Duality 2
(Boyd. Ch5.7~5.9)

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5.7.1 Introducing new variables and equality constraints

minimize $\quad f_0(Ax+b)$

dual function is constant: $g = \inf_x L(x) = \inf_x f_0(Ax+b) = p^\star$

d^*가 p^*와 같음을 쉽게 알 수 있지만 그 이상으로는 유용한 뭔가가 없다...!

dual functional $g(x^*)$

$$\begin{array}{ll} \text{minimize} & f_0(y) \\ \text{subject to} & Ax + b = y. \end{array}$$

Y=ax+b로 문제를 변환해본다

$$L(x, y, \nu) = f_0(y) + \nu^T (Ax + b - y).$$

Lagrangian은 equality constraint을 가진 형태가 된다.

$$g(v) = \inf_{x,y} \left(f_0(y) + A^T v x + b^T v - v^T y \right)$$

우선, $A^T\nu$ 가 0이 아니면 , dual function g의 값은 -oo 가 되므로, (Unbounded 되어버린다!)

$A^T\nu = 0$을 위의 식에 대입하면,

$$g(\nu) = b^T \nu + \inf_y (f_0(y) - \nu^T y) = b^T \nu - f_0^*(\nu).$$

3.3절 Conjugate

$$\sup\left(y^T v - f_0(y)\right) = f_0^*(v)$$

$$* \inf_x (f(x)) = -\sup_x (-f(x))$$

Dual problem is, maximize g subject to $A^T\nu$ =0

이러한 변형을 많은 경우에서 유용하게 사용할 수 있다.

Example : Unconstrained geometric program

$$\text{minimize} \quad \log\left(\sum_{i=1}^m \exp(a_i^T x + b_i)\right)$$

1. Reformulate

$$\begin{array}{ll} \text{minimize} & f_0(y) = \log\left(\sum_{i=1}^m \exp y_i\right) \\ \text{subject to} & Ax + b = y, \end{array}$$

equal to $= -f_0^*(v)$

2. Lagrangian

$$L = f_0(y) + v^T(Ax + b - y)$$

$$g = b^T v + \inf_y \left(f_0(y) - v^T y \right) + \underbrace{A^T v x}$$

$A^T v$ should be 0

3. conjugate(page 93)

$$f_0^*(\nu) = \begin{cases} \sum_{i=1}^m \nu_i \log \nu_i & \nu \succeq 0,\ \mathbf{1}^T \nu = 1 \\ \infty & \text{otherwise} \end{cases}$$

4. dual function g(v)

$$b^T \nu - \sum_{i=1}^m \nu_i \log \nu_i$$

dual problem is...

Example2 : Reformulation can be applied to constraints function as well

$$\begin{array}{ll} \text{minimize} & f_0(A_0x + b_0) \\ \text{subject to} & f_i(A_ix + b_i) \le 0, \quad i = 1, \dots, m, \end{array}$$

$y_i = A_i X + b_i$

$$L(x, y_0, \dots, y_m, \lambda, \nu_0, \dots, \nu_m) = f_0(y_0) + \sum_{i=1}^m \lambda_i f_i(y_i) + \sum_{i=0}^m \nu_i^T (A_ix + b_i - y_i).$$

if $\lambda_i = 0$, $\nu_i \neq 0$, $g = -\infty$

elif $\lambda_i = 0$, $\nu_i = 0$

$L = f_0(y_0)$ so g is still valid

for $\lambda \succ 0$, $g(\lambda, \nu_0, \dots, \nu_m)$, $\sum A_i^T \nu_i = 0$

$$\begin{aligned} &= \sum_{i=0}^m \nu_i^T b_i + \inf_{y_0,\dots,y_m} \left(f_0(y_0) + \sum_{i=1}^m \lambda_i f_i(y_i) - \sum_{i=0}^m \nu_i^T y_i \right) \\ &= \sum_{i=0}^m \nu_i^T b_i + \inf_{y_0} \left(f_0(y_0) - \nu_0^T y_0 \right) + \sum_{i=1}^m \lambda_i \inf_{y_i} \left(f_i(y_i) - (\nu_i/\lambda_i)^T y_i \right) \\ &= \sum_{i=0}^m \nu_i^T b_i - f_0^*(\nu_0) - \sum_{i=1}^m \lambda_i f_i^*(\nu_i/\lambda_i). \end{aligned}$$

$\lambda_i \ge 0$ 에서도 성립하다.

