BDA Ch.2 Excercise 8.

known. ⟨ numeric values ⟩

$Y \mid \theta \sim N(\theta, \sigma^2)$ | $\sigma^2 = 400$

$\theta \sim N(\mu_0, \tau_0^2)$ | $\mu_0 = 180$, $\tau_0^2 = 1600$

$\rightarrow \theta \mid y \sim N(\mu_n, \tau_n^2)$ | $\bar{y} = 150$

(a) $\theta \mid y \sim N(\mu_n, \tau_n^2)$

$$\mu_n = \frac{\frac{1}{\tau_0^2}}{\frac{1}{\tau_0^2} + \frac{n}{\sigma^2}} \mu_0 + \frac{\frac{n}{\sigma^2}}{\frac{1}{\tau_0^2} + \frac{n}{\sigma^2}} \bar{y}. \qquad \tau_n^2 = \frac{1}{\frac{1}{\tau_0^2} + \frac{n}{\sigma^2}}$$

Now, plug numeric values.

$$\theta \mid y \sim N\left(\frac{\frac{1}{1600}}{\frac{1}{1600} + \frac{n}{400}} 180 + \frac{\frac{n}{400}}{\frac{1}{1600} + \frac{n}{400}} \cdot 150 , \frac{1}{\frac{1}{1600} + \frac{n}{400}} \right)$$

$$= N\left(\frac{180 + 600n}{1+4n} , \frac{1600}{1+4n} \right)$$

(b) $\tilde{Y} \mid \theta \sim N(\theta, \sigma^2)$

$$\tilde{Y} \mid y \sim N(\mu_n, \sigma^2 + \tau_n^2)$$
$$= N\left(\frac{180 + 600n}{1+4n} , 400 + \frac{1600}{1+4n} \right)$$

(c) $n=10 \rightarrow \theta \mid y \sim N\left(\frac{6180}{41}, \frac{1600}{41}\right) \rightarrow$ 95% CI : (138.4, 162.9)

$\tilde{Y} \mid y \sim N\left(\frac{6180}{41}, 400 + \frac{1600}{41}\right) \rightarrow$ 95% CI : (109.6, 191.2)

(d) $n=100 \rightarrow \theta \mid y \sim N\left(\frac{60180}{401}, \frac{1600}{401}\right) \rightarrow$ 95% CI : (146.1, 153.9)

$\tilde{Y} \mid y \sim N\left(\frac{60180}{401}, 400 + \frac{1600}{401}\right) \rightarrow$ 95% CI : (110.6, 189.4)

Normal data with a conjugate prior distribution.

Likelihood

$$\begin{cases} Y_i \mid \mu, \sigma^2 \sim N(\mu, \sigma^2) \\ \text{data} \mid \mu, \sigma^2 \; : \; p(\text{data} \mid \mu, \sigma^2) = \prod p(y_i \mid \mu, \sigma^2) = \left(\frac{1}{2\pi\sigma^2}\right)^{\frac{n}{2}} \cdot e^{-\frac{\Sigma (y-\mu)^2}{2\sigma^2}} \end{cases}$$

prior

$$\begin{cases} \mu, \sigma^2 \; : \; p(\mu, \sigma^2) = p(\mu \mid \sigma^2) \cdot p(\sigma^2) \\ \mu \mid \sigma^2 \sim N\left(\mu_0, \frac{\sigma^2}{k_0}\right) \longrightarrow \text{pdf.} \; \overbrace{\left(\frac{k_0}{2\pi\sigma^2}\right)^{\frac{1}{2}} \cdot e^{-\frac{k_0(\mu-\mu_0)^2}{2\sigma^2}}}^{p(\mu \mid \sigma^2)} \\ \sigma^2 \sim \chi^{-2}(v_0, \sigma_0^2) = \Gamma^{-1}\left(\frac{v_0}{2}, \frac{v_0 \sigma_0^2}{2}\right) \longrightarrow \text{pdf.} \; \overbrace{\frac{\left(\frac{v_0\sigma_0^2}{2}\right)^{\frac{v_0}{2}}}{\Gamma(\frac{v_0}{2})} \cdot \left(\frac{1}{\sigma^2}\right)^{\frac{v_0}{2}+1} \cdot e^{-\frac{v_0\sigma_0^2}{2\sigma^2}}}^{p(\sigma^2)} \end{cases}$$

posterior

$$p(\mu, \sigma^2 \mid \text{data}) \propto p(\text{data} \mid \mu, \sigma^2) \cdot p(\mu \mid \sigma^2) \cdot p(\sigma^2)$$

