Week 3

1.

(a) $p(y|\theta) \sim N(\theta, 20^2)$

$p(\theta) \sim N(180, 40^2)$

$p(\theta|y) \propto p(\theta) \cdot p(y|\theta)$

$$\propto \exp\left\{-\frac{1}{800}\sum^{n}(y_i-\theta)^2\right\} \times \exp\left\{-\frac{1}{3200}(\theta-180)^2\right\}$$

$$= \exp\left\{-\frac{1}{3200}\left\{(1+4n)\theta^2 - 2\left(4\sum^{n} y_i + 180\right)\theta + C\right\}\right\}$$

$$\propto \exp\left\{-\frac{1}{2} \times \left(\frac{4n+1}{1600}\right)\left(\theta - \frac{4\sum_{i=1}^{n} y_i + 180}{4n+1}\right)^2\right\}$$

$$\Rightarrow \sigma^2 = \frac{1600}{4n+1} = \frac{1}{\frac{n}{400} + \frac{1}{1600}}, \quad \mu = \frac{4\sum^{n} y_i + 180}{4n+1} = \frac{180}{4n+1} + \frac{4n}{4n+1}\bar{y}$$

$$\Rightarrow p(\theta|y) \sim N\left(\frac{600n}{4n+1} + \frac{180}{4n+1}, \frac{1600}{4n+1}\right)$$

(b) $p(\tilde{y}|y) \Rightarrow E[\tilde{y}|y] = E[E[\tilde{y}|\theta]|y] = E[\theta|y] = \dfrac{600n+100}{4n+1}$

$$\sim N\left(\frac{600n+100}{4n+1}, \frac{1600}{4n+1}+400\right)$$

$$Var(\tilde{y}|y) = E[Var(\tilde{y}|\theta)|y] + Var[E[\tilde{y}|\theta]|y]$$

$$= E[20^2|y] + Var[\theta|y] = 400 + \frac{1600}{4n+1}$$

(c) $n=10 \rightarrow p(\theta|y) \sim N\left(\frac{6100}{41}, \frac{1600}{41}\right)$ + 95% CI $(138.4079, 162.9755)$

(d) $n=100 \rightarrow$ 95% CI $(1455.561, 1480.049)$

2. ✓likelihood)

$$p(y|\mu,\sigma^2)=\prod \times \frac{1}{\sqrt{2\pi\sigma^2}}\exp\left(-\frac{1}{2\sigma^2}(y_i-\mu)^2\right)$$

$$\propto \sigma^{-n}\cdot\exp\left(-\frac{1}{2\sigma^2}\Sigma(y_i-\mu)^2\right)$$

✓prior)

$$p(\mu,\sigma^2)=p(\mu|\sigma^2)\times p(\sigma^2)$$

$$\mu|\sigma^2\sim N\left(\mu_0,\frac{\sigma_0^2}{k_0}\right),\ p(\mu|\sigma^2)\propto\sigma^{-1}\exp\left(\frac{k_0}{-2\sigma^2}(\mu-\mu_0)^2\right)$$

$$\sigma^2\sim\chi^{-2}(\nu_0,\sigma_0^2),\ p(\sigma^2)\propto\left(\frac{1}{\sigma^2}\right)^{\frac{\nu_0}{2}+1}\exp\left(-\frac{1}{2\sigma^2}\nu_0\sigma_0^2\right)$$

$$p(\mu,\sigma^2)\propto\sigma^{-1}\exp\left(-\frac{k_0}{2\sigma^2}(\mu-\mu_0)^2\right)\times(\sigma^{-2})^{\frac{\nu_0}{2}+1}\cdot\exp\left(-\frac{1}{2\sigma^2}\nu_0\sigma_0^2\right)$$

