Duality

2017년 12월 18일 월요일 오후 2:56

minimize $f(x)$

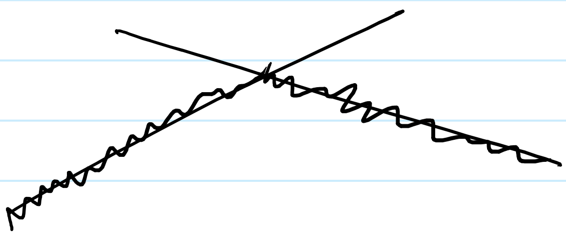
subject to $g_i(x) \le 0$

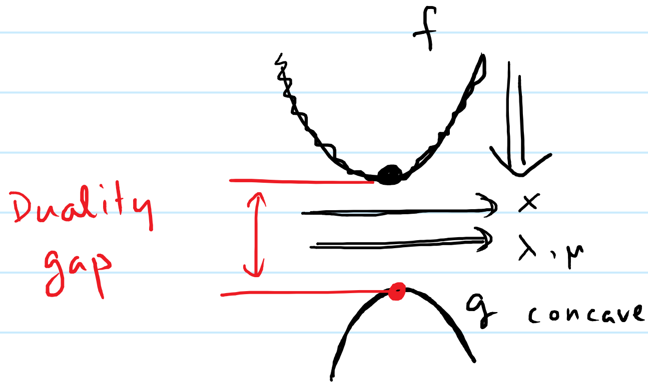
$h_j(x) = 0$

If Slater condition satisfied, KKT holds.

$$\mathcal{L} = f(x) + \underbrace{\Sigma \underbrace{\lambda_i}_{\ge 0} \underbrace{g_i}_{\le 0}}_{=0} + \Sigma \mu_j \underbrace{h_j}_{=0}$$

$$g(\lambda,\mu) \overset{def}{=} \inf_{x \in D} \mathcal{L}(x,\lambda,\mu) \le \mathcal{L}(x_*,\lambda,\mu) = f(x^*)$$

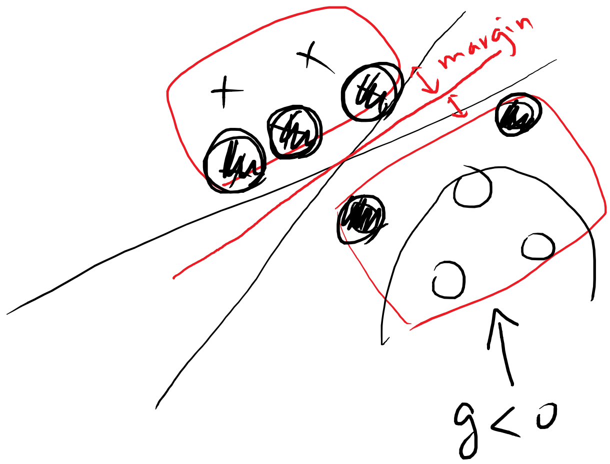




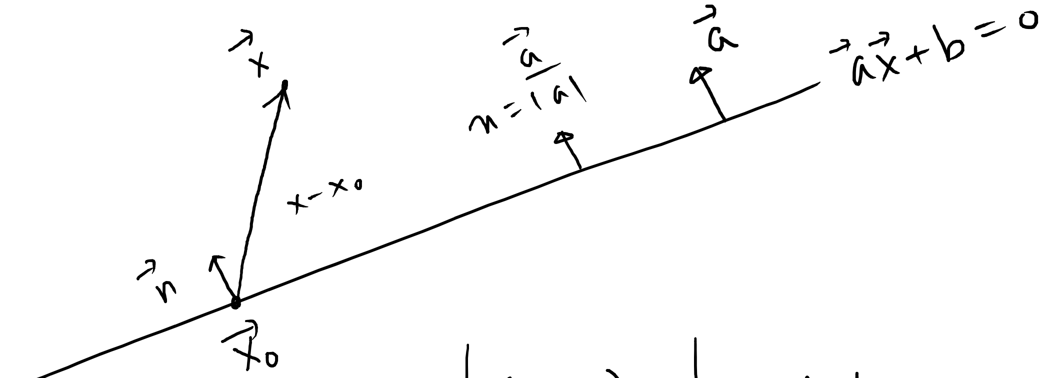
If Slater condition satisfied, Duality gap $=0$.

