Expectation and variance

Expectation

Definition

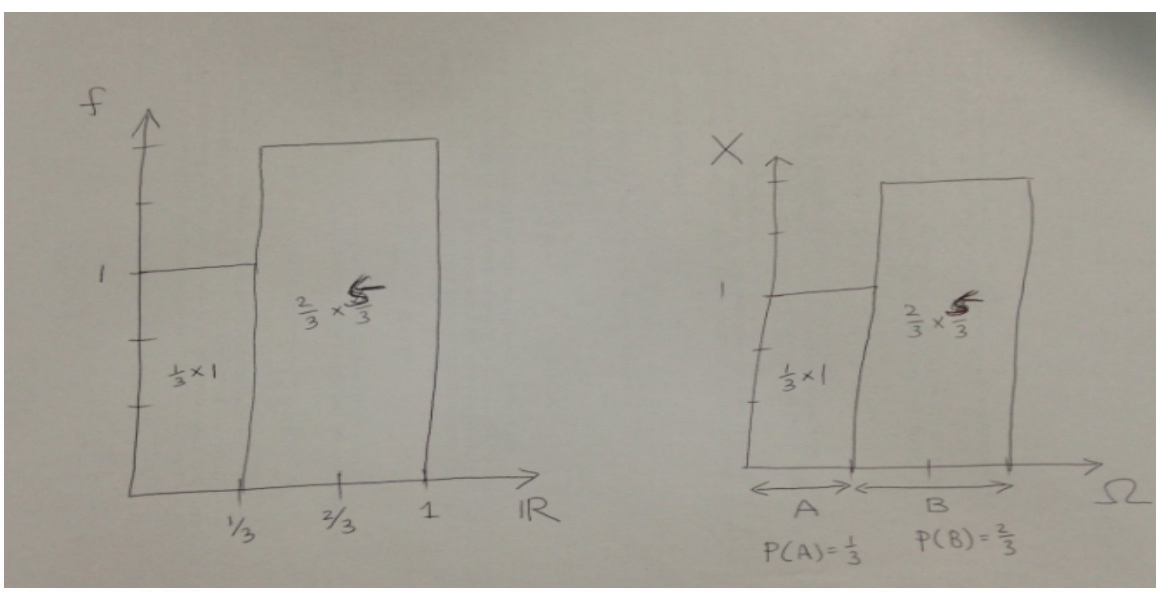
$$\mathbb{E}(X) = \sum_{x_i} x_i \times P(X = x_i) = \sum_{x_i} x_i \times p_{x_i}$$

Interpretation 1

$$\underbrace{\mathbb{E}(X)}_{\textbf{Expected payoff}} = \sum_{x_i} \underbrace{x_i}_{\textbf{Payoff}} \times \underbrace{p_{x_i}}_{\textbf{Probability}}$$

Interpretation 2

$$\underbrace{\mathbb{E}(X)}_{\textbf{Area under curve}} = \sum_{x_i} \underbrace{x_i}_{\textbf{Height}} \times \underbrace{P(X = x_i)}_{\textbf{Width}}$$



Properties of expectation

Expectation as a linear operator

$$(1) \qquad \mathbb{E}(X+Y) = \mathbb{E}(X) + \mathbb{E}(Y)$$
$$(2) \qquad \mathbb{E}(aX) = a\mathbb{E}(X)$$
$$(3) \qquad \mathbb{E}(a) = a$$

Change of variable - Recycle, save the earth

$$(4) \qquad \mathbb{E}[g(X)] = \sum_{x_i} g(x_i) p_{x_i}$$
$$(5) \qquad \mathbb{E}[g(X,Y)] = \sum_{x_i} \sum_{y_j} g(x_i, y_j) p_{x_i, y_j}$$

Product of independent random variables

$$(6) \qquad \mathbb{E}[XY] = \mathbb{E}[X]\,\mathbb{E}[Y] \qquad \text{if } X \text{ and } Y \text{ are independent}$$

$$(7) \qquad \mathbb{E}[g(X)h(Y)] = \mathbb{E}[g(X)]\,\mathbb{E}[h(Y)] \qquad \text{if } X \text{ and } Y \text{ are independent}$$

