# THE HIGHER ALGEBRAIC K-THEORY OF STABLE $\infty$-CATEGORIES

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ABSTRACT. We summarize the higher algebraic K-theory of stable $\infty$-categories.

CONTENTS



## 1. INTRODUCTION

This paper is a summary of the workshop on the higher algebraic K-theory held in Kyoto in September 2024. Although we do not plan to write a detailed introduction, we are making it available to the public. To make it easier to read, all proofs are included in Appendix (if we write).

### 1.1. Notation.

From here all categories are assumed to be $\infty$-categories. We let

- An denote the category of anima.
- Cat denote the category of small categories.
- $\mathrm{Cat}^{\mathrm{lex}}$ denote the category of small categories which admit finite limits, with left exact functors.
- $\mathrm{Cat}^{\mathrm{st}}$ denote the category of small stable categories with exact functors.
- $\mathrm{Cat}^{\mathrm{perf}}$ denote the category of small idempotent complete stable categories with exact functors.
- Sp denote the category of spectra.

## 2. Preliminaries

### 2.1. The Grothendieck Group.

**Definition 2.1.** Let $(\mathcal{C}, \oplus)$ be a stable category, and let $X$ and $Y$ be objects of $\mathcal{C}$. We let $[X]$ denote the connected component of $X$. The connected component set $\pi_0(\operatorname{core}\mathcal{C})$, together with the operation $+$ defined by

$$[X] + [Y] := [X \oplus Y]$$

forms an ordinary monoid $(\pi_0(\operatorname{core}\mathcal{C}), +)$. We define the *Grothendieck group* $\mathcal{K}_0(\mathcal{C})$ of $\mathcal{C}$ as

$$\mathcal{K}_0(\mathcal{C}) := (\pi_0(\operatorname{core}\mathcal{C}), +)/\sim$$

where $\sim$ is the equivalence relation generated by the following relation: $[X] = [X'] + [X'']$ whenever $X' \to X \to X''$ is a cofiber sequence in $\mathcal{C}$.

**Remark 2.2.** The connected component set $\pi_0(\operatorname{core}\mathcal{C})$ is the set of equivalence classes of objects of $\mathcal{C}$. Moreover, the Grothendieck group $\mathcal{K}_0(\mathcal{C})$ is actually abelian.

(1) The zero object $0$ of $\mathcal{C}$ is a unit object $[0]$ of $\mathcal{K}_0(\mathcal{C})$, since $X \to X \to 0$ is a cofiber sequence in $\mathcal{C}$ for every object $X$ of $\mathcal{C}$.
(2) For every object $X$ of $\mathcal{C}$, $[\Omega X]$ and $[\Sigma X]$ are inverse objects of $[X]$ in $\mathcal{K}_0(\mathcal{C})$, since $\Omega X \to 0 \to X$ and $X \to 0 \to X$ are cofiber sequences in $\mathcal{C}$.
(3) For every objects $X$ and $Y$ of $\mathcal{C}$, we have $[X] + [Y] = [Y] + [X]$, since $X \to X \oplus Y \to Y$ and $Y \to X \oplus Y \to X$ are cofiber sequences in $\mathcal{C}$.

**Remark 2.3** (Eilenberg swindle)**.** Let $\mathcal{C}$ be a stable category with countable coproduct. Then the Grothendieck group $\mathcal{K}_0(\mathcal{C})$ is trivial. It follows from that $X \to \oplus_{n\geq 0} X \to \oplus_{n\geq 1} X$ is a cofiber sequence in $\mathcal{C}$ for every object $X$ of $\mathcal{C}$, and that the last two terms are equivalent.

**Remark 2.4.** The construction $\mathcal{C} \mapsto \mathcal{K}_0(\mathcal{C})$ determine a functor $\mathcal{K}_0 : \mathrm{h}\,\mathrm{Cat}^{\mathrm{st}} \to \mathrm{Ab}$.

### 2.2. Arrow Categories and Twisted Arrow Categories.

**Definition 2.5.** Let $\mathcal{C}$ be a category. We define the *arrow category* $\mathrm{Ar}(\mathcal{C})$ of $\mathcal{C}$ as

$$\mathrm{Ar}(\mathcal{C}) := \mathrm{Fun}([1], \mathcal{C}).$$

**Definition 2.6.** Let $\mathcal{C}$ be a category. We define the *twisted arrow category* $\mathrm{TwAr}^{\mathrm{r}}(\mathcal{C})$ of $\mathcal{C}$ as a right fibration classifying the mapping anima functor. That is, the source-target projection

$$(s, t) : \mathrm{TwAr}^{\mathrm{r}}(\mathcal{C}) \to \mathcal{C} \times \mathcal{C}^{\mathrm{op}}$$

is the right fibration classifying the mapping anima functor $\mathrm{Map}_{\mathcal{C}} : \mathcal{C} \times \mathcal{C}^{\mathrm{op}} \to \mathrm{An}^{\mathrm{op}}$.

**Remark 2.7.** Let $\mathcal{C}$ be a category. Let see the objects and morphisms of $\mathrm{Ar}(\mathcal{C})$ and $\mathrm{TwAr}^{\mathrm{r}}(\mathcal{C})$.

- The objects of both are both morphisms in $\mathcal{C}$.
- A morphism from $f$ to $g$ in $\mathrm{Ar}(\mathcal{C})$ is a diagram, depicted as

$$\begin{tikzcd}
 & 1' & \\
 & & 1 \arrow[ul, "g"'] \\
0 \arrow[uur] \arrow[urr] \arrow[dr, "f"'] & & \\
 & 0' \arrow[uuu] \arrow[uur] &
\end{tikzcd}$$

- A morphism from $f$ to $g$ in $\mathrm{TwAr}^{\mathrm{r}}(\mathcal{C})$ is a diagram, depicted as

$$\begin{tikzcd}
& 1' \arrow[dd] & \\
& & 1 \arrow[ul, "g"'] \\
0 \arrow[uur] \arrow[urr] \arrow[r, "f"'] & 0' \arrow[ur] &
\end{tikzcd}$$

**Notation 2.8.** Let $\mathcal{C}$ be a stable category. We let $\mathrm{Seq}(\mathcal{C})$ denote the full subcategory of $\mathrm{Fun}(\Delta^1 \times \Delta^1, \mathcal{C})$ spanned by the bifiber sequences in $\mathcal{C}$. We have an equivalence $\mathrm{Seq}(\mathcal{C}) \simeq \mathrm{Ar}(\mathcal{C})$, so that the category $\mathrm{Seq}(\mathcal{C})$ is stable.

