CLT

Normal distribution $\mathcal{N}(\mu, \sigma^2)$

PDF

$$f(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$$

where

Mean	μ
Variance	σ^2

Intuition

(1) Flip a p-coin n times and record the number of heads

(2) Standardize the number of heads

(3) Report the standardized number of heads (CLT)

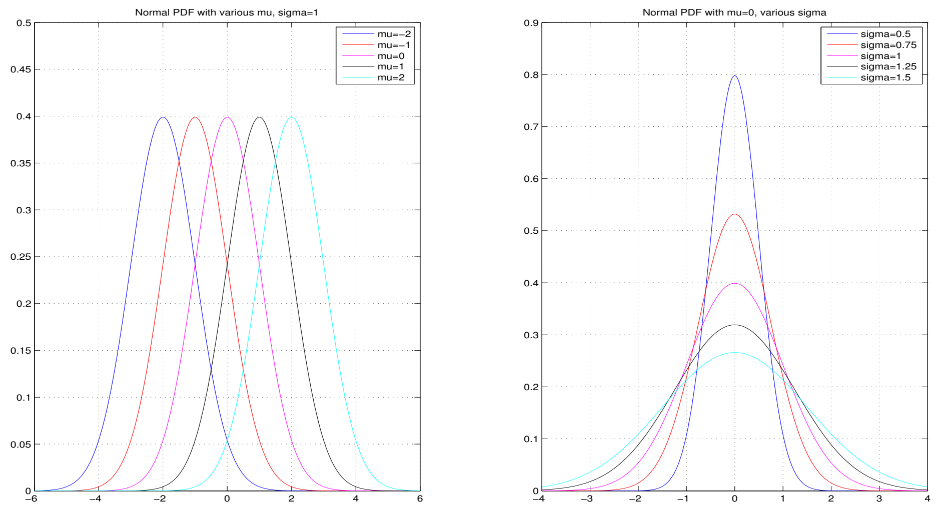


Figure 1: Normal PDF with various μ and σ.

```
clear all; close all; clc;

subplot(1,2,1)
mu=-2:2;
v=1;
x=-6:0.01:6;
color='brmkc';
for i=1:length(mu)
    y=pdf('Normal',x,mu(i),v);
    plot(x,y,color(i)); axis([-6 6 0 0.5]); hold on; grid on;
end
legend('mu=-2','mu=-1','mu=0','mu=1','mu=2');
title('Normal PDF with various mu, sigma=1')

subplot(1,2,2)
mu=0;
v=0.5:0.25:1.5;
x=-4:0.01:4;
color='brmkc';
for i=1:length(v)
    y=pdf('Normal',x,mu,v(i));
    plot(x,y,color(i)); axis([-4 4 0 0.9]); hold on; grid on;
end
legend('sigma=0.5','sigma=0.75','sigma=1','sigma=1.25','sigma=1.5');
title('Normal PDF with mu=0, various sigma')
```

Example - Standard normal distribution

The PDF of the standard normal distribution $\mathcal{N}(0, 1^2)$ is given by

$$f(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}$$

Show that

(1) Total mass is indeed 1

(2) Mean 0

(3) Variance 1

With $I := \int_{-\infty}^{\infty} e^{-\frac{x^2}{2}} dx$

$$\begin{aligned} I^2 &= \left(\int_{-\infty}^{\infty} e^{-\frac{x^2}{2}} dx\right)\left(\int_{-\infty}^{\infty} e^{-\frac{y^2}{2}} dy\right) \\ &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-\frac{x^2+y^2}{2}} dxdy = \int_0^{2\pi}\int_0^{\infty} e^{-\frac{r^2}{2}} rdrd\theta \\ &= 2\pi \int_0^{\infty} e^{-\frac{r^2}{2}} rdr = 2\pi \left[-e^{-\frac{r^2}{2}}\right]_0^{\infty} = 2\pi \end{aligned}$$

$$I^2 = 2\pi \quad \Rightarrow \quad I = \sqrt{2\pi} \quad \Rightarrow \quad \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} dx = 1$$

$$\text{Integrand is odd} \quad \Rightarrow \quad \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} x e^{-\frac{x^2}{2}} dx = 0$$

$$\begin{aligned} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} x^2 e^{-\frac{x^2}{2}} dx &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} (-x)\left(e^{-\frac{x^2}{2}}\right)' dx \qquad \text{(Integration by part)} \\ &= \frac{1}{\sqrt{2\pi}} \left[(-x)\left(e^{-\frac{x^2}{2}}\right)\right]_{-\infty}^{\infty} - \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} (-x)'\left(e^{-\frac{x^2}{2}}\right) dx \\ &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{x^2}{2}} dx = 1 \end{aligned}$$

