Week3_HW

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#8 Normal dist with unknown mean

a) $y|\theta \sim N(\theta, 20^2)$
$\theta \sim N(180, 40^2)$
$\theta|y \sim ?$

$\Rightarrow$ $y|\theta \sim N(\mu, \sigma^2)$
$\theta \sim N(\mu_0, \tau_0^2)$
$\theta|y \sim N(\mu_n, \tau_n^2)$

$$\mu_n = \frac{\frac{1}{\tau_0^2}}{\frac{1}{\tau_0^2} + \frac{n}{\sigma^2}} \mu_0 + \frac{\frac{n}{\sigma^2}}{\frac{1}{\tau_0^2} + \frac{n}{\sigma^2}} \bar{y} \quad / \quad \tau_n^2 = \frac{1}{\frac{n}{\sigma^2} + \frac{1}{\tau_0^2}}$$

$\mu_0 = 180$
$\tau_0 = 40^2$
$\bar{y} = 150$
$\sigma^2 = 20^2$

$$\Rightarrow \begin{cases} \mu_n = \dfrac{\frac{1}{40^2}}{\frac{1}{40^2} + \frac{n}{20^2}} \times 180 + \dfrac{\frac{n}{20^2}}{\frac{1}{40^2} + \frac{n}{20^2}} \times 150 = \dfrac{180 + 600n}{4n+1} \\ \tau_n^2 = \dfrac{1}{\frac{n}{20^2} + \frac{1}{40^2}} = \dfrac{1600}{4n+1} \end{cases}$$

b) $E[\tilde{y}|y] = E[E[\tilde{y}|\theta]|y] = E[\theta|y] = \frac{600n+180}{4n+1}$ (posterior dist의 mean)

$Var[\tilde{y}|y] = E[Var[\tilde{y}|\theta]|y] + Var[E[\tilde{y}|\theta]|y]$
$= E[20^2|y] + Var[\theta|y] = 400 + \frac{1600}{4n+1}$ (Data + posterior)

$\therefore \tilde{y}|y \sim N\left(\frac{600n+180}{4n+1}, 400 + \frac{1600}{4n+1}\right)$

c) $\theta|y \sim N\left(\frac{6180}{41}, \frac{1600}{41}\right)$

95% C.I. $\left[\frac{6180}{41} - 1.96 \times \sqrt{\frac{1600}{41}}, \frac{6180}{41} + 1.96 \times \sqrt{\frac{1600}{41}}\right]$

$\tilde{y}|y \sim N\left(\frac{6180}{41}, 400 + \frac{1600}{41}\right)$

95% C.I. $\left[\frac{6180}{41} - 1.96 \times \sqrt{400 + \frac{1600}{41}}, \frac{6180}{41} + 1.96 \sqrt{400 + \frac{1600}{41}}\right]$

d) $\theta|y$: 95% C.I. $\left[\frac{60180}{401} \pm 1.96 \times \sqrt{\frac{1600}{401}}\right]$

$\tilde{y}|y$: 95% C.I. $\left[\frac{60180}{401} \pm 1.96 \times \sqrt{400 + \frac{1600}{401}}\right]$

#2

$$p(\sigma^2|y) \propto p(\sigma^2)\,p(y|\sigma^2)$$

$$= p(\sigma^2)\int p(y|\mu,\sigma^2)\,p(\mu|\sigma^2)\,d\mu$$

$$\propto \left(\frac{1}{\sigma^2}\right)^{\frac{\nu_0}{2}+1}\exp\left(-\frac{\nu_0\sigma_0^2}{2\sigma^2}\right)\int\left(\frac{1}{\sigma^2}\right)^{\frac{n}{2}}\exp\left[-\frac{1}{2\sigma^2}\left\{(n-1)s^2+n(\bar{y}-\mu)^2\right\}\right]\left(\frac{k_0}{\sigma^2}\right)^{\frac{1}{2}}\exp\left[-\frac{k_0}{2\sigma^2}(\mu-\mu_0)^2\right]d\mu$$

