1.

(a) Prior : $\theta \sim N(180, 40^2)$

Likelihood : $y|\theta \sim N(150, 20^2)$

Posterior : $\theta|y \sim N(\mu_n, \tau_n^2)$

$$\mu_n = \frac{\frac{1}{\tau_0^2}}{\frac{n}{\sigma^2}+\frac{1}{\tau_0^2}} \cdot \mu_0 + \frac{\frac{n}{\sigma^2}}{\frac{n}{\sigma^2}+\frac{1}{\tau_0^2}} \times \bar{y} = \frac{\frac{1}{1600}\times 180}{\frac{n}{400}+\frac{1}{1600}} + \frac{\frac{n}{400}\times 150}{\frac{n}{400}+\frac{1}{1600}} = \frac{\frac{180+600n}{1600}}{\frac{4n+1}{1600}} = \frac{600n+180}{4n+1}$$

$$\tau_n^2 = \frac{1}{\frac{n}{\sigma^2}+\frac{1}{\tau_0^2}} = \frac{1}{\frac{n}{400}+\frac{1}{1600}} = \frac{1600}{4n+1}$$

$$\boxed{\therefore\ \theta|y \sim N\left(\frac{600n+180}{4n+1}, \frac{1600}{4n+1}\right)}$$

(b)

$$E[\tilde{y}|y] = E[E[\tilde{y}|\mu]|y]$$
$$= E[\mu|y] = \mu_n = \frac{600n+180}{4n+1}$$

$$V[\tilde{y}|y] = E[V[\tilde{y}|\mu]|y] + V[E[\tilde{y}|\mu]|y]$$
$$= E[\sigma^2|y] + V[\mu|y] = \sigma^2 + \tau_n^2 = 400 + \frac{1600}{4n+1} = \frac{1600n+2000}{4n+1}$$

$$\boxed{\therefore\ \tilde{y}|y \sim N\left(\frac{600n+180}{4n+1}, \frac{1600n+2000}{4n+1}\right)}$$

16000
2000

(c)

$$\theta|y \sim N\left(\frac{6180}{41}, \frac{1600}{41}\right)$$

95% Confidence interval $= \left[\frac{6180}{41} - 1.96\times\sqrt{\frac{1600}{41}}, \frac{6150}{41} + 1.96\times\sqrt{\frac{1600}{41}}\right] = \boxed{[138.49, 162.98]}$

$$\tilde{y}|y \sim N\left(\frac{6180}{41}, \frac{18000}{41}\right)$$

95% Confidence interval $= \left[\frac{6180}{41} - 1.96\times\sqrt{\frac{18000}{41}}, \frac{6180}{41} + 1.96\times\sqrt{\frac{18000}{41}}\right] = \boxed{[109.66, 191.80]}$

(d)

95% P.I for $\theta = \left[\frac{60180}{401} - 1.96\times\sqrt{\frac{1600}{401}}, \frac{60180}{401} + 1.96\times\sqrt{\frac{1600}{401}}\right] = \boxed{[146.16, 153.99]}$

95% P.P.I for $\tilde{y} = \left[\frac{60180}{401} - 1.96\times\sqrt{\frac{162000}{401}}, \frac{60180}{401} + 1.96\times\sqrt{\frac{162000}{401}}\right] = \boxed{[110.68, 189.41]}$

2.

$$P(\sigma^2|y) \propto P(\sigma^2)\cdot P(y|\sigma^2)$$

$$= P(\sigma^2)\int P(y|\mu,\sigma^2)\cdot P(\mu|\sigma^2)\,d\mu$$

$$\propto \left(\frac{1}{\sigma^2}\right)^{\frac{\nu_0}{2}+1}\exp\left(-\frac{\nu_0\sigma_0^2}{2\sigma^2}\right)\int\left(\frac{1}{\sigma^2}\right)^{\frac{n}{2}}\exp\left[-\frac{1}{2\sigma^2}\{(n-1)s^2+n(\bar{y}-\mu)^2\}\right]\left(\frac{k_0}{\sigma^2}\right)^{\frac{1}{2}}\cdot\exp\left[-\frac{k_0}{2\sigma^2}(\mu-\mu_0)^2\right]d\mu$$

