8.1

(a)

$Var(y_{i,j} \mid \theta_j, \sigma^2)$

$Var(y_{i,j} \mid \mu, \tau^2)$

$\begin{cases} \{Y_{1,j}, \cdots, Y_{n_j,j} \mid \theta_j, \sigma^2\} \sim iid \ N(\theta_j, \sigma^2) \Rightarrow \text{그룹 내 변동성} \\ \{\theta_1, \cdots, \theta_m \mid \mu, \tau^2\} \sim iid \ N(\mu, \tau^2) \Rightarrow \text{그룹 간 변동성} \end{cases}$

직관적으로 $\underbrace{Var(y_{i,j} \mid \theta_i, \sigma^2)}_{\text{그룹 내 변동성}} < \underbrace{Var(y_{i,j} \mid \mu, \tau^2)}_{\text{그룹 간 변동성 + 그룹내 변동성}}$

(b)

- $Cov[y_{i1,j}, y_{i2,j} \mid \theta_j, \sigma^2] = 0$ ∵ θ_j, σ^2 이 given 인 상태에서 $y_{i1,j}$ 와 $y_{i2,j}$ 는 conditionally independent
- $Cov[y_{i1,j}, y_{i2,j} \mid \mu, \tau^2] > 0$ ∵ θ_j의 값이 아직 정해지지 않은 상황. $N(\mu, \tau^2)$으로부터 sampling 된 θ_j를 $y_{i1,j}$ 와 $y_{i2,j}$가 공유하게 되므로 공분산의 값이 양수일 것이다.

(c)

$y_{i,j} = \theta_j + \varepsilon_{ij}$, $\varepsilon_{ij} \sim iid \ N(0, \sigma^2)$

$y_{ij} \mid \theta_j, \sigma^2 \sim iid \ N(\theta_j, \sigma^2)$

① $Var[Y_{ij} \mid \theta_j, \sigma^2] = Var[\varepsilon_{ij} \mid \theta_j, \sigma^2] = \sigma^2$

② $Var[\bar{y}_{\cdot j} \mid \theta_j, \sigma^2] = \sigma^2 / n_j$ ∵ $\bar{y}_{\cdot,j}$를 구성하는 표본의 개수가 n_j

③ $Cov[y_{i1,j}, y_{i2,j} \mid \theta_j, \sigma^2] = 0$ (b) 에서 설명

+) $Cov[y_{i1,j}, y_{i2,j} \mid \theta_j, \sigma^2] = E[y_{i1,j}, y_{i2,j} \mid \theta_j, \sigma^2] - E[y_{i1,j} \mid \theta_j, \sigma^2] \cdot E[y_{i2,j} \mid \theta_j, \sigma^2]$

$= \underbrace{E[y_{i1,j} \mid \theta_j, \sigma^2] \cdot E[y_{i2,j} \mid \theta_j, \sigma^2]}_{\text{conditionally independent}} - E[y_{i1,j} \mid \theta_j, \sigma^2] \cdot E[y_{i2,j} \mid \theta_j, \sigma^2] = 0$

④ $Var[y_{i,j} \mid \mu, \tau^2] = Var(E[y_{i,j} \mid \theta_j, \sigma^2] \mid \mu, \tau^2) + E(Var[y_{i,j} \mid \theta_j, \sigma^2] \mid \mu, \tau^2)$ ← 총분산법칙

$= Var(\theta_j \mid \mu, \tau^2) + E(\sigma^2 \underbrace{\mid \mu, \tau^2}_{\text{관계 X}}) = \tau^2 + \sigma^2$

⑤ $Var[\bar{y}_{\cdot,j} \mid \mu, \tau^2] = Var(E[\bar{y}_{\cdot,j} \mid \theta_j, \sigma^2] \mid \mu, \tau^2) + E(Var[\bar{y}_{\cdot,j} \mid \theta_j, \sigma^2] \mid \mu, \tau^2)$ ← 총분산법칙

$= Var(\theta_j \mid \mu, \tau^2) + E(\sigma^2/n_j \mid \mu, \tau^2)$

$= \tau^2 + \sigma^2/n_j$

⑥ $Cov[y_{i1,j}, y_{i2,j} \mid \mu, \tau^2] = Cov(E[y_{i1,j} \mid \theta_j, \sigma^2], E[y_{i2,j} \mid \theta_j, \sigma^2]) + E(Cov[y_{i1,j}, y_{i2,j} \mid \theta_j, \sigma^2] \mid \mu, \tau^2)$

$= Cov(\theta_j, \theta_j) + \underbrace{0}_{\text{③에서 설명}} = Var(\theta_j) + 0 = \tau^2$

(d)

Using Bayes' rule, show that

$p(\mu \mid \theta_1, \cdots, \theta_m, \sigma^2, \tau^2, y_1, \cdots, y_m) = p(\mu \mid \theta_1, \cdots, \theta_m, \tau^2)$

Sol) $$p(\mu \mid \theta_1, \cdots, \theta_m, \sigma^2, \tau^2, y_1, \cdots, y_m) \xrightarrow{\text{Bayes' Rule}} = \frac{p(\mu, \theta_1, \cdots, \theta_m, y_1, \cdots, y_m \mid \sigma^2 \tau^2)}{\int_{-\infty}^{\infty} p(\mu, \theta_1, \cdots, \theta_m, y_1 \cdots, y_m \mid \sigma^2 \tau^2) d\mu}$$

$$= \frac{p(y_1, \cdots, y_m \mid \theta_1 \cdots, \theta_m, \sigma^2) p(\theta_1, \cdots, \theta_m \mid \mu, \tau^2) p(\mu)}{\int_{-\infty}^{\infty} \underbrace{p(y_1, \cdots, y_m \mid \theta_1, \cdots, \theta_m, \sigma^2)}_{\mu\text{와 무관 ⇒ 상수취급}} p(\theta_1, \cdots, \theta_m \mid \mu, \tau^2) p(\mu) \boxed{d\mu}}$$

$$= \frac{p(\theta_1, \cdots, \theta_m \mid \mu, \tau^2) p(\mu)}{\int_{-\infty}^{\infty} p(\theta_1, \cdots, \theta_m \mid \mu, \tau^2) p(\mu) d\mu} \xrightarrow{\text{Bayes' Rule}} = p(\mu \mid \theta_1, \cdots, \theta_m, \tau^2)$$