Conditional expectation and variance

Conditional expectation

$\boxed{E(X|Y=y) \text{ is a number}}$

$$E(X) = \sum_{x_i} x_i P(X = x_i) \quad \Rightarrow \quad E(X|Y=y) = \sum_{x_i} x_i P(X = x_i|Y=y)$$

$$E(X) = \int_{-\infty}^{\infty} x f_X(x) dx \quad \Rightarrow \quad E(X|Y=y) = \int_{-\infty}^{\infty} x f_{X|Y}(x|y) dx$$

$\boxed{E(X|Y) \text{ is a random variable, not a number}}$

$$\omega \quad \rightarrow \quad y = Y(\omega) \quad \rightarrow \quad P(X = x|Y=y) \quad \rightarrow \quad E(X|Y=y)$$

or

$$E(X|Y)(\omega) \quad = \quad E(X|Y = Y(\omega))$$

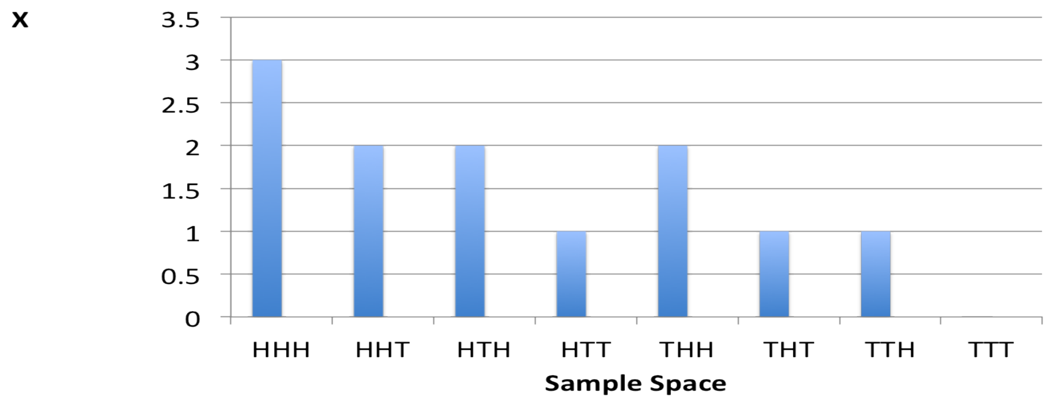


Figure 1: X is the number of heads in the three fair coin flips.

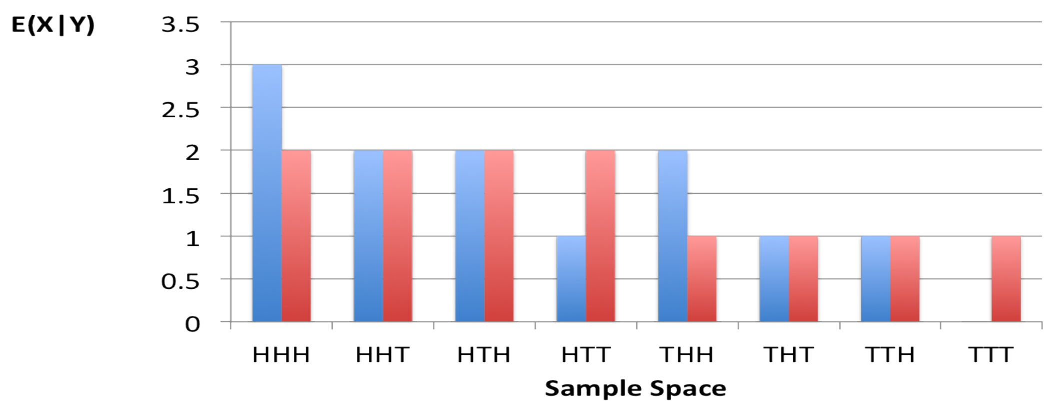


Figure 2: X is the number of heads in the three fair coin flips (blue). Y is the indicator of having the head in the first fair coin flip. $E(X|Y)$ is the red bars. $E(X|Y)$ is a kind of a local average.