SVM

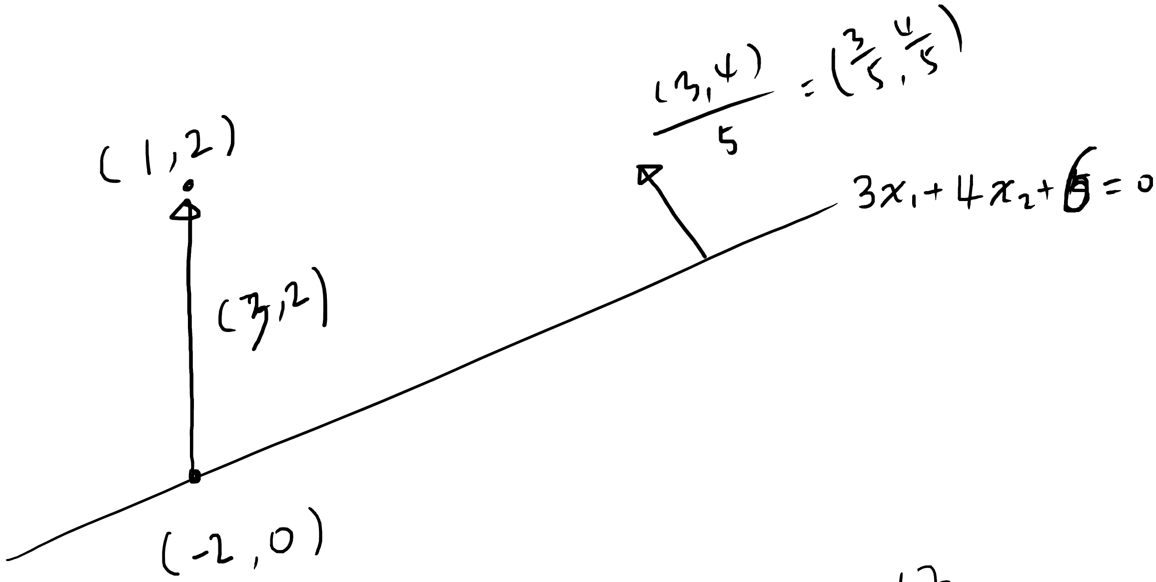
2018년 4월 7일 토요일 오후 1:29



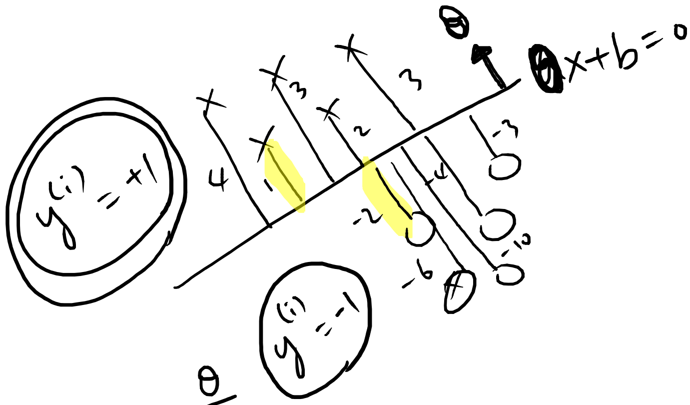
distance between x and P

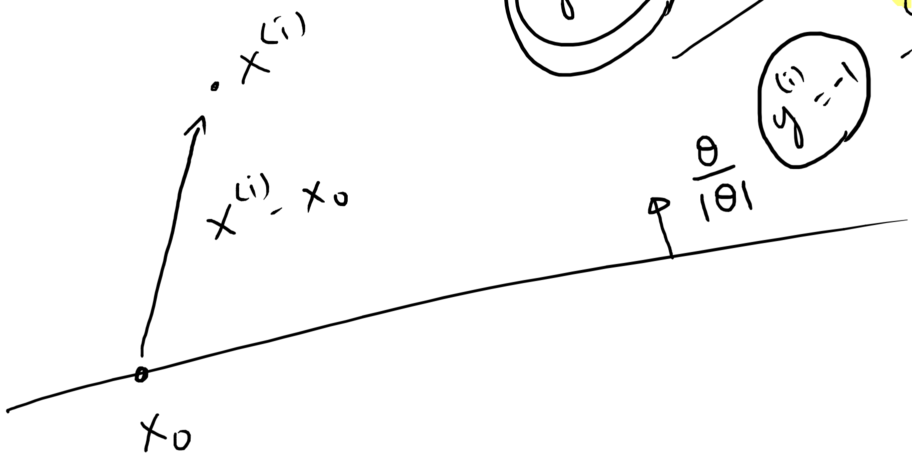


$$\left|(x-x_0)\cdot n\right| = \text{distance}.$$



$$(3,2)\cdot\left(\frac{3}{5},\frac{4}{5}\right)=\frac{17}{5}=3.4$$





$$\frac{(x^{(i)}-x_0)\cdot\theta}{|\theta|}=\frac{\overset{+b}{\theta x^{(i)}}-\underset{}{\overset{0\,=\,+b}{\theta x_0}}}{|\theta|}$$

$$=\frac{\theta x^{(i)}+b}{|\theta|}$$

$$=\begin{cases} + & \text{if } y^{(i)}=+1 \\ - & \text{if } y^{(i)}=-1 \end{cases}$$

$$\text{margin}=\min_i \; y^{(i)}\left[\frac{\theta x^{(i)}+b}{|\theta|}\right]$$

$$\underset{\theta,b}{\text{maximize}}$$

$$\min_i \; y^{(i)}[\theta x^{(i)}+b] = 0.5$$

$$\boxed{\theta_*, b_*}$$

$$\min_i \; y^{(i)}[\theta x^{(i)}+b]=1$$

$$\boxed{2\theta_*, 2b_*}$$

2

convex

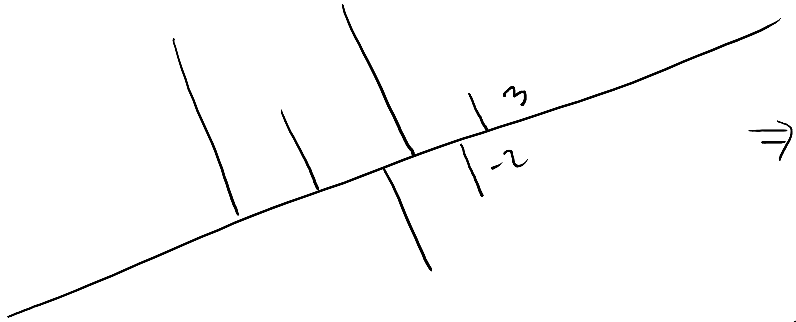