No free lunch

$$(8) \qquad X \geq 0 \quad \Rightarrow \quad \mathbb{E}(X) \geq 0$$

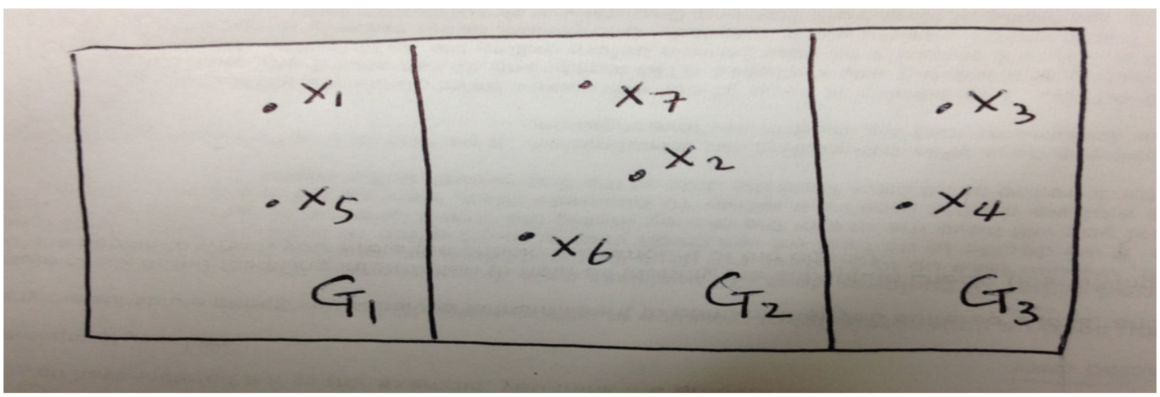
$$(9) \qquad X \geq Y \quad \Rightarrow \quad \mathbb{E}(X) \geq \mathbb{E}(Y)$$

$$(10) \qquad |\mathbb{E}(X)| \leq \mathbb{E}(|X|)$$

Cauchy-Schwartz inequality

$$(11) \qquad \mathbb{E}|XY| \leq (\mathbb{E}X^2)^{1/2}(\mathbb{E}Y^2)^{1/2}$$

$$
\begin{aligned}
\mathbb{E}(X+Y) &= \sum_{x_i}\sum_{y_j}(x_i+y_j)p_{x_i,y_j} \\
&= \sum_{x_i}\sum_{y_j} x_i p_{x_i,y_j} + \sum_{x_i}\sum_{y_j} y_j p_{x_i,y_j} \\
&= \sum_{x_i} x_i \left(\sum_{y_j} p_{x_i,y_j}\right) + \sum_{y_j} y_j \left(\sum_{x_i} p_{x_i,y_j}\right) \\
&= \sum_{x_i} x_i p_{x_i} + \sum_{y_j} y_j p_{y_j} \\
&= \mathbb{E}(X) + \mathbb{E}(Y)
\end{aligned}
$$



$$
\begin{aligned}
\mathbb{E}[g(X)] &= \sum_{g_k} g_k P(g(X)=g_k) \\
&= \sum_{g_k} g_k \left(\sum_{x_i \text{ with } g(x_i)=g_k} p_{x_i} \right) \\
&= \sum_{g_k} \left(\sum_{x_i \text{ with } g(x_i)=g_k} g(x_i) p_{x_i} \right) \\
&= \sum_{x_i} g(x_i) p_{x_i}
\end{aligned}
$$

$$\begin{aligned}\mathbb{E}[XY] &= \sum_{x_i}\sum_{y_j} x_i y_j p_{x_i,y_j} = \sum_{x_i}\sum_{y_j} x_i y_j p_{x_i} p_{y_j} \\ &= \left(\sum_{x_i} x_i p_{x_i}\right)\left(\sum_{y_j} y_j p_{y_j}\right) = \mathbb{E}[X]\mathbb{E}[Y]\end{aligned}$$

$$X \geq 0 \quad \Rightarrow \quad x_i \geq 0 \quad \Rightarrow \quad \mathbb{E}(X) = \sum_{x_i} x_i p_{x_i} \geq 0$$

$$\begin{aligned}X \geq Y \quad \Rightarrow \quad X - Y \geq 0 \quad &\Rightarrow \quad \mathbb{E}(X-Y) \geq 0 \\ &\Rightarrow \quad \mathbb{E}(X) - \mathbb{E}(Y) \geq 0 \quad \Rightarrow \quad \mathbb{E}(X) \geq \mathbb{E}(Y)\end{aligned}$$

Proof of Cauchy-Schwartz inequality

If $\mathbb{E}X^2 = 0$, $X = 0$ with probability 1 and hence $\mathbb{E}|XY| = 0$. By the same token, if $\mathbb{E}Y^2 = 0$, then $\mathbb{E}|XY| = 0$. In these two extreme cases (11) holds trivially. So, without loss of generality we assume that $\mathbb{E}X^2 > 0$ and $\mathbb{E}Y^2 > 0$.