Conditional variance

$\boxed{Var(X|Y=y) \text{ is a number}}$

$$Var(X) = E(X - E(X))^2 \quad \Rightarrow \quad Var(X|Y=y) \;=\; E((X - E(X|Y=y))^2|Y=y)$$

$$\begin{aligned} Var(X|Y=y) &= E((X - E(X|Y=y))^2|Y=y) \\ &= E(X^2|Y=y) - (E(X|Y=y))^2 \end{aligned}$$

$\boxed{Var(X|Y) \text{ is a random variable, not a number}}$

$$\omega \quad \rightarrow \quad y = Y(\omega) \quad \rightarrow \quad P(X=x|Y=y) \quad \rightarrow \quad Var(X|Y=y)$$

or

$$Var(X|Y)(\omega) \;=\; Var(X|Y=Y(\omega))$$

Properties of conditional expectation and conditional variance

Properties of conditional expectation

$$\begin{array}{ll}
(1) & \mathbb{E}(X+Y|Z) = \mathbb{E}(X|Z) + \mathbb{E}(Y|Z) \\
(2) & \mathbb{E}(aX|Y) = a\mathbb{E}(X|Y) \\
(3) & \mathbb{E}(g(Y)|Y) = g(Y) \\
(4) & \mathbb{E}(g(Y)X|Y) = g(Y)\mathbb{E}(X|Y) \\
(5) & \mathbb{E}(X|Y) = \mathbb{E}(X) \text{ if } X \text{ and } Y \text{ are independent} \\
(6) & \mathbb{E}(X) = \mathbb{E}\mathbb{E}(X|Y)
\end{array}$$

Properties of conditional variance

$$\begin{array}{ll}
(7) & Var(X) = \mathbb{E}Var(X|Y) + Var(\mathbb{E}(X|Y)) \\
(8) & Var(X) \geq \mathbb{E}Var(X|Y) \quad \text{and} \quad Var(X) \geq Var(\mathbb{E}(X|Y))
\end{array}$$

$$\mathbb{E}(X|Y) = \begin{cases} \mathbb{E}(X|Y=y_1) & \text{if } Y = y_1 \text{ with probability } P(Y=y_1) \\ \mathbb{E}(X|Y=y_2) & \text{if } Y = y_2 \text{ with probability } P(Y=y_2) \\ \cdots & \text{if } \cdots \\ \mathbb{E}(X|Y=y_n) & \text{if } Y = y_n \text{ with probability } P(Y=y_n) \end{cases}$$

$$\begin{aligned}
\mathbb{E}\mathbb{E}(X|Y) &= \sum_{y_j} \mathbb{E}(X|Y=y_j)\mathbb{P}(Y=y_j) \\
&= \sum_{y_j} \left(\sum_{x_i} x_i \mathbb{P}(X=x_i|Y=y_j) \right) \mathbb{P}(Y=y_j) \\
&= \sum_{x_i} x_i \left(\sum_{y_j} \mathbb{P}(X=x_i|Y=y_j)\mathbb{P}(Y=y_j) \right) \\
&= \sum_{x_i} x_i \mathbb{P}(X=x_i) = \mathbb{E}X
\end{aligned}$$

Example - Symmetry

Let X and Y be iid $B(n,p)$. Calculate $E(X|X+Y=m)$.

Known info

$$E(X+Y|X+Y=m) = E(m|X+Y=m) = m$$

Symmetry

$$\begin{aligned} E(X+Y|X+Y=m) &= E(X|X+Y=m) + E(Y|X+Y=m) \\ &\overset{\text{by symmetry}}{=} 2E(X|X+Y=m) \end{aligned}$$

Wrap up

$$E(X|X+Y=m) = \frac{m}{2}$$

Example - From joint to conditional

Compute $E(X|Y=y)$, where the joint PDF of X and Y is given by

$$f(x,y) = \frac{e^{-\frac{x}{y}}e^{-y}}{y} \quad 0<x<\infty,\ 0<y<\infty$$

$$\boxed{X|Y=y \sim Exp(\lambda),\ \lambda = \frac{1}{y}}$$

$$f_{X|Y}(x|y) \quad \propto \quad \frac{e^{-\frac{x}{y}}e^{-y}}{y} \quad \propto \quad e^{-\frac{x}{y}} \quad \propto \quad \frac{1}{y}e^{-\frac{x}{y}}$$

$$X|Y=y \quad \sim \quad Exp(\lambda),\ \lambda = \frac{1}{y}$$

$$\boxed{E(X|Y=y) = y,\ Var(X|Y=y) = y^2}$$

$$E(X|Y=y) = \frac{1}{\lambda} = y \quad \text{and} \quad Var(X|Y=y) = \frac{1}{\lambda^2} = y^2$$

$$\boxed{E(X|Y) = Y,\ Var(X|Y) = Y^2}$$

Expected amount of money spent in the store

Suppose that the number of people entering a department store on a given day is a random variable with mean 50 and variance 100. Suppose further that the amounts of money spent by these customers are iid with mean \$8 and standard deviation \$4. Finally, suppose also that the amount of money spent by a customer is also independent of the total number of customers who enter the store. What is the expected amount of money spent in the store on a given day?

