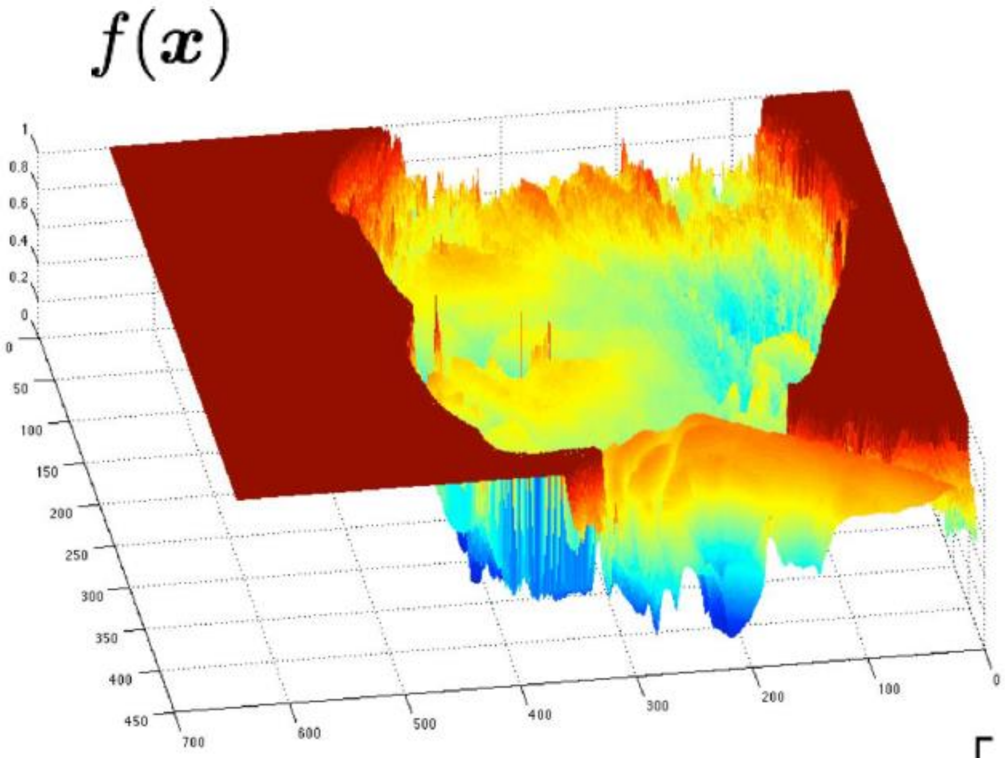
Convolution as Smoothing



domain $\boldsymbol{x} = \begin{bmatrix} x \\ y \end{bmatrix}$

Sang-wook Lee(Yonsei)

