ESC HW3(Due: Nov. 12th, 2019)

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8.1

(a) $Var[y_{ij}|\theta_i, \sigma^2]$: variance in i^{th}group sampling y

$Var[y_{ij}|\mu, \tau^2]$: variance

Thus, $Var[y_{ij}|\mu, \tau^2]$ will be bigger than $Var[y_{ij}|\mu, \tau^2]$.

(b) When θ_j is fixed, y_{ij}'s are $i.i.d$

$\therefore Cov[y_{i1,j}, y_{i2,j}|\theta_j, \sigma^2] = 0$

Information of $y_{i1,j}$ updates θ_j and offers $y_{i2,j}$'s information.

(c) $Var[y_{ij}|\theta_j, \sigma^2] = Var[\epsilon_{ij}|\theta_j, \sigma^2] = \sigma^2$

$Var[\bar{y_{ij}}|\theta_j, \sigma^2] = \dfrac{\sigma^2}{n_j}$

$$\begin{aligned}
Var[y_{ij}|\mu, \tau^2] &= E[Var(y_{ij}|\theta_i, \sigma^2)|\mu, \tau^2] + Var[E((y_{ij}|\theta_i, \sigma^2)|\mu, \tau^2)] \\
&= E[\sigma^2|\mu, \tau^2] + Var[\theta^2|\mu, \tau^2] \\
&= \sigma^2 + \tau^2
\end{aligned}$$

$$\begin{aligned}
Var[\bar{y}_{ij}|\mu, \tau^2] &= E[Var(\bar{y}_{ij}|\theta_j, \sigma^2)|\mu, \tau^2] + Var[E(\bar{y}_{ij}|\theta_j, \sigma^2)|\mu, \tau^2)] \\
&= E[\frac{\sigma^2}{n_i}|\mu, \tau^2] + Var[\frac{\sigma^2}{n_i}|\mu, \tau^2] \\
&= \frac{\sigma^2}{n_i} + \tau^2
\end{aligned}$$

$$\begin{aligned}
Cov(y_{i1j}, y_{i2j}|\theta_j, \sigma^2) &= E(y_{i1j}, y_{i2j}|\theta_j, \sigma^2) - E(y_{i1j}|\theta_j, \sigma^2)E(y_{i2j}|\theta_j, \sigma^2) \\
&= 0
\end{aligned}$$

$$
\begin{aligned}
Cov(y_{i1j}, y_{i2j}|\mu, \tau^2) &= E[Cov(y_{i1j}, y_{i2j}|\theta_j, \sigma^2)|\mu, \tau^2] + Cov[E(y_{i1j}, y_{i2j}|\theta_j, \sigma^2)|\mu, \tau^2] \\
&= Cov(\theta_j, \theta_j) \\
&= \tau^2
\end{aligned}
$$

(d)

$$
\begin{aligned}
P(\mu|\theta_1, \cdots, \theta_m, \sigma^2, \tau^2, y_1, \cdots, y_m) &= P(\mu|\tilde{\theta}, \sigma^2, \tau^2, \tilde{y}) \\
&= \frac{P(\mu, \tilde{\theta}, \sigma^2, \tau^2, \tilde{y})}{\int P(\mu, \tilde{\theta}, \sigma^2, \tau^2, \tilde{y}) d\mu} \\
&= \frac{P(\mu)P(\sigma^2)P(\tau^2)P(Y|\tilde{\theta}, \sigma^2)P(\tilde{\theta}|\mu, \tau^2)}{\int P(\mu)P(\sigma^2)P(\tau^2)P(Y|\tilde{\theta}, \sigma^2)P(\tilde{\theta}|\mu, \tau^2) d\mu}
\end{aligned}
$$