Let N be the number of people entering a department store on a given day, with mean 50 and variance 100, and let X_i be the amounts of money spent by the i-th customer, iid with mean \$8 and standard deviation \$4. Let T be the amount of money spent in the store on a given day. Then, T can be represented in terms of N and X_i:

$$T = \sum_{i=1}^{N} X_i$$

$\boxed{E(T|N) \text{ is a function of } N}$

$$E(T|N) = E(\sum_{i=1}^{N} X_i|N) = \sum_{i=1}^{N} E(X_i|N) = \sum_{i=1}^{N} E(X_i) = \sum_{i=1}^{N} 8 = 8N$$

$\boxed{ET = EE(T|N)}$

$$ET = EE(T|N) = E8N = 8EN = 8 \times 50 = 400$$

$\boxed{Var(T|N) \text{ is a function of } N}$

$$Var(T|N) = Var(\sum_{i=1}^{N} X_i|N) = \sum_{i=1}^{N} Var(X_i|N) = \sum_{i=1}^{N} Var(X_i) = \sum_{i=1}^{N} 16 = 16N$$

$\boxed{Var(T) = Var(E(T|N)) + EVar(T|N)}$

$$Var(E(T|N)) = Var(8N) = 64Var(N) = 64 \times 100 = 6400$$

$$EVar(T|N) = E16N = 16EN = 16 \times 50 = 800$$

$$Var(T) = Var(E(T|N)) + EVar(T|N) = 6400 + 800 = 7200$$

Trapped miner - Expectation

A miner is trapped in a mine containing 3 doors. The first door leads to a tunnel that will take him to safety after 3 hours of travel. The second door leads to a tunnel that will return him to the mine after 5 hours of travel. The third door leads to a tunnel that will return him to the mine after 7 hours. If we assume that the miner is at all times equally likely to choose any one of the doors, what is the expected time ET until he reaches safety?

$$\boxed{\mathbb{E}T = \mathbb{E}\mathbb{E}(T|X)}$$

Let T be the exit time and let X be the first choice of the door he takes.

$$\mathbb{E}(T|X=x) \quad = \quad \begin{cases} 3 & \text{for } x=1 \\ 5+\mathbb{E}T & \text{for } x=2 \\ 7+\mathbb{E}T & \text{for } x=3 \end{cases}$$

$$\mathbb{E}(T|X) \quad = \quad \begin{cases} 3 & \text{with probability } 1/3 \\ 5+\mathbb{E}T & \text{with probability } 1/3 \\ 7+\mathbb{E}T & \text{with probability } 1/3 \end{cases}$$

$$\mathbb{E}T = \mathbb{E}\mathbb{E}(T|X) = 3 * \frac{1}{3} + (5+\mathbb{E}T) * \frac{1}{3} + (7+\mathbb{E}T) * \frac{1}{3} \quad \Rightarrow \quad \mathbb{E}T = 15$$

Trapped miner - Variance

$\boxed{Var(\mathbb{E}(T|X))}$

$$\mathbb{E}(T|X) = \begin{cases} 3 & \text{with probability } 1/3 \\ 5 + \mathbb{E}T = 20 & \text{with probability } 1/3 \\ 7 + \mathbb{E}T = 22 & \text{with probability } 1/3 \end{cases}$$

$$\begin{aligned} Var(\mathbb{E}(T|X)) &= \mathbb{E}\mathbb{E}(T|X)^2 - (\mathbb{E}\mathbb{E}(T|X))^2 \\ &= \mathbb{E}\mathbb{E}(T|X)^2 - (\mathbb{E}T)^2 \\ &= (3)^2 * \frac{1}{3} + (20)^2 * \frac{1}{3} + (22)^2 * \frac{1}{3} - (15)^2 = 72.6667 \end{aligned}$$