CDF of standard normal distribution

CDF $N(x)$ of standard normal distribution

$$N(x) = \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}} e^{-\frac{s^2}{2}} ds$$

Properties of $N(x)$

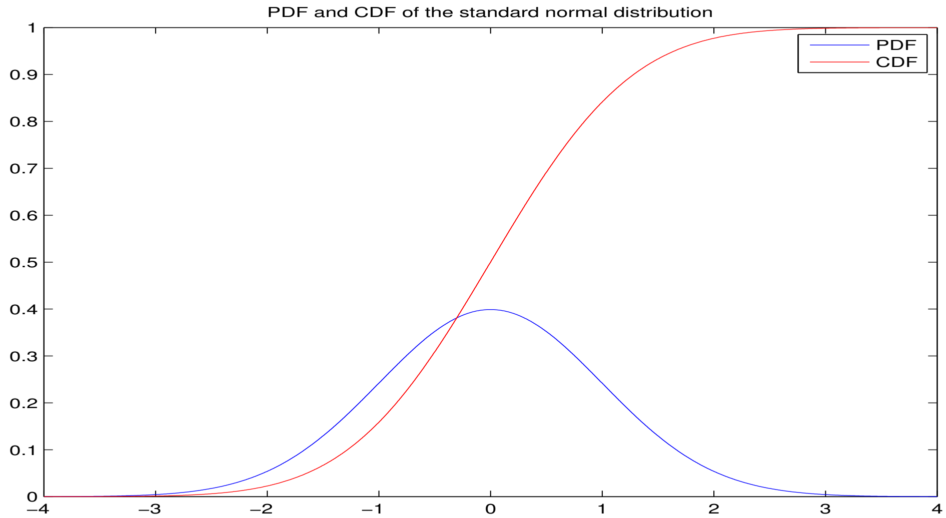
$$\begin{aligned}
&(1) \qquad \mathbb{P}(a \leq Z \leq b) = N(b) - N(a) \\
&(2) \qquad \mathbb{P}(Z \geq x) = \mathbb{P}(Z \leq -x) = N(-x) \\
&(3) \qquad \mathbb{P}(Z \geq x) = 1 - \mathbb{P}(Z \leq x) = 1 - N(x) \\
&(4) \qquad \mathbb{P}(Z \leq 0) = \mathbb{P}(Z \geq 0) = 0.5
\end{aligned}$$

Matlab

	Description
`cdf`	CDF
`icdf`	Inverse CDF
`pdf`	PDF/PMF

```
cdf('Distribution',Evaluation_Points,Parameter1,Parameter2)
```

$$\begin{aligned}
P(-1 \leq Z \leq 2) &= P(Z \leq 2) - P(Z \leq -1) = N(2) - N(-1) \\
&= \texttt{cdf('Normal',2,0,1) - cdf('Normal',-1,0,1)} \\
&= \texttt{0.9772 - 0.1587} \\
&= \texttt{0.8186}
\end{aligned}$$




```
clear all; close all; clc;

mu=0;
v=1;

x=-4:0.01:4;
y1=pdf('Normal',x,mu,v);
y2=cdf('Normal',x,mu,v);

plot(x,y1,'-',x,y2,'-r');
legend('PDF','CDF');
title('PDF and CDF of the standard normal distribution')
```

Quantile q_α in terms of z_α

$$\underbrace{q_\alpha}_{\alpha \text{ quantile of } \mathcal{N}(\mu,\sigma^2)} = \mu + \sigma \cdot \underbrace{z_\alpha}_{\alpha \text{ quantile of } \mathcal{N}(0,1)}$$

