2021년 3월 25일 목요일 오전 9:10

1.
(a)

prior : $\mu \sim N(\mu_0, \tau_0^2)$

likelihood : $y|\mu \sim N(\mu, \sigma^2)$

posterior : $\mu|y \sim N(\mu_n, \tau_n^2)$

$$\mu_n = \frac{\frac{1}{\tau_0^2}}{\frac{n}{\sigma^2}+\frac{1}{\tau_0^2}}\mu + \frac{\frac{n}{\sigma^2}}{\frac{n}{\sigma^2}+\frac{1}{\tau_0^2}}\bar{y}$$

$$\tau_n^2 = \frac{1}{\frac{n}{\sigma^2}+\frac{1}{\tau_0^2}}$$

$\theta \sim N(180, 40^2)$

$y|\theta \sim N(150, 20^2)$

$$\mu_n = \frac{\frac{1}{1600}}{\frac{n}{400}+\frac{1}{1600}}180 + \frac{\frac{n}{400}}{\frac{n}{400}+\frac{1}{1600}}150 = \frac{180+600n}{1+4n}$$

$$\tau_n^2 = \frac{1}{\frac{n}{400}+\frac{1}{1600}} = \frac{1600}{1+4n}$$

$$\theta|y \sim N\left(\frac{180+600n}{1+4n}, \frac{1600}{1+4n}\right)$$

(b) $\tilde{y}|y \sim N(\mu_n, \sigma^2+\tau_n^2)$

$\hookrightarrow \tilde{y}|y \sim N\left(\frac{180+600n}{1+4n}, 20^2+\frac{1600}{1+4n}\right)$

(c) $n=10$

$\theta|y \sim N\left(\frac{6180}{41}, \frac{1600}{41}\right)$

$\rightarrow$ 95% C.I : (138.5, 163)

$\tilde{y}|y \sim N\left(\frac{6180}{41}, 400+\frac{1600}{41}\right)$

$\rightarrow$ 95% C.I : (109.7, 191.8)

(d) $n=100$

$\theta|y \sim N\left(\frac{60180}{401}, \frac{1600}{401}\right)$

$\rightarrow$ 95% C.I : (146.2, 154)

$\tilde{y}|y \sim N\left(\frac{60180}{401}, 400+\frac{1600}{401}\right)$

$\rightarrow$ 95% C.I : (110.7, 189.5)

2.

prior : $P(\mu, \sigma^2) = P(\mu|\sigma^2) \cdot P(\sigma^2)$

$\mu | \sigma^2 \sim N(\mu_0, \sigma^2/k_0)$

$\sigma^2 \sim Inv-\chi^2(\nu_0, \sigma_0^2) = Inv \sim \Gamma(\nu_0/2, \nu_0\sigma_0^2/2)$

Joint Prior :

$$P(\mu, \sigma^2) \propto \left(\frac{1}{\sigma^2}\right)^{1/2} \left(\frac{1}{\sigma^2}\right)^{\frac{\nu_0}{2}+1} \cdot \exp\left\{-\frac{1}{2\sigma^2}\left(k_0(\mu-\mu_0)^2 + \nu_0\sigma_0^2\right)\right\}$$

$$\sim Normal-Inv-\chi^2(\mu_0, \sigma_0^2/k_0; \nu_0, \sigma_0^2)$$

likelihood : $P(y|\mu,\sigma^2) \propto (\sigma^2)^{-n/2} \cdot \exp\left\{-\frac{1}{2\sigma^2}\Sigma(y_i-\mu)^2\right\}$

posterior : $p(\mu,\sigma^2|y) \propto p(y|\mu,\sigma^2) \cdot p(\mu,\sigma^2)$

$$\propto (\sigma^2)^{-\frac{n+\nu_0}{2}-1} \cdot \exp\left\{-\frac{1}{2\sigma^2}\left(\nu_0^2\sigma_0^2 + (n-1)S^2 + \frac{nk_0(\bar{y}-\mu_0)^2}{n+k_0}\right)\right\}$$

$$\sim Normal-Inv-\chi^2(\mu_n, \sigma_n^2/k_n; \nu_n, \sigma_n^2)$$

$$\mu_n = \frac{k_0}{k_0+n}\mu_0 + \frac{n}{k_0+n}\bar{y}$$

$$k_n = k_0 + n$$

$$\nu_n = \nu_0 + n$$

$$\nu_n\sigma_n^2 = \nu_0\sigma_0^2 + (n-1)S^2 + \frac{k_0 n}{k_0+n}(\bar{y}-\mu_0)^2$$

