

# THE ALGEBRAIC K-THEORY OF STABLE $\infty$ -CATEGORIES

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ABSTRACT. We summarize the algebraic K-theory of small stable  $\infty$ -categories.

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## 1. INTRODUCTION

This paper is a summary of the workshop on the algebraic K-theory held in Kyoto in September 2024.

1.1. **Notations.** From here all categories are assumed to be small  $\infty$ -categories. We let

- $\mathbf{An}$  denote the category of small anima.
- $\mathbf{Cat}$  denote the category of small categories.
- $\mathbf{Cat}^{\text{lex}}$  denote the category of small categories which admit finite limits, with left exact functors.
- $\mathbf{Cat}^{\text{st}}$  denote the category of small stable categories with exact functors.
- $\mathbf{Cat}^{\text{perf}}$  denote the category of small idempotent complete stable categories with exact functors.
- $\mathbf{Sp}$  denote the category of spectra.

## 2. PRELIMINARIES

Throughout this paper, some notions can be defined under less restrictive conditions. For instance, the algebraic K-theory can be defined for small categories with finite (co)limits. However, in this paper, we primarily work within the framework of small stable categories.

2.1. **The Grothendieck Group.** In this section, we review the definition of the Grothendieck group for stable categories.

**Definition 2.1.** Let  $(\mathcal{C}, \oplus)$  be a stable category, and let  $X$  and  $Y$  be objects of  $\mathcal{C}$ . We let  $[X]$  denote the connected component of  $X$ . The connected component set  $\pi_0(\text{core } \mathcal{C})$ , together with the operation  $+$  defined by

$$[X] + [Y] := [X \oplus Y]$$

form an ordinary monoid  $(\pi_0(\text{core } \mathcal{C}), +)$ . We define the *Grothendieck group*  $\mathcal{K}_0(\mathcal{C})$  of  $\mathcal{C}$  as

$$\mathcal{K}_0(\mathcal{C}) := (\pi_0(\text{core } \mathcal{C}), +) / \sim,$$

where  $\sim$  is the equivalence relation generated by the following relation:  $[X] = [X'] + [X'']$  whenever  $X' \rightarrow X \rightarrow X''$  is a cofiber sequence in  $\mathcal{C}$ .

**Remark 2.2.** Let  $(\mathcal{C}, \oplus)$  be a stable category. Then the connected component set  $\pi_0(\text{core } \mathcal{C})$  is the set of equivalence classes of objects of  $\mathcal{C}$ . Moreover, the Grothendieck group  $\mathcal{K}_0(\mathcal{C})$  is actually abelian.

- (1) The zero object  $0$  of  $\mathcal{C}$  defines a unit element  $[0]$  in  $\mathcal{K}_0(\mathcal{C})$ , since  $X \rightarrow X \rightarrow 0$  is a cofiber sequence in  $\mathcal{C}$  for every object  $X$  of  $\mathcal{C}$ .
- (2) For every object  $X$  of  $\mathcal{C}$ ,  $[\Omega X]$  and  $[\Sigma X]$  are inverse element of  $[X]$  in  $\mathcal{K}_0(\mathcal{C})$ , since  $\Omega X \rightarrow 0 \rightarrow X$  and  $X \rightarrow 0 \rightarrow \Sigma X$  are cofiber sequences in  $\mathcal{C}$ .
- (3) For every objects  $X$  and  $Y$  of  $\mathcal{C}$ , we have  $[X] + [Y] = [Y] + [X]$ , since  $X \rightarrow X \oplus Y \rightarrow Y$  and  $Y \rightarrow X \oplus Y \rightarrow X$  are cofiber sequences in  $\mathcal{C}$ .

**Remark 2.3.** We can also define the Grothendieck group as follows: Let  $\mathcal{C}$  be a stable category, and let  $X$  be an object of  $\mathcal{C}$ . We let  $\mathcal{K}_0(\mathcal{C})$  denote the free abelian group on generators  $[X]$  modulo the relations given by  $[X] = [X'] + [X'']$  whenever  $X' \rightarrow X \rightarrow X''$  is a cofiber sequence in  $\mathcal{C}$ .

**Example 2.4.** We let  $\text{FinTop}_*$  denote the category of finite pointed spaces. Then the Grothendieck group  $\mathcal{K}_0(\text{FinTop}_*)$  is isomorphic to  $\mathbb{Z}$ . This isomorphism is given by the reduced Euler characteristic  $[X] \mapsto \chi(X) - 1$ .

**Example 2.5.** Let  $R$  be a ring. We let  $\text{Perf}(R)$  denote the category of perfect complexes over  $R$ . Then the Grothendieck group  $\mathcal{K}_0(\text{Perf}(R))$  is isomorphic to  $K_0(R)$ . The map  $K_0(R) \rightarrow \mathcal{K}_0(\text{Perf}(R))$  is defined as follows: It sends each finitely generated projective  $R$ -module  $P$  to the complex  $P$  concentrated in degree zero. The inverse  $\mathcal{K}_0(\text{Perf}(R)) \rightarrow K_0(R)$  sends each perfect complex  $[P_*]$  to its alternating sum  $\sum_n (-1)^n [P_n]$ .

**Remark 2.6** (Eilenberg Swindle). Let  $(\mathcal{C}, \oplus)$  be a stable category with countable coproducts. Then the Grothendieck group  $\mathcal{K}_0(\mathcal{C})$  is trivial. Indeed, for every object  $X$  of  $\mathcal{C}$ , we have  $[X] = 0$ , since

$$\bigoplus_{n \geq 1} X \rightarrow \bigoplus_{n \geq 0} X \rightarrow X$$

is a cofiber sequence in  $\mathcal{C}$  and the last two terms are equivalent. It can be generalized to the algebraic K-theory (see corollary 5.5).

**Remark 2.7.** Let  $F : \mathcal{C} \rightarrow \mathcal{D}$  be an exact functor between stable categories. Then for every object  $X$  of  $\mathcal{C}$ , the construction  $[X] \mapsto [F(X)]$  defines a group homomorphism  $\mathcal{K}_0(F) : \mathcal{K}_0(\mathcal{C}) \rightarrow \mathcal{K}_0(\mathcal{D})$ . The constructions  $\mathcal{C} \mapsto \mathcal{K}_0(\mathcal{C})$  and  $F \mapsto \mathcal{K}_0(F)$  determine a functor

$$\mathcal{K}_0 : \text{Cat}^{\text{st}} \rightarrow \text{Ab}.$$

**Remark 2.8.** Let  $\mathcal{C}$  be a stable category. Then the suspension  $\Sigma : \mathcal{C} \rightarrow \mathcal{C}$  induces the map  $\mathcal{K}_0(\Sigma) : \mathcal{K}_0(\mathcal{C}) \rightarrow \mathcal{K}_0(\mathcal{C})$  given by multiplication by  $-1$  (remark 2.2).

**2.2. Arrow Categories and Twisted Arrow Categories.** In this section, we recall the notions of (twisted) arrow categories and the category of sequences.

**Definition 2.9.** Let  $\mathcal{C}$  be a category. We define the *arrow category*  $\text{Ar}(\mathcal{C})$  of  $\mathcal{C}$  as

$$\text{Ar}(\mathcal{C}) := \text{Fun}([1], \mathcal{C}).$$

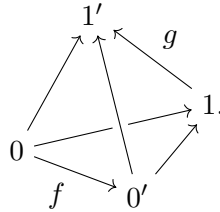
**Definition 2.10** ([kerodon] [Construction 03JG](#)). Let  $\mathcal{C}$  be a category. The *twisted arrow category*  $\text{TwAr}(\mathcal{C})$  of  $\mathcal{C}$  is the simplicial set defined by

$$\text{TwAr}(\mathcal{C})_n := \text{Hom}_{\text{sSet}}([n] \star [n]^{\text{op}}, \mathcal{C})$$

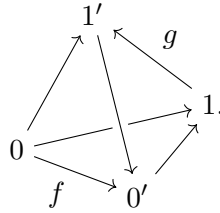
for every  $n \geq 0$ , where  $\star$  denotes the join operator.

**Remark 2.11.** Let  $\mathcal{C}$  be a category. We can describe objects and morphisms of  $\text{Ar}(\mathcal{C})$  and  $\text{TwAr}(\mathcal{C})$ .

- The objects of both are morphisms in  $\mathcal{C}$ .
- A morphism from  $f$  to  $g$  in  $\text{Ar}(\mathcal{C})$  is a diagram, depicted as



- A morphism from  $f$  to  $g$  in  $\text{TwAr}(\mathcal{C})$  is a diagram, depicted as



**Remark 2.12.** Let  $\mathcal{C}$  be a category. Then there are two projections

$$s : \text{TwAr}(\mathcal{C}) \rightarrow \mathcal{C} \quad \text{and} \quad t : \text{TwAr}(\mathcal{C}) \rightarrow \mathcal{C}^{\text{op}}$$

which are defined as follows: They send each  $n$ -simplex  $\sigma$  of  $\text{TwAr}(\mathcal{C})$  to the composition

$$[n] \hookrightarrow [n] \star [n]^{\text{op}} \xrightarrow{\sigma} \mathcal{C} \quad \text{and} \quad [n]^{\text{op}} \hookrightarrow [n] \star [n]^{\text{op}} \xrightarrow{\sigma} \mathcal{C}.$$

respectively. Then these projections induce a right fibration

$$(s, t) : \text{TwAr}(\mathcal{C}) \rightarrow \mathcal{C} \times \mathcal{C}^{\text{op}}.$$

As a consequence, for a category  $\mathcal{C}$ ,  $\text{TwAr}(\mathcal{C})$  is also a category.

**Notation 2.13.** Let  $\mathcal{C}$  be a stable category. We let  $\text{Seq}(\mathcal{C})$  denote the full subcategory of  $\text{Fun}(\Delta^1 \times \Delta^1, \mathcal{C})$  spanned by the bifiber sequences in  $\mathcal{C}$ .

**Remark 2.14.** Let  $\mathcal{C}$  be a stable category. Then we have an equivalence of categories  $\text{Seq}(\mathcal{C}) \simeq \text{Ar}(\mathcal{C})$ , which implies that the category  $\text{Seq}(\mathcal{C})$  is stable.