**Notation 2.9.** Let $\mathcal{C}$ be a stable category. We define functors from $\mathrm{Seq}(\mathcal{C})$ to $\mathcal{C}$ as

$$\begin{aligned}
\mathrm{fib} : \mathrm{Seq}(\mathcal{C}) \to \mathcal{C} : (X \to Y \to Z) \mapsto X \\
\mathrm{mid} : \mathrm{Seq}(\mathcal{C}) \to \mathcal{C} : (X \to Y \to Z) \mapsto Y \\
\mathrm{cofib} : \mathrm{Seq}(\mathcal{C}) \to \mathcal{C} : (X \to Y \to Z) \mapsto Z.
\end{aligned}$$

2.3. **Verdier Sequences and Squares.**

**Definition 2.10.** Let $\mathcal{C} \xrightarrow{f} \mathcal{D} \xrightarrow{p} \mathcal{E}$ be a sequence in $\mathrm{Cat}^{\mathrm{st}}$. We will say that the sequence has *vanishing composition* if the composition $pf$ is a zero object of $\mathrm{Cat}^{\mathrm{st}}$. In this case, the functor $pf$ is equivalent to the functor $\mathcal{C} \to 0 \to \mathcal{E}$, since the full subcategory of $\mathrm{Fun}^{\mathrm{ex}}(\mathcal{C}, \mathcal{E})$ spanned by the zero objects is contractible.

**Definition 2.11.** Let $\mathcal{C} \xrightarrow{f} \mathcal{D} \xrightarrow{p} \mathcal{E}$ be a sequence in $\mathrm{Cat}^{\mathrm{st}}$ with vanishing composition. We will say that this sequence is *Verdier* if it is a bifiber sequence in $\mathrm{Cat}^{\mathrm{st}}$. In this case, we will refer to the functor $f$ as the *Verdier inclusion* and to the functor $p$ as the *Verdier projection*.

**Definition 2.12.** Let $\mathcal{C} \xrightarrow{f} \mathcal{D} \xrightarrow{p} \mathcal{E}$ be a Verdier sequence. We will say that this sequence is *split* if the functor $p$ admits left and right adjoint functors. In this case, we will refer to the functor $f$ as the *split Verdier inclusion* and to the functor $p$ as the *split Verdier projection*.

**Definition 2.13.** Let $\mathcal{C} \xrightarrow{f} \mathcal{D} \xrightarrow{p} \mathcal{E}$ be a sequence in $\mathrm{Cat}^{\mathrm{st}}$ with vanishing composition. We will say that this sequence is *Karoubi* if its idempotent completion $\mathcal{C}^\natural \to \mathcal{D}^\natural \to \mathcal{E}^\natural$ is a bifiber sequence in $\mathrm{Cat}^{\mathrm{perf}}$. In this case, we will refer to the functor $f$ as the *Karoubi inclusion* and to the functor $p$ as the *Karoubi projection*.

We can characterize Verdier inclusions and projections.

**Definition 2.14.** Let $f : \mathcal{C} \to \mathcal{D}$ be an exact functor between stable categories. We will say that a morphism in $\mathcal{D}$ is an *equivalence modulo* $\mathcal{C}$ in $\mathcal{D}$ if its (co)fiber belongs in the essential image of $f$. We define the category $\mathcal{D}/\mathcal{C}$ as the localization of $\mathcal{D}$ with respect to the set of equivalences modulo $\mathcal{C}$ in $\mathcal{D}$. We will refer to the category $\mathcal{D}/\mathcal{C}$ as the *Verdier quotient* of $\mathcal{D}$ by $\mathcal{C}$.

The next proposition implies that the Verdier quotient is universal.

**Proposition 2.15** ([NS18] Theorem.1.3.3)**.** Let $f : \mathcal{C} \to \mathcal{D}$ be an exact functor between stable categories. Then

(1) The Verdier quotient $\mathcal{D}/\mathcal{C}$ is stable, and the localization functor $\mathcal{D} \to \mathcal{D}/\mathcal{C}$ is exact.
(2) For every stable category $\mathcal{E}$, the restriction functor

$$\mathrm{Fun}^{\mathrm{ex}}(\mathcal{D}/\mathcal{C}, \mathcal{E}) \to \mathrm{Fun}^{\mathrm{ex}}(\mathcal{D}, \mathcal{E})$$

is fully faithful, and its essential image consists of the functors which vanish after composing with $f$.

(3) The sequence $\mathcal{C} \to \mathcal{D} \to \mathcal{D}/\mathcal{C}$ is a cofiber sequence in $\mathrm{Cat}^{\mathrm{st}}$.

**Proposition 2.16.** Let $\mathcal{C} \xrightarrow{f} \mathcal{D} \xrightarrow{p} \mathcal{E}$ be a sequence in $\mathrm{Cat}^{\mathrm{st}}$ with vanishing composition. The following conditions are equivalent:

(1) The sequence is Verdier.

(2) The functor $f$ is fully faithful and its essential image is closed under retracts in $\mathcal{D}$, and the functor $p$ exhibits $\mathcal{E}$ as the Verdier quotient of $\mathcal{D}$ by $\mathcal{C}$.

(3) The functor $f$ exhibits $\mathcal{C}$ as the kernel of $p$, and the functor $p$ is a localization.

We can characterize split Verdier inclusions and projections.

**Proposition 2.17.** Let $\mathcal{C} \xrightarrow{f} \mathcal{D} \xrightarrow{p} \mathcal{E}$ be a sequence in $\mathrm{Cat}^{\mathrm{st}}$ with vanishing composition. The following conditions are equivalent:

(1) The sequence is split Verdier.

(2) The functor $p$ admits fully faithful left and right adjoint functors.

(3) The functor $f$ admits fully faithful left and right adjoint functors.

We can characterize Karoubi inclusions and projections.

**Proposition 2.18.** Let $\mathcal{C} \xrightarrow{f} \mathcal{D} \xrightarrow{p} \mathcal{E}$ be a sequence in $\mathrm{Cat}^{\mathrm{st}}$ with vanishing composition. The following conditions are equivalent:

(1) The sequence is Karoubi.

(2) The functor $f$ is fully faithful and the functor $p$ has the dense essential image, and the restriction to the essential image of $p$ is a Verdier projection.

We can describe Karoubi sequences using Ind-categories.

**Theorem 2.19** (Thomason-Neeman's localization theorem)**.** *Let $\mathcal{C} \xrightarrow{f} \mathcal{D} \xrightarrow{p} \mathcal{E}$ be a sequence in* $\mathrm{Cat}^{\mathrm{st}}$ *with vanishing composition. The following conditions are equivalent:*

*(1) The sequence is Karoubi.*

*(2) The sequence* $\mathrm{Ind}(\mathcal{C}) \to \mathrm{Ind}(\mathcal{D}) \to \mathrm{Ind}(\mathcal{E})$ *is Verdier (of non-necessarily small stable categories).*

We introduce the relative versions of these sequences.

