1. Normal distribution with unknown mean: a random sample of n students is drawn from a large population, and their weights are measured. The average weight of the n sampled students is $\overline{y} = 150$ pounds. Assume the weights in the population are normally distributed with unknown mean θ and known standard deviation 20 pounds. Suppose your prior distribution for θ is normal with mean 180 and standard deviation 40.

(a) prior $\theta \sim N(180, 40^2)$

likelihood $y|\theta \sim N(\theta, 20^2)$

$\sigma^2 = 20^2$

$\mu_0 = 180.$ $\tau_0^2 = 40^2$

$\bar{y} = 150.$

posterior $\theta|y \sim N(\mu_n, \tau_n^2)$

$$\mu_n = \frac{\frac{1}{\tau_0^2}}{\frac{1}{\tau_0^2}+\frac{n}{\sigma^2}}\mu_0 + \frac{\frac{n}{\sigma^2}}{\frac{1}{\tau_0^2}+\frac{n}{\sigma^2}}\bar{y}$$

$$= \frac{\frac{1}{1600}}{\frac{1}{1600}+\frac{n}{400}}\cdot 180 + \frac{\frac{n}{400}}{\frac{1}{1600}+\frac{n}{400}}\cdot 150$$

$$= \frac{1}{4n+1}\cdot 180 + \frac{4n}{4n+1}\cdot 150 = \frac{600n+180}{4n+1}$$

$$\tau_n^2 = \frac{1}{\frac{1}{\tau_0^2}+\frac{1}{\sigma^2}}$$

$$= \frac{1600}{4n+1} \quad \Rightarrow \theta|y \sim N\left(\frac{600n+180}{4n+1}, \frac{1600}{4n+1}\right)$$

(b) $E(\tilde{y}|y) = \mu_n = \frac{600n+180}{4n+1}.$

$V(\tilde{y}|y) = \sigma^2 + \tau_n^2 = 400 + \frac{1600}{4n+1}$

(c) let $n=10.$ 95% PI

① $\theta|y \sim N((600n+180)/(4n+1), 1600/(4n+1))$

② $\tilde{y}|y \sim N((600n+180)/(4n+1), (400+1600/(4n+1)))$

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n=10

post=st.norm((6000+180)/(40+1), (1600/(40+1))**0.5)

postpre=st.norm((6000+180)/(40+1), (400+1600/(40+1))**0.5)


lbA=post.ppf(0.025)
ubA=post.ppf(0.975)
print(lbA, ubA)
lbB=postpre.ppf(0.025)
ubB=postpre.ppf(0.975)
print(lbB, ubB)
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138.48790937180107 162.97550526234522
109.66476055447222 191.7986540796741
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(d)

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n=100

post=st.norm((600*n+180)/(4*n+1), (1600/(4*n+1))**0.5)

postpre=st.norm((600*n+180)/(4*n+1), (400+1600/(4*n+1))**0.5)