With $t = -\mathbb{E}|XY|/\mathbb{E}Y^2$

$$\begin{aligned}(|X| + t|Y|)^2 = X^2 + 2t|XY| + t^2Y^2 \geq 0 \quad &\Rightarrow \quad \mathbb{E}X^2 + 2t\mathbb{E}|XY| + t^2\mathbb{E}Y^2 \geq 0 \\ &\Rightarrow \quad (\mathbb{E}X^2)(\mathbb{E}Y^2) \geq (\mathbb{E}|XY|)^2\end{aligned}$$

Example - Expectation of coin related random variables

Distribution	Expectation	Variance
$B(p)$	p	pq
$B(n,p)$	np	npq
$Geo(p)$	$\frac{1}{p}$	$\frac{q}{p^2}$
$NB(r,p)$	$\frac{r}{p}$	$\frac{rq}{p^2}$

$$X \sim B(p) \quad \Rightarrow \quad \mathbb{E}(X) = 1 \times p + 0 \times q = p$$

$$\begin{aligned} X \sim B(n,p) \quad &\Rightarrow \quad X = \sum_{i=1}^{n} X_i, \;\; X_i \text{ iid } B(p) \\ &\Rightarrow \quad \mathbb{E}(X) = \sum_{i=1}^{n} \mathbb{E}(X_i) = np \end{aligned}$$

$$\begin{aligned} 1 + x + x^2 + \cdots = \frac{1}{1-x} \quad &\overset{\text{Diff wrt } x}{\Rightarrow} \quad 1 + 2x + 3x^2 + \cdots = \frac{1}{(1-x)^2} \\ &\overset{\text{Let } x = q}{\Rightarrow} \quad 1 + 2q + 3q^2 + \cdots = \frac{1}{p^2} \\ &\Rightarrow \quad X \sim Geo(p) \quad \Rightarrow \quad \mathbb{E}(X) = \left(\sum_{k=1}^{\infty} k q^{k-1} \right) p = \frac{1}{p} \end{aligned}$$

$$\begin{aligned} X \sim NB(r,p) \quad &\Rightarrow \quad X = \sum_{i=1}^{r} X_i, \;\; X_i \text{ iid } Geo(p) \\ &\Rightarrow \quad \mathbb{E}(X) = \sum_{i=1}^{r} \mathbb{E}(X_i) = \frac{r}{p} \end{aligned}$$

Example - Maximization of expected profit

A newsboy purchases papers at 10 cents and sells them at 15 cents. However, he is not allowed to return unsold papers. If his daily demand X is $B(n, p)$ with $n = 10$, $p = 0.4$, approximately how many papers should he purchase so as to maximize his expected profit?

With t purchase, his profit $Y(t)$ and expected profit $f(t) = EY(t)$ are

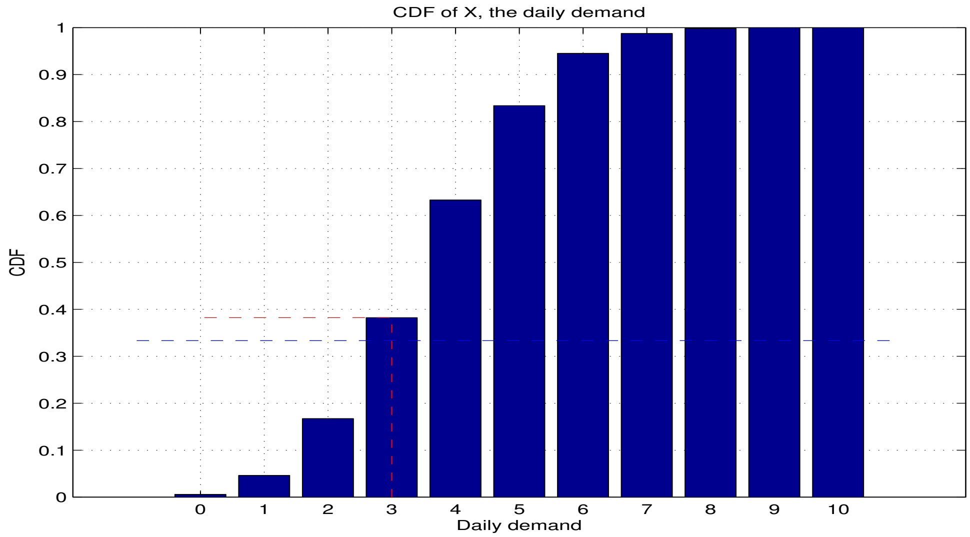
$$\begin{aligned} Y(t) &= \begin{cases} 5t & \text{if } X \geq t \\ 5X - 10(t - X) & \text{if } X < t \end{cases} \\ f(t) &= 5tP(X \geq t) + \sum_{i=0}^{t-1} (15i - 10t) P(X = i) \end{aligned}$$