**Definition 2.20.** A square in $\mathrm{Cat}^{\mathrm{st}}$ is called

- *Verdier* if it is (co)Cartesian and its both vertical maps are Verdier projections.
- *split Verdier* if it is (co)Cartesian and its both vertical maps are split Verdier projections.
- *Karoubi* if its idempotent completion is Verdier.

**Remark 2.21.** Every Verdier (split Verdier, Karoubi) square is also coCartesian in $\mathrm{Cat}^{\mathrm{st}}$.

**Remark 2.22.** Every Verdier square is a Karoubi square, since the idempotent completion preserves small limits. A Karoubi square is a Verdier square if and only if its both vertical maps are essentially surjective.

### 2.4. Additive and Grouplike Functors.

**Definition 2.23.** Let $\mathcal{E}$ be a category with a terminal object, and let $F : \mathrm{Cat}^{\mathrm{st}} \to \mathcal{E}$ be a functor. We will say that $F$ is *reduced* if the object $F(0)$ is equivalent to a terminal object of $\mathcal{E}$, where 0 is a zero object in $\mathrm{Cat}^{\mathrm{st}}$.

**Definition 2.24.** Let $\mathcal{E}$ be a category with finite limits, and let $F : \mathrm{Cat}^{\mathrm{st}} \to \mathcal{E}$ be a reduced functor. The functor $F$ is called

- *Verdier-localizing* if it takes every Verdier square in $\mathrm{Cat}^{\mathrm{st}}$ to a Cartesian square in $\mathcal{E}$.
- *additive* if it takes every split Verdier square in $\mathrm{Cat}^{\mathrm{st}}$ to a Cartesian square in $\mathcal{E}$.
- *Karoubi-localizing* if it takes every Karoubi square in $\mathrm{Cat}^{\mathrm{st}}$ to a Cartesian square in $\mathcal{E}$.

**Definition 2.25.** Let $\mathcal{E}$ be a category with finite limits, and let $F : \mathrm{Cat}^{\mathrm{st}} \to \mathcal{E}$ be an additive functor. We will say that $F$ is *grouplike* if it lifts to the category $\mathrm{Mon}_{\mathbb{E}_\infty}(\mathcal{E})$ takes values in $\mathrm{Grp}_{\mathbb{E}_\infty}(\mathcal{E})$.

**Definition 2.26.** Let $\mathcal{E}$ be a category with finite limits, and let $F : \mathrm{Cat}^{\mathrm{st}} \to \mathcal{E}$ be a reduced functor. We will say that $F$ is *extension-splitting* if, for every stable category $\mathcal{C}$, the fiber-cofiber map

$$(\mathrm{fib}, \mathrm{cofib}) : \mathrm{Seq}(\mathcal{C}) \to \mathcal{C}^2$$

induces an equivalence $F(\mathrm{Seq}(\mathcal{C})) \to F(\mathcal{C})^2$.

**Example 2.27.** We give some (counter)examples.

- The core functor $\mathrm{core} : \mathrm{Cat}^{\mathrm{st}} \to \mathrm{An}$ is additive grouplike.
- The algebraic K-theory $\mathcal{K} : \mathrm{Cat}^{\mathrm{st}} \to \mathrm{An}$ and the algebraic K-theory spectrum $\mathcal{K}_{\geq 0} : \mathrm{Cat}^{\mathrm{st}} \to \mathrm{Sp}$ are Verdier-localizing (theorem 6.1) and grouplike (corollary 4.5), but not Karoubi-localizing.
- The functor $\mathcal{K} \circ (-)^\natural : \mathrm{Cat}^{\mathrm{st}} \to \mathrm{An}$ is Karoubi-localizing (corollary 7.8).
- The functor $\mathcal{K}_{\geq 0} \circ (-)^\natural : \mathrm{Cat}^{\mathrm{st}} \to \mathrm{Sp}$ is additive, but not Verdier-localizing.

**Proposition 2.28.** The additive, Verdier-localizing, Karoubi-localizing, and extension-splitting functors preserve finite products.

The additivity can be verified for split Verdier sequences if a codomain category $\mathcal{E}$ is stable.

**Proposition 2.29.** Let $\mathcal{E}$ be a category with finite limits, and let $F : \mathrm{Cat}^{\mathrm{st}} \to \mathcal{E}$ be a reduced functor. If $\mathcal{E}$ is stable, then the following conditions are equivalent:

(1) The functor $F$ is additive (resp. Verdier-localizing, Karoubi-localizing).
(2) The functor $F$ takes every split Verdier sequence (resp. Verdier sequence, Karoubi sequence) in $\mathrm{Cat}^{\mathrm{st}}$ to a fiber sequence in $\mathcal{E}$.

The relationship between Verdier-localizing and Karoubi-localizing functors is as follows.

**Definition 2.30.** Let $f : \mathcal{C} \to \mathcal{D}$ be a functor between categories. We will say that $f$ *has the dense image* if, for every object $X$ of $\mathcal{D}$, there exists an object $Y$ in the essential image of $\mathcal{C}$ such that $Y$ is a retract of $X$.

**Definition 2.31.** Let $f : \mathcal{C} \to \mathcal{D}$ be an exact functor between stable categories. We will say that $f$ is a *Karoubi equivalence* if it is fully faithful and has the dense image.

**Proposition 2.32.** Let $\mathcal{E}$ be a category with finite limits, and let $F : \mathrm{Cat}^{\mathrm{st}} \to \mathcal{E}$ be a reduced functor. The following conditions are equivalent:

(1) The functor $F$ is Karoubi-localizing.

(2) The functor $F$ is Verdier-localizing and inverts Karoubi equivalences.

The next proposition implies that additive grouplike functors and extension-splitting functors are equivalent.

**Proposition 2.33.** Let $\mathcal{E}$ be a category with finite limits, and let $F : \mathrm{Cat}^{\mathrm{st}} \to \mathcal{E}$ be a reduced functor. The following conditions are equivalent:

(1) The functor $F$ is additive grouplike.

(2) The functor $F$ is extension-splitting.

We can construct Karoubi-localization functors from Verdier-localizing functors using the idempotent completion.

**Proposition 2.34.** Let $\mathcal{E}$ be a category with finite limits, and let $F : \mathrm{Cat}^{\mathrm{st}} \to \mathcal{E}$ be a Verdier-localizing functor. Suppose that $F$ takes every Cartesian square in $\mathrm{Cat}^{\mathrm{st}}$ whose vertical maps are dense inclusions, to a Cartesian square in $\mathcal{E}$. Then the functor $F \circ (-)^{\natural} : \mathrm{Cat}^{\mathrm{st}} \to \mathcal{E}$ is Karoubi-localizing.

2.5. **Waldhausen's Fibration Theorem.** In this section, we recall Waldhausen's fibration theorem. We will use this theorem in the proof of the localization theorem (theorem 6.1).

