Multivariate normal distribution

Bivariate normal distribution $\mathcal{N}(\mu, \Sigma)$

PDF

$$f(\mathbf{x}) = \frac{1}{\sqrt{(2\pi)^2|\Sigma|}} e^{-\frac{1}{2}(\mathbf{x}-\mu)'\Sigma^{-1}(\mathbf{x}-\mu)}$$

where

mean	μ
covariance matrix	Σ
determinant of the covariance matrix Σ	$\|\Sigma\|$

Covariance matrix and its inverse

$$\Sigma = \begin{pmatrix} \sigma_x^2 & \rho\sigma_x\sigma_y \\ \rho\sigma_x\sigma_y & \sigma_y^2 \end{pmatrix} \Rightarrow |\Sigma| = (1-\rho^2)\sigma_x^2\sigma_y^2$$

$$\Rightarrow \Sigma^{-1} = \frac{1}{(1-\rho^2)\sigma_x^2\sigma_y^2} \begin{pmatrix} \sigma_y^2 & -\rho\sigma_x\sigma_y \\ -\rho\sigma_x\sigma_y & \sigma_x^2 \end{pmatrix}$$

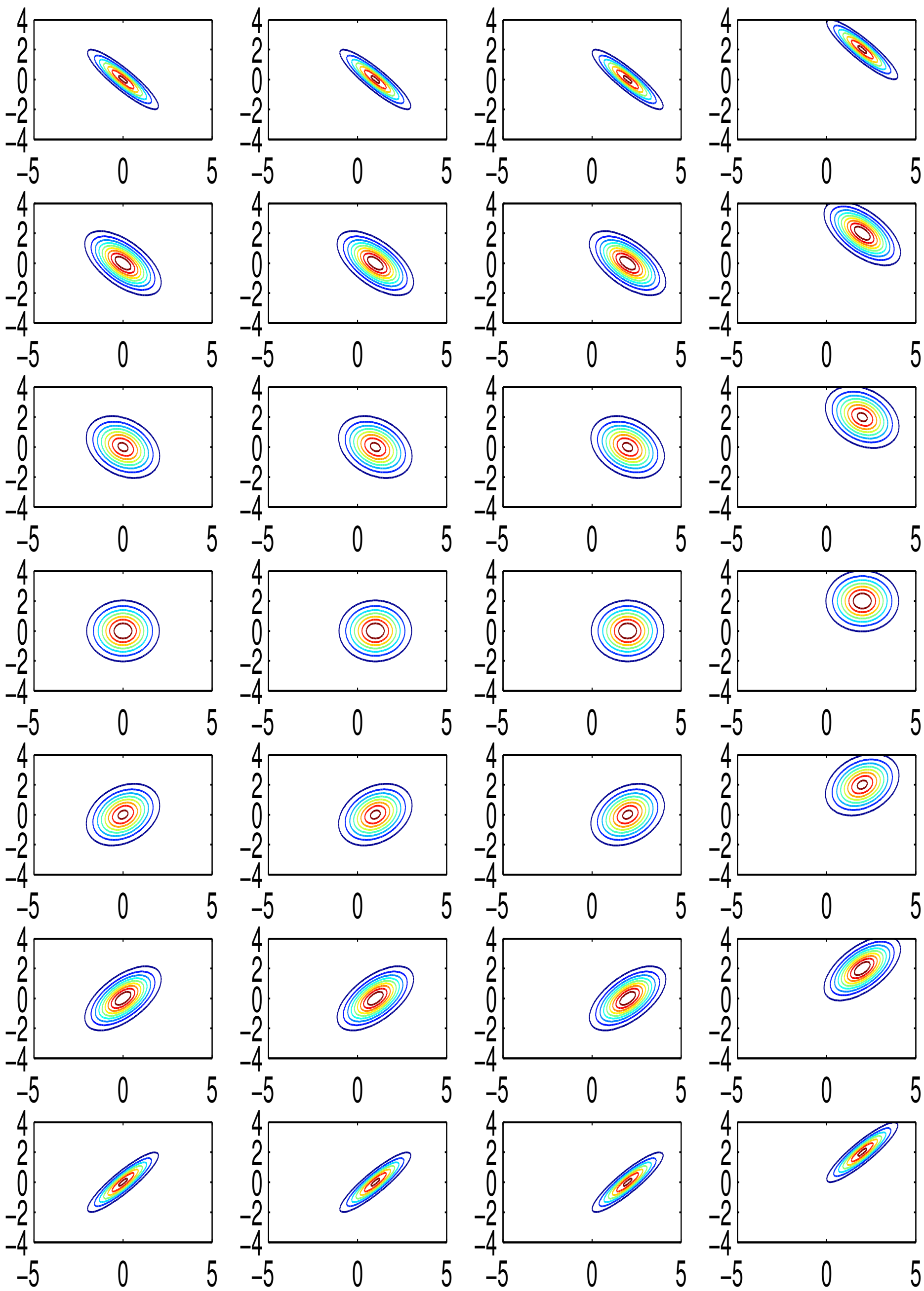
Another form of PDF

With $\tilde{x} = \frac{x-\mu_x}{\sigma_x}$, $\tilde{y} = \frac{y-\mu_y}{\sigma_y}$

