

# THE HIGHER ALGEBRAIC K-THEORY OF STABLE $\infty$ -CATEGORIES

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ABSTRACT. We summarize the higher algebraic K-theory of stable  $\infty$ -categories.

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## 1. INTRODUCTION

This paper is a summary of the workshop on the higher algebraic K-theory held in Kyoto in September 2024.

1.1. **Notation.** From here all categories are assumed to be  $\infty$ -categories. We let

- $\mathbf{An}$  denote the category of small anima.
- $\mathbf{Cat}$  denote the category of small categories.
- $\mathbf{Cat}^{\mathrm{lex}}$  denote the category of small categories which admit finite limits, with left exact functors.
- $\mathbf{Cat}^{\mathrm{st}}$  denote the category of small stable categories with exact functors.
- $\mathbf{Cat}^{\mathrm{perf}}$  denote the category of small idempotent complete stable categories with exact functors.
- $\mathbf{Sp}$  denote the category of spectra.

## 2. PRELIMINARIES

In this section, we recall some basic notions of  $\infty$ -categories.

2.1. **The Grothendieck Group.** In this section, we review the definition of the Grothendieck group for stable categories.

**Definition 2.1.** Let  $(\mathcal{C}, \oplus)$  be a stable category, and let  $X$  and  $Y$  be objects of  $\mathcal{C}$ . We let  $[X]$  denote the connected component of  $X$ . The connected component set  $\pi_0(\mathrm{core} \mathcal{C})$ , together with the operation  $+$  defined by

$$[X] + [Y] := [X \oplus Y]$$

forms an ordinary monoid  $(\pi_0(\text{core } \mathcal{C}), +)$ . We define the *Grothendieck group*  $\mathcal{K}_0(\mathcal{C})$  of  $\mathcal{C}$  as

$$\mathcal{K}_0(\mathcal{C}) := (\pi_0(\text{core } \mathcal{C}), +) / \sim$$

where  $\sim$  is the equivalence relation generated by the following relation:  $[X] = [X'] + [X'']$  whenever  $X' \rightarrow X \rightarrow X''$  is a cofiber sequence in  $\mathcal{C}$ .

**Remark 2.2.** Let  $(\mathcal{C}, \oplus)$  be a stable category. Then the connected component set  $\pi_0(\text{core } \mathcal{C})$  is the set of equivalence classes of objects of  $\mathcal{C}$ . Moreover, the Grothendieck group  $\mathcal{K}_0(\mathcal{C})$  is actually abelian.

- (1) The zero object  $0$  of  $\mathcal{C}$  is a unit object  $[0]$  of  $\mathcal{K}_0(\mathcal{C})$ , since  $X \rightarrow X \rightarrow 0$  is a cofiber sequence in  $\mathcal{C}$  for every object  $X$  of  $\mathcal{C}$ .
- (2) For every object  $X$  of  $\mathcal{C}$ ,  $[\Omega X]$  and  $[\Sigma X]$  are inverse objects of  $[X]$  in  $\mathcal{K}_0(\mathcal{C})$ , since  $\Omega X \rightarrow 0 \rightarrow X$  and  $X \rightarrow 0 \rightarrow X$  are cofiber sequences in  $\mathcal{C}$ .
- (3) For every objects  $X$  and  $Y$  of  $\mathcal{C}$ , we have  $[X] + [Y] = [Y] + [X]$ , since  $X \rightarrow X \oplus Y \rightarrow Y$  and  $Y \rightarrow X \oplus Y \rightarrow X$  are cofiber sequences in  $\mathcal{C}$ .

**Remark 2.3** (Eilenberg swindle). Let  $\mathcal{C}$  be a stable category with countable coproducts. Then the Grothendieck group  $\mathcal{K}_0(\mathcal{C})$  is trivial. Indeed, for every object  $X$  of  $\mathcal{C}$ ,

$$X \rightarrow \bigoplus_{n \geq 0} X \rightarrow \bigoplus_{n \geq 1} X$$

is a cofiber sequence in  $\mathcal{C}$ , and that the last two terms are equivalent. It can be generalized to the algebraic K-theory (see corollary 5.3).

**Remark 2.4.** The construction  $\mathcal{C} \mapsto \mathcal{K}_0(\mathcal{C})$  determine a functor  $\mathcal{K}_0 : \mathbf{hCat}^{\text{st}} \rightarrow \mathbf{Ab}$ .

**2.2. Arrow Categories and Twisted Arrow Categories.** In this section, we recall the notions of (twisted) arrow categories.

**Definition 2.5.** Let  $\mathcal{C}$  be a category. We define the *arrow category*  $\text{Ar}(\mathcal{C})$  of  $\mathcal{C}$  as

$$\text{Ar}(\mathcal{C}) := \text{Fun}([1], \mathcal{C}).$$

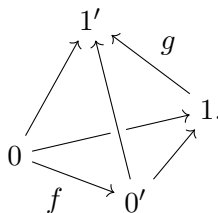
**Definition 2.6.** Let  $\mathcal{C}$  be a category. The *twisted arrow category*  $\text{TwAr}(\mathcal{C})$  of  $\mathcal{C}$  is the simplicial set which is defined by

$$\text{TwAr}(\mathcal{C})_n := \text{Hom}_{\text{Set}}([n] \star [n]^{\text{op}}, \mathcal{C})$$

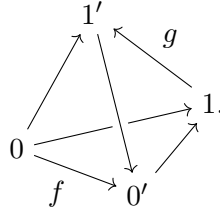
for every  $n \geq 0$ , where  $\star$  is the join operator. By [HA, Proposition 5.2.1.3],  $\text{TwAr}(\mathcal{C})$  is a category.

**Remark 2.7.** Let  $\mathcal{C}$  be a category. Let see the objects and morphisms of  $\text{Ar}(\mathcal{C})$  and  $\text{TwAr}(\mathcal{C})$ .

- The objects of both are morphisms in  $\mathcal{C}$ .
- A morphism from  $f$  to  $g$  in  $\text{Ar}(\mathcal{C})$  is a diagram, depicted as



- A morphism from  $f$  to  $g$  in  $\text{TwAr}(\mathcal{C})$  is a diagram, depicted as



**Notation 2.8.** Let  $\mathcal{C}$  be a stable category. We let  $\text{Seq}(\mathcal{C})$  denote the full subcategory of  $\text{Fun}(\Delta^1 \times \Delta^1, \mathcal{C})$  spanned by the bifiber sequences in  $\mathcal{C}$ .

**Remark 2.9.** Let  $\mathcal{C}$  be a stable category. Then we have an equivalence of categories  $\text{Seq}(\mathcal{C}) \simeq \text{Ar}(\mathcal{C})$ , which implies that the category  $\text{Seq}(\mathcal{C})$  is stable.

**Notation 2.10.** Let  $\mathcal{C}$  be a stable category. We define functors from  $\text{Seq}(\mathcal{C})$  to  $\mathcal{C}$  as follows:

$$\begin{aligned} \text{fib} : \text{Seq}(\mathcal{C}) &\rightarrow \mathcal{C} : (X \rightarrow Y \rightarrow Z) \mapsto X, \\ \text{mid} : \text{Seq}(\mathcal{C}) &\rightarrow \mathcal{C} : (X \rightarrow Y \rightarrow Z) \mapsto Y, \\ \text{cofib} : \text{Seq}(\mathcal{C}) &\rightarrow \mathcal{C} : (X \rightarrow Y \rightarrow Z) \mapsto Z. \end{aligned}$$

### 3. LOCALIZATION PROPERTIES OF FUNCTORS

In this section, we define various functors with localizing properties and recall their relations: additive, Verdier-localizing, Karoubi-localizing, grouplike functors.

We follow the terminology of [Cal+23]. In [Cal+23], these notions are defined for Poincaré-Verdier squares. We use the same terminology for Verdier squares.

**3.1. Verdier Sequences and Squares.** In this section, we recall the notions of (split) Verdier sequences and Karoubi sequences and their relative versions: (split) Verdier squares and Karoubi squares.

**Definition 3.1.** Let  $\mathcal{C} \xrightarrow{f} \mathcal{D} \xrightarrow{p} \mathcal{E}$  be a sequence in  $\text{Cat}^{\text{st}}$ . We will say that the sequence has *vanishing composition* if the composition  $pf$  is a zero object of  $\text{Cat}^{\text{st}}$ .

