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Project on Complex Analysis

Contents

§ 1 Modulus Estimate

Prove that there is a constant C, independent of n, such that if $\{z_j\}$ are complex numbers and if

$$\sum_{j=1}^{n} |z_j| \geqslant 1,$$

then there is a subcollection $\{z_{j_1}, ..., z_{j_k}\} \subseteq \{z_1, ..., z_n\}$ such that

$$\left| \sum_{m=1}^{k} z_{j_m} \right| \geqslant C.$$

Find the best constant C.

§ 1.1 Solution

We claim [1]

$$\left| \sum_{m=1}^{k} z_{j_m} \right| \geqslant \frac{1}{\pi} \sum_{j=1}^{n} |z_j| \geqslant \frac{1}{\pi} := C.$$

Write $z_j = |z_j| e^{i\alpha_j}$, where $0 \leqslant \alpha_j < 2\pi$, $\forall 1 \leqslant j \leqslant n$. For $0 \leqslant \theta \leqslant 2\pi$, $S(\theta) := \{k : \cos(\alpha_k - \theta) > 0, 1 \leqslant k \leqslant n\}$. Then we have the following estimate for $\left|\sum_{k\in S(\theta)} z_k\right|$:

$$\begin{aligned} \left| \sum_{k\in S(\theta)} z_k \right| &= \left| \sum_{k\in S(\theta)} e^{-i\theta} z_k \right| \geqslant Re \left(\sum_{k\in S(\theta)} e^{-i\theta} z_k \right) = \sum_{j=1}^{n} |z_j| \cos^{+}(\alpha_j - \theta) \\ &= \sum_{j=1}^{n} |z_j| \max\{\cos(\alpha_j - \theta), 0\} \\ &= \sum_{j=1}^{n} |z_j| \cdot \frac{|\cos(\alpha_j - \theta)| + \cos(\alpha_j - \theta)}{2} := f(\theta). \end{aligned}$$

Since $f(\theta) \in C([0, 2\pi])$, $\exists \theta_0 \in [0, 2\pi]$, such that $f(\theta_0) = \underset{0\leqslant\theta\leqslant 2\pi}{max} f(\theta)$. Observe that $f(\theta) \geqslant 0, \forall \theta \in [0, 2\pi]$, by the mean-value formula:

$$\frac{1}{2\pi} \int_0^{2\pi} f(\theta) d\theta = f(\theta_1) \leqslant f(\theta_0), \quad \exists \theta_1 \in [0, 2\pi].$$

Let $S = S(\theta_0)$. Note that $\cos^+(\theta) = \sum_{k=-\infty}^{\infty} \cos(\theta)\chi_{\{-\frac{\pi}{2}+k\pi \leqslant \theta \leqslant \frac{\pi}{2}+k\pi\}} \Rightarrow \cos^+(\theta) = \cos^+(\theta + 2\pi)$. Hence:

$$\begin{aligned} f(\theta_0) &\geqslant \frac{1}{2\pi}\int_0^{2\pi} f(\theta)d\theta = \frac{\sum_{j=1}^n |z_j|}{2\pi}\int_0^{2\pi} \cos^+(\alpha_j - \theta)d\theta \\ &= \frac{\sum_{j=1}^n |z_j|}{2\pi}\int_{-\alpha_j}^{2\pi-\alpha_j} \cos^+(\theta)d\theta \\ &= \frac{\sum_{j=1}^n |z_j|}{2\pi}\left(\int_{-\alpha_j}^{0} + \int_0^{2\pi} - \int_{2\pi-\alpha_j}^{2\pi}\right)\cos^+(\theta)d\theta \\ &= \frac{\sum_{j=1}^n |z_j|}{2\pi}\int_0^{2\pi} \cos^+(\theta)d\theta \\ &= \frac{\sum_{j=1}^n |z_j|}{2\pi}\left(\int_0^{\frac{\pi}{2}} + \int_{\frac{3\pi}{2}}^{2\pi}\right)\cos(\theta)d\theta = \frac{1}{\pi}\sum_{j=1}^n |z_j|. \end{aligned}$$

Therefore, we conclude $\left|\sum_{k\in S(\theta_0)} z_k\right| = f(\theta_0) \geqslant \dfrac{1}{\pi}\sum_{j=1}^n |z_j| \Rightarrow C = \dfrac{1}{\pi}$. where $\{z_{j_1}, ..., z_{j_k}\} = \{z_k\}_{k\in S(\theta_0)} \subseteq \{z_1, ..., z_n\}$.

Q.E.D.

§ 1.2 Remark

If we doesn't require the constant C to be independent of n, then we have the following version of this type of estimate [2]:

Suppose $\{z_j\}$ are complex numbers and

$$\sum_{j=1}^n |z_j| > 0,$$

then there is a subcollection $\{z_{j_1}, ..., z_{j_k}\} \subseteq \{z_1, ..., z_n\}$ such that

$$\left|\sum_{m=1}^k z_{j_m}\right| \geqslant \lambda_n \left(\sum_{j=1}^n |z_j|\right).$$

For some constant $\lambda_n > 0$, which depends merely on n.

Proof. We inherit the notations from above, namely $S(\theta), f(\theta)$.... Let $\mathcal{I} = \{\theta \in [0, 2\pi] : \exists 1 \leqslant j \leqslant n, s.t. \cos(\alpha_j - \theta) = 0\}$. For each j, $\cos(\alpha_j - \theta)$ has two zeros in $[0, 2\pi)$. For distinct j, these zeros might coincide. Then $2 \leqslant |\mathcal{I}| \leqslant 2n$. For any neighboring $\theta_j < \theta_{j+1} \in \mathcal{I}$, when θ ranges through $[\theta_j, \theta_{j+1}]$, $S(\theta) =: \mathcal{S}_j$ doesn't change, denote the collection of all the endpoints θ_j, where the index j is in the set $\mathcal{S}_j$, as $\mathcal{I}_j$, and $\sum_{k\in\mathcal{S}_j} z_k := r_j e^{i\beta_j}, r \geqslant 0$. Then $r_j = |\sum_{k\in\mathcal{S}_j} z_k| \leqslant \underset{1\leqslant j\leqslant|\mathcal{I}|-1}{max} r_j =: R$. When $\theta \in [\theta_j, \theta_{j+1}]$:

$$f(\theta) = Re\left(\sum_{k\in\mathcal{S}_j} e^{-i\theta} z_k\right) = Re\left(r_j e^{-i(\theta-\beta_j)}\right) = r_j\cos(\theta-\beta_j) \Rightarrow$$

$$\int_{\theta_j}^{\theta_{j+1}} f(\theta)d\theta = r_j\int_{\theta_j}^{\theta_{j+1}}\cos(\theta-\beta_j)d\theta = r_j\left(\sin(\theta-\beta_j)\right)|_{\theta_j}^{\theta_{j+1}}$$

$$= \quad 2r_j\cos\left(\frac{\theta_{j+1}+\theta_j}{2}-\beta_j\right)\sin\left(\frac{\theta_{j+1}-\theta_j}{2}\right) \leqslant 2r_j\sin\left(\frac{\theta_{j+1}-\theta_j}{2}\right).$$