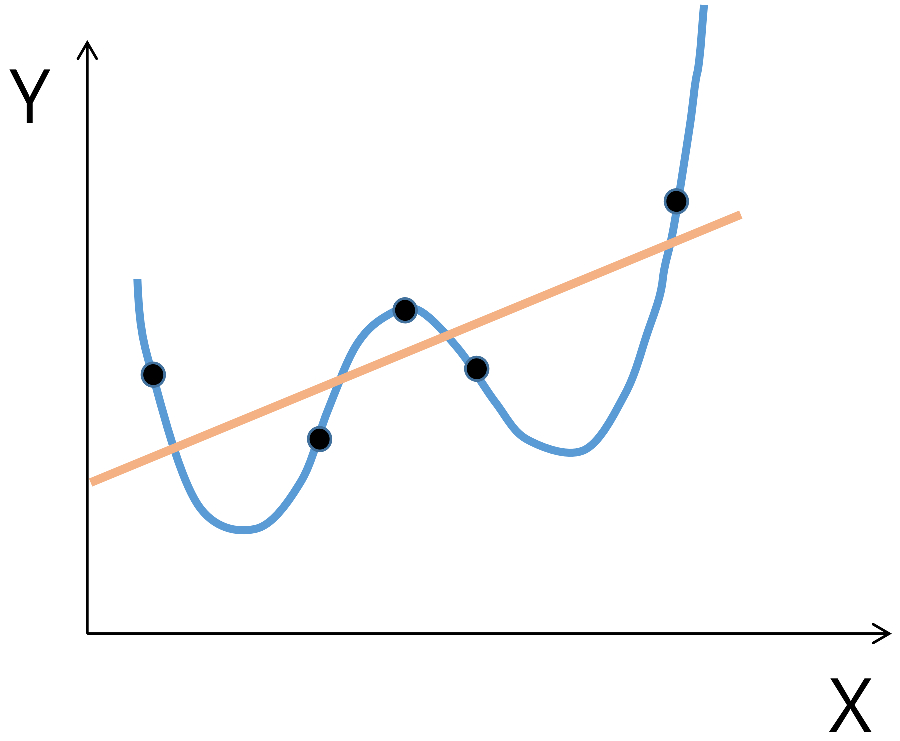
Convolution as Smoothing

What is convolution? Averaging

What is averaging? Smoothing

Smoothness

Smooth : Small f'' or less wiggly



