1. BDA chapter 2 Exercise 8

(a) prior: $\theta \sim N(\mu_0, \tau_0^2)$

likelihood: $y|\theta \sim N(\theta, \sigma^2)$ σ^2: known

posterior: $\theta|y \sim N(\mu_n, \tau_n^2)$

where $$\mu_n = \underbrace{\frac{\frac{1}{\tau_0^2}}{\frac{1}{\tau_0^2}+\frac{n}{\sigma^2}}\mu_0}_{\text{prior mean}} + \underbrace{\frac{\frac{n}{\sigma^2}}{\frac{1}{\tau_0^2}+\frac{n}{\sigma^2}}\bar{y}}_{\text{data mean}}$$

$$\tau_n^2 = \frac{1}{\frac{1}{\tau_0^2}+\frac{n}{\sigma^2}}$$

when we plug in the numbers,

$$\mu_n = \frac{\frac{1}{1600}}{\frac{1}{1600}+\frac{n}{400}}180 + \frac{\frac{n}{400}}{\frac{1}{1600}+\frac{n}{400}}\cdot 150 = \frac{180+600n}{1+4n}$$

$$\tau_n = \frac{1}{\frac{1}{1600}+\frac{n}{400}} = \frac{1600}{1+4n}$$

$$\Rightarrow \theta|y \sim N\left(\frac{180+600n}{1+4n}, \frac{1600}{1+4n}\right)$$

(b) $\tilde{y}|y \sim N(\mu_n, \sigma^2+\tau_n^2)$

$$= N\left(\frac{180+600n}{1+4n}, 400+\frac{1600}{1+4n}\right)$$

(c) $n=10$

$\theta|y \sim N\left(\frac{6180}{41}, \frac{1600}{41}\right) \longrightarrow$ 95% PI: (138.49, 162.97)

$\tilde{y}|y \sim N\left(\frac{6180}{41}, 40+\frac{1600}{41} = \frac{3240}{41}\right) \longrightarrow$ 95% PI: (143.31, 168.15)

(d) $n=100$

$\theta|y \sim N\left(\frac{60180}{401}, \frac{1600}{401}\right) \longrightarrow$ 95% PI: (146.15, 153.99)

$\tilde{y}|y \sim N\left(\frac{60180}{401}, 400+\frac{1600}{401}\right) \longrightarrow$ 95% PI: (110.67, 189.47)

2. Normal data w/ conjugate prior → σ^2의 marginal posterior 유도

Prior: $p(\mu, \sigma^2) = p(\mu|\sigma^2) \cdot p(\sigma^2)$

$$\begin{cases} \mu|\sigma^2 \sim N\left(\mu_0, \frac{\sigma^2}{\kappa_0}\right) \\ \sigma^2 \sim \text{Inv-}\chi^2(\nu_0, \sigma_0^2) = \text{Inv-gamma}\left(\frac{\nu_0}{2}, \frac{\nu_0\sigma_0^2}{2}\right) \end{cases}$$

Joint prior $p(\mu, \sigma^2) \propto \left(\frac{1}{\sigma^2}\right)^{\frac{1}{2}} \left(\frac{1}{\sigma^2}\right)^{\frac{\nu_0}{2}+1} \cdot \exp\left(-\frac{1}{2\sigma^2}\left[\kappa_0(\mu-\mu_0)^2 + \nu_0\sigma_0^2\right]\right)$

$$\Rightarrow \text{Normal-Inv-}\chi^2\left(\mu_0, \frac{\sigma_0^2}{\kappa_0}; \nu_0, \sigma_0^2\right)$$

Likelihood: $p(y|\mu, \sigma^2) \propto (\sigma^2)^{-\frac{n}{2}} \exp\left(-\frac{1}{2\sigma^2}\sum(y_i-\mu)^2\right)$

Posterior: $p(\mu, \sigma^2|y) \propto p(y|\mu, \sigma^2) \cdot p(\mu, \sigma^2)$

$$\propto (\sigma^2)^{-\frac{n+\nu_0}{2}-1} \exp\left(-\frac{1}{2\sigma^2}\left(\nu_0^2\sigma_0^2 + (n-1)s^2 + \frac{n\kappa_0(\bar{y}-\mu_0)^2}{n+\kappa_0}\right)\right)$$

$$\Rightarrow \text{Normal-Inv-}\chi^2\left(\mu_n, \frac{\sigma_n^2}{\kappa_n}; \nu_n, \sigma_n^2\right)$$