$$\underset{\theta, b}{\text{minimize}} \quad \frac{1}{2}\theta^2$$ convex

$$\boxed{\text{subject to} \quad y^{(i)}\left[\theta x^{(i)} + b\right] \geq 1}$$ convex

$$g_i \leq 0$$

$$g_i = 1 - y^{(i)}\left[\theta x^{(i)} + b\right]$$

"θ"



$\Rightarrow$

minimize $\frac{1}{2}\|w\|^2$

subject to $y^{(i)}(wx^{(i)}+b) \geq 1$ — $g_i \leq 0$

$$L = \frac{1}{2}\|w\|^2 + \sum \lambda_i \left[1 - y^{(i)}(wx^{(i)}+b)\right] \quad -\left(\sum_i \lambda_i y^{(i)}\right) b$$

$w^T w \longrightarrow w + w = 2w$

$$Ax \xrightarrow{\frac{\partial}{\partial x}} A^T$$
$$x^T A \xrightarrow{\frac{\partial}{\partial x}} A$$

KKT

$\frac{\partial L}{\partial w} = 0$: $\boxed{w + \sum \lambda_i \left[-y^{(i)} x^{(i)}\right] = 0}$

$\boxed{w = \sum \lambda_i y^{(i)} x^{(i)}}$

$\frac{\partial L}{\partial b} = 0$: $\boxed{\sum \lambda_i \left[-y^{(i)}\right] = 0}$