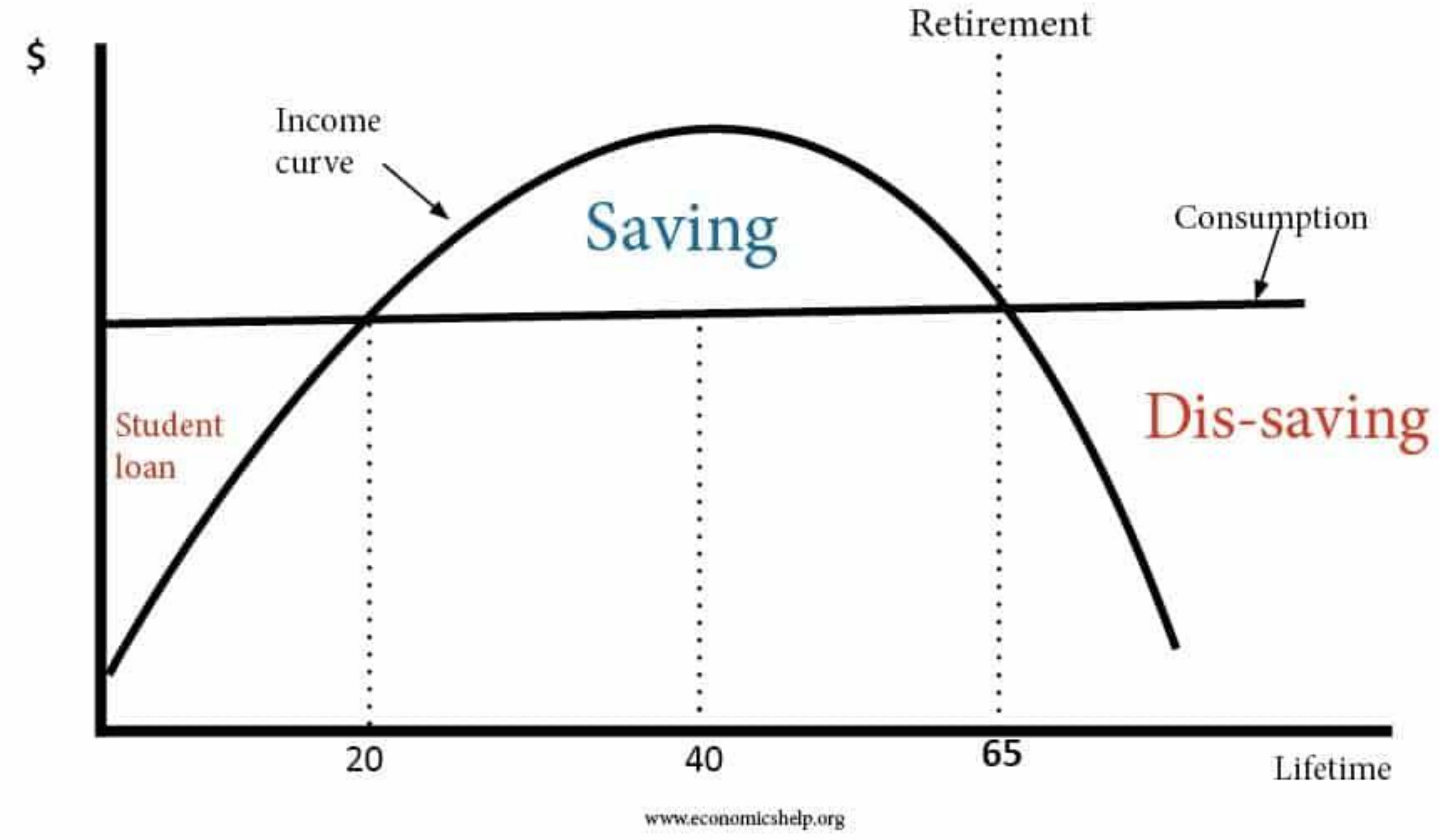
Small f'' and less wiggly

Big f'' and more wiggly

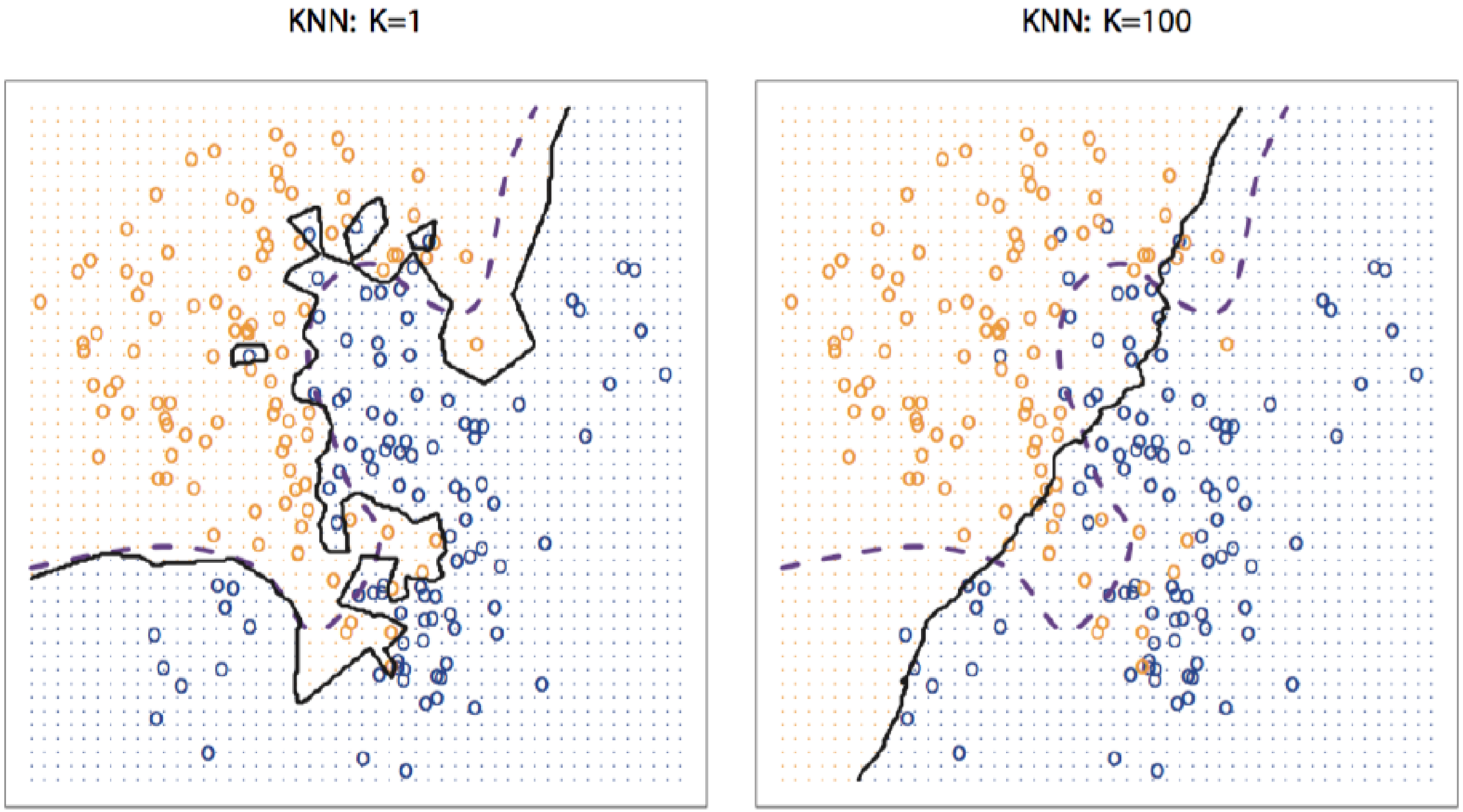
Averaging is smoothing?

