Week3_2019122064 Dagun Oh

2019122064 오다건

1. BDA ch 2. 8

→ known sd : 20

$\bar{y}=150$, unknown mean, normal dist, $\theta \sim N(180, 40^2)$

y : weight

→ 정리하면, $y|\theta \sim N(\theta, 20^2)$ (θ, σ^2)

$\theta \sim N(180, 40^2)$ (μ_0, τ_0)

$\theta|y \sim N(\mu_n, \tau_n^2)$

(a) $\theta|y \sim N(\mu_n, \tau_n^2)$

$$\mu_n = \frac{\frac{1}{\tau_0^2}}{\frac{n}{\sigma^2}+\frac{1}{\tau_0^2}}\mu_0 + \frac{\frac{n}{\sigma^2}}{\frac{n}{\sigma^2}+\frac{1}{\tau_0^2}}\bar{y}$$

$$= \frac{\frac{1}{1600}}{\frac{n}{400}+\frac{1}{1600}}180 + \frac{\frac{n}{400}}{\frac{1}{1600}+\frac{n}{400}} \times 150$$

$$= \frac{1}{4n+1}180 + \frac{4n}{4n+1}150 = \frac{600n+180}{4n+1}$$

$$\tau_n^2 = \frac{1}{\frac{n}{\sigma^2}+\frac{1}{\tau_0^2}} = \frac{1}{\frac{n}{400}+\frac{1}{1600}} = \frac{1600}{1+4n}$$

$$\therefore \theta|y \sim N\left(\frac{600n+180}{4n+1}, \frac{1600}{4n+1}\right)$$

(b) predictive $(\tilde{y})$

$$\tilde{y} \sim N(\mu_n, \tau_n^2+\sigma^2)$$

$$\sim N\left(\frac{600n+180}{4n+1}, 400+\frac{1600}{4n+1}\right)$$

(c) $n=10$, 95% posterior interval for θ ...하시는 R 돌려야함

〃 posterior interval for $\tilde{y}$

↳ Bayesian Credible interval

- $\theta \mid y \sim N\left(\frac{6180}{41}, \frac{1600}{41}\right) \rightarrow 95\%$

→ sqrt( ) 이용

$\text{qnorm}\left(0.025, \frac{6180}{41}, \sqrt{\frac{1600}{41}}\right) = 138.4879$ (138.4879, 162.9755)

$\text{qnorm}(0.975, \;〃\;\; 〃\;) = 162.9755$

- $\tilde{y} \mid y \sim N\left(\frac{6180}{41}, \frac{18000}{41}\right) \rightarrow 95\%$

$\text{qnorm}\left(0.025, \frac{6180}{41}, \sqrt{\frac{18000}{41}}\right) = 109.6648$ (109.6648, 191.7987)

$\text{qnorm}(0.975, \;〃\;\; 〃\;) = 191.7987$

(d) $n=100$, do the same 이하

- $\theta \mid y \sim N\left(\frac{60180}{401}, \frac{1600}{401}\right)$

$\text{qnorm}\left(0.025, \frac{60180}{40}, \sqrt{\frac{1600}{401}}\right) = 146.1598$ (146.1598, 153.9899)

$\text{qnorm}(0.975, \;〃\;\; 〃\;) = 153.9899$

- $\tilde{y} \mid y \sim N\left(\frac{60180}{401}, \frac{162000}{401}\right)$

$\text{qnorm}\left(0.025, \frac{60180}{401}, \sqrt{\frac{162000}{401}}\right) = 110.6805$ (110.6805, 189.4691)

$\text{qnorm}(0.975, \;〃,\;\; 〃\;) = 189.4691$

2. Two parameter models — Normal data with conjugate prior distribution에서 수식의 marginal posterior distribution 유도.

- likelihood)

$$p(y \mid \mu, \sigma^2) = \prod \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{1}{2\sigma^2}(y_i - \mu)^2\right)$$

$$\propto \sigma^{-n} \cdot \exp\left(-\frac{1}{2\sigma^2} \Sigma (y_i - \mu)^2\right)$$

