

5. Consider the fitted values that result from performing linear regression without an intercept. In this setting, the i th fitted value takes the form

$$\hat{y}_i = x_i \hat{\beta},$$

where

$$\hat{\beta} = \left(\sum_{i=1}^n x_i y_i \right) / \left(\sum_{i'=1}^n x_{i'}^2 \right). \quad (3.38)$$

Show that we can write

$$\hat{y}_i = \sum_{i'=1}^n a_{i'} y_{i'}.$$

What is $a_{i'}$?

By the above equation, $\hat{y}_i = x_i \hat{\beta}$, where $\hat{\beta} = \frac{\sum x_i y_i}{\sum x_i^2}$.

Then, $\hat{y}_i = x_i \frac{\sum_{j=1}^n x_j y_j}{\sum_{j=1}^n x_j^2} = \frac{\sum_{j=1}^n x_i x_j y_j}{\sum_{j=1}^n x_j^2}$. Since x_j are fixed for $j \in \{1, \dots, n\}$, $\sum_{j=1}^n x_j$ is a constant, say $\frac{1}{S} := \sum_{j=1}^n x_j$.

$$\text{Thus, } \hat{y}_i = \frac{\sum_{j=1}^n x_i x_j y_j}{\sum_{j=1}^n x_j^2} = S \sum_{j=1}^n x_i x_j y_j = \sum_{j=1}^n (S x_i x_j) y_j. \quad \therefore a_j = \frac{x_i x_j}{\sum_{k=1}^n x_k^2}$$

Also, we can write $\hat{y}_i = \sum_{i'=1}^n a_{i'} y_{i'} = \sum_{i'=1}^n \left\{ \left(\frac{x_i}{\sum_{j=1}^n x_j^2} \right) x_{i'} \right\} y_{i'}$

Ex. 3.4 Show how the vector of least squares coefficients can be obtained from a single pass of the Gram-Schmidt procedure (Algorithm 3.1). Represent your solution in terms of the QR decomposition of X .

Suppose $X = QR$, where $Q^T Q = I$, and R is the upper triangular matrix.

Define a linear regression model $y = f(x) + \varepsilon = X\beta + \varepsilon = (QR)\beta + \varepsilon$, where $\varepsilon \sim N(0, \sigma^2 I_n)$.

$$\text{Then. } \varepsilon^T \varepsilon = \sum_{i=1}^n \varepsilon_i^2 = (y - (QR)\beta)^T (y - (QR)\beta) = (y^T - \beta^T R^T Q^T)(y - QR\beta) = y^T y - y^T QR\beta - \beta^T R^T Q^T y + \beta^T R^T Q^T QR\beta$$

$$\text{Since } (y^T QR\beta)^T = y^T QR\beta, \text{ and } Q^T Q = I, \varepsilon^T \varepsilon = y^T y - 2\beta^T R^T Q^T y + \beta^T R^T R\beta$$

$$\therefore \text{By the normal equation, } -2R^T Q^T y + 2R^T R\hat{\beta} = 0. \quad \therefore R^T R\hat{\beta} = R^T Q^T y$$

$$\therefore \hat{\beta} = (R^T R)^{-1} R^T Q^T y$$

By our assumption, X has full rank. $\therefore R$ is Invertible since there is no zero element on the diagonal.

$$\therefore (R^T R)^{-1} = R^{-1} (R^T)^{-1} \quad \therefore \hat{\beta} = R^{-1} (R^T)^{-1} R^T Q^T y = \underline{R^{-1} Q^T y}$$

$$\therefore \underline{\hat{y}} = \underline{X\hat{\beta}} = (QR)\hat{\beta} = (QR)(R^{-1} Q^T y) = Q(RR^{-1}) Q^T y = \underline{Q Q^T y}$$