[Homework] Detailed properties of probability

1. We flip a fair coin many times. Find the probability that

   (a) we have exactly 5 heads in the first 11 flips.
   Sol. $\binom{11}{5}(\frac{1}{2})^5(\frac{1}{2})^6$

   (b) we have the very first head at the 11-th flip.
   Sol. $\frac{1}{2^{11}}$

   (c) we have the very second head at the 11-th flip.
   Sol. $\binom{10}{1}(\frac{1}{2})^1 \cdot (\frac{1}{2})^9 \cdot \frac{1}{2}$

   (d) we have the very third head at the 11-th flip.
   Sol. $\binom{10}{2}(\frac{1}{2})^2 \cdot (\frac{1}{2})^8 \cdot \frac{1}{2}$

   (e) we have more heads than tails for the first time at the 11-th flip.
   Sol. $\frac{\binom{8}{4}-\binom{8}{2}}{2^{11}}$

2. A fair dice is rolled five times. What is the probability that

   (a) the second dice lands on a higher value than the first?
   Sol. $\frac{15 \cdot 6^3}{6^5}$

   (b) 6 comes up at least once?
   Sol. (Complement) Let A be the event that there are at least one 6. A^c is the event that there are no 6 and

   $$P(A^c) = (\frac{5}{6})^5 \Rightarrow P(A) = 1 - P(A^c) = 1 - (\frac{5}{6})^5$$

   (c) all outcomes from the five rolls are identical?
   Sol. $\frac{6}{6^5}$

   (d) the five outcomes are different?
   Sol. $\frac{6 \cdot 5 \cdot 4 \cdot 3 \cdot 2}{6^5}$

3. We have a colored fair dice. Each of the two faces of this dice are colored red, green, and blue. We roll this fair dice 11 times. Find the probability that

   (a) we have exactly 5 reds in the first 11 flips.
   Sol. $\binom{11}{5} \cdot (\frac{1}{3})^5 \cdot (\frac{2}{3})^6$

   (b) we have the very first red at the 11-th flip.
   Sol. $\binom{10}{0} \cdot (\frac{1}{3})^0 \cdot (\frac{2}{3})^{10} \cdot (\frac{1}{3})$

(c) we have the very second red at the 11-th flip.
Sol. $\binom{10}{1} \cdot (\frac{1}{3})^1 \cdot (\frac{2}{3})^9 \cdot (\frac{1}{3})$

(d) we have the very third red at the 11-th flip.
Sol. $\binom{10}{2} \cdot (\frac{1}{3})^2 \cdot (\frac{2}{3})^8 \cdot (\frac{1}{3})$

4. A roulette wheel has 38 spaces: 18 red, 18 black, and 2 green. The wheel will be spun five times. Find the probability that

   (a) all the five land on red.
   Sol. $\binom{5}{5} \cdot (\frac{18}{38})^5$

   (b) the first two land on red and the last three land on other colors.
   Sol. $(\frac{18}{38})^2 \cdot (\frac{20}{38})^3$

   (c) two land on red and three land on other colors.
   Sol. $\binom{5}{2} \cdot (\frac{18}{38})^2 \cdot (\frac{20}{38})^3$

5. There are 10 hotels in a certain town. If 7 different group of people check into hotels in a day, what is the probability that each check into a different hotel?
   Sol. When the first person checks into the hotel, the next person will check into a different hotel with probability $\frac{9}{10}$. The next person will check into a different hotel with probability $\frac{8}{10}$.Thus the probability that we check into seven different hotels is given by
$$\frac{10}{10} \cdot \frac{9}{10} \cdot \frac{8}{10} \cdots \frac{4}{10} = \frac{\binom{10}{7} \cdot 7!}{10^7}$$

6. In a drawer there are 10 red, 3 green and 7 blue socks. Seven socks are withdrawn. Find the probability that

   (a) 3 red, 2 green, and 2 blue socks are withdrawn.
   Sol. $\frac{\binom{10}{3}\cdot\binom{3}{2}\cdot\binom{7}{2}}{\binom{20}{7}}$

   (b) at least 2 red socks are withdrawn.
   Sol. $1 - P(\text{no red socks withdrawn}) - P(\text{only 1 red socks withdrawn})$
   $=1 - \frac{\binom{10}{0}\cdot\binom{10}{7}}{\binom{20}{7}} - \frac{\binom{10}{1}\cdot\binom{10}{6}}{\binom{20}{7}}$

   (c) exactly one color is missing among the three withdrawn socks.
   Sol. $1 - \frac{\binom{10}{3}+\binom{3}{3}+\binom{7}{3}+\binom{10}{1}\cdot\binom{3}{1}\cdot\binom{7}{1}}{\binom{20}{3}}$

   (d) exactly two colors are missing among the three withdrawn socks.
   Sol. $\frac{\binom{10}{3}+\binom{3}{3}+\binom{7}{3}}{\binom{20}{3}}$

(e) no colors are missing among the three withdrawn socks.
Sol. $\frac{\binom{10}{1}\cdot\binom{3}{1}\cdot\binom{7}{1}}{\binom{20}{3}}$