Marginal Posterior for σ^2 : $p(\sigma^2|y)$

$$p(\sigma^2|y) \propto p(\sigma^2) \cdot p(y|\sigma^2) = p(\sigma^2) \int p(y|\mu,\sigma^2) \cdot p(\mu,\sigma^2) dy$$

$$\propto (1/\sigma^2)^{\nu_0/2+1} \exp\left(-\nu_0\sigma_0^2/2\sigma^2\right) \cdot \int \left(\frac{1}{\sigma^2}\right)^{n/2} \cdot \exp\left\{-\frac{1}{2\sigma^2}\left((n-1)s^2 + n(\bar{y}-\mu)^2\right)\right\}$$

$$\cdot \left(\frac{k_0}{\sigma^2}\right)^{1/2} \cdot \exp\left\{-\frac{k_0}{2\sigma^2}(\mu-\mu_0)^2\right\} d\mu$$

$$\propto (1/\sigma^2)^{\nu_0/2+1} \cdot \exp\left(-\frac{\nu_0\sigma_0^2}{2\sigma^2}\right) \cdot \left(\frac{1}{\sigma^2}\right)^{n/2} \cdot \exp\left\{-\frac{1}{2\sigma^2}(n-1)s^2\right\} \cdot \left(\frac{1}{\sigma^2}\right)^{1/2}$$

$$* \underline{\cdot \int \exp\left\{-\frac{1}{2\sigma^2} n(\bar{y}-\mu)^2 - \frac{k_0}{2\sigma^2}(\mu-\mu_0)^2\right\} d\mu}$$

$$* \int \exp\left\{-\frac{1}{2\sigma^2} n(\bar{y}-\mu)^2 - \frac{k_0}{2\sigma^2}(\mu-\mu_0)^2\right\} d\mu$$

$$= \int \exp\left\{-\frac{1}{2\sigma^2}\left(n(\bar{y}-\mu)^2 + k_0(\mu-\mu_0)^2\right)\right\} d\mu$$

$$= \int \exp\left\{-\frac{1}{2\sigma^2}\left(n\mu^2 - 2n\bar{y}\mu + n\bar{y}^2 + k_0\mu^2 + 2k_0\mu\mu_0 + k_0\mu_0^2\right)\right\} d\mu$$

$$= \int \exp\left\{-\frac{1}{2\sigma^2}\left((n+k_0)\left(\mu - \frac{n\bar{y}+k_0\mu_0}{n+k_0}\right)^2 + n\bar{y}^2 + k_0\mu_0^2 - \frac{(n\bar{y}+k_0\mu_0)^2}{n+k_0}\right\} d\mu$$

$$= \exp\left\{-\frac{1}{2\sigma^2} \cdot \left(\frac{nk_0(\bar{y}-\mu_0)^2}{n+k_0}\right)\right\} \int \exp\left\{-\frac{1}{2\sigma^2}\left((n+k_0)\left(\mu - \frac{n\bar{y}+k_0\mu_0}{n+k_0}\right)^2\right\} d\mu$$

