Q1

$$p(\alpha,\beta) = (\alpha+\beta)^{-\frac{5}{2}}$$

$$\theta_j \mid \alpha,\beta \sim \text{Beta}(\alpha,\beta)$$

$$y_j \mid \theta_j \sim \text{Binomial}(n_j,\theta_j)$$

marginal

$$p(\alpha,\beta \mid y) = \int p(\theta,\alpha,\beta \mid y)\, d\theta$$

$$\propto \int p(y\mid\theta)\, p(\theta\mid\alpha,\beta)\, p(\alpha,\beta)\, d\theta$$

$$= p(\alpha,\beta) \int p(y\mid\theta)\, p(\theta\mid\alpha,\beta)\, d\theta$$

$$= p(\alpha,\beta) \int \binom{n_j}{y_j} \theta_j^{y_j} (1-\theta_j)^{n_j-y_j}$$

$$\times \frac{\Gamma(\alpha,\beta)}{\Gamma(\alpha,\beta)} \theta_j^{\alpha-1} (1-\theta_j)^{\beta-1}\, d\theta$$

$$\propto p(\alpha,\beta)\frac{\Gamma(\alpha,\beta)}{\Gamma(\alpha)\Gamma(\beta)}\int \theta_j^{\alpha+y_j-1}(1-\theta_j)^{\beta+n_j-y_j-1}\,d\theta$$

$$\propto(\alpha+\beta)^{-\frac{5}{2}}\frac{\Gamma(\alpha,\beta)}{\Gamma(\alpha)\Gamma(\beta)}\,\frac{\Gamma(\alpha+y_j)\Gamma(\beta+n_j-y_j)}{\Gamma(\alpha+\beta+n_j)}$$

Q2