$\boxed{\lambda_i \geq 0}$

$1 - y^{(i)}(wx^{(i)}+b) \leq 0$

$\lambda_i \left[1 - y^{(i)}(wx^{(i)}+b)\right] = 0$

$- \sum \lambda_i y^{(i)} w^T x^{(i)}$

$x^{(i)T} w$

$$g(\lambda) = \inf_x \cancel{\max} \, L(w, b, \lambda)$$

$$\|w\|^2 = w^T w = \left(\sum_i \lambda_i y^{(i)} x^{(i)T}\right)\left(\sum_j \lambda_j y^{(j)} x^{(j)}\right)$$

$$= \sum_i \sum_j \lambda_i \lambda_j y^{(i)} y^{(j)} x^{(i)T} x^{(j)}$$

$$g(\lambda) = \frac{1}{2} \sum_i \sum_j \lambda_i \lambda_j y^{(i)} y^{(j)} x^{(i)T} x^{(j)} + \sum \lambda_i$$

$$\underset{\lambda}{\text{maximize}}\; g(\lambda) = \frac{1}{2}\sum_i\sum_j \lambda_i\lambda_j y^{(i)} y^{(j)} x^{(i)T} x^{(j)} + \sum \lambda_i$$

$$- \sum_i \lambda_i y^{(i)} x^{(i)T} \sum_j \lambda_j y^{(j)} x^{(j)} + 0$$

$$= \sum \lambda_i - \frac{1}{2}\sum_i\sum_j \lambda_i\lambda_j y^{(i)} y^{(j)} x^{(i)T} x^{(j)}$$

subject to $\lambda_i \geq 0$

$\sum \lambda_i y^{(i)} = 0$

	Primal	Dual
변수	w, b	λ_i
# 변수	$\#w + 1$ (小)	n (大)

In practice

almost all $\lambda_i = 0$

$\boxed{\text{Dual solved} \quad \lambda_i \text{ fixed.}}$

$\lambda_i = 0$

$\lambda_i > 0 \rightarrow$ support vector

new data

$x \rightarrow \text{sgn}\left[w x + b \right]$ support vector로 계산

$\sum \lambda_i y^{(i)} x^{(i)} = \sum \lambda_i y^{(i)} x^{(i)}$

$$\Big(\sum \lambda_i y^{(i)} x^{(i)} = \underbrace{\sum \lambda_i y \; x}_{\substack{\text{support vector} \\ (\lambda_i \neq 0)}} \Big)$$

support vector로 계산

$$\lambda_i \left[1 - y^{(i)} \left(w x^{(i)} + b \right) \right] = 0$$

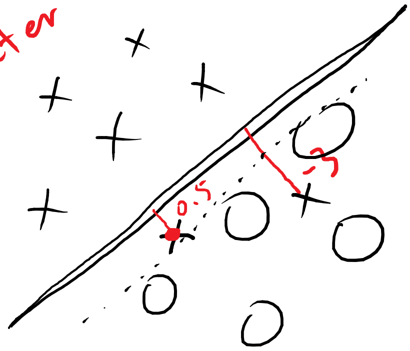
$i \rightarrow$ support vector $\rightarrow \lambda_i \neq 0$

$$1 = y^{(i)} \left(w x^{(i)} + b \right)$$

$$y^{(i)} = w x^{(i)} + b$$

$$b = y^{(i)} - w x^{(i)}$$

$$b = \frac{1}{\#\,\text{support vector}} \sum_{\substack{\text{support vector} \\ (\lambda_i \neq 0)}} y^{(i)} - w x^{(i)}$$



fixed ≥ 0
hyperparameter

minimize $\frac{1}{2}\theta^2 + C\sum \xi_i$

subject $y^{(i)}\left[\theta x^{(i)} + b\right] \ge 1 - \xi_i$

$\xi_i \ge 0$

$(w, b, \xi, \lambda_i, \lambda_{i+n})$

$$L = \frac{1}{2}\|w\|^2 + C\sum \xi_i + \sum \lambda_i\left[1 - \xi_i - y^{(i)}\left[w\cancel{x}^{(i)} + b\right]\right] + \sum \lambda_{n+i}\left[-\xi_i\right]$$

$\frac{\partial L}{\partial w} = 0$: $w + \sum \lambda_i\left[-y^{(i)}\cancel{x}^{(i)}\right] = 0 \rightarrow \boxed{w = \sum \lambda_i y^{(i)} x^{(i)}}$

$\frac{\partial L}{\partial b} = 0$: $\boxed{\sum \lambda_i\left(-y^{(i)}\right) = 0}$