[결론] prior $\mu, \sigma^2 \sim N\text{-Inv-}\chi^2\left(\mu_0, \frac{\sigma_0^2}{k_0}; v_0, \sigma_0^2\right)$

posterior $\mu, \sigma^2 \mid y \sim N\text{-Inv-}\chi^2\left(\mu_n, \frac{\sigma_n^2}{k_n}; v_n, \sigma_n^2\right).$

$$\begin{cases} \mu_n = \frac{k_0}{k_0+n}\mu_0 + \frac{n}{k_0+n}\bar{y} \\ k_n = k_0 + n \\ v_n = v_0 + n \\ v_n\sigma_n^2 = v_0\sigma_0^2 + (n-1)s^2 + \frac{k_0 n}{k_0+n}(\bar{y}-\mu_0)^2 \end{cases}$$

Marginal $\sigma^2 \mid y$

$$p(\sigma^2 \mid y) \propto p(\sigma^2) \cdot p(y \mid \sigma^2) = p(\sigma^2) \cdot \int p(y, \mu \mid \sigma^2)\, d\mu = p(\sigma^2) \cdot \int p(y \mid \mu, \sigma^2)\, p(\mu \mid \sigma^2)\, d\mu$$

$$\propto \underbrace{\int \left(\frac{1}{\sigma^2}\right)^{\frac{n}{2}} e^{-\frac{\Sigma(y-\mu)^2}{2\sigma^2}} \cdot \left(\frac{1}{\sigma^2}\right)^{\frac{1}{2}} e^{-\frac{k_0(\mu-\mu_0)^2}{2\sigma^2}} d\mu}_{(*)} \cdot \left(\frac{1}{\sigma^2}\right)^{\frac{v_0}{2}+1} \cdot e^{-\frac{v_0\sigma_0^2}{2\sigma^2}}$$

$$\propto \underline{\left(\frac{1}{\sigma^2}\right)^{\frac{n+1}{2}} \cdot e^{-\frac{(n-1)s^2}{2\sigma^2}} \cdot e^{-\frac{\frac{nk_0(\bar{y}-\mu_0)^2}{n+k_0}}{2\sigma^2}}} \cdot \left(\frac{1}{\sigma^2}\right)^{-\frac{1}{2}} \cdot \left(\frac{1}{\sigma^2}\right)^{\frac{v_0}{2}+1} \cdot e^{-\frac{v_0\sigma_0^2}{2\sigma^2}}$$

$$= \left(\frac{1}{\sigma^2}\right)^{\frac{v_0+n}{2}+1} \cdot \exp\left[-\frac{1}{2\sigma^2}\left\{v_0\sigma_0^2 + (n-1)s^2 + \frac{nk_0(\bar{y}-\mu_0)^2}{n+k_0}\right\}\right]$$

Equal.

Compare : $\chi^{-2}(v_n, \sigma_n^2) \to \text{pdf} \propto \left(\frac{1}{\sigma^2}\right)^{\frac{v_n}{2}+1} \cdot e^{-\frac{v_n\sigma_n^2}{2\sigma^2}}$

$= \Gamma^{-1}\left(\frac{v_n}{2}, \frac{v_n\sigma_n^2}{2}\right)$

$\therefore \; \sigma^2 \mid y \sim \chi^{-2}(v_n, \sigma_n^2)$, $\quad v_n = v_0 + n$

$\qquad v_n\sigma_n^2 = v_0\sigma_0^2 + (n-1)s^2 + \frac{nk_0(\bar{y}-\mu_0)^2}{n+k_0}$

$$\circledast \int \left(\frac{1}{\sigma^2}\right)^{\frac{n}{2}} \cdot e^{-\frac{\Sigma(y-\mu)^2}{2\sigma^2}} \cdot \left(\frac{1}{\sigma^2}\right)^{\frac{1}{2}} \cdot e^{-\frac{k_0(\mu-\mu_0)^2}{2\sigma^2}} \, d\mu$$

$$= \int \left(\frac{1}{\sigma^2}\right)^{\frac{n+1}{2}} \cdot \exp\left[-\frac{1}{2\sigma^2}\left\{\Sigma(y-\mu)^2 + k_0(\mu-\mu_0)^2\right\}\right] d\mu$$

$$= \left(\frac{1}{\sigma^2}\right)^{\frac{n+1}{2}} \int \exp\left[-\frac{1}{2\sigma^2}\left\{\Sigma(y-\bar{y})^2 + \Sigma(\bar{y}-\mu)^2 + \overset{\text{zero.}}{\cancel{2\Sigma(y-\bar{y})(\bar{y}-\mu)}} + k_0(\mu-\mu_0)^2\right\}\right] d\mu$$

$$= \left(\frac{1}{\sigma^2}\right)^{\frac{n+1}{2}} e^{\frac{-(n-1)s^2}{2\sigma^2}} \int \exp\left[-\frac{1}{2\sigma^2}\left\{\boxed{\Sigma(\mu-\bar{y})^2 + k_0(\mu-\mu_0)^2}\right\}\right] d\mu$$

$$\boxed{} = \Sigma(\mu-\bar{y})^2 + k_0(\mu-\mu_0)^2 = n\mu^2 - \cancel{2nx}\, 2n\bar{y}\mu + n\bar{y}^2 + k_0\mu^2 - 2k\mu_0\mu + k\mu_0^2$$