Note that $\theta_{j+1}-\theta_j \in (0,\pi]$, so it makes sense that the RHS ≥ 0. Suppose $\mathcal{I}_j = \{\theta_1, \theta_2, ..., \theta_{N+1}\}$, where $\theta_1 < \theta_2 < ... < \theta_{N+1}$. By the definition of $\mathcal{I}_j$, $\theta_{j+1}-\theta_j \in (0,\pi]$. Observe that $\sin(\cdot)$ is concave on $[0,\frac{\pi}{2}]$, so the Jensen's inequality implies that

$$\frac{1}{N}\sum_{j=1}^{N}\sin\left(\frac{\theta_{j+1}-\theta_j}{2}\right) \leqslant \sin\left(\frac{1}{N}\sum_{j=1}^{N}\frac{\theta_{j+1}-\theta_j}{2}\right).$$

Thus,

$$\int_0^{2\pi} f(\theta)d\theta = \sum_{j=1}^{N}\int_{\theta_j}^{\theta_{j+1}} f(\theta)d\theta \leqslant \sum_{j=1}^{N} 2r_j\sin\left(\frac{\theta_{j+1}-\theta_j}{2}\right)$$

$$\leqslant \quad 2R\sum_{j=1}^{N}\sin\left(\frac{\theta_{j+1}-\theta_j}{2}\right) = 2RN\frac{1}{N}\sum_{j=1}^{N}\sin\left(\frac{\theta_{j+1}-\theta_j}{2}\right)$$

$$\leqslant \quad 2RN\sin\left(\frac{1}{N}\sum_{j=1}^{N}\frac{\theta_{j+1}-\theta_j}{2}\right) = 2RN\sin\left(\frac{1}{N}\cdot\frac{\theta_{N+1}-\theta_1}{2}\right)$$

$$\leqslant \quad 2RN\sin\left(\frac{\pi}{N}\right) \leqslant 2R\cdot 2n\sin\left(\frac{\pi}{2n}\right) := 2R\lambda_n^{-1}.$$

The last inequality holds because $\frac{\sin x}{x}$ monotonely decreases on $(0,\pi]$. By what we have proved in the previous section:

$$2R\lambda_n^{-1} \geqslant \int_0^{2\pi} f(\theta)d\theta \geqslant 2\sum_{j=1}^{n}|z_j| \Rightarrow R \geqslant \lambda_n\sum_{j=1}^{n}|z_j|.$$

Therefore, $\exists\{z_{j_1}, ..., z_{j_k}\} \subseteq \{z_1, ..., z_n\}$, such that

$$\left|\sum_{m=1}^{k} z_{j_m}\right| \geqslant \lambda_n\left(\sum_{j=1}^{n}|z_j|\right), \quad \lambda_n = \frac{1}{2n\sin\left(\frac{\pi}{2n}\right)}.$$

Q.E.D.

Note that $\{\lambda_n\}_{n=1}^{\infty}$ decreases and $\lim\limits_{n\to\infty} \lambda_n = \dfrac{1}{\pi}$, which is what we have deduced in the previous section! This also verifies $\dfrac{1}{\pi}$ is the uniform lower bound for all λ_n, and $\dfrac{1}{\pi}$ doesn't depend on n.

§ 2 Injective Extension

Let f be holomorphic on a neighborhood of $\overline{D}(0,1)$. Assume that the restriction of f to $\overline{D}(0,1)$ is one-to-one and f' is nowhere zero on $\overline{D}(0,1)$. Prove that in fact f is one-to-one on a neighborhood of $\overline{D}(0,1)$.

§ 2.1 Lemma

Let f be a non trivial holomorphic function in an open set Ω. $z_0 \in \Omega$, $f'(z_0) \neq 0$, then there exists a neighborhood of z_0, namely $D(z_0, \delta) \subset \Omega$, such that f is injective and holomorphic in $D(z_0, \delta)$.[3]

Proof. Since f is holomorphic at z_0, and $f'(z_0) \neq 0$, $\phi(z) := f(z) - f(z_0)$ has a zero of order 1 at z_0. By the discreteness of zeros of non constant holomorphic functions, $\exists \mu > 0$, where $d(\partial\Omega, z_0) > \mu$, such that $\phi(z)$ has no zeros on $\overline{D}(z_0, \mu) - \{z_0\}$. Set $\Gamma = \{z : |z - z_0| = \mu\}$, $\varepsilon = \underset{z\in\Gamma}{min} |\phi(z)| = \underset{z\in\Gamma}{min} |f(z) - f(z_0)|$. Then $\varepsilon > 0$. $f(z) - f(z_0)$ is continuous at z_0, for this $\varepsilon > 0$, exists $\delta > 0$, such that whenever $z \in \overline{D}(z_0, \delta) \subset \overline{D}(z_0, \mu)$,

$$|f(z) - f(z_0)| < \varepsilon \tag{2.1}$$

We claim that $f(z)$ is injective and holomorphic on $\overline{D}(z_0, \delta)$. Otherwise, there exists $z_1, z_2 \in \overline{D}(z_0, \delta), z_1 \neq z_2$, such that $f(z_1) = f(z_2) =: w^*$. From euqation (1), we know $|\omega^* - f(z_0)| < \varepsilon$. Apply Rouché's theorem[4] to Γ, then $f(z) - \omega^* = (f(z) - f(z_0)) + (f(z_0) - \omega^*)$ has same number of zeros (counted with multiplicities) as $f(z) - f(z_0)$ on $D(z_0, \mu)$. Hence, $f(z) - \omega^*$ has only one zero on $D(z_0, \mu)$, which contardicts the assumption that $f(z) - \omega^*$ has at least two zeros on $D(z_0, \delta)$, namely z_1 and z_2.

