Uniform distribution $U(a, b)$

$$f(x) = \frac{1}{b-a} \quad \text{for } a < x < b$$

Intuition

$$\text{Position of Poisson point given } N_{PPP(\lambda)}([a, b]) = 1$$

Mean and variance

$$\mathbb{E}X = \frac{a+b}{2}, \qquad Var(X) = \frac{1}{12}(b-a)^2$$

$$a^2 - b^2 = (a+b)(a-b)$$

$$\mathbb{E}X = \int_a^b x f(x) dx = \frac{1}{b-a} \int_a^b x dx = \frac{1}{b-a} \left[\frac{x^2}{2} \right]_a^b = \frac{a+b}{2}$$

$$\begin{aligned} a^3 + b^3 &= (a+b)(a^2 + b^2 - ab) \\ a^3 - b^3 &= (a-b)(a^2 + b^2 + ab) \end{aligned}$$

$$\mathbb{E}X^2 = \int_a^b x^2 f(x) dx = \frac{1}{b-a} \int_a^b x^2 dx = \frac{1}{b-a} \left[\frac{x^3}{3} \right]_a^b = \frac{a^2 + b^2 + ab}{3}$$

Example - Break the stick

We break the stick of length L into two pieces by choosing the break point uniformly over the interval $[0, L]$. Let X be the length of the longer stick. Find its mean and variance.

$$X = \underbrace{\frac{L}{2}}_{\text{Half of the stick}} + \underbrace{Y}_{\text{The rest, Uniform on } [0, L/2]}$$

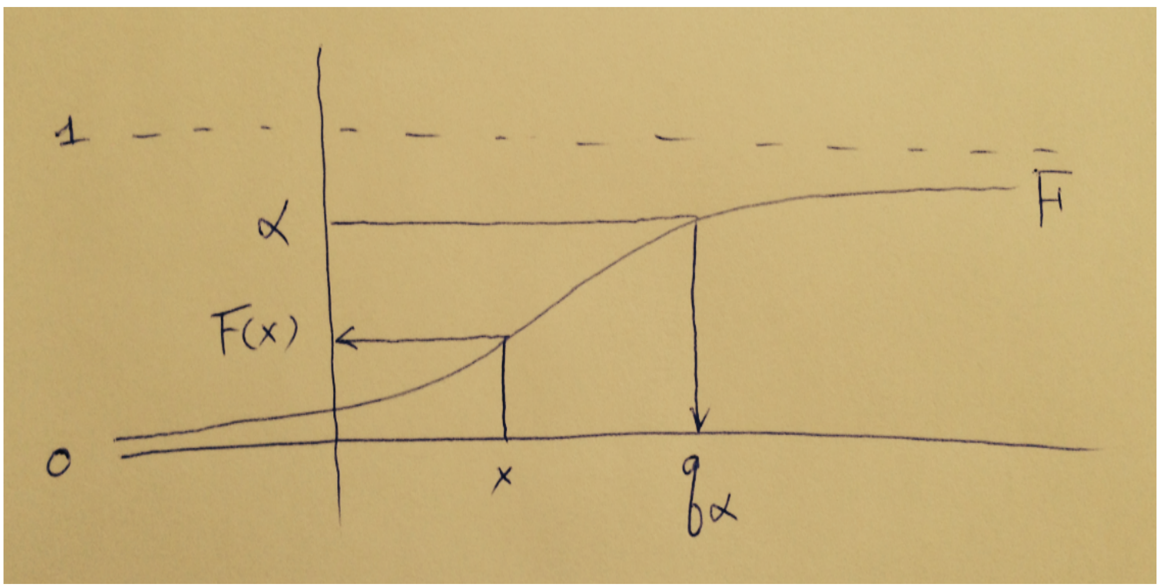
$$\begin{aligned}
\mathbb{E}[X] &= \frac{L}{2} + \mathbb{E}[Y] = \frac{L}{2} + \frac{L}{4} = \frac{3L}{4} \\
Var(X) &= Var(Y) = \frac{1}{12}\left(\frac{L}{2}\right)^2 = \frac{L^2}{48}
\end{aligned}$$

Simulation of a random variable X with a CDF F, using $U(0,1)$

[Step 1] Generate a random number U from $U(0,1)$.

[Step 2] $X = F^{-1}(U)$, or $X = \sup\{x \in \mathbf{R}; F(x) < U\}$ if F is not bijective.

$$\mathbb{P}(X \leq x) = \mathbb{P}(F^{-1}(U) \leq x) = \mathbb{P}(U \leq F(x)) = F(x)$$



Example - Simulation of a random variable $X \sim Exp(0.5)$

Suppose you have a random number generator which generates a random number U from $U(0,1)$. Generate a random number X from $Exp(0.5)$, using U.

$$\begin{aligned}\bar{F}(x) = e^{-0.5x} \quad \text{for } x \geq 0 \;\; &\Rightarrow \;\; F(x) = 1 - e^{-0.5x} \quad \text{for } x \geq 0 \\ &\Rightarrow \;\; X = F^{-1}(U) = -2\log(1-U) \sim Exp(0.5)\end{aligned}$$

$$U \sim U(0,1) \quad \Rightarrow \quad 1 - U \sim U(0,1) \quad \Rightarrow \quad X = -2\log(U) \sim Exp(0.5)$$