$\mu_n = \frac{\kappa_0}{\kappa_0+n}\mu_0 + \frac{n}{\kappa_0+n}\bar{y}$ (κ_0: prior sample size for μ_0)

$\kappa_n = \kappa_0 + n$

$\nu_n = \nu_0 + n$ — prior sample size for σ_0^2

$$\underbrace{\nu_n\sigma_n^2}_{\text{posterior sum of sqrs}} = \underbrace{\nu_0\sigma_0^2}_{\text{prior sum of sqrs.}} + \underbrace{(n-1)s^2}_{\text{sample sum of sqrs}} + \underbrace{\frac{\kappa_0 n}{\kappa_0+n}(\bar{y}-\mu_0)^2}_{\text{additional uncertainty from difference of } \bar{y} \text{ and } \mu_0}$$

$\bar{y}$: sample mean, μ_0: prior mean.

Marginal posterior for σ^2: $p(\sigma^2|y)$

$$p(\sigma^2|y) \propto p(\sigma^2)\, p(y|\sigma^2) = p(\sigma^2)\int p(y|\mu, \sigma^2)\, p(\mu, \sigma^2)\, d\mu$$

$$\propto \left(\frac{1}{\sigma^2}\right)^{\frac{\nu_0}{2}+1} \exp\left(-\frac{\nu_0\sigma_0^2}{2\sigma^2}\right) \cdot \int\left(\frac{1}{\sigma^2}\right)^{\frac{n}{2}} \cdot \exp\left(-\frac{1}{2\sigma^2}\left((n-1)s^2 + n(\bar{y}-\mu)^2\right)\right) \cdot \left(\frac{\kappa_0}{\sigma^2}\right)^{\frac{1}{2}} \cdot \exp\left(-\frac{\kappa_0}{2\sigma^2}(\mu-\mu_0)^2\right) d\mu$$

$$\propto \left(\frac{1}{\sigma^2}\right)^{\frac{\nu_0}{2}+1} \exp\left(-\frac{\nu_0\sigma_0^2}{2\sigma^2}\right) \cdot \left(\frac{1}{\sigma^2}\right)^{\frac{n}{2}} \cdot \exp\left(-\frac{1}{2\sigma^2}(n-1)s^2\right) \cdot \left(\frac{1}{\sigma^2}\right)^{\frac{1}{2}}$$

① $$\cdot \boxed{\int \exp\left[-\frac{1}{2\sigma^2}n(\bar{y}-\mu)^2 - \frac{\kappa_0}{2\sigma^2}(\mu-\mu_0)^2\right] d\mu}$$

$$
① \int \exp\left[-\frac{1}{2\sigma^2} n(\bar{y}-\mu)^2 - \frac{K_0}{2\sigma^2}(\mu-\mu_0)^2\right] d\mu
$$

$$
= \int \exp\left(-\frac{1}{2\sigma^2}\left\{ n(\bar{y}-\mu)^2 + K_0(\mu-\mu_0)^2 \right\}\right) d\mu
$$

$$
= \int \exp\left(-\frac{1}{2\sigma^2}\left(n\mu^2 - 2n\bar{y}\mu + n\bar{y}^2 + K_0\mu^2 + 2K_0\mu\mu_0 + K_0\mu_0^2 \right)\right) d\mu
$$

$$
= \int \exp\left(-\frac{1}{2\sigma^2}\left\{ (n+K_0)\left(\mu - \frac{n\bar{y}+K_0\mu_0}{n+k_0}\right)^2 + \underline{n\bar{y}^2 + K_0\mu_0^2 - \frac{(n\bar{y}+K_0\mu_0)^2}{n+K_0}} \right\} d\mu
$$

$$
= \exp\left(-\frac{1}{2\sigma^2}\cdot \underline{\left(\frac{nK_0(\bar{y}-\mu_0)^2}{n+K_0}\right)}\right) \int \exp\left(-\frac{1}{2\sigma^2}\left\{(n+K_0)\left(\mu - \frac{n\bar{y}+K_0\mu_0}{n+K_0}\right)^2\right\}\right) d\mu
$$