**Notation 2.35.** Let $\mathcal{D}$ be a stable category, let $\mathcal{C}$ be a stable full subcategory of $\mathcal{D}$, and let $\mathcal{I}$ be a category. We let $\mathrm{Fun}^{\mathcal{C}}(\mathcal{I}, \mathcal{D})$ denote the full subcategory of $\mathrm{Fun}(\mathcal{I}, \mathcal{D})$ spanned by the functors which take every maps in $\mathcal{I}$ to equivalences modulo $\mathcal{C}$.

**Theorem 2.36** (Waldhausen's fibration theorem)**.** *Let $\mathcal{C} \to \mathcal{D} \to \mathcal{E}$ be a Verdier sequence, and let $F : \mathrm{Cat}^{\mathrm{st}} \to \mathrm{An}$ be an additive grouplike functor. Then, for every $n \geq 0$, the constant map*

$$\mathrm{const} : \mathcal{D} \to \mathrm{Fun}^{\mathcal{C}}([n], \mathcal{D}) : X \mapsto (X \to \cdots \to X)$$

*induces a bifiber sequence of $\mathbb{E}_\infty$-groups*

$$F(\mathcal{C}) \to F(\mathcal{D}) \to |F \,\mathrm{Fun}^{\mathcal{C}}([-], \mathcal{D})|.$$

We can deduce when an additive functor becomes a Verdier-localizing functor.

**Corollary 2.37.** Let $F : \mathrm{Cat}^{\mathrm{st}} \to \mathrm{An}$ be an additive functor. These conditions are equivalent:

(1) The functor $F$ is Verdier-localizing.

(2) For every Verdier sequence $\mathcal{C} \to \mathcal{D} \to \mathcal{E}$, the canonical map $|F \,\mathrm{Fun}^{\mathcal{C}}([-], \mathcal{D})| \to F(\mathcal{E})$ is an equivalence of anima.

## 3. The Higher K-Theory of Stable $\infty$-Categories

### 3.1. Simplicial Objects.

**Definition 3.1.** The inclusion $\mathrm{N}(\Delta) \subseteq \mathrm{Cat}$ induces an adjunction

$$\mathrm{asscat} : \mathrm{Fun}(\mathrm{N}(\Delta)^{\mathrm{op}}, \mathrm{An}) \rightleftarrows \mathrm{Cat} : \mathrm{N}^r.$$

We will refer to the left adjoint as the *associated category functor*, and to the right adjoint as the *Rezk nerve.*

**Definition 3.2.** Let $\mathcal{C}$ be a category. We will refer to a functor

$$X : \mathrm{N}(\Delta)^{\mathrm{op}} \to \mathcal{C}$$

as a *simplicial object* of $\mathcal{C}$. We will say that $X$ is a *simplicial anima* if $\mathcal{C}$ is An.

**Remark 3.3.** Let $\mathcal{C}$ be a category. For every $n \geq 0$, we have an equivalence of anima

$$\mathrm{N}^r_n(\mathcal{C}) \simeq \mathrm{Map}_{\mathrm{Cat}}([n], \mathcal{C}) \simeq \mathrm{core}\,\mathrm{Fun}([n], \mathcal{C}).$$

**Notation 3.4.** We let $[n]$ denote the category the ordinary nerve $\mathrm{N}([n])$ of $[n]$, instead of $\Delta^n$. On the other hand, we let $\Delta^n$ denote the functor

$$\Delta^n := \mathrm{Map}_{\mathrm{Cat}}(-,[n]) : \mathrm{N}(\Delta)^{\mathrm{op}} \to \mathrm{An}.$$

Then we have an equivalence of functors $\mathrm{N}^r([n]) \simeq \Delta^n$.

We define the Segal condition and completeness specifically for simplicial anima, although these concepts are applicable to every category.

**Definition 3.5.** Let $X : \mathrm{N}(\Delta)^{\mathrm{op}} \to \mathrm{An}$ be a simplicial anima. We will say that $X$ is *Segal* if the $n$-spine inclusion $\mathrm{sp}^n \subseteq \Delta^n$ induces an equivalence of anima

$$X_n \simeq \mathrm{Map}_{\mathrm{Fun}(\mathrm{N}(\Delta)^{\mathrm{op}},\mathrm{An})}(\Delta^n, X) \to \mathrm{Map}_{\mathrm{Fun}(\mathrm{N}(\Delta)^{\mathrm{op}},\mathrm{An})}(\mathrm{sp}^n, X) \simeq X_1 \times_{X_0} \cdots \times_{X_0} X_1$$

for every $n \geq 0$.

The Segal condition can be interpreted as stating that a Segal simplicial anima has a unique spine lifting up to a choice of contractible spaces.

**Definition 3.6.** Let $X : \mathrm{N}(\Delta)^{\mathrm{op}} \to \mathrm{An}$ be a Segal simplicial anima. We will say that $X$ is *complete* if the following diagram is a Cartesian diagram in An.

$$\begin{tikzcd}
X_0 \arrow[r, "\mathrm{diag}"] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & X_0 \times X_0 \arrow[d, "{(s,s)}"] \\
X_3 \arrow[r, "{(d^{\{0,2\}}, d^{\{1,3\}})}"'] & X_1 \times X_1
\end{tikzcd}$$

The completeness condition can be understood as indicating that the higher simplices of a complete Segal simplicial anima correspond to equivalences related to its degenerate edges.

**Proposition 3.7.** The Rezk nerve $\mathrm{N}^r : \mathrm{Cat} \to \mathrm{Fun}(\mathrm{N}(\Delta)^{\mathrm{op}}, \mathrm{An})$ is fully faithful. Moreover, its essential image precisely consists of complete Segal simplicial anima.

### 3.2. **The algebraic K-Theory.**

**Definition 3.8.** Let $\mathcal{C}$ be a category with finite limits. For every $n \geq 0$, we let $Q_n(\mathcal{C})$ denote the full subcategory of $\mathrm{Fun}(\mathrm{TwAr}^{\mathrm{r}}[n], \mathcal{C})$ spanned by the diagrams which take every square in $\mathrm{TwAr}^{\mathrm{r}}[n]$ to a Cartesian square in $\mathcal{C}$.

The construction $n \mapsto Q_n(\mathcal{C})$ determines a functor

$$Q(\mathcal{C}) : \mathrm{N}(\Delta)^{\mathrm{op}} \to \mathrm{Cat}^{\mathrm{lex}}$$

and furthermore, the construction $\mathcal{C} \mapsto Q(\mathcal{C})$ defines a functor

$$Q : \mathrm{Cat}^{\mathrm{lex}} \to \mathrm{Fun}(\mathrm{N}(\Delta)^{\mathrm{op}}, \mathrm{Cat}^{\mathrm{lex}}).$$

We will refer to this functor as the *(Quillen's) Q-construction.*

**Proposition 3.9.** Let $\mathcal{C}$ be a category with finite limits. Then the simplicial object in $\mathrm{Cat}^{\mathrm{lex}}$

$$Q(\mathcal{C}) : \mathrm{N}(\Delta)^{\mathrm{op}} \to \mathrm{Cat}^{\mathrm{lex}}$$

is complete Segal. In particular, the simplicial anima

$$\mathrm{core}\, Q(\mathcal{C}) : \mathrm{N}(\Delta)^{\mathrm{op}} \to \mathrm{An}$$

is complete Segal.

