Canonical Variable & Canonical Correlation 계산 방법

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symmetric. 컴퓨터에 적합

	ith eigenvalue	ith eigenvector	Canonical Correlation	Canonical coefficient	canonical variable
$\Sigma_{11}^{-1/2}\Sigma_{12}\Sigma_{22}^{-1}\Sigma_{21}\Sigma_{11}^{-1/2}$	λ_i	e_i	$\sqrt{\lambda_i}=\rho_i^*$	$a_i=\Sigma_{11}^{-1/2}e_i$	$U_i=e_i^T\Sigma_{11}^{-1/2}X^{(1)}$
$\Sigma_{22}^{-1/2}\Sigma_{21}\Sigma_{11}^{-1}\Sigma_{12}\Sigma_{22}^{-1/2}$	λ_i	f_i	$\sqrt{\lambda_i}=\rho_i^*$	$b_i=\Sigma_{22}^{-1/2}f_i$	$V_i=f_i^T\Sigma_{22}^{-1/2}X^{(2)}$
$\rho_{11}^{-1/2}\rho_{12}\rho_{22}^{-1}\rho_{21}\rho_{11}^{-1/2}$	λ_i	e_i^*	$\sqrt{\lambda_i}=\rho_i^*$	$a_i^*=\rho_{11}^{-1/2}e_i^*$	$U_i=e_i^{*T}\rho_{11}^{-1/2}Z^{(1)}$
$\rho_{22}^{-1/2}\rho_{21}\rho_{11}^{-1}\rho_{12}\rho_{22}^{-1/2}$	λ_i	f_i^*	$\sqrt{\lambda_i}=\rho_i^*$	$b_i^*=\rho_{22}^{-1/2}f_i^*$	$V_i=f_i^{*T}\rho_{22}^{-1/2}Z^{(2)}$
$\Sigma_{11}^{-1}\Sigma_{12}\Sigma_{22}^{-1}\Sigma_{21}$	λ_i	a_i	$\sqrt{\lambda_i}=\rho_i^*$	a_i	$U_i=a_i^TX^{(1)}$
$\Sigma_{22}^{-1}\Sigma_{21}\Sigma_{11}^{-1}\Sigma_{12}$	λ_i	b_i	$\sqrt{\lambda_i}=\rho_i^*$	b_i	$V_i=b_i^TX^{(2)}$
$\rho_{11}^{-1}\rho_{12}\rho_{22}^{-1}\rho_{21}$	λ_i	a_i^*	$\sqrt{\lambda_i}=\rho_i^*$	a_i^*	$U_i=a_i^{*T}Z^{(1)}$
$\rho_{22}^{-1}\rho_{21}\rho_{11}^{-1}\rho_{12}$	λ_i	b_i^*	$\sqrt{\lambda_i}=\rho_i^*$	b_i^*	$V_i=b_i^{*T}Z^{(2)}$

계산량↓

e_i 가 구해졌을 때 f_i 구하는법 : $f_i \propto \rho_{22}^{-1/2} \rho_{21} \rho_{11}^{-1/2} e_i$ 활용 (발표자료 page 20)

$$e_1 = \begin{bmatrix} 0.8447 \\ 0.4466 \end{bmatrix}$$

$$\rho_{22} = \begin{bmatrix} 1.0 & 0.2 \\ 0.2 & 1.0 \end{bmatrix} \quad \rho_{21} = \begin{bmatrix} 0.5 & 0.3 \\ 0.6 & 0.4 \end{bmatrix} \quad \rho_{11} = \begin{bmatrix} 1.0 & 0.4 \\ 0.4 & 1.0 \end{bmatrix}$$