$$
= \exp\left(-\frac{1}{2\sigma^2}\left(\frac{nK_0(\bar{y}-\mu_0)^2}{n+K_0}\right)\right) \frac{\sqrt{2\pi}\,\sigma}{\sqrt{n+K_0}}
$$

$$
= \left(\frac{1}{\sigma^2}\right)^{\frac{\overset{\nu_n}{\nu_0+n}}{2}+\frac{\cancel{2}^{\,1}}{2}} \cdot \frac{\sqrt{2\pi}\,\cancel{\sigma}}{\sqrt{n+K_0}} \exp\left(-\frac{1}{2\sigma^2}\left(\underbrace{\nu_0\sigma_0^2 + (n-1)s^2 + \frac{nk_0}{n+K_0}(\bar{y}-\mu)^2}_{\nu_n\sigma_n^2}\right)\right)
$$

$$
\propto \left(\frac{1}{\sigma^2}\right)^{\frac{\nu_n}{2}+\frac{1}{2}} \exp\left(-\frac{1}{2\sigma^2}\left(\nu_n\sigma_n^2\right)\right) \sim \text{Inv-}\chi^2(\nu_n, \sigma_n^2)
$$

where $\nu_n = \nu_0 + n$

$$
\nu_n\sigma_n^2 = \nu_0\sigma_0^2 + (n-1)s^2 + \frac{nK_0}{n+K_0}(\bar{y}-\mu)^2
$$

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R-code example

```
library(ggplot2)
library(tidyverse)
library(dplyr)
library(invgamma)
library(LaplacesDemon)
library(ggpubr)
# data
D = c(1.64, 1.70, 1.72, 1.74, 1.82, 1.82, 1.82, 1.90, 2.08)
n = length(D); xbar = mean(D); s2 = var(D)

# prior
mu0 = 1.9; kappa0 = 1; s20 = 0.01; nu0 = 1

# posterior
kappa1 = kappa0 + n
nu1 = nu0 + n
mu1 = (kappa0 * mu0 + n * xbar) / kappa1
s21 = (1/ nu1) * (nu0*s20 + (n-1)*s2 + (kappa0*n/kappa1)*(xbar-mu0)^2 )
# visualize
prior = function(theta, sigma2)
  dnorm(theta, mu0, sqrt(sigma2/kappa0)) * dinvchisq(sigma2, nu0, s20)
posterior = function(theta, sigma2)
  dnorm(theta, mu1, sqrt(sigma2/kappa1)) * dinvchisq(sigma2, nu1, s21)
mu = seq(1.6, 2, length.out = 100)
sigma2 = seq(0.001, 0.04, length.out = 100)
cmu = rep(mu, each = length(sigma2))
csigma2 = rep(sigma2, length(mu))
prr_dens = mapply(prior, cmu, csigma2)
post_dens = mapply(posterior, cmu, csigma2)

title1 = "Joint prior"
p1 = data.frame(mu = cmu, sigma2 = csigma2, dens = prr_dens) %>%
  ggplot(aes(x=cmu, y=sigma2))+
  geom_raster(aes(fill = dens, alpha= dens), interpolate= T)+
  geom_contour(aes(z= dens), color = 'black', size= 0.2)+
  scale_fill_gradient(low= "cornflowerblue", high= "cornflowerblue",
guide= F)+
  scale_alpha(range= c(0,1), guide=F)+
  labs(title=title1)
title2 = "Joint posterior"
```

```
p2 = data.frame(mu = cmu, sigma2 = csigma2, dens = post_dens) %>%
  ggplot(aes(x=cmu, y=sigma2))+
  geom_raster(aes(fill = dens, alpha= dens), interpolate= T)+
  geom_contour(aes(z= dens), color = 'black', size= 0.2)+
  scale_fill_gradient(low= "cornflowerblue", high= "cornflowerblue",
guide= F)+
  scale_alpha(range= c(0,1), guide=F)+
  labs(title=title2)

ggarrange(p1, p2)
```