- prior)

$$p(\mu,\sigma^2) = p(\mu|\sigma^2)\cdot p(\sigma^2)$$

$$\mu|\sigma^2 \sim N(\mu_0, \frac{\sigma_0^2}{k_0}), \quad p(\mu|\sigma^2) \propto \sigma^{-1}\exp\left(\frac{k_0}{-2\sigma^2}(\mu-\mu_0)^2\right)$$

$$\sigma^2 \sim \chi^{-2}(\nu_0, \sigma_0^2), \quad p(\sigma^2) \propto \left(\frac{1}{\sigma^2}\right)^{\frac{\nu_0}{2}+1} \exp\left(-\frac{1}{2\sigma^2}\nu_0\sigma_0^2\right)$$

$$p(\mu,\sigma^2) \propto \sigma^{-1}\exp\left(-\frac{k_0}{2\sigma^2}(\mu-\mu_0)^2\right)\cdot(\sigma^{-2})^{\frac{\nu_0}{2}+1}\cdot\exp\left(-\frac{1}{2\sigma^2}\nu_0\sigma_0^2\right)$$

- marginal posterior distribution 유도..

$$p(\mu,\sigma^2|y) \Rightarrow p(y|\mu,\sigma^2)p(\mu,\sigma^2)$$

$$\propto \sigma^{-n}\cdot\exp\left(-\frac{1}{2\sigma^2}\Sigma(y_i-\mu)^2\right)\times\sigma^{-1}\exp\left(-\frac{k_0}{2\sigma^2}(\mu-\mu_0)^2\right)\cdot(\sigma^{-2})^{\frac{\nu_0}{2}+1}\cdot\exp(\text{''})$$

$$\propto (\sigma^2)^{-\frac{n}{2}}\cdot\exp\left(-\frac{1}{2\sigma^2}\{(n-1)S^2+n(\bar{y}-\mu)^2\}\right)\cdot$$

$$\Sigma(y_i-\bar{y}+\bar{y}-\mu)^2 = \underbrace{\Sigma(y_i-\bar{y})^2}_{(n-1)S^2} + \underbrace{\Sigma(\bar{y}-\mu)^2}_{n(\bar{y}-\mu)^2}$$

$$\times(\sigma^2)^{-\frac{1}{2}}\times(\sigma^2)^{-\frac{\nu_0}{2}-1}\times\exp\left(-\frac{1}{2\sigma^2}\{k_0(\mu-\mu_0)^2+\nu_0\sigma_0^2\}\right)$$

$$\rightarrow N\text{-}Inv\text{-}\chi^2\left(\mu_n, \frac{\sigma_n^2}{k_n}; \nu_n, \sigma_n^2\right)$$

$$\mu_n: \frac{k_0}{k_0+n}\mu_0 + \frac{n}{k_0+n}\cdot\bar{y}$$

$$k_n: k_0+n$$

$$\nu_n = \nu_0+n$$

$$\nu_n\sigma_n^2 = \nu_0\sigma_0^2 + (n-1)S^2 + \frac{k_0 n}{k_0+n}(\bar{y}-\mu_0)^2$$

diff between sample mean and prior mean.

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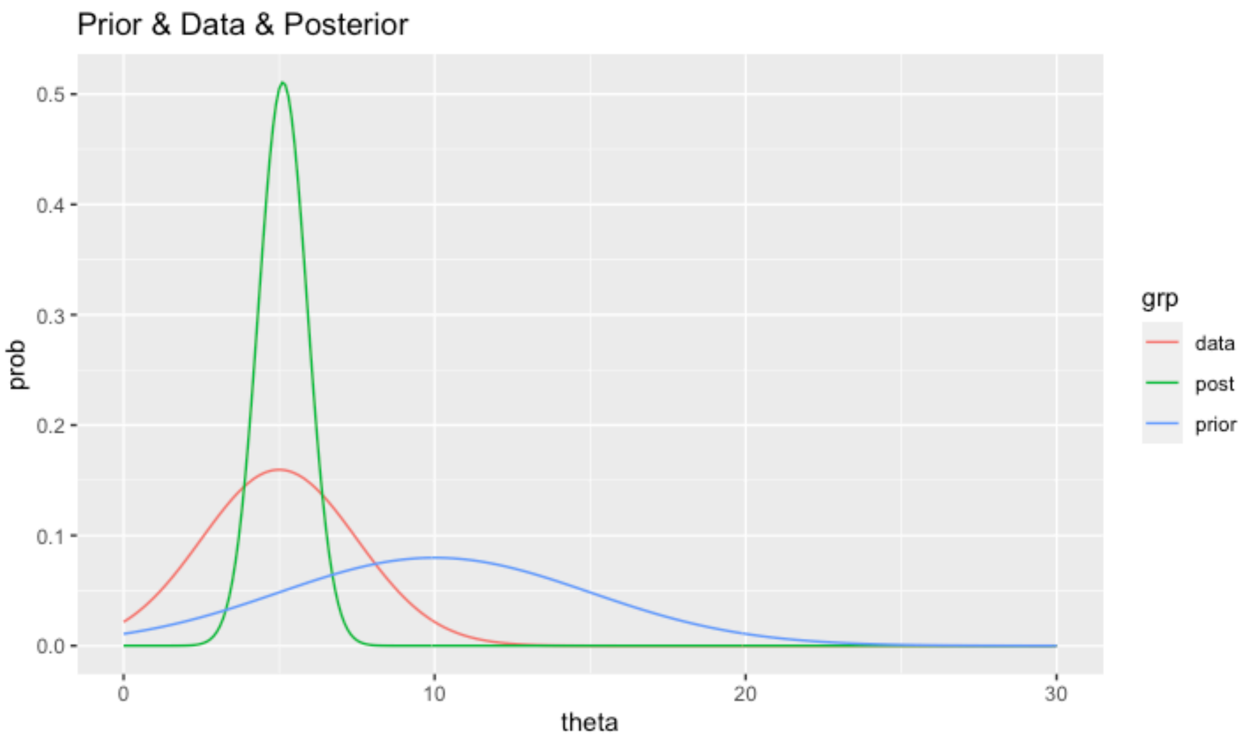
3. R-code simulation.