$$(\mathbf{x}-\mu)'\Sigma^{-1}(\mathbf{x}-\mu) = \frac{\tilde{x}^2 + \tilde{y}^2 - 2\rho\tilde{x}\tilde{y}}{1-\rho^2}$$

$$f(\mathbf{x}) = \frac{1}{2\pi\sigma_x\sigma_y} e^{-\frac{\tilde{x}^2+\tilde{y}^2}{2}} \qquad \text{if } X \text{ and } Y \text{ are independent}$$

$$f(\mathbf{x}) = \frac{1}{2\pi\sigma_x\sigma_y\sqrt{1-\rho^2}} e^{-\frac{\tilde{x}^2+\tilde{y}^2-2\rho\tilde{x}\tilde{y}}{2(1-\rho^2)}} \qquad \text{in general}$$



```
clear all; close all; clc;

for i=1:7
    for j=1:4

        % Choose mean
        Choose_mean = j;
        switch Choose_mean
            case 1; mu = [0 0];
            case 2; mu = [1 0];
            case 3; mu = [2 0];
            case 4; mu = [2 2];
        end

        % Choose covariance matrix
        Choose_covariance_matrix = i;
        switch Choose_covariance_matrix
            case 1; Sigma = [1 -0.9; -0.9 1];
            case 2; Sigma = [1 -0.6; -0.6 1];
            case 3; Sigma = [1 -0.3; -0.3 1];
            case 4; Sigma = [1 0; 0 1];
            case 5; Sigma = [1 0.3; 0.3 1];
            case 6; Sigma = [1 0.6; 0.6 1];
            case 7; Sigma = [1 0.9; 0.9 1];
        end

        x = -5:0.1:5;
        y = -4:0.1:4;
        [X,Y] = meshgrid_Lee(x,y);
        F = mvnpdf([X(:) Y(:)],mu,Sigma);
        F = reshape(F,length(x),length(y));

        subplot(7,4,4*i+j-4)
        % mesh(X,Y,F);
        contour(X,Y,F);

    end
end
```

Multivariate normal distribution $\mathcal{N}(\mu, \Sigma)$

PDF

$$f(\mathbf{x}) = \frac{1}{\sqrt{(2\pi)^{d}|\Sigma|}} e^{-\frac{1}{2}(\mathbf{x}-\mu)'\Sigma^{-1}(\mathbf{x}-\mu)}$$

where

mean	μ
covariance matrix	Σ
determinant of the covariance matrix Σ	$\lvert\Sigma\rvert$

Definition

Multivariate normal $\mathbf{x} = [x_1, x_2, \cdots, x_d]'$ is given by

$$\underbrace{\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_d \end{bmatrix}}_{\mathbf{x}} = \underbrace{\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{d1} & a_{d2} & \cdots & a_{dn} \end{bmatrix}}_{\mathbf{A}} \underbrace{\begin{bmatrix} z_1 \\ z_2 \\ \vdots \\ z_n \end{bmatrix}}_{\mathbf{z}} + \underbrace{\begin{bmatrix} \mu_1 \\ \mu_2 \\ \vdots \\ \mu_d \end{bmatrix}}_{\mu},$$

$$\mathbf{x} = \mathbf{A}\mathbf{z} + \mu,$$

where z_k are iid standard normal and where $\mathbf{A}$ and μ are constants.

Computation of mean and covariance matrix

$$\begin{aligned} \mathbb{E}\mathbf{x} &= \mathbb{E}(\mathbf{A}\mathbf{z} + \mu) = \mathbf{A}\mathbb{E}\mathbf{z} + \mu = \mathbf{A}\mathbf{0} + \mu = \mu \\ \Sigma &= \mathbb{E}(\mathbf{x}-\mu)(\mathbf{x}-\mu)^T = \mathbb{E}(\mathbf{A}\mathbf{z})(\mathbf{A}\mathbf{z})^T = \mathbf{A}(\mathbb{E}\mathbf{z}\mathbf{z}^T)\mathbf{A}^T = \mathbf{A}\mathbf{I}\mathbf{A}^T = \mathbf{A}\mathbf{A}^T \end{aligned}$$