**Notation 2.15.** Let  $\mathcal{C}$  be a stable category. We define functors from  $\text{Seq}(\mathcal{C})$  to  $\mathcal{C}$  as follows:

$$\begin{aligned} \text{fib} : \text{Seq}(\mathcal{C}) &\rightarrow \mathcal{C} : (X \rightarrow Y \rightarrow Z) \mapsto X, \\ \text{mid} : \text{Seq}(\mathcal{C}) &\rightarrow \mathcal{C} : (X \rightarrow Y \rightarrow Z) \mapsto Y, \\ \text{cofib} : \text{Seq}(\mathcal{C}) &\rightarrow \mathcal{C} : (X \rightarrow Y \rightarrow Z) \mapsto Z. \end{aligned}$$

### 3. LOCALIZATION PROPERTIES OF FUNCTORS

In this section, we define various functors with localization properties: additive, Verdier-localizing, Karoubi-localizing, and grouplike functors.

We follow the terminology of [Cal+23]. In [Cal+23], these notions are defined for Poincaré-Verdier squares. We use the same terminology for Verdier squares.

**3.1. Verdier Sequences and Squares.** In this section, we recall the notions of (split) Verdier sequences and Karoubi sequences and their relative versions: (split) Verdier squares and Karoubi squares.

**Definition 3.1.** Let  $\mathcal{C} \xrightarrow{f} \mathcal{D} \xrightarrow{p} \mathcal{E}$  be a sequence in  $\text{Cat}^{\text{st}}$ . We will say that the sequence *has vanishing composition* if the composition  $pf$  is a zero object of  $\text{Cat}^{\text{st}}$ .

In this case, the composition  $pf$  is equivalent to the functor  $\mathcal{C} \rightarrow 0 \rightarrow \mathcal{E}$ , since the full subcategory of  $\text{Fun}^{\text{ex}}(\mathcal{C}, \mathcal{E})$  spanned by the zero objects is contractible. That is, there exists the following commutative diagram:

$$\begin{array}{ccc} \mathcal{C} & \xrightarrow{f} & \mathcal{D} \\ \downarrow & & \downarrow p \\ 0 & \longrightarrow & \mathcal{E}. \end{array}$$

We will say that the sequence is a *fiber* (resp. *cofiber*) sequence if the above diagram is a cartesian (resp. cocartesian) diagram.

**Definition 3.2** ([Cal+23] Definition A.1.1). Let  $\mathcal{C} \xrightarrow{f} \mathcal{D} \xrightarrow{p} \mathcal{E}$  be a sequence in  $\text{Cat}^{\text{st}}$  with vanishing composition. We will say that it is *Verdier* if it is a bifiber sequence in  $\text{Cat}^{\text{st}}$ . In this case, we will refer to the functor  $f$  as the *Verdier inclusion* and to the functor  $p$  as the *Verdier projection*.

**Definition 3.3** ([Cal+23] Definition A.2.4). Let  $\mathcal{C} \xrightarrow{f} \mathcal{D} \xrightarrow{p} \mathcal{E}$  be a Verdier sequence. We will say that it is *split* if the functor  $p$  admits both adjoint functors. In this case, we will refer to the functor  $f$  as the *split Verdier inclusion* and to the functor  $p$  as the *split Verdier projection*.

**Definition 3.4** ([Cal+23] Definition A.3.5). Let  $\mathcal{C} \xrightarrow{f} \mathcal{D} \xrightarrow{p} \mathcal{E}$  be a sequence in  $\text{Cat}^{\text{st}}$  with vanishing composition. We will say that it is *Karoubi* if its idempotent completion  $\mathcal{C}^{\text{id}} \rightarrow \mathcal{D}^{\text{id}} \rightarrow \mathcal{E}^{\text{id}}$  is a bifiber sequence in  $\text{Cat}^{\text{perf}}$ . In this case, we will refer to the functor  $f$  as the *Karoubi inclusion* and to the functor  $p$  as the *Karoubi projection*.

We can characterize Verdier inclusions and projections (proposition 3.7). The fiber of an exact functor  $f : \mathcal{C} \rightarrow \mathcal{D}$  can be computed by its kernel  $\ker(f)$ . On the other hand, its cofiber is described by the Verdier quotient.

**Definition 3.5** ([Cal+23] Definition A.1.3). Let  $f : \mathcal{C} \rightarrow \mathcal{D}$  be an exact functor between stable categories. We will say that a morphism in  $\mathcal{D}$  is an *equivalence modulo  $\mathcal{C}$*  in  $\mathcal{D}$  if its fiber (or equivalently, its cofiber) belongs in the essential image of  $f$ .

We define the category  $\mathcal{D}/\mathcal{C}$  as the localization of  $\mathcal{D}$  with respect to the set of equivalences modulo  $\mathcal{C}$  in  $\mathcal{D}$ . We will refer to the category  $\mathcal{D}/\mathcal{C}$  as the *Verdier quotient* of  $\mathcal{D}$  by  $\mathcal{C}$ .

The next proposition implies that the Verdier quotient is universal.

**Proposition 3.6** ([NS18] Theorem.1.3.3). Let  $f : \mathcal{C} \rightarrow \mathcal{D}$  be an exact functor between stable categories. Then

- (1) The Verdier quotient  $\mathcal{D}/\mathcal{C}$  is stable, and the localization functor  $\mathcal{D} \rightarrow \mathcal{D}/\mathcal{C}$  is exact.

- (2) For every stable category  $\mathcal{E}$ , the restriction functor

$$\mathrm{Fun}^{\mathrm{ex}}(\mathcal{D}/\mathcal{C}, \mathcal{E}) \rightarrow \mathrm{Fun}^{\mathrm{ex}}(\mathcal{D}, \mathcal{E})$$

is fully faithful, and its essential image consists of the functors which vanish after composing with  $f$ .

- (3) The sequence  $\mathcal{C} \rightarrow \mathcal{D} \rightarrow \mathcal{D}/\mathcal{C}$  is a cofiber sequence in  $\mathrm{Cat}^{\mathrm{st}}$ .

**Proposition 3.7** ([Cal+23] Corollary A.1.10). Let  $\mathcal{C} \xrightarrow{f} \mathcal{D} \xrightarrow{p} \mathcal{E}$  be a sequence in  $\mathrm{Cat}^{\mathrm{st}}$  with vanishing composition. The following conditions are equivalent:

- (1) The sequence is Verdier.
- (2) The functor  $f$  is fully faithful and its essential image is closed under retracts in  $\mathcal{D}$ , and the functor  $p$  exhibits  $\mathcal{E}$  as the Verdier quotient of  $\mathcal{D}$  by  $\mathcal{C}$ .
- (3) The functor  $f$  exhibits  $\mathcal{C}$  as the kernel of  $p$ , and the functor  $p$  is a localization.

We can characterize split Verdier inclusions and projections.

**Proposition 3.8** ([Cal+23] Corollary A.2.6). Let  $\mathcal{C} \xrightarrow{f} \mathcal{D} \xrightarrow{p} \mathcal{E}$  be a sequence in  $\mathrm{Cat}^{\mathrm{st}}$  with vanishing composition. The following conditions are equivalent:

- (1) The sequence is split Verdier.
- (2) The functor  $p$  admits fully faithful both adjoint functors.
- (3) The functor  $f$  is fully faithful and admits both adjoint functors.

We can characterize Karoubi inclusions and projections.

**Proposition 3.9** ([Cal+23] Corollary A.3.8). Let  $\mathcal{C} \xrightarrow{f} \mathcal{D} \xrightarrow{p} \mathcal{E}$  be a sequence in  $\mathrm{Cat}^{\mathrm{st}}$  with vanishing composition. The following conditions are equivalent:

- (1) The sequence is Karoubi.
- (2) The functor  $f$  is fully faithful and the functor  $p$  has the dense essential image  $p(\mathcal{D}) \subseteq \mathcal{E}$ , and the induced functor  $\mathcal{D} \rightarrow p(\mathcal{D})$  is a Verdier projection.

We can describe Karoubi sequences using Ind-categories.

**Theorem 3.10** (Thomason-Neeman's localization theorem). *Let  $\mathcal{C} \xrightarrow{f} \mathcal{D} \xrightarrow{p} \mathcal{E}$  be a sequence in  $\mathrm{Cat}^{\mathrm{st}}$  with vanishing composition. The following conditions are equivalent:*

- (1) *The sequence is Karoubi.*
- (2) *The sequence  $\mathrm{Ind}(\mathcal{C}) \rightarrow \mathrm{Ind}(\mathcal{D}) \rightarrow \mathrm{Ind}(\mathcal{E})$  is Verdier (of non-necessarily small categories).*

We next introduce the relative versions of these sequences.

**Definition 3.11** ([Cal+23] Definition.1.5.1). A square in  $\mathrm{Cat}^{\mathrm{st}}$  is called

- *Verdier* if it is cartesian and its both vertical maps are Verdier projections.
- *split Verdier* if it is cartesian and its both vertical maps are split Verdier projections.
- *Karoubi* if it is cartesian after idempotent completion and its both vertical maps are Karoubi projections.

**Remark 3.12.** In definition 3.11, the condition that the square is cartesian can be replaced by the condition that it is cocartesian. (See proof A.1.)

**3.2. Additive and Grouplike Functors.** In this section, we define additive, Verdier-localizing, Karoubi-localizing, and grouplike functors.

**Definition 3.13.** Let  $\mathcal{E}$  be a category with a terminal object, and let  $F : \text{Cat}^{\text{st}} \rightarrow \mathcal{E}$  be a functor. We will say that  $F$  is *reduced* if  $F(0)$  is equivalent to a terminal object of  $\mathcal{E}$ , where  $0$  is a zero object in  $\text{Cat}^{\text{st}}$ .