Convolution

Definition

$F_X * F_Y$	CDF of $X+Y$, when X and Y are independent
$p_X * p_Y$	PMF of $X+Y$, when X and Y are independent
$f_X * f_Y$	PDF of $X+Y$, when X and Y are independent

Computation

$$
\begin{aligned}
(F_X * F_Y)(a) &= F_{X+Y}(a) = \int_{-\infty}^{\infty} F_Y(a-b) \underbrace{dF_X(b)}_{P(b \le X \le b+db)} \\
(p_X * p_Y)(a) &= p_{X+Y}(a) = \sum_b p_Y(a-b) \underbrace{p_X(b)}_{P(X=b)} \\
(f_X * f_Y)(a) &= f_{X+Y}(a) = \int_{-\infty}^{\infty} f_Y(a-b) \underbrace{f_X(b)db}_{P(b \le X \le b+db)}
\end{aligned}
$$

Convolution of two Poisson

$$Po(\lambda_1) * Po(\lambda_2) = Po(\lambda_1 + \lambda_2)$$

With two independent $X \sim Po(\lambda_1)$ and $Y \sim Po(\lambda_2)$, for a non-negative integer a

$$\begin{aligned}
p_{X+Y}(a) &= \sum_b p_X(b) p_Y(a-b) \quad (b \geq 0,\ a-b \geq 0 \Rightarrow 0 \leq b \leq a \text{ integer}) \\
&= \sum_{b=0}^{a} \frac{\lambda_1^b}{b!} e^{-\lambda_1} \frac{\lambda_2^{a-b}}{(a-b)!} e^{-\lambda_2} \\
&= e^{-(\lambda_1+\lambda_2)} \sum_{b=0}^{a} \frac{\lambda_1^b}{b!} \frac{\lambda_2^{a-b}}{(a-b)!} = \frac{e^{-(\lambda_1+\lambda_2)}}{a!} \sum_{b=0}^{a} \frac{a!}{b!(a-b)!} \lambda_1^b \lambda_2^{a-b} \\
&= \frac{(\lambda_1+\lambda_2)^a}{a!} e^{-(\lambda_1+\lambda_2)}
\end{aligned}$$

Convolution of two Uniform

$$U\left(-\frac{1}{2},\frac{1}{2}\right) * U\left(-\frac{1}{2},\frac{1}{2}\right) = (1-|x|)^{+}$$

With two independent $X \sim U\left(-\frac{1}{2},\frac{1}{2}\right)$ and $Y \sim U\left(-\frac{1}{2},\frac{1}{2}\right)$, for $0 \leq a \leq 1$ (By symmetry we can figure out the rest if we understand the region $0 \leq a \leq 1$)

$$f_{X+Y}(a) = \int_{-\infty}^{\infty} f_X(b) f_Y(a-b) db$$

$$\begin{aligned}
-\frac{1}{2} \leq b \leq \frac{1}{2},\ -\frac{1}{2} \leq a-b \leq \frac{1}{2} &\Rightarrow -\frac{1}{2} \leq b \leq \frac{1}{2},\ -\frac{1}{2} \leq a-b,\ a-b \leq \frac{1}{2} \\
&\Rightarrow -\frac{1}{2} \leq b \leq \frac{1}{2},\ b \leq a+\frac{1}{2},\ a-\frac{1}{2} \leq b \quad (0 \leq a \leq 1) \\
&\Rightarrow -\frac{1}{2} \leq b \leq \frac{1}{2},\ b \leq \frac{1}{2},\ a-\frac{1}{2} \leq b \\
&\Rightarrow a-\frac{1}{2} \leq b \leq \frac{1}{2}
\end{aligned}$$

$$f_{X+Y}(a) = \int_{-\infty}^{\infty} f_X(b) f_Y(a-b) db = \int_{a-\frac{1}{2}}^{\frac{1}{2}} 1 db = 1-a$$

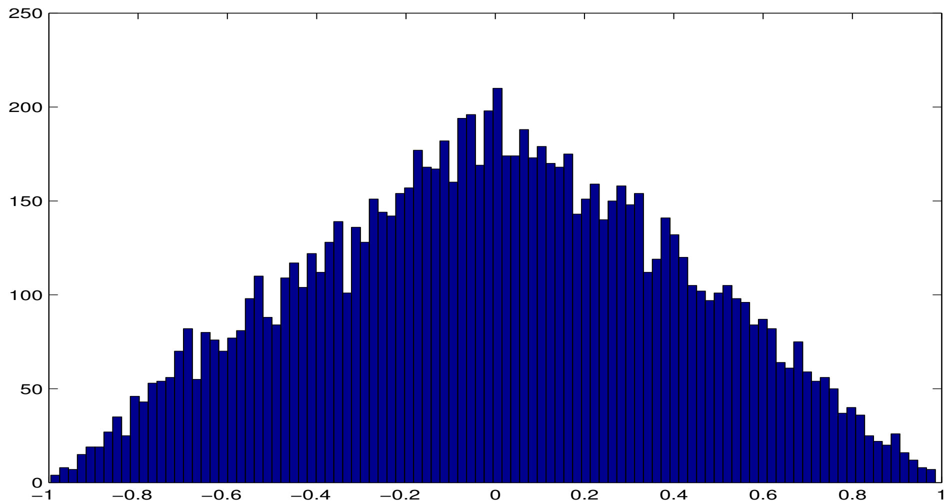


Figure 1: Empirical distribution of $X+Y$ where X and Y are iid $U\left(-\frac{1}{2},\frac{1}{2}\right)$.

```
clear all; close all; clc; rng('default')

n=2; % S_n
NumSimu=10000; % Number of simulations

x=rand(n,NumSimu)-0.5;
A=sum(x);
hist(A,100)
```

