Week 2 손지우

7. 코드 실행 완료

2. Data가 binomial distribution일때, Likelihood를 Exponential Families 형태로 변환해 보기. 또한 왜 Beta distribution이 Conjugacy인지 생각해 보기.

1) 변환하기

let $X \sim Bin(n, \theta)$ for n: 자연수, $0 \le \theta \le 1$ ($X_1, \dots, X_n \overset{iid}{\sim} Ber(\theta)$ → $X = X_1 + \cdots + X_n$ 으로 생각하면 편함)

$$f(x; n, \theta) = \binom{n}{x} \theta^x (1-\theta)^{n-x}$$

$$\propto \left(\frac{\theta}{1-\theta}\right)^x (1-\theta)^n$$

$$= \exp\left(x \log\frac{\theta}{1-\theta}\right)(1-\theta)^n$$

$$= \exp(x\phi)\left(\frac{1}{1+\exp(\phi)}\right)^n$$

exponential family form: $f(y|\phi) = h(y)\,c(\phi)\exp(\phi\, k(y))$ 라고 한다면

$$\phi = \log\frac{\theta}{1-\theta}, \quad h(x) = 1, \quad c(\phi) = \{1+\exp(\phi)\}^{-n}, \quad k(x) = x$$

2) Beta dist.이 왜 conjugacy?

Let $\theta \sim Beta(\alpha, \beta)$ for $\alpha > 0$, $\beta > 0$, $0 \le \theta \le 1$

$$p(\theta) = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} \theta^{\alpha-1}(1-\theta)^{\beta-1}$$

그렇다면 왜 Beta dist.이 conjugacy가 될 수 있을까?

우선 parameter space가 같고, $\theta^\alpha(1-\theta)^\beta$ format도 같기 때문이다. = exponential family form으로 변환가능

그래서 parameter 간의 의미공유도 가능해진다.

이때 $\theta \sim Beta(n_0 t_0, n_0(1-t_0))$

$\theta | X=x \sim Beta(n_0 t_0 + x, n_0(1-t_0) + n - x)$

$W \sim Beta(\alpha, \beta)$ 일때, $E(W) = \frac{\alpha}{\alpha+\beta}$ 인데.

$\theta \sim Beta(n_0 t_0, n_0(1-t_0))$로 놓으면 $E(\theta) = \frac{n_0 t_0}{n_0} = t_0$ 로 prior guess로서의 역할 잘 해준다.

개인적으로 n_0와 prior guess($E(\theta)$)를 미리 설정해두고 이것과 분포형태에 맞춰서 '잘' t_0를 푸는게 가장 효율적일듯.

3 Relationship between Poisson distribution and Negative Binomial Distribution

$$X \sim NB(r,p) \qquad \text{where } p(X=x) = \binom{r-1+x}{x}(1-p)^r p^x$$

$$\text{Let mean } \frac{pr}{1-p} = \lambda \qquad \rightarrow \qquad p = \frac{\lambda}{r+\lambda}$$

↳ X: r번째 실패하기 까지의 성공 횟수 ($x = 0, 1, 2, \cdots$)

r: 실패 횟수 , p: 성공확률

$$\sum_{x=0}^{\infty} \binom{r-1+x}{x}(1-p)^r p^x = 1$$ 이므로

$$E(X) = \sum_{x=0}^{\infty} x \binom{r-1+x}{x}(1-p)^r p^x$$

$$= \sum_{x=0}^{\infty} x \frac{(r-1+x)!}{x!\,(r-1)!}(1-p)^r p^x$$

$$= \sum_{x=1}^{\infty} \frac{(r-1+x)!}{(x-1)!\; r!}(1-p)^{r+1} p^{x-1} \cdot r \cdot \frac{p}{1-p}$$

$$= \frac{rp}{1-p} \overset{\text{let}}{=} \lambda$$

3.1 Prove the following.

$$Poi(\lambda) = \lim_{r\to\infty\,?} NB\left(r, \frac{\lambda}{r+\lambda}\right)$$

$$p(x) = \binom{r-1+x}{x} p^x (1-p)^r$$

$$= \frac{(r-1+x)!}{x!\,(r-1)!}\left(\frac{\lambda}{r+\lambda}\right)^x \left(\frac{r}{r+\lambda}\right)^r$$

$$= \frac{(r+x-1)(r+x-2)\cdots r\,\cancel{(r-1)!}}{x!\;\cancel{(r-1)!}} \cdot \left(\frac{\lambda}{r+\lambda}\right)^x \left(\frac{r}{r+\lambda}\right)^r$$

$$= \frac{(r+x-1)(r+x-2)\cdots r}{(r+\lambda)^x} \cdot \frac{\lambda^x}{x!}\left(\frac{r}{r+\lambda}\right)^r$$

$$= \left(\frac{r+x-1}{r+\lambda}\cdot\frac{r+x-2}{r+\lambda}\cdot \cdots \cdot \frac{r}{r+\lambda}\right)\frac{\lambda^x}{x!}\left(1-\frac{\lambda}{r+\lambda}\right)^r$$