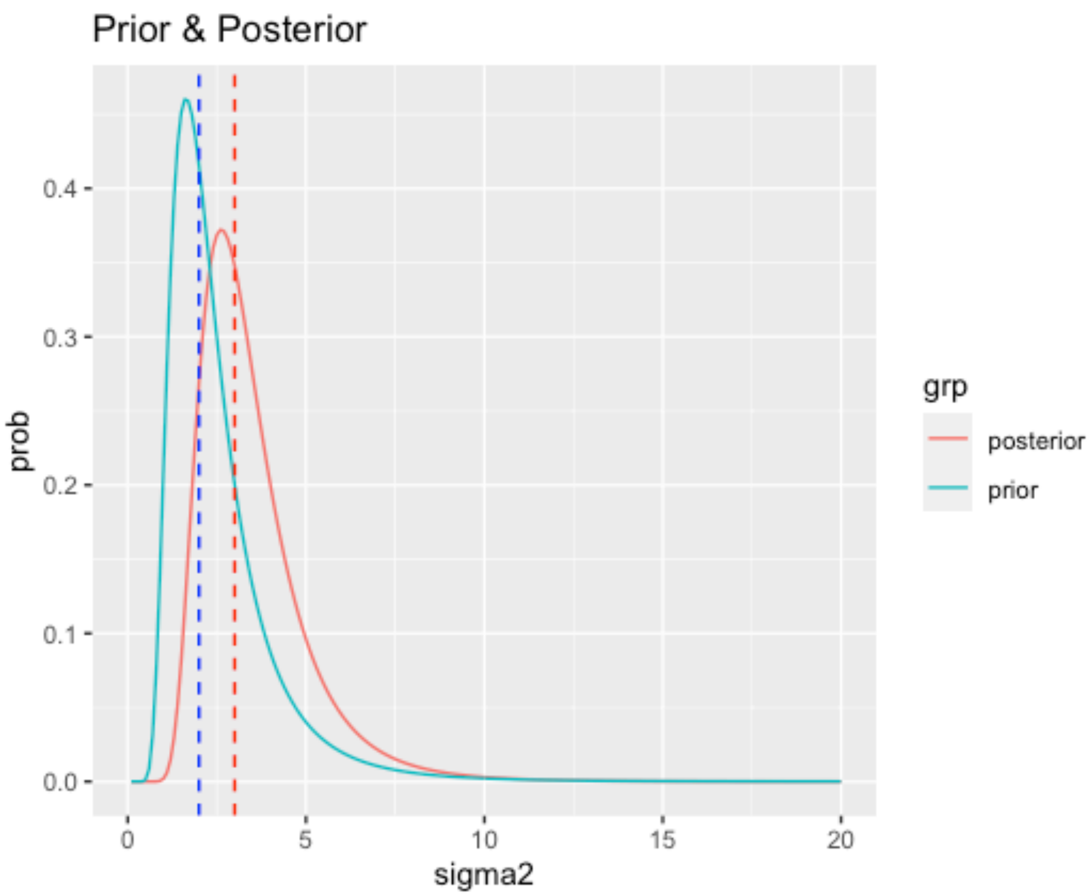
- One-parameter model

```
install.packages('ggplot2')
install.packages('tidyr')
install.packages("ggpubr")

library(ggplot2)
library(ggpubr)
library(tidyr)

## Normal model with unknown mu

## prior
```

```
mu_0 = 10
tau_0 = 5

## data
mu = 5
sd = 2.5
n = 10

## posterior(parameter update)
mu_n = ((1/tau_0^2)/(1/tau_0^2+n/sd^2))*mu_0+
  (n/sd^2/(1/tau_0^2+n/sd^2))*mu
tau_n = sqrt(1/(1/tau_0^2+n/sd^2))


title = "Prior & Data & Posterior"
theta = seq(0,30,0.1)
p = data.frame(theta = theta,
               prior = dnorm(theta, mu_0, tau_0),
               post = dnorm(theta, mu_n, tau_n),
               data = dnorm(theta, mu, sd)
)%>% gather(grp, prob, -theta) %>%
  ggplot(aes(x=theta, y=prob, color=grp))+geom_line()+labs(title=title)

ggarrange(p)


## Normal model with unknown sigma

# prior
sigma_0 = 2
nu_0 = 9

# data1
data = rnorm(5, 7, 3)
mu = mean(data)
sigma = var(data)
n = length(data)

# posterior
nu_n = nu_0 + n
sigma_n = (nu_0*sigma_0^2+sum((data-mu)^2))/nu_n


dist_inverse_chi = function(theta, v, tau2)
  ((v*tau2/2)^(v/2))/gamma(v/2) *(1/theta)^(v/2 +1) * exp(-v*tau2/(2*theta))

title ="Prior & Posterior"
sigma2 = seq(0,20,0.1)
p = data.frame(sigma2 = sigma2,
```

```
                prior = dist_inverse_chi(sigma2, nu_0, sigma_0),
                posterior = dist_inverse_chi(sigma2, nu_n, sigma_n)
) %>%
  gather(grp, prob, -sigma2) %>%
  ggplot(aes(x=sigma2, y=prob, color=grp))+geom_line()+labs(title=title)+
  geom_vline(xintercept=sigma_0, linetype="dashed", color="blue")+
  geom_vline(xintercept=sigma_n, linetype="dashed", color="red")


ggarrange(p)
```





