[Homework] IID coin flips

1. The joint PMF of X and Y are given by

y_j				
3	$\frac{2}{10}$	$\frac{1}{10}$	$\frac{1}{10}$	
2	$\frac{1}{10}$	0	$\frac{1}{10}$	
1	$\frac{2}{10}$	$\frac{1}{10}$	0	
0	0	$\frac{1}{10}$	0	
	0	**1**	**2**	x_i

(a) Find the marginal PMF of X.
Sol.

i	0	1	2	
$P(X=i)$	$\frac{5}{10}$	$\frac{3}{10}$	$\frac{1}{10}$	1

(b) Find the marginal PMF of Y.
Sol.

i	0	1	2	3	
$P(Y=i)$	$\frac{1}{10}$	$\frac{3}{10}$	$\frac{2}{10}$	$\frac{4}{10}$	1

(c) Find the conditional PMF of X given $Y=1$.
Sol.

$$P(X=0|Y=1) = \frac{P(x=0,Y=1)}{P(Y=1)} = \frac{\frac{2}{10}}{\frac{3}{10}} = \frac{2}{3}$$

$$P(X=1|Y=1) = \frac{P(x=1,Y=1)}{P(Y=1)} = \frac{\frac{1}{10}}{\frac{3}{10}} = \frac{1}{3}$$

$$P(X=2|Y=1) = \frac{P(x=2,Y=1)}{P(Y=1)} = \frac{0}{\frac{3}{10}} = 0$$

(d) Find the conditional PMF of Y given $X=2$.
Sol.

$$P(Y=0|X=2) = \frac{P(x=2,Y=0)}{P(X=2)} = \frac{0}{\frac{2}{10}} = 0$$

$$P(Y=1|X=2) = \frac{P(x=2,Y=1)}{P(X=2)} = \frac{0}{\frac{2}{10}} = 0$$

$$P(Y=2|X=2) = \frac{P(x=2, Y=2)}{P(X=2)} = \frac{\frac{1}{10}}{\frac{2}{10}} = \frac{1}{2}$$

$$P(Y=3|X=2) = \frac{P(x=2, Y=3)}{P(X=2)} = \frac{\frac{1}{10}}{\frac{2}{10}} = \frac{1}{2}$$

2. The CDF F is given by

$$F(x) = \begin{cases} 0 & \text{for } x < 0 \\ 0.2 & \text{for } 0 \leq x < 1 \\ 0.5 & \text{for } 1 \leq x < 2 \\ 0.9 & \text{for } 2 \leq x < 3 \\ 1 & \text{for } 3 \leq x \end{cases}$$

Compute the corresponding PMF, i.e., calculate $p_i = P(X = i)$, $i = 0, 1, 2, 3$.
Sol.

i	0	1	2	3	
$P(X = i)$	0.2	0.3	0.4	0.1	1

3. Two balls are chosen randomly from an urn containing 8 white, 4 black, and 2 orange balls. Suppose that we win \$2 for each black ball selected and we lose \$1 for each white ball selected. Let X denote our winnings.

(a) What are the possible values of X?
Sol. The possible results of the experiment are

$$\{W, W\}, \{W, B\}, \{W, O\}, \{B, B\}, \{B, O\}, \{O, O\}$$

Then X can take the values:
-2 for $\{W, W\}$, 1 for $\{W, B\}$
-1 for $\{W, O\}$, 4 for $\{B, B\}$
2 for $\{B, O\}$, 0 for $\{O, O\}$

(b) What are the probabilities associated with each value?
Sol.f

$$P(X = -2) = P(\{W, W\}) = \frac{\binom{8}{2}}{\binom{14}{2}} = \frac{28}{91}$$

$$P(X = 1) = P(\{W, B\}) = \frac{\binom{8}{1} \cdot \binom{4}{1}}{\binom{14}{2}} = \frac{32}{91}$$

$$P(X=-1) = P(\{W,O\}) = \frac{\binom{8}{1}\cdot\binom{2}{1}}{\binom{14}{2}} = \frac{16}{91}$$
$$P(X=4) = P(\{B,B\}) = \frac{\binom{4}{2}}{\binom{14}{2}} = \frac{6}{91}$$
$$P(X=2) = P(\{B,O\}) = \frac{\binom{4}{1}\cdot\binom{2}{1}}{\binom{14}{2}} = \frac{8}{91}$$
$$P(X=0) = P(\{O,O\}) = \frac{\binom{2}{2}}{\binom{14}{2}} = \frac{1}{91}$$