1. Consumption smoothing in Economics
2. KNN
3. CLT

Consumption smoothing



KNN



CLT

No matter how ugly the distribution is…

Philosophy of Averaging

Local info vs. global info

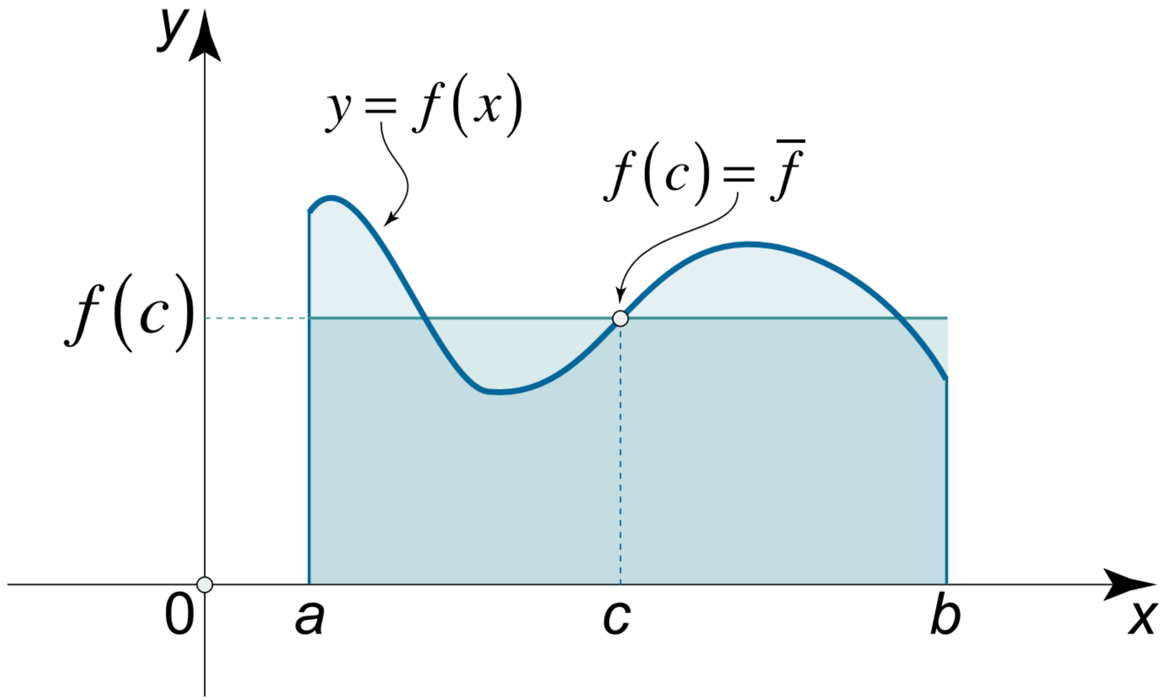
Convex combination(weighted average)

$$\bar{X} = \sum_{i=1}^{n} \frac{1}{n} X_i \text{ s.t. } \sum_{i=1}^{n} \frac{1}{n} = 1$$

$$X^* = \sum_{i=1}^{n} w_i X_i \text{ s.t. } \sum_{i=1}^{n} w_i = 1, w_i \geq 0$$

Integral as averaging

$$\bar{f} = \sum_{i=1}^{n} \frac{1}{n} f(x_i) \rightarrow \int_0^1 f(x)dx$$



Convolution as averaging

$$\sum_{i=1}^{n} \frac{1}{n} f(x_i) \to \int_0^1 f(t)dt = \int_0^1 1 \cdot f(t)\, dt$$

$$\sum_{i=1}^{n} w_i f(x_i) \to \int_0^1 g(x-t) \cdot f(t)dt = f * g(x)$$


GOD WHY
WHYYYYYY !!

Why need to use convolution?

1. Nice algebraic properties
2. Kernel smoothing
3. Image processing
4. Fourier transform

Nice algebraic properties

Commutativity

$$f * g = g * f$$

Proof: By definition

$$(f * g)(t) = \int_{-\infty}^{\infty} f(\tau) g(t - \tau)\, d\tau$$

Changing the variable of integration to $u = t - \tau$ the result follows.

Associativity

$$f * (g * h) = (f * g) * h$$

Proof: This follows from using Fubini's theorem (i.e., double integrals can be evaluated as i

Distributivity

$$f * (g + h) = (f * g) + (f * h)$$

Proof: This follows from linearity of the integral.

Associativity with scalar multiplication

$$a(f * g) = (af) * g$$

for any real (or complex) number a.

Multiplicative identity

No algebra of functions possesses an identity for the convolution. The lack of identity is typ
convolved with a delta distribution or, at the very least (as is the case of L^1) admit approxin
convolution. Specifically,

$$f * \delta = f$$

where δ is the delta distribution.

Inverse element

Some distributions have an inverse element for the convolution, $S^{(-1)}$, which is defined by

$$S^{(-1)} * S = \delta.$$

The set of invertible distributions forms an abelian group under the convolution.

Complex conjugation

$$\overline{f * g} = \overline{f} * \overline{g}$$

...그만 알아보자

Kernel Smoothing

Idea

Suppose we use a bandwidth $g(\neq h)$ and the kernel L to estimate θ_2.

