풀이

BDA Exercise 8

(a) $y|\theta \sim N(\theta, 20^2)$ $\theta \sim N(180, 4^2)$ $\rightarrow$ $\frac{1}{\tau_0^2} = \frac{1}{1600}$ $\frac{n}{\sigma^2} = \frac{n}{400}$ $\mu_0 = 180, \bar{y} = 150$

$\rightarrow \tau_n^2 = \frac{1600}{4n+1}$ $\mu_n = \frac{1600}{4n+1} \times \frac{1}{1600} \times 180 + \frac{1600}{4n+1} \times \frac{n}{400} \times 150 \rightarrow \frac{600n+180}{4n+1}$

$\theta|y \sim N\left(\frac{600n+180}{4n+1}, \frac{1600}{4n+1}\right)$

(b) $E(\tilde{y}|y) = \mu_n = \frac{600n+180}{4n+1}$

$Var(\tilde{y}|y) = \sigma^2 + \tau_n^2 = 400 + \frac{1600}{4n+1} = \frac{1600n+2000}{4n+1}$

$\tilde{y}|y \sim N\left(\frac{600n+180}{4n+1}, \frac{1600n+2000}{4n+1}\right)$

(c) $n=10 \rightarrow \theta|y \sim N\left(\frac{6180}{41}, \frac{1600}{41}\right)$

$\rightarrow$ 95% CI = (138.488, 162.975)

$\tilde{y}|y \sim N\left(\frac{6180}{41}, \frac{18000}{41}\right) \Rightarrow$ 95% CI is (109.665, 191.799)

(d) $n=100 \rightarrow \theta|y \sim N\left(\frac{60180}{401}, \frac{1600}{401}\right) \rightarrow$ 95% CI = (146.160, 153.990)

$\tilde{y}|y \sim N\left(\frac{60180}{401}, \frac{162000}{401}\right) \rightarrow$ 95% CI = (110.680, 189.469)

2. Marginal distribution of $\sigma^2 | y$

$$p(\sigma^2|y) \propto p(\sigma^2)\, p(y|\sigma^2) \Rightarrow$$

$$= p(\sigma^2) \int p(y|\mu,\sigma^2)\, p(\mu|\sigma^2)\, d\mu$$

$$\propto \left(\frac{1}{\sigma^2}\right)^{\frac{\nu_0}{2}+1} \exp\left(\frac{\nu_0\sigma_0^2}{2\sigma^2}\right) \int \left(\frac{1}{\sigma^2}\right)^{\frac{n}{2}} \exp\left(-\frac{1}{2\sigma^2}\left[(n-1)s^2 + n(\bar{y}-\mu)^2\right)\right] \left(\frac{k_0}{\sigma^2}\right)^{\frac{1}{2}}$$

$$\exp\left[\frac{\kappa_0}{2\sigma^2}(\mu-\mu_0)^2\right] d\mu$$

$$\propto \left(\frac{1}{\sigma^2}\right)^{\frac{\nu_0}{2}+1} \cdot \exp\left(-\frac{\nu_0\sigma_0^2}{2\sigma^2}\right) \left(\frac{1}{\sigma^2}\right)^{\frac{n}{2}} \exp\left[-\frac{1}{2\sigma^2}(n-1)s^2\right] \left(\frac{1}{\sigma^2}\right)^{\frac{1}{2}} \underbrace{\int \exp\left[-\frac{1}{2\sigma^2} n(\bar{y}-\mu)^2 - \frac{\kappa_0}{2\sigma^2}(\mu-\mu_0)^2\right] d\mu}$$

$$= \int \exp\left[-\frac{n+k_0}{2\sigma^2}\left(\mu + \frac{n\bar{y}+\kappa_0\mu_0}{n+k_0}\right)^2\right] d\mu$$

$$\exp\left[-\frac{1}{2\sigma^2}\left(\frac{nk_0(\bar{y}-\mu_0)^2}{n+k_0}\right)\right.$$

$$\propto \left(\frac{1}{\sigma^2}\right)^{\nu_0/2+1} \left(\frac{1}{\sigma^2}\right)^{\frac{n+1}{2}} \exp\left[-\frac{\nu_0\sigma_0^2}{2\sigma^2}\right) \exp\left[-\frac{1}{2\sigma^2}(n-1)s^2\right] \exp\left[-\frac{1}{2\sigma^2}\left[\frac{nk_0(\bar{y}-\mu_0)^2}{n+k_0}\right)\right] \int$$