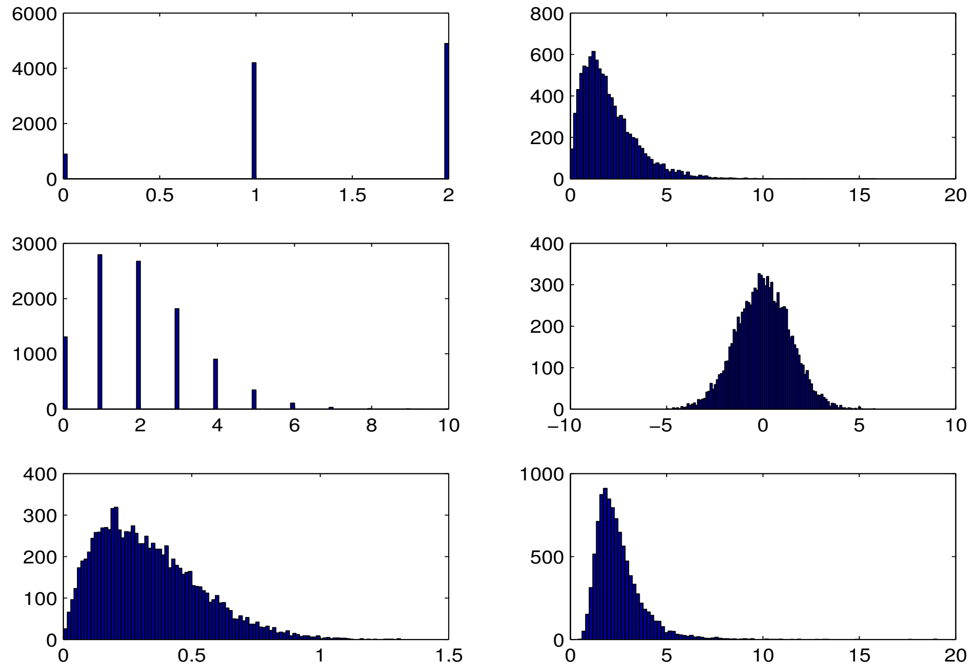


Figure 2: Empirical distribution of $X+Y$ where X and Y are iid $B(0.7)$ (top left), $Exp(1)$ (top right), $Po(1)$ (center left), $N(0,1)$ (center right), $Beta(1,5)$ (bottom left), and $F(20,10)$ (bottom right).

```
for i=1:6

    subplot(3,2,i)
    n=2; % Consider S_n=X_1+...+X_n
    N_Sim=10000; % Number of simulations

    if     (i==1), x=random('Binomial',1*ones(n,N_Sim),0.7*ones(n,N_Sim));
    elseif (i==2), x=random('exp',(1/la)*ones(n,N_Sim));
    elseif (i==3), x=random('Poisson',la*ones(n,N_Sim));
    elseif (i==4), x=random('Normal',0*ones(n,N_Sim),1*ones(n,N_Sim));
    elseif (i==5), x=random('Beta',1*ones(n,N_Sim),5*ones(n,N_Sim));
    elseif (i==6), x=random('F',20*ones(n,N_Sim),10*ones(n,N_Sim));
    end

    A=sum(x);
    hist(A,100)

end
```

Gamma function $\Gamma(\alpha)$

Definition

For $\alpha > 0$

$$\Gamma(\alpha) = \int_0^\infty x^{\alpha-1} e^{-x} dx$$

Properties

$$\begin{array}{ll} (1) & \Gamma(\alpha+1) = \alpha\Gamma(\alpha) \\ (2) & \Gamma(1/2) = \sqrt{\pi},\ \Gamma(1) = 1,\ \Gamma(2) = 1 \\ (3) & \Gamma(n+1) = n! \end{array}$$

$$\begin{aligned} \Gamma(\alpha+1) &= \int_0^\infty x^{(\alpha+1)-1} e^{-x} dx = \int_0^\infty -x^{(\alpha+1)-1} \left(e^{-x}\right)' dx \\ &= \left[-x^{(\alpha+1)-1} e^{-x}\right]_0^\infty - \int_0^\infty \left(-x^{(\alpha+1)-1}\right)' e^{-x} dx \\ &= \alpha \int_0^\infty x^{\alpha-1} e^{-x} dx = \alpha\Gamma(\alpha) \end{aligned}$$

With $s = \sqrt{x}$, $ds = \frac{dx}{2\sqrt{x}}$, using the integration technique on the normal distribution (You will see this integration technique in the chapter on the normal distribution)

$$\Gamma(1/2) = \int_0^\infty x^{-1/2} e^{-x} dx = 2\int_0^\infty e^{-s^2} ds = \sqrt{\pi}$$

Gamma distribution $\Gamma(\alpha, \lambda)$

PDF

$$f(x)dx = \frac{(\lambda x)^{\alpha-1}e^{-\lambda x}}{\Gamma(\alpha)}\lambda dx \quad \text{for } x > 0$$