Integration trick related to PDF of the normal distribution

Compute the following integral:

$$\int_{-\infty}^{\infty} e^{-x^2-2x} dx$$

$$-x^2 - 2x = -(x^2 + 2x + 1) + 1 = -(x+1)^2 + 1$$

$$\begin{aligned}
\int_{-\infty}^{\infty} e^{-x^2-2x} dx &= e \int_{-\infty}^{\infty} e^{-(x+1)^2} dx \\
&= e\sqrt{2\pi \cdot \frac{1}{2}} \int_{-\infty}^{\infty} \underbrace{\frac{1}{\sqrt{2\pi \cdot \frac{1}{2}}} e^{-\frac{(x+1)^2}{2\cdot\frac{1}{2}}}}_{\text{PDF of } N(-1,\frac{1}{2})} dx \\
&= e\sqrt{2\pi \cdot \frac{1}{2}}
\end{aligned}$$

Cheap way of generating normal samples from uniform samples - Box-Muller

[**Step 1**] Generate two iid U_1 and U_2 from $U(0,1)$.

[**Step 2**] Set

$$V_i = 2U_i - 1$$

[**Step 3**] Repeat [**Step 1**] and [**Step 2**] until (V_1, V_2) is in inside the unit circle.

[**Step 4**] With $r^2 = V_1^2 + V_2^2 \leq 1$, set

$$Z_i = V_i \sqrt{-2 \frac{\log r^2}{r^2}}$$

Z_i are iid $N(0,1)$.

Cheap way of generating normal samples from uniform samples

[Step 1] Generate two iid U_1 and U_2 from $U(0,1)$.

[Step 2] Set

$$\Theta = 2\pi U_1, \qquad R^2 = -2\log U_2$$

[Step 3] Set

$$Z_1 = R\cos\Theta, \qquad Z_2 = R\sin\Theta$$

Z_1 and Z_2 are iid $N(0,1)$.

Background fact 1

For an uniform random variable U on $[0,1]$, let $V = -2\log U$. Then,

$$(1) \qquad V \sim Exp\left(\frac{1}{2}\right)$$

Background fact 2

For two iid samples X and Y from $N(0,1)$, let $S = X^2 + Y^2$ and $\Theta = \tan^{-1}\frac{Y}{X}$. Then,

$$\begin{aligned} &(1) \qquad \Theta \sim U(0,2\pi) \\ &(2) \qquad S \sim Exp\left(\frac{1}{2}\right) \\ &(3) \qquad S \text{ and } \Theta \text{ are independent} \end{aligned}$$

$$P(V \le v) = P(-2\log U \le v) = P\left(\log U \ge -\frac{1}{2}v\right) = P\left(U \ge e^{-\frac{1}{2}v}\right) = 1 - e^{-\frac{1}{2}v}$$

$$\Rightarrow \quad f_V(v) = \frac{1}{2}e^{-\frac{1}{2}v}$$

With $x = r\cos\theta = \sqrt{s}\cos\theta$, $y = r\sin\theta = \sqrt{s}\sin\theta$, and $s = r^2$,

$$\left|\frac{\partial(x,y)}{\partial(s,\theta)}\right| = \left|\begin{pmatrix} \frac{1}{2\sqrt{s}}\cos\theta & \frac{1}{2\sqrt{s}}\sin\theta \\ -\sqrt{s}\sin\theta & \sqrt{s}\cos\theta \end{pmatrix}\right| = \frac{1}{2}$$

$$\begin{aligned}
f_{S,\Theta}(s,\theta) &= f_{X,Y}(x,y)\left|\frac{\partial(x,y)}{\partial(s,\theta)}\right| \\
&= f_X(x)f_Y(y)\left|\frac{\partial(x,y)}{\partial(s,\theta)}\right| \\
&= \frac{1}{\sqrt{2\pi}}e^{-\frac{x^2}{2}}\frac{1}{\sqrt{2\pi}}e^{-\frac{y^2}{2}}\frac{1}{2} \\
&= \frac{1}{4\pi}e^{-\frac{x^2+y^2}{2}} \\
&= \underbrace{\frac{1}{2\pi}}_{\Theta\sim U(0,2\pi)} \quad \cdot \quad \underbrace{\frac{1}{2}e^{-\frac{1}{2}s}}_{R^2\sim Exp\left(\frac{1}{2}\right)}
\end{aligned}$$