$$= (n+k_0)\Big(\mu - \underbrace{\frac{n\bar{y}+k_0\mu_0}{n_0+k_0}}_{A}\Big)^2 + n\bar{y}^2 + k_0\mu_0^2 - \frac{(n\bar{y}+k_0\mu_0)^2}{n_0+k_0}$$

$$= (n+k_0)(\mu-A)^2 + \frac{nk_0(\bar{y}-\mu_0)^2}{n+k_0}$$

$$\circledast = \left(\frac{1}{\sigma^2}\right)^{\frac{n+1}{2}} e^{\frac{-(n-1)s^2}{2\sigma^2}} \int \exp\left[-\frac{1}{2\sigma^2}\left\{(n+k_0)(\mu-A)^2 + \frac{nk_0(\bar{y}-\mu_0)^2}{n+k_0}\right\}\right] d\mu$$

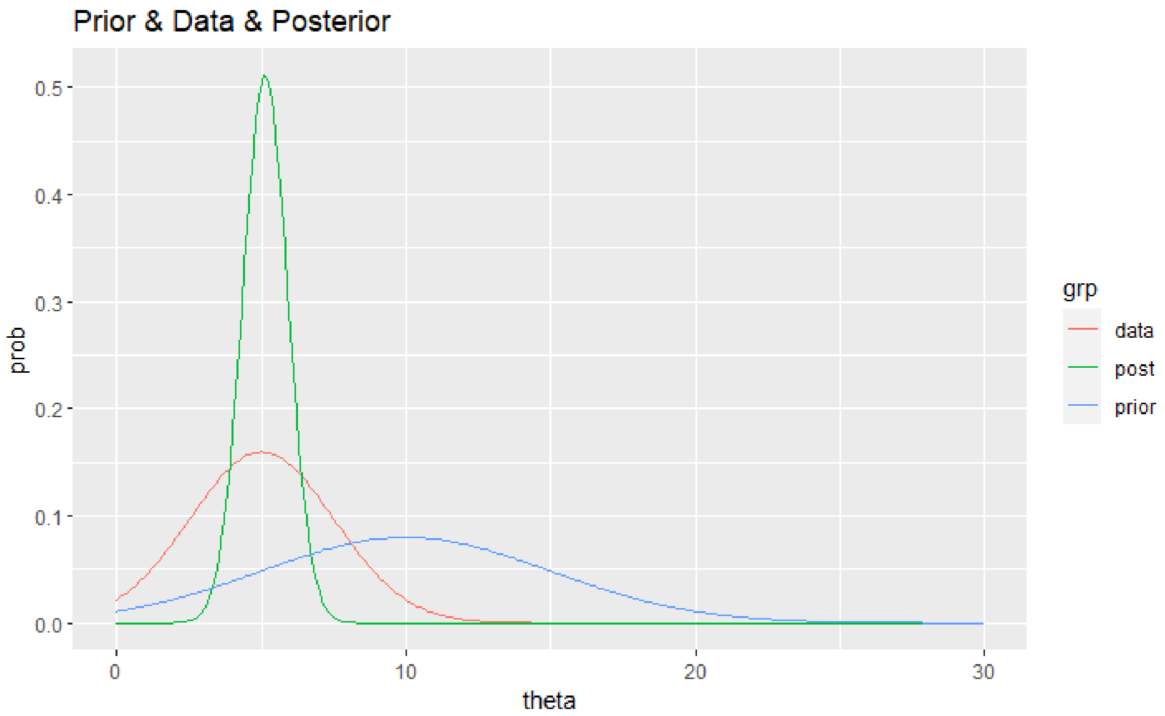
$$= \left(\frac{1}{\sigma^2}\right)^{\frac{n+1}{2}} \cdot e^{\frac{-(n-1)s^2}{2\sigma^2}} \cdot e^{-\frac{\frac{nk_0(\bar{y}-\mu_0)^2}{n+k_0}}{2\sigma^2}} \cdot \int e^{-\frac{(n+k_0)(\mu-A)^2}{2\sigma^2}} d\mu$$

$$= \left(\frac{1}{\sigma^2}\right)^{\frac{n+1}{2}} \cdot e^{\frac{-(n-1)s^2}{2\sigma^2}} \cdot e^{-\frac{\frac{nk_0(\bar{y}-\mu_0)^2}{n+k_0}}{2\sigma^2}} \cdot \frac{\sqrt{2\pi}\ \sigma}{\sqrt{n+k_0}}$$

$$\propto \left(\frac{1}{\sigma^2}\right) e^{-\frac{(n-1)s^2}{2\sigma^2}} \cdot e^{-\frac{\frac{nk_0(\bar{y}-\mu_0)^2}{n+k_0}}{2\sigma^2}} \cdot \left(\frac{1}{\sigma^2}\right)^{-\frac{1}{2}}$$

3. R코드 시각화 따라해보기

Normal model with known variance (결과만)



Normal model with known mean (결과만)