Joint MGF of multivariate normal distribution $\mathcal{N}(\mu, \Sigma)$

$$\begin{aligned} \phi(t) &= e^{\mu t + \frac{1}{2}\sigma^2 t^2} && \text{for } x \sim \mathcal{N}(\mu, \sigma^2) \\ \phi(t) &= e^{\mathbf{t}^T \mu + \frac{1}{2}\mathbf{t}^T \Sigma \mathbf{t}} && \text{for } \mathbf{x} \sim \mathcal{N}(\mu, \Sigma) \end{aligned}$$

With $\mathbf{z} = (z_k)^T$ iid $N(0, 1^2)$

$$\mathbf{x} \sim \mathcal{N}(\mu, \Sigma) \quad \Rightarrow \quad \mathbf{x} = \mathbf{A}\mathbf{z} + \mu \;\; \text{where } \mathbf{A}\mathbf{A}^T = \Sigma$$

$$\mathbf{t}^T \mathbf{x} = \mathbf{t}^T(\mathbf{A}\mathbf{z} + \mu) = \textstyle\sum_k a_k z_k + b \sim \mathcal{N}(\mu_1, \sigma_1^2)$$

where

$$\begin{aligned} \mu_1 &= \mathbb{E}\mathbf{t}^T(\mathbf{A}\mathbf{z} + \mu) = \mathbf{t}^T(\mathbf{A}\mathbb{E}\mathbf{z} + \mu) = \mathbf{t}^T(\mathbf{A}\mathbf{0} + \mu) = \mathbf{t}^T \mu \\ \sigma_1^2 &= \mathbb{E}(\mathbf{t}^T \mathbf{A}\mathbf{z})(\mathbf{t}^T \mathbf{A}\mathbf{z})^T = \mathbf{t}^T \mathbf{A} E(\mathbf{z}\mathbf{z}^T)\mathbf{A}^T \mathbf{t} = \mathbf{t}^T \mathbf{A}\mathbf{I}\mathbf{A}^T \mathbf{t} = \mathbf{t}^T \mathbf{A}\mathbf{A}^T \mathbf{t} = \mathbf{t}^T \Sigma \mathbf{t} \end{aligned}$$

$$\phi_{\mathbf{x}}(\mathbf{t}) = \mathbb{E}e^{\mathbf{t}^T \mathbf{x}} = \mathbb{E}e^{\mathbf{t}^T(\mathbf{A}\mathbf{z} + \mu)} = \phi_{\mathcal{N}(\mu_1, \sigma_1^2)}(1) = e^{\mu_1 + \frac{1}{2}\sigma_1^2} = e^{\mathbf{t}^T \mu + \frac{1}{2}\mathbf{t}^T \Sigma \mathbf{t}}$$

Properties of multivariate normal distribution $\mathcal{N}(\mu, \Sigma)$

(1) μ and Σ completely determine the multivariate normal distribution

(2) If off diagonals of Σ are all 0, then all the components of $\mathbf{x}$ are independent

(3) If for fixed i, $\Sigma_{ij} = 0$ for all $j \neq i$, then $\mathbf{x}_i$ is independent to $\mathbf{x}_j$, $j \neq i$

Suppose two multivariate normal random variables $\mathbf{x}$ and $\mathbf{y}$ have common mean μ and covariance matrix Σ. Then, their joint MGFs of $\mathbf{x}$ and $\mathbf{y}$ are identical:

$$\phi_{\mathbf{x}}(\mathbf{t}) = e^{\mathbf{t}^T\mu + \frac{1}{2}\mathbf{t}^T\Sigma\mathbf{t}} = \phi_{\mathbf{y}}(\mathbf{t})$$

Hence, they have same distribution.

Suppose a multivariate normal random variable $\mathbf{x}$ have a covariance matrix Σ, whose off diagonals are all 0. Then, joint MGFs of $\mathbf{x}$ and $\mathbf{y}$, where $\mathbf{y}_i$ are independent $\mathcal{N}(\mu_i, \sigma_i^2)$, are identical:

$$\phi_{\mathbf{x}}(\mathbf{t}) = \phi_{\mathcal{N}(\mu,\Sigma)}(\mathbf{t}) = e^{\mathbf{t}^T\mu + \frac{1}{2}\mathbf{t}^T\Sigma\mathbf{t}} = \underbrace{e^{\mu_1 t_1 + \frac{1}{2}\sigma_1^2 t_1^2}}_{\phi_{N(\mu_1,\sigma_1^2)}(t_1)} \underbrace{e^{\mu_2 t_2 + \frac{1}{2}\sigma_2^2 t_2^2}}_{\phi_{N(\mu_2,\sigma_2^2)}(t_2)} \cdots \underbrace{e^{\mu_d t_d + \frac{1}{2}\sigma_d^2 t_d^2}}_{\phi_{N(\mu_d,\sigma_d^2)}(t_d)} = \phi_{\mathbf{y}}(\mathbf{t})$$

Hence, they have same distribution. In particular, all the components x_i of the multivariate normal random variable $\mathbf{x}$ are independent.

