8. Normal distribution with unknown mean: a random sample of n students is drawn from a large population, and their weights are measured. The average weight of the n sampled students is $\overline{y} = 150$ pounds. Assume the weights in the population are normally distributed with unknown mean θ and known standard deviation 20 pounds. Suppose your prior distribution for θ is normal with mean 180 and standard deviation 40.

   (a) Give your posterior distribution for θ. (Your answer will be a function of n.)

   (b) A new student is sampled at random from the same population and has a weight of $\tilde{y}$ pounds. Give a posterior predictive distribution for $\tilde{y}$. (Your answer will still be a function of n.)

   (c) For $n = 10$, give a 95% posterior interval for θ and a 95% posterior predictive interval for $\tilde{y}$.

   (d) Do the same for $n = 100$.

(a) $\bar{y} = 150$, $y|\theta \sim N(\theta, 400)$, $\theta \sim N(180, 40^2)$

$$\theta | y \propto p(\theta) p(y|\theta)$$
$$\propto \exp\left(-\frac{1}{2\times 400}\Sigma(y_i-\theta)^2\right)\exp\left(-\frac{1}{2\times 1600}(\theta-180)^2\right)$$
$$\propto \exp\left(-\frac{1}{2}\left(\frac{4n+1}{1600}\left(\theta - \frac{4\Sigma y_i+180}{4n+1}\right)^2\right)\right)$$
$$\sim N\left(\frac{\frac{1}{1600}\times 180}{\frac{1}{40^2}+\frac{n}{20^2}} + \frac{\frac{n}{400}\times 150}{\frac{1}{40^2}+\frac{n}{20^2}}, \frac{1}{\frac{1}{1600}+\frac{n}{400}}\right)$$

(b) $E(\tilde{y}|y) = E[E(\tilde{y}|\mu_0)|y]] = E[\mu_0|y] = \mu = \frac{180+800n}{4n+1}$

$V(\tilde{y}|y) = E[V(\tilde{y}|\mu)|y) + V(E(\tilde{y}|\mu)|y) = \sigma^2 + \tau_n^2 = \frac{1600}{4n+1}+400$

$$\tilde{y}|y \sim N\left(\frac{180+800n}{4n+1}, \frac{1600}{4n+1}+400\right)$$

(c) $\theta | y \sim N\left(\frac{180+8000}{41}, \frac{1600}{41}\right)$

① $P(\text{LowerBound} \le \theta \le \text{Upperbd} \mid y) \ge 0.95$

$\therefore$ U = top 2.5% quantile of $N\left(\frac{6180}{41}, \frac{1600}{41}\right)$

L = bottom 2.5% " " " "

$U = 162.97$, $L = 138.48$

② $\tilde{y}|y \sim N\left(\frac{6180}{41}, \frac{3240}{41}\right)$ in this case, $(L, U) = (109, 191)$

(d) plug in $n=100$ in the same formula.