**Definition 3.14** ([HLS23] Definition 2.1). Let  $\mathcal{E}$  be a category with finite limits, and let  $F : \text{Cat}^{\text{st}} \rightarrow \mathcal{E}$  be a reduced functor. The functor  $F$  is called

- *Verdier-localizing* if it takes every Verdier square in  $\text{Cat}^{\text{st}}$  to a cartesian square in  $\mathcal{E}$ .
- *additive* if it takes every split Verdier square in  $\text{Cat}^{\text{st}}$  to a cartesian square in  $\mathcal{E}$ .
- *Karoubi-localizing* if it takes every Karoubi square in  $\text{Cat}^{\text{st}}$  to a cartesian square in  $\mathcal{E}$ .

Every additive (resp. Verdier-localizing, Karoubi-localizing) functor  $F : \text{Cat}^{\text{st}} \rightarrow \mathcal{E}$  sends split Verdier sequences (resp. Verdier sequences, Karoubi sequences) to fiber sequences in  $\mathcal{E}$ . If  $\mathcal{E}$  is stable, the converse holds.

**Proposition 3.15** ([Cal+23] Proposition 1.5.5). Let  $\mathcal{E}$  be a category with finite limits, and let  $F : \text{Cat}^{\text{st}} \rightarrow \mathcal{E}$  be a reduced functor. If  $\mathcal{E}$  is stable, then the following conditions are equivalent:

- (1) The functor  $F$  is additive (resp. Verdier-localizing, Karoubi-localizing).
- (2) The functor  $F$  takes every split Verdier sequence (resp. Verdier sequence, Karoubi sequence) in  $\text{Cat}^{\text{st}}$  to a fiber sequence in  $\mathcal{E}$ . (See proof A.2.)

**Definition 3.16** ([HLS23] Definition 2.1). Let  $\mathcal{E}$  be a category with finite limits, and let  $F : \text{Cat}^{\text{st}} \rightarrow \mathcal{E}$  be an additive functor. We will say that  $F$  is *grouplike* if it lifts to the category  $\text{Mon}_{\mathbb{E}_{\infty}}(\mathcal{E})$  takes values in the full subcategory  $\text{Grp}_{\mathbb{E}_{\infty}}(\mathcal{E})$  of  $\text{Mon}_{\mathbb{E}_{\infty}}(\mathcal{E})$ .

**Example 3.17** ([Cal+23] Example 1.5.10). We give some (counter)examples.

- (1) The core functor  $\text{core} : \text{Cat}^{\text{st}} \rightarrow \text{An}$  is additive, but not grouplike.
- (2) The algebraic K-theory  $\mathcal{K} : \text{Cat}^{\text{st}} \rightarrow \text{An}$  (definition 4.12) is Verdier-localizing (theorem 6.1) and grouplike (corollary 5.4), but not Karoubi-localizing.
- (3) The connective K-theory spectrum  $\mathcal{K} \circ (-)^{\natural} : \text{Cat}^{\text{st}} \rightarrow \text{Sp}_{\geq 0}$  (definition 9.2) is Karoubi-localizing (example 8.9), thus is Verdier-localizing (proposition 3.28).
- (4) The non-connective K-theory spectrum  $\mathbb{K} : \text{Cat}^{\text{st}} \rightarrow \text{Sp}$  (definition 9.9) is Karoubi-localizing.

**Proposition 3.18** ([HLS23] Observation 2.2). The additive, Verdier-localizing, Karoubi-localizing functors preserve finite products. (See proof A.3.)

**Proposition 3.19.** Let  $F : \text{Cat}^{\text{st}} \rightarrow \mathcal{E}$  be an additive functor. If  $\mathcal{E}$  is additive (TBA), then  $F$  is grouplike.

**3.3. Additive Grouplike vs. Extension-splitting.** We can characterize additive grouplike functors by extension-splitting functors. We will use lemma 3.21 and proposition 3.22 to show that the algebraic K-theory is additive grouplike (corollary 5.4).

**Definition 3.20.** Let  $\mathcal{E}$  be a category with finite limits, and let  $F : \text{Cat}^{\text{st}} \rightarrow \mathcal{E}$  be a reduced functor. We will say that  $F$  is *extension-splitting* if, for every stable category  $\mathcal{C}$ , the fiber-cofiber map

$$(\text{fib}, \text{cofib}) : \text{Seq}(\mathcal{C}) \rightarrow \mathcal{C}^2$$

induces an equivalence  $F(\text{Seq}(\mathcal{C})) \rightarrow F(\mathcal{C})^2$ .

We show that additive grouplike functors and extension-splitting functors are equivalent (proposition 3.22).

**Lemma 3.21** ([HLS23] Lemma 2.5). Let  $\mathcal{E}$  be a category with finite limits, and let  $F : \text{Cat}^{\text{st}} \rightarrow \mathcal{E}$  be a reduced and product-preserving functor. The following conditions are equivalent:

- (1) The functor  $F$  is extension-splitting.
- (2) The functor  $F$  sends the source-target projection  $(s, t) : \text{Ar}(\mathcal{C}) \rightarrow \mathcal{C}^2$  for every object stable category  $\mathcal{C}$  to an equivalence in  $\mathcal{E}$ .

**Proposition 3.22** ([HLS23] Proposition 2.4). Let  $\mathcal{E}$  be a category with finite limits, and let  $F : \text{Cat}^{\text{st}} \rightarrow \mathcal{E}$  be a reduced functor. The following conditions are equivalent:

- (1) The functor  $F$  is additive grouplike.
- (2) The functor  $F$  is extension-splitting.

**3.4. Additive vs. Verdier-localizing.** In this section, we recall Waldhausen's fibration theorem. We will use this theorem in the proof of the localization theorem (theorem 6.1).

**Notation 3.23.** Let  $\mathcal{D}$  be a stable category, let  $\mathcal{C}$  be a stable full subcategory of  $\mathcal{D}$ , and let  $\mathcal{J}$  be a category. We let  $\text{Fun}^{\mathcal{C}}(\mathcal{J}, \mathcal{D})$  denote the full subcategory of  $\text{Fun}(\mathcal{J}, \mathcal{D})$  spanned by the functors which take every map in  $\mathcal{J}$  to an equivalence modulo  $\mathcal{C}$ .

**Theorem 3.24** (Waldhausen's fibration theorem). *Let  $\mathcal{C} \rightarrow \mathcal{D} \rightarrow \mathcal{E}$  be a Verdier sequence, and let  $F : \text{Cat}^{\text{st}} \rightarrow \text{An}$  be an additive grouplike functor. Then, for every  $n \geq 0$ , the constant map*

$$\text{const} : \mathcal{D} \rightarrow \text{Fun}^{\mathcal{C}}([n], \mathcal{D}) : X \mapsto (X \rightarrow \cdots \rightarrow X)$$

*induces a bifiber sequence of  $\mathbb{E}_{\infty}$ -groups*

$$F(\mathcal{C}) \rightarrow F(\mathcal{D}) \rightarrow |F \text{Fun}^{\mathcal{C}}([-], \mathcal{D})|.$$

We can deduce when an additive functor becomes a Verdier-localizing functor.

**Corollary 3.25** ([HLS23] Corollary 2.10). Let  $F : \text{Cat}^{\text{st}} \rightarrow \text{An}$  be an additive functor. These conditions are equivalent:

- (1) The functor  $F$  is Verdier-localizing.
- (2) For every Verdier sequence  $\mathcal{C} \rightarrow \mathcal{D} \rightarrow \mathcal{E}$ , the canonical map  $|F \text{Fun}^{\mathcal{C}}([-], \mathcal{D})| \rightarrow F(\mathcal{E})$  is an equivalence of anima.

**3.5. Verdier-localizing vs. Karoubi-localizing.** The relationship between Verdier-localizing and Karoubi-localizing functors is as follows.

**Definition 3.26.** Let  $f : \mathcal{C} \rightarrow \mathcal{D}$  be a functor between categories. We will say that  $f$  *has the dense image* if, for every object  $X$  of  $\mathcal{D}$ , there exists an object  $Y$  in the essential image of  $\mathcal{C}$  such that  $Y$  is a retract of  $X$ .

**Definition 3.27.** Let  $f : \mathcal{C} \rightarrow \mathcal{D}$  be an exact functor between stable categories. We will say that  $f$  is a *Karoubi equivalence* if it is fully faithful and has the dense image.

**Proposition 3.28** ([HLS23] Observation 2.12). Let  $\mathcal{E}$  be a category with finite limits, and let  $F : \text{Cat}^{\text{st}} \rightarrow \mathcal{E}$  be a reduced functor. The following conditions are equivalent:

- (1) The functor  $F$  is Karoubi-localizing.
- (2) The functor  $F$  is Verdier-localizing and inverts Karoubi equivalences.

We can construct Karoubi-localization functors from Verdier-localizing functors using the idempotent completion.



**Proposition 3.29** ([HLS23] Lemma 2.13). Let  $\mathcal{E}$  be a category with finite limits, and let  $F : \text{Cat}^{\text{st}} \rightarrow \mathcal{E}$  be a Verdier-localizing functor. Suppose that  $F$  takes every cartesian square in  $\text{Cat}^{\text{st}}$  whose vertical maps are dense inclusions, to a cartesian square in  $\mathcal{E}$ . Then the functor  $F \circ (-)^{\natural} : \text{Cat}^{\text{st}} \rightarrow \mathcal{E}$  is Karoubi-localizing.

#### 4. THE ALGEBRAIC K-THEORY OF STABLE $\infty$ -CATEGORIES

In this section, we recall the Q-construction and define the algebraic K-theory. Furthermore, we introduce the S-construction and demonstrate that the definitions of the algebraic K-theory given by these two constructions are equivalent.

**4.1. Simplicial Objects.** In this section, we recall the basic notions of simplicial objects.

**Definition 4.1.** The inclusion  $N(\Delta) \subseteq \text{Cat}$  induces an adjunction

$$\text{asscat} : \text{Fun}(N(\Delta)^{\text{op}}, \text{An}) \rightleftarrows \text{Cat} : N^r.$$

We will refer to the left adjoint as the *associated category functor*, and to the right adjoint as the *Rezk nerve*.