Q.E.D.

§ 2.2 Proof of the problem

Suppose otherwise, f is not injective on any neighborhood of $\overline{D}(0,1)$. Let f be holomorphic in $\Omega \supsetneq \overline{D}(0,1)$. Choose $1 > \varepsilon$ sufficiently small such that $\overline{D}(0, 1+\varepsilon) \subset \Omega$. Then, $\forall n \in \mathbb{N}^*, \exists z_n, \omega_n \in D(0, 1 + \frac{\varepsilon}{n})$, such that $f(z_n) = f(\omega_n)$ and $z_n \neq \omega_n$.

Since $\{z_n\}, \{\omega_n\}$ are bounded, Bolzano's theorem suggests that one can choose two converging subsequences of $\{z_n\}$, and $\{\omega_n\}$ with the same index set, denoted $\{z_{n_k}\}$ and $\{\omega_{n_k}\}$. Suppose $\{z_{n_k}\} \to z$ and $\{\omega_{n_k}\} \to \omega$. Note that f is holomorphic in Ω, and $z, \omega \in \overline{D}(0, 1+\varepsilon) \subset \Omega$, hence $f(z) = f(\lim\limits_{n_k \to \infty} z_{n_k}) = \lim\limits_{n_k \to \infty} f(z_{n_k}) = \lim\limits_{n_k \to \infty} f(\omega_{n_k}) = f(\lim\limits_{n_k \to \infty} \omega_{n_k}) = f(\omega)$.

Observe that $z_n, \omega_n \in D(0, 1+\frac{\varepsilon}{n}) \subset D(0, 1+\frac{1}{n})$, then $|z_n|, |\omega_n| \leqslant 1+\frac{1}{n}$. As a result, $|z|, |\omega| \leqslant 1$. Because f is injective on $\overline{D}(0,1)$, then $z = \omega$. Moreover, f' doesn't vanish on $\overline{D}(0,1)$ implies $f'(z) \neq 0$. Our lemma yields that f is injective on a neighborhood of z, namely $D(z, \delta)$. However, since $\{z_{n_k}\}, \{\omega_{n_k}\} \to z = \omega$, $\exists K > 0$, such that $\forall k > K$, $|z_{n_k} - z| < \delta, |\omega_{n_k} - z| < \delta$. Then $z_{n_k} \neq \omega_{n_k} \Rightarrow f(z_{n_k}) \neq f(\omega_{n_k})$, which contradicts our assumption that $f(z_{n_k}) = f(\omega_{n_k})$!

Q.E.D.

§ 2.3 Remark

In fact, we can further weaken the conditions provided in the problem, once one realizes the fact that if f is injective on $\overline{D}(0,1)$, then $f'(z) \neq 0$, for all $z \in D(0,1)$. Hnece, one doesn't have to assume f is injective on $\overline{D}(0,1)$ and f' is nowhere zero on $\overline{D}(0,1)$ to reach the same conclusion. We shall reduce $f'(z) \neq 0$ on $\overline{D}(0,1)$ to $f'(z) \neq 0$ on $\partial D(0,1)$:

Let f be holomorphic on a neighborhood of $\overline{D}(0,1)$, namely Ω. Assume that $f|_{\overline{D}(0,1)}$ is injective and $f' \neq 0$ on $\partial D(0,1)$. Then f is injective on a neighborhood of $\overline{D}(0,1)$.

Proof. **Step 1**, by modifying the trick we used in the proof of the lemma, we shall prove our claim that $f'(z) \neq 0$, for all $z \in D(0,1)$. Suppose otherwise, there exists $z_0 \in D(0,1)$, such that $f'(z_0) = 0$. Then z_0 is a zero of $f(z) - f(z_0)$, having an order $m (m \geqslant 2)$. Since f is injective, f is non constant, by the discreteness of the zeros of non constant holomorphic functions, there exists a sufficiently small $\mu > 0$, such that $\overline{D}(z_0, \mu) \subset D(0,1)$ and $f(z) - f(z_0), f'(z)$ has no zeros in $\overline{D}(z_0, \mu) - \{z_0\}$. Set $\varepsilon = \frac{1}{2} \underset{|z - z_0| = \mu}{min} |f(z) - f(z_0)|$. Then $|f(z) - f(z_0)| > \varepsilon$ on $\partial D(z_0, \mu)$.

By Rouché's theorem, $f(z) - f(z_0) - \varepsilon$ has the same number of zeros (counted with multiplicities) on $D(z_0, \mu)$ as $f(z) - f(z_0)$. Since $(f(z) - f(z_0) - \varepsilon)' = f'(z) \neq 0$ in $D(z_0, \mu)$, all the zeros of $f(z) - f(z_0) - \varepsilon$ are simple. Hence, $f(z) - f(z_0) - \varepsilon$ has m-many distinct zeros on $D(z_0, \mu) \subset D(0,1)$, which conflicts the fact that $f|_{\overline{D}(0,1)}$ is injective.

Step 2, repeat the the same argument in the proof of the problem above. Notice that in the proof of the problem, the condition that $f'(z) \neq 0$ on $\overline{D}(0,1)$ is applied only when we deduced $f'(z) \neq 0$, where z is the mutual limit of $\{z_{n_k}\}$ and $\{\omega_{n_k}\}$. Now we've deduced that $f'(z) \neq 0$ on $D(0,1)$, combined with the condition that $f'(z) \neq 0$ on $\partial D(0,1)$, we could reach the same conclusion with the same argument.

Q.E.D.

§ 3 Zeros Analysis

Prove that the equation $az^3 - z + b = e^{-z}(z+2)$, where $a > 0$ and $b > 2$, has exactly 2 roots in $\{z \in \mathbb{C} : Re(z) \geqslant 0\}$.

§ 3.1 Proof

Step 1: Apply Rouché's theorem to show that $az^3 - z + b$ and $e^{-z}(z+2)$ have the same number of roots in $\{z \in \mathbb{C} : Re(z) \geqslant 0\}$.

Let $\Gamma = \Gamma_1 \cup \Gamma_2$, where $\Gamma_1 = \{z \in \mathbb{C} : Re(z) = 0, |Im(z)| \leqslant R\}$, $\Gamma_2 = \{z \in \mathbb{C} : Re(z) > 0, |z| = R\}$. $f(z) := az^3 - z + b$, $g(z) := e^{-z}(z+2)$ are both entire functions. We wish to show that $f(z)$ and $g(z)$ satisfy the conditions for Rouché's theorem on Γ, namely $|f(z)| > |g(z)|$, provided R is sufficiently large, then pass $R \to \infty$ to conclude.