This proposition follows from the following lemmata.

**Notation 3.10.** For every $n \geq 0$, we let $\mathcal{J}_n$ denote the full subcategory of $\mathrm{TwAr^r}[n]$ spanned by the images of objects $(i \leq j)$ in $[n]$ satisfying $j \leq i+1$.

**Lemma 3.11.** Let $\mathcal{C}$ be a category with finite limits, and let $F : \mathrm{TwAr^r}[n] \to \mathcal{E}$ be a functor. The following conditions are equivalent:

(1) The functor $F$ belongs to $Q_n(\mathcal{C})$.

(2) The functor $F$ is a right Kan extension of its restriction to $\mathcal{J}_n$ along the inclusion $\mathcal{J}_n \subseteq \mathrm{TwAr^r}[n]$.

**Lemma 3.12.** Let $\mathcal{C}$ be a category with finite limits. Then the restriction of $\mathrm{Fun}(\mathrm{TwAr^r}[n], \mathcal{C})$ along the inclusion $\mathcal{J}_n \subseteq \mathrm{TwAr^r}[n]$ induces an equivalence of categories

$$Q_n(\mathcal{C}) \to \mathrm{Fun}(\mathcal{J}_n, \mathcal{C}).$$

**Remark 3.13.** Lemma 3.12 implies that, if $\mathcal{C}$ is stable, then $Q_n(\mathcal{C})$ is stable. Therefore we obtain functors

$$Q(\mathcal{C}) : \mathrm{N}(\Delta)^{\mathrm{op}} \to \mathrm{Cat^{st}} \text{ and } Q : \mathrm{Cat^{st}} \to \mathrm{Fun}(\mathrm{N}(\Delta)^{\mathrm{op}}, \mathrm{Cat^{st}}).$$

Moreover, for every stable category $\mathcal{C}$, the category $Q_n(\mathcal{C})$ is a complete Segal simplicial anima, since $\mathrm{Cat^{st}}$ is stable under finite limits in Cat.

**Definition 3.14.** Let $\mathcal{C}$ be a category with finite limits. Then we define the *category of spans* in $\mathcal{C}$ as

$$\mathrm{Span}(\mathcal{C}) := \mathrm{asscat}\,\mathrm{core}\, Q(\mathcal{C}).$$

The construction $\mathcal{C} \mapsto \mathrm{Span}(\mathcal{C})$ determines a functor

$$\mathrm{Span} : \mathrm{Cat^{lex}} \to \mathrm{Cat}.$$

**Definition 3.15.** Let $\mathcal{C}$ be a stable category. Then we define the *algebraic K-anima* (or *algebraic K-theory anima*, or *projective class anima*) as

$$\mathcal{K}(\mathcal{C}) := \Omega|\,\mathrm{Span}(\mathcal{C})| \simeq \Omega|\,\mathrm{core}\, Q(\mathcal{C})|$$

where the base object of the loop space is given by the zero object of $\mathrm{Span}(\mathcal{C})$.

The construction $\mathcal{C} \mapsto \mathcal{K}(\mathcal{C})$ determines a functor

$$\mathcal{K} : \mathrm{Cat^{st}} \to \mathrm{An}.$$

We will refer to this functor as the *algebraic K-theory* (or *algebraic K-functor*).

**Definition 3.16.** Let $\mathcal{C}$ be a stable category. For every $n \geq 1$, we define the *n-th K-group* of $\mathcal{C}$ as the abelian group

$$\mathcal{K}_n(\mathcal{C}) := \pi_n \mathcal{K}(\mathcal{C}).$$

**Remark 3.17.** Let $\mathcal{C}$ be a stable category. Then we have an equivalence

$$\pi_0 \mathcal{K}(\mathcal{C}) \simeq \mathcal{K}_0(\mathcal{C})$$

where $\mathcal{K}_0(\mathcal{C})$ is the Grothendieck group of $\mathcal{C}$.

3.3. **Waldhausen's S-Construction.** In this section, we construct the algebraic K-theory using Waldhausen's S-construction.

**Definition 3.18.** Let $\mathcal{C}$ be a stable category. An $[n]$-*gapped object* of $\mathcal{C}$ is a functor $F : \mathrm{Ar}[n] \to \mathcal{C}$ which satisfies the following properties:

(1) For every $0 \leq i \leq n$, $F(i,i)$ is a zero object of $\mathcal{C}$.
(2) For every $i \leq j \leq k$, the following diagram is a (co)Cartesian diagram in $\mathcal{C}$.

$$\begin{tikzcd} F(i,j) \arrow[r] \arrow[d] & F(i,k) \arrow[d] \\ 0 \simeq F(j,j) \arrow[r] \arrow[ur, phantom, "\ulcorner"] & F(j,k) \end{tikzcd}$$

We let $S_n(\mathcal{C})$ denote the full subcategory of $\mathrm{Fun}(\mathrm{Ar}[n], \mathcal{C})$ spanned by the $[n]$-gapped objects of $\mathcal{C}$.

**Remark 3.19.** Let $\mathcal{C}$ be a stable category. We can describe the low-dimensional simplices of $S_n(\mathcal{C})$.

- The category $S_0(\mathcal{C})$ is the full subcategory of $\mathcal{C}$ spanned by the zero objects of $\mathcal{C}$. Thus $S_0(\mathcal{C})$ is contractible.
- The category $S_1(\mathcal{C})$ is equivalent to $\mathcal{C}$, since every object of $S_1(\mathcal{C})$ is of the form $0 \to X \to 0$, where $X$ is an object of $\mathcal{C}$.
- The category $S_2(\mathcal{C})$ is equivalent to the arrow category $\mathrm{Ar}(\mathcal{C})$ of $\mathcal{C}$, since every object of $S_2(\mathcal{C})$ is of the form $0 \to X' \to X \to X'' \to 0$, where $X' \to X \to X''$ is a cofiber sequence in $\mathcal{C}$.

**Remark 3.20.** Let $\mathcal{C}$ be a stable category. We have an equivalence of categories

$$S_n(\mathcal{C}) \simeq \mathrm{Fun}([n-1], \mathcal{C})$$

for every $n \geq 0$. Thus, if $\mathcal{C}$ is stable, then $S_n(\mathcal{C})$ is stable.