Suppose a multivariate normal random variable $\mathbf{x}$ have a covariance matrix Σ such that for fixed i, $\Sigma_{ij} = 0$ for all $j \neq i$. Then, joint MGFs of $\mathbf{x}$ and $\mathbf{y}$, where the mean and covariance matrix of $\mathbf{y}$ are identical to those of $\mathbf{x}$ and where in addition $\mathbf{y}_i$ is independent to $\mathbf{y}_j$, $j \neq i$, are identical. Hence, they have same distribution. In particular, $\mathbf{x}_i$ is independent to $\mathbf{x}_j$, $j \neq i$.

X and Y from the same multivariate normal $\mathbf{x}$, where $Cov(X, Y) = 0$

$$X,\, Y \text{ independent} \;\Rightarrow\; Cov(X, Y) = 0$$

Converse is not true in general. However, if $[X, Y]^T$ bivariate normal, its a completely different story.

$$\begin{array}{rrcl} & Cov(X,Y) = 0 & \Rightarrow & X,\, Y \text{ independent (wrong)} \\ X,\, Y \text{ normal}, & Cov(X,Y) = 0 & \Rightarrow & X,\, Y \text{ independent (wrong)} \\ [X,Y]^T \text{ bivariate normal}, & Cov(X,Y) = 0 & \Rightarrow & X,\, Y \text{ independent (correct)} \end{array}$$

Let X be standard normal. Independent to X we flip a fair coin and record its out put S as 1 if head and -1 if tail. Now, define Y as

$$Y = S \cdot X$$

Y is standard normal.

$$\begin{aligned} P(Y \le y) &= P(S=1)P(X \le y) + P(S=-1)P(X \ge -y) \\ &= \frac{1}{2}\int_{-\infty}^{y} \frac{1}{\sqrt{2\pi}} e^{-\frac{s^2}{2}} ds + \frac{1}{2}\int_{-y}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{s^2}{2}} ds \end{aligned}$$

$$f_Y(y) = \frac{1}{2}\frac{1}{\sqrt{2\pi}} e^{-\frac{y^2}{2}} + \frac{1}{2}\frac{1}{\sqrt{2\pi}} e^{-\frac{y^2}{2}} = \frac{1}{\sqrt{2\pi}} e^{-\frac{y^2}{2}}$$

$$\begin{aligned} EXY &= P(S=1)E(XY|S=1) + P(S=-1)E(XY|S=-1) \\ &= P(S=1)E(X^2) + P(S=-1)E(-X^2) = \frac{1}{2}E(X^2) - \frac{1}{2}E(X^2) = 0 \end{aligned}$$

$$Cov(X, Y) = 0$$

But, by the construction of Y, X and Y are not independent. For example, if $X = 2$, then Y is either 2 or -2.

How to generate random samples from $\mathcal{N}(\mu, \Sigma)$

Cholesky decomposition of covariance or correlation matrix Σ

$$\Sigma = \mathbf{LU}$$

where $\mathbf{L}$ is an lower triangular matrix, $\mathbf{U}$ is an upper triangular matrix, and

$$\mathbf{L} = \mathbf{U}^T$$

[Matlab] chol

```
U = chol(Sigma)
```

How to generate random samples from $\mathcal{N}(\mu, \Sigma)$

[**Step 1**] Do Cholesky decomposition, i.e., `U = chol(Sigma)`.

[**Step 2**] With iid standard normal samples $\mathbf{z}$

$$\mathbf{x} = \mu + \mathbf{Lz} = \mu + \mathbf{U}^T\mathbf{z}$$

$$\mathbb{E}\mathbf{x} = \mathbb{E}(\mu + \mathbf{Lz}) = \mu + \mathbf{L}\mathbb{E}\mathbf{z} = \mu + \mathbf{L0} = \mu$$

$$\mathbb{E}(\mathbf{x} - \mu)(\mathbf{x} - \mu)^T = \mathbb{E}(\mathbf{Lz})(\mathbf{Lz})^T = \mathbf{L}(\mathbb{E}\mathbf{z}\mathbf{z}^T)\mathbf{L}^T = \mathbf{LIU} = \mathbf{LU} = \Sigma$$

