[Homework] Distributions related to the Poisson point process

1. The county hospital is located at the center of a square whose sides are 4 km wide. If an accident occurs within this square, then the hospital sends out an ambulance. The road network is rectangular, so the travel distance from the hospital, whose coordinates are $(0, 0)$, to the point (x, y) is $|x| + |y|$. If an accident occurs at a point that is uniformly distributed in the square, find the expected travel distance of the ambulance.
   Sol. We can assume that the coordinates X and Y of the accident are independent and uniformly distributed over $(-2, 2)$. Hence the expected travel distance of the ambulance is

$$E(|X| + |Y|) = E(|X|) + E(|Y|) = 2 \cdot \int_{-2}^{2} |x| \cdot \frac{1}{3} dx = \frac{8}{3}$$

2. Buses arrive at a specified stop at 15-minute intervals starting at 7 A.M. That is, they arrive at 7, 7:15, 7:30, 7:45, and so on. If a passenger arrives at the stop at a time that is uniformly distributed between 7 and 7:30, find the probability that he waits

   (a) less than 5 minutes for a bus.
   Sol. X = number of minutes past 7 the passenger arrives.
   X is a uniform random variable over the interval $(0, 30)$.

$$P(10 < X < 15) + P(25 < X < 30) = \int_{10}^{15} \frac{1}{30} dx + \int_{25}^{30} \frac{1}{30} dx = \frac{1}{3}$$

   (b) more than10 minutes for a bus.
   Sol.

$$P(0 < X < 5) + P(15 < X < 20) = \int_{0}^{5} \frac{1}{30} dx + \int_{15}^{20} \frac{1}{30} dx = \frac{1}{3}$$

3. The joint PDF $f_{X,Y}(x, y)$ of X and Y is given by

$$f(x, y) = 12xy(1 - x), \qquad \text{for } 0 < x < 1 \text{ and } 0 < y < 1.$$

   (a) Find the PDF $f_X(x)$ of X and the PDF $f_Y(y)$ of Y.
   Sol.

$$f_X(x) = \int_{0}^{1} f(x, y) dy = 6x - 6x^2$$

$$f_Y(y) = \int_0^1 f(x,y)dx = 2y$$

(b) Are X and Y independent?
Sol. Since $f(x,y) = f_X(x)f_Y(y)$, X and Y are independent.

(c) Calculate EX, EY, $Var(X)$, $Var(Y)$, $Cov(X,Y)$.
Sol. (c)

$$EX = \int_0^1 x f_X(x)dx = \frac{1}{2}, \quad EY = \int_0^1 y f_Y(y)dy = \frac{2}{3}$$
$$EX^2 = \int_0^1 x^2 f_X(x)dx = \frac{3}{10}, \quad EY^2 = \int_0^1 y^2 f_Y(y)dy = \frac{1}{2}$$
$$VarX = EX^2 - (EX)^2 = \frac{1}{20}, \quad VarY = EY^2 - (EY)^2 = \frac{1}{18}$$
$$EXY = \int_0^1 \int_0^1 xyf(x,y)dxdy = \frac{1}{3}$$
$$Cov(X,Y) = EXY - EX \cdot EY = 0$$

4. The joint PDF $f(x,y)$ of X and Y is given by

$$f(x,y) = x + y, \qquad \text{for } 0 \le x \le 1 \text{ and } 0 \le y \le 1.$$

(a) Are X and Y independent?
Sol. Since $f(x,y) \neq f_X(x)f_Y(y)$, X and Y are not independent.

(b) Find the PDF $f_X(x)$ of X.
Sol.

$$f_X(x) = \int_0^1 f(x,y)dy = x + \frac{1}{2}$$
$$f_Y(y) = \int_0^1 f(x,y)dx = y + \frac{1}{2}$$

(c) Calculate $P(X + Y < 1)$.
Sol.
$$P(X+Y<1) = \int_0^1 \int_0^{1-y} f(x,y)dxdy = \frac{1}{3}$$

5. The joint PDF $f(x,y)$ of X and Y is given by

$$f(x,y) = xe^{-(x+y)}, \qquad \text{for } x > 0 \text{ and } y > 0.$$

(a) Identify the distribution of X and report its mean and variance.
Sol.

$$f_X(x) = \int_0^\infty f(x,y)dy = xe^{-x}$$
$$X \sim \Gamma(2,1)$$
$$EX = \frac{n}{\lambda} = 2, VarX = \frac{n}{\lambda^2} = 2$$

(b) Are X and Y independent?
Sol.

$$f_Y(y) = \int_0^\infty f(x,y)dx = e^{-y}$$
$$Y \sim \exp(1), EX = \frac{1}{\lambda} = 1, VarY = \frac{1}{\lambda^2} = 1$$

Since $f(x,y) = f_X(x)f_Y(y)$, X and Y are independent.