**Definition 3.21.** The construction $n \mapsto S_n(\mathcal{C})$ determines a functor

$$S(\mathcal{C}) : \mathrm{N}(\Delta)^{\mathrm{op}} \to \mathrm{Cat}^{\mathrm{st}}$$

and furthermore, the construction $\mathcal{C} \mapsto S(\mathcal{C})$ determines a functor

$$S : \mathrm{Cat}^{\mathrm{st}} \to \mathrm{Fun}(\mathrm{N}(\Delta)^{\mathrm{op}}, \mathrm{Cat}^{\mathrm{st}}).$$

We will refer to this functor as *(Waldhausen's) S-construction.*

**Definition 3.22.** Let $\mathcal{C}$ be a stable category. Then we define the *algebraic K-anima* as

$$\mathcal{K}_S(\mathcal{C}) := \Omega|\operatorname{core} S(\mathcal{C})|.$$

The construction $\mathcal{C} \mapsto \mathcal{K}_S(\mathcal{C})$ determines a functor

$$\mathcal{K}_S : \mathrm{Cat}^{\mathrm{st}} \to \mathrm{An}.$$

We will refer to this functor as the *algebraic K-theory.*

**Remark 3.23.** Let $\mathcal{C}$ be a stable category. Then the anima $|\operatorname{core} S(\mathcal{C})|$ admits a canonical base point given by a map

$$0 \simeq \operatorname{core} S_0(\mathcal{C}) \to |\operatorname{core} S(\mathcal{C})|.$$

Moreover, $|\operatorname{core} S(\mathcal{C})|$ is connected, since the canonical map

$$0 \simeq \pi_0 \operatorname{core} S_0(\mathcal{C}) \to \pi_0|\operatorname{core} S(\mathcal{C})|$$

is surjective.

**Proposition 3.24.** The two definitions of algebraic K-anima (definitions 3.15 and 3.22) induce an equivalence of anima

$$\mathcal{K}(\mathcal{C}) \simeq \mathcal{K}_S(\mathcal{C})$$

for every stable category $\mathcal{C}$.

## 4. The Additivity Theorem

### 4.1. The Additivity Theorem.

The goal of this section is to prove the additivity theorem.

**Theorem 4.1** ([HLS23] Theorem.4.1 (The Additivity Theorem))**.** *Let $\mathcal{C}$ be a stable category. Then the source-target projection induces an equivalence of anima*

$$(s,t) : |\operatorname{Span}(\operatorname{Ar}(\mathcal{C}))| \to |\operatorname{Span}(\mathcal{C})|^2.$$

The proof of theorem 4.1 follows from the next two propositions.

**Proposition 4.2.** Let $\mathcal{C}$ be a stable category. Then there are canonical equivalences of categories

$$\operatorname{Span}(\mathcal{C}) \to \operatorname{Span}(\mathcal{C}^{\mathrm{op}}) \quad \text{and} \quad \operatorname{Span}(\operatorname{Ar}(\mathcal{C})) \simeq \operatorname{Span}(\operatorname{TwAr}^{\mathrm{r}}(\mathcal{C})).$$

Moreover, they fit together into a natural commutative diagram

$$\begin{tikzcd}
\operatorname{Span}(\operatorname{Ar}(\mathcal{C})) \arrow[r, "\simeq"] \arrow[d, "{(s,t)}"'] & \operatorname{Span}(\operatorname{TwAr}^{\mathrm{r}}(\mathcal{C})) \arrow[d, "{(s,t)}"] \\
\operatorname{Span}(\mathcal{C}) \times \operatorname{Span}(\mathcal{C}) \arrow[r, "\simeq"'] & \operatorname{Span}(\mathcal{C}) \times \operatorname{Span}(\mathcal{C}^{\mathrm{op}}).
\end{tikzcd}$$

**Proposition 4.3.** Let $\mathcal{C}$ be a stable category. Then the source-target projection

$$(s,t) : \operatorname{Span}(\operatorname{TwAr}^{\mathrm{r}}(\mathcal{C})) \to \operatorname{Span}(\mathcal{C}) \times \operatorname{Span}(\mathcal{C}^{\mathrm{op}})$$

is cofinal.

We introduce some corollaries of the additivity theorem.

**Corollary 4.4.** Let $\mathcal{C}$ be a stable category. Then the source-target projection $(s,t) : \operatorname{Ar}(\mathcal{C}) \to \mathcal{C}$ induces an equivalence of anima

$$\mathcal{K}(\operatorname{Ar}(\mathcal{C})) \to \mathcal{K}(\mathcal{C})^2.$$

**Corollary 4.5.** The algebraic K-theory $\mathcal{K} : \mathrm{Cat}^{\mathrm{st}} \to \mathrm{An}$ is additive grouplike.

**Proposition 4.6** (Eilenberg swindle)**.** Let $\mathcal{C}$ be a stable category with countable coproducts. For every additive grouplike functor $F : \mathrm{Cat}^{\mathrm{st}} \to \mathrm{An}$, we have

$$F(\mathcal{C}) \simeq 0$$

for every stable category $\mathcal{C}$. In particular, $i$-th K-groups vanish for every $i \geq 0$.

### 4.2. The algebraic K-Theory Spectrum.

We can define an algebraic K-theory spectrum $\mathcal{K}_{\geq 0} : \mathrm{Cat}^{\mathrm{st}} \to \mathrm{Sp}$.

**Definition 4.7.** Corollary 4.5 implies that the K-theory functor lifts to a functor

$$\mathcal{K} : \mathrm{Cat}^{\mathrm{st}} \to \mathrm{Grp}_{\mathbb{E}_\infty}(\mathrm{An}).$$

Since we have the equivalence $\mathrm{Grp}_{\mathbb{E}_\infty}(\mathrm{An}) \simeq \mathrm{Sp}_{\geq 0}$, we obtain a functor

$$\mathcal{K}_{\geq 0} : \mathrm{Cat}^{\mathrm{st}} \to \mathrm{Sp}_{\geq 0} \subseteq \mathrm{Sp}.$$

We will refer to this functor as the *algebraic K-theory spectrum.*

**Remark 4.8.** There is the equivalence $\Omega^\infty : \mathrm{Sp}_{\geq 0} \rightleftarrows \mathrm{Grp}_{\mathbb{E}_\infty}(\mathrm{An}) : \Sigma^\infty$. We can recover the algebraic K-functor from algebraic K-theory spectrum as

$$\mathcal{K} \simeq \Omega^\infty \mathcal{K}_{\geq 0} : \mathrm{Cat}^{\mathrm{st}} \to \mathrm{Sp}_{\geq 0} \simeq \mathrm{Grp}_{\mathbb{E}_\infty}(\mathrm{An})$$

since $\Sigma^\infty$ is fully faithful.

## 5. The Universality Theorem

The goal of this section is to prove the universality theorem.