Then,

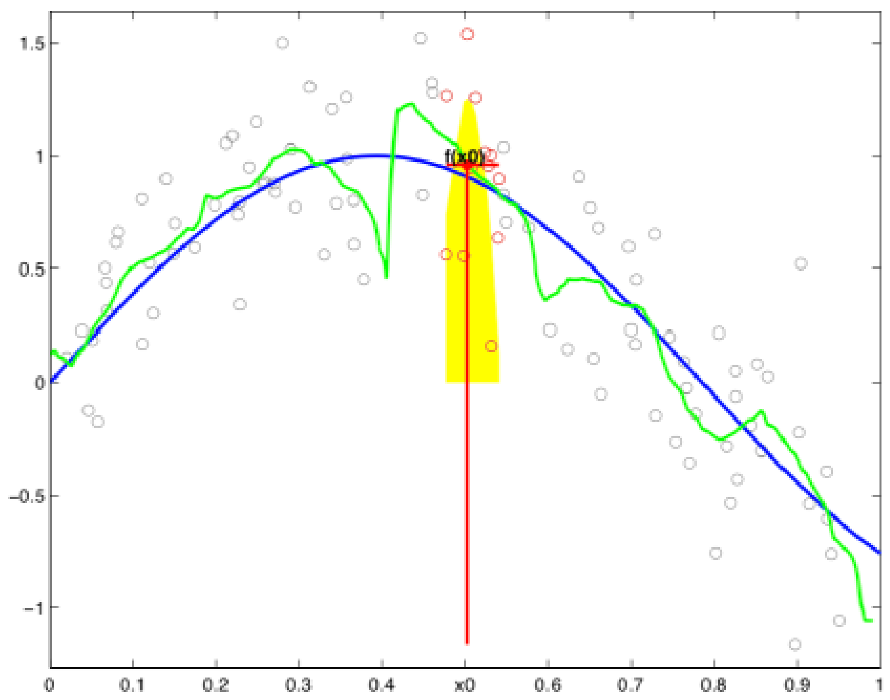
$$\mathrm{Bias}(\widetilde{\theta}_{2,0}(g)) = -\frac{1}{2}\ \mu_2(L*L)\ \theta_3\ g^2 + o(n^{-1}g^{-5} + g^2)$$

$$\mathrm{var}(\widetilde{\theta}_{2,0}(g)) = 2n^{-2}g^{-9}\int (L''*L'')^2\theta_0 + O(n^{-1}) + o(n^{-2}g^{-9})$$

$$g_{\mathrm{opt}} = \operatorname*{argmin}_{g>0} \mathrm{MSE}(\widetilde{\theta}_{2,0}(g))$$

$$= \left[\frac{18\int (L''*L'')^2}{\mu_2(L*L)^2}\right]^{1/13} \left(\frac{\theta_0}{\theta_3^2}\right)^{1/13} n^{-2/13}$$