Skewness - Measure of symmetry of distribution wrt its mean

Definition

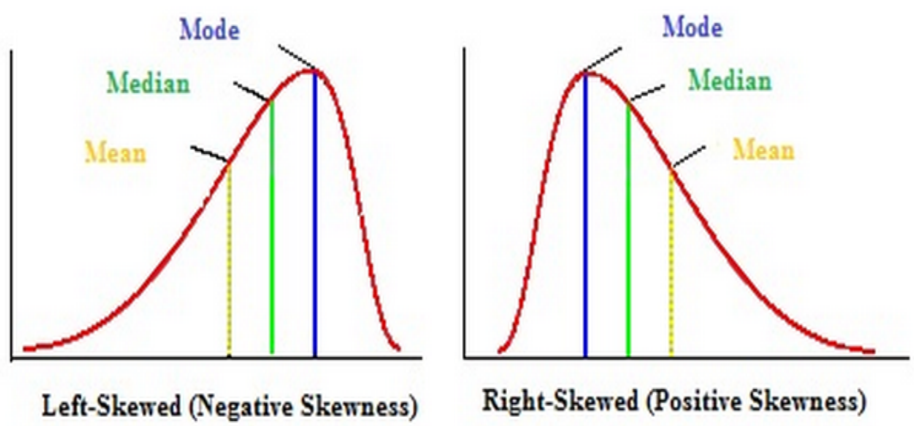
$$\text{Skewness}(X) = \mathbb{E}\left[\left(\frac{X-\mu}{\sigma}\right)^3\right] = \begin{cases} \sum_x \left(\frac{x-\mu}{\sigma}\right)^3 p(x) & \text{if } X \text{ is discrete} \\ \int_{-\infty}^{\infty} \left(\frac{x-\mu}{\sigma}\right)^3 f(x)dx & \text{if } X \text{ is continuous} \end{cases}$$

Interpretation

Negative skewness $\Rightarrow$ Left skewed

Zero skewness $\Rightarrow$ Balanced

Positive skewness $\Rightarrow$ Right skewed



Kurtosis - Measure of thickness of the tail of the distribution

Definition

$$\text{Kurtosis}(X) = \mathbb{E}\left[\left(\frac{X-\mu}{\sigma}\right)^4\right] = \begin{cases} \sum_x \left(\frac{x-\mu}{\sigma}\right)^4 p(x) & \text{if } X \text{ is discrete} \\ \int_{-\infty}^{\infty} \left(\frac{x-\mu}{\sigma}\right)^4 f(x)dx & \text{if } X \text{ is continuous} \end{cases}$$

and

$$\text{Excessive Kurtosis}(X) \quad = \quad \text{Kurtosis}(X) - 3$$

Interpretation

$$\begin{aligned} \text{Kurtosis} > 3 \text{ or Excessive Kurtosis} > 0 \quad &\Rightarrow \quad \text{Fat tail} \\ \text{Kurtosis} = 3 \text{ or Excessive Kurtosis} = 0 \quad &\Rightarrow \quad \text{Like normal distribution} \\ \text{Kurtosis} < 3 \text{ or Excessive Kurtosis} < 0 \quad &\Rightarrow \quad \text{Light tail} \end{aligned}$$

MGF (moment generating function)

$$\phi(t) = \phi_X(t) = Ee^{tX}$$

$\boxed{\text{Reason for name}}$

$$\begin{array}{llll}
\phi(t) = Ee^{tX} & \Rightarrow & \phi'(t) = EXe^{tX} & \Rightarrow \phi'(0) = EX \\
\phi(t) = Ee^{tX} & \Rightarrow & \phi''(t) = EX^2e^{tX} & \Rightarrow \phi''(0) = EX^2 \\
\vdots & \vdots & \vdots & \\
\phi(t) = Ee^{tX} & \Rightarrow & \phi^{(n)}(t) = EX^ne^{tX} & \Rightarrow \phi^{(n)}(0) = EX^n
\end{array}$$

Why MGF

(1) $\phi_X(t) = \phi_Y(t) \quad \Rightarrow \quad$ Distribution of $X \quad = \quad$ Distribution of Y

(2) $\phi_{X_n}(t) \to \phi_Y(t) \quad \Rightarrow \quad$ Distribution of $X_n \quad \to \quad$ Distribution of Y

