Categories in Tokyo 1st

2024/11/23

An Introduction to Stable ∞-Categories

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Goals

(1) Answer the question " What is a stable ∞-cat ? "

(2) Understand the def of spectra !

Que : What is a stable ∞-cat ?

Ans : It is an ∞-version of $\left(\begin{array}{l} \text{an abelian cat} \\ \text{a triangulated cat } (\Delta\text{-cat}). \end{array} \right.$

Plan

(1) Review the classical theory of Δ-cats.
In particular, bad behaviors of Δ-cats.

(2) Definition of stable ∞-cats.

(3) Constructions and properties of stable ∞-cats.

(4) How to construct stable ∞-cats ? : Stabilization.

(5) The most important example Sp : the cat of spectra.

References

1. J. Lurie. Higher Algebra Sec 1, 2017.

2. Y. Harpaz, Introduction to stable ∞-cats, 2013.

1. Review the classical theory of Δ-cats.

Def. 1.1.

A Δ-cat $(\mathcal{T}, \Sigma, \mathcal{E})$ consists of the following data:

- An additive cat $\mathcal{T}$,
- An equiv of cats $\Sigma : \mathcal{T} \to \mathcal{T} : X \mapsto \Sigma X$.
- $\mathcal{E}$ = the set of diagrams $X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} \Sigma X$, called distinguished triangles.

satisfying some axioms.

(TR3) For $\forall$ diagram
$$\begin{tikzcd} X \arrow[r] \arrow[d] \arrow[dr, phantom, "\circlearrowright"] & Y \arrow[d] \\ X' \arrow[r] & Y' \end{tikzcd}$$
, $\exists$ morph $Z \to Z' \in \mathcal{C}$

s.t.
$$\begin{tikzcd} X \arrow[r] \arrow[d] & Y \arrow[r] \arrow[d] & Z \arrow[r] \arrow[d] & \Sigma X \arrow[d] & \in \mathcal{E} \\ X' \arrow[r] & Y' \arrow[r] & Z' \arrow[r] & \Sigma X' & \in \mathcal{E}. \end{tikzcd}$$

not unique. non-canonical.

Rem. 1.2. $(\mathcal{T}, \Sigma, \mathcal{E})$: Δ-cat

For $\forall f : X \to Y \in \mathcal{T}$, $\exists$ obj $Z \in \mathcal{T}$. s.t.

$X \xrightarrow{f} Y \to Z \to \Sigma X \in \mathcal{E}$, where Z is uniquely determined only up to isomorphism.

$\rightsquigarrow$ We cannot define a func $\mathrm{Arr}(\mathcal{C}) \to \mathcal{C}$
$f \longmapsto Z$.

Rem. 1.3.

For example, in the derived cat $D(R)$ of complexes,
the cone is defined only up to homotopy.
However, data of homotopies is not contained in $D(R)$.

Consider a new cat which have higher information:
objects, morphisms, homotopies, homotopies b/w homotopies, ...

2. Definitions of stable ∞-cats.

Def. 2.1. $\mathcal{C}$: ∞-cat

- An obj $\phi \in \mathcal{C}$ is an initial obj
  $\overset{\text{def}}{\iff}$ For $^\forall X \in \mathcal{C}$, $\mathrm{Map}_{\mathcal{C}}(\phi, X)$ is contractible.

- An obj $1 \in \mathcal{C}$ is an terminal obj
  $\overset{\text{def}}{\iff}$ For $^\forall X \in \mathcal{C}$, $\mathrm{Map}_{\mathcal{C}}(X, 1)$ is contractible.

- An obj $0 \in \mathcal{C}$ is a zero obj
  $\overset{\text{def}}{\iff}$ It is both an initial and a terminal obj.

- $\mathcal{C}$ is pointed (pted) $\overset{\text{def}}{\iff}$ $\mathcal{C}$ contains a zero obj 0.

Def. 2.2. $\mathcal{C}$: pted ∞-cat, 0 : zero obj of $\mathcal{C}$.

- A triangle in $\mathcal{C}$.
  $\overset{\text{def}}{\iff}$ A func $\Delta^1 \times \Delta^1 \to \mathcal{C}$ depicted as

$$\begin{tikzcd} X \arrow[r] \arrow[d] & Y \arrow[d] \\ 0 \arrow[r] & Z \end{tikzcd}.$$

- A triangle is a cofiber seq $\overset{\text{def}}{\iff}$ It is a cocart. square.

- A triangle is a fiber seq $\overset{\text{def}}{\iff}$ It is a cart. square.