Let $f_1 = c \cdot \rho_{22}^{-1/2} \rho_{21} \rho_{11}^{-1/2} e_1$

Then $b_1 = \rho_{22}^{-1/2} f_1 = c \, \rho_{22}^{-1} \rho_{21} \rho_{11}^{-1/2} e_1$

$$\rho_{22}^{-1} \rho_{21} \rho_{11}^{-1/2} e_1 = \begin{bmatrix} 0.4026 \\ 0.5443 \end{bmatrix}$$

$$b_1 = c \begin{bmatrix} 0.4026 \\ 0.5443 \end{bmatrix}$$

$$V_1 = b_1^T z^{(2)}$$

$$Var(b_1^T z^{(2)}) = c \begin{bmatrix} 0.4026 & 0.5443 \end{bmatrix} \rho_{22} \, c \begin{bmatrix} 0.4026 \\ 0.5443 \end{bmatrix}$$

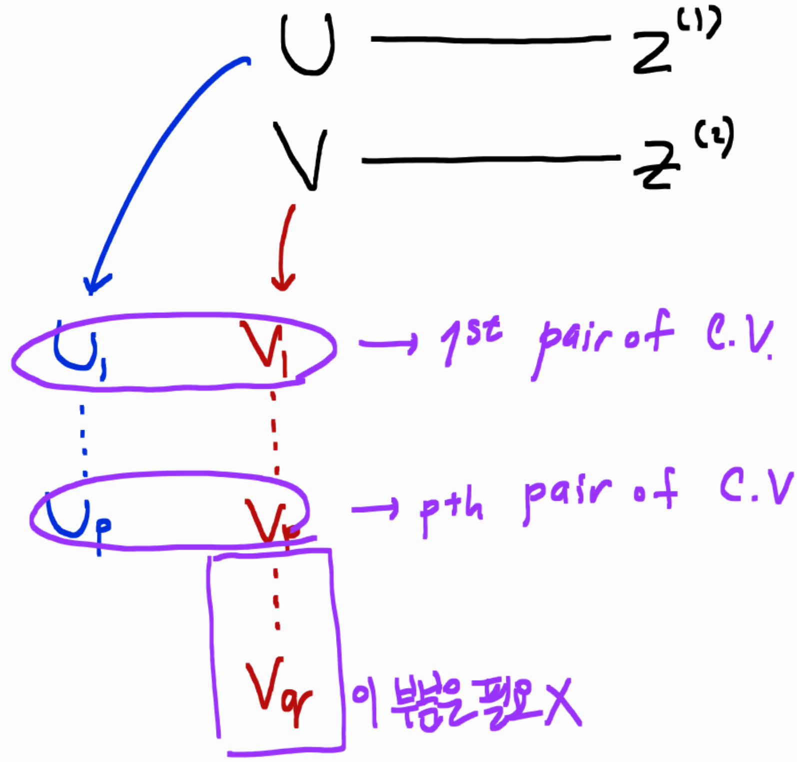
$$= c^2 \begin{bmatrix} 0.4026 & 0.5443 \end{bmatrix} \begin{bmatrix} 1 & 0.2 \\ 0.2 & 1 \end{bmatrix} \begin{bmatrix} 0.4026 \\ 0.5443 \end{bmatrix}$$

$$= c^2 \times \frac{1}{0.5460} = 1$$

$$\therefore c = \frac{1}{0.7389}$$

Canonical Variable에 대한 해석 (발표자료 page 27)

· Canonical Variables 와 그것들의 component 간의 상관관계를 파악



알고싶은건 $\mathrm{Corr}(\underset{(p\times1)}{U}, \underset{(p\times1)}{Z^{(1)}})$ & $\mathrm{Corr}(\underset{(q\times1)}{V}, \underset{(q\times1)}{Z^{(2)}})$

$$\mathrm{Corr}(U, Z^{(1)}) = \frac{\mathrm{Cov}(U, Z)}{\sqrt{\mathrm{Cov}(U)}\sqrt{\mathrm{Cov}(Z)}}$$