On Γ_1, $z = iy$, $|g(z)|^2 = |e^{iy}|^2 \cdot |iy + 2|^2 = y^2 + 4$, $|f(z)|^2 = |-aiy^3 - iy + b|^2 = b^2 + (ay^3 + y)^2$. Hence, $|f(z)|^2 - |g(z)|^2 = (ay^3+y)^2 - y^2 + b^2 - 4 = ay^4(ay^2+2) + b^2 - 4 \geqslant b^2 - 4 > 0$.

On Γ_2, Let $z = Re^{i\theta}$, $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$. Then $|g(z)| = |e^{-Re^{i\theta}}| \cdot |Re^{i\theta} + 2| = e^{-R\cos\theta}|Re^{i\theta} + 2| \leqslant R + 2$. $|f(z)| = |aR^3 e^{3i\theta} - Re^{i\theta} + b| \geqslant aR^3 - R - 2$, where R is sufficiently large such that $aR^3 - R - 2 > 0$. Note that there exists $R_0 > 0$, such that $aR^3 - R - 2 > R + 2$ when $R > R_0$.

Hence, when $R > R_0$, $|f(z)| > |g(z)|$ on Γ, by Rouché's theorem [4], $f(z) - g(z)$ has the same number of roots as $f(z)$ in $\{z \in \mathbb{C} : Re(z) > 0, |z| < R\}$, for all $R > R_0$. Passing $R \to \infty$, then $f(z) - g(z)$ and $g(z)$ has the same number of roots in $\{z \in \mathbb{C} : Re(z) > 0\}$. Since $|f(z)| > |g(z)|$ on $\{z \in \mathbb{C} : Re(z) = 0\}$. We conclude that $f(z) - g(z)$ has the same number of roots as $f(z)$ in $\{z \in \mathbb{C} : Re(z) \geqslant 0\}$.

Step 2: We shall show that $f(z) = az^3 - z + b$ has exactly two roots in $\{z \in \mathbb{C} : Re(z) \geqslant 0\}$.

Observe that $f(z)$ has a root z_1 on $\{z \in \mathbb{R} : Re(z) < 0\}$, since $\lim\limits_{z \to -\infty} f(z) = -\infty$ and $f(0) = b > 0$. Suppose $f(z) = a(z - z_1)(z - z_2)(z - z_3)$, by Vieta's theorem[5], $z_1 + z_2 + z_3 = 0$, and $z_1 z_2 z_3 = -\frac{b}{a} < 0$.

We claim neither z_2 nor z_3 can be on the negative real axis. Otherwise, if z_2, $z_3 \in \{z \in \mathbb{R} : Re(z) < 0\}$, then $z_1 + z_2 + z_3 < 0$, a contradiction! If either z_2 or $z_3 \in \{z \in \mathbb{R} : Re(z) < 0\}$, assume $z_2 \in \{z \in \mathbb{R} : Re(z) < 0\}$ and $z_3 \in \{z \in \mathbb{R} : Re(z) \geqslant 0\} \cup \{z \in \mathbb{C} : Im(z) \neq 0\}$, then either $z_1 z_2 z_3 > 0$ or $z_1 z_2 z_3$ has a none zero imaginary part, which is a contradiction!

Therefore, since $z_1 + z_2 + z_3 = 0$ requires $Im(z_2) = -Im(z_3)$, it follows from our claim that either z_2, $z_3 \in \{z \in \mathbb{R} : Re(z) \geqslant 0\}$ or z_2, $z_3 \in \{z \in \mathbb{C} : Im(z) \neq 0\}$. We are done in the former case. In the latter case, since $z_2 z_3 = -\frac{b}{az_1} > 0$, $z_2 |z_3|^2 = -\frac{b}{az_1}\overline{z_3}$, then $Re(z_2)Re(z_3) \geqslant 0$. Note that $z_1 + z_2 + z_3 = 0$ implies $Re(z_1) + Re(z_2) + Re(z_3) = 0$, hence $Re(z_2)$, $Re(z_3) \geqslant 0$, i.e. z_2, $z_3 \in \{z \in \mathbb{C} : Re(z) \geqslant 0\}$. Now we conclude that $f(z)$ has exactly two roots in $\{z \in \mathbb{C} : Re(z) \geqslant 0\}$.

Above all, $az^3 - z + b - e^{-z}(z+2) = f(z) - g(z)$ has exactly two roots in $\{z \in \mathbb{C} : Re(z) \geqslant 0\}$.

§ 4 Gauss-Lucas Theorem

If f is a polynomial on $\mathbb{C}$, then the zeros of f' are contained in the closed convex hull of the zeros of f.

§ 4.1 Proof

The assertion follows from the Gauss-Lucas Theorem. Suppose $\mathcal{S} = \{z_1, z_2, ..., z_n\} \subset \mathbb{C}$. Then the closed convex hull of $\mathcal{S}$ is defined as

$$\mathcal{C}(\mathcal{S}) := \{\sum_{j=1}^{n} c_j z_j : \sum_{j=1}^{n} c_j = 1, 0 \leqslant c_j \in \mathbb{R}, \text{for all } 1 \leqslant j \leqslant n\} \tag{4.1}$$

Now suppose f is a polynomial with $n = \deg(f) \geqslant 2$, otherwise $\deg(f') = 0$ and there is nothing to prove. Let $\mathcal{S} = \{z \in \mathbb{C} : f(z) = 0\} = \{z_1, z_2, ..., z_n\}$, $\mathcal{S}' = \{z \in \mathbb{C} : f'(z) = 0\}$. We wish to show that $\mathcal{C}(\mathcal{S}') \subset \mathcal{C}(\mathcal{S})$.