To find an optimal t_0 that maximize $f(t)$, differentiate $f(t)$ discretely: Find t_0 with

$$\begin{aligned} f(t_0) - f(t_0 - 1) &= 15P(X \geq t_0) - 10 \geq 0 \\ f(t_0 + 1) - f(t_0) &= 15P(X \geq t_0 + 1) - 10 \leq 0 \end{aligned}$$

or

$$P(X \leq t_0 - 1) \leq \frac{1}{3} \qquad \text{and} \qquad P(X \leq t_0) \geq \frac{1}{3}$$




```
n=10;
p=0.4;

i=0:n;
pmf=binomial(n,i).*(p.^i).*((1-p).^(n-i));
cdf=cumsum(pmf);

% Find the smallest t0 such that P(X<=t0) exceeds 1/3
t=find(cdf>=1/3,1);
t0=t-1 % i starts from 0, but MATLAB index starts from 1

% Plot of CDF of X, the daily demand
bar(i,cdf); grid on; hold on;                       % Plot of CDF of X
plot(-1:0.1:n+1,1/3,'--b')                          % Critical level 1/3
plot([t0 t0 0],[0 cdf(t0+1) cdf(t0+1)],'--r')       % F(t0) >= 1/3
title('CDF of X, the daily demand')
xlabel('Daily demand'); ylabel('CDF');
```

Variance, covariance, correlation coefficient

Variance

$$Var(X) = \mathbb{E}(X - \mathbb{E}X)^2 = \mathbb{E}X^2 - (\mathbb{E}X)^2$$

Covariance

$$Cov(X,Y) = \mathbb{E}[(X - \mathbb{E}X)(Y - \mathbb{E}Y)] = \mathbb{E}(XY) - (\mathbb{E}X)(\mathbb{E}Y)$$

Correlation coefficient

$$-1 \le \rho = \frac{Cov(X,Y)}{\sqrt{Var(X)}\sqrt{Var(Y)}} \le 1$$

$$\begin{aligned} Var(X) &= \mathbb{E}(X - \mathbb{E}X)^2 \\ &= \mathbb{E}(X^2 - 2(\mathbb{E}X)X + (\mathbb{E}X)^2) \\ &= \mathbb{E}X^2 - 2(\mathbb{E}X)\mathbb{E}X + (\mathbb{E}X)^2 \\ &= \mathbb{E}X^2 - (\mathbb{E}X)^2 \end{aligned}$$

$$\begin{aligned} Cov(X,Y) &= \mathbb{E}[(X - \mathbb{E}X)(Y - \mathbb{E}Y)] \\ &= \mathbb{E}[XY - (\mathbb{E}Y)X - (\mathbb{E}X)Y + (\mathbb{E}X)(\mathbb{E}Y)] \\ &= \mathbb{E}(XY) - (\mathbb{E}Y)(\mathbb{E}X) - (\mathbb{E}X)(\mathbb{E}Y) + (\mathbb{E}X)(\mathbb{E}Y) \\ &= \mathbb{E}(XY) - (\mathbb{E}Y)(\mathbb{E}X) \end{aligned}$$

$$\begin{aligned} |Cov(X,Y)| &\le \mathbb{E}\,|(X - \mathbb{E}X)(Y - \mathbb{E}Y)| \\ &\le (\mathbb{E}(X - \mathbb{E}X)^2)^{1/2}(\mathbb{E}(Y - \mathbb{E}Y)^2)^{1/2} = \sqrt{Var(X)}\sqrt{Var(Y)} \end{aligned}$$

Properties of variance and covariance

$$\begin{array}{ll}
(1) & Var(X) = Cov(X,X) \\
(2) & Cov(aX+bY,Z) = aCov(X,Z) + bCov(Y,Z) \\
(2) & Cov(Z,aX+bY) = aCov(Z,X) + bCov(Z,Y) \\
(3) & Cov(X,Y) = Cov(Y,X) \\
(4) & Cov(X,a) = Cov(a,X) = 0 \\
(5) & Cov(X,Y) = 0 \qquad \text{if } X \text{ and } Y \text{ are independent}
\end{array}$$

$$Var(X) = \mathbb{E}(X-\mathbb{E}X)^2 = \mathbb{E}[(X-\mathbb{E}X)(X-\mathbb{E}X)] = Cov(X,X)$$

$$\begin{aligned}
Cov(aX+bY,Z) &= \mathbb{E}(aX+bY-a\mathbb{E}X-b\mathbb{E}Y)(Z-\mathbb{E}Z) \\
&= \mathbb{E}[a(X-\mathbb{E}X)+b(Y-\mathbb{E}Y)](Z-\mathbb{E}Z) \\
&= a\mathbb{E}[(X-\mathbb{E}X)(Z-\mathbb{E}Z)] + b\mathbb{E}[(Y-\mathbb{E}Y)(Z-\mathbb{E}Z)] \\
&= aCov(X,Z) + bCov(Y,Z)
\end{aligned}$$

$$Cov(X,Y) = \mathbb{E}[(X-\mathbb{E}X)(Y-\mathbb{E}Y)] = \mathbb{E}[(Y-\mathbb{E}Y)(X-\mathbb{E}X)] = Cov(Y,X)$$

$$Cov(X,a) = \mathbb{E}[(X-\mathbb{E}X)(a-a)] = 0$$

$$\begin{aligned}
X,\ Y \text{ independent} &\Rightarrow \mathbb{E}(XY) = \mathbb{E}(X)\mathbb{E}(Y) \\
&\Rightarrow Cov(X,Y) = \mathbb{E}(XY) - \mathbb{E}(X)\mathbb{E}(Y) = 0
\end{aligned}$$