$$= \left(\frac{1}{\sigma^2}\right)^{\frac{\nu_0}{2}+1}\exp\left(-\frac{\nu_0\sigma_0^2}{2\sigma^2}\right)\left(\frac{1}{\sigma^2}\right)^{\frac{n}{2}}\exp\left[-\frac{1}{2\sigma^2}(n-1)s^2\right]\cdot\left(\frac{1}{\sigma^2}\right)^{\frac{1}{2}}\underbrace{\int\exp\left[-\frac{1}{2\sigma^2}n(\bar{y}-\mu)^2-\frac{k_0}{2\sigma^2}(\mu-\mu_0)^2\right]d\mu}_{\Downarrow}$$

$$= \int\exp\left[-\frac{1}{2\sigma^2}\left\{n(\bar{y}-\mu)^2+k_0(\mu-\mu_0)^2\right\}\right]d\mu$$

$$= \int\exp\left[-\frac{1}{2\sigma^2}\left\{(n+k_0)\mu^2-2(n\bar{y}+k_0\mu_0)\mu+k_0\mu_0^2+n\bar{y}^2\right\}\right]d\mu$$

$$= \int\exp\left[\frac{n+k_0}{-2\sigma^2}\left\{\mu^2-2\frac{(n\bar{y}+k_0\mu_0)}{n+k_0}\mu+\left(\frac{n\bar{y}+k_0\mu_0}{n+k_0}\right)^2\right\}-\frac{1}{2\sigma^2}\left\{k_0\mu_0^2+n\bar{y}^2-\left(\frac{n\bar{y}+k_0\mu_0}{n+k_0}\right)^2\right\}\right]d\mu$$

$$= \int\exp\left[-\frac{n+k_0}{2\sigma^2}\left(\mu-\frac{n\bar{y}+k_0\mu_0}{n+k_0}\right)^2\right]d\mu\cdot\exp\left[-\frac{1}{2\sigma^2}\left(\frac{nk_0(\bar{y}-\mu_0)^2}{n+k_0}\right)\right]$$

$$= \left(\frac{1}{\sigma^2}\right)^{\frac{\nu_0}{2}+1}\cdot\left(\frac{1}{\sigma^2}\right)^{\frac{n+1}{2}}\cdot\exp\left(\frac{-\nu_0\sigma_0^2}{2\sigma^2}\right)\exp\left(-\frac{1}{2\sigma^2}(n-1)s^2\right)\exp\left[-\frac{1}{2\sigma^2}\left(\frac{nk_0(\bar{y}-\mu_0)^2}{n+k_0}\right)\right]\times$$

$$\cancelto{1}{\exp\left[-\frac{n+k_0}{2\sigma^2}\left(\mu-\frac{n\bar{y}+k_0\mu_0}{n+k_0}\right)^2\right]d\mu}$$

$$= (\sigma^2)^{-\frac{\nu_0+n}{2}-1}\exp\left[-\frac{1}{2\sigma^2}\left\{\nu_0\sigma_0^2+(n-1)s^2+\frac{nk_0}{n+k_0}(\bar{y}-\mu_0)^2\right\}\right]$$

$$= (\sigma^2)^{-\frac{\nu_0+n}{2}-1}\exp\left[-\frac{\nu_n}{2\sigma^2}\left\{\frac{1}{\nu_n}\left(\nu_0\sigma_0^2+(n-1)s^2+\frac{nk_0}{n+k_0}(\bar{y}-\mu_0)^2\right\}\right]$$