5.7.2

Transforming the objective

$$\text{minimize} \quad \|Ax - b\|, \quad \Longrightarrow \quad \begin{array}{ll} \text{minimize} & (1/2)\|y\|^2 \\ \text{subject to} & Ax - b = y. \end{array}$$

이렇게 변환을 해도 문제는 같지만,
dual problem의 형태는 바뀐다.

$$L = \frac{1}{2}\|y\|^2 + v^T(y - Ax + b)$$

$$g = \inf_{x,y} \left(\frac{1}{2}\|y\|^2 + v^T y - v^T A x + b^T v \right)$$

$$= b^T v + \inf_y \left(\frac{1}{2}\|y\|^2 + v^T y \right) \qquad A^T v = 0$$

$$= -\infty \qquad \text{O.W}$$

$$\inf_y \left(\frac{1}{2}\|y\|^2 + v^T y\right) = -\sup_y \left(-v^T y - \frac{1}{2}\|y\|^2\right) = -f^*(v)$$

라그랑지안을 세울 때,
v 대신 $-v$를 써도 무관하다. ($v \neq 0$ 이면되는 Equality Constraint)

2nd Conjugate of $\frac{1}{2}\|\cdot\|^2 = \frac{1}{2}\|\cdot\|_*^2$

Definition
(dual Norm
$\|v\|_* = \sup\{v^T x \mid \|x\| \le 1\}$)

We can write dual problem as

maximize $-\frac{1}{2}\|v\|_*^2 + b^T v$

subject to $A^T v = 0$

(page 255) 참고

$\|Ax-b\|$의 dual problem과 형태가 다르다. (제곱텀에 의해 차이 발생)

$L: \quad \|y\| + v^T(b+y-Ax)$

$g: \quad \inf_y (\|y\| + v^T y) + v^T b - \inf_x (A^T v x)$

$A^T v = 0$

$= b^T v - \sup_y (-y^T v - \|y\|)$

$= b^T v - f^*(v)$ where $f^*(v) = 0 \quad (\|v\|_* \le 1)$
$= \infty \quad$ o.w

So dual problem g is maximize $b^T v$

subject to $f^*(v) = 0$ (or $\|v\|_* \le 1$)

$A^T v = 0$

5.7.3

Implicit constraints

$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & Ax = b \\ & l \preceq x \preceq u \end{array}$$

방법 1

L = $c^T(x) + v^T(Ax - b) + \lambda_1(x - l) - \lambda_2(x - u)$

x가 affine하면 상한, 하한이 존재하지 않으므로

$$A^T V + \lambda_1 - \lambda_2 + c = 0$$

$$\lambda_1, \lambda_2 \geq 0$$

방법 2(문제를 이렇게 바꿔도 무방하다는 아이디어!)

$$f_0(x) = \begin{cases} c^T x & l \preceq x \preceq u \\ \infty & \text{otherwise.} \end{cases}$$

$$\begin{aligned} g(\nu) &= \inf_{l \preceq x \preceq u} \left(c^T x + \nu^T (Ax - b)\right) \\ &= -b^T \nu - u^T (A^T \nu + c)^- + l^T (A^T \nu + c)^+ \end{aligned}$$

rewrite here

where $y_i^+ = \max\{y_i, 0\}$, $y_i^- = \max\{-y_i, 0\}$

$$g = \inf_{l \le x \le u} \left(-b^T v - (A^T v + c)x\right)$$

$(A^T v + c)x$ is affine,

so, achieve infimum at l or u
(최대하게)

if $A^T v + c > 0$, x는 작아야함 $\Rightarrow x = l$

$A^T v + c < 0$, x는 커야함 $\Rightarrow x = u$

e.g. $\inf_{1 \le x \le 2} (3 + ax)$ if 기울기 $a = -2 < 0$, x는 2에서 최대하게 -1을 갖는다.

if 기울기 $a = 1 > 0$, x는 1에서 최대하게 4를 갖는다.

5.8.1

Weak alternatives via the dual function

$\mathcal{D}: \bigcap_{i=1}^{m} \mathrm{dom}\, f_i \cap \bigcap_{i=1}^{p} \mathrm{dom}\, h_i \neq \emptyset$

$$f_i(x) \le 0, \quad i = 1, \dots, m, \qquad h_i(x) = 0, \quad i = 1, \dots, p. \qquad \text{(A)}$$

위의 식처럼 제약을 설정할 때, constraint f와 h를 만족하는 영역이 항상 존재한다고 가정하였다. 그리고 이는 아래와 같은 최적화문제로 재정의할 수 있다.