**Theorem 5.1** ([HLS23] Theorem.5.1 (The Universality Theorem))**.** *The algebraic K-theory* $\mathcal{K} : \mathrm{Cat}^{\mathrm{st}} \to \mathrm{An}$ *is an initial additive grouplike functor under the core functor* $\mathrm{core} : \mathrm{Cat}^{\mathrm{st}} \to \mathrm{An}$. *That is, the natural map* $\tau : \mathrm{core} \Rightarrow \mathcal{K}$ *is an initial object in the category* $\mathrm{Fun}(\mathrm{Cat}^{\mathrm{st}}, \mathrm{An})^{\mathrm{add,grp}}_{\mathrm{core}\,/}$.

**Notation 5.2.** Let $F : \mathrm{Cat}^{\mathrm{st}} \to \mathrm{An}$ be a reduced functor. We denote a functor

$$GF(-) := \Omega|FQ(-)| : \mathrm{Cat}^{\mathrm{st}} \to \mathrm{An}.$$

For example, the functor $G\,\mathrm{core}$ is equivalent to the algebraic K-theory $\mathcal{K}$.

*Proof.* We want to show that the natural transformation $\tau : \mathrm{core} \Rightarrow \mathcal{K}$ induces an equivalence

$$\tau^* : \mathrm{Nat}(\mathcal{K}, F) \to \mathrm{Nat}(\mathrm{core}, F)$$

for every additive grouplike functor $F : \mathrm{Cat}^{\mathrm{st}} \to \mathrm{An}$.

Now consider the following diagram

$$\begin{tikzcd}
\mathrm{Nat}(G\,\mathrm{core}, F) \arrow[r, "(\eta_{\mathrm{core}})^*"] \arrow[d, "G"] \arrow[dd, bend right=60, "(\eta_F)^*"'] & \mathrm{Nat}(\mathrm{core}, F) \arrow[d, "G"'] \arrow[dd, bend left=60, "(\eta_F)^*"] \\
\mathrm{Nat}(GG\,\mathrm{core}, GF) \arrow[r, "(G\eta_{\mathrm{core}})^*"] \arrow[d, "(\eta_{G\,\mathrm{core}})^*"] & \mathrm{Nat}(G\,\mathrm{core}, GF) \arrow[d, "(\eta_{\mathrm{core}})^*"'] \\
\mathrm{Nat}(G\,\mathrm{core}, GF) \arrow[r, "(\eta_{\mathrm{core}})^*"] & \mathrm{Nat}(\mathrm{core}, GF)
\end{tikzcd}$$

where the upper square commutes since $G$ is a functor, and the other there parts commute since $\eta$ is natural. Suppose the red-colored maps are equivalent, then we can show that the upper horizontal map $(\eta_{\mathrm{core}})^*$ is an equivalence. If we apply this to the case $F \simeq \mathrm{core}$, then we obtain the desired result. This assumption follows from the next two propositions. □

The next proposition implies that $(\eta_F)_*$ and $(G\eta_{\mathrm{core}})^*$ are equivalences.

**Proposition 5.3.** Let $F : \mathrm{Cat}^{\mathrm{st}} \to \mathrm{An}$ be an additive grouplike functor. Then the natural transformation

$$\eta_F : F \Rightarrow GF$$

is an equivalence.

The next proposition implies that $(\eta_{G\,\mathrm{core}})$ is an equivalence.

**Proposition 5.4.** Let $F : \mathrm{Cat}^{\mathrm{st}} \to \mathrm{An}$ be an additive functor. Then the two natural transformations

$$\eta_{GF}, G\eta_F : GF \Rightarrow GGF$$

differ by an automorphism of the target. That is, the following diagram commutes

$$\begin{tikzcd} GF \arrow[r, "G\eta_F"] \arrow[dr, "\eta_{GF}"'] & GGF \arrow[d, "\simeq"] \\ & GGF. \end{tikzcd}$$

## 6. The Localization Theorem

The goal of this section is to prove the localization theorem.

**Theorem 6.1** ([HLS23] Theorem.6.1 (The Localization Theorem))**.** *The algebraic K-theory* $\mathcal{K} : \mathrm{Cat}^{\mathrm{st}} \to \mathrm{An}$ *and the algebraic K-theory spectrum* $\mathcal{K}_{\geq 0} : \mathrm{Cat}^{\mathrm{st}} \to \mathrm{Sp}$ *are Verdier-localizing.*

By the corollary of theorem 2.36, an additive grouplike functor $F : \mathrm{Cat}^{\mathrm{st}} \to \mathrm{An}$ is Verdier-localizing if and only if it satisfies the following condition:

($*$) For every Verdier sequence $\mathcal{C} \to \mathcal{D} \to \mathcal{E}$, the canonical map $|F\,\mathrm{Fun}^{\mathcal{C}}([-], \mathcal{D})| \to F(\mathcal{E})$ is an equivalence of anima.

The next proposition implies that it is enough to prove that the core functor satisfies ($*$).

**Proposition 6.2.** Let $F : \mathrm{Cat}^{\mathrm{st}} \to \mathrm{An}$ be an additive functor. If $F$ satisfies ($*$), then $|FQ(-)|$ and $\Omega|FQ(-)|$ also satisfy ($*$).

**Remark 6.3.** If the core functor satisfies ($*$), then the K-theory functor also satisfies ($*$). Indeed, we can write

$$\mathcal{K}(-) \simeq \Omega|\,\mathrm{Span}(-)| \simeq \Omega|\,\mathrm{asscat}\,\mathrm{core}\,Q(-)| \simeq \Omega|\,\mathrm{core}\,Q(-)|.$$

To prove the proposition, we need the following lemma.

**Lemma 6.4.** Verdier sequences are stable under applying the functor

$$\mathrm{Fun}(\mathcal{I}, -) : \mathrm{Cat}^{\mathrm{st}} \to \mathrm{Cat}^{\mathrm{st}}$$

for every finite poset $\mathcal{I}$.

From the above discussion, we need to show that the core functor satisfies ($*$). Let $\mathcal{D}$ be a stable category, and let $\mathcal{C}$ be a stable subcategory of $\mathcal{D}$. We let $\mathcal{D}_{\mathcal{C}}$ denote the full subcategory of $\mathcal{D}$ spanned by the equivalences modulo $\mathcal{C}$ in $\mathcal{D}$. Then we obtain

$$\mathrm{core}\,\mathrm{Fun}^{\mathcal{C}}([-], \mathcal{D}) \simeq \mathrm{core}\,\mathrm{Fun}([-], \mathcal{D}_{\mathcal{C}}) \simeq \mathrm{Map}_{\mathrm{Cat}}([-], \mathcal{D}_{\mathcal{C}}) \simeq \mathrm{N}^r(\mathcal{D}_{\mathcal{C}}).$$

Since the canonical map $|\,\mathrm{N}^r(\mathcal{D}_{\mathcal{C}})| \to |\mathcal{D}_{\mathcal{C}}|$ is an equivalence of anima, we have an equivalence

$$|\,\mathrm{core}\,\mathrm{Fun}^{\mathcal{C}}([-], \mathcal{D})| \simeq |\mathcal{D}_{\mathcal{C}}|.$$

Thus it suffices to show the following proposition.