Def. 2.3. $\mathcal{C}$: pted ∞-cat. $f : X \to Y \in \mathcal{C}$.

- A **cofiber** of f $\overset{\text{def}}{\iff}$ A cofiber seq
$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d] & Y \arrow[d] \\ 0 \arrow[r] & \mathrm{cofib}(f) \arrow[ul, phantom, "\ulcorner", very near start] \end{tikzcd}$$

- A **fiber** of f $\overset{\text{def}}{\iff}$ A fiber seq
$$\begin{tikzcd} \mathrm{fib}(f) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & X \arrow[d, "f"] \\ 0 \arrow[r] & Y \end{tikzcd}$$

Rem. 2.4. $\mathcal{C}$: pted ∞-cat.

- A cofiber of f, if it exists, is determined unique up to equiv.

- $\mathcal{E} = \{ \text{cofiber seqs} \} \subseteq \mathrm{Fun}(\Delta^1 \times \Delta^1, \mathcal{C})$

Define $\theta : \mathcal{E} \longrightarrow \mathrm{Fun}(\Delta^1, \mathcal{C})$

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d] & Y \arrow[d] \\ 0 \arrow[r] & Z \arrow[ul, phantom, "\ulcorner", very near start] \end{tikzcd} \longmapsto (X \xrightarrow{f} Y)$$

If $^\forall$ morph in $\mathcal{C}$ admits a cofiber, θ is a trivial fibration.

$\leadsto$ θ admits a section $\mathrm{cofib} : \mathrm{Fun}(\Delta^1, \mathcal{C}) \to \mathrm{Fun}(\Delta^1 \times \Delta^1, \mathcal{C})$

We also denote $\mathrm{cofib} : \mathrm{Fun}(\Delta^1, \mathcal{C}) \to \mathrm{Fun}(\Delta^1 \times \Delta^1, \mathcal{C}) \xrightarrow{ev_{(1,1)}} \mathcal{C}$

It is a functor!

Def. 2.5.

An ∞-cat $\mathcal{C}$ is stable $\overset{\text{def}}{\iff}$ It satisfies :

(0) $\mathcal{C}$ is pted. i.e. $\exists$ zero obj $0 \in \mathcal{C}$.

(1) $\forall$ morphism in $\mathcal{C}$ admits a fiber and a cofiber.

(2) A triangle in $\mathcal{C}$ is a fiber seq iff a cofiber seq.

Rem. 2.6.

- Having a Δ-str of additive cat is " structure."

  That is, we cannot ask whether an additive cat is a Δ-cat without specifying Σ and dist. triangles.

$\updownarrow$

- Stability of ∞-cats is " property."

  We can check the given ∞-cat is stable or not !

3. Constructions and properties of stable ∞-cats.

Every obj in stable ∞-cats has "suspension" and "looping."

Nota. 3.1. $\mathcal{C}$: stable ∞-cats

For every obj $X \in \mathcal{C}$, define objs ΩX and ΣX as :

$$\begin{tikzcd} X \arrow[r] \arrow[d] & 0 \arrow[d] \\ 0 \arrow[r] & \Sigma X \arrow[ul, phantom, "\ulcorner", very near start] \end{tikzcd} \qquad \begin{tikzcd} \Omega X \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & 0 \arrow[d] \\ 0 \arrow[r] & X \end{tikzcd}$$

Rem. 3.2.

As stated in Rem. 2.4, the constructions

$X \mapsto \Sigma X$ and $X \mapsto \Omega X$ define functors

$\Sigma : \mathcal{C} \to \mathcal{C}$ and $\Omega : \mathcal{C} \to \mathcal{C}$.

Rem. 3.3.

- The previous constructions can be defined for ∞-cats $\mathcal{C}$ with pushouts and pullbacks.
  Then two funcs determine an adj $\Sigma \dashv \Omega$.
- If $\mathcal{C}$ is stable, the adj $\Sigma \dashv \Omega$ is an equiv.
  ($\because$ pushouts $\Leftrightarrow$ pullbacks)

The htpy cat $h\mathcal{C}$ of a stable ∞-cat $\mathcal{C}$ admits a Δ-str.

We need to show ;

(i) An additive str on $h\mathcal{C}$.

(ii) The translation func $\Sigma : h\mathcal{C} \to h\mathcal{C}$.

(iii) The set of dist. triangles $X \to Y \to Z \to X[1]$.

Lem. 3.4. $\mathcal{C}$; stable ∞-cat

The htpy cat $h\mathcal{C}$ is an additive cat.

proof.