In this case, the composition  $pf$  is equivalent to the functor  $\mathcal{C} \rightarrow 0 \rightarrow \mathcal{E}$ , since the full subcategory of  $\text{Fun}^{\text{ex}}(\mathcal{C}, \mathcal{E})$  spanned by the zero objects is contractible. That is, there exists the following commutative diagram:

$$\begin{array}{ccc} \mathcal{C} & \xrightarrow{f} & \mathcal{D} \\ \downarrow & & \downarrow p \\ 0 & \longrightarrow & \mathcal{E}. \end{array}$$

We will say that the sequence is a fiber (resp. cofiber) sequence if the above diagram is a Cartesian (resp. coCartesian) diagram.

**Definition 3.2** ([Cal+23] Definition A.1.1). Let  $\mathcal{C} \xrightarrow{f} \mathcal{D} \xrightarrow{p} \mathcal{E}$  be a sequence in  $\text{Cat}^{\text{st}}$  with vanishing composition. We will say that this sequence is *Verdier* if it is a bifiber sequence in  $\text{Cat}^{\text{st}}$ . In this case, we will refer to the functor  $f$  as the *Verdier inclusion* and to the functor  $p$  as the *Verdier projection*.

**Definition 3.3** ([Cal+23] Definition A.2.4). Let  $\mathcal{C} \xrightarrow{f} \mathcal{D} \xrightarrow{p} \mathcal{E}$  be a Verdier sequence. We will say that this sequence is *split* if the functor  $p$  admits both adjoint functors. In this case, we will refer to the functor  $f$  as the *split Verdier inclusion* and to the functor  $p$  as the *split Verdier projection*.

**Definition 3.4** ([Cal+23] Definition A.3.5). Let  $\mathcal{C} \xrightarrow{f} \mathcal{D} \xrightarrow{p} \mathcal{E}$  be a sequence in  $\text{Cat}^{\text{st}}$  with vanishing composition. We will say that this sequence is *Karoubi* if its idempotent completion  $\mathcal{C}^{\natural} \rightarrow \mathcal{D}^{\natural} \rightarrow \mathcal{E}^{\natural}$  is a bifiber sequence in  $\text{Cat}^{\text{perf}}$ . In this case, we will refer to the functor  $f$  as the *Karoubi inclusion* and to the functor  $p$  as the *Karoubi projection*.

We can characterize Verdier inclusions and projections (??). The fiber of exact functor  $f : \mathcal{C} \rightarrow \mathcal{D}$  can be computed by its kernel category  $\ker(f)$ . On the other hand, its cofiber is described by the Verdier quotient.

**Definition 3.5** ([Cal+23] Definition A.1.3). Let  $f : \mathcal{C} \rightarrow \mathcal{D}$  be an exact functor between stable categories. We will say that a morphism in  $\mathcal{D}$  is an *equivalence modulo  $\mathcal{C}$*  in  $\mathcal{D}$  if its fiber (or equivalently, its cofiber) belongs in the essential image of  $f$ .

We define the category  $\mathcal{D}/\mathcal{C}$  as the localization of  $\mathcal{D}$  with respect to the set of equivalences modulo  $\mathcal{C}$  in  $\mathcal{D}$ . We will refer to the category  $\mathcal{D}/\mathcal{C}$  as the *Verdier quotient* of  $\mathcal{D}$  by  $\mathcal{C}$ .

The next proposition implies that the Verdier quotient is universal.

**Proposition 3.6** ([NS18] Theorem.1.3.3). Let  $f : \mathcal{C} \rightarrow \mathcal{D}$  be an exact functor between stable categories. Then

- (1) The Verdier quotient  $\mathcal{D}/\mathcal{C}$  is stable, and the localization functor  $\mathcal{D} \rightarrow \mathcal{D}/\mathcal{C}$  is exact.
- (2) For every stable category  $\mathcal{E}$ , the restriction functor

$$\text{Fun}^{\text{ex}}(\mathcal{D}/\mathcal{C}, \mathcal{E}) \rightarrow \text{Fun}^{\text{ex}}(\mathcal{D}, \mathcal{E})$$

is fully faithful, and its essential image consists of the functors which vanish after composing with  $f$ .

- (3) The sequence  $\mathcal{C} \rightarrow \mathcal{D} \rightarrow \mathcal{D}/\mathcal{C}$  is a cofiber sequence in  $\text{Cat}^{\text{st}}$ .

**Proposition 3.7** ([Cal+23] Corollary A.1.10). Let  $\mathcal{C} \xrightarrow{f} \mathcal{D} \xrightarrow{p} \mathcal{E}$  be a sequence in  $\text{Cat}^{\text{st}}$  with vanishing composition. The following conditions are equivalent:

- (1) The sequence is Verdier.
- (2) The functor  $f$  is fully faithful and its essential image is closed under retracts in  $\mathcal{D}$ , and the functor  $p$  exhibits  $\mathcal{E}$  as the Verdier quotient of  $\mathcal{D}$  by  $\mathcal{C}$ .
- (3) The functor  $f$  exhibits  $\mathcal{C}$  as the kernel of  $p$ , and the functor  $p$  is a localization.

We can characterize split Verdier inclusions and projections.

**Proposition 3.8** ([Cal+23] Corollary A.2.6). Let  $\mathcal{C} \xrightarrow{f} \mathcal{D} \xrightarrow{p} \mathcal{E}$  be a sequence in  $\text{Cat}^{\text{st}}$  with vanishing composition. The following conditions are equivalent:

- (1) The sequence is split Verdier.
- (2) The functor  $p$  admits fully faithful both adjoint functors.
- (3) The functor  $f$  is fully faithful and admits both adjoint functors.

We can characterize Karoubi inclusions and projections.

**Proposition 3.9** ([Cal+23] Corollary A.3.8). Let  $\mathcal{C} \xrightarrow{f} \mathcal{D} \xrightarrow{p} \mathcal{E}$  be a sequence in  $\text{Cat}^{\text{st}}$  with vanishing composition. The following conditions are equivalent:

- (1) The sequence is Karoubi.
- (2) The functor  $f$  is fully faithful and the functor  $p$  has the dense essential image  $p(\mathcal{D}) \subseteq \mathcal{E}$ , and the induced functor  $\mathcal{D} \rightarrow p(\mathcal{D})$  is a Verdier projection.

We can describe Karoubi sequences using Ind-categories.

**Theorem 3.10** (Thomason-Neeman's localization theorem). *Let  $\mathcal{C} \xrightarrow{f} \mathcal{D} \xrightarrow{p} \mathcal{E}$  be a sequence in  $\text{Cat}^{\text{st}}$  with vanishing composition. The following conditions are equivalent:*

- (1) *The sequence is Karoubi.*
- (2) *The sequence  $\text{Ind}(\mathcal{C}) \rightarrow \text{Ind}(\mathcal{D}) \rightarrow \text{Ind}(\mathcal{E})$  is Verdier (of non-necessarily small categories).*

We next introduce the relative versions of these sequences.

**Definition 3.11** ([Cal+23] Definition.1.5.1). A square in  $\text{Cat}^{\text{st}}$  is called

- *Verdier* if it is Cartesian and its both vertical maps are Verdier projections.
- *split Verdier* if it is Cartesian and its both vertical maps are split Verdier projections.
- *Karoubi* if it is Cartesian after idempotent completion and its both vertical maps are Karoubi projections.

**Remark 3.12.** In definition 3.11, the condition that the square is Cartesian can be replaced by the condition that it is coCartesian. (See proof A.1.)

**3.2. Additive and Grouplike Functors.** In this section, we define additive, Verdier-localizing, Karoubi-localizing, and grouplike functors.

**Definition 3.13.** Let  $\mathcal{E}$  be a category with a terminal object, and let  $F : \text{Cat}^{\text{st}} \rightarrow \mathcal{E}$  be a functor. We will say that  $F$  is *reduced* if  $F(0)$  is equivalent to a terminal object of  $\mathcal{E}$ , where  $0$  is a zero object in  $\text{Cat}^{\text{st}}$ .

**Definition 3.14** ([HLS23] Definition 2.1). Let  $\mathcal{E}$  be a category with finite limits, and let  $F : \text{Cat}^{\text{st}} \rightarrow \mathcal{E}$  be a reduced functor. The functor  $F$  is called

- *Verdier-localizing* if it takes every Verdier square in  $\text{Cat}^{\text{st}}$  to a Cartesian square in  $\mathcal{E}$ .
- *additive* if it takes every split Verdier square in  $\text{Cat}^{\text{st}}$  to a Cartesian square in  $\mathcal{E}$ .
- *Karoubi-localizing* if it takes every Karoubi square in  $\text{Cat}^{\text{st}}$  to a Cartesian square in  $\mathcal{E}$ .