$$\lim_{r\to\infty} p(x) = (1\cdot \cdots \cdot 1)\cdot \frac{\lambda^x}{x!} \lim_{r\to\infty}\left(1-\frac{\lambda}{r+\lambda}\right)^r$$

$$\lim_{r\to\infty}\left(1-\frac{\lambda}{r+\lambda}\right)^{\left(-\frac{r+\lambda}{\lambda}\right)\cdot\left(-\frac{r\lambda}{r+\lambda}\right)} = e^{-\lambda} \qquad \left(\because\ e = \lim_{n\to\infty}\left(1+\frac{a}{n}\right)^{\frac{n}{a}}\right)$$

$$= \frac{e^{-\lambda}\lambda^x}{x!} \quad \text{for } x = 0, 1, 2, \cdots$$

↳ $Pois(\lambda)$

3.2 Compare the variance of each distribution. Show that the Negative Binomial distribution is always overdispersed.

① NB

$$Var(X) = E(X^2) - E^2(X)$$

$$= r \cdot \frac{p}{(1-p)^2}$$

$$E(X^2) = \sum_{x=0}^{\infty} x^2 \binom{r-1+x}{x} p^x (1-p)^r$$

$$= \sum_{x=1}^{\infty} x \frac{(r-1+x)!}{(x-1)!\, r!} p^{x-1} (1-p)^{r+1} \cdot \frac{rp}{1-p}$$

$x-1=k$로 치환

$$= \frac{rp}{1-p} \sum_{k=0}^{\infty} (k+1) \frac{(r+k)!}{k!\, r!} p^k (1-p)^{r+1}$$

$$= \frac{rp}{1-p} \left\{ \underbrace{\sum_{k=0}^{\infty} k \frac{(r+k)!}{k!\, r!} p^k (1-p)^{r+1}}_{NB(r+1,\, p)\text{의 평균} \,=\, \frac{(r+1)p}{1-p}} + \underbrace{\sum_{k=0}^{\infty} \frac{(r+k)!}{k!\, r!} p^k (1-p)^{r+1}}_{NB(r+1,\, p) \text{ 확률합} \,=\, 1} \right\}$$

$$= \frac{rp}{1-p} \left(\frac{(r+1)p}{1-p} + 1 \right)$$

$$= \frac{rp}{1-p} \left(\frac{rp + \cancel{p} + 1 - \cancel{p}}{1-p} \right)$$

$$= \frac{rp(rp+1)}{(1-p)^2}$$

$$\rightarrow Var(X) = E(X^2) - E^2(X)$$

$$= \frac{rp(rp+1)}{(1-p)^2} - \left(\frac{rp}{1-p} \right)^2$$

$$= \frac{rp}{(1-p)^2}$$

② Pois

$Y \sim Pois(\lambda)$ where $\frac{rp}{1-p} = \lambda$

$$Var(Y) = \lambda = \frac{rp}{1-p}$$

$\rightarrow Var(Y) \leq Var(X)$ 즉, Pois 보다 NB일 때 Variance가 더 크다.

즉, NB로 보고 variance가 조금 더 큰 형태로 바꿔서 (one param → two param)

Variance fitting을 조금 더 안정적으로 해줄 수 있다.

3.3 Likewise, prove the following.

$$Y \sim Binom(n,p) \qquad \text{where } p(y) = \binom{n}{x} p^y p^{n-y}$$

$$\text{Let mean } np = \lambda \quad \rightarrow \quad p = \frac{\lambda}{n}$$

$$Poi(\lambda) = \lim_{n\to\infty} Binom(n, \frac{\lambda}{n})$$

$$p(y) = \binom{n}{y} p^y (1-p)^{n-y}$$

$$= \frac{n!}{y!(n-y)!} p^y (1-p)^{n-y}$$

$$= \frac{n \cdot (n-1) \cdot \cdots \cdot (n-y+1) \cancel{(n-y)!}}{y! \cancel{(n-y)!}} \left(\frac{\lambda}{n}\right)^y \left(1-\frac{\lambda}{n}\right)^{n-y}$$

$$= \frac{n(n-1) \cdots (n-y+1)}{n^y} \frac{\lambda^y}{y!} \left(1-\frac{\lambda}{n}\right)^n \left(1-\frac{\lambda}{n}\right)^{-y}$$

$$\lim_{n\to\infty} p(y) = 1 \cdot \cdots \cdot 1 \frac{\lambda^y}{y!} \lim_{n\to\infty}\left(1-\frac{\lambda}{n}\right)^n \cdot 1$$

$$\lim\left(1-\frac{\lambda}{n}\right)^{(-\frac{n}{\lambda})^{-\lambda}} = e^{-\lambda}$$

$$= \frac{e^{-\lambda}\lambda^y}{y!}$$

$$\sim Pois(\lambda)$$

결론: 음이항분포나 이항분포 모두 포아송분포를 극한분포로 갖는다.