- For $^\forall X, Y \in \mathcal{C}$, $\mathrm{Map}_{(h)\mathcal{C}}(X, Y)$ has the base pt, given by the zero map $X \to 0 \to Y$.
- Since $\mathrm{Map}_{\mathcal{C}}(-, Y)$ sends colims to lims,

$$\mathrm{Map}_{\mathcal{C}}(\Sigma X, Y) \underset{(1)}{\xrightarrow{\sim}} \Omega\,\mathrm{Map}_{\mathcal{C}}(X, Y)$$

- $$\begin{aligned} \pi_0 \mathrm{Map}_{\mathcal{C}}(X, Y) &\simeq \pi_0 \mathrm{Map}_{\mathcal{C}}(\Sigma^2 X', Y) && (\Sigma \text{ is an equiv}) \\ &\simeq \pi_0 \Omega^2 \mathrm{Map}_{\mathcal{C}}(X', Y) && (1) \\ &\simeq \pi_2 \mathrm{Map}_{\mathcal{C}}(X', Y). && (\text{def of } \pi_n) \end{aligned}$$

$\rightsquigarrow$ Hom-set of $h\mathcal{C}$ admits an abelian group str. $\square$

Def. 3.5. $\mathcal{C}$: stable ∞-cat.

- Define the func $\Sigma : \mathcal{C} \to \mathcal{C}$

$$\begin{array}{ccc} \mathcal{C} & \to & \mathcal{C} \\ \rotatebox{90}{$\in$} & & \rotatebox{90}{$\in$} \\ X & \mapsto & \Sigma X \end{array} \quad (\text{p.o. of } 0 \leftarrow X \to 0)$$

- $\rightsquigarrow$ It induces the func $\Sigma : h\mathcal{C} \to h\mathcal{C}$.
  (an equiv of 1-cats).

Def. 3.6. $\mathcal{C}$: stable ∞-cat

- A distinguished diagram in $\mathcal{C}$

  $\overset{\text{def}}{\iff}$ A diagram $\Delta^1 \times \Delta^2 \to \mathcal{C}$ depicted as

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d] & Y \arrow[r] \arrow[d, "g"] & 0 \arrow[d] \\ 0 \arrow[r] \arrow[ur, phantom, "\ulcorner"] & Z \arrow[r, "h"'] \arrow[ur, phantom, "\ulcorner"] & W \end{tikzcd}.$$

- By def,

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d] & Y \arrow[r] \arrow[d, "g"] & 0 \arrow[r, equal] \arrow[d] & 0 \arrow[d] \\ 0 \arrow[r] & Z \arrow[r, "h"'] \arrow[ur, phantom, "\ulcorner"] & W \arrow[r, "\sim", "\phi"'] \arrow[ur, phantom, "\ulcorner"] & \Sigma X \end{tikzcd} \quad \text{in } \mathcal{C}.$$

  $\rightsquigarrow$ we obtain a diagram $X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{\phi h} \Sigma X$ (✱) in $h\mathcal{C}$.

- A diagram of the form $\bullet \to \bullet \to \bullet \longrightarrow \bullet$ is a distinguished triangle in $h\mathcal{C}$.

  $\overset{\text{def}}{\iff}$ It is isomorphic in $h\mathcal{C}$ to (✱) for $\exists\, f : \Delta^1 \to \mathcal{C}$.

Rem. 3.7.

We can construct long (co)fiber sequences.

$$\begin{tikzcd} X \arrow[r] \arrow[d] & Y \arrow[r] \arrow[d] \arrow[dl, phantom, "\ulcorner", very near start] & 0 \arrow[d] \arrow[dl, phantom, "\ulcorner", very near start] \\ 0 \arrow[r] & Z \arrow[r] & W \end{tikzcd} \longrightarrow \begin{tikzcd} X \arrow[r] \arrow[d] & Y \arrow[r] \arrow[d] & 0 \arrow[r, equal] \arrow[d] & 0 \arrow[d] \arrow[dl, phantom, "\ulcorner", very near start] \\ 0 \arrow[r] & Z \arrow[r] & W \arrow[r, phantom, "\simeq"] & \Sigma X \end{tikzcd}$$