By the fundamental theorem of Algebra,

$$f(z) = c \prod_{j=1}^{n} (z - z_j) \tag{4.2}$$

where $c \neq 0$ is the leading term of f. For any $z \in \mathcal{S}'$, if $z \in \mathcal{S}$, then $z \in \mathcal{C}(\mathcal{S})$ and there is nothing prove. If $z \notin \mathcal{S}$, the logarithmic derivative of f vanishes at z. By Eq.(4.2):

$$0 = \frac{f'(z)}{f(z)} = \sum_{j=1}^{n} \frac{1}{z - z_j} = \sum_{j=1}^{n} \frac{\overline{z} - \overline{z}_j}{|z - z_j|^2} \tag{4.3}$$

where $|z - z_j| \neq 0$. Hence,

$$\sum_{j=1}^{n} \frac{\overline{z}}{|z - z_j|^2} = \sum_{j=1}^{n} \frac{\overline{z}_j}{|z - z_j|^2} \tag{4.4}$$

Take the complex conjugate on both sides, and we obtain

$$z \left(\sum_{j=1}^{n} \frac{1}{|z - z_j|^2} \right) = \sum_{j=1}^{n} \frac{z}{|z - z_j|^2} = \sum_{j=1}^{n} \frac{z_j}{|z - z_j|^2} \tag{4.5}$$

Therefore,

$$z = \sum_{j=1}^{n} \left(\frac{|z - z_j|^{-2}}{\sum_{k=1}^{n} |z - z_k|^{-2}} \right) z_j := \sum_{j=1}^{n} c_j z_j \tag{4.6}$$

Note that

$$c_j = \frac{|z - z_j|^{-2}}{\sum_{k=1}^{n} |z - z_k|^{-2}} \geqslant 0 \tag{4.7}$$

and

$$\sum_{j=1}^{n} c_j = \sum_{j=1}^{n} \left(\frac{|z - z_j|^{-2}}{\sum_{k=1}^{n} |z - z_k|^{-2}} \right) = \frac{\sum_{j=1}^{n} |z - z_j|^{-2}}{\sum_{k=1}^{n} |z - z_k|^{-2}} = 1 \tag{4.8}$$

Hence, $z \in \mathcal{C}(\mathcal{S})$, where $\mathcal{C}(\mathcal{S})$ is given in Eq.(4.1). Since $z \in \mathcal{C}(\mathcal{S}')$ is arbitrary, we conclude that $\mathcal{C}(\mathcal{S}') \subset \mathcal{C}(\mathcal{S})$, which completes our proof.

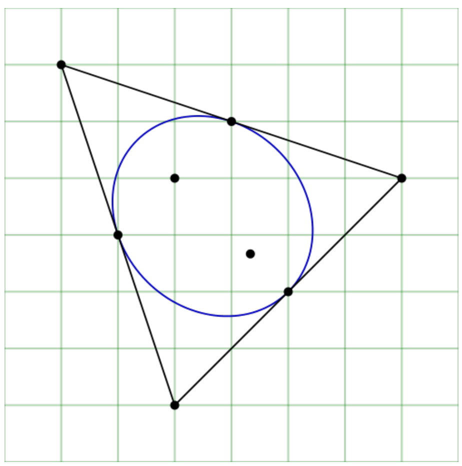

Figure 1: Sketch of a Steiner inellipse, from [6]. According to Marden's theorem[7] , given the triangle with vertices $(1,7), (7,5), (3,1)$, the foci of the inellipse are $(3,5)$ and $(13/3, 11/3)$, since $D_x(1+7i-x)(7+5i-x)(3+i-x)$ $= -3\left(\frac{13}{3}+\frac{11}{3}i-x\right)(3+5i-x)$

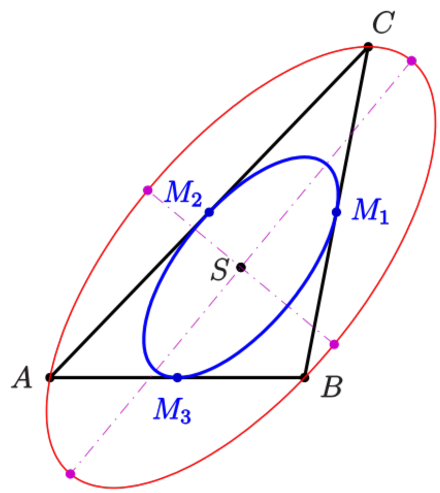

Figure 2: Another sketch of Steiner ellipses with respect to a triangle Δ ABC, from[6].
— Arbitrary triangle ΔABC
— Steiner inellipse
— Steiner ellipse
— Major and minor axes

§ 4.2 Remark

In fact, we can further narrow down the range of $\mathcal{S}'$ in special cases.[8]

If a polynomial of degree n of real coefficients has n distinct real zeros $x_1 < x_2 < \cdots < x_n$, we see, using Rolle's theorem, that the zeros of the derivative polynomial are in the interval $[x_1, x_n]$ which is the convex hull of the set of roots. The convex hull of the roots of the polynomial $P(x) = p_n x^n + p_{n-1}x^{n-1} + \cdots + p_0$ particularly includes the point $-\dfrac{p_{n-1}}{n \cdot p_n}$.

It is easy to see that if $P(x) = ax^2 + bx + c$ is a second degree polynomial, the zero of $P'(x) = 2ax + b$ is the average of the roots of P. In that case, the convex hull is the line segment with the two roots as endpoints and it is clear that the average of the roots is the middle point of the segment.

For a fourth degree complex polynomial P (quartic function) with four distinct zeros forming a concave quadrilateral, one of the zeros of P lies within the convex hull of the other three; all three zeros of P' lie in two of the three triangles formed by the interior zero of P and two others zeros of P.[9]

For a third degree complex polynomial P (cubic function) with three distinct zeros, Marden's theorem[7] states that the zeros of P' are the foci of the Steiner inellipse which is the unique ellipse tangent to the midpoints of the triangle formed by the zeros of P:

§ 5 Strict Harmonicity

If $f : U \to \mathbb{R}$ is merely continuous, we might call f strictly subharmonic if whenever $\overline{D}(z,r) \subseteq U$, then

$$f(z) < \frac{1}{2\pi}\int_0^{2\pi} f(z+re^{i\theta})d\theta \tag{5.1}$$

For C^2 functions, is this equivalent to the assertion that $\nabla^2 f > 0$? (we use ∇^2 instead of $\triangle$ for the Laplace operator). Does one definition imply the other? Can you think of a definition that applies to continuous functions and is equivalent to $\nabla^2 f > 0$ when f is C^2?

§ 5.1 Solution

We divide the problem into two parts:
⋆ Whether Eq.(5.1) is equivalent to $\nabla^2 f > 0$, provided f is C^2?
⋆ Is there a definition of subharmonicity that applies to continuous functions and is equivalent to $\nabla^2 f > 0$ when f is C^2?

§ 5.2 Are the definitions equivalent?