$$\propto \left(\frac{1}{\sigma^2}\right)^{\frac{\nu_0}{2}+1}\cdot\exp\left(-\frac{\nu_0\sigma_0^2}{2\sigma^2}\right)\cdot\left(\frac{1}{\sigma^2}\right)^{\frac{n}{2}}\exp\left[-\frac{1}{2\sigma^2}(n-1)s^2\right]\cdot\left(\frac{1}{\sigma^2}\right)^{\frac{1}{2}}\underbrace{\int\exp\left[-\frac{1}{2\sigma^2}n(\bar{y}-\mu)^2-\frac{k_0}{2\sigma^2}(\mu-\mu_0)^2\right]d\mu}_{A}$$

$$A = \int\exp\left[-\frac{1}{2\sigma^2}\{n(\bar{y}-\mu)^2+k_0(\mu-\mu_0)^2\}\right]d\mu = \int\exp\left[-\frac{1}{2\sigma^2}\{n\mu^2-2n\bar{y}\mu+n\bar{y}^2+k_0\mu^2-2k_0\mu\cdot\mu_0+k_0\mu_0^2\}\right]d\mu$$

$$= \int\exp\left[-\frac{1}{2\sigma^2}\{(n+k_0)\mu^2-2(n\bar{y}+k_0\mu_0)\mu+k_0\mu_0^2+n\bar{y}^2\}\right]d\mu$$

$$= \int\exp\left[-\frac{n+k_0}{2\sigma^2}\left\{\mu^2-2\frac{(n\bar{y}+k_0\mu_0)}{n+k_0}\mu+\left(\frac{n\bar{y}+k_0\mu_0}{n+k_0}\right)^2\right\}-\frac{1}{2\sigma^2}\left\{k_0\mu_0^2+n\bar{y}^2-\left(\frac{n\bar{y}+k_0\mu_0}{n+k_0}\right)^2\right\}\right]d\mu$$

$$= \int\exp\left[-\frac{n+k_0}{2\sigma^2}\left(\mu-\frac{n\bar{y}+k_0\mu_0}{n+k_0}\right)^2\right]d\mu \times \exp\left[-\frac{1}{2\sigma^2}\left(\frac{nk_0(\bar{y}-\mu_0)^2}{n+k_0}\right)\right]$$

$$\propto \left(\frac{1}{\sigma^2}\right)^{\frac{\nu_0}{2}+1}\cdot\left(\frac{1}{\sigma^2}\right)^{\frac{n+1}{2}}\cdot\exp\left(-\frac{\nu_0\sigma_0^2}{2\sigma^2}\right)\cdot\exp\left[-\frac{1}{2\sigma^2}(n-1)s^2\right]\cdot\exp\left[-\frac{1}{2\sigma^2}\left(\frac{nk_0(\bar{y}-\mu_0)^2}{n+k_0}\right)\right]\cdot\underbrace{\int\exp\left[-\frac{n+k_0}{2\sigma^2}\left(\mu-\frac{n\bar{y}+k_0\mu_0}{n+k_0}\right)^2\right]d\mu}_{N\left(\frac{n\bar{y}+k_0\mu_0}{n+k_0},\ \frac{\sigma^2}{n+k_0}\right)}$$

$$= (\sigma^2)^{-\frac{\nu_0+n}{2}-1} \times \exp\left[-\frac{1}{2\sigma^2}\underbrace{\left\{\nu_0\sigma_0^2+(n-1)s^2+\frac{nk_0}{n+k_0}(\bar{y}-\mu_0)^2\right\}}_{=\nu_n\sigma_n^2}\right]$$

$$= (\sigma^2)^{-\frac{\nu_0+n}{2}-1} \times \exp\left[-\frac{\nu_n}{2\sigma^2}\left\{\underbrace{\frac{1}{\nu_n}\times\left(\nu_0\sigma_0^2+(n-1)s^2+\frac{nk_0}{n+k_0}(\bar{y}-\mu_0)^2\right)}_{=\sigma_n^2}\right\}\right]$$

