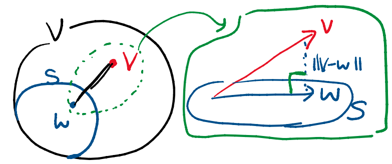
0604

Wednesday, January 20, 2021 3:35 PM

6.4. The Projection Theorem



- 대부분의 Approximation 문제는 ... $S \subset V$.

find $w \in S$ s.t. $\|v - w\| \le \|v - z\|$ $\forall z \in S$.

The best approximation w always exists, and it is characterized by the orthogonality condition as in

Thm 289 (The projection theorem) V: inp space / $\mathbb{R}$, $S \subset V$

$\|\cdot\|$: inp induced

fin dim

S가 fin dim 아니라도, V: Hilbert; S: closed subspace of V 이면 projection thm 성립

1. $\forall v \in V$, $\exists!\, w \in S$ s.t. $\|v-w\| = \min\{\|v-z\| \mid z \in S\}$

   Such w is called $w = \mathrm{proj}_S v$. ∘ best approximation to v from S.
   ∘ projection of v onto S.

1: 유일하게 있다!

2: 이 조건 만족하는 거다!

3: 그래서 구하는 것은 이거다!

2. $w = \mathrm{proj}_S v$ iff $\langle v-w, z\rangle = 0$ $\forall z \in S$.

3. $S = \mathrm{sp}\{u_1, u_2, \cdots, u_n\}$ $\rightarrow$ $\mathrm{proj}_S v = \sum_{i=1}^{n} x_i u_i$

   where $x = [x_1, \cdots, x_n]^T$ satisfies $Gx = b$,

   where $G_{ij} = \langle u_j, u_i\rangle$, $b_i = \langle v, u_i\rangle$.

pf) 2 증명 $\rightarrow$ 3 증명 $\rightarrow$ 1 증명.

Consider any fixed $w \in S$.

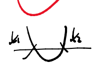
"Any other vector in S" can be written as $y = w + tz$ for some $t \in \mathbb{R}$, $z \in S$.

pf) $y \in S \Leftrightarrow y = w + tz$. ($\Rightarrow$) let $t = 1$, $z = y - w \in S$.
($\Leftarrow$) trivial.

any other vector in S.

$$\|v - (w+tz)\|^2 = \langle (v-w) - tz, (v-w) - tz\rangle$$
$$= \|v-w\|^2 + t^2\|z\|^2 - 2t\langle v-w, z\rangle$$
$$= \|v-w\|^2 + \phi(t).$$

어떻게 해야 $\phi(t) \ge 0$ $\forall t \in \mathbb{R}$? $\Leftrightarrow$ 어떻게 해야 $w = \mathrm{proj}_S v$. ?



$\phi(t)$에서 $t_1 = 0$, $t_2 = \dfrac{2\langle z, v-w\rangle}{\|y\|^2}$) $t_2 = t_1$ 이어야 $\rightarrow$ $\langle z, v-w\rangle = 0$.

$\therefore$ $\|v - (w+tz)\|^2 \ge \|v-w\|^2$ $\forall t \in \mathbb{R}$, $z \in S$. $\Leftrightarrow$ $\boxed{\langle z, v-w\rangle = 0 \ \forall z \in S}$ (2 증명)

이걸 이용해 $w = \mathrm{proj}_S v$ 찾는 방법?

- $\langle v-w, z\rangle = 0$ $\forall z \in S$. $\Leftrightarrow$ $\langle v-w, u_i\rangle = 0$ ($\mathrm{sp}\{u_1 \cdots u_n\} = S$.)
- Since $w \in S$, $w = \sum_{j=1}^{n} x_j u_j$. ($x = [w]_u$)

대입해, 해당 i에 대해서 $\langle v - \sum_{j=1}^{n} x_j u_j, u_i\rangle = 0$ $\Leftrightarrow$ $\sum_{j=1}^{n} x_j \langle u_j, u_i\rangle = \langle v, u_i\rangle$

그러면 $i=1,2,\cdots,n$ 에 대해서 $Gx=b. \to x=G^{-1}b.$

$(b_i = \langle v, u_i \rangle)$

$\therefore \ \mathrm{proj}_S v = \sum_{i=1}^{n} x_i u_i. \quad x = G^{-1}b.$ (unique) (3.1 참고). □ example 290 HW.