$\frac{\partial L}{\partial \xi_i} = 0$: $\boxed{C - \lambda_i - \lambda_{n+i} = 0}$

$\lambda_{n+i} > 0 \rightarrow \xi_i = 0$

$0 \le \lambda_i < C$

$\lambda_i \ge 0$, $y^{(i)}\left[wx^{(i)} + b\right] \ge 1 - \xi_i$, $\lambda_i\left[1 - \xi_i - y^{(i)}\left[wx^{(i)} + b\right]\right] = 0$

$\lambda_{n+i} \ge 0$, $\xi_i \ge 0$, $\lambda_{n+i}\left[-\xi_i\right] = 0$

$$g = -\frac{1}{2}\sum_i\sum_j \lambda_i\lambda_j y^{(i)} y^{(j)} x^{(i)T} x^{(j)}$$

$$+\sum_i \lambda_i$$

$$+ C\sum \xi_i + \sum \lambda_i(-\xi_i) + \sum \lambda_{n+i}(-\xi_i)$$

$$= \sum \xi_i \underbrace{(C - \lambda_i - \lambda_{n+i})}_{=0} = 0$$

subject to

$\lambda_i \geq 0$

$\lambda_{n+i} \geq 0$ $\Rightarrow$ $0 \leq \lambda_i \leq C$

$\sum \lambda_i y^{(i)} = 0$

$C = \lambda_i + \lambda_{n+i}$

$\boxed{\text{dual solved} \quad \lambda_i \quad \text{fixed}}$ $\Rightarrow$

$\lambda_i = 0$

$0 < \lambda_i < C$

$\lambda_i = C$

support vector

$W = \sum \lambda_i y^{(i)} x^{(i)}$ support vector가 결정

support vector

$$\lambda_i \left[1 - \xi_i - y^{(i)}\left[W x^{(i)} + b \right] \right] = 0$$

support vector $\lambda_i > 0$

$$y^{(i)}\left[W x^{(i)} + b \right] = 1 - \xi_i$$

$\lambda_i < C$

for C

$$\cancel{y^{(i)}}\left[W x^{(i)} + b \right] = y^{(i)}$$

for $0 < \lambda_i < C$

$$\cancel{y^{(i)}}\left[w x^{(i)} + b \right] = y^{(i)}$$

$$b = y^{(i)} - w x^{(i)}$$

$$b = \frac{1}{} \sum y^{(i)} - w x^{(i)}$$

summation $0 < \lambda_i < C$

new data x $\longrightarrow$ $\mathrm{sgn}\left[w x + b \right]$

primal dual

w, b, ξ λ_i, λ_{n+i}

$\boxed{\text{identify support vector.}}$

$\boxed{\text{kernel trick}}$

$\boxed{\text{kernel trick}}$

$$(x^{(i)}, y^{(i)}) \xrightarrow{\phi} (\phi(x^{(i)}), y^{(i)})$$

$$g = \sum \lambda_i - \frac{1}{2} \sum \lambda_i \lambda_j y^{(i)} y^{(j)} \cancel{\phi(x^{(i)})^T \phi(x^{(j)})} \quad \boxed{K(x^{(i)}, x^{(j)})}$$

kernel

subject $0 \leq \lambda_i \leq C$

$$\sum \lambda_i y^{(i)} = 0$$

$\cancel{K(x, x')} \leftarrow x \Longrightarrow \phi(x)$

dim=3 dim = 1,000,000,000,000

$$(\vec{x}^{(1)} \cdot \vec{x}^{(2)})^2 = (ax+by)^2 = a^2x^2 + b^2y^2 + 2axby$$

$x_1^{(1)}$ $x_2^{(1)}$

$$= \phi(x,y)^T \phi(a,b)$$

$\cancel{x_1^{(i)}, x_2^{(i)}}$

(x, y)

(a, b)

$$= \phi(x, y) \cdot \phi(a, b)$$

$$\begin{pmatrix} x^2 \\ y^2 \\ \sqrt{2}xy \end{pmatrix} \cdot \begin{pmatrix} a^2 \\ b^2 \\ \sqrt{2}ab \end{pmatrix}$$

$$(x, y) \xrightarrow{\phi} (x^2, y^2, \sqrt{2}xy)$$