$$\begin{array}{ll} \text{minimize} & 0 \\ \text{subject to} & f_i(x) \le 0, \quad i = 1, \dots, m \\ & h_i(x) = 0, \quad i = 1, \dots, p. \end{array}$$

$$g(\lambda, \nu) = \inf_{x \in \mathcal{D}} \left(\sum_{i=1}^{m} \lambda_i f_i(x) + \sum_{i=1}^{p} \nu_i h_i(x) \right),$$

$$p^\star = \begin{cases} 0 & \text{If} \quad \begin{array}{ll} f_i(x) \le 0, & i=1,\dots,m \\ h_i(x) = 0, & i=1,\dots,p. \end{array} \quad \text{is feasible} \\ \infty & \text{If} \quad \begin{array}{ll} f_i(x) \le 0, & i=1,\dots,m \\ h_i(x) = 0, & i=1,\dots,p. \end{array} \quad \text{is not feasible} \end{cases}$$

$$d^\star = \begin{cases} \infty & \lambda \succeq 0,\ g(\lambda,\nu) > 0 \text{ is feasible} \\ 0 & \lambda \succeq 0,\ g(\lambda,\nu) > 0 \text{ is infeasible.} \end{cases}$$

* positive homogeneous g

If $g(\lambda,\nu) > 0$ is feasible, $\sum \lambda_i f_i(x) > 0$

as $\lambda \to \infty$, $\sum \lambda_i f_i(x) \to \infty$ so infimum is ∞.
(최소상계)

Combining these result says that if $\lambda \succeq 0, \quad g(\lambda,\nu) > 0$ (B)

is feasible, (A) is not feasible

1. (B)가 feasible 하면, d* 는 무한대로 정의된다.
2. Weak duality 에 의해 p* 역시 무한대로 정의된다.
3. p*가 무한대인 경우는 (A) 가 not feasible할 때이다.

반대로 (A) 가 feasible 하면,
$p^* = 0$ 이고 Weak duality에 의해 $d^* = 0$이다.
그리고 이것은 (B)가 infeasible 할때이다.

\# 이 때 (A)와 (B)식을 **weak alternatives** 라고 부르고, 최대 하나의 식만 feasible하다는 특징을 가지고 있습니다. 따라서, (A)가 feasible 하다면 우리는 (B)의 infeasibility를 proof 혹은 certificate 한다고 할 수 있습니다.(역도 성립)

Strict inequalities(등호가 빠진 상황)

$$f_i(x) < 0, \quad i = 1, \ldots, m, \qquad h_i(x) = 0, \quad i = 1, \ldots, p.$$

Alternative inequality is $\quad \lambda \succeq 0, \qquad \lambda \neq 0, \qquad g(\lambda, \nu) \geq 0.$

Proof that both are weak alternatives

첫번째 식이 hold한다고 가정하고, lambda 역시 조건을 만족한다고 가정한다면,

$$L = \lambda_1 f_1(\tilde{x}) + \cdots + \lambda_m f_m(\tilde{x}) + \nu_1 h_1(\tilde{x}) + \cdots + \nu_p h_p(\tilde{x}) < 0.$$ 이 항상 성립한다.

하지만,
$$\begin{aligned} g(\lambda, \nu) &= \inf_{x \in \mathcal{D}} \left(\sum_{i=1}^{m} \lambda_i f_i(x) + \sum_{i=1}^{p} \nu_i h_i(x) \right) \\ &\leq \sum_{i=1}^{m} \lambda_i f_i(\tilde{x}) + \sum_{i=1}^{p} \nu_i h_i(\tilde{x}) \\ &< 0. \end{aligned}$$

for any $\tilde{x}$,

$\lambda_i < 0$ for some i, $f_i(x) < 0$

Which is Contradict to $g(\lambda, \nu) \geq 0$

5.8.2

Strong alternatives

i) Strict inequality에서의 strong alternative

조건1 f_i are convex

조건2 h_i are affine

조건3 $x \in relint\ D$ with $Ax = b$($Ax = b$의 해가 D space(보통 R^n)에 있다)

→ Slater's Condition

$\Updownarrow$

$d^* = p^*$

조건1~3을 추가적으로 만족하면, 아래의 두식은 **strong alternative**를 만족한다.

$$f_i(x) < 0, \quad i = 1, \dots, m, \qquad Ax = b, \qquad \text{(C)}$$

$$\lambda \succeq 0, \qquad \lambda \neq 0, \qquad g(\lambda, \nu) \geq 0. \qquad \text{(D)}$$

proof

$$\begin{array}{ll} \text{minimize} & s \\ \text{subject to} & f_i(x) - s \le 0, \quad i = 1, \dots, m \\ & Ax = b \end{array}$$

1. subjection을 만족하지 못한다면((C)가 infeasible) → p^*는 positive하다.

$\left(\begin{array}{l} i=1,\dots,m \\ m=1\text{일때} \end{array}\right)$ 예시로, $f_i(x)$=1이라면(infeasible), s는 1이 된다.

반대로, $f_i(x)$ = -1 이면(feasible), s는 -1이 된다.