Example of MGF

$$
\begin{array}{ll}
B(p) & \phi(t) = 1 + p(e^t - 1) \\[2ex]
B(n,p) & \phi(t) = \left[1 + p(e^t - 1)\right]^n \\[2ex]
Po(\lambda) & \phi(t) = e^{\lambda(e^t - 1)} \\[2ex]
\mathcal{N}(\mu, \sigma^2) & \phi(t) = e^{\mu t + \frac{1}{2}\sigma^2 t^2}
\end{array}
$$

$$
\phi_{B(p)}(t) = Ee^{tX} = e^t \times p + 1 \times (1-p) = 1 + p(e^t - 1)
$$

$$
\phi_{B(n,p)}(t) = \prod_{k=1}^{n} Ee^{tX_k} = \prod_{k=1}^{n} (1 + p(e^t - 1)) = (1 + p(e^t - 1))^n
$$

$$
\phi_{Po(\lambda)}(t) = \sum_{k=0}^{\infty} e^{tk} \frac{\lambda^k}{k!} e^{-\lambda} = \left(\sum_{k=0}^{\infty} \frac{(\lambda e^t)^k}{k!} \right) e^{-\lambda} = e^{\lambda e^t} e^{-\lambda} = e^{\lambda(e^t - 1)}
$$

$$
\begin{aligned}
\phi_{\mathcal{N}(\mu,\sigma^2)}(t) &= \int_{-\infty}^{\infty} e^{tx} \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\mu)^2}{2\sigma^2}} dx \\
&= e^{\mu t + \frac{1}{2}\sigma^2 t^2} \int_{-\infty}^{\infty} \underbrace{\frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-(\mu+\sigma^2 t))^2}{2\sigma^2}}}_{\text{PDF of } N(\mu+\sigma^2 t, \sigma^2)} dx = e^{\mu t + \frac{1}{2}\sigma^2 t^2}
\end{aligned}
$$

Poisson approximation revisited

If n is large, p is small, $np = \lambda$ is medium,

$$B(n,p) \approx Po(\lambda)$$

meaning, with $X \sim B(n,p)$, $Y \sim Po(\lambda)$, for any $k = 0, 1, 2, \ldots$

$$P(X = k) = \binom{n}{k} p^k q^{n-k} \quad \approx \quad P(Y = k) = e^{-\lambda} \frac{\lambda^k}{k!}$$

When n large, p small ($q \approx 1$), $np = \lambda$ medium	Exact		Approximate
Distribution of $\sum_{i=1}^{n} X_i$, where X_i are iid $B(p)$	$B(n,p)$	$\approx$	$Po(\lambda)$
(Exact) Mean match	np	$=$	λ
(Approximate) Variance match	npq	$\approx$	λ

$$\phi_{B(n,p)}(t) = (1 + p(e^t - 1))^n \approx \left(e^{p(e^t-1)}\right)^n = e^{\lambda(e^t-1)} = \phi_{Po(\lambda)}(t)$$

Properties of normal distribution

(1) X Normal $\Rightarrow$ $aX+b$ Normal

(2) $X,\ Y$ Normal $\Rightarrow$ $X+Y$ Normal if they are independent

With $X \sim \mathcal{N}(\mu, \sigma^2)$

$$\begin{aligned}\phi_{aX+b}(t) &= \mathbb{E}e^{t(aX+b)} = e^{bt}\mathbb{E}e^{atX} = e^{bt}\phi_X(at) = e^{bt}e^{\mu(at)+\frac{1}{2}\sigma^2(at)^2} \\ &= e^{(a\mu+b)t+\frac{1}{2}a^2\sigma^2t^2} = \phi_{N(a\mu+b,a^2\sigma^2)}(t)\end{aligned}$$

With two independent random variables $X \sim N(\mu_1, \sigma_1^2)$ and $Y \sim N(\mu_2, \sigma_2^2)$

$$\phi_{X+Y}(t) = \phi_X(t)\phi_Y(t) = e^{\mu_1 t+\frac{1}{2}\sigma_1^2t^2}e^{\mu_2 t+\frac{1}{2}\sigma_2^2t^2} = e^{(\mu_1+\mu_2)t+\frac{1}{2}(\sigma_1^2+\sigma_2^2)t^2} = \phi_{N(\mu_1+\mu_2,\sigma_1^2+\sigma_2^2)}(t)$$

Standardization and reverse standardization of normal distribution

Standardization

If X has mean μ and standard deviation σ, then

$$\frac{X-\mu}{\sigma} \quad \text{has mean 0 and standard deviation 1}$$

If X is normal in addition, $\frac{X-\mu}{\sigma}$ is also normal.