→ 이부분 계산이 애매..
그래서 다르게 접근

p×p matrix $\mathrm{Corr}(U, Z^{(1)})$의 i행 j열의 원소

$$= \mathrm{Corr}(U_i, Z_j^{(1)}) = \frac{\mathrm{Cov}(U_i, Z_j^{(1)})}{\sqrt{\underbrace{\mathrm{Var}(U_i)}_{\text{unit variance}}\ \underbrace{\mathrm{Var}(Z_j^{(1)})}_{\text{standardized}}}}$$

$$= \frac{\mathrm{Cov}(U_i, Z_j^{(1)})}{1\cdot 1} = \mathrm{Cov}(U_i, Z_j^{(1)})$$

$$\therefore \mathrm{Corr}(U, Z^{(1)}) = \mathrm{Cov}(U, Z^{(1)}) = \mathrm{Cov}(AX^{(1)}, V_{11}^{-1/2}X^{(1)})$$

$$= A\Sigma_{11}V_{11}^{-1/2}$$

$$\underbrace{A V_{11}^{1/2}}\ \underbrace{V_{11}^{-1/2}\Sigma_{11}V_{11}^{-1/2}}_{\rho_{11}} = A_z \rho_{11}$$

$$\begin{bmatrix} -a_1^T- \\ -a_2^T- \\ \vdots \\ -a_p^T- \end{bmatrix} V_{11}^{1/2} = \begin{bmatrix} -a_1^T V_{11}^{1/2}- \\ -a_2^T V_{11}^{1/2}- \\ \vdots \\ -a_p^T V_{11}^{1/2}- \end{bmatrix}$$

각 열벡터들은 standardized variables로부터 canonical variables를 구할때 사용되는 coefficient vector

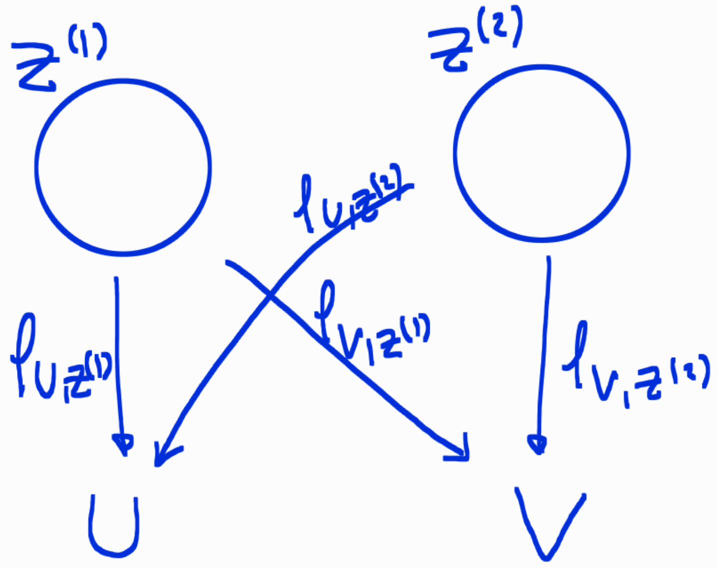
그래서 정리하면

$$\rho_{U,z^{(1)}} = A\Sigma_{11}V_{11}^{-1/2} = A_z\rho_{11}$$

$$\rho_{V,z^{(2)}} = B\Sigma_{22}V_{22}^{-1/2} = B_z\rho_{22}$$

$$\rho_{U,z^{(2)}} = A\Sigma_{12}V_{22}^{-1/2} = A_z\rho_{12}$$

$$\rho_{V,z^{(1)}} = B\Sigma_{21}V_{11}^{-1/2} = B_z\rho_{21}$$



The first canonical correlation is larger than the absolute value of any entry in ρ_{12} (발표자료 page 31)

$p\times q$ matrix ρ_{12}의 i행 j열의 원소

$$= Corr(X_i^{(1)}, X_j^{(2)})$$

이 또한 $Corr(\vec{a}^T \mathbb{X}^{(1)}, \vec{b}^T \mathbb{X}^{(2)})$의 한 형태임

모든 선형결합의 correlation 중에

$Corr(\vec{a}_1^T \mathbb{X}^{(1)}, \vec{b}^T \mathbb{X}^{(2)}) = \rho_1^*$가 가장 크므로

$$Corr(X_i^{(1)}, X_j^{(2)}) \le \rho_1^*$$

즉 ρ_1^*은 ρ_{12}의 원소들에 대한 upper bound를 제공한다.