0606

Wednesday, January 20, 2021 10:03 PM

6.6. Orthogonal Complement

- 추상적인 V.sp을 알기 쉬운 subspace 등으로 "분해" 해보자...

Def 298. V: inp$/\mathbb{R}$. $\quad S^\perp = \{u \in V : \underbrace{\langle u, s\rangle}_{} = 0,\ \forall s \in S\}$.

$S \subset V$ $(S \neq \emptyset)$.

↳ 어떤 inproduct 에 대해 $S^\perp$도 달라질 것!

Thm 299. $S^\perp$ is a vector space.

pf) HW. ① $0 \in S^\perp$ ② $u, v \in S^\perp \to u+v \in S^\perp$ ③ $u \in S^\perp,\ k \in \mathbb{R} \to ku \in S^\perp$

Ex. 301. $V = C[0,1]$, $L^2(0,1)$ inp.

$S = \{v \in V \mid \int_0^1 v(x)\,dx = 0\}$. mean-zero functions.

$S^\perp = ?$ 모든 S의 원소들과 직교하는 애들의 집합을 찾자.

↳ 이게 핵심.

sol) $S^\perp = \{u \in C[0,1] \mid \int_0^1 u(x)s(x)\,dx = 0\ \ \forall s \in S\}$

WTS $S^\perp = \{\text{constant functions}\}$, i.e. $u(x) = C$, $u \in S^\perp$.

($\Leftarrow$) Sps $u(x) = C$, $s \in S$. $\to \int_0^1 u(x)s(x)\,dx = C\int_0^1 s(x)\,dx = 0$. $\therefore u \in S^\perp$.

($\Rightarrow$) Sps $u \in S^\perp$.

Define $\bar{u} = \int_0^1 u(x)\,dx$ ... constant function. $\to \int_0^1 u(x) - \bar{u}\,dx = 0$. $\therefore u - \bar{u} \in S$.

$$\int_0^1 (u(x)-\bar{u})^2\,dx = \underbrace{\int_0^1 \underset{S^\perp}{u(x)}\,(\underset{S}{u(x)-\bar{u}})\,dx}_{=0} - \underbrace{\int \bar{u}\,(u(x)-\bar{u})\,dx}_{=\ \bar{u}\int u-\bar{u}\,dx\ =\ 0} = 0.$$

Since $(u(x)-\bar{u})^2$ is continuous, we must have $u(x) = \bar{u}$ ... const ft. $\square$.

Lemma 302 V: inp$/\mathbb{R}$. S, T orthogonal subspaces of V. $\to S \cap T = \{0\}$.

↳ $\forall s \in S, \forall t \in T$, $\langle s, t\rangle = 0$.

pf) Sps $x \in S$, $x \in T$ $\to \langle x, x\rangle = 0 \to x = 0$. $\square$.

Thm 303. V: inp$/\mathbb{R}$. fin dim. $S < V \longrightarrow (S^\perp)^\perp = S$.

↳ inf dim 이면 S closed 라는 조건이 있어야. 그래야 proj_S를 알 수 있으므로. (몰라도 ok)

pf) ($\Leftarrow$). Let $s \in S$, $y \in S^\perp \to \langle s, y\rangle = 0$, i.e. s is orthogonal to $y \in S^\perp$

$\to s \in (S^\perp)^\perp$. $\quad \therefore S \subset (S^\perp)^\perp$.

($\Rightarrow$) Let $x \in (S^\perp)^\perp$, $x_S = \mathrm{proj}_S x \in S \subset (S^\perp)^\perp$.

① $x_S = \mathrm{proj}_S x \to x_S - x \in S^\perp$

② $x \in (S^\perp)^\perp$, $x_S \in (S^\perp)^\perp \to x - x_S \in (S^\perp)^\perp$.

$\Big\}\to x_S = x \in S$. $\quad \therefore (S^\perp)^\perp \subset S$.

p.276.