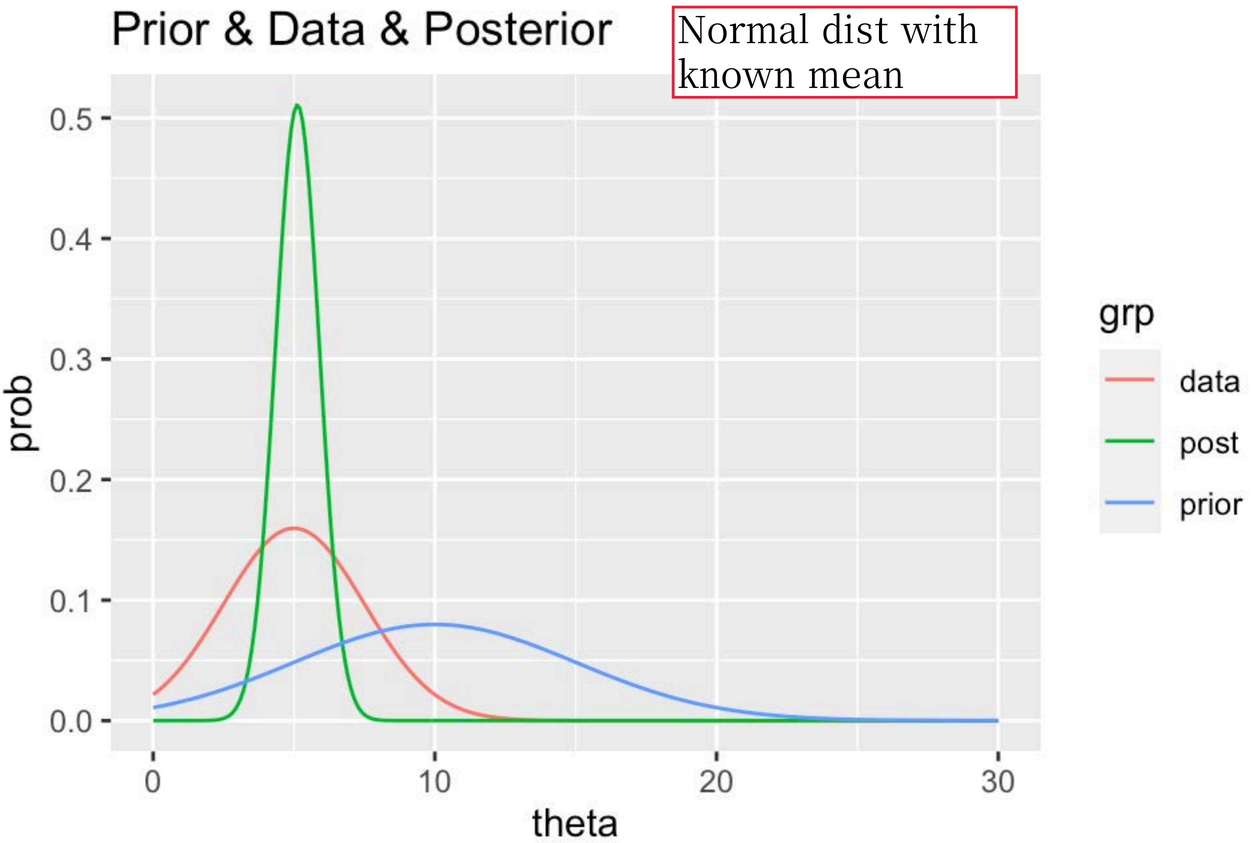
$$= \exp\left\{-\frac{1}{2\sigma^2}\left(\frac{nk_0(\bar{y}-\mu_0)^2}{n+k_0}\right) \frac{\sqrt{2\pi}\,\sigma}{\sqrt{n+k_0}}\right\}$$

$$\therefore \; = \left(1/\sigma^2\right)^{v_0+n/2+\frac{1}{2}} \cdot \frac{\sqrt{2\pi}\,\sigma}{\sqrt{n+k_0}} \exp\left\{-\frac{1}{2\sigma^2}\left(v_0\sigma_0^2+(n-1)s^2+\frac{nk_0}{n+k_0}(\bar{y}-\mu)^2\right)\right\}$$

