1. $y|\theta \sim N(\theta, 20^2)$ ← unknown (θ), $\bar{y} = 150$.

prior: $\theta \sim N(180, 40^2)$

(a) posterior

$$p(\theta|y) \propto \underbrace{p(y|\theta)}_{\text{likelihood}} \cdot \underbrace{p(\theta)}_{\text{prior}}$$

$$\propto \exp\left\{-\frac{1}{2\cdot 20^2}\sum_{i=1}^{n}(y_i-\theta)^2 - \frac{1}{2\cdot 40^2}(\theta-180)^2\right\}$$

$$= \exp\left\{-\frac{1}{3200}\left(4\Sigma y_i^2 - 8\theta\Sigma y_i + \underline{4n\theta^2} + \underline{\theta^2} - 360\theta + 180^2\right)\right\}$$

$$\propto \exp\left\{-\frac{1}{3200}(4n+1)\left(\theta^2 - 2\times\frac{4\Sigma y_i + 180}{4n+1}\theta + \left(\frac{4\Sigma y_i + 180}{4n+1}\right)^2\right)\right\}$$

$$= \exp\left\{-\frac{1}{2\cdot\left(\frac{1600}{4n+1}\right)}\left(\theta - \frac{4\Sigma y_i + 180}{4n+1}\right)^2\right\}$$

유도과정

$$\therefore \theta|y \sim N\left(\frac{4\Sigma y_i + 180}{4n+1}, \frac{1600}{4n+1}\right)$$

$$\underset{\bar{y}=150}{=} \boxed{N\left(\frac{600n+180}{4n+1}, \frac{1600}{4n+1}\right)}$$

(b) prediction

$$p(\tilde{y}|y) = \int p(\tilde{y}|\theta)p(\theta|y)\,d\theta \quad : \text{정규분포}$$

⇒ $E[\tilde{y}|y]$, $Var(\tilde{y}|y)$ 만 알면 정의 분포 OK!

-반복기대의 법칙-

$$E[\tilde{y}|y] = E[E[\tilde{y}|\theta]|y]$$

$$= E[\theta|y] = \frac{600n+180}{4n+1}$$

$$Var(\tilde{y}|y) = E[Var(\tilde{y}|\theta)|y] + Var(E[\tilde{y}|\mu]|y)$$

$$= E[20^2|y] + Var(\theta|y)$$

$$= 400 + \frac{1600}{4n+1}$$

$$\Rightarrow \boxed{\tilde{y}|y \sim N\left(\frac{600n+180}{4n+1}, 400 + \frac{1600}{4n+1}\right)}$$

(c) $n=10$.

① $\theta \mid y \sim N\left(\frac{6180}{41}, \frac{1600}{41}\right)$

$\Rightarrow$ 95% C.I. for θ

$$\frac{6180}{41} \pm 1.96 \times \sqrt{\frac{1600}{41}}$$

$$= \boxed{(138.49,\ 162.98)}$$

② $\tilde{y} \mid y \sim N\left(\frac{6180}{41}, 400+\frac{1600}{41}\right)$

$\Rightarrow$ 95% C.I for $\tilde{y}$

$$\frac{6180}{41} \pm 1.96 \times \sqrt{400+\frac{1600}{41}}$$

$$\therefore \boxed{(109.66,\ 191.80)}$$

(d) $n=100$

① $\theta \mid y \sim N\left(\frac{60180}{401}, \frac{1600}{401}\right)$

$\Rightarrow$ 95% C.I. for θ

$$\frac{60180}{401} \pm 1.96 \times \sqrt{\frac{1600}{401}}$$

$$\therefore \boxed{(146.16,\ 153.99)}$$

② $\tilde{y} \mid y \sim N\left(\frac{60180}{401}, 400+\frac{1600}{401}\right)$

$\Rightarrow$ 95% C.I. for $\tilde{y}$

$$\frac{60180}{401} \pm 1.96 \times \sqrt{400+\frac{1600}{401}}$$

$$\therefore \boxed{(110.68,\ 189.47)}$$

2. $P(\sigma^2|y) \propto P(\sigma^2, y) = \underbrace{\overset{㉠}{P(\sigma^2)}\,\overset{㉡}{P(y|\sigma^2)}}$

㉠ $P(\sigma^2) \propto \left(\frac{1}{\sigma^2}\right)^{\frac{\nu_0}{2}+1} \exp\left(-\frac{\nu_0\sigma_0^2}{2\sigma^2}\right)$ $\quad \because \sigma^2 \sim \text{Inv-}\chi^2(\nu_0, \sigma_0^2)$

㉡ $P(y|\sigma^2) = \int \underbrace{P(y|\mu,\sigma^2)}_{\text{Likelihood.}} P(\mu|\sigma^2)\, d\mu$ $\quad \propto \sigma^{-1}\exp\left\{-\frac{k_0}{2\sigma^2}(\mu-\mu_0)^2\right\}$

$\propto \sigma^{-n}\exp\left(-\frac{1}{2\sigma^2}\Sigma(y_i-\mu)^2\right)$ $\quad \because \mu|\sigma^2 \sim N\left(\mu_0, \frac{\sigma_0^2}{k_0}\right)$