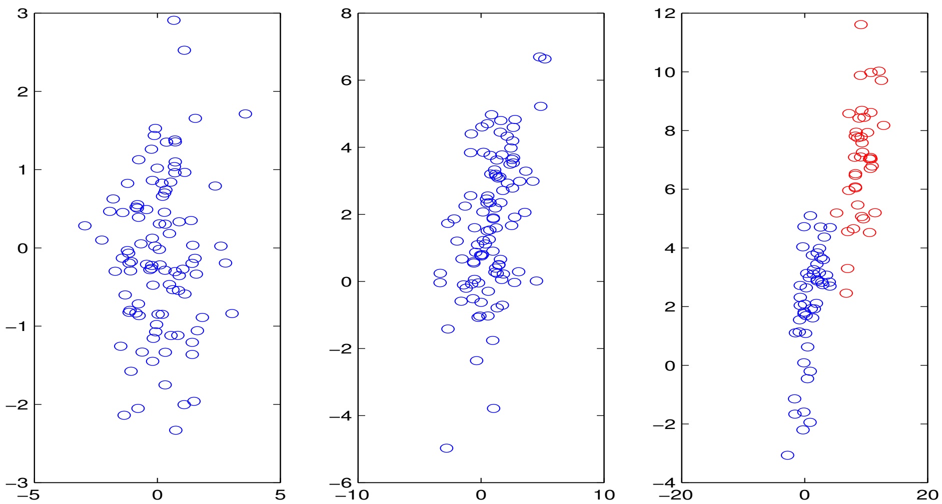


Figure 1: 100 standard normal samples in the plane (left), 100 samples from $N(\mu, \Sigma)$ (center), and 50 (blue) samples from $N(\mu_1, \Sigma_1)$ and 40 (blue) samples from $N(\mu_2, \Sigma_2)$ (right).

```
clear all; close all; clc; rng('default');

subplot(131)
n=100; x=randn(n,2); plot(x(:,1),x(:,2),'o')

subplot(132)
n=100;
Mu=[1 2]'; Si=[3 2; 2 5];
U=chol(Si);
x=repmat(Mu,1,n)+U'*randn(2,n);
plot(x(1,:),x(2,:),'o')

subplot(133)
n1=50;
Mu1=[1 2]'; Si1=[3 2; 2 5];
U1=chol(Si1); x1=repmat(Mu1,1,n1)+U1'*randn(2,n1);
plot(x1(1,:),x1(2,:),'o'); hold on
n2=40;
Mu2=[9 7]'; Si2=[3 1; 2 3];
U2=chol(Si2); x2=repmat(Mu2,1,n2)+U2'*randn(2,n2);
plot(x2(1,:),x2(2,:),'or')
```

Properties of Σ and Λ

$$\Sigma = \begin{bmatrix} \Sigma_{11} & \Sigma_{12} \\ \Sigma_{21} & \Sigma_{22} \end{bmatrix}, \quad \Lambda = \begin{bmatrix} \Lambda_{11} & \Lambda_{12} \\ \Lambda_{21} & \Lambda_{22} \end{bmatrix}$$

$\boxed{\Lambda_{11}^{-1} = \Sigma_{11} - \Sigma_{12}\Sigma_{22}^{-1}\Sigma_{21}}$

$$\Sigma\Lambda = \begin{bmatrix} \Sigma_{11}\Lambda_{11} + \Sigma_{12}\Lambda_{21} & \Sigma_{11}\Lambda_{12} + \Sigma_{12}\Lambda_{22} \\ \Sigma_{21}\Lambda_{11} + \Sigma_{22}\Lambda_{21} & \Sigma_{21}\Lambda_{12} + \Sigma_{22}\Lambda_{22} \end{bmatrix} = \begin{bmatrix} \mathbf{I}_{11} & \mathbf{0}_{12} \\ \mathbf{0}_{21} & \mathbf{I}_{22} \end{bmatrix}$$

$$\overset{\text{Remove } \Lambda_{21}}{\Rightarrow} \quad \Lambda_{11}^{-1} = \Sigma_{11} - \Sigma_{12}\Sigma_{22}^{-1}\Sigma_{21}$$

$\boxed{\Sigma_{11}\Lambda_{12} = -\Sigma_{12}\Lambda_{22}}$

$$\Sigma\Lambda = \begin{bmatrix} \Sigma_{11}\Lambda_{11} + \Sigma_{12}\Lambda_{21} & \Sigma_{11}\Lambda_{12} + \Sigma_{12}\Lambda_{22} \\ \Sigma_{21}\Lambda_{11} + \Sigma_{22}\Lambda_{21} & \Sigma_{21}\Lambda_{12} + \Sigma_{22}\Lambda_{22} \end{bmatrix} = \begin{bmatrix} \mathbf{I}_{11} & \mathbf{0}_{12} \\ \mathbf{0}_{21} & \mathbf{I}_{22} \end{bmatrix}$$

$\boxed{\Sigma_{21}\Lambda_{12} = -\Sigma_{22}\Lambda_{22} + \mathbf{I}_{22}}$

$$\Sigma\Lambda = \begin{bmatrix} \Sigma_{11}\Lambda_{11} + \Sigma_{12}\Lambda_{21} & \Sigma_{11}\Lambda_{12} + \Sigma_{12}\Lambda_{22} \\ \Sigma_{21}\Lambda_{11} + \Sigma_{22}\Lambda_{21} & \Sigma_{21}\Lambda_{12} + \Sigma_{22}\Lambda_{22} \end{bmatrix} = \begin{bmatrix} \mathbf{I}_{11} & \mathbf{0}_{12} \\ \mathbf{0}_{21} & \mathbf{I}_{22} \end{bmatrix}$$