Intuition

α-th arrival time of Poisson point of intensity λ

Mean and variance

	mean	variance
$Geo(p)$	$\frac{1}{p}$	$\frac{q}{p^2}$
$\frac{1}{n}Geo(p)$	$\frac{1}{np}$	$\frac{q}{(np)^2}$
$Exp(\lambda)$	$\frac{1}{\lambda}$	$\frac{1}{\lambda^2}$
$\Gamma(n, \lambda)$	$\frac{n}{\lambda}$	$\frac{n}{\lambda^2}$
$\Gamma(\alpha, \lambda)$	$\frac{\alpha}{\lambda}$	$\frac{\alpha}{\lambda^2}$

Related distributions

Exponential distribution $Exp(\lambda)$	$\Gamma(1, \lambda)$
Erlang distribution	$\Gamma(2, \lambda)$
Chi-square distribution χ_1^2	$\Gamma\left(\frac{1}{2}, \frac{1}{2}\right)$
Chi-square distribution χ_d^2	$\Gamma\left(\frac{d}{2}, \frac{1}{2}\right)$
Inverse gamma distribution $IG(\alpha, \lambda)$	Distribution of $\frac{1}{X}$, $X \sim \Gamma(\alpha, \lambda)$

$$
\begin{aligned}
\mathbb{E}[X] &= \int_0^\infty x \frac{\lambda(\lambda x)^{\alpha-1} e^{-\lambda x}}{\Gamma(\alpha)} dx \\
&= \frac{\Gamma(\alpha+1)}{\lambda\Gamma(\alpha)} \int_0^\infty \underbrace{\frac{\lambda(\lambda x)^{(\alpha+1)-1} e^{-\lambda x}}{\Gamma(\alpha+1)}}_{\text{PDF of } \Gamma(\alpha+1,\lambda)} dx = \frac{\alpha\Gamma(\alpha)}{\lambda\Gamma(\alpha)} = \frac{\alpha}{\lambda}
\end{aligned}
$$

$$
\begin{aligned}
\mathbb{E}[X^2] &= \int_0^\infty x^2 \frac{\lambda(\lambda x)^{\alpha-1} e^{-\lambda x}}{\Gamma(\alpha)} dx \\
&= \frac{\Gamma(\alpha+2)}{\lambda^2\Gamma(\alpha)} \int_0^\infty \underbrace{\frac{\lambda(\lambda x)^{(\alpha+2)-1} e^{-\lambda x}}{\Gamma(\alpha+2)}}_{\text{PDF of } \Gamma(\alpha+2,\lambda)} dx = \frac{(\alpha+1)\alpha\Gamma(\alpha)}{\lambda^2\Gamma(\alpha)} = \frac{\alpha(\alpha+1)}{\lambda^2}
\end{aligned}
$$

$$
Var(X) = \frac{\alpha}{\lambda^2}
$$

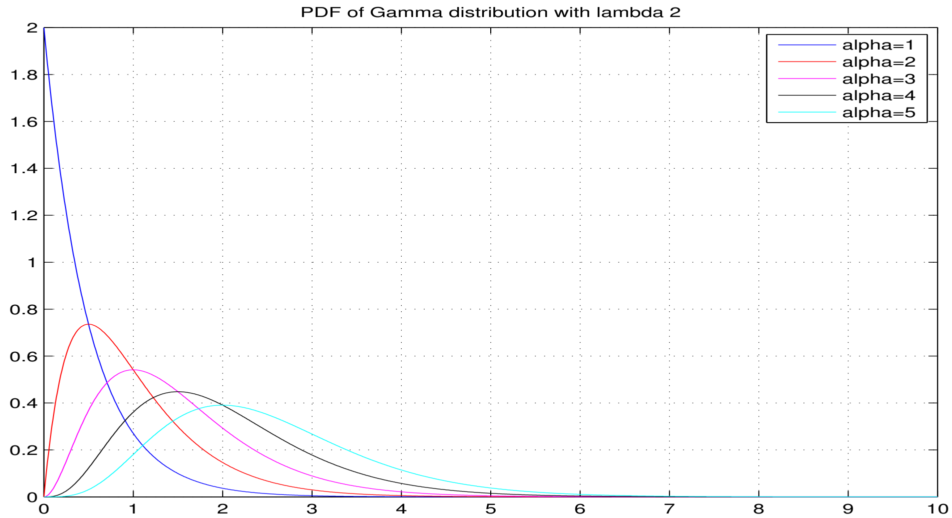


Figure 3: PDF of Gamma distribution with $\lambda = 2$.

```
clear all; close all; clc;

la=2;
x=0:0.01:10;

al=[1 2 3 4 5];
color='brmkc';

for i=1:length(al)

    y=pdf('gam',x,al(i),1/la);
    plot(x,y,color(i));
    axis([0 10 0 2]); hold on; grid on;
    lcontrol=legend('alpha=1','alpha=2','alpha=3','alpha=4','alpha=5');
    title('PDF of Gamma distribution with lambda 2')

end
```

Properties of Gamma distribution

$$
\begin{array}{ll}
(1) & Exp(\lambda) \stackrel{d}{=} \Gamma(1,\lambda) \\
(2) & Exp(\lambda) * Exp(\lambda) \stackrel{d}{=} \Gamma(2,\lambda) \\
(3) & Exp(\lambda) * Exp(\lambda) * \cdots * Exp(\lambda) \stackrel{d}{=} \Gamma(n,\lambda) \\
(4) & \Gamma(\alpha,\lambda) * \Gamma(\beta,\lambda) \stackrel{d}{=} \Gamma(\alpha+\beta,\lambda)
\end{array}
$$

$$
\underbrace{\lambda e^{-\lambda x}}_{Exp(\lambda)} = \underbrace{\frac{\lambda(\lambda x)^{1-1}e^{-\lambda x}}{\Gamma(1)}}_{\Gamma(1,\lambda)} \qquad \text{for } x \geq 0
$$