Note: $S, T \subset V$ subspaces. Algebraic sum of S, T : $\underbrace{S+T}_{\text{also a subspace.}} = \{s+t \mid s \in S, t \in T\}$

① 여기서 만약 each $v \in V$ 가 $v = s+t$ 이 unique 하다면? Direct Sum.

② (Algebraic sum 에서) S, T가 서로 orthogonal subspaces 라면? $V = \underbrace{S \oplus T}_{\text{이 기호를 쓴다.}}$

Lemma 304. $S, T \subset V$. orthogonal subspaces. $\rightarrow$ $S \oplus T$ is a direct sum!

i.e. $v \in S \oplus T \rightarrow v = s+t$ ($s \in S$, $t \in T$ unique)

pf) Since $v \in S \oplus T$, $v = s+t$, for some $s \in S$, $t \in T$.

WTS such s, t is unique.

Sps $s' \in S$, $t' \in T$ s.t. $v = s'+t'$. $\rightarrow$ $s+t = s'+t'$ $\rightarrow$ $\underbrace{s-s'}_{\in S} = \underbrace{t'-t}_{\in T}$ $\rightarrow$ $t = t'$, $s = s'$ ($\because S \cap T = \{0\}$.) $\square$

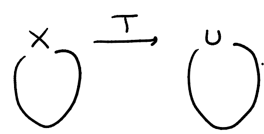
Lemma 305. $V = S \oplus T \rightarrow \dim(V) = \dim(S) + \dim(T)$.

Thm 306. V: inp/$\mathbb{R}$, fin.dim. $\rightarrow$ $V = S \oplus S^{\perp}$. ($S \subset V$ subspace)

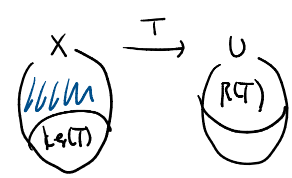
pf) HW. $v \in V \rightarrow v = \underbrace{v - \mathrm{proj}_S v}_{S^{\perp} \text{ proj. thm}} + \underbrace{\mathrm{proj}_S v}_{\in S}$.

Ex 307. HW.

6.6.1. Fundamental Theorem of Linear Algebra



타원에 $T: \underset{\substack{\text{fin} \\ \langle , \rangle}}{X} \xrightarrow{\text{linear}} \underset{\substack{\text{fin} \\ \langle , \rangle}}{U}$ 가 있을 거다...



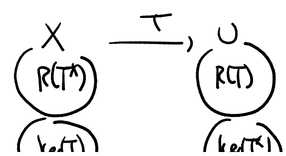
거기서 $R(T) \subset U$, $\ker(T) \subset X$ 라는 걸 그렸었다...

이 부분은 뭘까... 하고 봤더니

- $\ker(T) \cap R(T^*) = \{0\}$ 여기에 $R(T^*)$가 있겠구나...
- $\dim(X) - \mathrm{nullity}(T) = \mathrm{rank}(T)$
  Fund. Thm of L.A. (Thm 104)
- $\mathrm{rank}(T) = \mathrm{rank}(T^*)$
  Rank Theorem. (6.2)

$R(T^*)$로 잘 되겠구나...

이건 T^*에 대해서도 마찬가지이므로...



이렇구나...

이건 T^*에 대해서도 마찬가지니까... $R(T^*)$ $ker(T)$ $R(T)$ $ker(T^*)$ 이렇구나...

이제 우리는 $R(T^*)$ 와 $ker(T)$ 가 orthogonal subspaces 라는 것을 보일 거다...

Thrm. 308. $R(T^*)^{\perp} = ker(T)$. 즉 $R(T)^{\perp} = ker(T^*)$

pf) ($\Rightarrow$) $x \in R(T^*)^{\perp}$ $\rightarrow$ $\langle x, T^*(u)\rangle_X = 0$ $\forall u \in U$.

$\rightarrow$ $\langle T(x), u\rangle_U = 0$. $\forall u \in U$.

