1(a) $p(y|\theta) \sim N(\theta, 20^2)$

$p(\theta) \sim N(180, 40^2)$

$p(\theta|y) \propto p(\theta) \cdot p(y|\theta)$

$$\propto \exp\left\{-\frac{1}{800}\sum^n (y_i-\theta)^2\right\} \times \exp\left\{-\frac{1}{3200}(\theta-180)^2\right\}$$

$$= \exp\left\{-\frac{1}{3200}\left\{(1+4n)\theta^2 - 2(4\sum^n y_i + 180)\theta + C\right\}\right\}$$

$$\propto \exp\left\{-\frac{1}{2}\times\left(\frac{4n+1}{1600}\right)\left(\theta - \frac{4\sum^n y_i + 180}{4n+1}\right)^2\right\}$$

$$\Rightarrow \sigma^2 = \frac{1600}{4n+1} = \frac{1}{\frac{n}{400}+\frac{1}{1600}}, \quad \mu = \frac{4\sum^n y_i + 180}{4n+1} = \frac{1}{4n+1}180 + \frac{4n}{4n+1}\bar{y}$$

$$\Rightarrow p(\theta|y) \sim N\left(\frac{600n}{4n+1} + \frac{180}{4n+1}, \frac{1600}{4n+1}\right)$$

(b) $p(\tilde{y}|y) \Rightarrow E[\tilde{y}|y] = E\left[E[\tilde{y}|\theta]|y\right] = E[\theta|y] = \frac{600n+180}{4n+1}$

$$Var(\tilde{y}|y) = E\left[Var(\tilde{y}|\theta)|y\right] + Var\left[E[\tilde{y}|\theta]|y\right]$$

$$= E[20^2|y] + Var[\theta|y] = 400 + \frac{1600}{4n+1}$$

$$\rightarrow \sim N\left(\frac{600n+180}{4n+1}, \frac{1600}{4n+1}+400\right)$$

(c) $n=10 \rightarrow p(\theta|y) \sim N\left(\frac{6180}{41}, \frac{1600}{41}\right) \rightarrow$ 95% C.I (138.4879, 162.9755)

(d) $n=100 \rightarrow$ 95% C.I (1455.561, 1480.049)

CamScanner로 스캔하기

HW 2

$$p(\sigma^2) \propto \left(\frac{1}{\sigma^2}\right)^{\frac{\nu_0}{2}+1} \cdot \exp\left[-\frac{\nu_0\sigma_0^2}{2\sigma^2}\right]$$

$$p(y|\mu,\sigma^2) \propto \left(\frac{1}{\sigma^2}\right)^{\frac{n}{2}} \cdot \exp\left[-\frac{1}{2\sigma^2}\{(n-1)s^2+n(\bar{y}-\mu)^2\}\right]$$

$$p(\mu|\sigma^2) \propto \left(\frac{k_0}{\sigma^2}\right)^{\frac{1}{2}} \cdot \exp\left\{-\frac{k_0}{2\sigma^2}(\mu-\mu_0)^2\right\}$$

$$\Rightarrow p(\sigma^2|y) \propto p(\sigma^2)\cdot p(y|\sigma^2) = p(\sigma^2)\int p(y|\mu,\sigma^2)\cdot p(\mu|\sigma^2)\,d\mu$$

$$\propto \left(\frac{1}{\sigma^2}\right)^{\frac{\nu_0}{2}+1} \cdot \exp\left[-\frac{\nu_0\sigma_0^2}{2\sigma^2}\right] \int \left(\frac{1}{\sigma^2}\right)^{\frac{n}{2}} \cdot \exp\left[-\frac{1}{2\sigma^2}\{(n-1)s^2+n(\bar{y}-\mu)^2\}\right] \cdot \left(\frac{k_0}{\sigma^2}\right)^{\frac{1}{2}} \exp\left\{-\frac{k_0}{2\sigma^2}(\mu-\mu_0)^2\right\} d\mu$$

$$\propto \left(\frac{1}{\sigma^2}\right)^{\frac{\nu_0}{2}+1} \exp\left[-\frac{\nu_0\sigma_0^2}{2\sigma^2}\right] \left(\frac{1}{\sigma^2}\right)^{\frac{n}{2}} \cdot \exp\left[-\frac{1}{2\sigma^2}(n-1)s^2\right] \cdot \left(\frac{k_0}{\sigma^2}\right)^{\frac{1}{2}} \cdot \underline{\int \exp\left[-\frac{1}{2\sigma^2}\{n(\bar{y}-\mu)^2+k_0(\mu-\mu_0)^2\}\right] d\mu}$$

$$* \int \exp\left[-\frac{1}{2\sigma^2}\{n(\bar{y}-\mu)^2+k_0(\mu-\mu_0)^2\}\right] d\mu = \int \exp\left[-\frac{1}{2\sigma^2}\{n\mu^2-2n\bar{y}\mu+n\bar{y}^2+k_0\mu^2-2k_0\mu_0\mu+k_0\mu_0^2\}\right] d\mu$$

$$= \int \exp\left[-\frac{1}{2\sigma^2}\{(n+k_0)\mu^2-2(n\bar{y}+k_0\mu_0)\mu+k_0\mu_0^2+n\bar{y}^2\}\right] d\mu =$$

$$= \int \exp\left[-\frac{(n+k_0)}{2\sigma^2}\left\{\mu-\frac{n\bar{y}+k_0\mu_0}{n+k_0}\right\}^2\right] d\mu \cdot \exp\left(-\frac{1}{2\sigma^2}\left\{\frac{nk_0(\bar{y}-\mu)^2}{n+k_0}\right\}\right)$$

Normalizing constant of $N\left(\frac{n\bar{y}+k_0\mu_0}{n+k_0}, \frac{\sigma^2}{n+k_0}\right)$

$$\to \propto \left(\frac{1}{\sigma^2}\right)^{\frac{\nu_0}{2}+1} \cdot \left(\frac{1}{\sigma}\right)^{\frac{n+1}{2}} \cdot \exp\left(-\frac{\nu_0\sigma_0^2}{2\sigma^2}\right) \exp\left(-\frac{1}{2\sigma^2}(n-1)s^2\right) \cdot \exp\left(-\frac{1}{2\sigma^2}\left(\frac{nk_0(\bar{y}-\mu)^2}{n+k_0}\right)\right) \cdot \underline{\int \exp\left[-\frac{n+k_0}{2\sigma^2}\left(\mu-\frac{n\bar{y}+k_0\mu_0}{n+k_0}\right)^2\right] d\mu}$$

$$\propto \left(\frac{1}{\sigma^2}\right)^{\frac{\nu_0+n_0}{2}+1} \cdot \exp\left[-\frac{1}{2\sigma^2}\{\underbrace{\nu_0\sigma_0^2+(n-1)s^2+\frac{nk_0}{n+k_0}(\bar{y}-\mu)^2}_{=\nu_n\sigma_n^2}\}\right]$$

$$= \left(\frac{1}{\sigma^2}\right)^{\frac{\nu_0+n_0}{2}+1} \cdot \exp\left[-\frac{\nu_n}{2\sigma^2}\{\underbrace{\frac{1}{\nu_n}\{\nu_0\sigma_0^2+(n-1)s^2+\frac{nk_0}{n+k_0}(\bar{y}-\mu)^2\}}_{=\sigma_n^2}\}\right]$$

$$\underline{\sigma^2|y \sim \text{Inv-}\chi^2(\nu_0+n, \sigma_n^2)}$$