Kernel Smoothing



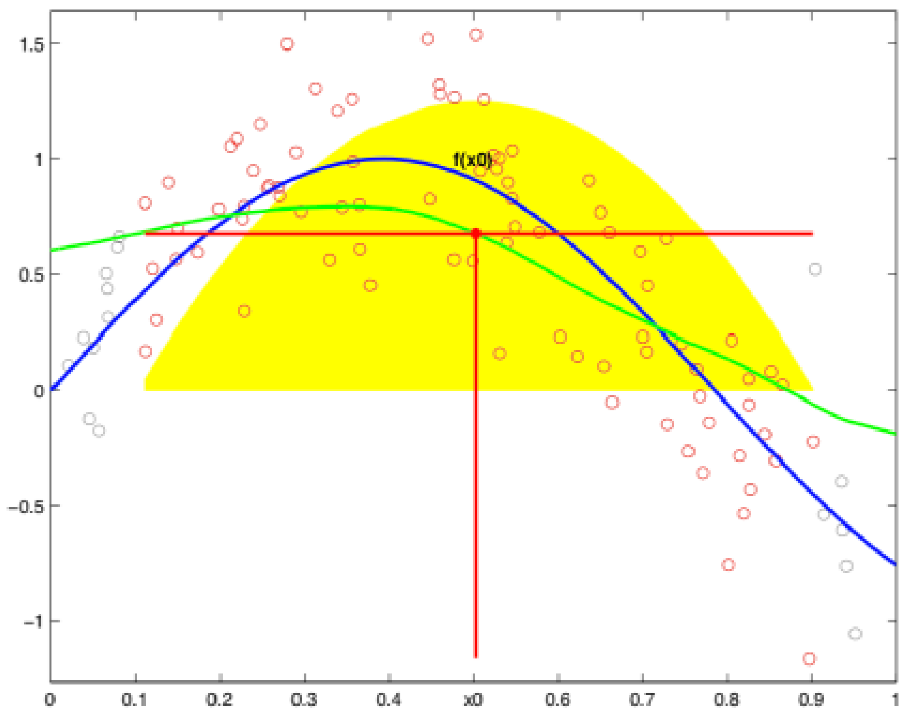


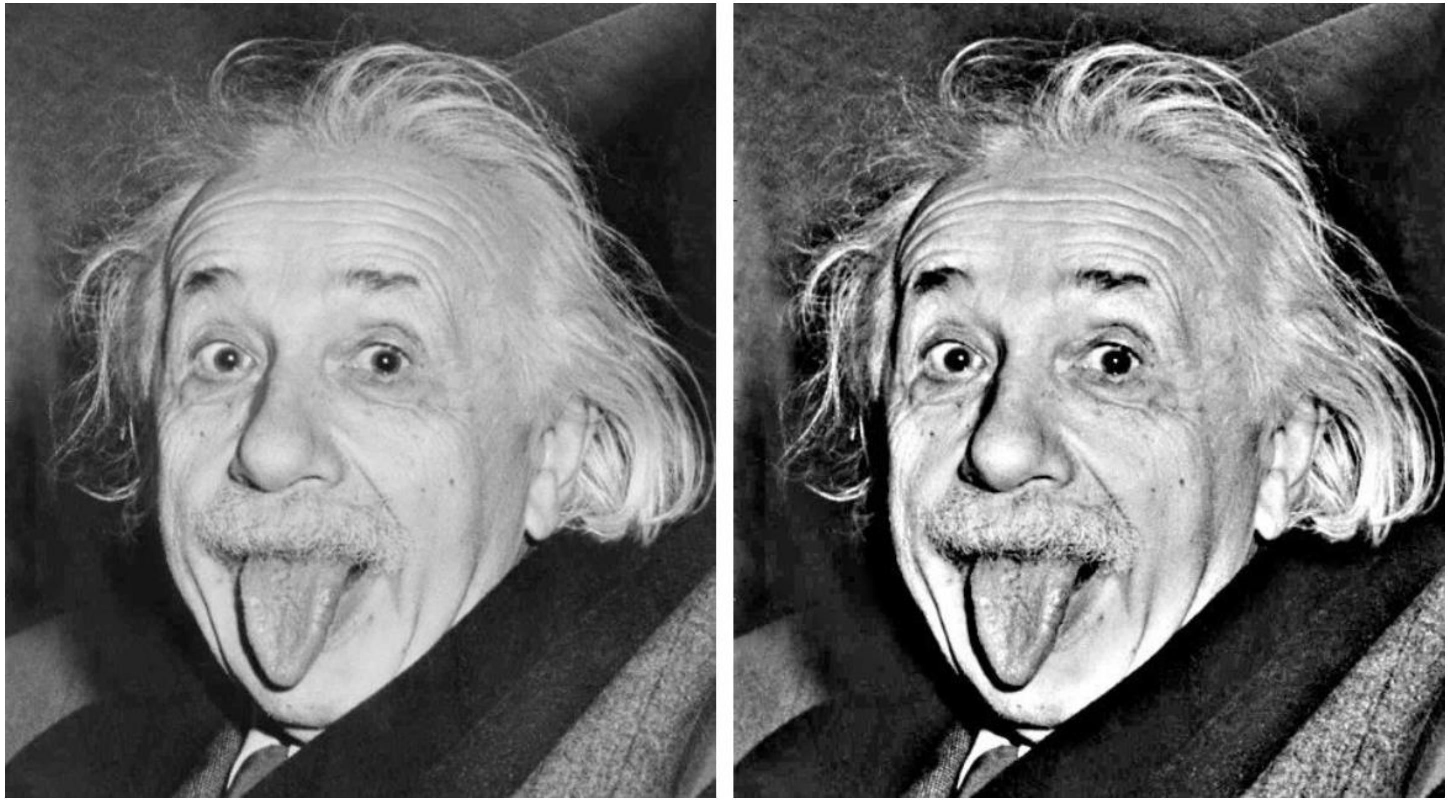
Image processing

$$(f * g)(x) = \int_{-\infty}^{\infty} f(y)g(x-y)dy$$

filtered signal

filter

input signal



Fourier Transform

$$\mathcal{F}\{g * h\} = \mathcal{F}\{g\}\mathcal{F}\{h\}$$

What is Fourier transform?

