Week 3 HW HyeondoOh

Week 3 HW [illegible]

Date

1 BDA Ch2 ex 8

8. Normal distribution with unknown mean: a random sample of n students is drawn from a large population, and their weights are measured. The average weight of the n sampled students is $\bar{y} = 150$ pounds. Assume the weights in the population are normally distributed with unknown mean θ and known standard deviation 20 pounds. Suppose your prior distribution for θ is normal with mean 180 and standard deviation 40.

(a) Give your posterior distribution for θ. (Your answer will be a function of n.)

(b) A new student is sampled at random from the same population and has a weight of $\tilde{y}$ pounds. Give a posterior predictive distribution for $\tilde{y}$. (Your answer will still be a function of n.)

(c) For $n = 10$, give a 95% posterior interval for θ and a 95% posterior predictive interval for $\tilde{y}$.

(d) Do the same for $n = 100$.

n sample $\rightarrow \bar{y} = 150$

$y|\theta \sim N(\theta, 20^2)$ prior $\theta \sim N(180, 40^2)$

a) posterior dist^n

$$\mu_n = \frac{\frac{1}{\tau_0^2}}{\frac{1}{\tau_0^2} + \frac{n}{\sigma^2}} \mu_0 + \frac{\frac{n}{\sigma^2}}{\frac{1}{\tau_0^2} + \frac{n}{\sigma^2}} \bar{y}$$

$$\tau_n^2 = \frac{1}{\frac{1}{\tau_0^2} + \frac{n}{\sigma^2}} = \frac{1}{\frac{1}{1600} + \frac{n}{400}} = \frac{1600}{4n+1}$$

$$\mu_n = \frac{1600}{4n+1} \frac{1}{1600} \cdot 180 + \frac{n}{400} \frac{1600}{4n+1} 150 = \frac{180 + 600n}{4n+1}$$

$$\therefore \theta|y \sim N\left(\frac{600n+180}{4n+1}, \frac{1600}{4n+1}\right)$$

b) posterior predictive dist^n for $\tilde{y}$

$$E(\tilde{y}|y) = \mu_n = \frac{600n+180}{4n+1}$$

$$Var(\tilde{y}|y) = \sigma^2 + \tau_n^2 = 400 + \frac{1600}{4n+1} = \frac{2000 + 1600n}{4n+1}$$

$$\therefore \tilde{y}|y \sim N\left(\frac{600n+180}{4n+1}, \frac{1600n + 2000}{4n+1}\right)$$

c) $n = 10$ posterior 95% interval for $\theta, \tilde{y}$

$$\theta|y \sim N\left(\frac{6180}{41}, \frac{1600}{41}\right), \quad \tilde{y}|y \sim N\left(\frac{6180}{41}, \frac{18000}{41}\right)$$

$\Rightarrow (138.49, 162.98)$, $(109.67, 191.80)$

d) $n = 100$

$$\theta|y \sim N\left(\frac{6180}{401}, \frac{1600}{401}\right), \quad \tilde{y}|y \sim N\left(\frac{6180}{401}, \frac{162000}{401}\right)$$

$(146.16, 153.99)$ $(110.68, 189.47)$

2 Two-par-Normal data with conjugate prior dist^n 에서 σ^2 marginal posterior dist^n 유도

* Likelihood

$$p(y|\mu, \sigma^2) = \prod_{i=1}^{n} \left(\frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(y_i-\mu)^2}{2\sigma^2}} \right)$$

$$\propto \frac{1}{\sigma^n} \exp\left\{ -\frac{\Sigma(y_i-\mu)^2}{2\sigma^2} \right\}$$

* prior

$$p(\mu, \sigma^2) = p(\mu|\sigma^2)\, p(\sigma^2)$$

$$\begin{cases} \mu|\sigma^2 \sim N\left(\mu_0, \frac{\sigma^2}{k_0}\right) \\ \sigma^2 \sim \text{Inv-}\chi^2(\nu_0, \sigma_0^2) = \text{Inv-}\Gamma\left(\frac{\nu_0}{2}, \frac{\nu_0\sigma_0^2}{2}\right) \end{cases}$$