따라서 p^*가 negative이다 는 위의 solution이 존재할 때((C) is feasible)과 동치이다.(iff 관계)

반대로, $p^* \geq 0$ 이기 위해서는 위의 solution이 존재하지 않아야 한다.((C) is infeasible)(iff 관계)

dual function is defined as

$$\inf_{x \in \mathcal{D},\, s} \left(s + \sum_{i=1}^{m} \lambda_i (f_i(x) - s) + \nu^T (Ax - b) \right) = \begin{cases} g(\lambda, \nu) & \mathbf{1}^T \lambda = 1 \\ -\infty & \text{otherwise.} \end{cases}$$

s의 기울기 $\left(1 - \sum_{i=1}^{m} \lambda_i\right) = 0$

$\sum \lambda_i = 1 \iff \mathbf{1}^T \lambda = 1$

so, the dual problem is expressed like

$$\begin{array}{ll} \text{maximize} & g(\lambda, \nu) \\ \text{subject to} & \lambda \succeq 0, \quad \mathbf{1}^T \lambda = 1. \end{array}$$

2. 조건1, 2,3 에 의해 convexity와 Ax=b의 해가 D 내부에 존재함을 알 수 있고, 이는 Slater's condition을 만족한다. 즉 d*=p*인 strong duality를 보장받는다. 따라서,

$$d^* = g(\lambda^\star, \nu^\star) = p^\star, \qquad \lambda^\star \succeq 0, \qquad \mathbf{1}^T\lambda^\star = 1.$$

를 만족하는 λ^*, v^*가 존재하게 된다.

$1^T\lambda^* = 1$ is equal to $\lambda_1 + \cdots + \lambda_n = 1$ so $\lambda_1, \ldots, \lambda_n$ 이 모두 0일수는 없다.

대입

3. 1번에서 알수 있듯이 p*≥0 이면, 2번의 식을 $g \geq 0, \lambda^* \geq 0, \lambda^* \neq 0$ 으로 변경할 수 있고, 이는 (D)의 조건과 동일하다.(즉, (C)가 infeasible 이면 (D)는 feasible)

동치

반대로 (D)가 feasible하다면, $d^* \geq 0$ 이므로 $p^* \geq 0$ 이다.(strong duality, p*=d*)

동치

따라서, 1번에서 알 수 있듯, (C)는 infeasible하다.

$$f_i(x) < 0, \quad i = 1, \ldots, m, \qquad Ax = b, \qquad \text{(C)}$$

$$\lambda \succeq 0, \qquad \lambda \neq 0, \qquad g(\lambda, \nu) \geq 0. \qquad \text{(D)}$$

ii) Nonstrict inequalities에서의 strong alternative

inequality system

$$f_i(x) \leq 0, \quad i = 1, \ldots, m, \qquad Ax = b,$$

alternative

$$\lambda \succeq 0, \qquad g(\lambda, \nu) > 0.$$

두 식은 strong alternative하다.

5.8.3

Examples

We consider m ellipsoids, described as

$$\mathcal{E}_i = \{x \mid f_i(x) \le 0\},$$

with $f_i(x) = x^T A_i x + 2b_i^T x + c_i$, $i = 1, \ldots, m$, where $A_i \in \mathbf{S}^n_{++}$. We ask when the intersection of these ellipsoids has nonempty interior. This is equivalent to feasibility of the set of strict quadratic inequalities

$$f_i(x) = x^T A_i x + 2b_i^T x + c_i < 0, \quad i = 1, \ldots, m. \tag{5.85}$$

The dual function g is

$$\begin{aligned} g(\lambda) &= \inf_x \left(x^T A(\lambda) x + 2b(\lambda)^T x + c(\lambda) \right) \\ &= \begin{cases} -b(\lambda)^T A(\lambda)^\dagger b(\lambda) + c(\lambda) & A(\lambda) \succeq 0, \quad b(\lambda) \in \mathcal{R}(A(\lambda)) \\ -\infty & \text{otherwise,} \end{cases} \end{aligned}$$

where

$$A(\lambda) = \sum_{i=1}^m \lambda_i A_i, \qquad b(\lambda) = \sum_{i=1}^m \lambda_i b_i, \qquad c(\lambda) = \sum_{i=1}^m \lambda_i c_i.$$

조건 f_i : convex

h_i (제약 없음)

$x \in \text{relint}\, D \,(= S_{++}^n)$ with h_i

By, Strong alternatives Theorem, we can simply write that m intersections of ellipsoids are ~~nonempty~~ empty if and only if …

$g(\lambda) \ge 0, \quad \lambda \succeq 0, \quad \lambda \ne 0$

$\Leftrightarrow \; -b(\lambda)^T A(\lambda)^\dagger b(\lambda) + c(\lambda) \ge 0$

$\lambda \succeq 0, \; \lambda \ne 0$

따라서 위 조건을 만족하지 않으면

nonempty 라고 할수있다.