**Definition 4.2.** Let  $\mathcal{C}$  be a category. We will refer to a functor

$$X : N(\Delta)^{\text{op}} \rightarrow \mathcal{C}$$

as a *simplicial object* of  $\mathcal{C}$ . We will say that  $X$  is a *simplicial anima* if  $\mathcal{C}$  is  $\text{An}$ .

**Remark 4.3.** Let  $\mathcal{C}$  be a category. For every  $n \geq 0$ , we have an equivalence of anima

$$N_n^r(\mathcal{C}) \simeq \text{Map}_{\text{Cat}}([n], \mathcal{C}) \simeq \text{core Fun}([n], \mathcal{C}).$$

**Notation 4.4.** We let  $[n]$  denote the category the ordinary nerve  $N([n])$ , instead of  $\Delta^n$ . On the other hand, we let  $\Delta^n$  denote the functor

$$\Delta^n := \text{Map}_{\text{Cat}}(-, [n]) : N(\Delta)^{\text{op}} \rightarrow \text{An}.$$

Then we have an equivalence of simplicial anima  $N^r([n]) \simeq \Delta^n$ .

We define the Segal condition and completeness specifically for simplicial anima, although these concepts are applicable to every category.

**Definition 4.5.** Let  $X : N(\Delta)^{\text{op}} \rightarrow \text{An}$  be a simplicial anima. We will say that  $X$  is *Segal* if the  $n$ -spine inclusion  $\text{sp}^n \subseteq \Delta^n$  induces an equivalence of anima

$$X_n \simeq \text{Map}_{\text{Fun}(N(\Delta)^{\text{op}}, \text{An})}(\Delta^n, X) \rightarrow \text{Map}_{\text{Fun}(N(\Delta)^{\text{op}}, \text{An})}(\text{sp}^n, X) \simeq X_1 \times_{X_0} \cdots \times_{X_0} X_1$$

for every  $n \geq 0$ .

The Segal condition can be interpreted as stating that a Segal simplicial anima has a unique spine lifting up to a choice of contractible spaces.

**Definition 4.6.** Let  $X : N(\Delta)^{\text{op}} \rightarrow \text{An}$  be a Segal simplicial anima. We will say that  $X$  is *complete* if the following diagram is a cartesian diagram in  $\text{An}$ .

$$\begin{array}{ccc} X_0 & \xrightarrow{\text{diag}} & X_0 \times X_0 \\ \downarrow \lrcorner & & \downarrow (s, s) \\ X_3 & \xrightarrow{(d^{\{0,2\}}, d^{\{1,3\}})} & X_1 \times X_1 \end{array}$$

The completeness condition can be understood as indicating that the higher simplices of a complete Segal simplicial anima correspond to equivalences related to its degenerate edges.



**Proposition 4.7.** The Rezk nerve  $N^r : \text{Cat} \rightarrow \text{Fun}(N(\Delta)^{\text{op}}, \text{An})$  is fully faithful. Moreover, its essential image precisely consists of complete Segal simplicial anima.

#### 4.2. The Algebraic K-Theory.

**Definition 4.8** ([HLS23] Definition 3.1). Let  $\mathcal{C}$  be a category with finite limits. For every  $n \geq 0$ , we let  $Q_n(\mathcal{C})$  denote the full subcategory of  $\text{Fun}(\text{TwAr}[n], \mathcal{C})$  spanned by the diagrams which take every square in  $\text{TwAr}[n]$  to a cartesian square in  $\mathcal{C}$ .

The construction  $n \mapsto Q_n(\mathcal{C})$  determines a functor

$$Q(\mathcal{C}) : N(\Delta)^{\text{op}} \rightarrow \text{Cat}^{\text{lex}}$$

and furthermore, the construction  $\mathcal{C} \mapsto Q(\mathcal{C})$  defines a functor

$$Q : \text{Cat}^{\text{lex}} \rightarrow \text{Fun}(N(\Delta)^{\text{op}}, \text{Cat}^{\text{lex}}).$$

We will refer to this functor as the (*Quillen's*) *Q-construction*.

**Proposition 4.9** ([HLS23] Proposition 3.2). Let  $\mathcal{C}$  be a category with finite limits. Then the simplicial object in  $\text{Cat}^{\text{lex}}$

$$Q(\mathcal{C}) : N(\Delta)^{\text{op}} \rightarrow \text{Cat}^{\text{lex}}$$

is complete Segal. In particular, the simplicial anima

$$\text{core } Q(\mathcal{C}) : N(\Delta)^{\text{op}} \rightarrow \text{An}$$

is complete Segal. (See proof A.7.)

**Remark 4.10.** Corollary A.6 implies that, if  $\mathcal{C}$  is stable, so is  $Q_n(\mathcal{C})$ . Therefore we obtain functors

$$Q(\mathcal{C}) : N(\Delta)^{\text{op}} \rightarrow \text{Cat}^{\text{st}} \quad \text{and} \quad Q : \text{Cat}^{\text{st}} \rightarrow \text{Fun}(N(\Delta)^{\text{op}}, \text{Cat}^{\text{st}}).$$

Moreover, for every stable category  $\mathcal{C}$ , the category  $Q(\mathcal{C})$  is a complete Segal simplicial object in  $\text{Cat}^{\text{st}}$ , since  $\text{Cat}^{\text{st}}$  is stable under finite limits in  $\text{Cat}$ .

**Definition 4.11** ([HLS23] Definition 3.3). Let  $\mathcal{C}$  be a category with finite limits. Then we define the *category of spans* in  $\mathcal{C}$  as

$$\text{Span}(\mathcal{C}) := \text{asscat core } Q(\mathcal{C}).$$

The construction  $\mathcal{C} \mapsto \text{Span}(\mathcal{C})$  determines a functor

$$\text{Span} : \text{Cat}^{\text{lex}} \rightarrow \text{Cat}.$$

**Definition 4.12** ([HLS23] Definition 3.4). Let  $\mathcal{C}$  be a stable category. Then we define the *algebraic K-anima* (or *algebraic K-theory anima*, or *projective class anima*) as

$$\mathcal{K}(\mathcal{C}) := \Omega |\text{Span}(\mathcal{C})| \simeq \Omega |\text{core } Q(\mathcal{C})|$$

where the base object of the loop space is given by the zero object of  $\text{Span}(\mathcal{C})$ .

The construction  $\mathcal{C} \mapsto \mathcal{K}(\mathcal{C})$  determines a functor

$$\mathcal{K} : \text{Cat}^{\text{st}} \rightarrow \text{An}.$$

We will refer to this functor as the *algebraic K-theory* (or *algebraic K-functor*).

**Definition 4.13.** Let  $\mathcal{C}$  be a stable category. For every  $n \geq 0$ , we define the *n-th K-group*  $\mathcal{K}_n(\mathcal{C})$  of  $\mathcal{C}$  as

$$\mathcal{K}_n(\mathcal{C}) := \pi_n \mathcal{K}(\mathcal{C}).$$

**Proposition 4.14.** Let  $\mathcal{C}$  be a stable category. Then we have an isomorphism of groups

$$\pi_0 \mathcal{K}(\mathcal{C}) \simeq \mathcal{K}_0(\mathcal{C}),$$

where  $\mathcal{K}_0(\mathcal{C})$  is the Grothendieck group of  $\mathcal{C}$  (see definition 2.1).

**4.3. Waldhausen's S-Construction.** In this section, we construct the algebraic K-theory using Waldhausen's S-construction.

**Definition 4.15.** Let  $\mathcal{C}$  be a stable category. An  $[n]$ -gapped object of  $\mathcal{C}$  is a functor  $F : \text{Ar}[n] \rightarrow \mathcal{C}$  which satisfies the following conditions:

- (1) For every  $0 \leq i \leq n$ ,  $F(i, i)$  is a zero object of  $\mathcal{C}$ .
- (2) For every  $i \leq j \leq k$ , the following diagram is a (co)cartesian diagram in  $\mathcal{C}$ .

$$\begin{array}{ccc} F(i, j) & \longrightarrow & F(i, k) \\ \downarrow & \lrcorner & \downarrow \\ 0 \simeq F(j, j) & \longrightarrow & F(j, k) \end{array}$$

We let  $S_n(\mathcal{C})$  denote the full subcategory of  $\text{Fun}(\text{Ar}[n], \mathcal{C})$  spanned by the  $[n]$ -gapped objects of  $\mathcal{C}$ .

**Remark 4.16.** Let  $\mathcal{C}$  be a stable category. We can describe the low-dimensional simplices of  $S_n(\mathcal{C})$ .

- The category  $S_0(\mathcal{C})$  is the full subcategory of  $\mathcal{C}$  spanned by the zero objects of  $\mathcal{C}$ . Thus  $S_0(\mathcal{C})$  is contractible.
- The category  $S_1(\mathcal{C})$  is equivalent to  $\mathcal{C}$ , since every object of  $S_1(\mathcal{C})$  is of the form  $0 \rightarrow X \rightarrow 0$ , where  $X$  is an object of  $\mathcal{C}$ .
- The category  $S_2(\mathcal{C})$  is equivalent to the arrow category  $\text{Ar}(\mathcal{C})$  of  $\mathcal{C}$ , since every object of  $S_2(\mathcal{C})$  is of the form  $0 \rightarrow X' \rightarrow X \rightarrow X'' \rightarrow 0$ , where  $X' \rightarrow X \rightarrow X''$  is a cofiber sequence in  $\mathcal{C}$ .

**Remark 4.17.** Let  $\mathcal{C}$  be a stable category. We have an equivalence of categories

$$S_n(\mathcal{C}) \simeq \text{Fun}([n-1], \mathcal{C})$$

for every  $n \geq 0$ . Thus, if  $\mathcal{C}$  is stable, so is  $S_n(\mathcal{C})$ .

**Definition 4.18.** The construction  $n \mapsto S_n(\mathcal{C})$  determines a functor

$$S(\mathcal{C}) : \mathbf{N}(\Delta)^{\text{op}} \rightarrow \text{Cat}^{\text{st}}$$

and furthermore, the construction  $\mathcal{C} \mapsto S(\mathcal{C})$  defines a functor

$$S : \text{Cat}^{\text{st}} \rightarrow \text{Fun}(\mathbf{N}(\Delta)^{\text{op}}, \text{Cat}^{\text{st}}).$$

We will refer to this functor as *(Waldhausen's) S-construction*.