$$p(\mu|\sigma^2)\, p(\sigma^2) \propto \left(\frac{k_0}{\sigma^2}\right)^{\frac{1}{2}} \exp\left\{ -\frac{k_0}{2\sigma^2} (\mu-\mu_0)^2 \right\} \times \frac{\left(\frac{\nu_0\sigma_0^2}{2}\right)^{\frac{\nu_0}{2}}}{\Gamma\left(\frac{\nu_0}{2}\right)} \left(\frac{1}{\sigma^2}\right)^{\frac{\nu_0}{2}+1} \exp\left\{ -\frac{\nu_0\sigma_0^2}{2\sigma^2} \right\}$$

$$\propto \left(\frac{1}{\sigma^2}\right)^{\frac{1}{2}} \left(\frac{1}{\sigma^2}\right)^{\frac{\nu_0}{2}+1} \cdot \exp\left\{ -\frac{1}{2\sigma^2} \left(k_0(\mu-\mu_0)^2 + \nu_0\sigma_0^2 \right) \right\}$$

* $p(\sigma^2|y)$ marginal post dist^n

$$p(\sigma^2|y) \propto p(\sigma^2)\, p(y|\sigma^2) = p(\sigma^2) \int p(y|\mu, \sigma^2)\, p(\mu|\sigma^2)\, d\mu$$

$$\propto \left(\frac{1}{\sigma^2}\right)^{\frac{\nu_0}{2}+1} \exp\left\{ -\frac{\nu_0\sigma_0^2}{2\sigma^2} \right\} \int \left(\frac{1}{\sigma^2}\right)^{\frac{n}{2}} \exp\left\{ -\frac{1}{2\sigma^2} \left[(n-1)s^2 + n(\bar{y}-\mu)^2 \right] \right\} \left(\frac{k_0}{\sigma^2}\right)^{\frac{1}{2}} \cdot \exp\left\{ -\frac{k_0}{2\sigma^2} \cdot (\mu-\mu_0)^2 \right\} d\mu$$

$$\propto \underbrace{\left(\frac{1}{\sigma^2}\right)^{\frac{\nu_0}{2}+1} \cdot \exp\left(-\frac{\nu_0\sigma_0^2}{2\sigma^2}\right) \cdot \left(\frac{1}{\sigma^2}\right)^{\frac{n}{2}} \exp\left(-\frac{1}{2\sigma^2}(n-1)s^2\right) \left(\frac{1}{\sigma^2}\right)^{\frac{1}{2}}}_{①}$$

$$\cdot \underbrace{\int \exp\left[-\frac{1}{2\sigma^2} n(\bar{y}-\mu)^2 - \frac{k_0}{2\sigma^2} (\mu-\mu_0)^2 \right] d\mu}_{②}$$

② $\int \exp\left[-\frac{1}{2\sigma^2} \left(n(\bar{y}-\mu)^2 + k_0(\mu-\mu_0)^2 \right) \right] d\mu$

$$= \int \exp\left[-\frac{1}{2\sigma^2} \left\{ n\mu^2 - 2n\bar{y}\mu + n\bar{y}^2 + k_0\mu^2 - 2k_0\mu_0\mu + k_0\mu_0^2 \right\} \right] d\mu$$

$$= \int \exp\left[-\frac{1}{2\sigma^2} \left\{ (n+k_0)\mu^2 - 2(n\bar{y} + k_0\mu_0)\mu + k_0\mu_0^2 + n\bar{y}^2 \right\} \right] d\mu$$

$$= \int \exp\left[-\frac{n+k_0}{2\sigma^2} \left\{ \mu^2 - 2\frac{n\bar{y}+k_0\mu_0}{n+k_0}\mu + \left(\frac{n\bar{y}+k_0\mu_0}{n+k_0}\right)^2 \right\} - \frac{1}{2\sigma^2} \underbrace{\left\{ k_0\mu_0^2 + n\bar{y}^2 - \frac{(n\bar{y}+k_0\mu_0)^2}{n+k_0} \right\}}_{= \frac{nk_0(\bar{y}-\mu_0)^2}{n+k_0}} \right] d\mu$$

$$= \exp\left(-\frac{1}{2\sigma^2} \left(\frac{nk_0(\bar{y}-\mu_0)^2}{n+k_0} \right) \right) \cdot \int \exp\left[-\frac{n+k_0}{2\sigma^2} \left(\mu - \frac{n\bar{y}+k_0\mu_0}{n+k_0} \right)^2 \right] d\mu$$