Jensen's inequality

Definition

$\varphi : \mathbf{R} \to \mathbf{R}$ is convex if for any x, y and $0 < \lambda < 1$

$$\varphi(\lambda x + (1-\lambda)y) \leq \lambda\varphi(x) + (1-\lambda)\varphi(y)$$

$\varphi : \mathbf{R} \to \mathbf{R}$ is strictly convex if for any x, y and $0 < \lambda < 1$

$$\varphi(\lambda x + (1-\lambda)y) < \lambda\varphi(x) + (1-\lambda)\varphi(y)$$

Jensen's inequality

$$\text{If } \varphi \text{ convex,} \qquad \mathbb{E}\varphi(X) \geq \varphi(\mathbb{E}X)$$

$$\text{If } \varphi \text{ strictly convex,} \qquad \mathbb{E}\varphi(X) = \varphi(\mathbb{E}X) \quad \Leftrightarrow \quad X = \mathbb{E}X$$

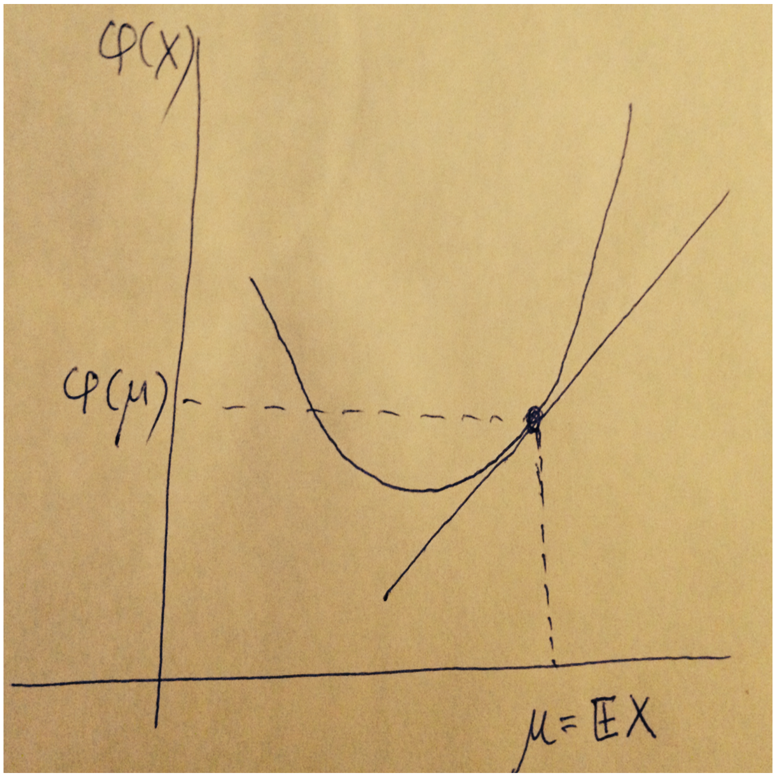
Example

$$\varphi(x) = x^2 \qquad \Rightarrow \quad \mathbb{E}X^2 \geq (\mathbb{E}X)^2$$

$$\varphi(x) = |x| \qquad \Rightarrow \quad \mathbb{E}|X| \geq |\mathbb{E}X|$$

$$\varphi(x) = e^x \qquad \Rightarrow \quad \mathbb{E}e^X \geq e^{\mathbb{E}X}$$

$$\varphi(x) = -\log x \quad \Rightarrow \quad \mathbb{E}\log X \leq \log \mathbb{E}X \quad \text{for } X > 0$$



$$\varphi(X) \geq \alpha(X-\mu)+\varphi(\mu) \overset{\text{Take expectation}}{\Rightarrow} \mathbb{E}\varphi(X) \geq \alpha\mathbb{E}(X-\mu)+\varphi(\mu) = \varphi(\mu) = \varphi(\mathbb{E}X)$$