With independent X and Y, where $X \sim \Gamma(\alpha,\lambda)$, $Y \sim \Gamma(\beta,\lambda)$, for $x \geq 0$

$$
\begin{aligned}
f_{X+Y}(x) &= \int_{-\infty}^{\infty} f_X(s) f_Y(x-s)ds \qquad (x \geq 0,\ s \geq 0,\ x-s \geq 0 \ \Rightarrow\ 0 \leq s \leq x) \\
&= \int_0^x \frac{\lambda(\lambda s)^{\alpha-1}e^{-\lambda s}}{\Gamma(\alpha)} \cdot \frac{\lambda(\lambda(x-s))^{\beta-1}e^{-\lambda(x-s)}}{\Gamma(\beta)} ds \\
&= \frac{1}{\Gamma(\alpha)\Gamma(\beta)} \left[\int_0^x \lambda(\lambda s)^{\alpha-1}\lambda(\lambda(x-s))^{\beta-1} ds\right] e^{-\lambda x} \\
&= \frac{1}{\Gamma(\alpha)\Gamma(\beta)} \left[\int_0^x \left(\frac{\lambda s}{\lambda x}\right)^{\alpha-1} \left(\frac{\lambda(x-s)}{\lambda x}\right)^{\beta-1} \frac{ds}{x}\right] \lambda(\lambda x)^{\alpha+\beta-1}e^{-\lambda x} \\
&= \underbrace{\frac{1}{\Gamma(\alpha)\Gamma(\beta)} \left[\int_0^1 t^{\alpha-1}(1-t)^{\beta-1}dt\right]}_{\text{Constant; should be } \frac{1}{\Gamma(\alpha+\beta)}} \lambda(\lambda x)^{\alpha+\beta-1}e^{-\lambda x}
\end{aligned}
$$

Paradox of inter arrival times

We run the Poisson point process with intensity λ from $t=-\infty$ to $t=\infty$. T_1 is the first arrival time after $t=0$, T_2 is the inter arrival time between the first and second arrival,..., and T_n is the inter arrival time between the $(n-1)$-th and n-th arrival. Let τ be the inter arrival time containing 04/15/2013. Then,

$$
\begin{aligned}
&(1) \qquad T_i \text{ are iid } Exp(\lambda)\\
&(2) \qquad ET_i=\frac{1}{\lambda},\ Var(T_i)=\frac{1}{\lambda^2}\\
&(3) \qquad \tau \text{ is } Exp(\lambda)*Exp(\lambda)\stackrel{d}{=}\Gamma(2,\lambda), \text{ not } Exp(\lambda)\\
&(4) \qquad E\tau=\frac{2}{\lambda},\ Var(\tau)=\frac{2}{\lambda^2}
\end{aligned}
$$

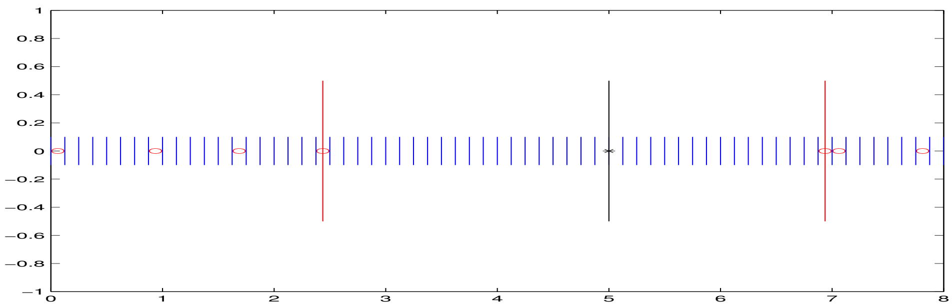


Figure 4: Inter arrival time containing 04/15/2013. The black line is 04/15/2013.

```
clear all; close all; clc; rng(7)

la=1;
n=8; % Range 0:n considered
i=4; % Poisson point process layer

% Generate PPP using coin flip instead of Exp coin flip
x=rand(n*2^(i-1),1);
p=la/2^(i-1);
index=find(x<p);
position=(index-0.5)/2^(i-1);
plot(position,zeros(length(position),1),'or'); hold on;
axis([0 8 -1 1]);

for j=0:2^(-i+1):n
    line([j j],[-0.1 0.1]);
end

% 04/15/2013
plot(5,0,'*k',[5 5],[-0.5 0.5],'Color','k');

% Poisson point left to 04/15/2013
i_left=find(position<5,1,'last');
x_left=position(i_left);
line([x_left x_left],[-0.5 0.5],'Color','r');

% Poisson point right to 04/15/2013
i_right=find(position>5,1,'first');
x_right=position(i_right);
line([x_right x_right],[-0.5 0.5],'Color','r');
```