	-2	-1	0	1	2	4	
p	$\frac{28}{91}$	$\frac{16}{91}$	$\frac{1}{91}$	$\frac{32}{91}$	$\frac{8}{91}$	$\frac{6}{91}$	1

4. A salesman has scheduled two appointments to sell encyclopedias. His first appointment will lead to a sale with probability 0.3, and his second will lead independently to a sale with probability 0.6. Any sale made is equally likely to be either for the deluxe model, which costs \$1000, or the standard model, which costs \$500. Determine the probability mass function of X, the total dollar value of all sales.

   Sol. For $i = 1, 2$ consider the events $S_i := \{\text{sale on the } i\text{th appointment}\}$. We know that S_1 and S_2 are independent, $P(S_1) = 0.3, P(S_2) = 0.6$. Let $D_i :=$ deluxe on ith, also. We know that $P(D_i|S_i) = P(D_i^c|S_i) = \frac{1}{2}$.
   Consequently, $P(S_i \cap D_i) = \frac{P(S_i)}{2}$ and $P(S_i \cap D_i^c) = \frac{P(S_i)}{2}$.
   The possible values of X are:
   2000 dollars. In this case, we have

$$P\{X = 2000\} = P(S_1 \cap D_1)P(S_2 \cap D_2) = \frac{0.3}{2}\cdot\frac{0.6}{2} = 0.045$$

   1500 dollars. In this case, we have

$$P\{X = 1500\} = P(S_1 \cap D_1)P(S_2 \cap D_2^c) + P(S_1 \cap D_1^c)P(S_2 \cap D_2) = 0.09$$

   1000 dollars. In this case, we have

$$P\{X = 1000\} = P(S_1 \cap D_1)P(S_2^c) + P(S_1^c)P(S_2 \cap D_2) + P(S_1 \cap D_1^c)P(S_2 \cap D_2^c) = 0.315$$

500 dollars. In this case, we have

$$P\{X = 500\} = P(S_1 \cap D_1^c)P(S_2^c) + P(S_1^c)P(S_2 \cap D_2^c) = 0.27$$

0 dollars. In this case, we have

$$P\{X = 0\} = P(S_1^c)P(S_2^c) = 0.28$$

5. Five distinct numbers are randomly distributed to players numbered 1 through 5. Whenever two players compare their numbers, the one with the higher one is declared the winner. Initially, players 1 and 2 compare their numbers; the winner then compares her number with that of player 3, and so on. Let X denote the number of times player 1 is a winner. Find $P(X = i)$, $i = 0, 1, 2, 3, 4$.

Sol. Let Y_j denote the number distributed to player j. Note that $(Y_1, \cdots, Y_5)$ is a random permutation of $(1, \cdots, 5)$, all permutations being equally likely. Therefore, $p(0) = P\{X = 0\} = P(Y_1 < Y_2)$. But half of all permutations of $(1, \cdots, 5)$ have $Y_1 < Y_2$, whereas half have $Y_1 > Y_2$. Therefore,

$$p(0) = \frac{\frac{1}{2} \cdot 5!}{5!} = \frac{1}{2}$$

Next note that $p(1) = P\{Y_2 < Y_1 < Y_3\}$. The number of ways to end up with $Y_2 < Y_1 < Y_3$ is the same as the number of ways to get $Y_1 < Y_2 < Y_3$. This is the same as $\cdots$. Therefore, the number of ways to get $Y_2 < Y_1 < Y_3$ is $\frac{1}{3!}$ times the total number of permutations. That is,

$$p(1) = \frac{\frac{1}{3!} \cdot 5!}{5!} = \frac{1}{6}$$

$$\begin{aligned}
p(2) &= P\{Y_2 < Y_1, Y_3 < Y_1, Y_4 > Y_1\} \\
&= P\{Y_1 = 3, Y_2 = 1, Y_3 = 2\} + P\{Y_1 = 3, Y_2 = 2, Y_3 = 1\} \\
&\quad + P\{Y_1 = 4, Y_2 = 1, Y_3 = 2, Y_4 = 5\} + P\{Y_1 = 4, Y_2 = 2, Y_3 = 1, Y_4 = 5\} \\
&\quad + P\{Y_1 = 4, Y_2 = 1, Y_3 = 3, Y_4 = 5\} + \cdots \\
&= (\frac{2}{5!} + \frac{2}{5!}) + (\frac{1}{5!} + \frac{1}{5!} + \cdots) \\
&= (2 \cdot \frac{2}{5!}) + (6 \cdot \frac{1}{5!}) = \frac{1}{12}
\end{aligned}$$