$$\mathfrak{L}(f) = \int_{-\infty}^{+\infty} e^{-xt} f(x) dx = F_1(t)$$

$$\mathfrak{F}(f) = \int_{-\infty}^{+\infty} e^{-ixt} f(x) dx = F_2(t)$$

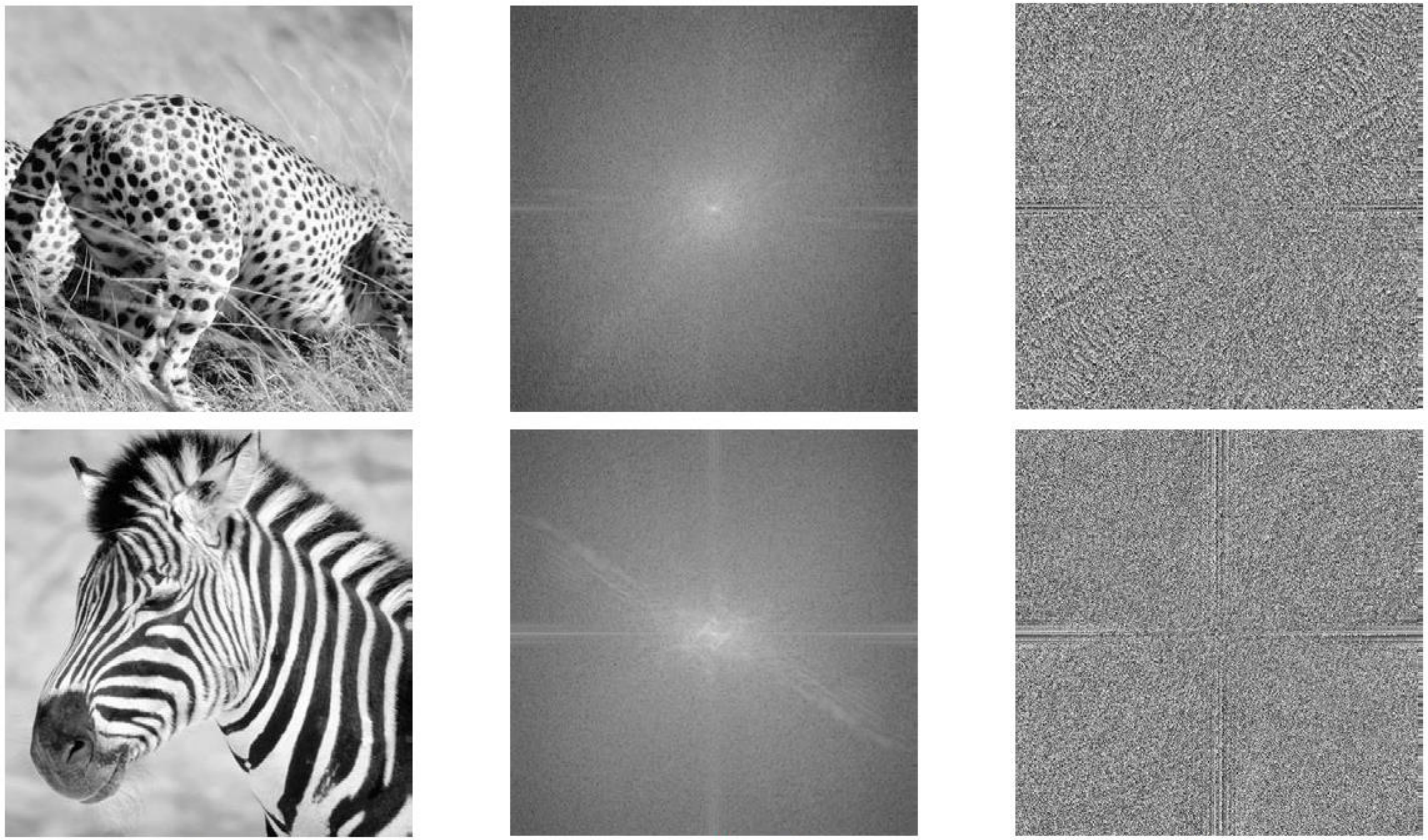
$$f(x) \to F(t)$$

Why need to know Fourier transform?