Every additive (resp. Verdier-localizing, Karoubi-localizing) functor  $F : \text{Cat}^{\text{st}} \rightarrow \mathcal{E}$  sends split Verdier sequences (resp. Verdier sequences, Karoubi sequences) to fiber sequences. If  $\mathcal{E}$  is stable, the converse holds.

**Proposition 3.15** ([Cal+23] Proposition 1.5.5). Let  $\mathcal{E}$  be a category with finite limits, and let  $F : \text{Cat}^{\text{st}} \rightarrow \mathcal{E}$  be a reduced functor. If  $\mathcal{E}$  is stable, then the following conditions are equivalent:

- (1) The functor  $F$  is additive (resp. Verdier-localizing, Karoubi-localizing).
- (2) The functor  $F$  takes every split Verdier sequence (resp. Verdier sequence, Karoubi sequence) in  $\text{Cat}^{\text{st}}$  to a fiber sequence in  $\mathcal{E}$ . (See proof A.2.)

**Definition 3.16** ([HLS23] Definition 2.1). Let  $\mathcal{E}$  be a category with finite limits, and let  $F : \text{Cat}^{\text{st}} \rightarrow \mathcal{E}$  be an additive functor. We will say that  $F$  is *grouplike* if it lifts to the category  $\text{Mon}_{\mathbb{E}_{\infty}}(\mathcal{E})$  takes values in the full subcategory  $\text{Grp}_{\mathbb{E}_{\infty}}(\mathcal{E})$  of  $\text{Mon}_{\mathbb{E}_{\infty}}(\mathcal{E})$ .

**Example 3.17** ([Cal+23] Example 1.5.10). We give some (counter)examples.

- (1) The core functor  $\text{core} : \text{Cat}^{\text{st}} \rightarrow \text{An}$  is additive, but not grouplike.
- (2) The algebraic K-theory  $\mathcal{K} : \text{Cat}^{\text{st}} \rightarrow \text{An}$  and the algebraic K-theory spectrum  $\mathcal{K}_{\geq 0} : \text{Cat}^{\text{st}} \rightarrow \text{Sp}$  are Verdier-localizing (theorem 6.1) and grouplike (??), but not Karoubi-localizing.
- (3) The non-connective K-theory spectrum  $\mathbb{K} : \text{Cat}^{\text{st}} \rightarrow \text{Sp}$  is Karoubi-localizing. (TBA)
- (4) The functor  $\mathcal{K} \circ (-)^{\natural} : \text{Cat}^{\text{st}} \rightarrow \text{An}$  is Karoubi-localizing (example 8.9), thus is Verdier-localizing (proposition 3.27).

(5) The functor  $\mathcal{K}_{\geq 0} \circ (-)^{\natural} : \text{Cat}^{\text{st}} \rightarrow \text{Sp}$  is additive, but not Verdier-localizing.

**Proposition 3.18** ([HLS23] Observation 2.2). The additive, Verdier-localizing, Karoubi-localizing functors preserve finite products. (See proof A.3.)

**3.3. Additive Grouplike vs. Extension-splitting.** We can characterize additive grouplike functors by extension-splitting functors. We will use lemma 3.20 and proposition 3.21 in the proof of ??.

**Definition 3.19.** Let  $\mathcal{E}$  be a category with finite limits, and let  $F : \text{Cat}^{\text{st}} \rightarrow \mathcal{E}$  be a reduced functor. We will say that  $F$  is *extension-splitting* if, for every stable category  $\mathcal{C}$ , the fiber-cofiber map

$$(\text{fib}, \text{cofib}) : \text{Seq}(\mathcal{C}) \rightarrow \mathcal{C}^2$$

induces an equivalence  $F(\text{Seq}(\mathcal{C})) \rightarrow F(\mathcal{C})^2$ .

We show that additive grouplike functors and extension-splitting functors are equivalent (proposition 3.21).

**Lemma 3.20** ([HLS23] Lemma 2.5). Let  $\mathcal{E}$  be a category with finite limits, and let  $F : \text{Cat}^{\text{st}} \rightarrow \mathcal{E}$  be a reduced and product-preserving functor. The following conditions are equivalent:

- (1) The functor  $F$  is extension-splitting.
- (2) The functor  $F$  sends the source-target projection  $(s, t) : \text{Ar}(\mathcal{C}) \rightarrow \mathcal{C}^2$  for every object  $\mathcal{C}$  of  $\text{Cat}^{\text{st}}$  to an equivalence in  $\mathcal{E}$ .

**Proposition 3.21** ([HLS23] Proposition 2.4). Let  $\mathcal{E}$  be a category with finite limits, and let  $F : \text{Cat}^{\text{st}} \rightarrow \mathcal{E}$  be a reduced functor. The following conditions are equivalent:

- (1) The functor  $F$  is additive grouplike.
- (2) The functor  $F$  is extension-splitting.

**3.4. Additive vs. Verdier-localizing.** In this section, we recall Waldhausen's fibration theorem. We will use this theorem in the proof of the localization theorem (theorem 6.1).

**Notation 3.22.** Let  $\mathcal{D}$  be a stable category, let  $\mathcal{C}$  be a stable full subcategory of  $\mathcal{D}$ , and let  $\mathcal{J}$  be a category. We let  $\text{Fun}^{\mathcal{C}}(\mathcal{J}, \mathcal{D})$  denote the full subcategory of  $\text{Fun}(\mathcal{J}, \mathcal{D})$  spanned by the functors which take every maps in  $\mathcal{J}$  to equivalences modulo  $\mathcal{C}$ .

**Theorem 3.23** (Waldhausen's fibration theorem). *Let  $\mathcal{C} \rightarrow \mathcal{D} \rightarrow \mathcal{E}$  be a Verdier sequence, and let  $F : \text{Cat}^{\text{st}} \rightarrow \text{An}$  be an additive grouplike functor. Then, for every  $n \geq 0$ , the constant map*

$$\text{const} : \mathcal{D} \rightarrow \text{Fun}^{\mathcal{C}}([n], \mathcal{D}) : X \mapsto (X \rightarrow \cdots \rightarrow X)$$

*induces a bifiber sequence of  $\mathbb{E}_{\infty}$ -groups*

$$F(\mathcal{C}) \rightarrow F(\mathcal{D}) \rightarrow |F \text{Fun}^{\mathcal{C}}([-], \mathcal{D})|.$$

We can deduce when an additive functor becomes a Verdier-localizing functor.

**Corollary 3.24** ([HLS23] Corollary 2.10). Let  $F : \text{Cat}^{\text{st}} \rightarrow \text{An}$  be an additive functor. These conditions are equivalent:

- (1) The functor  $F$  is Verdier-localizing.
- (2) For every Verdier sequence  $\mathcal{C} \rightarrow \mathcal{D} \rightarrow \mathcal{E}$ , the canonical map  $|F \text{Fun}^{\mathcal{C}}([-], \mathcal{D})| \rightarrow F(\mathcal{E})$  is an equivalence of anima.

**3.5. Verdier-localizing vs. Karoubi-localizing.** The relationship between Verdier-localizing and Karoubi-localizing functors is as follows.

**Definition 3.25.** Let  $f : \mathcal{C} \rightarrow \mathcal{D}$  be a functor between categories. We will say that  $f$  *has the dense image* if, for every object  $X$  of  $\mathcal{D}$ , there exists an object  $Y$  in the essential image of  $f$  such that  $Y$  is a retract of  $X$ .

**Definition 3.26.** Let  $f : \mathcal{C} \rightarrow \mathcal{D}$  be an exact functor between stable categories. We will say that  $f$  is a *Karoubi equivalence* if it is fully faithful and has the dense image.

**Proposition 3.27** ([HLS23] Observation 2.12). Let  $\mathcal{E}$  be a category with finite limits, and let  $F : \text{Cat}^{\text{st}} \rightarrow \mathcal{E}$  be a reduced functor. The following conditions are equivalent:

- (1) The functor  $F$  is Karoubi-localizing.
- (2) The functor  $F$  is Verdier-localizing and inverts Karoubi equivalences.

We can construct Karoubi-localization functors from Verdier-localizing functors using the idempotent completion.

**Proposition 3.28** ([HLS23] Lemma 2.13). Let  $\mathcal{E}$  be a category with finite limits, and let  $F : \text{Cat}^{\text{st}} \rightarrow \mathcal{E}$  be a Verdier-localizing functor. Suppose that  $F$  takes every Cartesian square in  $\text{Cat}^{\text{st}}$  whose vertical maps are dense inclusions, to a Cartesian square in  $\mathcal{E}$ . Then the functor  $F \circ (-)^{\natural} : \text{Cat}^{\text{st}} \rightarrow \mathcal{E}$  is Karoubi-localizing.