Next we note that

$$\begin{aligned} p(3) &= P\{Y_2 < Y_1, Y_3 < Y_1, Y_4 < Y_1, Y_5 > Y_1\} \\ &= P\{Y_1 = 4, Y_1 = 1, Y_2 = 2, Y_3 = 3, Y_4 = 4, Y_5 = 5\} + \cdots \\ &= \frac{3!}{5!} = \frac{1}{20} \end{aligned}$$

Finally,

$$p(4) = P\{Y_1 = 5\} = \frac{1}{5}$$

6. On a multiple-choice exam of 20 questions with 4 possible answers for each question, let S be the number of correct answers obtained just by guessing and let X_i, $1 \le i \le 20$, be the indicator of choosing the correct answer for the problem i.

   (a) Represent S in terms of X_i.
   Sol.
   $$S = X_1 + X_2 + \cdots + X_{20} = \sum_{i=1}^{20} X_i$$

   (b) What is the distribution of X_i?
   Sol. X_i:Bernoulli distribution with $p = \frac{1}{4}$. $X_i \sim B(\frac{1}{4})$

   (c) Discuss the independence, pairwise independence, or dependence of X_i.
   Sol. Since
   $$P(X_i = 1, X_j = 1) = P(X_i = 1)P(X_j = 1)$$
   $$P(X_i = 1, X_j = 0) = P(X_i = 1)P(X_j = 0)$$
   $$P(X_i = 0, X_j = 1) = P(X_i = 0)P(X_j = 1)$$
   $$P(X_i = 0, X_j = 0) = P(X_i = 0)P(X_j = 0)$$
   So, pairwise independence.
   Since
   $$P(X_1 = a_1, X_2 = a_2, \cdots X_{20} = a_{20}) = P(X_1 = a_1)P(X_2 = a_2) \cdots P(X_{20} = a_{20})$$
   where $a_1, \cdots a_{20} = 0$ or 1
   independet.

(d) Can we declare the distribution of S is binomial? Why or why not.

Sol.

$$S = X_1 + X_2 + \cdots + X_{20} = \sum_{i=1}^{20} X_i \sim B(20, \frac{1}{4})$$

7. We flip the fair coin 5 times independently and let X be the number of heads. Let D be the number of heads minus the number of tails.

(a) Represent D in terms of X.

Sol.

$$D = 2X - 5$$

(b) What is the distribution of X?

Sol.

$$X_i = \begin{cases} 1 & \text{head} \\ 0 & \text{tail} \end{cases}$$

$$X = X_1 + X_2 + \cdots + X_5$$

$$X_i \sim B(\frac{1}{2}), X \sim B(5, \frac{1}{2})$$

(c) Calculate the PMF of D.

Sol.

$$P(X = k) = P(D = 2k - 5) = \binom{5}{k}(\frac{1}{2})^k(\frac{1}{2})^{5-k}, \ (k = 0, 1, \cdots, 4, 5)$$

d	-5	-3	-1	1	3	5	
$P(D = d)$	$\binom{5}{0}(\frac{1}{2})^5$	$\binom{5}{1}(\frac{1}{2})^5$	$\binom{5}{2}(\frac{1}{2})^5$	$\binom{5}{3}(\frac{1}{2})^5$	$\binom{5}{4}(\frac{1}{2})^5$	$\binom{5}{5}(\frac{1}{2})^5$	1

8. We flip coins n times independently and let X be the number of heads. For the i-th flip we use the p_i-coin which lands on head with probability p_i. Let A_i be the event that the i-coin lands on head and let 1_{A_i} be its indicator.

(a) Represent X in terms of 1_{A_i}.