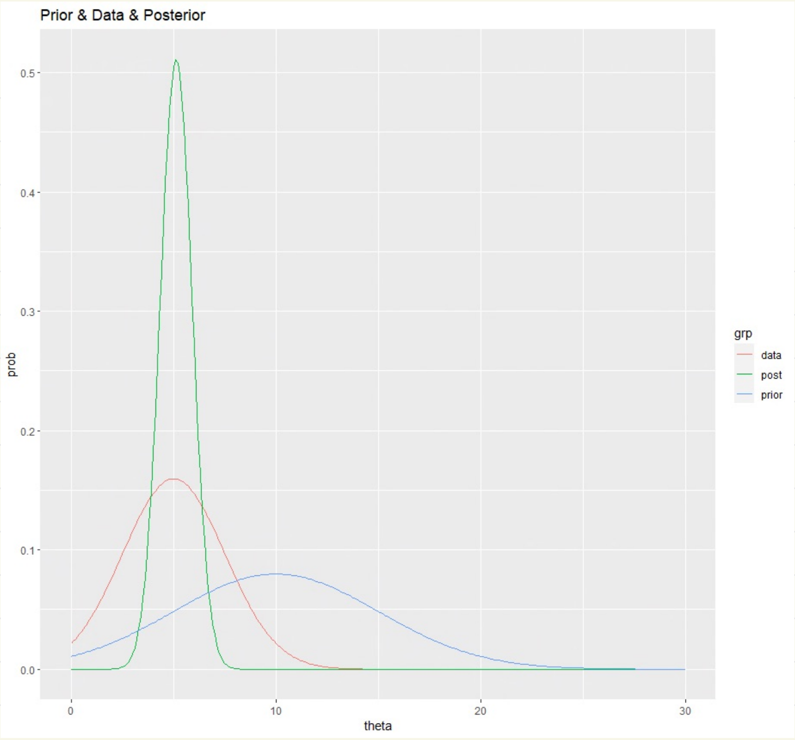
$$\therefore\ \sigma^2|y \sim \text{Inv-}\chi^2(\nu_0+n,\ \sigma_n^2)$$

$$= \text{Inv-}\chi^2(\nu_n,\ \sigma_n^2)$$

3.

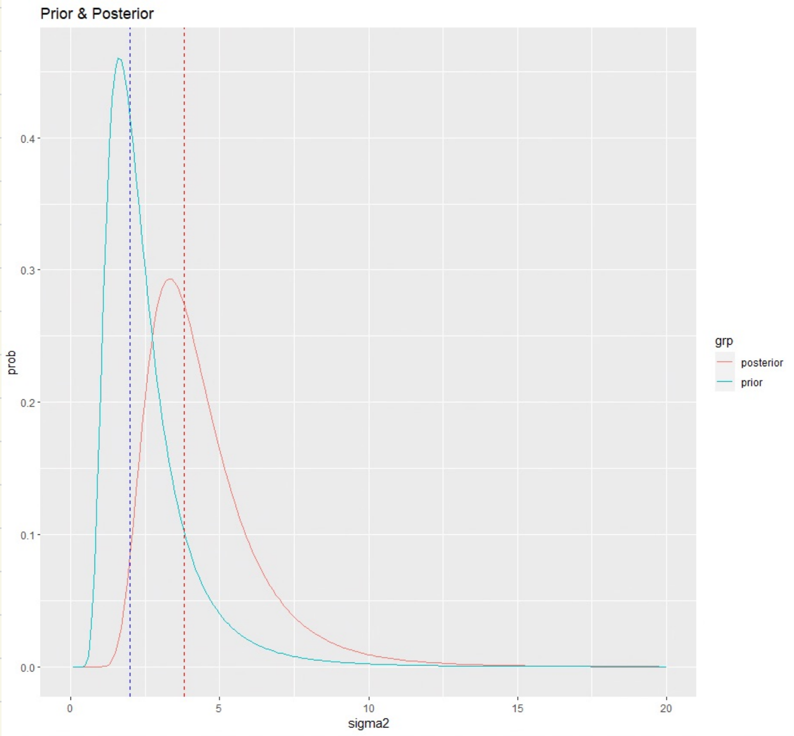
```r
install.packages('ggplot2')
install.packages('tidyr')
install.packages("ggpubr")

library(ggplot2)
library(ggpubr)
library(tidyr)

## Normal model with unknown mu

## prior
mu_0 = 10
tau_0 = 5

## data
mu = 5
sd = 2.5
n = 10

## posterior(parameter update)
mu_n = ((1/tau_0^2)/(1/tau_0^2+n/sd^2))*mu_0+
  (n/sd^2/(1/tau_0^2+n/sd^2))*mu
tau_n = sqrt(1/(1/tau_0^2+n/sd^2))


title = "Prior & Data & Posterior"
theta = seq(0,30,0.1)
p = data.frame(theta = theta,
               prior = dnorm(theta, mu_0, tau_0),
               post = dnorm(theta, mu_n, tau_n),
               data = dnorm(theta, mu, sd)
)%>% gather(grp, prob, -theta) %>%
  ggplot(aes(x=theta, y=prob, color=grp))+geom_line()+labs(title=title)

ggarrange(p)
```