$\boxed{\Lambda_{11}^{-1}\Lambda_{12} = -\Sigma_{12}\Sigma_{22}^{-1}}$

$$\begin{aligned} \Lambda_{11}^{-1}\Lambda_{12} &= (\Sigma_{11} - \Sigma_{12}\Sigma_{22}^{-1}\Sigma_{21})\Lambda_{12} \\ &= \Sigma_{11}\Lambda_{12} - \Sigma_{12}\Sigma_{22}^{-1}\Sigma_{21}\Lambda_{12} \\ &= -\Sigma_{12}\Lambda_{22} - \Sigma_{12}\Sigma_{22}^{-1}(-\Sigma_{22}\Lambda_{22} + \mathbf{I}_{22}) \\ &= -\Sigma_{12}\Sigma_{22}^{-1} \end{aligned}$$

Joint, marginal, and conditional distribution of $\mathcal{N}(\mu, \Sigma)$ - Part 1

Joint

$$\mathbf{x} = \begin{bmatrix} \mathbf{x}_1 \\ \mathbf{x}_2 \end{bmatrix} \sim \mathcal{N}\left(\begin{bmatrix} \mu_1 \\ \mu_2 \end{bmatrix}, \begin{bmatrix} \Sigma_{11} & \Sigma_{12} \\ \Sigma_{21} & \Sigma_{22} \end{bmatrix} \right)$$

Marginal

$$\mathbf{x}_1 \sim \mathcal{N}(\mu_1, \Sigma_{11})$$

Conditional

$$\mathbf{x}_1 | \mathbf{x}_2 \sim \mathcal{N}\left(\mu_{1|2}, \Sigma_{1|2}\right)$$

where

$$\begin{aligned} \mu_{1|2} &= \mu_1 - \Lambda_{11}^{-1}\Lambda_{12}(\mathbf{x}_2 - \mu_2) &&= \mu_1 + \Sigma_{12}\Sigma_{22}^{-1}(\mathbf{x}_2 - \mu_2) \\ \Sigma_{1|2} &= \Lambda_{11}^{-1} &&= \Sigma_{11} - \Sigma_{12}\Sigma_{22}^{-1}\Sigma_{21} \end{aligned}$$

With $\mathbf{y}_i = \mathbf{x}_i - \mu_i$

$$\begin{aligned} (\mathbf{x} - \mu)'\Sigma^{-1}(\mathbf{x} - \mu) &= \begin{bmatrix} \mathbf{y}_1^T & \mathbf{y}_2^T \end{bmatrix} \begin{bmatrix} \Lambda_{11} & \Lambda_{12} \\ \Lambda_{21} & \Lambda_{22} \end{bmatrix} \begin{bmatrix} \mathbf{y}_1 \\ \mathbf{y}_2 \end{bmatrix} \\ &= \mathbf{y}_1^T\Lambda_{11}\mathbf{y}_1 + \mathbf{y}_1^T\Lambda_{12}\mathbf{y}_2 + \mathbf{y}_2^T\Lambda_{21}\mathbf{y}_1 + \mathbf{y}_2^T\Lambda_{22}\mathbf{y}_2 \\ &\propto \mathbf{y}_1^T\Lambda_{11}\mathbf{y}_1 + \mathbf{y}_1^T\Lambda_{12}\mathbf{y}_2 + \mathbf{y}_2^T\Lambda_{21}\mathbf{y}_1 \\ &\propto (\mathbf{y}_1 - \alpha)^T\Lambda_{11}(\mathbf{y}_1 - \alpha) \end{aligned}$$

$$-\mathbf{y}_1^T\Lambda_{11}\alpha = \mathbf{y}_1^T\Lambda_{12}\mathbf{y}_2 \quad \Rightarrow \quad -\Lambda_{11}\alpha = \Lambda_{12}\mathbf{y}_2 \quad \Rightarrow \quad \alpha = -\Lambda_{11}^{-1}\Lambda_{12}\mathbf{y}_2$$

$$\mathbf{y}_1 - \alpha = \mathbf{x}_1 - \mu_1 + \Lambda_{11}^{-1}\Lambda_{12}\mathbf{y}_2 = \mathbf{x}_1 - \left(\mu_1 - \Lambda_{11}^{-1}\Lambda_{12}(\mathbf{x}_2 - \mu_2)\right) := \mathbf{x}_1 - \mu_{1|2}$$

$$\Sigma_{1|2} = \Lambda_{11}^{-1} = \Sigma_{11} - \Sigma_{12}\Sigma_{22}^{-1}\Sigma_{21}$$