#### 4. THE HIGHER ALGEBRAIC K-THEORY OF STABLE $\infty$ -CATEGORIES

In this section, we recall the Q-construction and define the algebraic K-theory. Moreover we introduce the S-construction, and show that two definitions of algebraic K-theory via two constructions are equivalent.

**4.1. Simplicial Objects.** In this section, we recall the basic notions of simplicial objects.

**Definition 4.1.** The inclusion  $N(\Delta) \subseteq \text{Cat}$  induces an adjunction

$$\text{asscat} : \text{Fun}(N(\Delta)^{\text{op}}, \text{An}) \rightleftarrows \text{Cat} : N^r.$$

We will refer to the left adjoint as the *associated category functor*, and to the right adjoint as the *Rezk nerve*.

**Definition 4.2.** Let  $\mathcal{C}$  be a category. We will refer to a functor

$$X : N(\Delta)^{\text{op}} \rightarrow \mathcal{C}$$

as a *simplicial object* of  $\mathcal{C}$ . We will say that  $X$  is a *simplicial anima* if  $\mathcal{C}$  is  $\text{An}$ .

**Remark 4.3.** Let  $\mathcal{C}$  be a category. For every  $n \geq 0$ , we have an equivalence of anima

$$N_n^r(\mathcal{C}) \simeq \text{Map}_{\text{Cat}}([n], \mathcal{C}) \simeq \text{core Fun}([n], \mathcal{C}).$$

**Notation 4.4.** We let  $[n]$  denote the category the ordinary nerve  $N([n])$  of  $[n]$ , instead of  $\Delta^n$ . On the other hand, we let  $\Delta^n$  denote the functor

$$\Delta^n := \text{Map}_{\text{Cat}}(-, [n]) : N(\Delta)^{\text{op}} \rightarrow \text{An}.$$

Then we have an equivalence of functors  $N^r([n]) \simeq \Delta^n$ .

We define the Segal condition and completeness specifically for simplicial anima, although these concepts are applicable to every category.

**Definition 4.5.** Let  $X : N(\Delta)^{\text{op}} \rightarrow \mathbf{An}$  be a simplicial anima. We will say that  $X$  is *Segal* if the  $n$ -spine inclusion  $\text{sp}^n \subseteq \Delta^n$  induces an equivalence of anima

$$X_n \simeq \text{Map}_{\text{Fun}(N(\Delta)^{\text{op}}, \mathbf{An})}(\Delta^n, X) \rightarrow \text{Map}_{\text{Fun}(N(\Delta)^{\text{op}}, \mathbf{An})}(\text{sp}^n, X) \simeq X_1 \times_{X_0} \cdots \times_{X_0} X_1$$

for every  $n \geq 0$ .

The Segal condition can be interpreted as stating that a Segal simplicial anima has a unique spine lifting up to a choice of contractible spaces.

**Definition 4.6.** Let  $X : N(\Delta)^{\text{op}} \rightarrow \mathbf{An}$  be a Segal simplicial anima. We will say that  $X$  is *complete* if the following diagram is a Cartesian diagram in  $\mathbf{An}$ .

$$\begin{array}{ccc} X_0 & \xrightarrow{\text{diag}} & X_0 \times X_0 \\ \downarrow & \lrcorner & \downarrow (s, s) \\ X_3 & \xrightarrow{(d^{\{0,2\}}, d^{\{1,3\}})} & X_1 \times X_1 \end{array}$$

The completeness condition can be understood as indicating that the higher simplices of a complete Segal simplicial anima correspond to equivalences related to its degenerate edges.

**Proposition 4.7.** The Rezk nerve  $N^r : \mathbf{Cat} \rightarrow \text{Fun}(N(\Delta)^{\text{op}}, \mathbf{An})$  is fully faithful. Moreover, its essential image precisely consists of complete Segal simplicial anima.

#### 4.2. The algebraic K-Theory.

**Definition 4.8** ([HLS23] Definition 3.1). Let  $\mathcal{C}$  be a category with finite limits. For every  $n \geq 0$ , we let  $Q_n(\mathcal{C})$  denote the full subcategory of  $\text{Fun}(\text{TwAr}[n], \mathcal{C})$  spanned by the diagrams which take every square in  $\text{TwAr}[n]$  to a Cartesian square in  $\mathcal{C}$ .

The construction  $n \mapsto Q_n(\mathcal{C})$  determines a functor

$$Q(\mathcal{C}) : N(\Delta)^{\text{op}} \rightarrow \mathbf{Cat}^{\text{lex}}$$

and furthermore, the construction  $\mathcal{C} \mapsto Q(\mathcal{C})$  defines a functor

$$Q : \mathbf{Cat}^{\text{lex}} \rightarrow \text{Fun}(N(\Delta)^{\text{op}}, \mathbf{Cat}^{\text{lex}}).$$

We will refer to this functor as the (*Quillen's*) *Q-construction*.

**Proposition 4.9** ([HLS23] Proposition 3.2). Let  $\mathcal{C}$  be a category with finite limits. Then the simplicial object in  $\mathbf{Cat}^{\text{lex}}$

$$Q(\mathcal{C}) : N(\Delta)^{\text{op}} \rightarrow \mathbf{Cat}^{\text{lex}}$$

is complete Segal. In particular, the simplicial anima

$$\text{core } Q(\mathcal{C}) : N(\Delta)^{\text{op}} \rightarrow \mathbf{An}$$

is complete Segal. (See proof B.4.)

**Remark 4.10.** Corollary B.3 implies that, if  $\mathcal{C}$  is stable, so is  $Q_n(\mathcal{C})$ . Therefore we obtain functors

$$Q(\mathcal{C}) : N(\Delta)^{\text{op}} \rightarrow \mathbf{Cat}^{\text{st}} \quad \text{and} \quad Q : \mathbf{Cat}^{\text{st}} \rightarrow \text{Fun}(N(\Delta)^{\text{op}}, \mathbf{Cat}^{\text{st}}).$$

Moreover, for every stable category  $\mathcal{C}$ , the category  $Q_n(\mathcal{C})$  is a complete Segal simplicial anima, since  $\mathbf{Cat}^{\text{st}}$  is stable under finite limits in  $\mathbf{Cat}$ .



**Definition 4.11** ([HLS23] Definition 3.3). Let  $\mathcal{C}$  be a category with finite limits. Then we define the *category of spans* in  $\mathcal{C}$  as

$$\text{Span}(\mathcal{C}) := \text{asscat core } Q(\mathcal{C}).$$

The construction  $\mathcal{C} \mapsto \text{Span}(\mathcal{C})$  determines a functor

$$\text{Span} : \text{Cat}^{\text{lex}} \rightarrow \text{Cat}.$$

**Definition 4.12** ([HLS23] Definition 3.4). Let  $\mathcal{C}$  be a stable category. Then we define the *algebraic K-anima* (or *algebraic K-theory anima*, or *projective class anima*) as

$$\mathcal{K}(\mathcal{C}) := \Omega |\text{Span}(\mathcal{C})| \simeq \Omega |\text{core } Q(\mathcal{C})|$$

where the base object of the loop space is given by the zero object of  $\text{Span}(\mathcal{C})$ .

The construction  $\mathcal{C} \mapsto \mathcal{K}(\mathcal{C})$  determines a functor

$$\mathcal{K} : \text{Cat}^{\text{st}} \rightarrow \text{An}.$$

We will refer to this functor as the *algebraic K-theory* (or *algebraic K-functor*).

**Definition 4.13.** Let  $\mathcal{C}$  be a stable category. For every  $n \geq 1$ , we define the *n-th K-group* of  $\mathcal{C}$  as the abelian group

$$\mathcal{K}_n(\mathcal{C}) := \pi_n \mathcal{K}(\mathcal{C}).$$

**Proposition 4.14.** Let  $\mathcal{C}$  be a stable category. Then we have an isomorphism of (abelian) groups

$$\pi_0 \mathcal{K}(\mathcal{C}) \simeq \mathcal{K}_0(\mathcal{C}),$$

where  $\mathcal{K}_0(\mathcal{C})$  is the Grothendieck group of  $\mathcal{C}$ .

**4.3. Waldhausen's S-Construction.** In this section, we construct the algebraic K-theory using Waldhausen's S-construction.