Provided f is C^2, we claim Eq.(5.1) is not equivalent to $\nabla^2 f > 0$. In fact, $\nabla^2 f > 0$ imply Eq.(5.1), but Eq.(5.1) doesn't imply $\nabla^2 f > 0$. Now, for all $z \in U$, define

$$\phi(r) := \frac{1}{2\pi}\int_0^{2\pi} f(z+re^{i\theta})d\theta = \frac{1}{2\pi r}\int_{\partial D(z,r)} f(y)dS(y), \tag{5.2}$$

where $r < \mathrm{dist}(z, \partial U) = \inf\{|z-p| : p \in \partial U\}$. Note that the RHS of Eq.(5.2):

$$\frac{1}{2\pi r}\int_{\partial D(z,r)} f(y)dS(y) = \frac{1}{2\pi}\int_{\partial D(0,1)} f(z+r\omega)dS(\omega). \tag{5.3}$$

Hence, Eq.(5.2) and Eq.(5.3) gives

$$\phi(r) = \frac{1}{2\pi}\int_{\partial D(0,1)} f(z+r\omega)dS(\omega). \tag{5.4}$$

Now, differentiate Eq.(5.4) with respect to r:

$$\begin{aligned}\phi'(r) &= \frac{1}{2\pi}\int_{\partial D(0,1)} \nabla f(z+r\omega)\cdot\omega dS(\omega)\\ &= \frac{1}{2\pi r}\int_{\partial D(z,r)} \nabla f(y)\frac{y-z}{r}dS(y)\\ &= \frac{1}{2\pi r}\int_{\partial D(z,r)} \frac{\partial f}{\partial n}dS = \frac{1}{2\pi r}\int_{D(z,r)} \nabla^2 f(y)dy,\end{aligned} \tag{5.5}$$

where the last equality is due to the Divergence theorem. Since f is continuous, take the limit as $r \to 0^+$ on both sides of Eq.(5.2), we obtain

$$\lim_{r\to 0}\phi(r) = \lim_{r\to 0}\left(\frac{1}{2\pi r}\int_{\partial D(z,r)} f(y)dS(y)\right) = f(x) \tag{5.6}$$

Now, if $\nabla^2 f > 0$, Eq.(5.6) yields that $\phi'(r) > 0$. Then, whenever $\overline{D}(z,r) \subseteq U$,

$$\frac{1}{2\pi}\int_0^{2\pi} f(z+re^{i\theta})d\theta = \phi(r) > \lim_{r\to 0}\phi(r) = f(z). \tag{5.7}$$

However, in general, Eq.(5.1) holds whenever $\overline{D}(z,r) \subseteq U$ doesn't imply $\nabla^2 f(z) > 0$ on U. Consider

$$f : U \to \mathbb{R}, \quad f(x+iy) = x^4, \quad U = \mathbb{C}. \tag{5.8}$$

For any $z = x+iy \in \mathbb{C}$, for all $r > 0$,

$$\frac{1}{2\pi}\int_0^{2\pi} f(z+re^{i\theta})d\theta = x^4 + 3r^2x^2 + \frac{3r^4}{8} > x^4 = f(z). \tag{5.9}$$

Then, Eq.(5.9) verifies that f satisfies Eq.(5.1), whenever $\overline{D}(z,r) \subseteq U = \mathbb{C}$. But $\nabla^2 f(z) = 12x^2$, which vanishes on $\{z \in \mathbb{C} : \mathrm{Re}(z) = 0\}$.

§ 5.3 An equivalent definition of strict subharmonicity

Now, we shall give an equivalent definition that applies to continuous functions and is equivalent to $\nabla^2 f > 0$ when f is C^2. We claim that: f is called strictly subharmonic if whenever $\overline{D}(z,r) \subseteq U$, then $\exists\delta > 0$ such that

$$f(z) - \delta|z|^2 \leqslant \frac{1}{2\pi}\int_0^{2\pi} f(z+re^{i\theta}) - \frac{\delta}{2\pi}\int_0^{2\pi} |z+re^{i\theta}|^2 d\theta. \tag{5.10}$$

Let's first prove this is equivalent to $\nabla^2 f > 0$ when $f \in C^2$. If $f \in C^2$ satisfies $\nabla^2 f > 0$, it has to satisfy Eq.(5.10). Since $|z|^2 \in C^2$ and $\nabla^2|z|^2 = 4 > 0$, it follows from subsection.(5.2) that whenever $\overline{D}(z,r) \subseteq U$,

$$|z|^2 < \frac{1}{2\pi}\int_0^{2\pi} |z+re^{i\theta}|^2 d\theta. \tag{5.11}$$

By subsection (5.2), Eq.(5.1) holds because $\nabla^2 f > 0$:

$$f(z) < \frac{1}{2\pi}\int_0^{2\pi} f(z+re^{i\theta})d\theta. \tag{5.12}$$

Hence, there exists some $\delta > 0$, such that

$$\delta\left(\frac{1}{2\pi}\int_0^{2\pi}|z+re^{i\theta}|^2 d\theta - |z|^2\right) \leqslant \frac{1}{2\pi}\int_0^{2\pi} f(z+re^{i\theta})d\theta - f(z). \tag{5.13}$$

which gives Eq.(5.10).

Conversely, if $f \in C^2$ satisfies Eq.(5.10): For all $z \in U$, $\overline{D}(z,r) \subseteq U$, there exists some $\delta > 0$ that Eq.(5.10) holds. If there exists $z \in U$, such that $\nabla^2 f(z) \leqslant 0$, then for any $\delta > 0$, there exists $\rho > 0$ and such that $D(z,\rho) \subseteq U$ and $\nabla^2(f(\zeta) - \delta|\zeta|^2) < 0$ on $D(z,\rho)$. Similar to Eq.(5.5),

$$\begin{aligned}&\frac{d}{dr}\left(\frac{1}{2\pi}\int_0^{2\pi} f(z+re^{i\theta}) - \delta|z+re^{i\theta}|^2 d\theta\right)\\ =&\frac{1}{2\pi r}\int_{D(z,r)} \nabla^2(f(\omega)-\delta|\omega|^2)d\omega < 0, \quad r<\rho.\end{aligned} \tag{5.14}$$

Besides, similar to Eq.(5.6),

$$f(z) - \delta|z|^2 = \lim_{r\to 0}\left(\frac{1}{2\pi}\int_0^{2\pi} f(z+re^{i\theta}) - \delta|z+re^{i\theta}|^2 d\theta\right) \tag{5.15}$$