Joint, marginal, and conditional distribution of $\mathcal{N}(\mu, \Sigma)$ - Part 2

Joint

$$\mathbf{x} \sim \mathcal{N}\left(\mu_{\mathbf{x}}, \Sigma_{\mathbf{x}}\right)$$

$$\mathbf{y} = \mathbf{A}\mathbf{x} + \mathbf{b} + \varepsilon, \qquad \varepsilon \sim \mathcal{N}\left(\mathbf{0}, \Sigma_{\varepsilon}\right) \qquad \varepsilon \text{ independent to } \mathbf{x}$$

$$\left[\begin{array}{c} \mathbf{x} \\ \mathbf{y} \end{array}\right] \sim \mathcal{N}\left(\left[\begin{array}{c} \mu_{\mathbf{x}} \\ \mathbf{A}\mu_{\mathbf{x}} + \mathbf{b} \end{array}\right], \left[\begin{array}{cc} \Sigma_{\mathbf{x}} & \Sigma_{\mathbf{x}}\mathbf{A}^T \\ \mathbf{A}\Sigma_{\mathbf{x}} & \mathbf{A}\Sigma_{\mathbf{x}}\mathbf{A}^T + \Sigma_{\varepsilon} \end{array}\right]\right)$$

Precison

$$\Lambda = \left[\begin{array}{cc} \Lambda_{11} & \Lambda_{12} \\ \Lambda_{21} & \Lambda_{22} \end{array}\right] = \left[\begin{array}{cc} \Sigma_{\mathbf{x}}^{-1} + \mathbf{A}^T\Sigma_{\varepsilon}^{-1}\mathbf{A} & -\mathbf{A}^T\Sigma_{\varepsilon}^{-1} \\ -\Sigma_{\varepsilon}^{-1}\mathbf{A} & \Sigma_{\varepsilon}^{-1} \end{array}\right]$$

Marginal

$$\mathbf{y} \sim \mathcal{N}\left(\mathbf{A}\mu_{\mathbf{x}} + \mathbf{b}, \mathbf{A}\Sigma_{\mathbf{x}}\mathbf{A}^T + \Sigma_{\varepsilon}\right)$$

Conditional

$$\mathbf{x}|\mathbf{y} \sim \mathcal{N}\left(\mu_{1|2}, \Sigma_{1|2}\right)$$

where

$$\begin{aligned} \mu_{1|2} &= \mu_1 + \Sigma_{12}\Sigma_{22}^{-1}(\mathbf{x}_2 - \mu_2) \\ &= \mu_{\mathbf{x}} + \Sigma_{\mathbf{x}}\mathbf{A}^T(\mathbf{A}\Sigma_{\mathbf{x}}\mathbf{A}^T + \Sigma_{\varepsilon})^{-1}(\mathbf{y} - (\mathbf{A}\mu_{\mathbf{x}} + \mathbf{b})) \end{aligned}$$

$$\begin{aligned} \Sigma_{1|2} &= \Sigma_{11} - \Sigma_{12}\Sigma_{22}^{-1}\Sigma_{21} \\ &= \Sigma_{\mathbf{x}} - \Sigma_{\mathbf{x}}\mathbf{A}^T(\mathbf{A}\Sigma_{\mathbf{x}}\mathbf{A}^T + \Sigma_{\varepsilon})^{-1}\mathbf{A}\Sigma_{\mathbf{x}} \end{aligned}$$

or using Woodbury identity $(\mathbf{A} + \mathbf{UCV})^{-1} = \mathbf{A}^{-1} - \mathbf{A}^{-1}\mathbf{U}(\mathbf{C}^{-1} + \mathbf{V}\mathbf{A}^{-1}\mathbf{U})^{-1}\mathbf{V}\mathbf{A}^{-1}$

$$\mu_{1|2} = \Sigma_{1|2}\left(\mathbf{A}^T\Sigma_{\varepsilon}^{-1}(\mathbf{y} - \mathbf{b}) + \Sigma_{\mathbf{x}}^{-1}\mu_{\mathbf{x}}\right)$$

$$\Sigma_{1|2}^{-1} = \Sigma_{\mathbf{x}}^{-1} + \mathbf{A}^T\Sigma_{\varepsilon}^{-1}\mathbf{A}$$

