{core} : \mathrm{Cat}^{\mathrm{st}} \to \mathrm{An}$ is equivalent to the algebraic K-theory $\mathcal{K} : \mathrm{Cat}^{\mathrm{st}} \to \mathrm{An}$.

Example 8.9. By corollary 8.6 and example 8.8, the algebraic K-theory $\mathcal{K} : \mathrm{Cat}^{\mathrm{st}} \to \mathrm{An}$ is additive Karoubian. By theorem 6.1, $\mathcal{K}$ is Verdier-localizing. Then the functor

$$\mathcal{K} \circ (-)^{\natural} : \mathrm{Cat}^{\mathrm{st}} \to \mathrm{An}$$

is Karoubi-localizing.

9. The Non-connective K-Theory Spectrum

In this section, we introduce both the connective and non-connective K-theory spectra. Our primary reference is [KNP24], though it contains some minor mistakes and typographical errors. We correct and adjust these as necessary.

9.1. **The Connective K-Theory Spectrum.**

Remark 9.1. As stated in example 8.9, the functor

$$\mathcal{K} \circ (-)^{\natural} : \mathrm{Cat}^{\mathrm{st}} \to \mathrm{An}$$

is Karoubi-localizing. In particular, it is an additive functor. Moreover, by **??**, it is also grouplike. Consequently, it lifts to a functor

$$\mathcal{K} \circ (-)^{\natural} : \mathrm{Cat}^{\mathrm{st}} \to \mathrm{Grp}_{\mathbb{E}_\infty}(\mathrm{An}).$$

Since there exists an equivalence $\mathrm{Grp}_{\mathbb{E}_\infty}(\mathrm{An}) \simeq \mathrm{Sp}_{\geq 0}$, we obtain a functor

$$\mathcal{K} \circ (-)^{\natural} : \mathrm{Cat}^{\mathrm{st}} \to \mathrm{Sp}_{\geq 0}.$$

Definition 9.2 ([KNP24] Definition 3.1.2)**.** We will refer to the functor constructed above

$$\mathcal{K} \circ (-)^{\natural} : \mathrm{Cat}^{\mathrm{st}} \to \mathrm{Sp}_{\geq 0}$$

as the *connective K-theory*. We often simply denote it by $\mathcal{K} : \mathrm{Cat}^{\mathrm{st}} \to \mathrm{Sp}_{\geq 0}$.

Remark 9.3. For every Karoubi square

$$\begin{tikzcd} \mathcal{C}' \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathcal{C} \arrow[d] \\ \mathcal{D}' \arrow[r] & \mathcal{D} \end{tikzcd}$$

in $\mathrm{Cat}^{\mathrm{st}}$, we obtain a cartesian square

$$\begin{tikzcd} \mathcal{K}(\mathcal{C}') \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathcal{K}(\mathcal{C}) \arrow[d] \\ \mathcal{K}(\mathcal{D}') \arrow[r] & \mathcal{K}(\mathcal{D}) \end{tikzcd}$$

in $\mathrm{Sp}_{\geq 0}$. However, in general, it is not a cartesian square in Sp since the inclusion $\mathrm{Sp}_{\geq 0} \subseteq \mathrm{Sp}$ does not preserve (finite) limits.

In particular, for every Karoubi sequence $\mathcal{C} \to \mathcal{D} \to \mathcal{E}$, we obtain a long exact sequence

$$\cdots \to \mathcal{K}_1(\mathcal{E}) \to \mathcal{K}_0(\mathcal{C}) \to \mathcal{K}_0(\mathcal{D}) \to \mathcal{K}_0(\mathcal{E}).$$

However, in general, the last map $\mathcal{K}_0(\mathcal{D}) \to \mathcal{K}_0(\mathcal{E})$ is not surjective. To extend this sequence, we need to define the non-connective K-theory spectrum.

9.2. **The Non-connective K-Theory Spectrum.** In this section, let κ be an uncountable regular cardinal. For a category $\mathcal{C}$, we let $\mathrm{Ind}_\kappa(\mathcal{C})$ denote the category of Ind-objects. If $\kappa = \omega$, we denote it by $\mathrm{Ind}(\mathcal{C})$.