**Definition 4.19.** Let  $\mathcal{C}$  be a stable category. Then we define the *algebraic K-anima* as

$$\mathcal{K}_S(\mathcal{C}) := \Omega | \text{core } S(\mathcal{C}) |.$$

The construction  $\mathcal{C} \mapsto \mathcal{K}_S(\mathcal{C})$  determines a functor

$$\mathcal{K}_S : \text{Cat}^{\text{st}} \rightarrow \text{An}.$$

We will refer to this functor as the *algebraic K-theory*.

**Remark 4.20.** Let  $\mathcal{C}$  be a stable category. Then the anima  $|\mathrm{core} S(\mathcal{C})|$  admits the canonical base point given by the map

$$0 \simeq \mathrm{core} S_0(\mathcal{C}) \rightarrow |\mathrm{core} S(\mathcal{C})|.$$

Moreover,  $|\mathrm{core} S(\mathcal{C})|$  is connected, since the canonical map

$$0 \simeq \pi_0 \mathrm{core} S_0(\mathcal{C}) \rightarrow \pi_0 |\mathrm{core} S(\mathcal{C})|$$

is surjective.

**Proposition 4.21.** For every stable category  $\mathcal{C}$ , two definitions of algebraic K-anima (definitions 4.12 and 4.19) induce an equivalence of anima

$$\mathcal{K}(\mathcal{C}) \simeq \mathcal{K}_S(\mathcal{C}).$$

## 5. THE ADDITIVITY THEOREM

The goal of this section is to prove the additivity theorem.

**Theorem 5.1** ([HLS23] Theorem 4.1: The Additivity Theorem). *Let  $\mathcal{C}$  be a stable category. Then the source-target projection induces an equivalence of anima*

$$|\mathrm{Span}(s, t)| : |\mathrm{Span}(\mathrm{Ar}(\mathcal{C}))| \rightarrow |\mathrm{Span}(\mathcal{C})|^2.$$

Before proving theorem 5.1, we show some corollaries. We need some lemmata.

**Lemma 5.2.** Let  $\mathcal{C}$  and  $\mathcal{D}$  be stable categories, and let  $F' \rightarrow F \rightarrow F''$  be a cofiber sequence of exact functors from  $\mathcal{C}$  to  $\mathcal{D}$ . Then we have

$$\mathcal{K}(F) = \mathcal{K}(F') + \mathcal{K}(F'').$$

*Proof.* Consider the following functors

$$\mathrm{mid}, \mathrm{fib} + \mathrm{cofib} : \mathrm{Seq}(\mathrm{Fun}^{\mathrm{ex}}(\mathcal{C}, \mathcal{D})) \rightarrow \mathrm{Fun}^{\mathrm{ex}}(\mathcal{C}, \mathcal{D}).$$

Since  $\mathcal{K}$  is extension-splitting (see proposition 3.22), by Waldhausen's Additivity Theorem (TBA), we have an equivalence

$$\mathcal{K}(\mathrm{mid}) \simeq \mathcal{K}(\mathrm{fib}) + \mathcal{K}(\mathrm{cofib}).$$

We obtain the assertion by applying it to the cofiber sequence  $F' \rightarrow F \rightarrow F''$ .  $\square$

**Lemma 5.3.** The algebraic K-theory  $\mathcal{K} : \mathrm{Cat}^{\mathrm{st}} \rightarrow \mathrm{An}$  preserves finite products.

*Proof.* Let  $\mathcal{C}$  and  $\mathcal{D}$  be two stable categories. Two projections  $\mathcal{C} \leftarrow \mathcal{C} \times \mathcal{D} \rightarrow \mathcal{D}$  induce a natural morphism

$$\mathcal{K}(\mathcal{C} \times \mathcal{D}) \rightarrow \mathcal{K}(\mathcal{C}) \times \mathcal{K}(\mathcal{D}).$$

Unwinding the definition, it suffices to show that the Q-construction  $Q : \mathrm{Cat}^{\mathrm{st}} \rightarrow \mathrm{Fun}(\mathrm{N}(\Delta), \mathrm{Cat}^{\mathrm{st}})$  preserves finite products. It follows from that we have an equivalence

$$\mathrm{Fun}(\mathrm{TwAr}[n], \mathcal{C} \times \mathcal{D}) \times \mathrm{Fun}(\mathrm{TwAr}[n], \mathcal{C}) \times \mathrm{Fun}(\mathrm{TwAr}[n], \mathcal{D})$$

and it restricts an equivalence

$$Q(\mathcal{C} \times \mathcal{D}) \simeq Q(\mathcal{C}) \times Q(\mathcal{D}).$$

$\square$

**Corollary 5.4** ([HLS23] Corollary 4.2). The algebraic K-theory  $\mathcal{K} : \mathrm{Cat}^{\mathrm{st}} \rightarrow \mathrm{An}$  is additive grouplike.

*Proof.* By proposition 3.22, it suffices to show that  $\mathcal{K}$  is a reduced functor and it is extension splitting. We have

$$\mathcal{K}(0) \simeq \Omega |\mathrm{Span}(0)| \simeq \Omega |\mathrm{core} Q(0)| \simeq 0.$$

Then by lemmas 3.21 and 5.3, it is enough to show that  $\mathcal{K}$  sends the source-target projection  $(s, t) : \mathrm{Ar}(\mathcal{C}) \rightarrow \mathcal{C}^2$  to an equivalence of anima. That is, there is an equivalence  $\mathcal{K}(s, t) : \mathcal{K}(\mathrm{Ar}(\mathcal{C})) \rightarrow \mathcal{K}(\mathcal{C})^2$ .

By theorem 5.1, we have an equivalence  $|\mathrm{Span}(\mathrm{Ar}(\mathcal{C}))| \rightarrow |\mathrm{Span}(\mathcal{C})|^2$ . Since the loop functor  $\Omega$  preserves limits, we obtain an equivalence

$$\mathcal{K}(s, t) : \mathcal{K}(\mathrm{Ar}(\mathcal{C})) \rightarrow \mathcal{K}(\mathcal{C})^2.$$

$\square$

**Corollary 5.5** (Eilenberg swindle). Let  $\mathcal{C}$  be a stable category with countable coproducts. Then the algebraic K-theory anima vanishes. That is, we have an equivalence of anima

$$\mathcal{K}(\mathcal{C}) \simeq 0.$$

*Proof.* Consider the following exact functor

$$F : \mathcal{C} \rightarrow \mathcal{C} : X \mapsto \bigoplus_{n \in \mathbb{N}} X_n.$$

Then there exists a cofiber sequence

$$\mathrm{id}_{\mathcal{C}} \rightarrow F \rightarrow F$$

of exact functors on  $\mathcal{C}$ . By lemma 5.2, we have an equivalence

$$\mathcal{K}(F) \simeq \mathcal{K}(\mathrm{id}_{\mathcal{C}}) + \mathcal{K}(\mathcal{F}).$$

Thus we have  $\mathcal{K}(\mathrm{id}_{\mathcal{C}}) \simeq 0$  and, in particular  $\mathcal{K}(\mathcal{C}) \simeq 0$ .  $\square$

**Corollary 5.6.** Let  $\mathcal{C}$  be a stable category. Then the suspension  $\Sigma : \mathcal{C} \rightarrow \mathcal{C}$  induces the map

$$\mathcal{K}(\Sigma) : \mathcal{K}(\mathcal{C}) \rightarrow \mathcal{K}(\mathcal{C})$$

given by multiplication by  $-1$ .

*Proof.* Applying lemma 5.2 to the cofiber sequence  $\mathrm{id} \rightarrow 0 \rightarrow \Sigma$ , we obtain

$$\mathcal{K}(0) \simeq \mathcal{K}(\mathrm{id}) + \mathcal{K}(\Sigma).$$

Then we have  $\mathcal{K}(\mathrm{id}) \simeq \mathcal{K}(\Sigma)$ . It implies that  $\mathcal{K}(\Sigma) : \mathcal{K}(\mathcal{C}) \rightarrow \mathcal{K}(\mathcal{C})$  is given by multiplication by  $-1$ .  $\square$

The proof of theorem 5.1 follows from the next two propositions.

**Proposition 5.7** ([HLS23] Proposition 4.3). Let  $\mathcal{C}$  be a stable category. Then there are canonical equivalences of categories

$$\mathrm{Span}(\mathcal{C}) \rightarrow \mathrm{Span}(\mathcal{C}^{\mathrm{op}}) \quad \text{and} \quad \mathrm{Span}(\mathrm{Ar}(\mathcal{C})) \simeq \mathrm{Span}(\mathrm{TwAr}(\mathcal{C})).$$

Moreover, they fit together into a natural commutative diagram

$$\begin{array}{ccc} \mathrm{Span}(\mathrm{Ar}(\mathcal{C})) & \xrightarrow{\simeq} & \mathrm{Span}(\mathrm{TwAr}(\mathcal{C})) \\ \mathrm{Span}(s, t) \downarrow & & \downarrow \mathrm{Span}(s, t) \\ \mathrm{Span}(\mathcal{C}) \times \mathrm{Span}(\mathcal{C}) & \xrightarrow[\simeq]{} & \mathrm{Span}(\mathcal{C}) \times \mathrm{Span}(\mathcal{C}^{\mathrm{op}}). \end{array}$$

**Proposition 5.8** ([HLS23] Proposition 4.4). Let  $\mathcal{C}$  be a stable category. Then the source-target projection

$$\mathrm{Span}(s, t) : \mathrm{Span}(\mathrm{TwAr}(\mathcal{C})) \rightarrow \mathrm{Span}(\mathcal{C}) \times \mathrm{Span}(\mathcal{C}^{\mathrm{op}})$$

is cofinal.

## 6. THE LOCALIZATION THEOREM

The goal of this section is to prove the localization theorem.