✓ marginal post dist)

$$p(\sigma^2|y)\propto p(\sigma^2)\,p(y|\sigma^2)=p(\sigma^2)\int p(y|\mu,\sigma^2)\times p(\mu|\sigma^2)\,d\mu$$

$$\propto\left(\frac{1}{\sigma^2}\right)^{\frac{\nu_0}{2}+1}\exp\left\{-\frac{2\nu_0\sigma_0^2}{2\sigma^2}\right\}$$

$$\times\int\left(\frac{1}{\sigma^2}\right)^{\frac{n}{2}}\exp\left\{-\frac{1}{2\sigma^2}\left\{(n-1)s^2+n(\bar{y}-\mu)^2\right\}\right\}$$

$$\left(\frac{k_0}{\sigma^2}\right)^{\frac{1}{2}}\times\exp\left[-\frac{k_0}{2\sigma^2}\times(\mu-\mu_0)^2\right]d\mu\Big\}$$

$$\propto \left(\frac{1}{\sigma^2}\right)^{\frac{\nu_0}{2}+1} \exp\left(-\frac{\nu_0\sigma_0^2}{2\sigma^2}\right) \times \left(\frac{1}{\sigma^2}\right)^{\frac{n}{2}} \exp\left(-\frac{1}{2\sigma^2}(n-1)S^2\right)\left(\frac{1}{\sigma^2}\right)^{\frac{1}{2}} \cdots A$$

$$\times \int \exp\left[-\frac{1}{2\sigma^2} n(\bar{y}-\mu)^2 - \frac{k_0}{2\sigma^2}(\mu-\mu_0)^2\right] d\mu \qquad \cdots B$$

B. $\int \exp\left[-\frac{1}{2\sigma^2}\left(n(\bar{y}-\mu)^2 + k_0(\mu-\mu_0)^2\right)\right] d\mu$

$$= \int \exp\left[-\frac{1}{2\sigma^2}\left\{n\mu^2 - 2n\bar{y}\mu + n\bar{y}^2 + k_0\mu^2 - 2k_0\mu_0\mu + k_0\mu_0^2\right\}\right] d\mu$$

$$= \int \exp\left[-\frac{1}{2\sigma^2}\left\{(n+k_0)\mu^2 - 2(n\bar{y}+k_0\mu_0)\mu + k_0\mu_0^2 + n\bar{y}^2\right\}\right] d\mu$$

$$= \int \exp\left[-\frac{n+k_0}{2\sigma^2}\left\{\mu^2 - 2\frac{n\bar{y}+k_0\mu_0}{n+k_0}\mu + \left(\frac{n\bar{y}+k_0\mu_0}{n+k_0}\right)^2\right\}\right.$$

$$\left. -\frac{1}{2\sigma^2}\left\{k_0\mu_0^2 + n\bar{y}^2 - \frac{(n\bar{y}+k_0\mu_0)^2}{n+k_0}\right\}\right] d\mu$$

$$= \exp\left(-\frac{1}{2\sigma^2}\left(\frac{nk_0(\bar{y}-\mu_0)^2}{n+k_0}\right)\right) \cdot \int \exp\left[-\frac{n+k_0}{2\sigma^2}\left(\mu - \frac{n\bar{y}+k_0\mu_0}{n+k_0}\right)^2\right] d\mu$$

$$= A \quad \left(\frac{1}{\sigma^2}\right)^{\frac{\nu_0}{2}+1}\left(\frac{1}{\sigma^2}\right)^{\frac{n+1}{2}} \exp\left(-\frac{1}{2\sigma^2}\left(\nu_0\sigma_0^2 + (n-1)\sigma^2\right)\right)$$

$$B \quad \exp\left(-\frac{1}{2\sigma^2}\left(\frac{nk_0(\bar{y}-\mu_0)^2}{n+k_0}\right)\right) \int \exp\left[-\frac{n+k_0}{2\sigma^2}\left(\mu - \frac{n\bar{y}+k_0\mu_0}{n+k_0}\right)^2\right] d\mu$$

$\sigma^2 | y \sim \chi^2(\nu_n, \sigma_n^2)$, $\nu_n = \nu_0 + n$

$$\nu_n\sigma_n^2 = \nu_0\sigma_0^2 + (n-1)\sigma^2 + \frac{nk_0}{n+k_0}(\bar{y}-\mu_0)^2$$