6.4.1. Overdetermined Linear Systems.

$Ax = y, \ A \in \mathbb{R}^{m\times n} \ (m>n)$ 인 경우.

$\to V = \mathbb{R}^m, \ S = \mathrm{col}(A) \subset \mathbb{R}^m,$ best approximation $w = Ax.$!!
$(y \in \mathbb{R}^m)$

Thm 291. $x \in \mathbb{R}^n = \arg\min_z \{\|Az-y\|_2 : z \in \mathbb{R}^n\} \iff A^TAx = A^Ty.$

pf). The best approximation Ax satisfies $(y-Ax)\circ z = 0, \ \forall z \in \mathrm{col}(A)$

Since $z \in \mathrm{col}(A)$, $z = Au$ for some $u \in \mathbb{R}^n$.

$\iff (y-Ax)\cdot Au = 0, \ \forall u \in \mathbb{R}^n.$

$\iff A^T(y-Ax)\circ u = 0 \ \forall u \in \mathbb{R}^n$

$\therefore A^Ty = A^TAx.$ □.

proof: Ex. 6.4.1

$\overset{m\times n}{A}$ full rank $(\mathrm{rank}(A) = n) \longrightarrow \overset{n\times n}{A^TA}$ non-singular. $\to x = (A^TA)^{-1}A^Ty$ unique. HW: 6.4의 4,5,6

but A^TA singular 이어도 $A^Ty = A^TAx$ 하는 (무수히 많이) 존재 (존재성은 projection theorem 에 의해 보장!!)

심화주제

1. Projection Operator.

V : Inp space $/\mathbb{R}$, fin dim, $S \subset V$. subspace.

Define $P : V \to V$ by $P(v) = \mathrm{proj}_S v. \ (v \in V)$

$\hookrightarrow$ orthogonal projection operator onto S.

$\iff$ for $v \in V$, $\langle v - P(v), s \rangle = 0 \ \forall s \in S.$

$\hookrightarrow$ v에 대해서 $P(v)$을 정의하는 식. (by projection theorem)
이를 이용해서 다음의 성질들을 보인다.

Ex 16.

① P is linear.

pf) i) $\alpha \in \mathbb{R}, \ P(\alpha v) = \alpha P(v)$?

$\langle v - P(v), s \rangle = 0 \ \forall s \in S. \ \to \langle \alpha v - \alpha P(v), s \rangle = 0 \ \forall s \in S.$

$\therefore P(\alpha v) = \alpha P(v).$

ii) $u, v \in V, \ P(u+v) = P(u) + P(v)$?

$\left.\begin{array}{l} \langle u - P(u), s \rangle = 0 \\ \langle v - P(v), s \rangle = 0 \end{array}\right\} \forall s \in S. \Rightarrow \langle u+v - P(u) - P(v), s \rangle = 0 \ \forall s \in S.$ □

② $P^2 = P$ (P is Idempotent) pf) $\|P(v) - u\|$ is minimum if $u = P(v)$ □
for $P(P(v))$, $(u \in S)$

$\hookrightarrow$ i.e. $P(P(v)) = P(v)$

③ $P^* = P$ self adjoint.

pf) Let $u, v \in V$. WTS $\langle u, P(v)\rangle = \langle P(u), v\rangle$. → 의미: 각도가 같다. 왜? $V = R(P) \oplus R(I-P)$ 이니까!

Since $\langle u - P(u), P(v)\rangle = 0 \rightarrow \langle u, P(v)\rangle = \langle P(u), P(v)\rangle$

$\langle v - P(v), P(u)\rangle = 0 \rightarrow \langle v, P(u)\rangle = \langle P(v), P(u)\rangle$ □.

위에서 보인 것은 fin dim inp space에서 orthogonal projection operator는 ① lin, ② idem, ③ SA.
반대로 ①, ②, ③을 만족하는 operator는 바로 OPO라는 것을 보일 수 있다.

Ex 17

pf) Suppose $P: V \to V$ ① linear ② idempotent ③ self-adjoint.

Define $S = R(P)$ so that $P: V \to S$.