1. Solving differential equation
2. Image process
3. Fourier series
4. MGF, characteristic function

Image process

Fourier transforms of natural images



Example

Seolgwangeun.jpg 750x750 jpeg image

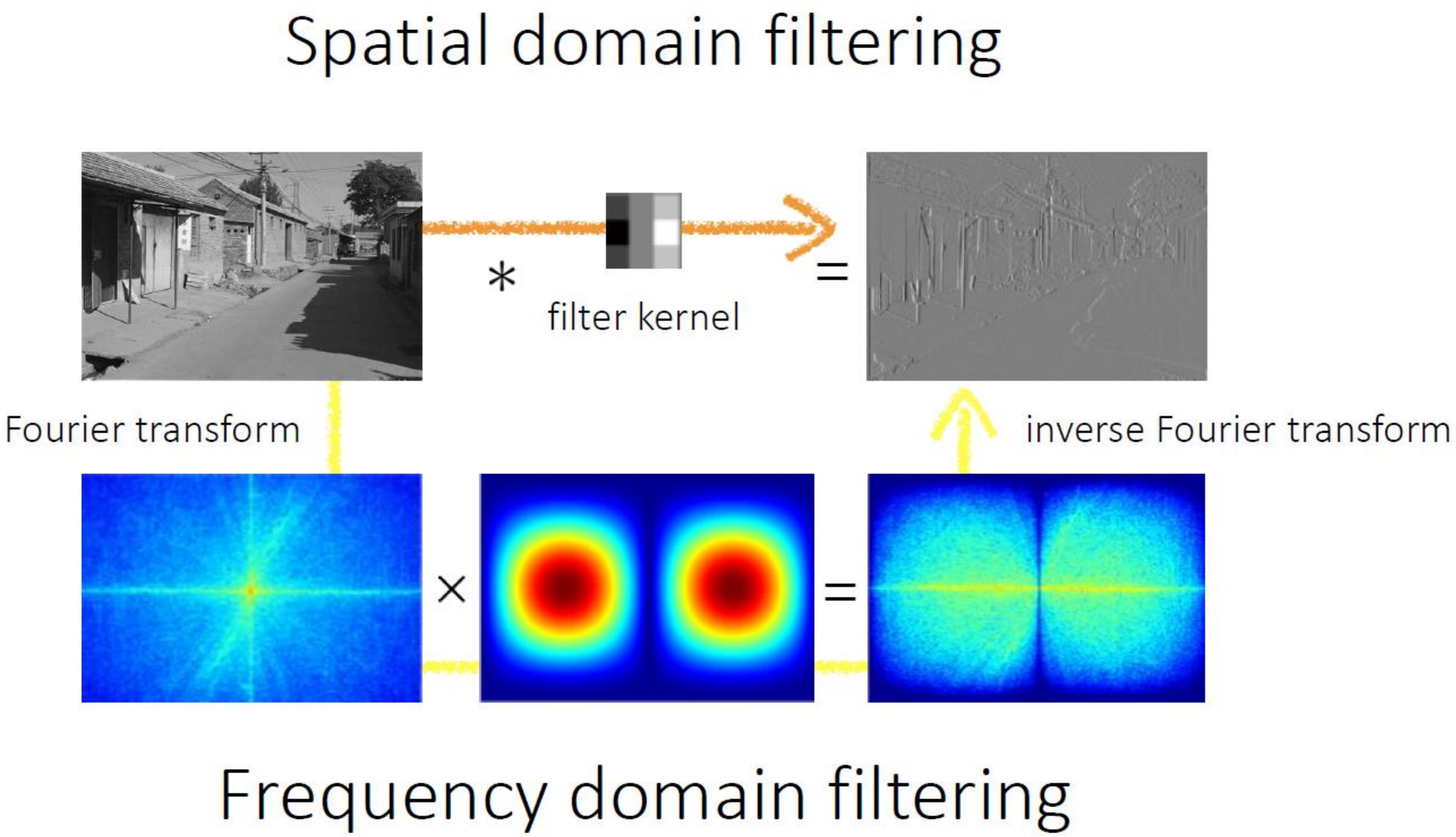
Thus the SVD of the image has 750 summands

rank 3

rank 45

rank 297

Image process



Taylor series vs. Fourier series

Taylor series: $f(x) = \sum_{n=0}^{\infty} c_n x^n$

Fourier series: $f(x) = \sum_{n=-\infty}^{\infty} c_n e^{inx}$

where $c_n = \int_{-\infty}^{+\infty} e^{-int} f(t) dt$

???


여긴 어디지?!
난 누구지!!

MGF, CF

$$M_X(t) = E(e^{Xt}) = \int_{-\infty}^{+\infty} e^{-xt} f(x)dx = \mathfrak{L}(f)$$

$$\phi_X(t) = E\left(e^{iXt}\right) = \int_{-\infty}^{+\infty} e^{-ixt} f(x)dx = \mathfrak{F}(f)$$

Summary

What is convolution? Averaging

What is averaging? Smoothing

What is Fourier? Awesome thing

Reference

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[3] Bumsup Ham, (2018), “*Introductory Artificial Intelligence*” lecture note

[4] Kwangeun Seol, (2018), “*Seoulkwangeun.jpg*” 존잘 사진

질문 받아요

감사합니다