Mean and variance of the sum of random variables

In general

$$\begin{aligned}
\mathbb{E}\left(\sum_{i=1}^{n} X_i\right) &= \sum_{i=1}^{n} \mathbb{E}(X_i) \\
Var\left(\sum_{i=1}^{n} X_i\right) &= \sum_{i=1}^{n} Var(X_i) + \sum_{i \neq j} Cov(X_i, X_j) \\
&= \sum_{i=1}^{n} Var(X_i) + 2 \sum_{1 \leq i < j \leq n} Cov(X_i, X_j)
\end{aligned}$$

If X_i are independent

$$\begin{aligned}
\mathbb{E}\left(\sum_{i=1}^{n} X_i\right) &= \sum_{i=1}^{n} \mathbb{E}(X_i) \\
Var\left(\sum_{i=1}^{n} X_i\right) &= \sum_{i=1}^{n} Var(X_i)
\end{aligned}$$

If X_i are iid

$$\begin{aligned}
\mathbb{E}\left(\sum_{i=1}^{n} X_i\right) &= \sum_{i=1}^{n} \mathbb{E}(X_i) = n\mathbb{E}(X_1) \\
Var\left(\sum_{i=1}^{n} X_i\right) &= \sum_{i=1}^{n} Var(X_i) = nVar(X_1)
\end{aligned}$$

Mean and variance of the weighted sum of random variables

In general

$$\begin{aligned}
\mathbb{E}\left(\sum_{i=1}^{n} a_i X_i\right) &= \sum_{i=1}^{n} a_i \mathbb{E}(X_i) \\
Var\left(\sum_{i=1}^{n} a_i X_i\right) &= \sum_{i=1}^{n} a_i^2 Var(X_i) + \sum_{i \neq j} a_i a_j Cov(X_i, X_j) \\
&= \sum_{i=1}^{n} Var(X_i) + 2 \sum_{1 \leq i < j \leq n} a_i a_j Cov(X_i, X_j)
\end{aligned}$$

If X_i are independent

$$\begin{aligned}
\mathbb{E}\left(\sum_{i=1}^{n} a_i X_i\right) &= \sum_{i=1}^{n} a_i \mathbb{E}(X_i) \\
Var\left(\sum_{i=1}^{n} a_i X_i\right) &= \sum_{i=1}^{n} a_i^2 Var(X_i)
\end{aligned}$$

If X_i are iid

$$\begin{aligned}
\mathbb{E}\left(\sum_{i=1}^{n} a_i X_i\right) &= \sum_{i=1}^{n} a_i \mathbb{E}(X_i) = \left(\sum_{i=1}^{n} a_i\right) \mathbb{E}(X_1) \\
Var\left(\sum_{i=1}^{n} a_i X_i\right) &= \sum_{i=1}^{n} a_i^2 Var(X_i) = \left(\sum_{i=1}^{n} a_i^2\right) Var(X_1)
\end{aligned}$$

Example - How to compute the variance

Let $Var(X) = 2$, $Var(Y) = 2$, $Var(Z) = 3$, $Cov(X, Y) = 0.25$. If Z is independent to both X and Y, compute the variance of V, where V is given by

$$V = X + 2Y - 3Z - 2$$

$$\begin{aligned}
Var(V) &\overset{(1)}{=} Cov(V,V) = Cov(X + 2Y - 3Z - 2, X + 2Y - 3Z - 2) \\
&\overset{(2)}{=} \begin{array}{rrrr} Cov(X,X) & +2Cov(X,Y) & -3Cov(X,Z) & -2Cov(X,1) \\ 2Cov(Y,X) & +4Cov(Y,Y) & -6Cov(Y,Z) & -4Cov(Y,1) \\ -3Cov(Z,X) & -6Cov(Z,Y) & +9Cov(Z,Z) & +6Cov(Z,1) \\ -2Cov(1,X) & -4Cov(1,Y) & +6Cov(1,Z) & +4Cov(1,1) \end{array} \\
&\overset{(3)}{=} \begin{array}{rrrr} Cov(X,X) & +2Cov(X,Y) & -3Cov(X,Z) & -2Cov(X,1) \\ 2Cov(X,Y) & +4Cov(Y,Y) & -6Cov(Y,Z) & -4Cov(Y,1) \\ -3Cov(X,Z) & -6Cov(Y,Z) & +9Cov(Z,Z) & +6Cov(Z,1) \\ -2Cov(X,1) & -4Cov(Y,1) & +6Cov(Z,1) & +4Cov(1,1) \end{array} \\
&= \begin{array}{rrrr} Cov(X,X) & +4Cov(X,Y) & -6Cov(X,Z) & -4Cov(X,1) \\ & +4Cov(Y,Y) & -12Cov(Y,Z) & -8Cov(Y,1) \\ & & +9Cov(Z,Z) & +12Cov(Z,1) \\ & & & +4Cov(1,1) \end{array} \\
&\overset{(4)}{=} \begin{array}{rrr} Cov(X,X) & +4Cov(X,Y) & -6Cov(X,Z) \\ & +4Cov(Y,Y) & -12Cov(Y,Z) \\ & & +9Cov(Z,Z) \end{array} \\
&\overset{(5)}{=} \begin{array}{rrr} Cov(X,X) & +4Cov(X,Y) & \\ & +4Cov(Y,Y) & \\ & & +9Cov(Z,Z) \end{array} \\
&\overset{(1)}{=} \begin{array}{rrr} Var(X) & +4Cov(X,Y) & \\ & +4Var(Y) & \\ & & +9Var(Z) \end{array} \\
&= \begin{array}{rrr} 2 & +4 \cdot 0.25 & \\ & +4 \cdot 2 & \\ & & +9 \cdot 3 \end{array} \\
&= 38
\end{aligned}$$