How to get PDF

From CDF to PDF

$$P(X \le x) \quad \overset{\text{Differentiate}}{\Rightarrow} \quad f_X(x)$$

From Jacobian to PDF

$$\begin{aligned} f_Y(y) &= f_X(x)\left|\frac{dx}{dy}\right| \\ f_{U,V}(u,v) &= f_{X,Y}(x,y)\left|\frac{\partial(x,y)}{\partial(u,v)}\right| \\ f_{Y_1,\ldots,Y_n}(y_1,\ldots,y_n) &= f_{X_1,\ldots,X_n}(x_1,\ldots,x_n)\left|\frac{\partial(x_1,\ldots,x_n)}{\partial(y_1,\ldots,y_n)}\right| \end{aligned}$$

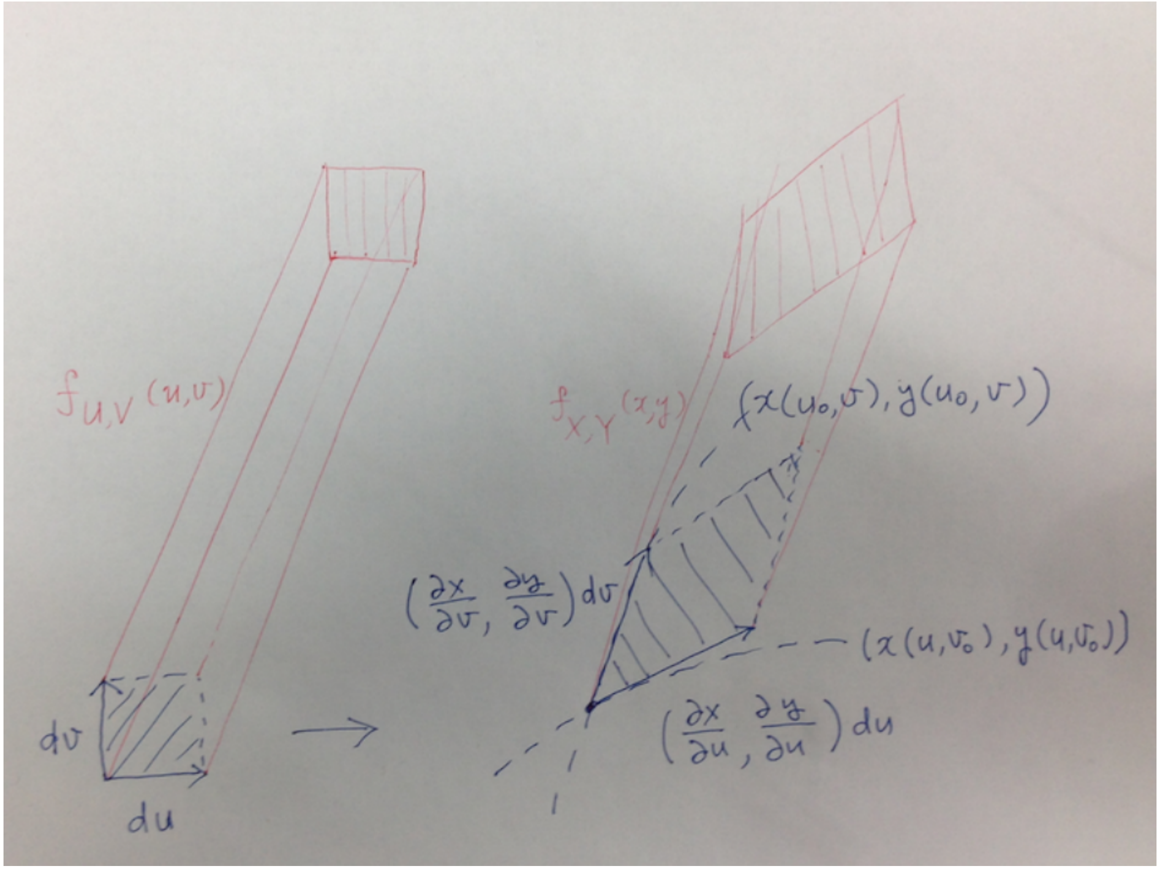
where

$$\left|\frac{\partial(x_1,\ldots,x_n)}{\partial(y_1,\ldots,y_n)}\right| = \left| det \begin{pmatrix} \frac{\partial x_1}{\partial y_1} & \cdots & \frac{\partial x_1}{\partial y_n} \\ \cdots & \cdots & \cdots \\ \frac{\partial x_n}{\partial y_1} & \cdots & \frac{\partial x_n}{\partial y_n} \end{pmatrix} \right|$$

Property of Jacobian

$$\left|\frac{\partial(x_1,\ldots,x_n)}{\partial(y_1,\ldots,y_n)}\right| = \frac{1}{\left|\frac{\partial(y_1,\ldots,y_n)}{\partial(x_1,\ldots,x_n)}\right|}$$

$$P(Y \le y) = P(X \le x) \quad \Rightarrow \quad f_Y(y) = f_X(x)\left|\frac{dx}{dy}\right|$$



Hight on left	$f_{U,V}(u_0, v_0)$
Area on left	$dudv$
Volumn on left	$f_{U,V}(u_0, v_0)dudv$
Hight on right	$f_{X,Y}(x_0, x_0)$
Area on right	$\left\lvert\frac{\partial(x,y)}{\partial(u,v)}\right\rvert dudv$
Volumn on right	$f_{X,Y}(x_0, x_0)\left\lvert\frac{\partial(x,y)}{\partial(u,v)}\right\rvert dudv$

$$f_{U,V}(u,v)dudu = f_{X,Y}(x,y)\left\lvert\frac{\partial(x,y)}{\partial(u,v)}\right\rvert dudv \quad \Rightarrow \quad f_{U,V}(u,v) = f_{X,Y}(x,y)\left\lvert\frac{\partial(x,y)}{\partial(u,v)}\right\rvert$$