$$g(\lambda) = \inf_x \left(\underbrace{x^T A(\lambda) x + 2b(\lambda)^T x}_{\text{stationarity}} + c(\lambda) \right)$$

미분 $$2A(\lambda)x + 2b(\lambda) = 0$$

$$x^* = -A(\lambda)^+ b(\lambda)$$

x^* 대입 $$g(\lambda) = -\left[-A(\lambda)^+ b(\lambda)\right]^T \underbrace{A(\lambda) A(\lambda)^+}_{1} b(\lambda) - 2b(\lambda)^T A(\lambda)^+ b(\lambda) + c(\lambda)$$

$$= b(\lambda)^T A(\lambda)^+ b(\lambda) - 2b(\lambda)^T A(\lambda)^+ b(\lambda) + c(\lambda)$$

$$= -b(\lambda)^T A(\lambda)^+ b(\lambda) + c(\lambda)$$

(If exists X^*, We can define $A(\lambda)^{-1}$.
But $A(\lambda)^{-1}$ might not exist (Not square matrix or X^* Not exist)
However, infimum of L can still exist. $\Rightarrow$ $A(\lambda)^+$ (pseudo Inverse)로 표현

Farkas' lemma

iii) Strict + nonstrict inequalities and strong alternatives

$$Ax \preceq 0, \qquad c^T x < 0,$$

$$A^T y + c = 0, \qquad y \succeq 0,$$

두 식은 strong alternative하다.

경제학에서의 응용

- p를 투자당시 가격, x를 투자자의 포트폴리오 목록이라고 정의합니다.

(x_1이 -2면, p_1 자산에 대해 2개만큼의 short을 선언한 것으로 정의합니다.)

- 그리고 $v^T x$ 를 특정 시점 이후의 자산가치라고 정의합니다.
- 포트폴리오를 구성하기 위해서, 초기 비용이 발생하므로, 이를 $p^T x < 0$이라고 설정할 수 있습니다.

모든 상황에서 특정 시점 이후의 자산가치가 늘어나는 경우는 일반적으로 불가능하다고 가정하므로(no Arbitrage 가정) 이를 inequality system으로 간략히 표현할 수 있습니다.

$$\underbrace{Vx \succeq 0, \qquad p^T x < 0}_{\leftarrow \text{ infeasible}} \xrightarrow{\text{Farkas' lemma}} -V^T y + p = 0, \qquad y \succeq 0.$$

그리고 위를 활용하여, no arbitrage 라는 제약을 만족하는, 특정 상품의 가격 p_n의 상한, 혹은 하한을 제시할 수 있습니다.

$$\begin{array}{ll} \text{minimize} & p_n \\ \text{subject to} & V^T y = p, \quad y \succeq 0, \end{array}$$

ESC 2023 WINTER WEEK5
Duality: Generalized version

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Something to Recall

A cone $K \subseteq \mathbf{R}^n$ is called a *proper cone* if it satisfies the following:

- K is convex.
- K is closed.
- K is *solid*, which means it has nonempty interior.
- K is *pointed*, which means that it contains no line (or equivalently, $x \in K,\ -x \in K \implies x = 0$).

A proper cone K can be used to define a *generalized inequality*, which is a partial ordering on $\mathbf{R}^n$ that has many of the properties of the standard ordering on $\mathbf{R}$. We associate with the proper cone K the partial ordering on $\mathbf{R}^n$ defined by

$$x \preceq_K y \iff y - x \in K.$$

We also write $x \succeq_K y$ for $y \preceq_K x$. Similarly, we define an associated strict partial ordering by

$$x \prec_K y \iff y - x \in \mathbf{int}\, K,$$

$$\boxed{K^* = \{y \mid x^T y \geq 0 \text{ for all } x \in K\}}$$

Assumption

$$
\begin{aligned}
\underset{x}{\text{minimize}} \quad & f_0(x) \\
\text{subject to} \quad & f_i(x) \preceq_{K_i} 0, \ \ i = 1, \ldots, m. \\
& h_i(x) = 0, \ \ i = 1, \ldots, p.
\end{aligned}
$$

- $K_i \subseteq R^{k_i}$: *proper cone*
- Domain:nonempty

Lagrange Dual and Weak Duality

- Lagrangian:$L(x, \lambda, v) = f_0(x) + \sum_{i=1}^{m} \lambda_i^T f_i(x) + \sum_{i=1}^{p} \nu_i h_i(x)$
- Dual:$g(\lambda, v) = \inf_{x \in D} L(x, \lambda, v)$
- nonnegativity requirement:$\lambda_i \succeq_{K_i^*} 0, \; i = 1, \ldots, m.$

위의 3가지로, weak duality가 증명가능하다.