**Theorem 6.1** ([HLS23] Theorem 6.1: The Localization Theorem). *The algebraic K-theory  $\mathcal{K} : \text{Cat}^{\text{st}} \rightarrow \text{An}$  is Verdier-localizing.*

**Remark 6.2.** By corollary 3.25, an additive grouplike functor  $F : \text{Cat}^{\text{st}} \rightarrow \text{An}$  is Verdier-localizing if and only if it satisfies the following condition:

- (\*) For every Verdier sequence  $\mathcal{C} \rightarrow \mathcal{D} \rightarrow \mathcal{E}$ , the canonical map  $|F \text{Fun}^{\mathcal{C}}([-], \mathcal{D})| \rightarrow F(\mathcal{E})$  is an equivalence of anima.

The next proposition implies that it is enough to prove that the core functor satisfies (\*).

**Proposition 6.3** ([HLS23] Proposition 6.2). Let  $F : \text{Cat}^{\text{st}} \rightarrow \text{An}$  be an additive functor. If  $F$  satisfies (\*), then  $|FQ(-)|$  and  $\Omega|FQ(-)|$  also satisfy (\*).

**Remark 6.4.** If the core functor satisfies (\*), then the algebraic K-theory also satisfies (\*) since  $\mathcal{K}(-) \simeq \Omega|\text{core } Q(-)|$ .

**Notation 6.5.** Let  $\mathcal{D}$  be a stable category, and let  $\mathcal{C}$  be a stable subcategory of  $\mathcal{D}$ . We let  $\mathcal{D}_{\mathcal{C}}$  denote the full subcategory of  $\mathcal{D}$  spanned by the equivalences modulo  $\mathcal{C}$  in  $\mathcal{D}$ .

**Remark 6.6.** Let  $\mathcal{D}$  be a stable category, and let  $\mathcal{C}$  be a stable subcategory of  $\mathcal{D}$ . We obtain an equivalence of anima

$$\text{core Fun}^{\mathcal{C}}([-], \mathcal{D}) \simeq \text{core Fun}([-], \mathcal{D}_{\mathcal{C}}) \simeq \text{Map}_{\text{Cat}}([-], \mathcal{D}_{\mathcal{C}}) \simeq N^r(\mathcal{D}_{\mathcal{C}}).$$

Then we have an equivalence

$$|\text{core Fun}^{\mathcal{C}}([-], \mathcal{D})| \simeq |\mathcal{D}_{\mathcal{C}}|,$$

since the canonical map  $|N^r(\mathcal{D}_{\mathcal{C}})| \rightarrow |\mathcal{D}_{\mathcal{C}}|$  is an equivalence

Thus it suffices to show the following proposition.

**Proposition 6.7** ([HLS23] Proposition 6.6). Let  $\mathcal{D}$  be a stable category, and let  $\mathcal{C}$  be a stable subcategory of  $\mathcal{D}$ . Then the map

$$|\mathcal{D}_{\mathcal{C}}| \rightarrow \text{core}(\mathcal{D}/\mathcal{C})$$

is faithful. If the inclusion  $\mathcal{C} \subseteq \mathcal{D}$  is a Verdier inclusion, then the above map is an equivalence.

Proposition 6.7 is a special case of the following proposition.

**Proposition 6.8** ([HLS23] Proposition 6.8). Let  $\mathcal{C}$  be a category, and let  $S$  be a subcategory of  $\mathcal{C}$ . If  $S$  is closed under 2-out-of-3 and pushouts in  $\mathcal{C}$ , then the map

$$|S| = S[S^{-1}] \rightarrow \mathcal{C}$$

is faithful. Moreover, the following conditions are equivalent:

- (1) The inclusion  $|S| \subseteq \text{core } \mathcal{C}[S^{-1}]$  is fully faithful.
- (2) The category  $S$  is closed under 2-out-of-6 in  $\mathcal{C}$ .
- (3) A morphism in  $\mathcal{D}$  belongs to  $S$  if and only if its source and target are in  $S$  and it is invertible in  $\mathcal{C}[S^{-1}]$ .

## 7. THE UNIVERSALITY THEOREM

The goal of this section is to prove the universality theorem.

**Theorem 7.1** ([HLS23] Theorem 5.1: The Universality Theorem). *The algebraic K-theory  $\mathcal{K} : \text{Cat}^{\text{st}} \rightarrow \text{An}$  is an initial additive grouplike functor under the core functor  $\text{core} : \text{Cat}^{\text{st}} \rightarrow \text{An}$ . That is, the natural map  $\tau : \text{core} \Rightarrow \mathcal{K}$  is an initial object in  $\text{Fun}(\text{Cat}^{\text{st}}, \text{An})_{\text{core}/}^{\text{add,grp}}$ .*

**Notation 7.2.** Let  $F : \text{Cat}^{\text{st}} \rightarrow \text{An}$  be a reduced functor. We denote a functor

$$GF(-) := \Omega|FQ(-)| : \text{Cat}^{\text{st}} \rightarrow \text{An}.$$

**Example 7.3.** the functor  $G \text{core}$  is equivalent to the algebraic K-theory  $\mathcal{K} : \text{Cat}^{\text{st}} \rightarrow \text{An}$ .

*Proof of theorem 7.1.* We want to show that the natural transformation  $\tau : \text{core} \Rightarrow \mathcal{K}$  induces an equivalence

$$\tau^* : \text{Nat}(\mathcal{K}, F) \rightarrow \text{Nat}(\text{core}, F)$$

for every additive grouplike functor  $F : \text{Cat}^{\text{st}} \rightarrow \text{An}$ .

Now consider the following diagram

$$\begin{array}{ccc}
 \text{Nat}(G \text{core}, F) & \xrightarrow{(\eta_{\text{core}})^*} & \text{Nat}(\text{core}, F) \\
 \downarrow G & & \downarrow G \\
 \text{Nat}(GG \text{core}, GF) & \xrightarrow{(G\eta_{\text{core}})^*} & \text{Nat}(G \text{core}, GF) \\
 \downarrow (\eta_{G \text{core}})^* & & \downarrow (\eta_{\text{core}})^* \\
 \text{Nat}(G \text{core}, GF) & \xrightarrow{(\eta_{\text{core}})^*} & \text{Nat}(\text{core}, GF)
 \end{array}$$

(The diagram is enclosed in a large red circle with labels  $(\eta_F)^*$  on the left and right sides.)

The upper square commutes since  $G$  is a functor, and the other three parts commute since  $\eta$  is natural. If the red-colored maps are equivalent, then we can show that the upper horizontal map  $(\eta_{\text{core}})^*$  is an equivalence. If we apply this to the case  $F$  is  $\text{core}$ , then we obtain the desired result. This assumption follows from the next two propositions.  $\square$

The next proposition implies that  $(\eta_F)_*$  and  $(G\eta_{\text{core}})^*$  are equivalences.

**Proposition 7.4** ([HLS23] Proposition 5.2). Let  $F : \text{Cat}^{\text{st}} \rightarrow \text{An}$  be an additive grouplike functor. Then the natural transformation

$$\eta_F : F \Rightarrow GF$$

is an equivalence.

The next proposition implies that  $(\eta_{G \text{core}})$  is an equivalence.

**Proposition 7.5** ([HLS23] Proposition 5.3). Let  $F : \text{Cat}^{\text{st}} \rightarrow \text{An}$  be an additive functor. Then the two natural transformations

$$\eta_{GF}, G\eta_F : GF \Rightarrow GGF$$

differ by an automorphism of the target. That is, the following diagram commutes

$$\begin{array}{ccc}
 GF & \xrightarrow{G\eta_F} & GGF \\
 \searrow \eta_{GF} & & \downarrow \simeq \\
 & & GGF
 \end{array}$$



## 8. THE COFINALITY THEOREM

The goal of this section is to prove the cofinality theorem.

**Theorem 8.1** ([HLS23] Theorem 7.1: The Cofinality Theorem). *Let  $f : \mathcal{C} \subseteq \mathcal{D}$  be a dense inclusion of stable categories. Then it induces a fiber sequence*

$$\mathcal{K}(\mathcal{C}) \rightarrow \mathcal{K}(\mathcal{D}) \rightarrow \mathcal{K}_0(\mathcal{D})/\mathcal{K}_0(\mathcal{C}).$$

*In particular, maps of abelian groups*

$$\mathcal{K}_i(\mathcal{C}) \rightarrow \mathcal{K}_i(\mathcal{D})$$

*are isomorphisms for every  $i \geq 1$ , and there exists a short exact sequence*

$$0 \rightarrow \mathcal{K}_0(\mathcal{C}) \rightarrow \mathcal{K}_0(\mathcal{D}) \rightarrow \mathcal{K}_0(\mathcal{D})/\mathcal{K}_0(\mathcal{C}) \rightarrow 0.$$

Theorem 8.1 holds for a broader class of additive Karoubian functors (theorem 8.5).

**Definition 8.2** ([HLS23] Definition 7.4). Let  $f : X \rightarrow Y$  be a map of  $\mathbb{E}_\infty$ -monoids. We will say that  $f$  is *cofinal* if it satisfies the following conditions:

- (1) The map  $\pi_0(f) : \pi_0(X) \rightarrow \pi_0(Y)$  is an inclusion.
- (2) For every element  $x$  in  $\pi_0(X)$ , there exists an element  $x'$  in  $\pi_0(X)$  such that  $x + x'$  is in  $\pi_0(Y)$ .

We will say that a cofinal map is *dense* if it satisfies the following condition:

- (3) The sequence of  $\mathbb{E}_\infty$ -monoids  $0 \rightarrow \pi_0(X) \rightarrow \pi_0(Y) \rightarrow \pi_0(Y)/\pi_0(X) \rightarrow 0$  is exact.

Or equivalently,

- (3') An element  $y$  in  $\pi_0(Y)$  belongs to  $\pi_0(X)$  if there exists an element  $x$  in  $\pi_0(X)$  such that  $x + y$  is in  $\pi_0(X)$ .

**Definition 8.3** ([HLS23] Definition 7.6). Let  $F : \text{Cat}^{\text{st}} \rightarrow \text{An}$  be an additive functor. We will say that  $F$  is *Karoubian* if it satisfies the following conditions:

- (1) The functor  $F$  takes every dense inclusion between stable categories to a dense map of  $\mathbb{E}_\infty$ -monoids.
- (2) The functor  $F$  preserves every cartesian square in  $\text{Cat}^{\text{st}}$  whose vertical maps are dense.