**Proposition 6.5.** Let $\mathcal{D}$ be a stable category, and let $\mathcal{C}$ be a stable subcategory of $\mathcal{D}$. We let $\mathcal{D}_{\mathcal{C}}$ denote the full subcategory of $\mathcal{D}$ spanned by the equivalences modulo $\mathcal{C}$ in $\mathcal{D}$. Then the map

$$|\mathcal{D}_{\mathcal{C}}| \to \mathrm{core}(\mathcal{D}/\mathcal{C})$$

is faithful. If the inclusion $\mathcal{C} \subseteq \mathcal{D}$ is a Verdier inclusion, then this map is an equivalence.

This proposition is a special case of the following proposition.

**Proposition 6.6.** Let $\mathcal{C}$ be a category, and let $S$ be a subcategory of $\mathcal{C}$. If $S$ is closed under 2-out-of-3 and pushouts in $\mathcal{C}$, then a map

$$|S| = S[S^{-1}] \to \mathcal{C}$$

is faithful. Moreover, the following conditions are equivalent:

(1) The inclusion $|S| \subseteq \operatorname{core}\mathcal{C}[S^{-1}]$ is fully faithful.

(2) The category $S$ is closed under 2-out-of-6 in $\mathcal{C}$.

(3) A morphism in $\mathcal{D}$ belongs to $S$ if and only if its source and target are in $S$ and it is invertible in $\mathcal{C}[S^{-1}]$.

## 7. The Cofinality Theorem

The goal of this section is to prove the cofinality theorem.

**Theorem 7.1** ([HLS23] Theorem.7.1 (The Cofinality Theorem))**.** *Let $f : \mathcal{C} \to \mathcal{D}$ be a functor between stable categories. If $f$ is a dense inclusion, then it induces a fiber sequence*

$$\mathcal{K}(\mathcal{C}) \to \mathcal{K}(\mathcal{D}) \to \mathcal{K}_0(\mathcal{D})/\mathcal{K}_0(\mathcal{C}).$$

*In particular, maps of abelian groups*

$$\mathcal{K}_i(\mathcal{C}) \to \mathcal{K}_i(\mathcal{D})$$

*are isomorphisms for every $i \geq 1$, and there exists a short exact sequence*

$$0 \to \mathcal{K}_0(\mathcal{C}) \to \mathcal{K}_0(\mathcal{D}) \to \mathcal{K}_0(\mathcal{D})/\mathcal{K}_0(\mathcal{C}) \to 0.$$

**Definition 7.2.** Let $f : X \to Y$ be a map of $\mathbb{E}_\infty$-monoids in An. We will say that $f$ is *cofinal* if it satisfies the following conditions:

(1) The map $\pi_0(f) : \pi_0(X) \to \pi_0(Y)$ is an inclusion.

(2) For every object $y$ in $\pi_0(Y)$, there exists an object $y'$ in $\pi_0(Y)$ such that $y + y'$ in $\pi_0(X)$.

We will say that a cofinal map is *dense* if it satisfies the following conditions:

(3) An object $y$ in $\pi_0(Y)$ belongs to $\pi_0(X)$ if there exists an object $x$ in $\pi_0(X)$ such that $x + y$ in $\pi_0(X)$.

**Definition 7.3.** Let $F : \mathrm{Cat}^{\mathrm{st}} \to \mathrm{An}$ be an additive functor. We will say that $F$ is *Karoubian* if it satisfies the following conditions:

(1) The functor $F$ takes every dense inclusion between stable categories to a dense map of $\mathbb{E}_\infty$-monoids.

(2) The functor $F$ preserves every Cartesian square in $\mathrm{Cat}^{\mathrm{st}}$ whose vertical maps are dense.

**Example 7.4.** The core functor $\operatorname{core} : \mathrm{Cat}^{\mathrm{st}} \to \mathrm{An}$ is Karoubian.

Theorem 7.1 holds for a broader class of additive Karoubian functors.

**Notation 7.5.** Let $F : \mathrm{Cat}^{\mathrm{st}} \to \mathrm{An}$ be an additive Karoubian functor. We will refer to the functor

$$F^{\mathrm{grp}} := \Omega|FQ - |$$

as the *group completion* of $F$. For example, the functor $(\operatorname{core})^{\mathrm{grp}}$ is equivalent to the algebraic K-theory $\mathcal{K}$.

**Theorem 7.6.** *Let $F : \mathrm{Cat}^{\mathrm{st}} \to \mathrm{An}$ be an additive Karoubian functor. For every dense inclusion $\mathcal{C} \subseteq \mathcal{D}$ between stable categories, the canonical map of $\mathbb{E}_\infty$-monoids*

$$F(\mathcal{D})/F(\mathcal{C}) \to F^{\mathrm{grp}}(\mathcal{D})/F^{\mathrm{grp}}(\mathcal{C})$$

*is an equivalence. Hence maps of abelian groups*

$$\pi_i F^{\mathrm{grp}}(\mathcal{C}) \to \pi_i F^{\mathrm{grp}}(\mathcal{D})$$

*are isomorphisms for every $i \geq 1$, and there exists a short exact sequence*

$$0 \to \pi_0 F^{\mathrm{grp}}(\mathcal{C}) \to \pi_0 F^{\mathrm{grp}}(\mathcal{D}) \to \pi_0 F^{\mathrm{grp}}(\mathcal{D})/\pi_0 F^{\mathrm{grp}}(\mathcal{C}) \to 0.$$

**Corollary 7.7.** If a functor $F : \mathrm{Cat}^{\mathrm{st}} \to \mathrm{An}$ is additive Karoubian, then so is the group completion $F^{\mathrm{grp}}$ of $F$.

**Corollary 7.8.** Let $F : \mathrm{Cat}^{\mathrm{st}} \to \mathrm{An}$ be an additive Karoubian functor. If the functor $F^{\mathrm{grp}}$ is Verdier-localizing, then the functor

$$F^{\mathrm{grp}} \circ (-)^\natural : \mathrm{Cat}^{\mathrm{st}} \to \mathrm{An}$$

is Karoubi-localizing. In particular, the functor $\mathcal{K} \circ (-)^\natural : \mathrm{Cat}^{\mathrm{st}} \to \mathrm{An}$ is Karoubi-localizing.

**Corollary 7.9.** Let $\mathcal{D}$ be a stable category, and let $\mathcal{C}$ be a dense stable subcategory of $\mathcal{D}$. Then the canonical maps of abelian groups

$$\mathcal{K}_i(\mathcal{C}) \to \mathcal{K}_i(\mathcal{D})$$

are isomorphisms for every $i \geq 1$.

## References


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