**Definition 4.15.** Let  $\mathcal{C}$  be a stable category. An  $[n]$ -gapped object of  $\mathcal{C}$  is a functor  $F : \text{Ar}[n] \rightarrow \mathcal{C}$  which satisfies the following properties:

- (1) For every  $0 \leq i \leq n$ ,  $F(i, i)$  is a zero object of  $\mathcal{C}$ .
- (2) For every  $i \leq j \leq k$ , the following diagram is a (co)Cartesian diagram in  $\mathcal{C}$ .

$$\begin{array}{ccc} F(i, j) & \longrightarrow & F(i, k) \\ \downarrow & \lrcorner & \downarrow \\ 0 \simeq F(j, j) & \longrightarrow & F(j, k) \end{array}$$

We let  $S_n(\mathcal{C})$  denote the full subcategory of  $\text{Fun}(\text{Ar}[n], \mathcal{C})$  spanned by the  $[n]$ -gapped objects of  $\mathcal{C}$ .

**Remark 4.16.** Let  $\mathcal{C}$  be a stable category. We can describe the low-dimensional simplices of  $S_n(\mathcal{C})$ .

- The category  $S_0(\mathcal{C})$  is the full subcategory of  $\mathcal{C}$  spanned by the zero objects of  $\mathcal{C}$ . Thus  $S_0(\mathcal{C})$  is contractible.
- The category  $S_1(\mathcal{C})$  is equivalent to  $\mathcal{C}$ , since every object of  $S_1(\mathcal{C})$  is of the form  $0 \rightarrow X \rightarrow 0$ , where  $X$  is an object of  $\mathcal{C}$ .
- The category  $S_2(\mathcal{C})$  is equivalent to the arrow category  $\text{Ar}(\mathcal{C})$  of  $\mathcal{C}$ , since every object of  $S_2(\mathcal{C})$  is of the form  $0 \rightarrow X' \rightarrow X \rightarrow X'' \rightarrow 0$ , where  $X' \rightarrow X \rightarrow X''$  is a cofiber sequence in  $\mathcal{C}$ .

**Remark 4.17.** Let  $\mathcal{C}$  be a stable category. We have an equivalence of categories

$$S_n(\mathcal{C}) \simeq \text{Fun}([n-1], \mathcal{C})$$

for every  $n \geq 0$ . Thus, if  $\mathcal{C}$  is stable, then  $S_n(\mathcal{C})$  is stable.

**Definition 4.18.** The construction  $n \mapsto S_n(\mathcal{C})$  determines a functor

$$S(\mathcal{C}) : \mathbf{N}(\Delta)^{\text{op}} \rightarrow \text{Cat}^{\text{st}}$$

and furthermore, the construction  $\mathcal{C} \mapsto S(\mathcal{C})$  determines a functor

$$S : \text{Cat}^{\text{st}} \rightarrow \text{Fun}(\mathbf{N}(\Delta)^{\text{op}}, \text{Cat}^{\text{st}}).$$

We will refer to this functor as *(Waldhausen's) S-construction*.

**Definition 4.19.** Let  $\mathcal{C}$  be a stable category. Then we define the *algebraic K-anima* as

$$\mathcal{K}_S(\mathcal{C}) := \Omega | \text{core } S(\mathcal{C}) |.$$

The construction  $\mathcal{C} \mapsto \mathcal{K}_S(\mathcal{C})$  determines a functor

$$\mathcal{K}_S : \text{Cat}^{\text{st}} \rightarrow \text{An}.$$

We will refer to this functor as the *algebraic K-theory*.

**Remark 4.20.** Let  $\mathcal{C}$  be a stable category. Then the anima  $| \text{core } S(\mathcal{C}) |$  admits a canonical base point given by a map

$$0 \simeq \text{core } S_0(\mathcal{C}) \rightarrow | \text{core } S(\mathcal{C}) |.$$

Moreover,  $| \text{core } S(\mathcal{C}) |$  is connected, since the canonical map

$$0 \simeq \pi_0 \text{core } S_0(\mathcal{C}) \rightarrow \pi_0 | \text{core } S(\mathcal{C}) |$$

is surjective.

**Proposition 4.21.** The two definitions of algebraic K-anima (?? and definition 4.19) induce an equivalence of anima

$$\mathcal{K}(\mathcal{C}) \simeq \mathcal{K}_S(\mathcal{C})$$

for every stable category  $\mathcal{C}$ .

## 5. THE ADDITIVITY THEOREM

The goal of this section is to prove the additivity theorem.

**Theorem 5.1** ([HLS23] Theorem.4.1: The Additivity Theorem). *Let  $\mathcal{C}$  be a stable category. Then the source-target projection induces an equivalence of anima*

$$| \text{Span}(s, t) | : | \text{Span}(\text{Ar}(\mathcal{C})) | \rightarrow | \text{Span}(\mathcal{C}) |^2.$$

Before proving theorem 5.1, we show some corollaries.

**Corollary 5.2** ([HLS23] Corollary 4.2). The algebraic K-theory  $\mathcal{K} : \text{Cat}^{\text{st}} \rightarrow \text{An}$  is additive grouplike.

*Proof.* By proposition 3.21, it suffices to show that  $\mathcal{K}$  is a reduced functor and it is extension splitting. We have

$$\mathcal{K}(0) \simeq \Omega | \text{Span}(0) | \simeq \Omega | \text{core } Q(0) | \simeq 0.$$

By ??, the algebraic K-theory  $\mathcal{K}$  preserves finite products. Then by ??, it is enough to show that  $\mathcal{K}$  sends the source-target projection  $(s, t) : \text{Ar}(\mathcal{C}) \rightarrow \mathcal{C}$  to an equivalence of anima. That is, there is an equivalence  $\mathcal{K}(s, t) : \mathcal{K}(\text{Ar}(\mathcal{C})) \rightarrow \mathcal{K}(\mathcal{C})^2$ .

By theorem 5.1, we have an equivalence  $|\mathrm{Span}(\mathrm{Ar}(\mathcal{C}))| \rightarrow |\mathrm{Span}(\mathcal{C})|^2$ . Since the loop functor  $\Omega$  preserves limits, we obtain an equivalence

$$\mathcal{K}(s, t) : \mathcal{K}(\mathrm{Ar}(\mathcal{C})) \rightarrow \mathcal{K}(\mathcal{C})^2.$$

□

**Corollary 5.3** (Eilenberg swindle). Let  $\mathcal{C}$  be a stable category with countable coproducts. Then the algebraic K-theory anima vanishes.

$$\mathcal{K}(\mathcal{C}) \simeq 0.$$

*Proof.* We first prove the following proposition: Let  $\mathcal{C}$  and  $\mathcal{D}$  be stable categories, and let  $F' \rightarrow F \rightarrow F''$  be a cofiber sequence of exact functors from  $\mathcal{C}$  to  $\mathcal{D}$ . Then we have

$$\mathcal{K}(F) = \mathcal{K}(F') + \mathcal{K}(F'').$$

Consider the following functors

$$\mathrm{mid}, \mathrm{fib} + \mathrm{cofib} : \mathrm{Seq}(\mathrm{Fun}^{\mathrm{ex}}(\mathcal{C}, \mathcal{D})) \rightarrow \mathrm{Fun}^{\mathrm{ex}}(\mathcal{C}, \mathcal{D}).$$

Since  $\mathcal{K}$  is extension splitting (see ??), by Waldhausen's Additivity Theorem (reference to be added), we have an equivalence

$$\mathcal{K}(\mathrm{mid}) \simeq \mathcal{K}(\mathrm{fib}) + \mathcal{K}(\mathrm{cofib}).$$

We obtain the assertion by applying to this proposition to the cofiber sequence  $F' \rightarrow F \rightarrow F''$ .

Consider the functor

$$F : \mathcal{C} \rightarrow \mathcal{C} : X \mapsto \bigoplus_{n \in \mathbb{N}} X_n.$$

Then there exists a cofiber sequence

$$\mathrm{id}_{\mathcal{C}} \rightarrow F \rightarrow F$$

of exact functors on  $\mathcal{C}$ . By the above proposition, we have an equivalence

$$\mathcal{K}(F) \simeq \mathcal{K}(\mathrm{id}_{\mathcal{C}}) + \mathcal{K}(F).$$

Thus we have  $\mathcal{K}(\mathrm{id}_{\mathcal{C}}) \simeq 0$  and  $\mathcal{K}(\mathcal{C}) \simeq 0$ . □

The proof of theorem 5.1 follows from the next two propositions.