(c) Identify the distribution of $X+Y$ and report its mean and variance.
Sol. Since X and Y are independent, $X+Y \sim \Gamma(3,1)$

$$E(X+Y) = EX + EY = 3$$

$$Var(X+Y) = VarX + VarY + 2Cov(X,Y) = 3$$

6. Show that
$$B(n,p) * B(m,p) = B(n+m,p)$$

Sol.

$$\begin{aligned}
f_Z(z) &= f_{X+Y}(z) \\
&= \sum_{x=0}^{z} f_X(x)f_Y(z-x) \\
&= \sum_{x=0}^{z} \binom{n_1}{x} p^x(1-p)^{n_1-x} \binom{n_2}{z-x} p^{z-x}(1-p)^{n_2-(z-x)} \\
&= p^z(1-p)^{n_1+n_2-z} \sum_{x=0}^{z} \binom{n_1}{x}\binom{n_2}{z-x} \\
&= \binom{n_1+n_2}{z} p^z(1-p)^{n_1+n_2-z}
\end{aligned}$$

7. Show that

$$Exp(\lambda) * Exp(\lambda) = \Gamma(2, \lambda)$$

Sol. Let X, Y, and $Z = X+Y$ denote the relevant random variables, and f_X, f_Y, and f_Z their densities. Then

$$f_X(x) = f_Y(x) = \begin{cases} \lambda e^{-\lambda x} & x \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

for $z > 0$

$$\begin{aligned} f_Z(z) &= \int_{-\infty}^{\infty} f_X(z-y) f_Y(y) \\ &= \lambda^2 z e^{-\lambda z} \\ &= \frac{1}{\Gamma(2)} \lambda \cdot (\lambda z)^{2-1} e^{-\lambda z} \end{aligned}$$

Therfore,

$$Exp(\lambda) * Exp(\lambda) = \Gamma(2, \lambda)$$

8. The lifetime in hours of an electronic tube is a random variable having a PDF given by

$$f(x) = xe^{-x}, \quad x > 0$$

Compute the expected lifetime of such a tube.
Sol.

$$f(x) = xe^{-x} = \frac{1}{\Gamma(2)} \cdot 1 \cdot (1 \cdot x)^{2-1} e^{-1 \cdot x}$$

$$X \sim \Gamma(\alpha = 2, \lambda = 1)$$

$$EX = \frac{n}{\lambda} = 2$$

9. When I enter the bank, there are already two people in line waiting for the service and I join the queue. In the bank there are four service desks and we assume the service time is iid $Exp(\lambda_1)$, $\lambda_1 = 2$ (in minutes). After I got serviced at bank, I visit the post office. When I enter the post office, there are already three people in line waiting for the service and I join the queue. In the post office there are two service desks and we assume the service time is iid $Exp(\lambda_2)$, $\lambda_2 = 4$ (in minutes). Let F be the fraction of waiting time in post office among the total waiting time

in both the bank and the post office. Calculate the mean and variance of F.

Sol. X_i : the $i(i = 1, 2)$th person waiting time in the bank, then $X_i \sim Exp(4\lambda_1)$
X_3 : my waiting time in the bank, then $X_3 \sim Exp(4\lambda_1)$
$X = X_1 + X_2 + X_3 \sim \Gamma(3, 4\lambda_1)$

Y_i : the $i(i = 1, 2, 3)$th person waiting time in the post office, then $Y_i \sim Exp(2\lambda_2)$
Y_4 : my waiting time in the post office, then $Y_3 \sim Exp(2\lambda_2)$
$Y = Y_1 + Y_2 + Y_3 + Y_4 \sim \Gamma(4, 2\lambda_2)$

$F = \frac{Y}{X+Y}$ is the fraction of waiting time spent in post office among the total waiting time in both the bank and the post office. Then

$$F \sim Beta(\beta, \alpha)$$

$$EF = \frac{\beta}{\alpha + \beta} = \frac{4}{7}$$
$$VarF = \frac{\alpha\beta}{(\alpha + \beta)^2(\alpha + \beta + 1)} = \frac{3}{98}$$

10. Let T be the inter arrival time containing 05/05/2013 of $PPP(2)$. Find its mean and variance.

    Sol. $T \sim \Gamma(2, 2)$
    $$ET = \frac{n}{\lambda} = 1, \quad VarT = \frac{n}{\lambda^2} = \frac{1}{2}$$

[Extra]

1. An accident occurs at a point X that is uniformly distributed on a road of length L. At the time of the accident, an ambulance is at a location Y that is also uniformly distributed on the road. Assuming that X and Y are independent, find the expected distance between the ambulance and the point of the accident.