Lagrange Dual and Weak Duality

$\lambda_i \succeq_{K_i^*} 0,\ f_i(\tilde{x}) \preceq_{K_i} 0$
Recall:$K^* = \{y | x^T y \geq 0 \mathit{for\ all}\ x \in K\}$
Thus, $\lambda^T f_i(\tilde{x}) \leq 0$
Therefore,$f_0(\tilde{x}) + \sum_{i=1}^{m} \lambda_i^T f_i(\tilde{x}) + \sum_{i=1}^{p} \nu_i h_i(\tilde{x}) \leq f_0(\tilde{x})$
$g(\lambda, \nu) \leq p^*$
So, the problem becomes...

$$\begin{array}{ll} \underset{x}{\text{maximize}} & g(\lambda, \nu) \\ \text{subject to} & \lambda_i \succeq_{K_i^*} 0,\ i = 1, \ldots, m. \end{array}$$

$d^* \leq p^*$

Strong Duality

Slater's condition and strong duality

As might be expected, *strong* duality ($d^\star = p^\star$) holds when the primal problem is convex and satisfies an appropriate constraint qualification. For example, a generalized version of Slater's condition for the problem

$$\begin{array}{ll} \text{minimize} & f_0(x) \\ \text{subject to} & f_i(x) \preceq_{K_i} 0, \quad i = 1, \dots, m \\ & Ax = b, \end{array}$$

where f_0 is convex and f_i is K_i-convex, is that there exists an $x \in \textbf{relint}\,\mathcal{D}$ with $Ax = b$ and $f_i(x) \prec_{K_i} 0$, $i = 1, \dots, m$. This condition implies strong duality (and also, that the dual optimum is attained).

Lagrange dual of semidefinite program

$$
\begin{aligned}
\underset{x}{\text{minimize}} \quad & c^T x \\
\text{subject to} \quad & x_1 F_1 + \ldots + x_n F_n + G \preceq_{K_1} 0 \\
\text{where} \quad & F_1, \ldots, F_n, G \in S^k, K_1 \in S^k_+
\end{aligned}
$$

Lagrange dual of semidefiniete program

$$\begin{aligned} L(x,Z) &= c^T x + \mathbf{tr}\left((x_1F_1 + \cdots + x_nF_n + G)Z\right) \\ &= x_1(c_1 + \mathbf{tr}(F_1Z)) + \cdots + x_n(c_n + \mathbf{tr}(F_nZ)) + \mathbf{tr}(GZ), \end{aligned}$$

which is affine in x. The dual function is given by

$$g(Z) = \inf_x L(x,Z) = \begin{cases} \mathbf{tr}(GZ) & \mathbf{tr}(F_iZ) + c_i = 0, \quad i = 1,\ldots,n \\ -\infty & \text{otherwise.} \end{cases}$$

The dual problem can therefore be expressed as

$$\begin{array}{ll} \text{maximize} & \mathbf{tr}(GZ) \\ \text{subject to} & \mathbf{tr}(F_iZ) + c_i = 0, \quad i = 1,\ldots,n \\ & Z \succeq 0. \end{array}$$

(We use the fact that $\mathbf{S}_+^k$ is self-dual, *i.e.*, $(\mathbf{S}_+^k)^* = \mathbf{S}_+^k$; see §2.6.)

Strong duality obtains if the semidefinite program (5.93) is strictly feasible, *i.e.*, there exists an x with

$$x_1F_1 + \cdots + x_nF_n + G \prec 0.$$

see Appendix 1.1

Complementary slackness:Generalized inequality version

$$
\begin{aligned}
f_0(x^*) &= g(\lambda^*, \nu^*) \\
&\leq f_0(x^*) + \sum_{i=1}^{m} \lambda_i^{*T} f_i(x^*) + \sum_{i=1}^{p} \nu_i h_i(x^*) \\
&= f_0(x^*)
\end{aligned}
$$

$$\therefore \lambda_i^{*T} f_i(x^*) = 0$$

결론:

- $\lambda_i^* \succ_{K_i^*} 0 \Rightarrow f_i(x^*) = 0$
- $f_i(x^*) \prec_{K_i} 0 \Rightarrow \lambda_i^* = 0$
- unlike scalar inequality case, there can be a case where $\lambda_i^* \neq 0$ and $f_i(x^*) \neq 0$