- Two parameter model

```
# data
D = c(1.64, 1.70, 1.72, 1.74, 1.82, 1.82, 1.82, 1.90, 2.08)
n = length(D); xbar = mean(D); s2 = var(D)

# prior
mu0 = 1.9; kappa0 = 1; s20 = 0.01; nu0 = 1

# posterior
kappa1 = kappa0 + n
nu1 = nu0 + n
mu1 = (kappa0 * mu0 + n * xbar) / kappa1
s21 = (1/ nu1) * (nu0*s20 + (n-1)*s2 + (kappa0*n/kappa1)*(xbar-mu0)^2 )

# visualize
prior = function(theta, sigma2)
  dnorm(theta, mu0, sqrt(sigma2/kappa0))*dsinvchisq(sigma2, nu0, s20)
posterior = function(theta, sigma2)
  dnorm(theta, mu1, sqrt(sigma2/kappa1))*dsinvchisq(sigma2, nu1, s21)

dsinvchisq = function(theta,v,tau2)
    ((v*tau2)^(v/2))/gamma(v/2)*(1/theta)^(v/2+1)*exp(-v*tau2/(2*theta))
```

```
mu = seq(1.6, 2, length.out = 100)
sigma2 = seq(0.001, 0.04, length.out = 100)
cmu = rep(mu, each = length(sigma2))
csigma2 = rep(sigma2, length(mu))
prr_dens = mapply(prior, cmu, csigma2)
post_dens = mapply(posterior, cmu, csigma2)

title1 = "Joint prior"
p1 = data.frame(mu = cmu, sigma2 = csigma2, dens = prr_dens) %>%
  ggplot(aes(x=cmu, y=sigma2))+
  geom_raster(aes(fill = dens, alpha= dens), interpolate= T)+
  geom_contour(aes(z= dens), color = 'black', size= 0.2)+
  scale_fill_gradient(low= "cornflowerblue", high= "cornflowerblue", guide= F)+
  scale_alpha(range= c(0,1), guide=F)+
  labs(title=title1)

title2 = "Joint posterior"
p2 = data.frame(mu = cmu, sigma2 = csigma2, dens = post_dens) %>%
  ggplot(aes(x=cmu, y=sigma2))+
  geom_raster(aes(fill = dens, alpha= dens), interpolate= T)+
  geom_contour(aes(z= dens), color = 'black', size= 0.2)+
  scale_fill_gradient(low= "cornflowerblue", high= "cornflowerblue", guide= F)+
  scale_alpha(range= c(0,1), guide=F)+
  labs(title=title2)

ggarrange(p1, p2)
```