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8.1 Consider the hierarchical model:

$\theta_1, \ldots, \theta_m \mid \mu, \tau^2 \sim iid\ N(\mu, \tau^2)$

$y_{1,j}, \ldots, y_{n_j,j} \mid \theta_j, \sigma^2 \sim iid\ N(\theta_j, \sigma^2)$

$Var[y_{i,j} \mid \theta_j, \sigma^2]$, $Var[\bar{y}_{\cdot,j} \mid \theta_j, \sigma^2]$, $Cov[y_{i1,j}, y_{i2,j} \mid \theta_j, \sigma^2]$

$Var[y_{i,j} \mid \mu, \tau^2]$, $Var[\bar{y}_{\cdot,j} \mid \mu, \tau^2]$, $Cov[y_{i1,j}, y_{i2,j} \mid \mu, \tau^2]$

ⓐ $Var[y_{i,j} \mid \theta_i, \sigma^2]$: ith group 에서 y를 sampling 할 때의 변동성 (group이 fixed)

$Var[y_{i,j} \mid \mu, \tau^2]$: $N(\mu, \tau^2)$를 따르는 $\theta_1, \ldots, \theta_m$ 를 하나 sampling 하고 그 그룹의 θ_i에 대해 y를 sampling 할 때의 변동성 (위의 경우에서 그룹 간 변동성이 추가 됨)

∴ $Var[y_{i,j} \mid \mu, \tau^2]$이 더 클 것이다.

ⓑ ① $Cov[y_{i1,j}, y_{i2,j} \mid \theta_j, \sigma^2] = 0$

θ_j가 고정된 상태에서 $y_{i,j}$들은 iid 이므로 $y_{i1,j}$, $y_{i2,j}$의 cov는 0 이다.

② $Cov[y_{i1,j}, y_{i2,j} \mid \mu, \tau^2]$

$y_{i1,j}$의 정보가 θ_j를 update 하고 $y_{i2,j}$에 대해 정보를 제공한다.

같은 그룹에서 온 $y_{i1,j}$ 와 $y_{i2,j}$는 positively correlated 되어있을 것이다.

ⓒ

① $Var[y_{i,j} \mid \theta_j, \sigma^2] = \sigma^2$ $\quad (\because y_{1,j}, \ldots, y_{n_j,j} \mid \theta_j, \sigma^2 \sim iid\ N(\theta_j, \sigma^2))$

② $Var[\bar{y}_{\cdot j} \mid \theta_j, \sigma^2] = \frac{\sigma^2}{n_j}$ (sample mean의 variance)

③ $$Var[y_{ij} \mid \mu, \tau^2] = E[Var(y_{ij} \mid \theta_i, \sigma^2) \mid \mu, \tau^2] + Var[E(y_{ij} \mid \theta_i, \sigma^2) \mid \mu, \tau^2]$$
$$= E[\sigma^2 \mid \mu, \tau^2] + Var[\theta_j \mid \mu, \tau^2]$$
$$= \sigma^2 + \tau^2$$

∴ ⓐ에서 생각한대로 ①분산보다 ③분산이 크다

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④ $Var[\bar{y}_{\cdot j} \mid \mu, \tau^2] = E[Var(\bar{y}_{\cdot j} \mid \theta_j, \sigma^2) \mid \mu, \tau^2]$
$+ Var[E(\bar{y}_{\cdot j} \mid \theta_j, \sigma^2) \mid \mu, \tau^2]$
$= E[\frac{\sigma^2}{n_j} \mid \mu, \tau^2] + Var[\theta_j \mid \mu, \tau^2]$
$= \frac{\sigma^2}{n_j} + \tau^2$

⑤ $Cov[y_{i_1 j}, y_{i_2 j} \mid \theta_j, \sigma^2]$
$= E[y_{i_1 j}, y_{i_2 j} \mid \theta_j, \sigma^2] - E[y_{i_1 j} \mid \theta_j, \sigma^2] E[y_{i_2 j} \mid \theta_j, \sigma^2]$
$= 0$

⑥ $Cov[y_{i_1 j}, y_{i_2 j} \mid \mu, \tau^2]$
$= E[\cancel{Cov(y_{i_1 j}, y_{i_2 j} \mid \theta_j, \sigma^2)}^{0} \mid \mu, \tau^2] + Cov[E(y_{i_1 j} \mid \theta_j, \sigma^2), E(y_{i_2 j} \mid \theta_j, \sigma^2)]$
$= Cov[\theta_j, \theta_j]$
$= \tau^2$

∴ ⓑ에서 예상한대로 ⑤공분산은 0, ⑥공분산은 양수가 나왔다.

ⓓ A B

$P(\mu \mid \theta_1, \dots, \theta_m, \sigma^2, \tau^2, y_1, \dots, y_m)$

$$= \frac{P(Y \mid \underset{\sim}{\theta}, \sigma^2) \cdot P(\underset{\sim}{\theta} \mid \tau^2, \mu) P(\sigma^2) P(\tau^2) P(\mu)}{\int P(Y \mid \underset{\sim}{\theta}, \sigma^2) \cdot P(\underset{\sim}{\theta} \mid \tau^2, \mu) \cdot P(\sigma^2) P(\tau^2) P(\mu)\, d\mu} \quad (\because \text{Bayes Rule})$$

$$= \frac{\cancel{P(Y \mid \underset{\sim}{\theta}, \sigma^2)} P(\underset{\sim}{\theta} \mid \tau^2, \mu) \cancel{P(\sigma^2)} \cancel{P(\tau^2)} P(\mu)}{\cancel{P(Y \mid \underset{\sim}{\theta}, \sigma^2)} \cancel{P(\sigma^2)} \cancel{P(\tau^2)} \int P(\underset{\sim}{\theta} \mid \tau^2, \mu) P(\mu)\, d\mu} = \frac{P(\underset{\sim}{\theta} \mid \tau^2, \mu)}{\int P(\underset{\sim}{\theta} \mid \tau^2, \mu) P(\mu)\, d\mu}$$

$= P(\mu \mid \underset{\sim}{\theta}, \tau^2)$

$\therefore P(\mu \mid \underset{\sim}{\theta}, \sigma^2, \tau^2, \underset{\sim}{y}) = P(\mu \mid \underset{\sim}{\theta}, \tau^2)$