$$\exp\left[-\frac{n+k_0}{2\sigma^2}\left(\mu - \frac{n\bar{y}+k_0\mu_0}{n+k_0}\right)^2\right] d\mu$$

$$= [\sigma^2]^{-\frac{\nu_0+n}{2}-1} \exp\left[-\frac{1}{2\sigma^2}\left[\nu_0\sigma_0^2 + (n-1)s^2 + \frac{nk_0}{n+k_0}(\bar{y}+\mu_0)^2\right)\right]$$

$$= (\sigma^2)^{-\frac{\nu_0+n}{2}} \exp\left[-\frac{\nu_n}{\sigma^2}\left[\frac{1}{\nu_n}\left(\nu_0\sigma_0^2 (n-1)s^2\right) + \frac{n_0k_0}{n_0+n}(\bar{y}-\mu_0)^2\right]\right\}\right]$$

$\rightarrow$ $\sigma^2 | y \sim \text{Inv}\chi^2(\nu_0+n, \sigma_n^2)$ $\longrightarrow$ $\text{Inv}\chi^2(\nu_n, \sigma_n^2)$

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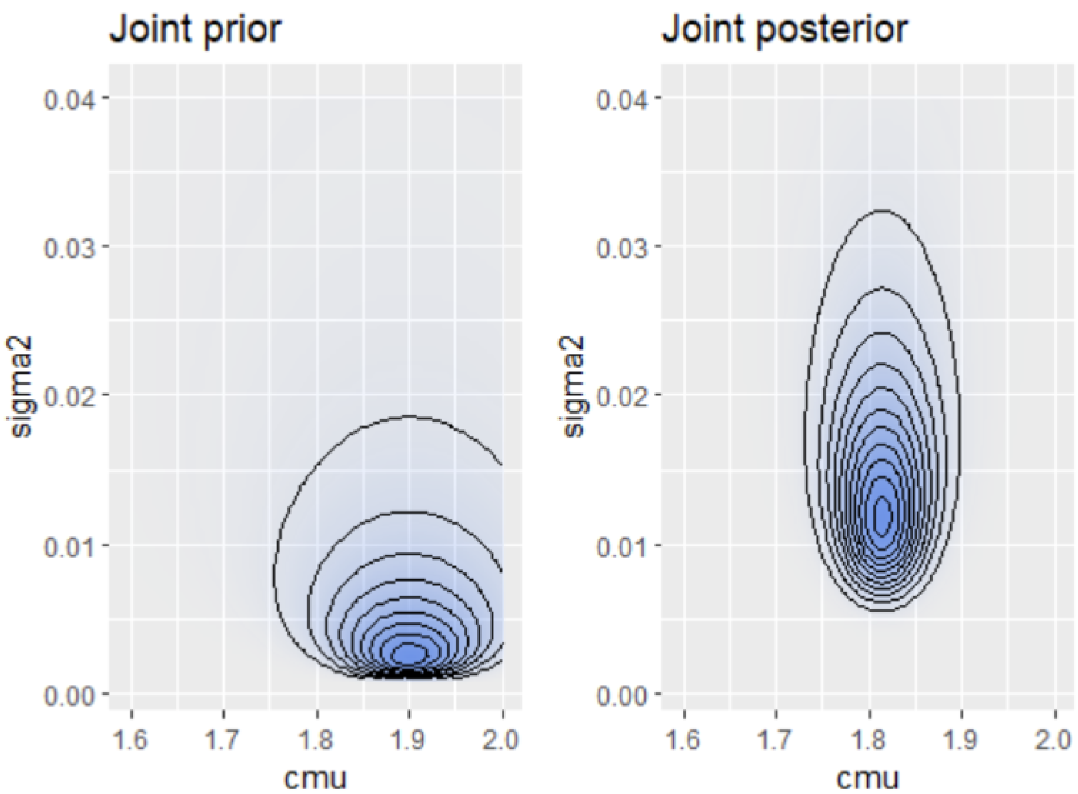
Joint prior
Joint posterior
sigma2
cmu
0.00
0.01
0.02
0.03
0.04
1.6
1.7
1.8
1.9
2.0