Reverse standardization

If X has mean 0 and standard deviation 1, then

$$\mu + \sigma X \quad \text{has mean } \mu \text{ and standard deviation } \sigma$$

If X is normal in addition, $\mu + \sigma X$ is also normal.

CLT

Properties of normal, not a CLT

X_i iid $\mathcal{N}(\mu, \sigma^2)$	Mean	Variance	Distribution
X_i	μ	σ^2	Normal
S_n	μn	$\sigma^2 n$	Normal
$\frac{S_n - \mu n}{\sigma\sqrt{n}}$	0	1	Normal

CLT

iid	Mean	Variance	Distribution
X_i	μ	σ^2	Not Normal
S_n	μn	$\sigma^2 n$	Not Normal, but approximately normal for large n
$\frac{S_n - \mu n}{\sigma\sqrt{n}}$	0	1	Not Normal, but approximately normal for large n

Let X_k be iid with mean μ and variance σ^2 and let $S_n = X_1 + X_2 + \cdots + X_n$. Then,

(1) S_n has mean μn and variance $\sigma^2 n$

(2) Dist of $S_n \approx$ Normal dist with same mean and variance if n is large

$$\text{Dist of } S_n \quad \approx \quad N(\mu n, \sigma^2 n) \qquad \text{if } n \text{ is large}$$

(3) Dist of $\frac{S_n - \mu n}{\sigma\sqrt{n}} \approx$ Normal dist with same mean and variance if n is large

[CLT] $$\text{Dist of } \frac{S_n - n\mu}{\sigma\sqrt{n}} \quad \approx \quad N(0,1) \qquad \text{if } n \text{ is large}$$

meaning, for any x

$$P\left(\frac{S_n - \mu n}{\sigma\sqrt{n}} \le x\right) \quad \to \quad N(x) = \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}} e^{-\frac{s^2}{2}}\, ds$$

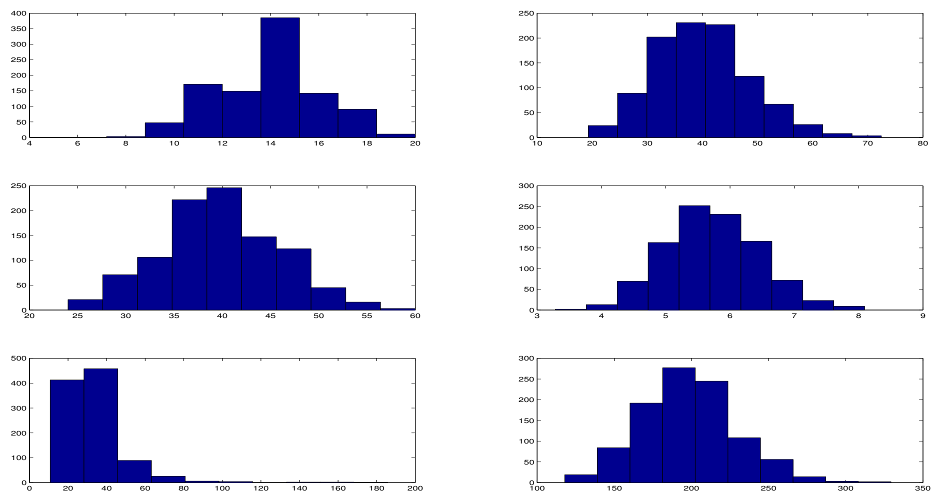


Figure 2: Empirical distribution of 1,000 samples of $S_{20} = X_1 + ... + X_{20}$, where X_i are iid $B(0.7)$ (top left), $Exp(2)$ (top right), $Po(2)$ (middle left), $Beta(2,5)$ (middle right), $F(2,5)$ (bottom left), $Gamma(2,5)$ (bottom right).

```
clear all; close all; clc; rng('default')

for i=1:6

    subplot(3,2,i)

    n=20; % Consider S_n=X_1+...+X_n
    N_Sim=1000; % Number of simulations

    if     (i==1), x=random('Binomial',1*ones(n,N_Sim),0.7*ones(n,N_Sim));
    elseif (i==2), x=random('exp',2*ones(n,N_Sim));
    elseif (i==3), x=random('Poisson',2*ones(n,N_Sim));
    elseif (i==4), x=random('Beta',2*ones(n,N_Sim),5*ones(n,N_Sim));
    elseif (i==5), x=random('F',2*ones(n,N_Sim),5*ones(n,N_Sim));
    elseif (i==6), x=random('Gamma',2*ones(n,N_Sim),5*ones(n,N_Sim));
    end

    A=sum(x);
    hist(A)

end
```

