[Homework] Counting

1. How many different letter arrangements can be made from

   (a) cat?
   Sol. Since "cat" has three unique letters we have $3! = 6$ possible arrangements.

   (b) bob?
   Sol. Since "bob" has three characters with the "b "repeated two times, so we have $\frac{3!}{2!}$ possible arrangements.

   (c) paper?
   Sol. Since "paper" has five characters with the "p "repeated two times, so we have $\frac{5!}{2!}$ possible arrangements.

   (d) pepper?
   Sol. Since "pepper" has six characters with the "p "repeated three times, the "e "repeated two times so we have $\frac{6!}{2! \cdot 3!}$ possible arrangements.

   (e) Mississippi?
   Sol. Since "Mississippi" has eleven characters with the "i "repeated four times, the "s "repeated four times and the "p "repeated two times, so we have $\frac{11!}{4! \cdot 4! \cdot 2!}$ possible arrangements.

2. Delegates from 10 countries, including Korea, Japan, China, and the United States, are to be seated in a row. How many different seating arrangements are possible if

   (a) there is no constraint?
   Sol. $10!$

   (b) Korea and Japan delegates are to be seated next to each other?
   Sol. $9! \cdot 2!$

   (c) Korea and Japan delegates are to be seated next to each other and the China and U.S. delegates are not to be next to each other?
   Sol. If the Korea and Japan delegates are to be seated next to each other, they can be placed in $2!$ ways. Then this pair constitutes a new object which we can place anywhere among the remaining eight people, i.e. there are $9!$ arrangements of the eight remaining people and the Korea and Japan pair. Thus we have $2 \cdot 9! = 725760$ possible combinations. Since in some of

these the China and U.S. delegates are next to each other, this number over counts the true number we are looking for by $2 \cdot 2 \cdot 8! = 161280$ (the first two is for the number of arrangements of the China and U.S. pair). Combining these two criterion we have $2 \cdot 9! - 2 \cdot 2 \cdot 8!$.

3. An elevator starts at the basement of the 12 story building with 10 people and each one choose one's stop. How many different stop arrangements are possible if

   (a) there is no constraint?
   Sol. Let x_i equal the number of people getting off at floor i, where $i = 1, \cdots, 12$. Then the constraint that all people are off at the twelveth floor means that $x_1 + \cdots + x_{12} = 10$ with $x_i \geq 0$.
   This has $\binom{n+r-1}{r-1} = \binom{10+12-1}{12-1} = \binom{21}{11}$ possible distribution people.

   (b) three are from the same family and they move as a group?
   Sol. Three people are from the same family means that $x_1 + \cdots + x_{12} = 8$.
   This has $\binom{n+r-1}{r-1} = \binom{8+12-1}{12-1} = \binom{19}{11}$ possible distribution people.

   (c) three move as a group and another four move as a group, but these two groups stop at different floors?
   Sol. This means that
   (the number of non-negative integer solution $x_1 + \cdots + x_{12} = 5$) - (the number of non-negative integer solution $x_1 + \cdots + x_{12} = 4$)
   $\binom{5+12-1}{12-1} - \binom{4+12-1}{12-1} = \binom{16}{11} - \binom{15}{11}$

4. 12 people are to be divided into 3 groups. How many different group formations are possible if the size of 3 groups are

   (a) 3, 4, and 5?
   Sol. There are $\binom{12}{3} \cdot \binom{9}{4} \cdot \binom{5}{5} = \binom{12}{3,4,5}$ possible divisions.

   (b) 3, 3, and 6? Here, we assume two groups of size 3 are not distinguishable.
   Sol. There are $\frac{\binom{12}{3} \cdot \binom{9}{3} \cdot \binom{6}{6}}{2!}$ possible divisions.

   (c) 3, 3, and 6? Here, we assume (different tasks are assigned to each group and hence) two groups of size 3 are distinguishable.
   Sol. There are $\binom{12}{3} \cdot \binom{9}{3} \cdot \binom{6}{6}$ possible divisions.