7. In a drawer there are 10 red, 3 green and 7 blue socks. Starting from Author, Author and Bob withdraw socks from the draw consecutively until a green sock is selected. Find the probability that Author selects the green sock.
Sol. $\frac{3}{20}+\frac{17}{20}\cdot\frac{16}{19}\cdot\frac{3}{18}+\frac{17}{20}\cdot\frac{16}{19}\cdot\frac{15}{18}\cdot\frac{14}{17}\cdot\frac{3}{16}+\frac{17}{20}\cdot\frac{16}{19}\cdot\frac{15}{18}\cdot\frac{14}{17}\cdot\frac{13}{16}\cdot\frac{12}{15}\cdot\frac{3}{14}+\frac{17}{20}\cdot\frac{16}{19}\cdot\frac{15}{18}\cdot\frac{14}{17}\cdot\frac{13}{16}\cdot\frac{12}{15}\cdot\frac{11}{14}\cdot\frac{10}{13}\cdot\frac{3}{12}+\frac{17}{20}\cdot\frac{16}{19}\cdot\frac{15}{18}\cdot\frac{14}{17}\cdot\frac{13}{16}\cdot\frac{12}{15}\cdot\frac{11}{14}\cdot\frac{10}{13}\cdot\frac{9}{12}\cdot\frac{8}{11}\cdot\frac{3}{10}+\frac{17}{20}\cdot\frac{16}{19}\cdot\frac{15}{18}\cdot\frac{14}{17}\cdot\frac{13}{16}\cdot\frac{12}{15}\cdot\frac{11}{14}\cdot\frac{10}{13}\cdot\frac{9}{12}\cdot\frac{8}{11}\cdot\frac{7}{10}\cdot\frac{6}{9}\cdot\frac{3}{8}+\frac{17}{20}\cdot\frac{16}{19}\cdot\frac{15}{18}\cdot\frac{14}{17}\cdot\frac{13}{16}\cdot\frac{12}{15}\cdot\frac{11}{14}\cdot\frac{10}{13}\cdot\frac{9}{12}\cdot\frac{8}{11}\cdot\frac{7}{10}\cdot\frac{6}{9}\cdot\frac{5}{8}\cdot\frac{4}{7}\cdot\frac{3}{6}+\frac{17}{20}\cdot\frac{16}{19}\cdot\frac{15}{18}\cdot\frac{14}{17}\cdot\frac{13}{16}\cdot\frac{12}{15}\cdot\frac{11}{14}\cdot\frac{10}{13}\cdot\frac{9}{12}\cdot\frac{8}{11}\cdot\frac{7}{10}\cdot\frac{6}{9}\cdot\frac{5}{8}\cdot\frac{4}{7}\cdot\frac{3}{6}\cdot\frac{2}{5}\cdot\frac{3}{4}=\frac{41}{76}$

8. In a certain region there are many elks. We captured, tagged, and then released 5 elks. A certain time later, we captured 4 elks and 2 of these 4 have been tagged.

(a) Compute the probability that this happens as a function of the total number n of the elks in the region.
Sol. The number of total events is $\binom{n}{4}$. And there are $\binom{5}{2}\cdot\binom{n-5}{2}$ different ways to choose 2 from the previously tagged elks, and 2 from the elks which have not been tagged before.
Therefore, the probability is $\frac{\binom{5}{2}\cdot\binom{n-5}{2}}{\binom{n}{4}}$ different ways to choose 2 from the previously tagged elks, and 2 from the elks which have not been tagged before.

(b) Find n which maximizes this probability.
Sol. Let $f(n)=\frac{\binom{5}{2}\cdot\binom{n-5}{2}}{\binom{n}{4}}=\frac{120(n-5)(n-6)}{n(n-1)(n-2)(n-3)}$
We want to find n such that

$$\frac{f(n)}{f(n-1)}\geq 1, \frac{f(n+1)}{f(n)}\leq 1$$

$$9\leq n\leq 10$$

So n is 9 or 10, $f(9)=\frac{10}{21}, f(10)=\frac{10}{21}$

9. An instructor gives her class a set of 20 problems with the information that the final exam will consist of a random selection of 10 of them. If a student has figured out how to do 15 of the problems, what is the probability that he or she will answer correctly

(a) all 10 problems?
Sol. $\frac{\binom{15}{10}}{\binom{20}{10}}$

(b) at least 8 of the problems?
Sol. $\frac{\binom{15}{8}\cdot\binom{5}{2}}{\binom{20}{10}} + \frac{\binom{15}{9}\cdot\binom{5}{1}}{\binom{20}{10}} + \frac{\binom{15}{10}\cdot\binom{5}{0}}{\binom{20}{10}} = \frac{1}{2}$

10. A closet contains 10 pairs of shoes. If 8 shoes are randomly selected, what is the probability that there will be

(a) no complete pair?
Sol. 10 pairs = 20 shoes
There are $\binom{20}{8}$ ways to select the 8 shoes.
There are $\binom{10}{8}$ ways to choose which pairs contribute one shoe each, and two possibilities for picking the shoe from each of the pairs (left or right).
This gives a total of $\binom{10}{8}\cdot 2^8$ ways to choose 8 shoe that are unpaired

$$\frac{\binom{10}{8}\cdot 2^8}{\binom{20}{8}}$$

(b) exactly 1 complete pair?
Sol. There are 10 ways to choose the single pair of shoes and $\binom{9}{6}\cdot 2^6$ ways to choose the remaining six shoes so that they unpaired

$$\frac{10\cdot\binom{9}{6}\cdot 2^6}{\binom{20}{8}}$$

[Extra] Detailed properties of probability

1. We choose balls successively without replacement from an urn containing 20 red and 10 blue balls. What is the probability that all the red balls are removed before all the blue ones are removed from the urn?