lbA=post.ppf(0.025)
ubA=post.ppf(0.975)
print(lbA, ubA)
lbB=postpre.ppf(0.025)
ubB=postpre.ppf(0.975)
print(lbB, ubB)
```

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146.1597757402296 153.98985019493247
110.68051078098051 189.46911515418157
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2. marginal posterior distribution of $\sigma^2|y$

$$p(\sigma^2|y) \propto p(\sigma^2)p(y|\sigma^2) = p(\sigma^2)\int p(y,\mu|\sigma^2)\,d\mu$$

$$= \underbrace{p(\sigma^2)}_{\chi^2(\nu_0,\sigma_0^2)} \int \underbrace{p(y|\mu,\sigma^2)}_{\text{likelihood}}\, \underbrace{p(\mu|\sigma^2)}_{N(\mu_0,\frac{\sigma^2}{k_0})}\,d\mu$$

$$\propto \left(\frac{1}{\sigma^2}\right)^{\frac{\nu_0}{2}+1}\cdot \exp\left(-\frac{\nu_0\sigma_0^2}{2\sigma^2}\right)\cdot \int \left(\frac{1}{\sigma^2}\right)^{\frac{n}{2}}\cdot \exp\left[-\frac{1}{2\sigma^2}\left\{(n-1)s^2+n(\bar{y}-\mu)^2\right\}\right]\cdot\left(\frac{k_0}{\sigma^2}\right)^{\frac{1}{2}}\cdot\exp\left[-\frac{k_0}{2\sigma^2}(\mu-\mu_0)^2\right]d\mu$$

$$\propto \underbrace{\text{''} \cdot \text{''}}_{③} \cdot \underbrace{\left(\frac{1}{\sigma^2}\right)^{\frac{n}{2}}\cdot \exp\left(-\frac{1}{2\sigma^2}(n-1)s^2\right)\cdot\left(\frac{k_0}{\sigma^2}\right)^{\frac{1}{2}}}_{①}\int \exp\left[-\frac{1}{2\sigma^2}n(\bar{y}-\mu)^2-\frac{k_0}{2\sigma^2}(\mu-\mu_0)^2\right]d\mu$$

$$= \int \exp\left[-\frac{1}{2\sigma^2}\left(n\mu^2-2n\bar{y}\mu+n\bar{y}^2+k_0\mu^2-2k_0\mu_0\mu+k_0\mu_0^2\right)\right]d\mu$$

$$= \int \exp\left[-\frac{1}{2\sigma^2}\left\{(n+k_0)\mu^2-2(n\bar{y}+k_0\mu_0)\mu+k_0\mu_0^2+n\bar{y}^2\right\}\right]d\mu$$

$$= \int \exp\left[-\frac{n+k_0}{2\sigma^2}\left\{\mu^2-2\frac{n\bar{y}+k_0\mu_0}{n+k_0}\mu+\left(\frac{n\bar{y}+k_0\mu_0}{n+k_0}\right)^2\right\}-\frac{1}{2\sigma^2}\left\{k_0\mu_0^2+n\bar{y}^2-\frac{(n\bar{y}+k_0\mu_0)^2}{n+k_0}\right\}\right]d\mu$$

$$= \underbrace{\exp\left(-\frac{1}{2\sigma^2}\left(\frac{nk_0(\bar{y}-\mu_0)^2}{n+k_0}\right)\right)}_{②}\cdot \int \exp\left[-\frac{n+k_0}{2\sigma^2}\left(\mu-\frac{n\bar{y}+k_0\mu_0}{n+k_0}\right)^2\right]d\mu$$

$\sim$ pdf of N : $N\left(\frac{n\bar{y}+k_0\mu_0}{n+k_0}, \frac{\sigma^2}{n+k_0}\right)$

$\propto \sigma$

①.②.③

$$\propto \left(\frac{1}{\sigma^2}\right)^{\frac{\nu_0}{2}+\frac{n}{2}+1} \exp\left[-\frac{1}{2\sigma^2}\underbrace{\left\{\nu_0\sigma_0^2+(n-1)s^2+\frac{nk_0}{n+k_0}(\bar{y}-\mu_0)^2\right\}}_{=\nu_n\sigma_n^2}\right]$$

$$\propto \left(\frac{1}{\sigma^2}\right)^{\frac{\nu_0+n}{2}+1} \exp\left[-\frac{\nu_n}{2\sigma^2}\cdot \underbrace{\frac{1}{\nu_n}\left(\nu_0\sigma_0^2+(n-1)s^2+\frac{nk_0}{n+k_0}(\bar{y}-\mu_0)^2\right\}}_{\sigma_n^2}\right]$$

$$\sigma^2|y \sim \chi^{-2}(\nu_n, \sigma_n^2)$$

$$\nu_n = \nu_0 + n$$

$$\nu_n\sigma_n^2 = \nu_0\sigma_0^2 + (n-1)s^2 + \frac{nk_0}{n+k_0}(\bar{y}-\mu_0)^2$$