참고: Exercise 2.31 d

KKT conditon:Generalized Inequality version

Assumption:f_i,h_i differentiable

From complementary slackness; x^* minimizes L(x,λ^*, ν^*)

$$\nabla f_0(x^\star) + \sum_{i=1}^{m} Df_i(x^\star)^T \lambda_i^\star + \sum_{i=1}^{p} \nu_i^\star \nabla h_i(x^\star) = 0,$$

where $Df_i(x^\star) \in \mathbf{R}^{k_i \times n}$ is the derivative of f_i evaluated at $x^\star$ (see §A.4.1). Thus, if strong duality holds, any primal optimal $x^\star$ and any dual optimal $(\lambda^\star, \nu^\star)$ must satisfy the optimality conditions (or KKT conditions)

$$\begin{array}{rcll} f_i(x^\star) & \preceq_{K_i} & 0, & i=1,\ldots,m \\ h_i(x^\star) & = & 0, & i=1,\ldots,p \\ \lambda_i^\star & \succeq_{K_i^*} & 0, & i=1,\ldots,m \\ \lambda_i^{\star T} f_i(x^\star) & = & 0, & i=1,\ldots,m \\ \nabla f_0(x^\star) + \sum_{i=1}^m Df_i(x^\star)^T \lambda_i^\star + \sum_{i=1}^p \nu_i^\star \nabla h_i(x^\star) & = & 0. & \end{array} \tag{5.95}$$

opposite holds when converse

pertubed problem

$$\begin{aligned} \underset{x}{\text{minimize}} \quad & f_0(x) \\ \text{subject to} \quad & f_i(x) \leq u_i, \ i = 1, \ldots, m. \\ & h_i(x) = v_i, \ i = 1, \ldots, p. \end{aligned}$$

Terms:
-$p^*(u, v)$: optimal value of the pertubed problem
-u_i positive: relaxed the ith inequality constraint
-u_i negative: tightened the ith inequality constraint
-$p^*(0, 0)$:optimal value of the original problem

Global Inequality

Assumption:original problem is convex, slater's condition satisfied

$$p^*(u,v) \geq p^*(0,0) - {\lambda^*}^T u - {\nu^*}^T v$$

proof:

$$\begin{aligned} p^*(0,0) = g(\lambda^*, \nu^*) &\leq f_0(x) + \sum_{i=1}^{m} {\lambda_i^*}^T f_i(x) + \sum_{i=1}^{p} {\nu_i^*}^T h_i(x) \\ &\leq f_0(x) + {\lambda^*}^T u + {\nu^*}^T v \end{aligned}$$

$$\therefore f_0(x) \geq p^*(0,0) - {\lambda^*}^T u - {\nu^*}^T v$$

This inequality gives the lower bound of $p*(u,v)$

Sensitivity Analysis

- If $\lambda_i^\star$ is large and we tighten the ith constraint ($i.e.$, choose $u_i < 0$), then the optimal value $p^\star(u, v)$ is guaranteed to increase greatly.
- If $\nu_i^\star$ is large and positive and we take $v_i < 0$, or if $\nu_i^\star$ is large and negative and we take $v_i > 0$, then the optimal value $p^\star(u, v)$ is guaranteed to increase greatly.
- If $\lambda_i^\star$ is small, and we loosen the ith constraint ($u_i > 0$), then the optimal value $p^\star(u, v)$ will not decrease too much.
- If $\nu_i^\star$ is small and positive, and $v_i > 0$, or if $\nu_i^\star$ is small and negative and $v_i < 0$, then the optimal value $p^\star(u, v)$ will not decrease too much.

Local Sensitivity Analysis

$p^*(u, v)$: differentiable at $u = 0, v = 0$

$$\lambda_i^* = -\frac{\partial p^*(0,0)}{\partial u_i}, \ \nu_i^* = -\frac{\partial p^*(0,0)}{\partial v_i}$$

Interpretation:

- u_i tightened&small $\rightarrow$ p^* increase of $-\lambda_i^* u_i$
- u_i loosen&small $\rightarrow$ p^* decrease of $\lambda_i^* u_i$
- λ_i^* small$\rightarrow$ contraint가 optimal value에 영향을 미치는 부분이 작다.
- λ_i^* big $\rightarrow$ contraint가 optimal value에 영향을 미치는 부분이 크다.

Shadow Pricing

$$\begin{aligned} \underset{x}{\text{minimize}} \quad & f_0(x) \\ \text{subject to} \quad & f_i(x) \le u_i, \;\; i = 1, \ldots, m. \end{aligned}$$

f_0:price, $f_i(x) \le u_i$: resource constraint, $-p^*(u)$: optimal profit

Shadow Pricing

$$\lambda_i^* = -\frac{\partial p^*(0)}{\partial u_i}$$

Interpretation:

- $\lambda_i^* \rightarrow$ increase in profit for a small increase in resource i
- $\lambda_i^* \rightarrow$ equilibrium price for resource i

EX)
이 균형가격보다 자원을 더 싼 가격에 구한다 $\rightarrow$ 자원 많이 삼 $\rightarrow$ 자원가격상승
이 균형가격보다 자원이 더 비싸다. $\rightarrow$ 가지고 있는 자원도 팖$\rightarrow$ 자원가격하락

Pertubation: Generalized Inequality

5.9.3 Perturbation and sensitivity analysis

The results of §5.6 can be extended to problems involving generalized inequalities. We consider the associated perturbed version of the problem,

$$\begin{array}{ll} \text{minimize} & f_0(x) \\ \text{subject to} & f_i(x) \preceq_{K_i} u_i, \quad i = 1, \dots, m \\ & h_i(x) = v_i, \quad i = 1, \dots, p, \end{array}$$

where $u_i \in \mathbf{R}^{k_i}$, and $v \in \mathbf{R}^p$. We define $p^\star(u, v)$ as the optimal value of the perturbed problem. As in the case with scalar inequalities, $p^\star$ is a convex function when the original problem is convex.