$$\therefore \sigma^2|y \sim \text{Inv-}\chi^2(\nu_0+n,\sigma_n^2)$$

$$= \text{Inv-}\chi^2(\nu_n,\sigma_n^2)$$

시각화 돌려보기 : one-parameter

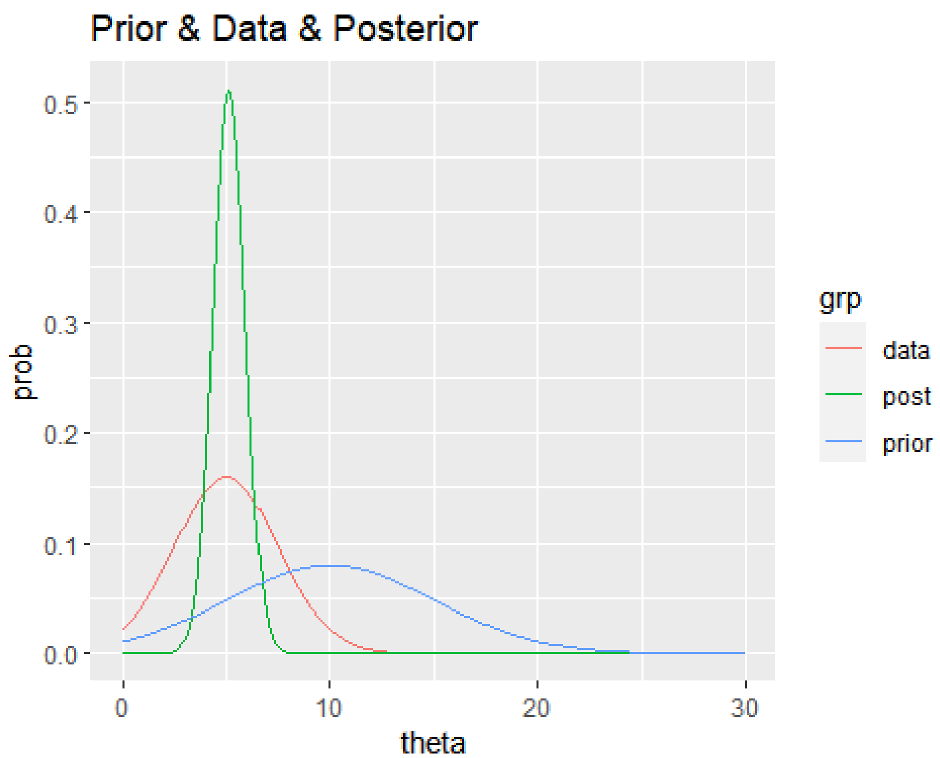
```r
library(ggplot2)
```

```
## Warning: package 'ggplot2' was built under R version 3.6.3
```

```
## Warning: replacing previous import 'vctrs::data_frame' by 'tibble::data_fr
ame'
## when loading 'dplyr'
```