Thus, by Eq.(5.14) and Eq.(5.15) and the Fundamental theorem of calculus, we obtain

$$\begin{aligned}f(z) - \delta|z|^2 =& \frac{1}{2\pi}\int_0^{2\pi} f(z+re^{i\theta}) - \delta|z+re^{i\theta}|^2 d\theta\\ &-\lim_{s\to 0}\int_s^r \frac{d}{d\tau}\left(\frac{1}{2\pi}\int_0^{2\pi} f(z+\tau e^{i\theta}) - \delta|z+\tau e^{i\theta}|^2 d\theta\right)d\tau\\ >& \frac{1}{2\pi}\int_0^{2\pi} f(z+re^{i\theta}) - \delta|z+re^{i\theta}|^2 d\theta,\end{aligned} \tag{5.16}$$

which contradicts the condition in Eq.(5.10). Hence, we must have $\nabla^2 f > 0$ on U.

Therefore, the definition of strict subharmonicity given by Eq.(5.10) is equivalent to $\nabla^2 f > 0$ when f is C^2. Next, we shall show it agrees with the definition given by Eq.(5.1) when $f : U \to \mathbb{R}$ is merely continuous.

§ 5.4 The new definition agrees with the original definition

Note that Eq.(5.10) automatically imply Eq.(5.1). Whenever $\overline{D}(z,r) \subseteq U$, there exists $\delta > 0$ such that Eq.(5.10) holds. Let's multiply δ on both sides of Eq.(5.11)

$$\delta|z|^2 < \frac{\delta}{2\pi}\int_0^{2\pi}|z+re^{i\theta}|^2 d\theta \tag{5.17}$$

and add it to Eq.(5.10):

$$f(z) < \frac{1}{2\pi}\int_0^{2\pi} f(z+re^{i\theta})d\theta. \tag{5.18}$$

Conversely, Eq.(5.1) imply Eq.(5.10). Whenever $\overline{D}(z,r) \subseteq U$ such that Eq.(5.1) holds, there exists $\delta > 0$ but sufficiently small, such that

$$\delta\left(\frac{1}{2\pi}\int_0^{2\pi} |z+re^{i\theta}|^2 d\theta - |z|^2\right) \leqslant \frac{1}{2\pi}\int_0^{2\pi} f(z+re^{i\theta})d\theta - f(z) \tag{5.19}$$

due to Eq.(5.11). But this is exactly Eq.(5.10):

$$f(z) - \delta|z|^2 \leqslant \frac{1}{2\pi}\int_0^{2\pi} f(z+re^{i\theta}) - \frac{\delta}{2\pi}\int_0^{2\pi} |z+re^{i\theta}|^2 d\theta. \tag{5.20}$$

This shows that the definition for strict subharmonicity agrees with the original definition when $f : U \to \mathbb{R}$ is merely continuous. Hence, Eq.(5.10) indeed applies to continuous functions and we have already verified it is equivalent to $\nabla^2 f > 0$ when $f \in C^2$.

§ 6 The Swiss Cheese

Let $\overline{D}(a_j, r_j)$ be pair wise disjoint closed discs in $D(0,1)$ such that the union of discs $\bigcup_{j=1}^{\infty} \overline{D}(a_j, r_j)$ is dense in $\overline{D}(0,1)$. Let $K = \overline{D}(0,1) - (\bigcup_{j=1}^{\infty} \overline{D}(a_j, r_j))$. Prove that such discs can be chosen so that $\sum_{j=1}^{\infty} r_j < 1$ and that in this case the conclusion of Mergelyan's theorem fails on K. (This is the famous ”swiss cheese” example from Alice Roth 1938, p.96.)

§ 6.1 Definition of the Swiss Cheese

Definition. Suppose $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$, if $\{\overline{D}(a_j, r_j)\}_{j=1}^{\infty}$ is a sequence of closed discs that satisfies the following properties:

(1) $\overline{D}(a_j, r_j) \subset \mathbb{D}$ are pairwise disjoint;
(2) $\sum_{j=1}^{\infty} r_j < 1$;
(3) $K = \overline{\mathbb{D}} \setminus (\bigcup_{j=1}^{\infty} D(a_j, r_j))$ has empty interior.

Then the set K is called Roth's Swiss Cheese.[10]

§ 6.2 Construction of the Swiss Cheese

Note that the third condition $\mathrm{int}(K) = \emptyset$ in above definition is equivalent to that $\bigcup_{j=1}^{\infty} \overline{D}(a_j, r_j)$ is dense in $\overline{\mathbb{D}}$. Denote $\mathbb{Q}^2 = \{z \in \mathbb{C} : \mathrm{Re}(z) \in \mathbb{Q}, \mathrm{Im}(z) \in \mathbb{Q}\}$,

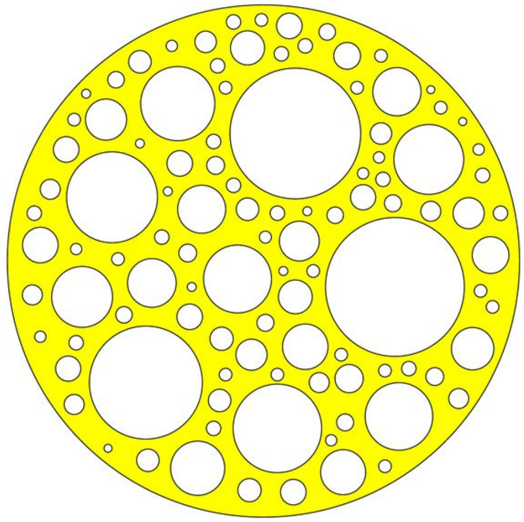

Figure 3: Roth's Swiss Cheese [11]

and $\mathcal{R}_1 = \mathbb{D} \cap \mathbb{Q}^2$. Choose $0 = a_1 \in \mathcal{R}_1$ and $r_1 < 1/2$ such that $\overline{D}(a_1, r_1) \subset \mathbb{D}$. Let $\mathcal{R}_2 = \mathcal{R}_1 - \overline{D}(a_1, r_1)$, choose $a_2 \in \mathcal{R}_2$, $r_2 < 2^{-2}$ such that $\overline{D}(a_2, r_2) \subset \mathbb{D}$ and $\overline{D}(a_2, r_2) \cap \overline{D}(a_1, r_1) = \emptyset$. ... Let $\mathcal{R}_n = \mathcal{R}_1 - \bigcup_{j=1}^{n-1} \overline{D}(a_j, r_j)$, choose $a_n \in \mathcal{R}_n$ and $r_n < 2^{-n}$ such that $\overline{D}(a_n, r_n) \subset \mathbb{D}$ and $\overline{D}(a_n, r_n) \cap (\bigcup_{j=1}^{n-1} \overline{D}(a_j, r_j)) = \emptyset$...