Now let $(\lambda^\star, \nu^\star)$ be optimal for the dual of the original (unperturbed) problem, which we assume has zero duality gap. Then for all u and v we have

$$p^\star(u, v) \geq p^\star - \sum_{i=1}^{m} \lambda_i^{\star T} u_i - \nu^{\star T} v,$$

the analog of the global sensitivity inequality (5.57). The local sensitivity result holds as well: If $p^\star(u, v)$ is differentiable at $u = 0$, $v = 0$, then the optimal dual variables $\lambda_i^\star$ satisfies

$$\lambda_i^\star = -\nabla_{u_i} p^\star(0, 0),$$

the analog of (5.58).

Weak Alternatives in generalized inequality

$$f_i(x) \preceq_{K_i} 0,\ i = 1, \ldots, m, \quad h_i(x) = 0,\ i = 1, \ldots, p \tag{1}$$

$$\lambda_i \succeq_{K_i} 0,\ i = 1, \ldots, m, \quad g(\lambda, \nu) > 0 \tag{2}$$

- 1과 2는 동시에 feasible할 수 없으므로, 위 둘은 weak alternatives다.

Weak Alternatives in generalized inequality

$$0 < g(\lambda, \nu) \leq \sum_{i=1}^{m} \lambda_i^T f_i(x) + \sum_{i=1}^{p} \nu_i h_i(x) \leq 0$$

$g(\lambda, \nu)$가 0보다 커야하는 것과 모순이 생긴다!

Weak Alternatives in generalized inequality

$$f_i(x) \prec_{K_i} 0,\ \ i = 1, \ldots, m, \quad Ax = b, \tag{3}$$

$$\lambda_i \succeq_{K_i} 0,\ \ i = 1, \ldots, m, \quad , \lambda \neq 0 \quad g(\lambda, \nu) \geq 0 \tag{4}$$

마찬가지로, 3과 4는 weak alternative의 관계를 가진다.
exercise 2.31 d 참고

Strong Alternatives in generalized inequality

suppose $\tilde{x} \in relint D$ satisfying $A\tilde{x} = b$ exists
Consider:

$$\begin{array}{ll} \text{minimize} & s \\ \text{subject to} & f_i(x) \preceq_{K_i} se_i, \; i = 1, \dots, m \\ & Ax = b \\ \text{where} & e_i \succ_{K_i} 0, \; x, s \in R \end{array}$$

Strong Alternatives in generalized inequality

The dual of above is:

$$\begin{array}{ll} \text{maximize} & g(\lambda, \nu) \\ \text{subject to} & \lambda_i \succeq_{K_i^*} 0, \ \ i = 1, \ldots, m \\ & \displaystyle\sum_{i=1}^{m} e_i^T \lambda_i = 1 \end{array}$$

Strong Alternatives in generalized inequality

(3)이 infeasible할 경우, $-fi(x) \notin intK_i \rightarrow -\lambda^T f_i(x) \leq 0$
$\lambda^T(se_i - f_i(x)) \geq 0 \rightarrow s \geq 0$
slater's condition에 의해, $d^* = p^*$ 를 만족하는 $\tilde{\lambda}, \tilde{\nu}$ 존재
$d^* = p^* \geq 0$을 만족하는 $\tilde{\lambda}, \tilde{\nu}$ 가 존재한다는 의미이므로, (4)는 feasible 하다.
마찬가지로 (3)이 feasible할 경우 (3)도 infeasible 하다.
따라서 둘은 strong alternative이다.

Strong Alternatives in generalized inequality

As we noted in the case of scalar inequalities, existence of an $x \in \textbf{relint}\, \mathcal{D}$ with $Ax = b$ is not sufficient for the system of nonstrict inequalities

$$f_i(x) \preceq_{K_i} 0, \quad i = 1, \ldots, m, \qquad Ax = b$$

and its alternative

$$\lambda_i \succeq_{K_i^\star} 0, \quad i = 1, \ldots, m, \qquad g(\lambda, \nu) > 0$$

to be strong alternatives. An additional condition is required, *e.g.*, that the optimal value of (5.101) is attained.