Example - PDF of $Y = X^3$, where $X \sim U(0,1)$

Use CDF

$$P(Y \leq y) = P(X \leq y^{1/3}) = y^{1/3} \quad \overset{\text{Differentiate}}{\Rightarrow} \quad f_Y(y) = \frac{1}{3} y^{-2/3} \quad \text{for } 0 < y < 1$$

Use Jacobean

$$\frac{dy}{dx} = 3x^2 = 3\left(x^3\right)^{2/3} = 3y^{2/3} \quad \Rightarrow \quad \frac{dx}{dy} = 1/\left(\frac{dy}{dx}\right) = \frac{1}{3} y^{-2/3} \quad \Rightarrow \quad \left|\frac{dx}{dy}\right| = \frac{1}{3} y^{-2/3}$$

$$f_Y(y) = f_X(x) \left|\frac{dx}{dy}\right| = \frac{1}{3} y^{-2/3} \quad \text{for } 0 < y < 1$$

Beta function $B(\alpha, \beta)$

Definition

$$B(\alpha, \beta) = \int_0^1 x^{\alpha-1}(1-x)^{\beta-1}dx \qquad \text{for } \alpha > 0,\ \beta > 0$$

Properties of Beta function

$$B(\alpha, \beta) = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}$$

Beta distribution $Beta(\alpha, \beta)$

PDF

For $0 < x < 1$

$$f(x) \propto x^{\alpha-1}(1-x)^{\beta-1} \quad \Rightarrow \quad f(x) = \frac{x^{\alpha-1}(1-x)^{\beta-1}}{B(\alpha,\beta)}$$

Mean and variance of $Beta(\alpha, \beta)$

$$\frac{\alpha}{\alpha+\beta}, \qquad \frac{\alpha\beta}{(\alpha+\beta)^2(\alpha+\beta+1)}$$

Intuition - Fraction of waiting time

If $X \sim \Gamma(\alpha, \lambda)$ and $Y \sim \Gamma(\beta, \lambda)$ are independent, then

$$\begin{aligned}
&(1) \qquad T = X + Y \sim \Gamma(\alpha+\beta, \lambda) \\
&(2) \qquad F = \frac{X}{X+Y} \sim Beta(\alpha, \beta) \\
&(3) \qquad T \text{ and } F \text{ are independent} \\
&(4) \qquad B(\alpha,\beta) = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}
\end{aligned}$$

With $t = x + y$ and $f = \frac{x}{x+y}$,

$$\left|\frac{\partial(x,y)}{\partial(t,f)}\right| = \left|\frac{\partial(t,f)}{\partial(x,y)}\right|^{-1} = \left|det\begin{pmatrix} 1 & 1 \\ \frac{t-x}{t^2} & -\frac{x}{t^2}\end{pmatrix}\right|^{-1} = t$$

$$\begin{aligned}
f_{T,F}(t,f) &= f_{X,Y}(x,y)\left|\frac{\partial(x,y)}{\partial(t,f)}\right| \\
&= \frac{\lambda(\lambda x)^{\alpha-1}e^{-\lambda x}}{\Gamma(\alpha)} \cdot \frac{\lambda(\lambda y)^{\beta-1}e^{-\lambda y}}{\Gamma(\beta)} \cdot t \\
&= \underbrace{\left(\frac{f^{\alpha-1}(1-f)^{\beta-1}}{\Gamma(\alpha)\Gamma(\beta)/\Gamma(\alpha+\beta)}\right)}_{\text{a function of } f \text{ only!}} \underbrace{\left(\frac{1}{\Gamma(\alpha+\beta)}\lambda(\lambda t)^{(\alpha+\beta)-1}e^{-\lambda t}\right)}_{\text{a function of } t \text{ only!}} \\
&= \left(\frac{f^{\alpha-1}(1-f)^{\beta-1}}{B(\alpha,\beta)}\right)\left(\frac{\lambda(\lambda t)^{(\alpha+\beta)-1}e^{-\lambda t}}{\Gamma(\alpha+\beta)}\right)
\end{aligned}$$