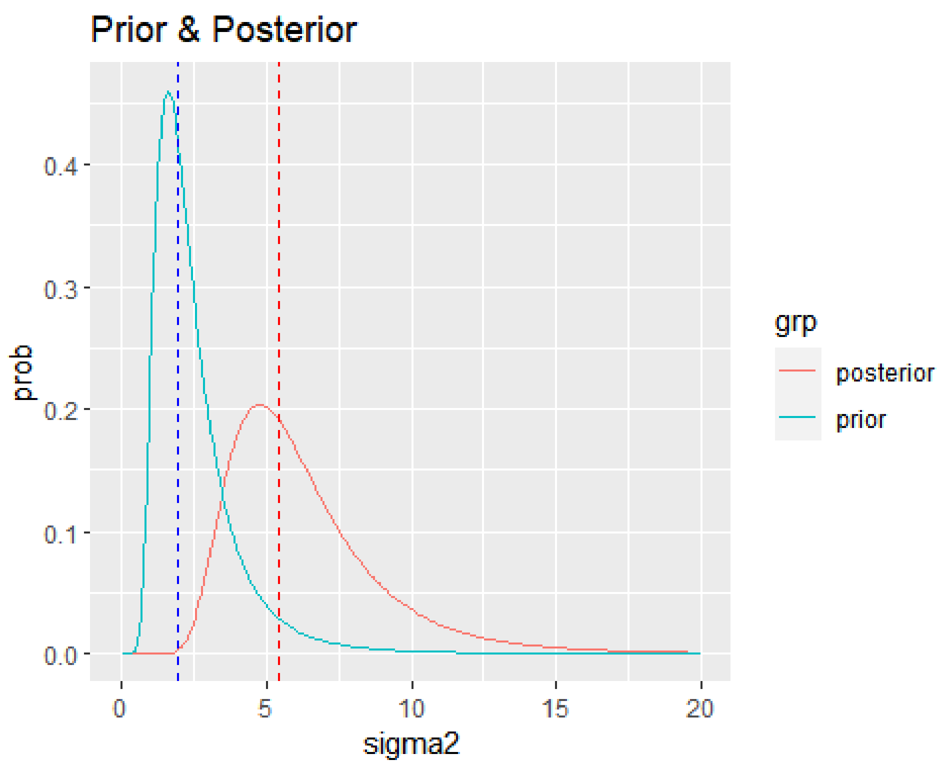
```r
library(ggpubr)
```

```
## Warning: package 'ggpubr' was built under R version 3.6.3

library(tidyr)

## Warning: package 'tidyr' was built under R version 3.6.3

## Normal model with unknown mu

## prior
mu_0 = 10
tau_0 = 5

## data
mu = 5
sd = 2.5
n = 10

## posterior(parameter update)
mu_n = ((1/tau_0^2)/(1/tau_0^2+n/sd^2))*mu_0+
  (n/sd^2/(1/tau_0^2+n/sd^2))*mu
tau_n = sqrt(1/(1/tau_0^2+n/sd^2))


title = "Prior & Data & Posterior"
theta = seq(0,30,0.1)
p = data.frame(theta = theta,
               prior = dnorm(theta, mu_0, tau_0),
               post = dnorm(theta, mu_n, tau_n),
               data = dnorm(theta, mu, sd)
)%>% gather(grp, prob, -theta) %>%
  ggplot(aes(x=theta, y=prob, color=grp))+geom_line()+labs(title=title)

ggarrange(p)
```




```
## Normal model with unknown sigma

# prior
sigma_0 = 2
nu_0 = 9

# data1
data = rnorm(5, 7, 3)
mu = mean(data)
sigma = var(data)
n = length(data)

# posterior
nu_n = nu_0 + n
sigma_n = (nu_0*sigma_0^2+sum((data-mu)^2))/nu_n


dist_inverse_chi = function(theta, v, tau2)
  ((v*tau2/2)^(v/2))/gamma(v/2) *(1/theta)^(v/2 +1) * exp(-v*tau2/(2*theta))

title ="Prior & Posterior"
sigma2 = seq(0,20,0.1)
p = data.frame(sigma2 = sigma2,
               prior = dist_inverse_chi(sigma2, nu_0, sigma_0),
               posterior = dist_inverse_chi(sigma2, nu_n, sigma_n)
```