With $\tilde{\mathbf{x}} = \mathbf{x} - \mu_{\mathbf{x}}$, $\tilde{\mathbf{y}} = \mathbf{y} - \mathbf{A}\mu_{\mathbf{x}} - \mathbf{b}$

$$
\begin{aligned}
\log p(\mathbf{x}, \mathbf{y}) &= \log p(\mathbf{x}) + \log p(\mathbf{y}|\mathbf{x}) \\
&\propto -\frac{1}{2}(\mathbf{x} - \mu_{\mathbf{x}})^T \Sigma_{\mathbf{x}}^{-1}(\mathbf{x} - \mu_{\mathbf{x}}) - \frac{1}{2}(\mathbf{y} - \mathbf{A}\mathbf{x} - \mathbf{b})^T \Sigma_{\varepsilon}^{-1}(\mathbf{y} - \mathbf{A}\mathbf{x} - \mathbf{b}) \\
&= -\frac{1}{2}\tilde{\mathbf{x}}^T \Sigma_{\mathbf{x}}^{-1}\tilde{\mathbf{x}} - \frac{1}{2}(\tilde{\mathbf{y}} - \mathbf{A}\tilde{\mathbf{x}})^T \Sigma_{\varepsilon}^{-1}(\tilde{\mathbf{y}} - \mathbf{A}\tilde{\mathbf{x}}) \\
&= -\frac{1}{2}\begin{bmatrix} \mathbf{x} \\ \mathbf{y} \end{bmatrix}^T \begin{bmatrix} \Sigma_{\mathbf{x}}^{-1} + \mathbf{A}^T\Sigma_{\varepsilon}^{-1}\mathbf{A} & -\mathbf{A}^T\Sigma_{\varepsilon}^{-1} \\ -\Sigma_{\varepsilon}^{-1}\mathbf{A} & \Sigma_{\varepsilon}^{-1} \end{bmatrix} \begin{bmatrix} \mathbf{x} \\ \mathbf{y} \end{bmatrix} \\
&:= -\frac{1}{2}\begin{bmatrix} \mathbf{x} \\ \mathbf{y} \end{bmatrix}^T \Sigma^{-1} \begin{bmatrix} \mathbf{x} \\ \mathbf{y} \end{bmatrix} \\
&:= -\frac{1}{2}\begin{bmatrix} \mathbf{x} \\ \mathbf{y} \end{bmatrix}^T \begin{bmatrix} \Lambda_{11} & \Lambda_{12} \\ \Lambda_{21} & \Lambda_{22} \end{bmatrix} \begin{bmatrix} \mathbf{x} \\ \mathbf{y} \end{bmatrix}
\end{aligned}
$$

$$
\Rightarrow \quad \Lambda = \begin{bmatrix} \Lambda_{11} & \Lambda_{12} \\ \Lambda_{21} & \Lambda_{22} \end{bmatrix} = \begin{bmatrix} \Sigma_{\mathbf{x}}^{-1} + \mathbf{A}^T\Sigma_{\varepsilon}^{-1}\mathbf{A} & -\mathbf{A}^T\Sigma_{\varepsilon}^{-1} \\ -\Sigma_{\varepsilon}^{-1}\mathbf{A} & \Sigma_{\varepsilon}^{-1} \end{bmatrix}
$$

$$
\Rightarrow \quad \Sigma_{1|2}^{-1} = \Lambda_{11} = \Sigma_{\mathbf{x}}^{-1} + \mathbf{A}^T\Sigma_{\varepsilon}^{-1}\mathbf{A}
$$

$$
\begin{aligned}
\Rightarrow \quad \mu_{1|2} &= \mu_1 - \Lambda_{11}^{-1}\Lambda_{12}(\mathbf{x}_2 - \mu_2) \\
&= \Sigma_{1|2}\Sigma_{1|2}^{-1}\mu_{\mathbf{x}} + \Sigma_{1|2}\mathbf{A}^T\Sigma_{\varepsilon}^{-1}(\mathbf{y} - \mathbf{A}\mu_{\mathbf{x}} - \mathbf{b}) \\
&= \Sigma_{1|2}(\Sigma_{\mathbf{x}}^{-1} + \mathbf{A}^T\Sigma_{\varepsilon}^{-1}\mathbf{A})\mu_{\mathbf{x}} + \Sigma_{1|2}\mathbf{A}^T\Sigma_{\varepsilon}^{-1}(\mathbf{y} - \mathbf{A}\mu_{\mathbf{x}} - \mathbf{b}) \\
&= \Sigma_{1|2}\left[(\Sigma_{\mathbf{x}}^{-1} + \mathbf{A}^T\Sigma_{\varepsilon}^{-1}\mathbf{A})\mu_{\mathbf{x}} + \mathbf{A}^T\Sigma_{\varepsilon}^{-1}(\mathbf{y} - \mathbf{A}\mu_{\mathbf{x}} - \mathbf{b})\right] \\
&= \Sigma_{1|2}\left[\Sigma_{\mathbf{x}}^{-1}\mu_{\mathbf{x}} + \mathbf{A}^T\Sigma_{\varepsilon}^{-1}(\mathbf{y} - \mathbf{b})\right]
\end{aligned}
$$