Sol.

$$1_{A_i} = \begin{cases} 1 & i\text{th head } p_i \\ 0 & \quad \text{tail } 1 - p_i \end{cases}$$

$$S = 1_{A_1} + 1_{A_2} + \cdots + 1_{A_n}$$

(b) What is the distribution of 1_{A_i}?
Sol.

$$1_{A_i} \sim B(p_i)$$

(c) Discuss the independence, pairwise independence, or dependence of 1_{A_i}.
Sol.

$$P(1_{A_1} \cdots 1_{A_n}) = P(1_{A_1})P(1_{A_2}) \cdots P(1_{A_n})$$

1_{A_i} is pairwise independence and independent.

(d) Is the distribution of X is binomial? Why or why not.
Sol. If all p_i's are same then $S \sim B(n, p)$.
If all p_i's are diffenent then S is not bibomial.
1_{A_i}:Bernolli distribution, independent
But, $S = \sum_{i=1}^{n} 1_{A_i}$: not binomial.

9. A fair coin is tossed thirteen times, independently. Let X be the number of times the coin lands heads in the first ten tosses, and let Y be the number of times the coin lands tails in the last ten tosses. Are X and Y independent?

(a) Provide an intuitive argument.
Sol. not independent.

$$P(X = 5) = \binom{10}{5}(\frac{1}{2})^{10}$$

$$P(Y = 7) = \binom{10}{7}(\frac{1}{2})^{10}$$

$$P(X = 3, Y = 10) = \frac{1}{2^{13}} \cdot \{\binom{7}{4} \cdot \binom{3}{1} + \binom{3}{2} \cdot \binom{7}{5}\}$$

$$P(X = 5)P(Y = 7) \neq P(X = 3, Y = 10)$$

(b) Provide a mathematical back up.
Sol.

$$P(X = 3) = \binom{10}{3}(\frac{1}{2})^{3}, P(Y = 10) = \binom{10}{10}(\frac{1}{2})^{10}$$

$$P(X = 3, Y = 10) = (\frac{1}{2})^{13}$$

But,

$$P(X = 3) \cdot P(Y = 10) \neq P(X = 3, Y = 10)$$

Therefore X and Y are not independent

10. Initially a jar contains n red candies and no white candies. A boy successively takes a candy from the jar and put a new white candy. The boy draws a candy until he has a white candy. Let X be the number of draws.

    (a) Can we declare the distribution of X is geometric? Why or why not.
    Sol. X is not geometric because it is neither independent and identical

    (b) Compute the PMF of X.
    Sol. Let E_i = the event of not drawing a previous chip on the ith draw.
    p_i = probability of stopping on the ith draw
    n = the number of chips
    X = the number of draws before ending
    Then,

$$\begin{aligned}
P(X=x) &= P(\text{probability of not stopping on any previous draws and stopping on the} x \text{t} \\
&= P(E_1 E_2 \cdots E_{x-1} (E_x)^c) \\
&= P(E_x^c | E_{x-1} E_{x-2} \cdots E_1) P(E_{x-1} | E_{x-2} E_{x-3} \cdots E_1) \cdots P(E_2 | E_1) P(E_1)
\end{aligned}$$

Now the probability of not drawing a previously seen chip given that there have been no repeat draws is just the number of unseen chips divided by the total number of chips. On draw i, this is just $\frac{n-i+1}{n}$.
Similarly, the probability of drawing a previously seen chip on draw x, given that there have been no repeats thus far, is just $\frac{x-1}{n}$. Thus the above becomes:

$$P(X=x) = \frac{x-1}{n} \cdot \prod_{i=1}^{x-1} \frac{n-i+1}{n} = \frac{x-1}{n} \cdot \frac{n!}{(n-x+1)! \cdot n^{x-1}}$$

Finally, since it is impossible to stop on the first draw, or draw a new one on the $(n+1)$ draw, we can give the following formula that includes the domain information:

$$P(X=x) = \begin{cases} \frac{x-1}{n} \cdot \prod_{i=1}^{x-1} \frac{n-i+1}{n} & x \in \{2, 3, \cdots, n+1\} \\ 0 & \text{otherwise} \end{cases}$$

[Extra] IID coin flips

1. Four buses carrying 148 students from the same school arrive at a football stadium. The buses carry, respectively, 40, 33, 25, and 50 students. One of the students is randomly selected. Let X denote the number of students that were

on the bus carrying the randomly selected student. One of the 4 bus drivers is also randomly selected. Let Y denote the number of students on her bus. Determine the PMFs of X and Y.