Example 9.4. Let $\mathcal{C}$ be an idempotent complete stable category. By proposition 3.9, the sequence

$$\mathcal{C} \to \mathrm{Ind}(\mathcal{C})^\kappa \to \mathrm{Ind}(\mathcal{C})^\kappa/\mathcal{C}$$

in $\mathrm{Cat}^{\mathrm{st}}$ is a Karoubi sequence. Then we obtain the fiber sequence

$$\mathcal{K}(\mathcal{C}) \to \mathcal{K}(\mathrm{Ind}(\mathcal{C})^\kappa) \to \mathcal{K}(\mathrm{Ind}(\mathcal{C})^\kappa/\mathcal{C}).$$

By the Eilenberg swindle (corollary 5.5), we have $\mathcal{K}(\mathrm{Ind}_\omega(\mathcal{C})^\kappa) \simeq 0$ since it has countable coproducts. Then we get an equivalence

$$\mathcal{K}(\mathcal{C}) \simeq \Omega\mathcal{K}(\mathrm{Ind}(\mathcal{C})^\kappa/\mathcal{C}).$$

However, in general, $\mathrm{Ind}(\mathcal{C})^\kappa/\mathcal{C}$ is not an idempotent complete. Thus we cannot iterate this delooping process.

Example 9.5. Let $\mathcal{C}$ be a stable category. By the cofinality theorem (theorem 8.1), the induced map

$$\mathcal{K}(\mathrm{Ind}(\mathcal{C})^\kappa/\mathcal{C}) \to \mathcal{K}((\mathrm{Ind}(\mathcal{C})^\kappa/\mathcal{C})^\natural)$$

is an equivalence on connective covers, namely as objects of $\mathrm{Sp}_{\geq 0}$. From example 9.4, we obtain an equivalence

$$\mathcal{K}(\mathcal{C}) \simeq \tau_{\geq 0}\Omega\mathcal{K}((\mathrm{Ind}(\mathcal{C})^\kappa/\mathcal{C})^\natural).$$

In the above discussion, we have $\mathcal{K}_0(\mathrm{Ind}(\mathcal{C})^\kappa/\mathcal{C}) = 0$, but $\mathcal{K}_0((\mathrm{Ind}(\mathcal{C})^\kappa/\mathcal{C})^\natural)$ is not necessarily zero. This is why the negative algebraic K-theory and its (negative) K-groups arise. We will define it as the (-1)-th K-group.

Definition 9.6 ([KNP24] Definition 3.2.7)**.** Let $\mathcal{C}$ be an idempotent complete stable category. Then we define the *Calkin category* $\mathrm{Calk}^{\natural}(\mathcal{C})$ of $\mathcal{C}$ by

$$\mathrm{Calk}^{\natural}(\mathcal{C}) := (\mathrm{Ind}(\mathcal{C})^{\kappa}/\mathcal{C})^{\natural}$$

Let $\mathcal{C}$ be a stable category. For every $n \geq 0$, we inductively define the *Calkin category* $\mathrm{Calk}(\mathcal{C})$ of $\mathcal{C}$ by

$$\mathrm{Calk}^{0}(\mathcal{C}) := \mathcal{C}^{\natural} \quad \text{and} \quad \mathrm{Calk}^{n+1}(\mathcal{C}) := \mathrm{Calk}^{\natural}(\mathrm{Calk}^{n}(\mathcal{C})).$$

Remark 9.7. For every $n \geq 0$, the construction $\mathcal{C} \mapsto \mathrm{Calk}^{n}(\mathcal{C})$ defines a functor

$$\mathrm{Calk}^{n} : \mathrm{Cat}^{\mathrm{st}} \to \mathrm{Cat}^{\mathrm{perf}}.$$

Definition 9.8 ([KNP24] Definition 3.2.9)**.** Let $\mathcal{C}$ be a stable category. We define the *non-connective K-theory spectrum* $\mathbb{K}(\mathcal{C})$ of $\mathcal{C}$ by

$$\mathbb{K}(\mathcal{C}) := \operatorname*{colim}_{n} \Omega^{\infty-n}\mathcal{K}(\mathrm{Calk}^{n}(\mathcal{C})),$$

where taking the colimit in Sp. Equivalently, we can define by

$$\tau_{\geq -n}\mathbb{K}(\mathcal{C}) := \Omega^{n}\mathcal{K}(\mathrm{Calk}^{n}(\mathcal{C})) \quad \text{and} \quad \mathbb{K}(C) := \operatorname*{colim}_{n} \tau_{\geq -n}\mathbb{K}(\mathcal{C}).$$

Definition 9.9. Let $\mathcal{C}$ be a stable category. The construction $\mathcal{C} \mapsto \mathbb{K}(\mathcal{C})$ determines a functor

$$\mathbb{K} : \mathrm{Cat}^{\mathrm{st}} \to \mathrm{Sp}.$$

We will refer to it as the *non-connective K-theory*.