$$\longrightarrow \begin{tikzcd} X \arrow[r] \arrow[d] & Y \arrow[r] \arrow[d] & 0 \arrow[d] \\ 0 \arrow[r] & Z \arrow[r] \arrow[d] & \Sigma X \arrow[d] \arrow[dl, phantom, "\ulcorner", very near start] \\ & 0 \arrow[r] & \Sigma Y \end{tikzcd} \longrightarrow \begin{tikzcd} X \arrow[r] \arrow[d] & Y \arrow[r] \arrow[d] & 0 \arrow[d] & \\ 0 \arrow[r] & Z \arrow[r] \arrow[d] & \Sigma X \arrow[r] \arrow[d] & 0 \arrow[d] \arrow[dl, phantom, "\ulcorner", very near start] \\ & 0 \arrow[r] & \Sigma Y \arrow[r] & \Sigma Z \end{tikzcd}$$

$$\longrightarrow \begin{tikzcd} X \arrow[r] \arrow[d] & Y \arrow[r] \arrow[d] & 0 \arrow[d] & & \\ 0 \arrow[r] & Z \arrow[r] \arrow[d] & \Sigma X \arrow[r] \arrow[d] & 0 \arrow[d] \arrow[r] & \cdots \\ & 0 \arrow[r] & \Sigma Y \arrow[r] & \Sigma Z & \end{tikzcd}$$

Prop. 3.8. $\mathcal{C}$; stable ∞-cat.

(1) $\mathcal{C}$ admits pushouts and pullbacks. (see next page)

$\downarrow$ cor.

(2) $\mathcal{C}$ admits finite colims and limits.

(3) A square in $\mathcal{C}$ is cocart $\Leftrightarrow$ cart.

$\downarrow$ cor.

(4) The coproducts in $\mathcal{C}$ coincide with products.

Stable ∞-cats are closed under many constructions.

Prop. 3.9. $\mathcal{C}$; stable ∞-cat.

(1) $\mathcal{C}^{op}$ is stable.

(2) For $\forall K \in sSet$, $Fun(K, \mathcal{C})$ is stable.

(3) For $\forall$ regular cardinal κ, $Ind_\kappa(\mathcal{C})$ is stable.

(4) The idempotent completion $Idem(\mathcal{C})$ is stable.

proof of prop. 3.8.

(1) We want to show $\exists$ pushout of
$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "g"'] & Y . \\ Z & \end{tikzcd}$$

- Since g admits a fiber
$$\begin{tikzcd} \mathrm{fib}(g) \arrow[r, "h"] \arrow[d] \arrow[dr, phantom, "\lrcorner" very near start] & X \arrow[r, "f"] \arrow[d, "g"] & Y . \\ 0 \arrow[r] & Z & \end{tikzcd}$$

- Since fh admits a cofiber
$$\begin{tikzcd} \mathrm{fib}(g) \arrow[r, "h"] \arrow[d] & X \arrow[r, "f"] \arrow[d] & Y \arrow[d] \\ 0 \arrow[r] & Z \arrow[r] \arrow[ur, phantom, "\ulcorner" very near end] & \mathrm{cofib}\,\mathrm{fib}(g) \end{tikzcd}$$

Since left and outer squares are cocartesian,

the right square is also cocartesian. $\square$.

(2) Since $\mathcal{C}$ admits an initial obj and pushouts (1),

$\mathcal{C}$ admits finite colims. $\square$

4. How to construct stable ∞-cats? ; Stabilization

We construct a stable ∞-cat from ∞-cats in two ways.

via (co) homology theory and spectrum objects.

Recall

A pted ∞-cat $\mathcal{C}$ is stable $\overset{\text{def}}{\Longleftrightarrow}$ It satisfies :

(1) $\forall$ morph in $\mathcal{C}$ admits a fiber and a cofiber.

(2) A triangle in $\mathcal{C}$ is a fiber seq iff a cofiber seq.

Thrm.4.1. $\mathcal{C}$: pted ∞-cat. TFAE

(1) $\mathcal{C}$ is stable.

(2) $\forall$ morph in $\mathcal{C}$ admits a fiber and the loop func $\Omega : \mathcal{C} \to \mathcal{C}$ is an equiv.

$\rightsquigarrow$ From a pted ∞-cat with finite lims $\mathcal{C}$,

We can formally add inverses $X \leftrightarrows \Omega X$.

i.e. formally invert the loop by taking

the limit of the tower $\cdots \xrightarrow{\Omega} \mathcal{C} \xrightarrow{\Omega} \mathcal{C}$.

Nota. 4.2. $\mathcal{C}$: pted ∞-cat with finite lims.

$$Sp^{\Omega}(\mathcal{C}) := \lim\left(\cdots \xrightarrow{\Omega} \mathcal{C} \xrightarrow{\Omega} \mathcal{C} \right) \text{ in } \mathrm{Cat}^{\mathrm{fin\,lim}}.$$

obj : a seq $\{X_n\}$ with $X_n \xrightarrow{\sim} \Omega X_{n+1}$!