**Proposition 5.4** ([HLS23] Proposition 4.3). Let  $\mathcal{C}$  be a stable category. Then there are canonical equivalences of categories

$$\mathrm{Span}(\mathcal{C}) \rightarrow \mathrm{Span}(\mathcal{C}^{\mathrm{op}}) \quad \text{and} \quad \mathrm{Span}(\mathrm{Ar}(\mathcal{C})) \simeq \mathrm{Span}(\mathrm{TwAr}(\mathcal{C})).$$

Moreover, they fit together into a natural commutative diagram

$$\begin{array}{ccc} \mathrm{Span}(\mathrm{Ar}(\mathcal{C})) & \xrightarrow{\simeq} & \mathrm{Span}(\mathrm{TwAr}(\mathcal{C})) \\ (s, t) \downarrow & & \downarrow (s, t) \\ \mathrm{Span}(\mathcal{C}) \times \mathrm{Span}(\mathcal{C}) & \xrightarrow[\simeq]{} & \mathrm{Span}(\mathcal{C}) \times \mathrm{Span}(\mathcal{C}^{\mathrm{op}}). \end{array}$$

**Proposition 5.5** ([HLS23] Proposition 4.4). Let  $\mathcal{C}$  be a stable category. Then the source-target projection

$$(s, t) : \mathrm{Span}(\mathrm{TwAr}(\mathcal{C})) \rightarrow \mathrm{Span}(\mathcal{C}) \times \mathrm{Span}(\mathcal{C}^{\mathrm{op}})$$

is cofinal.

## 6. THE LOCALIZATION THEOREM

The goal of this section is to prove the localization theorem.

**Theorem 6.1** ([HLS23] Theorem.6.1: The Localization Theorem). *The algebraic K-theory  $\mathcal{K} : \text{Cat}^{\text{st}} \rightarrow \text{An}$  and the algebraic K-theory spectrum  $\mathcal{K}_{\geq 0} : \text{Cat}^{\text{st}} \rightarrow \text{Sp}$  are Verdier-localizing.*

By the corollary of theorem 3.23, an additive grouplike functor  $F : \text{Cat}^{\text{st}} \rightarrow \text{An}$  is Verdier-localizing if and only if it satisfies the following condition:

- (\*) For every Verdier sequence  $\mathcal{C} \rightarrow \mathcal{D} \rightarrow \mathcal{E}$ , the canonical map  $|F \text{Fun}^{\mathcal{C}}([-], \mathcal{D})| \rightarrow F(\mathcal{E})$  is an equivalence of anima.

The next proposition implies that it is enough to prove that the core functor satisfies (\*).

**Proposition 6.2** ([HLS23] Proposition 6.2). Let  $F : \text{Cat}^{\text{st}} \rightarrow \text{An}$  be an additive functor. If  $F$  satisfies (\*), then  $|FQ(-)|$  and  $\Omega|FQ(-)|$  also satisfy (\*).

**Remark 6.3.** If the core functor satisfies (\*), then the K-theory functor also satisfies (\*). Indeed, we can write

$$\mathcal{K}(-) \simeq \Omega|\text{Span}(-)| \simeq \Omega|\text{asscat core } Q(-)| \simeq \Omega|\text{core } Q(-)|.$$

To prove the proposition, we need the following lemma.

**Lemma 6.4** ([HLS23] Lemma 6.4). Verdier sequences are stable under applying the functor

$$\text{Fun}(\mathcal{J}, -) : \text{Cat}^{\text{st}} \rightarrow \text{Cat}^{\text{st}}$$

for every finite poset  $\mathcal{J}$ .

From the above discussion, we need to show that the core functor satisfies (\*). Let  $\mathcal{D}$  be a stable category, and let  $\mathcal{C}$  be a stable subcategory of  $\mathcal{D}$ . We let  $\mathcal{D}_{\mathcal{C}}$  denote the full subcategory of  $\mathcal{D}$  spanned by the equivalences modulo  $\mathcal{C}$  in  $\mathcal{D}$ . Then we obtain

$$\text{core Fun}^{\mathcal{C}}([-], \mathcal{D}) \simeq \text{core Fun}([-], \mathcal{D}_{\mathcal{C}}) \simeq \text{Map}_{\text{Cat}}([-], \mathcal{D}_{\mathcal{C}}) \simeq N^r(\mathcal{D}_{\mathcal{C}}).$$

Since the canonical map  $|N^r(\mathcal{D}_{\mathcal{C}})| \rightarrow |\mathcal{D}_{\mathcal{C}}|$  is an equivalence of anima, we have an equivalence

$$|\text{core Fun}^{\mathcal{C}}([-], \mathcal{D})| \simeq |\mathcal{D}_{\mathcal{C}}|.$$

Thus it suffices to show the following proposition.

**Proposition 6.5** ([HLS23] Proposition 6.6). Let  $\mathcal{D}$  be a stable category, and let  $\mathcal{C}$  be a stable subcategory of  $\mathcal{D}$ . We let  $\mathcal{D}_{\mathcal{C}}$  denote the full subcategory of  $\mathcal{D}$  spanned by the equivalences modulo  $\mathcal{C}$  in  $\mathcal{D}$ . Then the map

$$|\mathcal{D}_{\mathcal{C}}| \rightarrow \text{core}(\mathcal{D}/\mathcal{C})$$

is faithful. If the inclusion  $\mathcal{C} \subseteq \mathcal{D}$  is a Verdier inclusion, then this map is an equivalence.

This proposition is a special case of the following proposition.

**Proposition 6.6** ([HLS23] Proposition 6.8). Let  $\mathcal{C}$  be a category, and let  $S$  be a subcategory of  $\mathcal{C}$ . If  $S$  is closed under 2-out-of-3 and pushouts in  $\mathcal{C}$ , then a map

$$|S| = S[S^{-1}] \rightarrow \mathcal{C}$$

is faithful. Moreover, the following conditions are equivalent:

- (1) The inclusion  $|S| \subseteq \text{core } \mathcal{C}[S^{-1}]$  is fully faithful.
- (2) The category  $S$  is closed under 2-out-of-6 in  $\mathcal{C}$ .
- (3) A morphism in  $\mathcal{D}$  belongs to  $S$  if and only if its source and target are in  $S$  and it is invertible in  $\mathcal{C}[S^{-1}]$ .

## 7. THE UNIVERSALITY THEOREM

The goal of this section is to prove the universality theorem.

**Theorem 7.1** ([HLS23] Theorem.5.1: The Universality Theorem). *The algebraic K-theory  $\mathcal{K} : \text{Cat}^{\text{st}} \rightarrow \text{An}$  is an initial additive grouplike functor under the core functor  $\text{core} : \text{Cat}^{\text{st}} \rightarrow \text{An}$ . That is, the natural map  $\tau : \text{core} \Rightarrow \mathcal{K}$  is an initial object in the category  $\text{Fun}(\text{Cat}^{\text{st}}, \text{An})_{\text{core}/}^{\text{add,grp}}$ .*

**Notation 7.2.** Let  $F : \text{Cat}^{\text{st}} \rightarrow \text{An}$  be a reduced functor. We denote a functor

$$GF(-) := \Omega|FQ(-)| : \text{Cat}^{\text{st}} \rightarrow \text{An}.$$

For example, the functor  $G \text{ core}$  is equivalent to the algebraic K-theory  $\mathcal{K}$ .

*proof of theorem 7.1.* We want to show that the natural transformation  $\tau : \text{core} \Rightarrow \mathcal{K}$  induces an equivalence

$$\tau^* : \text{Nat}(\mathcal{K}, F) \rightarrow \text{Nat}(\text{core}, F)$$

for every additive grouplike functor  $F : \text{Cat}^{\text{st}} \rightarrow \text{An}$ .

Now consider the following diagram

$$\begin{array}{ccc}
 \text{Nat}(G \text{ core}, F) & \xrightarrow{(\eta_{\text{core}})^*} & \text{Nat}(\text{core}, F) \\
 \downarrow G & & \downarrow G \\
 \text{Nat}(GG \text{ core}, GF) & \xrightarrow{(G\eta_{\text{core}})^*} & \text{Nat}(G \text{ core}, GF) \\
 \downarrow (\eta_{G \text{ core}})^* & & \downarrow (\eta_{\text{core}})^* \\
 \text{Nat}(G \text{ core}, GF) & \xrightarrow{(\eta_{\text{core}})^*} & \text{Nat}(\text{core}, GF)
 \end{array}$$

$(\eta_F)^*$  (curved red arrow from  $\text{Nat}(GG \text{ core}, GF)$  to  $\text{Nat}(G \text{ core}, GF)$ )      $(\eta_F)^*$  (curved red arrow from  $\text{Nat}(G \text{ core}, GF)$  to  $\text{Nat}(\text{core}, GF)$ )

where the upper square commutes since  $G$  is a functor, and the other three parts commute since  $\eta$  is natural. Suppose the red-colored maps are equivalent, then we can show that the upper horizontal map  $(\eta_{\text{core}})^*$  is an equivalence. If we apply this to the case  $F \simeq \text{core}$ , then we obtain the desired result. This assumption follows from the next two propositions.  $\square$

The next proposition implies that  $(\eta_F)_*$  and  $(G\eta_{\text{core}})^*$  are equivalences.