WTS $P(v) = \mathrm{proj}_S v$ for $v \in V$.

$$\begin{aligned} \langle v - P(v), s\rangle &= \langle v - P(v), P(u)\rangle \quad \text{for some } u \in V. \\ (s \in S) \quad &= \langle P(v - P(v)), u\rangle \quad \because \text{self adj.} \\ &= \langle 0, u\rangle = 0. \quad \because \text{idempotent.} \end{aligned}$$

$\therefore P(v) = \mathrm{proj}_S v$ □

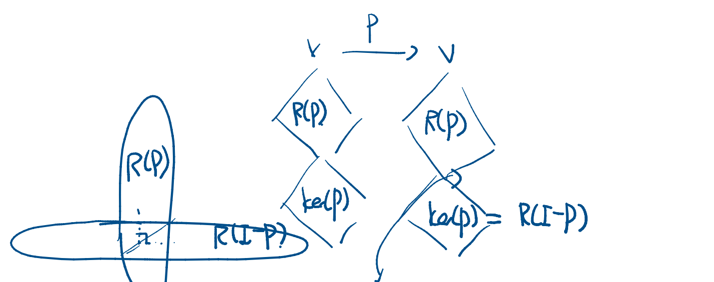
Ex 18.
그렇다면 OPO의 대표적인 예시는?

Define $P: \mathbb{R}^m \to \mathbb{R}^m$ s.t. $P(y) = A(A^TA)^{-1}A^Ty$. ($A \in \mathbb{R}^{m \times n}$ full rank)

P is OPO onto $\mathrm{col}(A)$.

pf) i) linear ii) idempotent iii) self-adjoint.

Ex 19. ③ 조건을 만족하면 "Orthogonal" PO. 조건을 만족하지 않으면 그냥 PO.



$\langle P(v), (I-P)(u)\rangle = \langle P(v), u - P(u)\rangle = 0$.
for any $u, v \in V$.

Ex 19. i) P is OPO onto $R(P)$ → $(I-P)$ is OPO onto $R(I-P)$.

ii) $\ker(P) = R(I-P)$

iii) $V = R(P) \oplus R(I-P)$ each $v \in V$, $v = s + t$ uniquely. (s: $R(P)$, t: $R(I-P)$)

pf) i) Sps P ① linear ② idempotent ③ SA

WTS $I-P$ ①, ②, ③.

① trivial.

② $$\begin{aligned} (I-P)^2(v) &= (I-P)((I-P)(v)) = (I-P)(v) - P((I-P)(v)) \\ (v \in V) \quad &= v - P(v) - P(v - P(v)) = v - P(v) = (I-P)(v). \end{aligned}$$

③ WTS $\langle (I-P)(u), v\rangle = \langle u, (I-P)(v)\rangle$.

$$\begin{aligned} \langle (I-P)(u), v\rangle = \langle u - P(u), v\rangle &= \langle u, v\rangle - \langle P(u), v\rangle \\ &= \langle u, v\rangle - \langle u, P(v)\rangle \\ &= \langle u, (I-P)(v)\rangle. \quad □ \end{aligned}$$

ii) WTS $\ker(P) = R(I-P)$

($\Rightarrow$) $v \in \ker(P) \rightarrow P(v)=0 \rightarrow v-P(v)=v \rightarrow (I-P)(v)=v \rightarrow v \in R(I-P)$

($\Leftarrow$) $v \in R(I-P) \rightarrow \underset{\text{for some } u\in V}{v=(I-P)(u)} \rightarrow v = u-P(u) \rightarrow P(v) = P(u)-P(u) = 0. \rightarrow v \in \ker(P)$

iii) $V = R(P) \oplus R(I-P)$

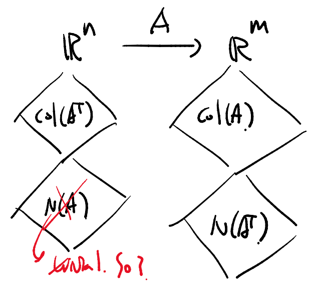
for any $v \in V$, $v = v-P(v)+P(v) = \underbrace{(I-P)(v)+P(v)}_{\text{each unique!}}$ $\square$

EX 20. $P: \mathbb{R}^m \to \mathbb{R}^m$, $P(y) = A(A^TA)^{-1}A^Ty$. ($A \in \mathbb{R}^{m\times n}$ full rank)

projection operator onto $\mathrm{col}(A)$

$(I-P): \mathbb{R}^m \to \mathbb{R}^m$, $(I-P)(y) = (I - A(A^TA)^{-1}A^T)y$.

projection operator onto $\mathrm{col}(A)^{\perp} = N(A^T)$



(a), (b), (c) HW.