With $Y_k = \frac{X_k - \mu}{\sigma}$, iid with mean 0 and variance 1

$$\phi_{\frac{S_n - \mu n}{\sigma\sqrt{n}}}(t) = \phi_{\frac{\sum_{k=1}^n Y_k}{\sqrt{n}}}(t) = \left(Ee^{\frac{t}{\sqrt{n}}Y_1}\right)^n = \left(\phi_{Y_1}\left(\frac{t}{\sqrt{n}}\right)\right)^n$$

$$\begin{aligned} \phi_{Y_1}(t) &= Ee^{tY_1} = E\left(1 + tY_1 + \frac{(tY_1)^2}{2!} + \cdots\right) \\ &\approx E\left(1 + tY_1 + \frac{(tY_1)^2}{2!}\right) = 1 + tEY_1 + \frac{t^2}{2}EY_1^2 = 1 + \frac{1}{2}t^2 \end{aligned}$$

$$\phi_{\frac{S_n - n\mu}{\sigma\sqrt{n}}}(t) = \left(\phi_{Y_1}\left(\frac{t}{\sqrt{n}}\right)\right)^n \approx \left(1 + \frac{t^2}{2n}\right)^n \approx e^{\frac{t^2}{2}} = \phi_{N(0,1)}(t)$$

Example - Number of students in a psychology course

The number of students who enroll in a psychology course is a Poisson random variable with mean 100. The professor in charge of the course has decided that if the number enrolling is 120 or more, he will teach the course in two separate sections, whereas if fewer than 120 students enroll, he will teach all of the students together in a single section. What is the probability that the professor will have to teach two sections?

Exact solution using Poisson distribution

With $X \sim Po(\lambda)$, $\lambda = 100$,

$$P(X \geq 120) = \sum_{k=120}^{\infty} \frac{\lambda^k}{k!} e^{-\lambda} = 0.0282$$

Approximate solution using normal approximation

With $X = \sum_{i=1}^{100} Y_i \sim Po(\lambda)$, $\lambda = 100$, where Y_i are iid $Po(1)$,

$$\begin{aligned}
P(X \geq 120) &\overset{\text{Continuity correction}}{=} P(X \geq 119.5) \\
&\overset{\text{Standardization}}{=} P\left(\frac{X-\lambda}{\sqrt{\lambda}} \geq \frac{119.5-\lambda}{\sqrt{\lambda}}\right) \\
&\overset{\text{CLT}}{\approx} P\left(Z \geq \frac{119.5-\lambda}{\sqrt{\lambda}}\right) \\
&= 1 - N\left(\frac{119.5-\lambda}{\sqrt{\lambda}}\right) = 1 - 0.9744 = 0.0256
\end{aligned}$$

Example - Time for grading

An instructor has 50 exam papers that will be graded in sequence. Time required to grade the 50 exams are iid with mean 20 and standard deviation 4 minutes. Approximate the probability that the instructor will grade at least 25 of the exams in the first 450 minutes of work.

With time X_k to grade the k^{th} exam paper, iid with mean 20 and variance 4^2

$$S = \sum_{k=1}^{25} X_k$$

$$\begin{aligned} P(S \leq 450) &\overset{\text{Standardization}}{=} P\left(\frac{S - 25 \cdot 20}{\sqrt{25 \cdot 4^2}} \leq \frac{450 - 25 \cdot 20}{\sqrt{25 \cdot 4^2}}\right) \\ &\overset{\text{CLT}}{\approx} P\left(Z \leq \frac{450 - 25 \cdot 20}{\sqrt{25 \cdot 4^2}}\right) = N\left(\frac{450 - 25 \cdot 20}{\sqrt{25 \cdot 4^2}}\right) = 0.0062 \end{aligned}$$

Example - 95% confidence interval

An astronomer is interested in measuring the distance, in light-years, from his observatory to a distant star. Although the astronomer has a measuring technique, he knows that, because of changing atmospheric conditions and normal error, each time a measurement is made it will not yield the exact distance, but merely an estimate. As a result, the astronomer plans to make a series of measurements and then use the average value of these measurements as his estimated value of the actual distance. If the astronomer believes that the values of the measurements are independent and identically distributed random variables having a common mean d (the actual distance) and a common variance of 4 (light-years), how many measurements need he make to be reasonably sure that his estimated distance is accurate to within ± 0.5 light-year?