**Proposition 7.3** ([HLS23] Proposition 5.2). Let  $F : \text{Cat}^{\text{st}} \rightarrow \text{An}$  be an additive grouplike functor. Then the natural transformation

$$\eta_F : F \Rightarrow GF$$

is an equivalence.

The next proposition implies that  $(\eta_{G \text{ core}})$  is an equivalence.

**Proposition 7.4** ([HLS23] Proposition 5.3). Let  $F : \text{Cat}^{\text{st}} \rightarrow \text{An}$  be an additive functor. Then the two natural transformations

$$\eta_{GF}, G\eta_F : GF \Rightarrow GGF$$

differ by an automorphism of the target. That is, the following diagram commutes

$$\begin{array}{ccc}
 GF & \xrightarrow{G\eta_F} & GGF \\
 \searrow \eta_{GF} & & \downarrow \simeq \\
 & & GGF
 \end{array}$$

## 8. THE COFINALITY THEOREM

The goal of this section is to prove the cofinality theorem.

**Theorem 8.1** ([HLS23] Theorem.7.1: The Cofinality Theorem). *Let  $f : \mathcal{C} \rightarrow \mathcal{D}$  be a functor between stable categories. If  $f$  is a dense inclusion, then it induces a fiber sequence*

$$\mathcal{K}(\mathcal{C}) \rightarrow \mathcal{K}(\mathcal{D}) \rightarrow \mathcal{K}_0(\mathcal{D})/\mathcal{K}_0(\mathcal{C}).$$

*In particular, maps of abelian groups*

$$\mathcal{K}_i(\mathcal{C}) \rightarrow \mathcal{K}_i(\mathcal{D})$$

*are isomorphisms for every  $i \geq 1$ , and there exists a short exact sequence*

$$0 \rightarrow \mathcal{K}_0(\mathcal{C}) \rightarrow \mathcal{K}_0(\mathcal{D}) \rightarrow \mathcal{K}_0(\mathcal{D})/\mathcal{K}_0(\mathcal{C}) \rightarrow 0.$$

Theorem 8.1 holds for a broader class of additive Karoubian functors (theorem 8.7).

**Definition 8.2** ([HLS23] Definition 7.4). Let  $f : X \rightarrow Y$  be a map of  $\mathbb{E}_\infty$ -monoids in  $\mathbf{An}$ . We will say that  $f$  is *cofinal* if it satisfies the following conditions:

- (1) The map  $\pi_0(f) : \pi_0(X) \rightarrow \pi_0(Y)$  is an inclusion.
- (2) For every element  $x$  in  $\pi_0(X)$ , there exists an element  $x'$  in  $\pi_0(X)$  such that  $x + x'$  is in  $\pi_0(Y)$ .

We will say that a cofinal map is *dense* if it satisfies the following condition:

- (3) The sequence of  $\mathbb{E}_\infty$ -monoids  $0 \rightarrow \pi_0(X) \rightarrow \pi_0(Y) \rightarrow \pi_0(Y)/\pi_0(X) \rightarrow 0$  is exact.

Or equivalently,

- (3') An element  $y$  in  $\pi_0(Y)$  belongs to  $\pi_0(X)$  if there exists an element  $x$  in  $\pi_0(X)$  such that  $x + y$  is in  $\pi_0(X)$ .

**Definition 8.3** ([HLS23] Definition 7.6). Let  $F : \mathbf{Cat}^{\text{st}} \rightarrow \mathbf{An}$  be an additive functor. We will say that  $F$  is *Karoubian* if it satisfies the following conditions:

- (1) The functor  $F$  takes every dense inclusion between stable categories to a dense map of  $\mathbb{E}_\infty$ -monoids.
- (2) The functor  $F$  preserves every Cartesian square in  $\mathbf{Cat}^{\text{st}}$  whose vertical maps are dense.

**Definition 8.4.** Let  $F : \mathbf{Cat}^{\text{st}} \rightarrow \mathbf{An}$  be an additive Karoubian functor. We will refer to the functor

$$F^{\text{grp}} := \Omega|FQ - | : \mathbf{Cat}^{\text{st}} \rightarrow \mathbf{An}$$

as the *group completion* of  $F$ .

**Example 8.5.** The core functor  $\text{core} : \mathbf{Cat}^{\text{st}} \rightarrow \mathbf{An}$  is Karoubian.

(1): Let  $\mathcal{C} \hookrightarrow \mathcal{D}$  be a dense inclusion of stable categories. Then  $\text{core } \mathcal{C} \rightarrow \text{core } \mathcal{D}$  is cofinal. Let  $d$  be an object of  $\pi_0(\text{core } \mathcal{D})$ , and let  $c$  be an object of  $\pi_0(\text{core } \mathcal{C})$  such that  $c + d$  is an object of  $\pi_0(\text{core } \mathcal{C})$ . Then we have an equivalence  $d \simeq \text{fib}(c \oplus d \rightarrow c)$ . Since  $\mathcal{C}$  is closed under fiber sequences,  $d$  is an object of  $\pi_0(\text{core } \mathcal{C})$ . That is,  $\text{core } \mathcal{C} \rightarrow \text{core } \mathcal{D}$  is dense.

(2): The core functor preserves limits, since it is a right adjoint functor.

**Example 8.6.** The group completion

$$(\text{core})^{\text{grp}} : \mathbf{Cat}^{\text{st}} \rightarrow \mathbf{An}$$

of  $\text{core} : \mathbf{Cat}^{\text{st}} \rightarrow \mathbf{An}$  is equivalent to the algebraic K-theory  $\mathcal{K} : \mathbf{Cat}^{\text{st}} \rightarrow \mathbf{An}$ .

**Theorem 8.7** ([HLS23] Theorem 7.7). *Let  $F : \text{Cat}^{\text{st}} \rightarrow \text{An}$  be an additive Karoubian functor. For every dense inclusion  $\mathcal{C} \subseteq \mathcal{D}$  of stable categories, the canonical map of  $\mathbb{E}_\infty$ -monoids*

$$F(\mathcal{D})/F(\mathcal{C}) \rightarrow F^{\text{grp}}(\mathcal{D})/F^{\text{grp}}(\mathcal{C})$$

*is an equivalence. Hence maps of abelian groups*

$$\pi_i F^{\text{grp}}(\mathcal{C}) \rightarrow \pi_i F^{\text{grp}}(\mathcal{D})$$

*are isomorphisms for every  $i \geq 1$ , and there exists a short exact sequence*

$$0 \rightarrow \pi_0 F^{\text{grp}}(\mathcal{C}) \rightarrow \pi_0 F^{\text{grp}}(\mathcal{D}) \rightarrow \pi_0 F^{\text{grp}}(\mathcal{D})/\pi_0 F^{\text{grp}}(\mathcal{C}) \rightarrow 0.$$

**Corollary 8.8** ([HLS23] Corollary 7.8). *Let  $F : \text{Cat}^{\text{st}} \rightarrow \text{An}$  be an additive Karoubian functor. Then the group completion*

$$F^{\text{grp}} : \text{Cat}^{\text{st}} \rightarrow \text{An}$$

*of  $F$  is also additive Karoubian. If  $F^{\text{grp}}$  is Verdier-localizing, then the functor*

$$F^{\text{grp}} \circ (-)^\natural : \text{Cat}^{\text{st}} \rightarrow \text{An}$$

*is Karoubi-localizing.*

**Example 8.9.** By example 8.6 and corollary 8.8, the algebraic K-theory  $\mathcal{K} : \text{Cat}^{\text{st}} \rightarrow \text{An}$  is additive Karoubian. By theorem 6.1,  $\mathcal{K}$  is Verdier-localizing. Then the functor

$$\mathcal{K} \circ (-)^\natural : \text{Cat}^{\text{st}} \rightarrow \text{An}$$

is Karoubi-localizing.

## 9. THE ALGEBRAIC K-THEORY SPECTRUM

We can define an algebraic K-theory spectrum  $\mathcal{K}_{\geq 0} : \text{Cat}^{\text{st}} \rightarrow \text{Sp}$ .

**Definition 9.1.** ?? implies that the K-theory functor lifts to a functor

$$\mathcal{K} : \text{Cat}^{\text{st}} \rightarrow \text{Grp}_{\mathbb{E}_\infty}(\text{An}).$$

Since we have the equivalence  $\text{Grp}_{\mathbb{E}_\infty}(\text{An}) \simeq \text{Sp}_{\geq 0}$ , we obtain a functor

$$\mathcal{K}_{\geq 0} : \text{Cat}^{\text{st}} \rightarrow \text{Sp}_{\geq 0} \subseteq \text{Sp}.$$

We will refer to this functor as the *algebraic K-theory spectrum*.