**Definition 8.4.** Let  $F : \text{Cat}^{\text{st}} \rightarrow \text{An}$  be an additive Karoubian functor. We will refer to the functor

$$F^{\text{grp}}(-) := \Omega|FQ(-)| : \text{Cat}^{\text{st}} \rightarrow \text{An}$$

as the *group completion* of  $F$ .

**Theorem 8.5** ([HLS23] Theorem 7.7). *Let  $F : \text{Cat}^{\text{st}} \rightarrow \text{An}$  be an additive Karoubian functor. For every dense inclusion  $\mathcal{C} \subseteq \mathcal{D}$  of stable categories, the canonical map of  $\mathbb{E}_\infty$ -monoids*

$$F(\mathcal{D})/F(\mathcal{C}) \rightarrow F^{\text{grp}}(\mathcal{D})/F^{\text{grp}}(\mathcal{C})$$

*is an equivalence. Hence maps of abelian groups*

$$\pi_i F^{\text{grp}}(\mathcal{C}) \rightarrow \pi_i F^{\text{grp}}(\mathcal{D})$$

*are isomorphisms for every  $i \geq 1$ , and there exists a short exact sequence*

$$0 \rightarrow \pi_0 F^{\text{grp}}(\mathcal{C}) \rightarrow \pi_0 F^{\text{grp}}(\mathcal{D}) \rightarrow \pi_0 F^{\text{grp}}(\mathcal{D})/\pi_0 F^{\text{grp}}(\mathcal{C}) \rightarrow 0.$$

**Corollary 8.6** ([HLS23] Corollary 7.8). Let  $F : \text{Cat}^{\text{st}} \rightarrow \text{An}$  be an additive Karoubian functor. Then the group completion

$$F^{\text{grp}} : \text{Cat}^{\text{st}} \rightarrow \text{An}$$

of  $F$  is also additive Karoubian. Moreover if  $F^{\text{grp}}$  is Verdier-localizing, then the functor

$$F^{\text{grp}} \circ (-)^{\natural} : \text{Cat}^{\text{st}} \rightarrow \text{An}$$

is Karoubi-localizing.

**Example 8.7.** The core functor  $\text{core} : \text{Cat}^{\text{st}} \rightarrow \text{An}$  is (additive) Karoubian.

(1): Let  $\mathcal{C} \hookrightarrow \mathcal{D}$  be a dense inclusion of stable categories. Then the map  $\text{core } f : \text{core } \mathcal{C} \rightarrow \text{core } \mathcal{D}$  is cofinal. Let  $d$  be an element in  $\pi_0(\text{core } \mathcal{D})$ , and let  $c$  be an element in  $\pi_0(\text{core } \mathcal{C})$  such that  $c + d$  is in  $\pi_0(\text{core } \mathcal{C})$ . Then we have an equivalence  $d \simeq \text{fib}(c \oplus d \rightarrow c)$ . The element  $d$  is in  $\pi_0(\text{core } \mathcal{C})$  since  $\mathcal{C}$  is closed under fibers. That is,  $\text{core } f$  is dense.

(2): The core functor preserves limits, since it is a right adjoint functor.

**Example 8.8.** The group completion

$$\text{core}^{\text{grp}} : \text{Cat}^{\text{st}} \rightarrow \text{An}$$

of  $\text{core} : \text{Cat}^{\text{st}} \rightarrow \text{An}$  is equivalent to the algebraic K-theory  $\mathcal{K} : \text{Cat}^{\text{st}} \rightarrow \text{An}$ .

**Example 8.9.** By corollary 8.6 and example 8.8, the algebraic K-theory  $\mathcal{K} : \text{Cat}^{\text{st}} \rightarrow \text{An}$  is additive Karoubian. By theorem 6.1,  $\mathcal{K}$  is Verdier-localizing. Then the functor

$$\mathcal{K} \circ (-)^{\natural} : \text{Cat}^{\text{st}} \rightarrow \text{An}$$

is Karoubi-localizing.

## 9. THE NON-CONNECTIVE K-THEORY SPECTRUM

In this section, we introduce the connective K-theory spectrum and non-connective K-theory spectrum. Our primary reference is [KNP24], though it contains some minor mistakes and typographical errors. We correct and adjust these as necessary.

### 9.1. The Connective K-Theory Spectrum.

**Remark 9.1.** As stated in example 8.9, the functor

$$\mathcal{K} \circ (-)^{\natural} : \text{Cat}^{\text{st}} \rightarrow \text{An}$$

is Karoubi-localizing. In particular, it is an additive functor. Moreover, by proposition 3.19, it is also grouplike. Consequently, it lifts to a functor

$$\mathcal{K} \circ (-)^{\natural} : \text{Cat}^{\text{st}} \rightarrow \text{Grp}_{\mathbb{E}_{\infty}}(\text{An}).$$

Since there exists an equivalence  $\text{Grp}_{\mathbb{E}_{\infty}}(\text{An}) \simeq \text{Sp}_{\geq 0}$ , we obtain a functor

$$\mathcal{K} \circ (-)^{\natural} : \text{Cat}^{\text{st}} \rightarrow \text{Sp}_{\geq 0}.$$

**Definition 9.2** ([KNP24] Definition 3.1.2). We will refer to the functor constructed above

$$\mathcal{K} \circ (-)^{\natural} : \text{Cat}^{\text{st}} \rightarrow \text{Sp}_{\geq 0}$$

as the *connective K-theory*. We often simply denote it by  $\mathcal{K} : \text{Cat}^{\text{st}} \rightarrow \text{Sp}_{\geq 0}$ .

**Remark 9.3.** For every Karoubi square

$$\begin{array}{ccc} \mathcal{C}' & \longrightarrow & \mathcal{C} \\ \downarrow \lrcorner & & \downarrow \\ \mathcal{D}' & \longrightarrow & \mathcal{D} \end{array}$$

in  $\text{Cat}^{\text{st}}$ , we obtain a cartesian square

$$\begin{array}{ccc} \mathcal{K}(\mathcal{C}') & \longrightarrow & \mathcal{K}(\mathcal{C}) \\ \downarrow \lrcorner & & \downarrow \\ \mathcal{K}(\mathcal{D}') & \longrightarrow & \mathcal{K}(\mathcal{D}) \end{array}$$

in  $\text{Sp}_{\geq 0}$ . However, in general, it is not a cartesian square in  $\text{Sp}$  since the inclusion  $\text{Sp}_{\geq 0} \subseteq \text{Sp}$  is not left exact.

**Remark 9.4.** For every Karoubi sequence  $\mathcal{C} \rightarrow \mathcal{D} \rightarrow \mathcal{E}$ , we obtain a long exact sequence

$$\cdots \rightarrow \mathcal{K}_1(\mathcal{E}) \rightarrow \mathcal{K}_0(\mathcal{C}) \rightarrow \mathcal{K}_0(\mathcal{D}) \rightarrow \mathcal{K}_0(\mathcal{E}).$$

However, in general, the last map  $\mathcal{K}_0(\mathcal{D}) \rightarrow \mathcal{K}_0(\mathcal{E})$  is not surjective. To extend this sequence, we need to define the non-connective K-theory spectrum.

**9.2. The Non-connective K-Theory Spectrum.** In this section, let  $\kappa$  be an uncountable regular cardinal. For a category  $\mathcal{C}$ , we let  $\text{Ind}_{\kappa}(\mathcal{C})$  denote the category of Ind-objects. If  $\kappa = \omega$ , we denote it by  $\text{Ind}(\mathcal{C})$ .

**Example 9.5.** Let  $\mathcal{C}$  be an idempotent complete stable category. By proposition 3.9, the sequence

$$\mathcal{C} \rightarrow \text{Ind}(\mathcal{C})^{\kappa} \rightarrow \text{Ind}(\mathcal{C})^{\kappa}/\mathcal{C}$$

in  $\text{Cat}^{\text{st}}$  is a Karoubi sequence. Then we obtain the fiber sequence

$$\mathcal{K}(\mathcal{C}) \rightarrow \mathcal{K}(\text{Ind}(\mathcal{C})^{\kappa}) \rightarrow \mathcal{K}(\text{Ind}(\mathcal{C})^{\kappa}/\mathcal{C}).$$

By the Eilenberg swindle (corollary 5.5), we have  $\mathcal{K}(\text{Ind}_{\omega}(\mathcal{C})^{\kappa}) = 0$  since it has countable coproducts. Then we get an equivalence

$$\mathcal{K}(\mathcal{C}) \simeq \Omega \mathcal{K}(\text{Ind}(\mathcal{C})^{\kappa}/\mathcal{C}).$$

However, in general,  $\text{Ind}(\mathcal{C})^{\kappa}/\mathcal{C}$  is not an idempotent complete. Thus we cannot iterate this delooping process.

**Example 9.6.** Let  $\mathcal{C}$  be a stable category. By the cofinality theorem (theorem 8.1), the induced map

$$\mathcal{K}(\text{Ind}(\mathcal{C})^{\kappa}/\mathcal{C}) \rightarrow \mathcal{K}((\text{Ind}(\mathcal{C})^{\kappa}/\mathcal{C})^{\natural})$$

is an equivalence on connective covers, namely as objects of  $\text{Sp}_{\geq 0}$ . Then we obtain an equivalence

$$\mathcal{K}(\mathcal{C}) \simeq \tau_{\geq 0} \Omega \mathcal{K}((\text{Ind}(\mathcal{C})^{\kappa}/\mathcal{C})^{\natural}).$$

In the above discussion, we have  $\mathcal{K}_0(\text{Ind}(\mathcal{C})^{\kappa}/\mathcal{C}) = 0$ , but  $\mathcal{K}_0((\text{Ind}(\mathcal{C})^{\kappa}/\mathcal{C})^{\natural})$  is not necessarily zero. This is why the negative algebraic K-theory and its (negative) K-groups arise. We will define it as the  $(-1)$ -th K-group.