For X_k, iid with mean d and variance 4

$$\frac{\sum_{k=1}^{n} X_k - nd}{\sqrt{4n}} \approx N(0,1) \quad \Rightarrow \quad \bar{X} = \frac{\sum_{k=1}^{n} X_k}{n} \approx \sqrt{\frac{4}{n}} \cdot N(0,1) + d$$

$$\Rightarrow \quad |\bar{X} - d| \leq 1.96\sqrt{\frac{4}{n}} \leq 0.5 \quad \text{with 95\% confidence}$$

$$\Rightarrow \quad n \geq 61.4656 \quad \text{Need at least 62 measurements!}$$

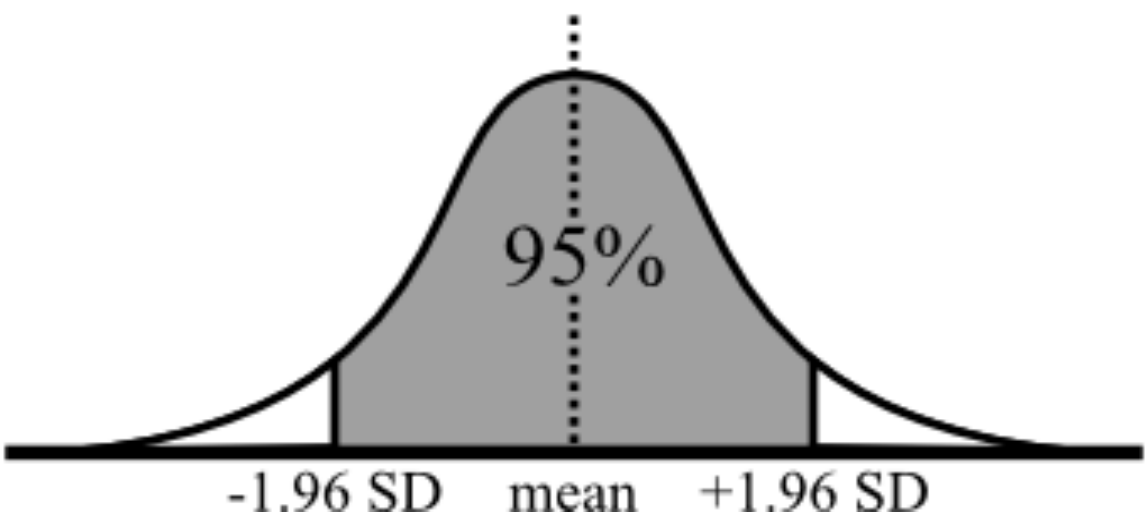


Figure 3: With 95% confidence we have $-1.96 \leq Z \leq 1.96$.

Example - Fair coin flips

We flips a fair coin many times and let X_i be the i-th flip record, where H and T are recorded as 1 and 0. Let Y_i be $Y_i = 2X_i - 1$, i.e., the i-th flip record where H and T are recorded as 1 and -1. Calculate the mean, variance, and approximate distribution of related random variables, i.e., fill up the blank in below table.

Random variable	Mean	Variance	Approximate distribution
Y_i			-
$\sum_{i=1}^{n} Y_i$			
$\frac{\sum_{i=1}^{n} Y_i}{\sqrt{n}}$			
$\frac{\sum_{i=1}^{nt} Y_i}{\sqrt{n}}$			
$\frac{\sum_{i=ns+1}^{nt} Y_i}{\sqrt{n}}$			

$$EY_i = 0, \quad EY_i^2 = 1, \quad Var(Y_i) = 1$$

Random variable	Mean	Variance	Approximate distribution
Y_i	0	1	-
$\sum_{i=1}^{n} Y_i$	0	n	$N(0,n)$
$\frac{\sum_{i=1}^{n} Y_i}{\sqrt{n}}$	0	1	$N(0,1)$
$\frac{\sum_{i=1}^{nt} Y_i}{\sqrt{n}}$	0	t	$N(0,t)$
$\frac{\sum_{i=ns+1}^{nt} Y_i}{\sqrt{n}}$	0	$t-s$	$N(0,t-s)$