$\because y|\mu,\sigma^2 \sim N(\mu, \sigma^2)$

$$= \left(\frac{1}{\sigma^2}\right)^{\frac{n+1}{2}} \boxed{\int \exp\left\{-\frac{1}{2\sigma^2}\left(\Sigma(\bar{y}-\mu)^2 + k_0(\mu-\mu_0)^2\right)\right\} d\mu} \cdot \exp\left(-\frac{1}{2\sigma^2}(n-1)s^2\right)$$

$$= \int \exp\left\{-\frac{1}{2\sigma^2}\left(n(\bar{y}-\mu)^2 + k_0(\mu-\mu_0)^2\right)\right\} d\mu$$

$$= \int \exp\left\{-\frac{1}{2\sigma^2}\left(n\bar{y}^2 - 2n\bar{y}\mu + n\mu^2 + k_0\mu^2 - 2k_0\mu_0\mu + k_0\mu_0^2\right)\right\} d\mu$$

$$= \int \exp\left\{-\frac{1}{2\sigma^2}\left((n+k_0)\mu^2 - 2(n\bar{y}+k_0\mu_0)\mu + k_0\mu_0^2 + n\bar{y}^2\right)\right\} d\mu$$

$$= \underbrace{\int \exp\left\{-\frac{(n+k_0)}{2\sigma^2}\left(\mu - \frac{n\bar{y}+k_0\mu_0}{n+k_0}\right)^2\right\} d\mu}_{= \frac{n+k_0}{\sigma^2} \propto \left(\frac{1}{\sigma^2}\right)^{-\frac{1}{2}} \text{ normalizing constant of } N\left(\frac{n\bar{y}+k_0\mu_0}{n+k_0}, \frac{\sigma^2}{n+k_0}\right)} \times \exp\left(-\frac{1}{2\sigma^2}\left(\frac{nk_0(\bar{y}-\mu_0)^2}{n+k_0}\right)\right)$$

$$\Rightarrow p(\sigma^2|y) \propto \left(\frac{1}{\sigma^2}\right)^{\frac{\nu_0}{2}+1} \exp\left(-\frac{\nu_0\sigma_0^2}{2\sigma^2}\right) \times \left(\frac{1}{\sigma^2}\right)^{\frac{n+1}{2}} \cdot \left(\frac{1}{\sigma^2}\right)^{-\frac{1}{2}} \times \exp\left\{-\frac{1}{2\sigma^2}\left(\frac{nk_0(\bar{y}-\mu_0)^2}{n+k_0}\right)\right\} \times \exp\left(-\frac{1}{2\sigma^2}(n-1)s^2\right)$$

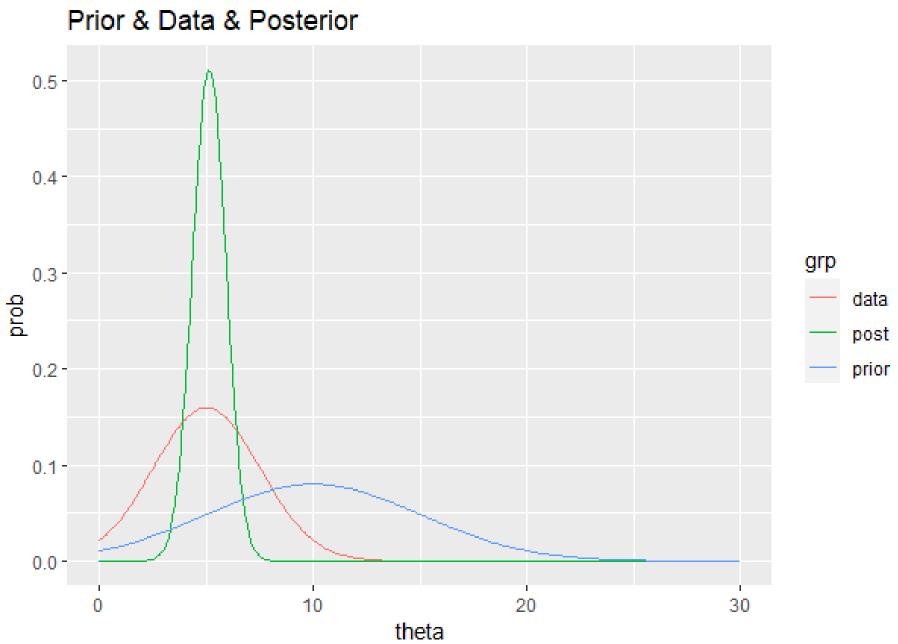
$$= \left(\frac{1}{\sigma^2}\right)^{\frac{\nu_0+n}{2}+1} \exp\left\{-\frac{1}{2\sigma^2}\left(\nu_0\sigma_0^2 + (n-1)s^2 + \frac{nk_0}{n+k_0}(\bar{y}-\mu_0)^2\right)\right\}$$

$$= \left(\frac{1}{\sigma^2}\right)^{\frac{\overbrace{\boxed{\nu_0+n}}^{=\nu_n}}{2}+1} \exp\left\{-\frac{\nu_n}{2\sigma^2}\left(\underbrace{\frac{1}{\nu_n}\left(\nu_0\sigma_0^2 + (n-1)s^2 + \frac{nk_0}{n+k_0}(\bar{y}-\mu_0)^2\right)}_{=\sigma_n^2}\right)\right\}$$

$$\Rightarrow \sigma^2|y \sim \text{Inv-}\chi^2(\nu_n, \sigma_n^2)$$

Normal model with unknown mu

- data) mu = 5(모른다 가정), sd = 2.5, n = 10
- prior) mu_0 = 10, tau_0 = 5 (normal)



Normal model with unknown sigma

- data) mu = 5(모른다 가정), sd = 2.5, n = 10
- prior) sigma_0 = 2, nu_0 = 9 (inverse chi-square)