Example - Variance of coin related random variables

Distribution	Expectation	Variance
$B(p)$	p	pq
$B(n,p)$	np	npq
$Geo(p)$	$\frac{1}{p}$	$\frac{q}{p^2}$
$NB(r,p)$	$\frac{r}{p}$	$\frac{rq}{p^2}$

$$\begin{aligned} X \sim B(p) \quad &\Rightarrow \quad \mathbb{E}(X^2) = \mathbb{E}(X) = p \\ &\Rightarrow \quad Var(X) = \mathbb{E}(X^2) - (\mathbb{E}X)^2 = p - p^2 = p(1-p) = pq \end{aligned}$$

$$\begin{aligned} X \sim B(n,p) \quad &\Rightarrow \quad X = \sum_{i=1}^{n} X_i, \; X_i \text{ iid } B(p) \\ &\Rightarrow \quad Var(X) = \sum_{i=1}^{n} Var(X_i) = npq \end{aligned}$$

$$\begin{aligned} 1 + x + x^2 + \cdots = \frac{1}{1-x} \quad &\overset{\text{Diff wrt } x}{\Rightarrow} \quad 1 + 2x + 3x^2 + \cdots = \frac{1}{(1-x)^2} \\ &\overset{\text{Diff wrt } x}{\Rightarrow} \quad 2 \cdot 1 + 3 \cdot 2x + 4 \cdot 3x^2 \cdots = \frac{2}{(1-x)^3} \\ &\overset{\text{Let } x = q}{\Rightarrow} \quad 2 \cdot 1 + 3 \cdot 2q + 4 \cdot 3q^2 \cdots = \frac{2}{p^3} \end{aligned}$$

$$\begin{aligned} X \sim Geo(p) \quad &\Rightarrow \quad \mathbb{E}[X(X-1)] = \left(\sum_{k=1}^{\infty} k(k-1) q^{k-2} \right) qp = \frac{2q}{p^2} \\ &\Rightarrow \quad \mathbb{E}X^2 = \mathbb{E}[X(X-1) + X] = \mathbb{E}[X(X-1)] + \mathbb{E}X = \frac{2q}{p^2} + \frac{1}{p} \\ &\Rightarrow \quad Var(X) = \mathbb{E}X^2 - (\mathbb{E}X)^2 = \frac{2q}{p^2} + \frac{1}{p} - \frac{1}{p^2} = \frac{q}{p^2} \end{aligned}$$

$$\begin{aligned} X \sim NB(r,p) \quad &\Rightarrow \quad X = \sum_{i=1}^{r} X_i, \; X_i \text{ iid } Geo(p) \\ &\Rightarrow \quad Var(X) = \sum_{i=1}^{r} Var(X_i) = \frac{rq}{p^2} \end{aligned}$$

How to measure the typical size of error or deviation from mean

First try

$$\underbrace{\mathbb{E}}_{\text{Average}} \underbrace{(X - \mathbb{E}X)}_{\text{Error}}$$

However, this try is fertile: $\mathbb{E}(X - \mathbb{E}X) = \mathbb{E}X - \mathbb{E}X = 0$.