Definition 9.10. Let $\mathcal{C}$ be a stable category. For every integer n, we define the n-th K-group $\mathbb{K}_n(\mathcal{C})$ of $\mathcal{C}$ by

$$\mathbb{K}_n(\mathcal{C}) := \pi_n\mathbb{K}(\mathcal{C}).$$

Remark 9.11. Let $\mathcal{C}$ be a stable category. For every $n > 0$, we obtain an isomorphism

$$\mathcal{K}_n(\mathcal{C}) = \mathbb{K}_n(\mathcal{C}),$$

and $\mathcal{K}_0(\mathcal{C}) \to \mathbb{K}_0(\mathcal{C})$ is injective. Moreover, we have an isomorphism

$$\mathbb{K}_{-1}(\mathcal{C}) = \mathcal{K}_0((\mathrm{Ind}(\mathcal{C})^{\kappa}/\mathcal{C})^{\natural}).$$

Appendix A. Proofs in Section 3 and Section 4

Proof A.1 (Remark 3.12)**.** We show that every Verdier square is a cocartesian diagram in $\mathrm{Cat}^{\mathrm{st}}$. Consider the following Verdier square

$$\begin{tikzcd}
\mathcal{D} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathcal{D}' \arrow[d] \\
\mathcal{E} \arrow[r] & \mathcal{E}'.
\end{tikzcd}$$

Since the vertical maps are Verdier projections, we can extend it to the following diagram.

$$\begin{tikzcd}
\mathcal{C} \arrow[r] \arrow[d] & \mathcal{D} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathcal{D}' \arrow[d] \\
0 \arrow[r] & \mathcal{E} \arrow[r] & \mathcal{E}'.
\end{tikzcd}$$

By definition, the left and outer squares are bicartesian squares. Then the right square is also a bicartesian square.

Proof A.2 (Proposition 3.15)**.** (1) $\Rightarrow$ (2) is obvious.

(2) $\Rightarrow$ (1): Consider the following Verdier square

$$\begin{tikzcd}
\mathcal{D} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathcal{D}' \arrow[d] \\
\mathcal{E} \arrow[r] & \mathcal{E}'.
\end{tikzcd}$$

Then we can extend it to the following diagram.

$$\begin{tikzcd}
\mathcal{C} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathcal{D} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathcal{D}' \arrow[d] \\
0 \arrow[r] & \mathcal{E} \arrow[r] & \mathcal{E}'.
\end{tikzcd}$$

By definition, the left, right, and outer squares are cartesian squares. Thus sequences $\mathcal{C} \to \mathcal{D} \to \mathcal{E}$ and $\mathcal{C} \to \mathcal{D}' \to E'$ are Verdier in $\mathrm{Cat}^{\mathrm{st}}$. By assumption, sequences $F(\mathcal{C}) \to F(\mathcal{D}) \to F(\mathcal{E})$ and $F(\mathcal{C}) \to F(\mathcal{D}') \to F(\mathcal{E}')$ are fiber sequences in $\mathcal{E}$. Then the left and outer squares in the following diagram are cartesian squares.

$$\begin{tikzcd}
F(\mathcal{C}) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & F(\mathcal{D}) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & F(\mathcal{D}') \arrow[d] \\
* \arrow[r] & F(\mathcal{E}) \arrow[r] & F(\mathcal{E}').
\end{tikzcd}$$

Then the right square is also a cartesian square.