$\Rightarrow$




```r
## Normal model with unknown sigma

# prior
sigma_0 = 2
nu_0 = 9

# data1
data = rnorm(5, 7, 3)
mu = mean(data)
sigma = var(data)
n = length(data)

# posterior
nu_n = nu_0 + n
sigma_n = (nu_0*sigma_0^2+sum((data-mu)^2))/nu_n


dist_inverse_chi = function(theta, v, tau2)
  ((v*tau2/2)^(v/2))/gamma(v/2) *(1/theta)^(v/2 +1) * exp(-v*tau2/(2*theta))

title ="Prior & Posterior"
sigma2 = seq(0,20,0.1)
p = data.frame(sigma2 = sigma2,
               prior = dist_inverse_chi(sigma2, nu_0, sigma_0),
               posterior = dist_inverse_chi(sigma2, nu_n, sigma_n)
) %>%
  gather(grp, prob, -sigma2) %>%
  ggplot(aes(x=sigma2, y=prob, color=grp))+geom_line()+labs(title=title)+
  geom_vline(xintercept=sigma_0, linetype="dashed", color="blue")+
  geom_vline(xintercept=sigma_n, linetype="dashed", color="red")

ggarrange(p)
```

$\Rightarrow$




```r
library(ggplot2)
library(ggpubr)
library(tidyr)


dsinvchi = function(theta, v, tau2)
  ((v*tau2/2)^(v/2))/gamma(v/2) *(1/theta)^(v/2 +1) * exp(-v*tau2/(2*theta))
# data
D = c(1.64, 1.70, 1.72, 1.74, 1.82, 1.82, 1.82, 1.90, 2.08)
n = length(D); xbar = mean(D); s2 = var(D)

# prior
mu0 = 1.9; kappa0 = 1; s20 = 0.01; nu0 = 1

# posterior
kappa1 = kappa0 + n
nu1 = nu0 + n
mu1 = (kappa0 * mu0 + n * xbar) / kappa1
s21 = (1/ nu1) * (nu0*s20 + (n-1)*s2 + (kappa0*n/kappa1)*(xbar-mu0)^2 )
# visualize
prior = function(theta, sigma2)
  dnorm(theta, mu0, sqrt(sigma2/kappa0)) * dsinvchi(sigma2, nu0, s20)
posterior = function(theta, sigma2)
  dnorm(theta, mu1, sqrt(sigma2/kappa1)) * dsinvchi(sigma2, nu1, s21)
mu = seq(1.6, 2, length.out = 100)
sigma2 = seq(0.001, 0.04, length.out = 100)
cmu = rep(mu, each = length(sigma2))
csigma2 = rep(sigma2, length(mu))
prr_dens = mapply(prior, cmu, csigma2)
post_dens = mapply(posterior, cmu, csigma2)

title1 = "Joint prior"
p1 = data.frame(mu = cmu, sigma2 = csigma2, dens = prr_dens) %>%
  ggplot(aes(x=cmu, y=sigma2))+
  geom_raster(aes(fill = dens, alpha= dens), interpolate= T)+
  geom_contour(aes(z= dens), color = 'black', size= 0.2)+
  scale_fill_gradient(low= "cornflowerblue", high= "cornflowerblue", guide= F)+
  scale_alpha(range= c(0,1), guide=F)+
  labs(title=title1)
title2 = "Joint posterior"
p2 = data.frame(mu = cmu, sigma2 = csigma2, dens = post_dens) %>%
  ggplot(aes(x=cmu, y=sigma2))+
  geom_raster(aes(fill = dens, alpha= dens), interpolate= T)+
  geom_contour(aes(z= dens), color = 'black', size= 0.2)+
  scale_fill_gradient(low= "cornflowerblue", high= "cornflowerblue", guide= F)+
  scale_alpha(range= c(0,1), guide=F)+
  labs(title=title2)

ggarrange(p1, p2)
```

$\Rightarrow$