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$$= ① \left(\frac{1}{\sigma^2}\right)^{\frac{v_0}{2}+1} \left(\frac{1}{\sigma^2}\right)^{\frac{n+1}{2}} \exp\left(-\frac{1}{2\sigma^2}\left(v_0\sigma_0^2 + (n-1)\sigma^2\right)\right)$$

$$② \exp\left(-\frac{1}{2\sigma^2}\left(\frac{nk_0(\bar{y}-\mu_0)^2}{n+k_0}\right)\right) \underbrace{\int \exp\left[-\frac{n+k_0}{2\sigma^2}\left(\mu - \frac{n\bar{y}+k_0\mu_0}{n+k_0}\right)^2\right] d\mu}_{N\left(\frac{n\bar{y}+k_0\mu_0}{n+k_0},\ \frac{\sigma^2}{n+k_0}\right)}$$

$$\propto \left(\frac{1}{\sigma^2}\right)^{\frac{v_0+n}{2}+1} \exp\left[-\frac{1}{2\sigma^2}\left\{v_0\sigma_0^2 + (n-1)\sigma^2 + \frac{nk_0}{n+k_0}(\bar{y}-\mu_0)^2\right\}\right]$$

$$\propto \left(\frac{1}{\sigma^2}\right)^{\frac{v_0+n}{2}+1} \exp\left[-\frac{v_0+n}{2\sigma^2} \cdot \underbrace{\frac{1}{v_0+n}\left\{v_0\sigma_0^2 + (n-1)\sigma^2 + \frac{nk_0}{n+k_0}(\bar{y}-\mu_0)^2\right\}}_{\sigma_n^2}\right]$$

$$\sigma^2 | y \sim \chi^{-2}(v_n, \sigma_n^2), \quad v_n = v_0 + n$$

$$v_n\sigma_n^2 = v_0\sigma_0^2 + (n-1)\sigma^2 + \frac{nk_0}{n+k_0}(\bar{y}-\mu_0)^2$$