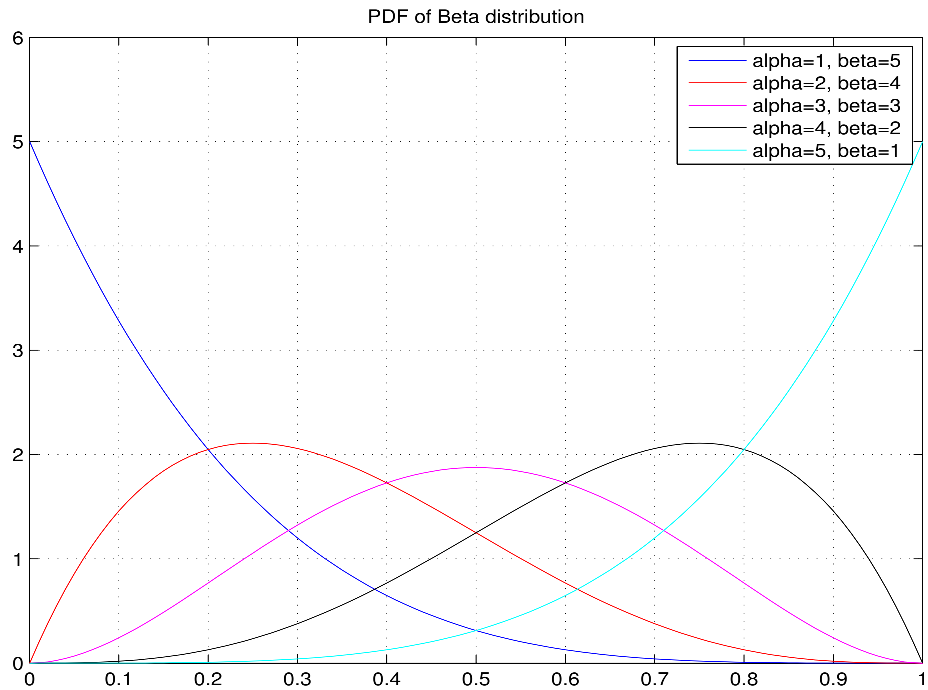


Figure 5: PDF of Beta distribution

```
clear all; close all; clc;

x=0:0.01:1;

al=[1 2 3 4 5];
bt=[5 4 3 2 1];
color='brmkc';

for i=1:length(al)
    y=pdf('Beta',x,al(i),bt(i));
    plot(x,y,color(i)); axis([0 1 0 6]); hold on; grid on;
end

legend('alpha=1, beta=5','alpha=2, beta=4','alpha=3, beta=3',...
    'alpha=4, beta=2','alpha=5, beta=1');
title('PDF of Beta distribution')
```

Example - Joint PDF of dependent random variables

The joint PDF $f(x,y)$ of X and Y is given by

$$f(x,y) = cxy, \qquad \text{for } 0 \le x \le 1,\ 0 \le y \le 1, \text{ and } 0 \le x+y \le 1$$

(a) Find c.
(b) Find the PDF $f_X(x)$ of X and the PDF $f_Y(y)$ of Y.
(c) Are X and Y independent?

$$
\begin{aligned}
(a)\quad \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)dxdy &= c\int_0^1\int_0^{1-y} xydxdy = \frac{c}{2}\int_0^1 y(1-y)^2dy \\
&= \frac{c}{2}B(2,3)\int_0^1 \underbrace{\frac{y^{2-1}(1-y)^{3-1}}{B(2,3)}}_{\text{PDF of } Beta(2,3)} dy \\
&= \frac{c}{2}B(2,3) = \frac{c}{2}\frac{\Gamma(2)\Gamma(3)}{\Gamma(5)} = \frac{c}{2}\frac{(1!)(2!)}{4!} \overset{\text{should be}}{=} 1 \\
&\Rightarrow \quad c = 4!
\end{aligned}
$$

(b) For $0 \le x \le 1$

$$
\begin{aligned}
f_X(x) &= \int_{-\infty}^{\infty} f(x,y)dy = \int_0^{1-x} f(x,y)dy \\
&= 4!\int_0^{1-x} xydy = 12x(1-x)^2 = \frac{x^{2-1}(1-x)^{3-1}}{B(2,3)} \\
\Rightarrow &\quad X \sim Beta(2,3) \quad \text{and by symmetry} \quad Y \sim Beta(2,3)
\end{aligned}
$$

(c) $0 \le X \le 1$ and $0 \le Y \le 1$. If they are independent, $X+Y$ can take values from 0 to 2. However, the joint PDF $f(x,y)$ does not put any mass on the region $x+y>1$. So, they cannot be independent, i.e., they are dependent.

You can see this dependency also from the joint PDF.

$$f(x,y) = 24xy1(0 \le x \le 1)1(0 \le y \le 1) \underbrace{1(0 \le x+y \le 1)}_{\text{Cannot decompose further}}$$

Example - Fraction of waiting time at bank

When I enter the bank, there is only one person in line waiting for the service and I join the queue. In the bank there are five service desks and we assume the service time is iid $Exp(\lambda_B)$, $\lambda_B^{-1} = 10$ (in minutes). After I got serviced at bank, I visit the post office. When I enter the post office, there are already two people in line waiting for the service and I join the queue. In the post office there are two service desks and we assume the service time is iid $Exp(\lambda_P)$, $\lambda_P^{-1} = 4$ (in minutes). Let F be the fraction of waiting time spent at bank among the total waiting time spent in both the bank and the post office. Calculate the mean and variance of F.

Waiting time T_B at bank is $T_B = X_1 + X_2$ where X_i are iid $Exp(5\lambda_B) = Exp(0.5)$. Hence

$$T_B = X_1 + X_2 \sim \Gamma(2, 0.5)$$

Waiting time T_P at post office is $T_P = Y_1 + Y_2 + Y_3$ where Y_i are iid $Exp(2\lambda_P) = Exp(0.5)$. Hence

$$T_P = Y_1 + Y_2 + Y_3 \sim \Gamma(3, 0.5)$$

$$T_B \sim \Gamma(2, 0.5), \quad T_P \sim \Gamma(3, 0.5) \quad \Rightarrow \quad F = \frac{T_B}{T_B + T_P} \sim Beta(2, 3)$$

With $\alpha = 2, \quad \beta = 3,$

$$EF = \frac{\alpha}{\alpha + \beta}, \qquad Var(F) = \frac{\alpha\beta}{(\alpha+\beta)^2(\alpha+\beta+1)}$$