**Remark 9.2.** There is the equivalence  $\Omega^\infty : \text{Sp}_{\geq 0} \rightleftarrows \text{Grp}_{\mathbb{E}_\infty}(\text{An}) : \Sigma^\infty$ . We can recover the algebraic K-functor from algebraic K-theory spectrum as

$$\mathcal{K} \simeq \Omega^\infty \mathcal{K}_{\geq 0} : \text{Cat}^{\text{st}} \rightarrow \text{Sp}_{\geq 0} \simeq \text{Grp}_{\mathbb{E}_\infty}(\text{An})$$

since  $\Sigma^\infty$  is fully faithful.

## APPENDIX A. PROOFS IN SECTION 3

**Proof A.1** (Remark 3.12). We show that every Verdier sequence square is a coCartesian diagram in  $\text{Cat}^{\text{st}}$ . Consider the following Verdier sequence.

$$\begin{array}{ccc} \mathcal{D} & \longrightarrow & \mathcal{D}' \\ \downarrow \lrcorner & & \downarrow \\ \mathcal{E} & \longrightarrow & \mathcal{E}', \end{array}$$

where the vertical maps are Verdier projections. Then we can extend to the following diagram.

$$\begin{array}{ccccc} \mathcal{C} & \longrightarrow & \mathcal{D} & \longrightarrow & \mathcal{D}' \\ \downarrow & & \downarrow \lrcorner & & \downarrow \\ 0 & \longrightarrow & \mathcal{E} & \longrightarrow & \mathcal{E}' \end{array}$$

By definition, the left and outer squares are biCartesian squares. Then the right square is also a biCartesian square.

**Proof A.2** (Proposition 3.15). (1)  $\Rightarrow$  (2) is obvious. (2)  $\Rightarrow$  (1): Consider the following Verdier sequence.

$$\begin{array}{ccc} \mathcal{D} & \longrightarrow & \mathcal{D}' \\ \downarrow \lrcorner & & \downarrow \\ \mathcal{E} & \longrightarrow & \mathcal{E}', \end{array}$$

Then we can extend to the following diagram.

$$\begin{array}{ccccc} \mathcal{C} & \longrightarrow & \mathcal{D} & \longrightarrow & \mathcal{D}' \\ \downarrow \lrcorner & & \downarrow \lrcorner & & \downarrow \\ 0 & \longrightarrow & \mathcal{E} & \longrightarrow & \mathcal{E}' \end{array}$$

By definition, the left, right, outer squares are Cartesian squares. Thus the sequences  $\mathcal{C} \rightarrow \mathcal{D} \rightarrow \mathcal{E}$  and  $\mathcal{C} \rightarrow \mathcal{D}' \rightarrow \mathcal{E}'$  are Verdier sequences in  $\text{Cat}^{\text{st}}$ . By assumption, the sequences  $F(\mathcal{C}) \rightarrow F(\mathcal{D}) \rightarrow F(\mathcal{E})$  and  $F(\mathcal{C}) \rightarrow F(\mathcal{D}') \rightarrow F(\mathcal{E}')$  are fiber sequences in  $\mathcal{E}$ . Then the left and outer squares in the following diagram are Cartesian squares.

$$\begin{array}{ccccc} F(\mathcal{C}) & \longrightarrow & F(\mathcal{D}) & \longrightarrow & F(\mathcal{D}') \\ \downarrow \lrcorner & & \downarrow \lrcorner & & \downarrow \\ * & \longrightarrow & F(\mathcal{E}) & \longrightarrow & F(\mathcal{E}') \end{array}$$

Then the right square is also a Cartesian square.

**Proof A.3** (Proposition 3.18). We show that every Verdier-localizing functor preserves finite products. Let  $\mathcal{E}$  be a category with finite limits, and let  $F : \text{Cat}^{\text{st}} \rightarrow \mathcal{E}$  be a Verdier-localizing functor. The following diagram is a Cartesian square in  $\text{Cat}^{\text{st}}$ .

$$\begin{array}{ccc} \mathcal{C} \times \mathcal{D} & \longrightarrow & \mathcal{D} \\ \downarrow \lrcorner & & \downarrow \\ \mathcal{C} & \longrightarrow & 0. \end{array}$$



Applying the functor  $F$ , we obtain the following Cartesian square in  $\mathcal{E}$ .

$$\begin{array}{ccc} F(\mathcal{C} \times \mathcal{D}) & \xrightarrow{\quad} & F(\mathcal{D}) \\ \downarrow & \lrcorner & \downarrow \\ F(\mathcal{C}) & \longrightarrow & * \end{array}$$

This implies that

$$F(\mathcal{C} \times \mathcal{D}) \simeq F(\mathcal{C}) \times F(\mathcal{D}).$$

#### APPENDIX B. PROOFS IN SECTION 4

To prove proposition 4.9, we need some preliminaries.

**Notation B.1.** For every  $n \geq 0$ , we let  $\mathcal{J}_n$  denote the full subcategory of  $\mathrm{TwAr}[n]$  spanned by the images of objects  $(i \leq j)$  in  $[n]$  satisfying  $j \leq i + 1$ .

**Lemma B.2.** Let  $\mathcal{C}$  be a category with finite limits, and let  $F : \mathrm{TwAr}[n] \rightarrow \mathcal{E}$  be a functor. The following conditions are equivalent:

- (1) The functor  $F$  belongs to  $Q_n(\mathcal{C})$ .
- (2) The functor  $F$  is the right Kan extension of its restriction to  $\mathcal{J}_n$  along the inclusion  $\mathcal{J}_n \subseteq \mathrm{TwAr}[n]$ .

*Proof.* The map

$$e_i : [1] \rightarrow [n] : 0 \mapsto i \text{ and } 1 \mapsto i + 1$$

in  $\Delta$  induces an equivalence of categories

$$\mathcal{J}_n \simeq \mathcal{J}_1 \coprod_{\mathcal{J}_0} \mathcal{J}_1 \cdots \coprod_{\mathcal{J}_0} \mathcal{J}_1 \simeq \mathrm{TwAr}[1] \coprod_{\mathrm{TwAr}[0]} \mathrm{TwAr}[1] \cdots \coprod_{\mathrm{TwAr}[0]} \mathrm{TwAr}[1]$$

in  $\mathrm{Cat}^{\mathrm{lex}}$ . Then the right Kan extension along the inclusion  $\mathcal{J}_n \subseteq \mathrm{TwAr}[n]$  factors through  $n(n-1)/2$ -times the right Kan extension along the inclusion  $\mathcal{J}_i \subseteq \mathrm{TwAr}[i]$  for  $2 \leq n$ . The right Kan extension along the inclusion  $\mathcal{J}_2 \subseteq \mathrm{TwAr}[2]$  correspondences the operation of taking the pullback. Then the equivalence of conditions follows from that a functor  $F$  belong to  $Q_n(\mathcal{C})$  if and only if each square is a Cartesian square.  $\square$

The next corollary follows from lemma B.2 immediately.

**Corollary B.3.** Let  $\mathcal{C}$  be a category with finite limits. Then the restriction of  $\mathrm{Fun}(\mathrm{TwAr}[n], \mathcal{C})$  along the inclusion  $\mathcal{J}_n \subseteq \mathrm{TwAr}[n]$  induces an equivalence of categories

$$Q_n(\mathcal{C}) \rightarrow \mathrm{Fun}(\mathcal{J}_n, \mathcal{C}).$$

**Proof B.4** (Proposition 4.9). (TBA)

#### REFERENCES

- [Cal+23] Baptiste Calmès et al. *Hermitian K-theory for stable  $\infty$ -categories II: Cobordism categories and additivity*. 2023. arXiv: 2009.07224 [math.KT]. URL: <https://arxiv.org/abs/2009.07224>.
- [HA] Jacob Lurie. *Higher Algebra*. 2017. URL: <https://people.math.harvard.edu/~lurie/papers/HA.pdf>.
- [HLS23] Fabian Hebestreit, Andrea Lachmann, and Wolfgang Steimle. *The localisation theorem for the K-theory of stable  $\infty$ -categories*. 2023. arXiv: 2205.06104 [math.KT]. URL: <https://arxiv.org/abs/2205.06104>.

- [NS18] Thomas Nikolaus and Peter Scholze. *On topological cyclic homology*. 2018. arXiv: [1707.01799](https://arxiv.org/abs/1707.01799) [math.AT]. URL: <https://arxiv.org/abs/1707.01799>.