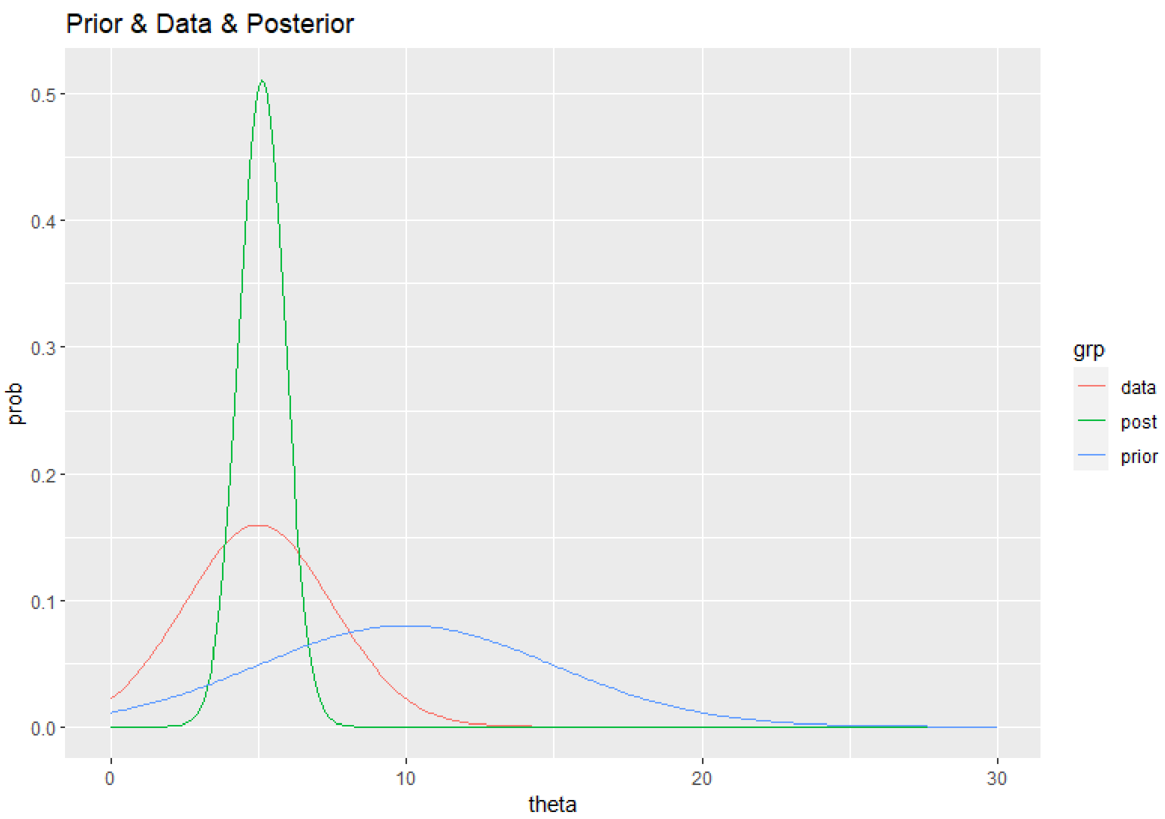
3. R코드

One par model

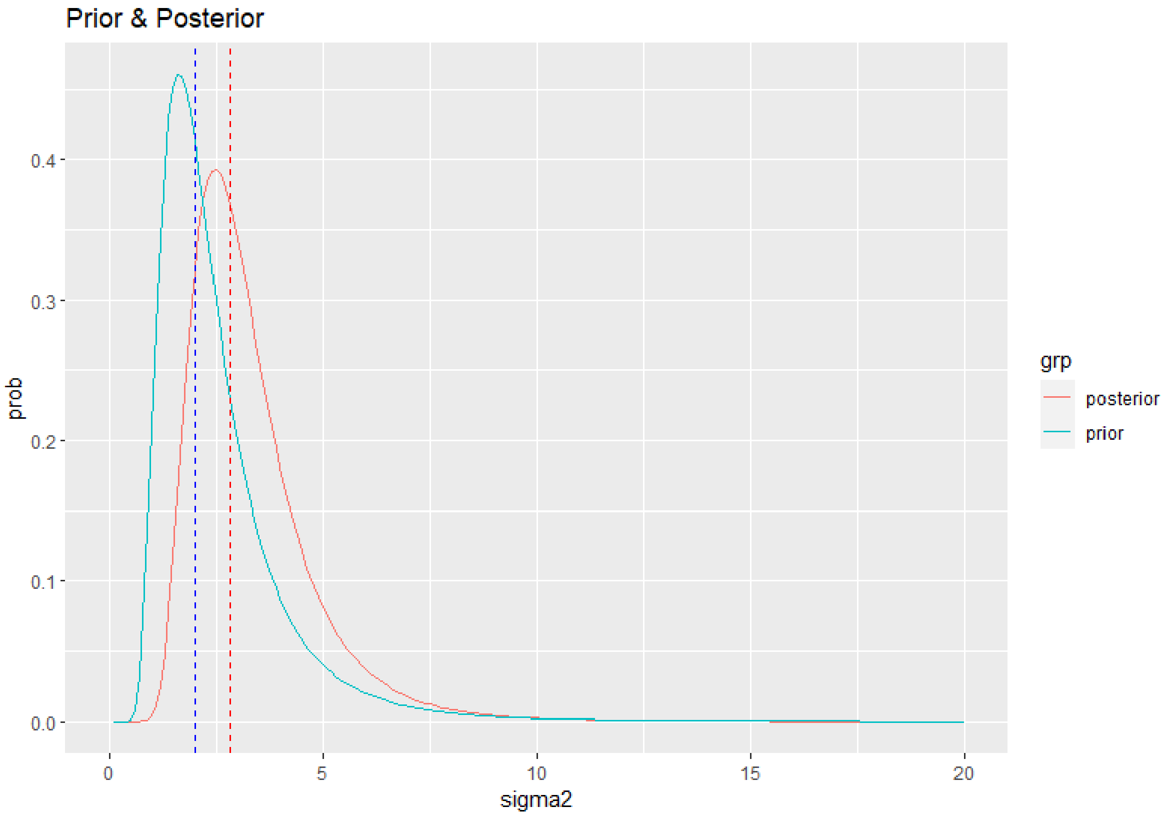
```
install.packages('ggplot2')
install.packages('tidyr')
install.packages("ggpubr")

library(ggplot2)
library(ggpubr)
library(tidyr)


## One parameter model
## Normal model with unknown mu

## prior
mu_0 = 10
tau_0 = 5

## data
mu = 5
sd = 2.5
n = 10

## posterior(parameter update)
mu_n = ((1/tau_0^2)/(1/tau_0^2+n/sd^2))*mu_0+
  (n/sd^2/(1/tau_0^2+n/sd^2))*mu
tau_n = sqrt(1/(1/tau_0^2+n/sd^2))


title = "Prior & Data & Posterior"
theta = seq(0,30,0.1)
p = data.frame(theta = theta,
               prior = dnorm(theta, mu_0, tau_0),
               post = dnorm(theta, mu_n, tau_n),
               data = dnorm(theta, mu, sd)
)%>% gather(grp, prob, -theta) %>%
  ggplot(aes(x=theta, y=prob, color=grp))+geom_line()+labs(title=title)

ggarrange(p)


## Normal model with unknown sigma

# prior
sigma_0 = 2
nu_0 = 9

# data1
data = rnorm(5, 7, 3)
mu = mean(data)
sigma = var(data)
n = length(data)

# posterior
nu_n = nu_0 + n
sigma_n = (nu_0*sigma_0^2+sum((data-mu)^2))/nu_n
```

```
dist_inverse_chi = function(theta, v, tau2)
  ((v*tau2/2)^(v/2))/gamma(v/2) *(1/theta)^(v/2 +1) * exp(-v*tau2/(2*theta))

title ="Prior & Posterior"
sigma2 = seq(0,20,0.1)
p = data.frame(sigma2 = sigma2,
               prior = dist_inverse_chi(sigma2, nu_0, sigma_0),
               posterior = dist_inverse_chi(sigma2, nu_n, sigma_n)
) %>%
  gather(grp, prob, -sigma2) %>%
  ggplot(aes(x=sigma2, y=prob, color=grp))+geom_line()+labs(title=title)+
  geom_vline(xintercept=sigma_0, linetype="dashed", color="blue")+
  geom_vline(xintercept=sigma_n, linetype="dashed", color="red")

ggarrange(p)
```