$\rightarrow$ $T(x) = 0$, $\rightarrow$ $x \in ker(T)$

($\Leftarrow$) $x \in ker(T)$.. Let $y \in R(T^*)$., then $\exists u \in U$ s.t. $T^*(u) = y$.

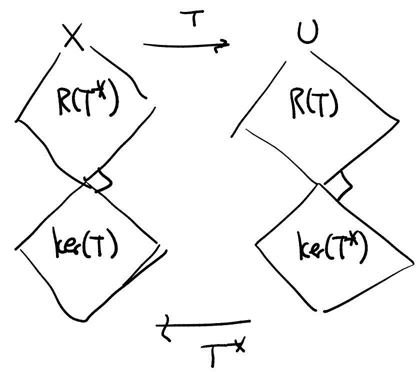
Then. $\langle x, y\rangle = \langle x, T^*(u)\rangle = \langle T(x), u\rangle = \langle 0, u\rangle = 0$.

Since this holds for any $y \in R(T^*)$, $x \in R(T^*)^{\perp}$ $\square$

우리는 드디어 Fundamental Subspaces of Linear Algebra 를 알게됐다!

linear

$T: X \rightarrow U$.
fin $\langle\cdot,\cdot\rangle$ fin $\langle\cdot,\cdot\rangle$

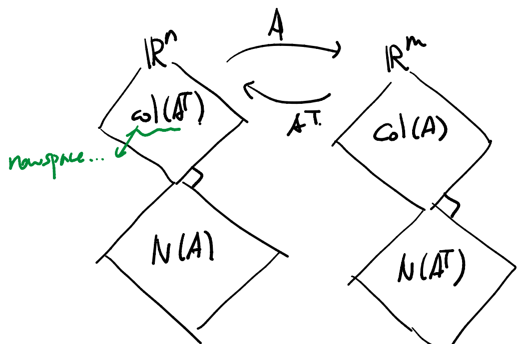


Conj 309
$\dim(X) = rank(T^*) + nullity(T)$ $\because$ orth subs
$= rank(T) + nullity(T)$ $\because$ F.D thrm
$\therefore rank(T) = rank(T^*)$

Conj 310.

만약 $T(x) = Ax$ 로 정의한다면?

($T: \mathbb{R}^n \rightarrow \mathbb{R}^m$, $A \in \mathbb{R}^{m \times n}$)
dot pdt dot pdt



Conj. 311. $rank(A) = rank(A^T)$
행렬에서 서로 독립인 열/행벡터 수는 같다...

Fundamental Subspaces 의 활용?

1. Compatibility Condition. $T(x) = u$ 에 solution 이 있는가?
$\Leftrightarrow$ $u \in R(T)$?

$Ax = y$ sol ?
$\Leftrightarrow$ $y \in col(A)$?
$\Leftrightarrow$ $y \notin null(A^T)$?

1. Compatibility condition.

$\Leftrightarrow \; u \in R(T)$?

$\Leftrightarrow \; u \notin \ker(T^*)$?

$\Leftrightarrow \; T^*(v) = 0 \longrightarrow \langle u, v \rangle = 0.$

$(v \in \ker(T^*))$

$\Leftrightarrow \; y \in col(A)$?

$\Leftrightarrow \; y \notin null(A^T)$?

$\Leftrightarrow \; A^T v = 0 \rightarrow v \cdot y = 0.$

2. $T(x) = y$, T singular. $\longrightarrow$ solution set $= \hat{x} + \ker(T)$

$\hat{x} \in X$.

Since $\hat{x} \in X$, $\hat{x} = \underset{\in R(T^*)}{\bar{x}} + \underset{\in \ker(T)}{z}$ uniquely.

Then $T(x) = y$ solution inf. many, but $\bar{x} \in R(T^*)$ is unique,

\+ minimum-norm LS solution.

$Ax = y$ 에서 unique minimum LS solution 인 $\bar{x}$을 찾는 연산을

$\bar{x} = A^+ y$ 라고 하며, A^+는 pseudo-inverse 라고 한다.

HW. 5, 6, 9.