By induction, we construct $\{\overline{D}(a_j, r_j)\}_{j=1}^{\infty}$, where $\overline{D}(a_j, r_j) \subset \mathbb{D}$ are pairwise disjoint, $\sum_{j=1}^{\infty} r_j < \sum_{j=1}^{\infty} 2^{-j} = 1$. Moreover, we claim $K = \overline{D} \setminus (\bigcup_{j=1}^{\infty} D(a_j, r_j))$ has empty interior, which is equivalent to show that $\bigcup_{j=1}^{\infty} \overline{D}(a_j, r_j)$ is dense in $\overline{\mathbb{D}}$. But since $\mathcal{R}_1$ is dense in $\overline{\mathbb{D}}$, by our construction, $\mathcal{R}_1 \subset \bigcup_{j=1}^{\infty} \overline{D}(a_j, r_j)$, it follows that the assertion $\mathrm{int}(K) = \emptyset$ holds.

§ 6.3 The conclusion of Mergelyan's theorem fails

Let K be an arbitray subset of $\mathbb{C}$, we follow the conventions of Rational Approximation theory[12] and write

$$\begin{aligned}
&C(K) := \{f : f \text{ is continuous on } K\},\\
&A(K) := \{f \in C(K) : f \text{ is holomorphic on } \mathrm{int}(K)\},\\
&R(K) := \{f \in A(K) : \forall \varepsilon > 0, \exists R_\varepsilon \text{ rational such that } \sup_{z \in K} |f(z) - R_\varepsilon(z)| < \varepsilon\},\\
&P(K) := \{f \in A(K) : \forall \varepsilon > 0, \exists P_\varepsilon \text{ polynomial such that } \sup_{z \in K} |f(z) - P_\varepsilon(z)| < \varepsilon\}.
\end{aligned}$$

Since any polynomial is also rational, we have that

$$P(K) \subset R(K) \subset A(K) \subset C(K).$$

(Mergelyan's theorem) For all $K \subset\subset \mathbb{C}$ compact, $\hat{\mathbb{C}} \setminus K$ connected. Let $f \in A(K)$, then $f(z)$ can be approxiamted by polynomials uniformly on K.[13] (in other words, $A(K) = R(K) = P(K)$.)

For the Roth's Swiss Cheese example, $K = \overline{D} \backslash (\bigcup_{j=1}^{\infty} D(a_j, r_j))$, which is bounded and closed, thus compact, and $\partial\mathbb{D} \cup \bigcup_{j=1}^{\infty} \partial D(a_j, r_j) \subset K$. We shall show that the Mergelyan's theorem fails by proving that $P(K) \neq A(K)$. Consider the fucntion $f(z) = |z|/z \in A(K)$ (note that $\mathrm{int}(K) = \emptyset$ and $0 \notin K$), we claim that f can not be approximated by any polynomials on K. Suppose $f(z)$ is approximated by polynomials on K, then for all $\varepsilon > 0$, there exists $p(z) \in P(K)$, such that

$$\sup_{z \in K} |f(z) - p(z)| < \varepsilon. \tag{6.1}$$

Since $p(z)$ is an entire function, for any finite index set J, we may we may apply the Cauchy's theorem to write

$$\int_{\partial\mathbb{D}} p(z)dz = \sum_{j \in J} \int_{\partial D(a_j, r_j)} p(z)dz, \tag{6.2}$$

where $\partial\mathbb{D}$ and $\partial D(a_j, r_j)$ are all equipped with counter-clockwise orientation.

Thus, we have that

$$\begin{aligned}
&\left| \int_{\partial\mathbb{D}} f(z)dz - \sum_{j \in J} \int_{\partial D(a_j, r_j)} f(z)dz \right| \\
=& \left| \int_{\partial\mathbb{D}} [f(z) - p(z)]dz - \sum_{j \in J} \int_{\partial D(a_j, r_j)} [f(z) - p(z)]dz \right| \\
\leqslant& \int_{\partial\mathbb{D}} \sup_{z \in K} |f(z) - p(z)| dz + \sum_{j \in J} \int_{\partial D(a_j, r_j)} \left(\sup_{z \in K} |f(z) - p(z)| \right) dz \\
<& \left(1 + \sum_{j \in J} r_j \right) 2\pi\varepsilon < 4\pi\varepsilon.
\end{aligned} \tag{6.3}$$

But

$$\begin{aligned}
&\left| \int_{\partial\mathbb{D}} f(z)dz \right| = \left| \int_{\partial\mathbb{D}} \frac{|z|}{z} dz \right| = \left| \int_0^{2\pi} \frac{1}{e^{i\theta}} ie^{i\theta} d\theta \right| = 2\pi, \\
&\left| \sum_{j \in J} \int_{\partial D(a_j, r_j)} f(z)dz \right| \leqslant \sum_{j \in J} \int_{\partial D(a_j, r_j)} |f(z)|dz = 2\pi \sum_{j \in J} r_j \leqslant 2\pi \sum_{j=1}^{\infty} r_j.
\end{aligned} \tag{6.4}$$

If we let $\varepsilon < \frac{1}{2}\left(1 - \sum_{j=1}^{\infty} r_j\right)$, it follows from Eq.(6.3) and Eq.(6.4) that

$$\begin{aligned}
2\pi \left(1 - \sum_{j=1}^{\infty} r_j \right) &\leqslant \left| \int_{\partial\mathbb{D}} f(z)dz \right| - \left| \sum_{j \in J} \int_{\partial D(a_j, r_j)} f(z)dz \right| \\
&\leqslant \left| \int_{\partial\mathbb{D}} f(z)dz - \sum_{j \in J} \int_{\partial D(a_j, r_j)} f(z)dz \right| < 4\pi\varepsilon < 2\pi \left(1 - \sum_{j=1}^{\infty} r_j \right).
\end{aligned} \tag{6.5}$$

This is a contradiction! Therefore, $P(K) \neq A(K)$, we conclude that the Mergelyan's theorem fails on K.

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