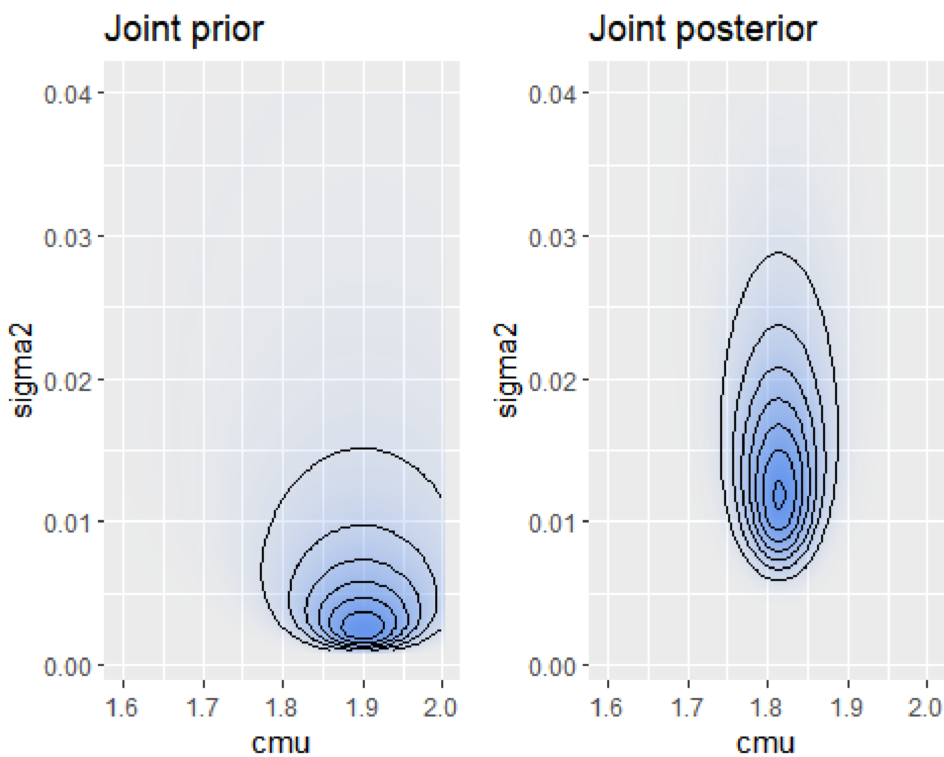
```
) %>%
  gather(grp, prob, -sigma2) %>%
  ggplot(aes(x=sigma2, y=prob, color=grp))+geom_line()+labs(title=title)+
  geom_vline(xintercept=sigma_0, linetype="dashed", color="blue")+
  geom_vline(xintercept=sigma_n, linetype="dashed", color="red")

ggarrange(p)

## Warning: Removed 2 row(s) containing missing values (geom_path).
```



시각화 돌려보기 : two-parameter

```
# data
D = c(1.64, 1.70, 1.72, 1.74, 1.82, 1.82, 1.82, 1.90, 2.08)
n = length(D); xbar = mean(D); s2 = var(D)

# prior
mu0 = 1.9; kappa0 = 1; s20 = 0.01; nu0 = 1

# posterior
kappa1 = kappa0 + n
nu1 = nu0 + n
mu1 = (kappa0 * mu0 + n * xbar) / kappa1
s21 = (1/ nu1) * (nu0*s20 + (n-1)*s2 + (kappa0*n/kappa1)*(xbar-mu0)^2 )
# visualize
```

```r
prior = function(theta, sigma2)
  dnorm(theta, mu0, sqrt(sigma2/kappa0)) * dist_inverse_chi(sigma2, nu0, s20)
posterior = function(theta, sigma2)
  dnorm(theta, mu1, sqrt(sigma2/kappa1)) * dist_inverse_chi(sigma2, nu1, s21)
mu = seq(1.6, 2, length.out = 100)
sigma2 = seq(0.001, 0.04, length.out = 100)
cmu = rep(mu, each = length(sigma2))
csigma2 = rep(sigma2, length(mu))
prr_dens = mapply(prior, cmu, csigma2)
post_dens = mapply(posterior, cmu, csigma2)

title1 = "Joint prior"
p1 = data.frame(mu = cmu, sigma2 = csigma2, dens = prr_dens) %>%
  ggplot(aes(x=cmu, y=sigma2))+
  geom_raster(aes(fill = dens, alpha= dens), interpolate= T)+
  geom_contour(aes(z= dens), color = 'black', size= 0.2)+
  scale_fill_gradient(low= "cornflowerblue", high= "cornflowerblue", guide= F)
+
  scale_alpha(range= c(0,1), guide=F)+
  labs(title=title1)
title2 = "Joint posterior"
p2 = data.frame(mu = cmu, sigma2 = csigma2, dens = post_dens) %>%
  ggplot(aes(x=cmu, y=sigma2))+
  geom_raster(aes(fill = dens, alpha= dens), interpolate= T)+
  geom_contour(aes(z= dens), color = 'black', size= 0.2)+
  scale_fill_gradient(low= "cornflowerblue", high= "cornflowerblue", guide= F)
+
  scale_alpha(range= c(0,1), guide=F)+
  labs(title=title2)

ggarrange(p1, p2)
```


Joint prior
Joint posterior
0.04
0.03
0.02
0.01
0.00
sigma2
1.6
1.7
1.8
1.9
2.0
cmu