$\boxed{\mathbb{E}Var(T|X)}$

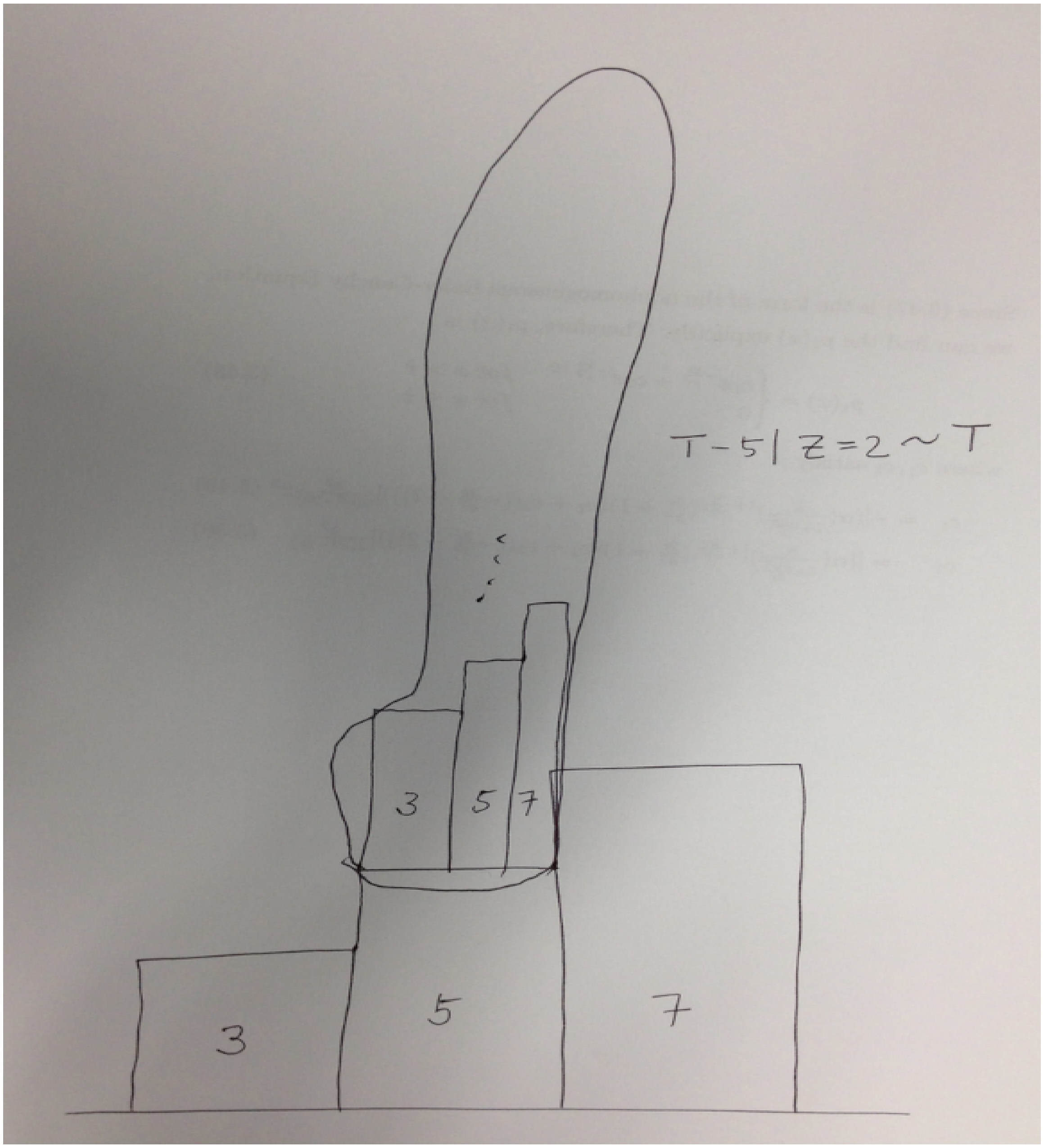
$$Var(T|X = x) = \begin{cases} 0 & \text{for } x = 1 \\ Var(T) & \text{for } x = 2 \\ Var(T) & \text{for } x = 3 \end{cases}$$

$$Var(T|X) = \begin{cases} 0 & \text{with probability } 1/3 \\ Var(T) & \text{with probability } 1/3 \\ Var(T) & \text{with probability } 1/3 \end{cases}$$

$$\mathbb{E}Var(T|X) = 0 * \frac{1}{3} + Var(T) * \frac{1}{3} + Var(T) * \frac{1}{3}$$

$\boxed{Var(T) = Var(\mathbb{E}(T|X)) + \mathbb{E}Var(T|X)}$

$$\begin{aligned} Var(T) &= Var(\mathbb{E}(T|X)) + \mathbb{E}Var(T|X) \\ &= 72.6667 + 0 * \frac{1}{3} + Var(T) * \frac{1}{3} + Var(T) * \frac{1}{3} = 218 \end{aligned}$$


T−5 | Z=2 ~ T
3
5
7
3
5
7

Waiting time for HT

Flip the fair coin until we get the pattern HT. Let W_{HT} be the number of flips to get the first HT. Calculate the mean and variance of W_{HT}.