**Definition 9.7** ([KNP24] Definition 3.2.7). Let  $\mathcal{C}$  be an idempotent complete stable category. Then we define the *Calkin category*  $\mathrm{Calk}^{\natural}(\mathcal{C})$  of  $\mathcal{C}$  by

$$\mathrm{Calk}^{\natural}(\mathcal{C}) := (\mathrm{Ind}(\mathcal{C})^{\kappa} / \mathcal{C})^{\natural}$$

Let  $\mathcal{C}$  be a stable category. For every  $n \geq 0$ , we inductively define the *Calkin category*  $\mathrm{Calk}(\mathcal{C})$  of  $\mathcal{C}$  by

$$\mathrm{Calk}^0(\mathcal{C}) := \mathcal{C}^{\natural} \quad \text{and} \quad \mathrm{Calk}^{n+1}(\mathcal{C}) := \mathrm{Calk}^{\natural}(\mathrm{Calk}^n(\mathcal{C})).$$

**Remark 9.8.** For every  $n \geq 0$ , the construction  $\mathcal{C} \mapsto \mathrm{Calk}^n(\mathcal{C})$  defines a functor

$$\mathrm{Calk}^n : \mathrm{Cat}^{\mathrm{st}} \rightarrow \mathrm{Cat}^{\mathrm{perf}}.$$

**Definition 9.9** ([KNP24] Definition 3.2.9). Let  $\mathcal{C}$  be a stable category. We define the *non-connective K-theory spectrum*  $\mathbb{K}(\mathcal{C})$

$$\mathbb{K}(\mathcal{C}) := \mathrm{colim}_n \Omega^{\infty-n} \mathcal{K}(\mathrm{Calk}^n(\mathcal{C})),$$

where taking the colimit in  $\mathrm{Sp}$ . Equivalently, we can define by

$$\tau_{\geq -n} \mathbb{K}(\mathcal{C}) := \Omega^n \mathcal{K}(\mathrm{Calk}^n(\mathcal{C})) \quad \text{and} \quad \mathbb{K}(\mathcal{C}) := \mathrm{colim}_n \tau_{\geq -n} \mathbb{K}(\mathcal{C}).$$

**Definition 9.10.** Let  $\mathcal{C}$  be a stable category. The construction  $\mathcal{C} \mapsto \mathbb{K}(\mathcal{C})$  determines a functor

$$\mathbb{K} : \mathrm{Cat}^{\mathrm{st}} \rightarrow \mathrm{Sp}.$$

We will refer to it as the *non-connective K-theory*.

**Definition 9.11.** Let  $\mathcal{C}$  be a stable category. For every integer  $n$ , we define the  $n$ -th K-group  $\mathbb{K}_n(\mathcal{C})$  of  $\mathcal{C}$

$$\mathbb{K}_n(\mathcal{C}) := \pi_n \mathbb{K}(\mathcal{C}).$$

**Remark 9.12.** Let  $\mathcal{C}$  be a stable category. For every  $n > 0$ , we obtain an isomorphism

$$\mathcal{K}_n(\mathcal{C}) = \mathbb{K}_n(\mathcal{C}),$$

and

$$\mathcal{K}_0(\mathcal{C}) \rightarrow \mathbb{K}_0(\mathcal{C})$$

is injective. Moreover, we have an isomorphism

$$\mathbb{K}_{-1}(\mathcal{C}) = \mathcal{K}_0((\mathrm{Ind}(\mathcal{C})^{\kappa} / \mathcal{C})^{\natural}).$$

## APPENDIX A. PROOFS IN SECTION 3 AND SECTION 4

**Proof A.1** (Remark 3.12). We show that every Verdier square is a cocartesian diagram in  $\text{Cat}^{\text{st}}$ . Consider the following Verdier square

$$\begin{array}{ccc} \mathcal{D} & \longrightarrow & \mathcal{D}' \\ \downarrow \lrcorner & & \downarrow \\ \mathcal{E} & \longrightarrow & \mathcal{E}'. \end{array}$$

Since the vertical maps are Verdier projections, we can extend it to the following diagram.

$$\begin{array}{ccccc} \mathcal{C} & \longrightarrow & \mathcal{D} & \longrightarrow & \mathcal{D}' \\ \downarrow & & \downarrow \lrcorner & & \downarrow \\ 0 & \longrightarrow & \mathcal{E} & \longrightarrow & \mathcal{E}'. \end{array}$$

By definition, the left and outer squares are bicartesian squares. Then the right square is also a bicartesian square.

**Proof A.2** (Proposition 3.15). (1)  $\Rightarrow$  (2) is obvious.

(2)  $\Rightarrow$  (1): Consider the following Verdier square

$$\begin{array}{ccc} \mathcal{D} & \longrightarrow & \mathcal{D}' \\ \downarrow \lrcorner & & \downarrow \\ \mathcal{E} & \longrightarrow & \mathcal{E}'. \end{array}$$

Then we can extend it to the following diagram.

$$\begin{array}{ccccc} \mathcal{C} & \longrightarrow & \mathcal{D} & \longrightarrow & \mathcal{D}' \\ \downarrow \lrcorner & & \downarrow \lrcorner & & \downarrow \\ 0 & \longrightarrow & \mathcal{E} & \longrightarrow & \mathcal{E}'. \end{array}$$

By definition, the left, right, and outer squares are cartesian squares. Thus sequences  $\mathcal{C} \rightarrow \mathcal{D} \rightarrow \mathcal{E}$  and  $\mathcal{C} \rightarrow \mathcal{D}' \rightarrow \mathcal{E}'$  are Verdier in  $\text{Cat}^{\text{st}}$ . By assumption, sequences  $F(\mathcal{C}) \rightarrow F(\mathcal{D}) \rightarrow F(\mathcal{E})$  and  $F(\mathcal{C}) \rightarrow F(\mathcal{D}') \rightarrow F(\mathcal{E}')$  are fiber sequences in  $\mathcal{E}$ . Then the left and outer squares in the following diagram are cartesian squares.

$$\begin{array}{ccccc} F(\mathcal{C}) & \longrightarrow & F(\mathcal{D}) & \longrightarrow & F(\mathcal{D}') \\ \downarrow \lrcorner & & \downarrow \lrcorner & & \downarrow \\ * & \longrightarrow & F(\mathcal{E}) & \longrightarrow & F(\mathcal{E}'). \end{array}$$

Then the right square is also a cartesian square.

**Proof A.3** (Proposition 3.18). We show that every Verdier-localizing functor preserves finite products. Let  $\mathcal{E}$  be a category with finite limits, and let  $F : \text{Cat}^{\text{st}} \rightarrow \mathcal{E}$  be a Verdier-localizing functor. The following diagram is a cartesian square in  $\text{Cat}^{\text{st}}$ .

$$\begin{array}{ccc} \mathcal{C} \times \mathcal{D} & \longrightarrow & \mathcal{D} \\ \downarrow \lrcorner & & \downarrow \\ \mathcal{C} & \longrightarrow & 0. \end{array}$$

Applying the functor  $F$ , we obtain the following cartesian square in  $\mathcal{E}$ .

$$\begin{array}{ccc} F(\mathcal{C} \times \mathcal{D}) & \xrightarrow{\quad} & F(\mathcal{D}) \\ \downarrow & \lrcorner & \downarrow \\ F(\mathcal{C}) & \longrightarrow & * \end{array}$$

This implies that

$$F(\mathcal{C} \times \mathcal{D}) \simeq F(\mathcal{C}) \times F(\mathcal{D}).$$

To prove proposition 4.9, we need some preliminaries.

**Notation A.4.** For every  $n \geq 0$ , we let  $\mathcal{J}_n$  denote the full subcategory of  $\mathrm{TwAr}[n]$  spanned by the images of objects  $(i \leq j)$  satisfying  $j \leq i + 1$ .

**Lemma A.5.** Let  $\mathcal{C}$  be a category with finite limits, and let  $F : \mathrm{TwAr}[n] \rightarrow \mathcal{E}$  be a functor. The following conditions are equivalent:

- (1) The functor  $F$  belongs to  $Q_n(\mathcal{C})$ .
- (2) The functor  $F$  is the right Kan extension of its restriction to  $\mathcal{J}_n$  along the inclusion  $\mathcal{J}_n \subseteq \mathrm{TwAr}[n]$ .

*Proof.* The map

$$e_i : [1] \rightarrow [n] : 0 \mapsto i \text{ and } 1 \mapsto i + 1$$

in  $\Delta$  induces an equivalence of categories

$$\mathcal{J}_n \simeq \mathcal{J}_1 \coprod_{\mathcal{J}_0} \mathcal{J}_1 \cdots \coprod_{\mathcal{J}_0} \mathcal{J}_1 \simeq \mathrm{TwAr}[1] \coprod_{\mathrm{TwAr}[0]} \mathrm{TwAr}[1] \cdots \coprod_{\mathrm{TwAr}[0]} \mathrm{TwAr}[1]$$

in  $\mathrm{Cat}^{\mathrm{lex}}$ . Then the right Kan extension along the inclusion  $\mathcal{J}_n \subseteq \mathrm{TwAr}[n]$  factors through  $n(n-1)/2$ -times the right Kan extension along the inclusion  $\mathcal{J}_i \subseteq \mathrm{TwAr}[i]$  for every  $n \geq 2$ . The right Kan extension along the inclusion  $\mathcal{J}_2 \subseteq \mathrm{TwAr}[2]$  correspondences the operation of taking the pullback. Then the equivalence of conditions follows from that a functor  $F$  belong to  $Q_n(\mathcal{C})$  if and only if each square is a cartesian square.  $\square$

The next corollary follows from lemma A.5 immediately.

**Corollary A.6.** Let  $\mathcal{C}$  be a category with finite limits. Then the restriction of  $\mathrm{Fun}(\mathrm{TwAr}[n], \mathcal{C})$  along the inclusion  $\mathcal{J}_n \subseteq \mathrm{TwAr}[n]$  induces an equivalence of categories

$$Q_n(\mathcal{C}) \rightarrow \mathrm{Fun}(\mathcal{J}_n, \mathcal{C}).$$

**Proof A.7** (Proposition 4.9). (TBA)

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