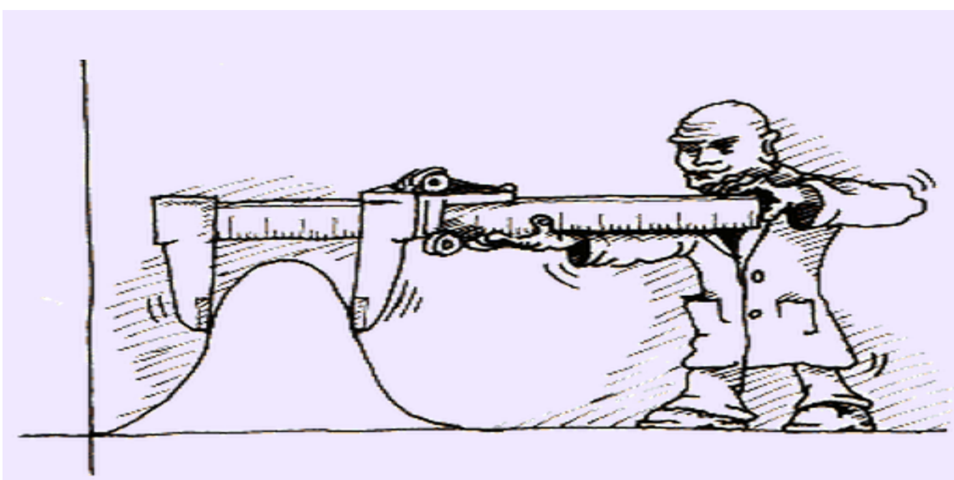
Second try

$$\underbrace{\mathbb{E}}_{\text{Average}} \left|\underbrace{X - \mathbb{E}X}_{\text{Error}}\right|$$

Due to computational difficulties, this measure of typical error size is not popular.

Standard way to measure typical error size

$$\underbrace{SD(X)}_{\text{Standard deviation}} = \sqrt{\underbrace{\mathbb{E}}_{\text{Average}} \left(\underbrace{X - \mathbb{E}X}_{\text{Error}}\right)^2} = \sqrt{Var(X)}$$



Standardization and reverse standardization

$\boxed{\text{Mean and variance lemma for standardization and reverse standardization}}$

$$
\begin{aligned}
\mathbb{E}(aX+b) &= a\mathbb{E}(X)+b \\
Var(aX+b) &= Var(aX) = a^2Var(X)
\end{aligned}
$$

$\boxed{\text{Standardization}}$

If X has mean μ and standard deviation σ, then

$$\frac{X-\mu}{\sigma} \quad \text{has mean 0 and standard deviation 1}$$

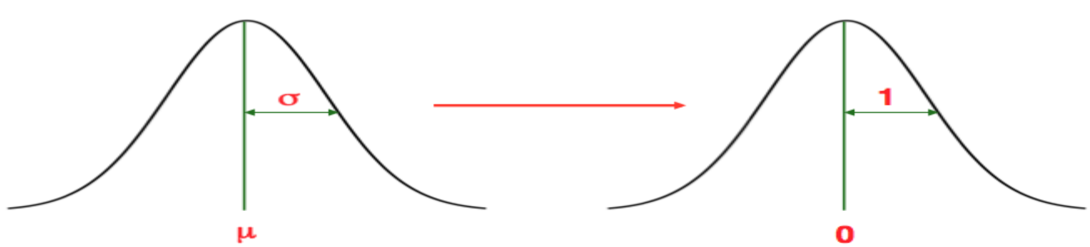
$\boxed{\text{Reverse standardization}}$

If X has mean 0 and standard deviation 1, then

$$\mu + \sigma * X \quad \text{has mean } \mu \text{ and standard deviation } \sigma$$

$$
\begin{aligned}
Var(aX+b) &= Cov(aX+b, aX+b) \\
&= a^2Var(X) + 2abCov(X,1) + b^2Var(1) \\
&= a^2Var(X)
\end{aligned}
$$

$$Var(aX) = Cov(aX, aX) = a^2Var(X)$$



Example - Standardization

We flips a fair coin many times.

X_i	i^{th} flip record, where H and T are recorded as 1 and 0
$Y_i := 2X_i - 1$	i^{th} flip record, where H and T are recorded as 1 and -1

Calculate the mean and variance of the following related random variables, i.e., fill up blanks of the below table.

Random variable	Mean	Variance
Y_i		
$\sum_{i=1}^n Y_i$		
$\frac{\sum_{i=1}^n Y_i}{\sqrt{n}}$		

$$
\begin{aligned}
X_i \text{ iid } B(0.5) \quad &\Rightarrow \quad \mathbb{E}X_i = 0.5,\ Var(X_i) = 0.5 * (1 - 0.5) = 0.25 \\
&\Rightarrow \quad Y_i \text{ iid with } \mathbb{E}Y_i = 0,\ Var(Y_i) = 1 \\
&\Rightarrow \quad \mathbb{E}\left(\sum_{i=1}^n Y_i\right) = 0,\ Var\left(\sum_{i=1}^n Y_i\right) = n \\
&\overset{\text{Standardization}}{\Rightarrow} \quad \frac{\sum_{i=1}^n Y_i}{\sqrt{n}} \quad \text{has mean 0, variance 1}
\end{aligned}
$$

Random variable	Mean	Variance
Y_i	0	1
$\sum_{i=1}^n Y_i$	0	n
$\frac{\sum_{i=1}^n Y_i}{\sqrt{n}}$	0	1