Thrm. 4.3.

There is the following adj :

$$\begin{tikzcd} \mathrm{Cat}^{st} \arrow[d, shift right=2, "\mathrm{inc}"'] \arrow[d, phantom, "\dashv"] \\ \mathrm{Cat}^{\mathrm{fin\,lim}} \arrow[u, shift right=2, "Sp^{\Omega}"'] \end{tikzcd}$$

$\mathrm{Cat}^{st} := \{\text{stable } \infty\text{-cats}\}$

$\mathrm{Cat}^{\mathrm{fin\,lim}} := \{\text{pted } \infty\text{-cat with finite lims}\}$

Rem. 4.4.

We get the counit $\Omega^{\infty} : Sp^{\Omega}(\mathcal{C}) \to \mathcal{C} ; \{X_n\} \mapsto X_0$.

Thrm. 4.5 $\mathcal{C}$: pted ∞-cat with finite lims. TFAE

(1) $\mathcal{C}$ is stable.

(2) The func $\Omega^{\infty} : Sp^{\Omega}(\mathcal{C}) \to \mathcal{C}$ is an equiv.

We can construct $Sp(\mathcal{C})$ in a different way.

Def. 4.6. $F : \mathcal{C} \to \mathcal{D}$: func of ∞-cats.

- F is reduced $\overset{\text{def}}{\iff}$ F pres terminal objs.
- F is excisive $\overset{\text{def}}{\iff}$ F sends pushouts to pullbacks.
- $\mathrm{Exc}_*(\mathcal{C}, \mathcal{D}) \overset{\text{full}}{\subseteq} \mathrm{Fun}(\mathcal{C}, \mathcal{D})$ of reduced excisive funcs.

Nota. 4.7. $\mathcal{C}$: ∞-cat with finite lims.

- $Sp(\mathcal{C}) := \mathrm{Exc}_*(\mathrm{An}_*^{\mathrm{fin}}, \mathcal{C})$; the cat of spectrum objs of $\mathcal{C}$.

"pted finite spaces" (see next page)

Prop. 4.8. $\mathcal{C}$: ∞-cat with finite lims.

(1) $Sp(\mathcal{C})$ is stable.

(2) $Sp(\mathcal{C}) \simeq Sp^{\Omega}(\mathcal{C}) := \lim(\cdots \xrightarrow{\Omega} \mathcal{C} \xrightarrow{\Omega} \mathcal{C})$.

sketch of proof.

- $F : \mathrm{An}_*^{\mathrm{fin}} \to \mathcal{C}$; reduced excisive $\in Sp(\mathcal{C})$.

- $$\begin{tikzcd} S^n \arrow[r] \arrow[d] & * \arrow[d] \\ * \arrow[r] & S^{n+1} \arrow[ul, phantom, "\ulcorner"] \end{tikzcd} \xrightarrow{\quad F \quad} \begin{tikzcd} F(S^n) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & 0 \arrow[d] \\ 0 \arrow[r] & F(S^{n+1}) \end{tikzcd}$$

$\longrightarrow$ $\exists$ equiv $\varphi_n : F(S^n) \xrightarrow{\simeq} \Omega F(S^{n+1})$.

$\rightsquigarrow$ $\{F(S^n), \varphi_n\}_{n \geq 0} \in Sp^{\Omega}(\mathcal{C})$. $\square$

5. The most important example Sp : the cat of spectra.

Nota. 5.1.

- An : the ∞-cat of anima (∞-groupoids, Kan-cpxs)
- $An_* := An_{*/}$: the ∞-cat of "pted" anima
- $An_*^{fin} \subseteq An_* \overset{def}{\iff}$ the minimal pted full subcat which contains $S^0 = *$ and is closed under finite colims.

Recall.

In classical homotopy theory,

A spectrum $\overset{def}{\iff}$ A seq $\{X_n\}_{n \geq 0}$ of pted top. spaces with $X_n \xrightarrow{\sim} \Omega X_{n+1}$ for $\forall n \geq 0$.

Def. 5.2.

- The category of spectra Sp

$$Sp := \lim(\cdots \xrightarrow{\Omega} An_{(*)} \xrightarrow{\Omega} An_{(*)}) \overset{\text{Prop.4.7.}}{\simeq} Exc_*(An_*^{fin}, An_{(*)})$$

- A spectrum $\overset{def}{\iff}$ A reduced excisive func $An_*^{fin} \to An_{(*)}$.