Normal model with unknown μ



Normal model with unknown σ^2



Two par model

```
## Two parameter model
# data
D = c(1.64, 1.70, 1.72, 1.74, 1.82, 1.82, 1.82, 1.90, 2.08)
n = length(D); xbar = mean(D); s2 = var(D)

# prior
mu0 = 1.9; kappa0 = 1; s20 = 0.01; nu0 = 1

# posterior
kappa1 = kappa0 + n
nu1 = nu0 + n
mu1 = (kappa0 * mu0 + n * xbar) / kappa1
s21 = (1/ nu1) * (nu0*s20 + (n-1)*s2 + (kappa0*n/kappa1)*(xbar-mu0)^2 )
# visualize
prior = function(theta, sigma2)
  dnorm(theta, mu0, sqrt(sigma2/kappa0)) * dsinvchisq(sigma2, nu0, s20)
posterior = function(theta, sigma2)
  dnorm(theta, mu1, sqrt(sigma2/kappa1)) * dsinvchisq(sigma2, nu1, s21)
dsinvchisq = function(theta,v,tau2)
  ((v*tau2)^(v/2))/gamma(v/2)*(1/theta)^(v/2+1)*exp(-v*tau2/(2*theta))
mu = seq(1.6, 2, length.out = 100)
sigma2 = seq(0.001, 0.04, length.out = 100)
cmu = rep(mu, each = length(sigma2))
csigma2 = rep(sigma2, length(mu))
prr_dens = mapply(prior, cmu, csigma2)
post_dens = mapply(posterior, cmu, csigma2)


title1 = "Joint prior"
```

```r
p1 = data.frame(mu = cmu, sigma2 = csigma2, dens = prr_dens) %>%
  ggplot(aes(x=cmu, y=sigma2))+
  geom_raster(aes(fill = dens, alpha= dens), interpolate= T)+
  geom_contour(aes(z= dens), color = 'black', size= 0.2)+
  scale_fill_gradient(low= "cornflowerblue", high= "cornflowerblue", guide= F)+
  scale_alpha(range= c(0,1), guide=F)+
  labs(title=title1)
title2 = "Joint posterior"
p2 = data.frame(mu = cmu, sigma2 = csigma2, dens = post_dens) %>%
  ggplot(aes(x=cmu, y=sigma2))+
  geom_raster(aes(fill = dens, alpha= dens), interpolate= T)+
  geom_contour(aes(z= dens), color = 'black', size= 0.2)+
  scale_fill_gradient(low= "cornflowerblue", high= "cornflowerblue", guide= F)+
  scale_alpha(range= c(0,1), guide=F)+
  labs(title=title2)

ggarrange(p1, p2)
```