Proof A.3 (Proposition 3.18)**.** We show that every Verdier-localizing functor preserves finite products. Let $\mathcal{E}$ be a category with finite limits, and let $F : \mathrm{Cat}^{\mathrm{st}} \to \mathcal{E}$ be a Verdier-localizing functor. The following diagram is a cartesian square in $\mathrm{Cat}^{\mathrm{st}}$.

$$\begin{tikzcd}
\mathcal{C} \times \mathcal{D} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathcal{D} \arrow[d] \\
\mathcal{C} \arrow[r] & 0.
\end{tikzcd}$$

Applying the functor F, we obtain the following cartesian square in $\mathcal{E}$.

$$\begin{tikzcd} F(\mathcal{C}\times\mathcal{D}) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & F(\mathcal{D}) \arrow[d] \\ F(\mathcal{C}) \arrow[r] & *. \end{tikzcd}$$

This implies that

$$F(\mathcal{C}\times\mathcal{D}) \simeq F(\mathcal{C}) \times F(\mathcal{D}).$$

To prove proposition 4.9, we need some preliminaries.

Notation A.4. For every $n \geq 0$, we let $\mathcal{J}_n$ denote the full subcategory of $\mathrm{TwAr}[n]$ spanned by the images of objects $(i \leq j)$ satisfying $j \leq i+1$.

Lemma A.5. Let $\mathcal{C}$ be a category with finite limits, and let $F : \mathrm{TwAr}[n] \to \mathcal{E}$ be a functor. The following conditions are equivalent:

(1) The functor F belongs to $Q_n(\mathcal{C})$.

(2) The functor F is the right Kan extension of its restriction to $\mathcal{J}_n$ along the inclusion $\mathcal{J}_n \subseteq \mathrm{TwAr}[n]$.

Proof. The map

$$e_i : [1] \to [n] : 0 \mapsto i \text{ and } 1 \mapsto i+1$$

in Δ induces an equivalence of categories

$$\begin{aligned} \mathcal{J}_n &\simeq \mathcal{J}_1 \coprod_{\mathcal{J}_0} \mathcal{J}_1 \cdots \coprod_{\mathcal{J}_0} \mathcal{J}_1 \\ &\simeq \mathrm{TwAr}[1] \coprod_{\mathrm{TwAr}[0]} \mathrm{TwAr}[1] \cdots \coprod_{\mathrm{TwAr}[0]} \mathrm{TwAr}[1] \end{aligned}$$

in $\mathrm{Cat}^{\mathrm{lex}}$. Then the right Kan extension along the inclusion $\mathcal{J}_n \subseteq \mathrm{TwAr}[n]$ factors through $n(n-1)/2$-times the right Kan extension along the inclusion $\mathcal{J}_i \subseteq \mathrm{TwAr}[i]$ for every $n \geq 2$. The right Kan extension along the inclusion $\mathcal{J}_2 \subseteq \mathrm{TwAr}[2]$ correspondences the operation of taking the pullback. Then the equivalence of conditions follows from that a functor F belong to $Q_n(\mathcal{C})$ if and only if each square is a cartesian square. □

The next corollary follows from lemma A.5 immediately.

Corollary A.6. Let $\mathcal{C}$ be a category with finite limits. Then the restriction of $\mathrm{Fun}(\mathrm{TwAr}[n], \mathcal{C})$ along the inclusion $\mathcal{J}_n \subseteq \mathrm{TwAr}[n]$ induces an equivalence of categories

$$Q_n(\mathcal{C}) \to \mathrm{Fun}(\mathcal{J}_n, \mathcal{C}).$$

Proof A.7 (Proposition 4.9)**.** (TBA)

References

[Cal+23] Baptiste Calmès et al. *Hermitian K-theory for stable ∞-categories II: Cobordism categories and additivity.* 2023. arXiv: 2009.07224 [math.KT]. URL: https://arxiv.org/abs/2009.07224.

[HLS23] Fabian Hebestreit, Andrea Lachmann, and Wolfgang Steimle. *The localisation theorem for the* K*-theory of stable ∞-categories.* 2023. arXiv: 2205.06104 [math.KT]. URL: https://arxiv.org/abs/2205.06104.

[kerodon] Jacob Lurie. *Kerodon.* https://kerodon.net. 2024.

[KNP24] Achim Krause, Thomas Nikolaus, and Phil Pützstück. *Sheaves on Manifolds*. Nov. 2024. URL: https://www.uni-muenster.de/IVV5WS/WebHop/user/nikolaus/Papers/sheaves-on-manifolds.pdf.

[NS18] Thomas Nikolaus and Peter Scholze. *On topological cyclic homology*. 2018. arXiv: 1707.01799 [math.AT]. URL: https://arxiv.org/abs/1707.01799.