5. When we expand $(x + 2y + 3z + 4w)^{100}$, find the coefficient of

(a) $x^{10}y^{20}z^{30}w^{39}$.
Sol. 0

(b) $x^{10}y^{20}z^{30}w^{40}$.
Sol. by multinomial theorem
$\binom{100}{10,20,30,40}2^{20}\cdot 3^{30}\cdot 4^{40} = \frac{100!}{10!20!30!40!}2^{20}\cdot 3^{30}\cdot 4^{40} = \binom{100}{10}\cdot\binom{90}{20}\cdot\binom{70}{30}\cdot\binom{40}{40}\cdot 2^{20}\cdot 3^{30}\cdot 4^{40}$

(c) $x^{10}y^{20}z^{30}w^{41}$.
Sol. 0

6. If 20 balls are to be distributed into 4 distinct urns, how many different configurations are possible if

(a) 20 balls are all different.
Sol. 4^{20}

(b) 20 balls are all identical.
Sol. If we let $x_i, i = 1, 2, 3, 4$, i-th urn, then x_1, x_2, x_3, x_4 are integers satisfying the equation $x_1 + x_2 + x_3 + x_4 = 20, x_i \geq 0$
So we have $\binom{20+4-1}{4-1} = \binom{23}{3}$ possible configurations.

(c) 20 balls are all identical and each urn has at least one ball.
Sol. If we let $x_i, i = 1, 2, 3, 4$, i-th urn, then x_1, x_2, x_3, x_4 are integers satisfying the equation $x_1 + x_2 + x_3 + x_4 = 20, x_i > 0$
let $x_i^{'} = x_i - 1$
$\Leftrightarrow x_1^{'} + x_2^{'} + x_3^{'} + x_4^{'} = 16, x_i \geq 0$
So we have $\binom{16+4-1}{4-1} = \binom{19}{3}$ possible configurations.

(d) 12 balls are identically red and 8 balls are identically white.
Sol. (the number of non-negative integer solutions of $x_1+x_2+x_3+x_4 = 12$)$\cdot$ (the number of non-negative integer solutions of $y_1 + y_2 + y_3 + y_4 = 8$)
So we have $\binom{12+4-1}{4-1}\cdot\binom{8+4-1}{4-1} = \binom{15}{3}\cdot\binom{11}{3}$

7. Prove

$$\binom{2n}{n} = \sum_{k=0}^{n}\binom{n}{k}^2$$

Sol. From problem 8 we have that when $m = n$ and $r = n$ that

$$\binom{n+n}{n} = \binom{n}{0}\binom{n}{n} + \binom{n}{1}\binom{n}{n-1} + \cdots + \binom{n}{n}\binom{n}{0}$$

Using the fact that $\binom{n}{k} = \binom{n}{n-k}$ the above is becomes

$$\binom{2n}{n} = \binom{n}{0}^2 + \binom{n}{1}^2 + \cdots + \binom{n}{n}^2$$

8. Prove that

$$\binom{n+m}{r} = \binom{n}{0}\binom{m}{r} + \binom{n}{1}\binom{m}{r-1} + \cdots + \binom{n}{r}\binom{m}{0}$$

Sol. We can do this in a combinatorial way by considering subgroups of size r from a group of n men and m women. The left hand side of the above represents one way of obtaining this identity. Another way to count the number of subsets of size r is to consider the number of possible groups can be found by considering a subproblem of how many men chosen to be included in the subset of size r. This number can range from zero men to r men. When we have a subset of size r with zero men we must have all women. This can be done in $\binom{n}{0}\binom{m}{r}$ ways. If we select one man and $r-1$ women the number of subsets that meet this criterion is given by $\binom{n}{1}\binom{m}{r-1}$. Continuing this logic for all possible subset of the men we have the right hand side of the above expression.