$\boxed{W_{HT} = X + Y}$

$$\underbrace{TTTTTTTTTTTTTTH}_{X \text{ is } Geo(1/2)}\underbrace{HHHHHHHHHT}_{Y \text{ is } Geo(1/2)}$$

(1) X is $Geo(1/2)$ and Y is also $Geo(1/2)$

(2) X and Y are independent

(3) $W_{HT} = X + Y$

$\boxed{EW_{HT} = EX + EY}$

$$EW_{HT} = EX + EY = \frac{1}{1/2} + \frac{1}{1/2} = 4$$

$\boxed{Var(W_{HT}) = Var(X) + Var(Y)}$

$$Var(W_{HT}) = Var(X) + Var(Y) = \frac{1/2}{(1/2)^2} + \frac{1/2}{(1/2)^2} = 4$$

Waiting time for HH - Expectation

Flip the fair coin until we get the pattern HH. Let W_{HH} be the number of flips to get the first HH. Calculate the mean and variance of W_{HH}.

$\boxed{\mathbb{E}W_{HH} = \mathbb{E}\mathbb{E}(W_{HH}|Y)}$

$$\underbrace{TTTTTTTTTTTTTH}_{X \text{ is } Geo(1/2)}$$

After we have the first head at the X-th flip, we record the next coin flip result as Y.

$$\mathbb{E}(W_{HH}|Y=y) = \begin{cases} \mathbb{E}X + 1 = 3 & \text{for } y = 1 \\ \mathbb{E}X + 1 + \mathbb{E}W_{HH} = 3 + \mathbb{E}W_{HH} & \text{for } y = 0 \end{cases}$$

$$\mathbb{E}(W_{HH}|Y) = \begin{cases} 3 & \text{with probability } 1/2 \\ 3 + \mathbb{E}W_{HH} & \text{with probability } 1/2 \end{cases}$$

$$\mathbb{E}W_{HH} = \mathbb{E}\mathbb{E}(W_{HH}|Y) = (3) * \frac{1}{2} + (3 + \mathbb{E}W_{HH}) * \frac{1}{2} \quad \Rightarrow \quad \mathbb{E}W_{HH} = 6$$

Waiting time for HH - Variance

$\boxed{Var(\mathbb{E}(W_{HH}|Y))}$

$$\mathbb{E}(W_{HH}|Y) = \begin{cases} 3 & \text{with probability } 1/2 \\ 9 & \text{with probability } 1/2 \end{cases}$$

$$\begin{aligned} Var(\mathbb{E}(W_{HH}|Y)) &= \mathbb{E}\mathbb{E}(W_{HH}|Y)^2 - (\mathbb{E}\mathbb{E}(W_{HH}|Y))^2 \\ &= \mathbb{E}\mathbb{E}(W_{HH}|Y)^2 - (\mathbb{E}W_{HH})^2 \\ &= (3)^2 * \frac{1}{2} + (9)^2 * \frac{1}{2} - (6)^2 = 9 \end{aligned}$$

$\boxed{\mathbb{E}Var(W_{HH}|Y)}$

$$Var(W_{HH}|Y = y) = \begin{cases} Var(X) = 2 & \text{for } y = 1 \\ Var(X) + Var(W_{HH}) = 2 + Var(W_{HH}) & \text{for } y = 0 \end{cases}$$

$$Var(W_{HH}|Y) = \begin{cases} 2 & \text{with probability } 1/2 \\ 2 + Var(W_{HH}) & \text{with probability } 1/2 \end{cases}$$

$$\mathbb{E}Var(W_{HH}|Y) = (2) * \frac{1}{2} + (2 + Var(W_{HH})) * \frac{1}{2}$$

$\boxed{Var(W_{HH}) = Var(\mathbb{E}(W_{HH}|Y)) + \mathbb{E}Var(W_{HH}|Y)}$

$$\begin{aligned} Var(W_{HH}) &= Var(\mathbb{E}(W_{HH}|Y)) + \mathbb{E}Var(W_{HH}|Y) \\ &= 9 + (2) * \frac{1}{2} + (2 + Var(W_{HH})) * \frac{1}{2} = 22 \end{aligned}$$