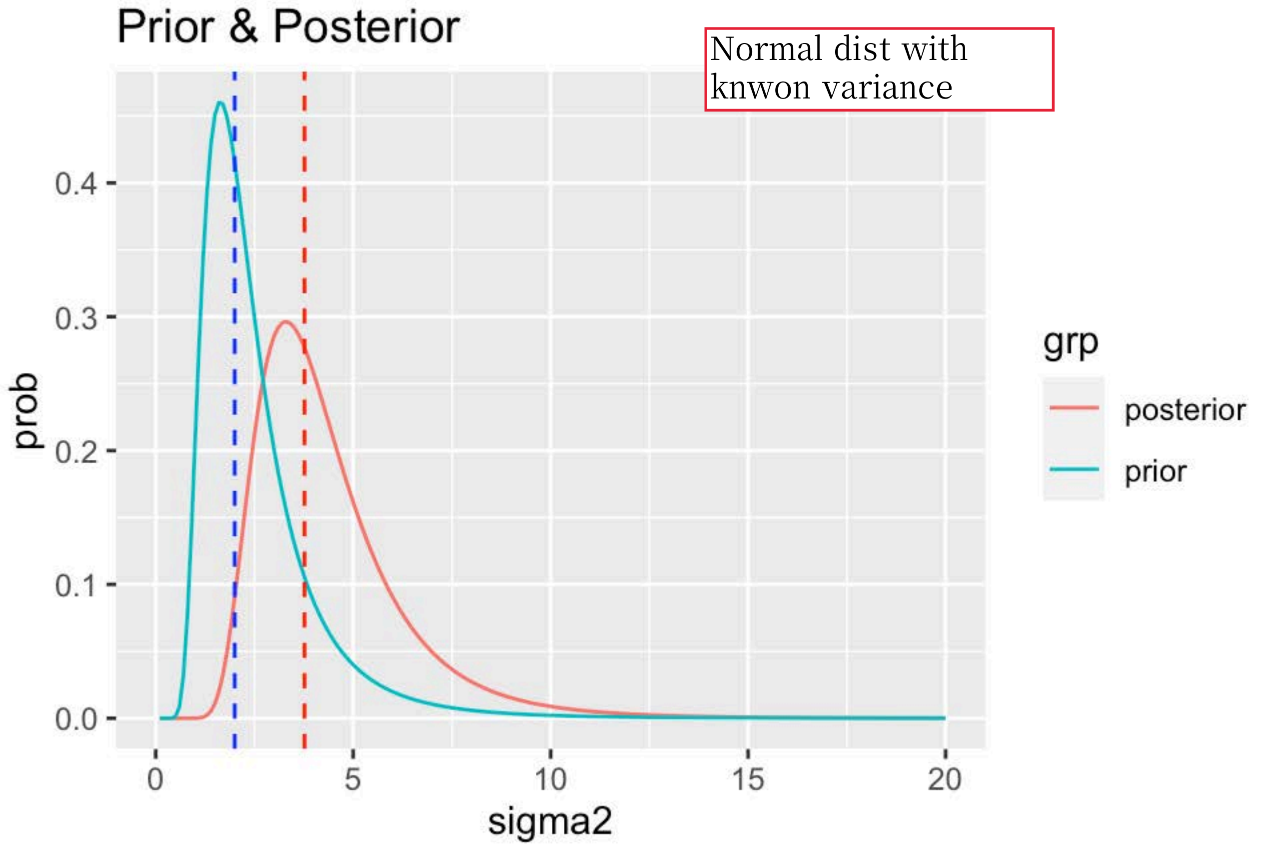
$$\propto \left(1/\sigma^2\right)^{v_n+1/2} \cdot \exp\left\{-\frac{1}{2\sigma^2}\left(v_n\sigma_n^2\right)\right\} \sim \text{Inv-}\chi^2(v_n, \sigma_n^2)$$

where, $v_n = v_0 + n$

$v_n\sigma_n^2 = v_0\sigma_0^2 + (n-1)s^2 + \frac{nk_0}{n+k_0}(\bar{y}-\mu)^2$



Prior & Data & Posterior
Normal dist with known mean
prob
0.5
0.4
0.3
0.2
0.1
0.0
0
10
20
30
theta
grp
data
post
prior



Prior & Posterior
Normal dist with knwon variance
prob
0.0
0.1
0.2
0.3
0.4
sigma2
0
5
10
15
20
grp
posterior
prior