9. Show the following combinatorial identities

$$\begin{aligned}(a) \sum_{k=1}^{n} k\binom{n}{k} &= n2^{n-1} \\ (b) \sum_{k=0}^{n} k(k-1)\binom{n}{k} &= n(n-1)2^{n-2}\end{aligned}$$

Sol. (a) Consider n people from which we want to count the total number of committees of any size with a chairman. For a committee of size $k=1$ we have $1 \cdot \binom{n}{0} = n$ possible choices. For a committee of size $k=2$ we have $\binom{n}{2}$ subsets of two people and two choices for the person who is the chair. This gives $2 \cdot \binom{n}{2}$ possible choices. For a committee of size $k=3$ we have $3 \cdot \binom{n}{3}$ etc. Summing all of these possible choices we find that the total number of committees with a chair is $\sum_{k=1}^{n} k\binom{n}{k}$
Another way to count the total number of all committees with a chair, is to consider first selecting the chairperson from which we have n choices and then

considering all possible subsets of size $n-1$ (which is 2^{n-1}) from which to construct the remaining committee members. The product then gives $n2^{n-1}$.

(b) Consider again n people where now we want to count the total number of committees of size k with a chairperson and a secretary. We can select all subsets of size k in $\binom{n}{k}$ ways. Given a subset of size k, there are k choices for the chairperson and $k-1$ choices for the secretary giving $k(k-1)\binom{n}{k}$ committees of size k with a chair and a secretary. The total number of these is then given by summing this result or

$$\sum_{k=0}^{n} k(k-1)\binom{n}{k}$$

Now consider first selecting the chairman which can be done in n ways. And selecting for the secretary which can be done in $n-1$ ways and then we look for all subsets of a set with $n-2$ elements (i.e. 2^{n-2})
So we have

$$\sum_{k=0}^{n} k(k-1)\binom{n}{k} = n(n-1)2^{n-2}$$

10. Let A_n, B_n, and C_n be the number of n coin flip outcomes such that, when we divide the number of heads by 3, the remainder is 0, 1, 2, respectively. Show that

$$A_n = A_{n-1} + C_{n-1}, \quad \text{for } n \geq 2$$

$$B_n = B_{n-1} + A_{n-1}, \quad \text{for } n \geq 2$$

$$C_n = C_{n-1} + B_{n-1}, \quad \text{for } n \geq 2$$

or

$$\begin{bmatrix} A_n \\ B_n \\ C_n \end{bmatrix} = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} A_{n-1} \\ B_{n-1} \\ C_{n-1} \end{bmatrix}$$

Since $A_1 = 1,\ B_1 = 1,\ C_1 = 0$,

$$\begin{bmatrix} A_n \\ B_n \\ C_n \end{bmatrix} = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}^{n-1} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$

Sol.

(1) $A_n =$ the number of heads is the form of $3k$ If nth flip is head, the number of

heads is the form of $3k+2$ in $n-1$ flips. If nth flip is tail, the number of heads is the form of $3k$ in $n-1$ flips. So we have $A_n = A_{n-1} + C_{n-1}$
(2) $B_n =$ the number of heads is the form of $3k+1$ If nth flip is head, the number of heads is the form of $3k$ in $n-1$ flips. If nth flip is tail, the number of heads is the form of $3k+1$ in $n-1$ flips. So we have $B_n = B_{n-1} + A_{n-1}$
(3) $C_n =$ the number of heads is the form of $3k+2$ If nth flip is head, the number of heads is the form of $3k+1$ in $n-1$ flips. If nth flip is tail, the number of heads is the form of $3k+2$ in $n-1$ flips. So we have $C_n = C_{n-1} + B_{n-1}$

11. An elevator starts at the basement of the 12 story building with 10 people and each one choose one's stop. Count

   (a) all the possible red stop button number configurations.
   Sol.
   $$\binom{12}{1} + \binom{12}{2} \cdots + \binom{12}{12}$$

   (b) all the possible stopping configurations of all 10 people.
   Sol.
   $$12^10$$

   (c) all the possible stopping configurations of all 10 people if we cannot identify (or don't care) one from another.
   Sol.
   $$\binom{10+12-1}{12-1} = \binom{21}{11}$$