Representing Function Values Through Integrals: Applications in Harmonic and Potential Theory

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Abstract

In this project, we investigate methods to represent the value of a function at a specific point using integrals. Inspired by analogies between Cauchy Integral Formula in Complex Analysis and fundamental results in Mathematical Analysis, we explore key results in harmonic and potential theory. Specifically, we analyze representation formulas for harmonic functions and their connection to boundary integrals, using tools such as Green's identities and spherical coordinate transformations. The methods are demonstrated through detailed proofs and examples.

1 Introduction

The concept of representing a function's value at a specific point through integrals forms a cornerstone in mathematical analysis. A prime example of this is the Cauchy Integral Formula in Complex Analysis, where the value of a holomorphic function inside a closed contour is expressed as a boundary integral. Similar ideas permeate other fields of analysis, particularly in harmonic and potential theory, where solutions to the Laplace equation (harmonic functions) are expressed in terms of their boundary data.

This project delves into these integral representations in the context of harmonic functions. The discussion centers around proving formulas that express the value of a harmonic function at a point within a domain as a boundary integral. These formulas involve the function values and normal derivatives on the boundary, demonstrating the interplay between boundary conditions and the behavior of harmonic functions in the interior.

We also explore examples that highlight the use of spherical coordinates and Green's identities to derive these representation formulas. Through rigorous proofs and careful analysis, the results provide a deeper understanding of the mathematical structures underlying harmonic and potential theory. This project aims to extend these ideas to broader contexts and applications.

2 Problem 1

$$u \in \mathbb{C}^2(\mathbb{R}^3), supp(u) := \left\{x \in \mathbb{R}^3 \mid \quad u(x) \neq 0\right\}$$
$$supp(u) \subset \mathbb{B}^3, \mathbb{B}^3 := \left\{x \in \mathbb{R}^3 \mid \quad \|x\| < 1\right\}, y \in \mathbb{B}^3, x \in \mathbb{R}^3$$

prove that

$$x \mapsto \frac{1}{\|x-y\|}$$

is a harmonic function and

$$u(y) = \int_{\mathbb{B}^3} -\frac{\Delta u(x) dx}{4\pi \|x-y\|}$$

Proof

$$v : \mathbb{R}^3 \to \mathbb{R}, x \mapsto \|x-y\|, B_y(\varepsilon) := \left\{x \in \mathbb{R}^3 \mid \|x-y\| < \varepsilon\right\},$$
$$\Omega_\varepsilon := \mathbb{B}^3 - B_y(\varepsilon), x = (x_1, x_2, x_3)^T, y = (y_1, y_2, y_3)^T$$
$$r = \|x-y\| = \sqrt{(x_1-y_1)^2 + (x_2-y_2)^2 + (x_3-y_3)^2}$$

Then we have

$$\nabla v(x) = \left(\frac{x_1-y_1}{r^3}, \frac{x_2-y_2}{r^3}, \frac{x_3-y_3}{r^3}\right)$$

then

$$\frac{\partial^2 v}{\partial x_1^2} = -\frac{r^3 - 3r(x_1-y_1)^2}{r^6}$$

similarly,we have

$$\frac{\partial^2 v}{\partial x_2^2} = -\frac{r^3 - 3r(x_2-y_2)^2}{r^6}, \frac{\partial^2 v}{\partial x_3^2} = -\frac{r^3 - 3r(x_3-y_3)^2}{r^6}$$

then we could prove

$$\Delta v(x) = -\frac{3r^3 - 3r \cdot r^2}{r^6} = 0$$

By

$$\int_{\mathbb{B}^3} -\frac{\Delta u(x)dx}{4\pi\|x-y\|} = -\frac{1}{4\pi}\int_{\mathbb{B}^3} v(x)\Delta u(x)dx$$

We have

$$\begin{aligned} -\frac{1}{4\pi}\int_{\mathbb{B}^3} v(x)\Delta u(x)dx &= -\frac{1}{4\pi}\left(\int_{\Omega_\varepsilon} + \int_{B_y(\varepsilon)} v(x)\Delta u(x)dx\right), \\ I = -\frac{1}{4\pi}\int_{\Omega_\varepsilon} v(x)\Delta u(x)dx, &\qquad J = \frac{1}{4\pi}\int_{B_y(\varepsilon)} v(x)\Delta u(x)dx \end{aligned} \tag{1}$$

By Green Equality

$$\int_{\Omega_\varepsilon}(u\Delta v - v\Delta u) = -\int_{\Omega_\varepsilon}(v\Delta u) = \int_{\partial\Omega_\varepsilon}\left(u\frac{\partial v}{\partial \boldsymbol{n}} - v\frac{\partial u}{\partial \boldsymbol{n}}\right)$$

$$= \frac{1}{4\pi}\left(\int_{\partial\mathbb{B}^3} u\frac{\partial v}{\partial \boldsymbol{n}} - \int_{\partial B_y(\varepsilon)} u\frac{\partial v}{\partial \boldsymbol{n}} - \int_{\partial\mathbb{B}^3} v\frac{\partial u}{\partial \boldsymbol{n}} + \int_{\partial B_y(\varepsilon)} v\frac{\partial u}{\partial \boldsymbol{n}}\right)$$

Now that

$$supp(u) \subset \mathbb{B}^3$$

instantly,we have

$$u(x) = 0, \frac{\partial u}{\partial \boldsymbol{n}} = 0, \forall x \in \partial\mathbb{B}^3$$

Then

$$I = -\frac{1}{4\pi}\int_{\Omega_\varepsilon} v\Delta u = \frac{1}{4\pi}\left(-\int_{\partial B_y(\varepsilon)} u\frac{\partial v}{\partial \boldsymbol{n}} + \int_{\partial B_y(\varepsilon)} v\frac{\partial u}{\partial \boldsymbol{n}}\right)$$

While

$$\left|\int_{\partial B_y(\varepsilon)} v\frac{\partial u}{\partial \boldsymbol{n}}\right| \le \max_{x\in\partial B_y(\varepsilon)} |\nabla u(x)| \cdot \left|\int_{\partial B_y(\varepsilon)} v\right|$$

$$= \max_{x\in\partial B_y(\varepsilon)} |\nabla u(x)| \cdot \int_{\partial B_y(\varepsilon)} \frac{1}{\varepsilon} = \max_{x\in\partial B_y(\varepsilon)} |\nabla u(x)| \cdot 4\pi\varepsilon = O(\varepsilon)$$

and

$$-\frac{1}{4\pi}\int_{\partial B_y(\varepsilon)} u\frac{\partial v}{\partial \boldsymbol{n}} = -\frac{1}{4\pi}\int_{\partial B_y(\varepsilon)} u\langle\nabla v, \vec{n}\rangle = \frac{1}{4\pi}\int_{\partial B_y(\varepsilon)} u\frac{r}{r^3} = \frac{1}{4\pi}\int_{\partial B_y(\varepsilon)} \frac{u}{\varepsilon^2}$$

By Integral Mean Value Theorem

$$\lim_{\varepsilon\to0^+}\frac{1}{4\pi}\int_{\partial B_y(\varepsilon)}\frac{u}{\varepsilon^2}=\frac{1}{4\pi}u(y)\frac{4\pi\varepsilon^2}{\varepsilon^2}=u(y)$$

Then we consider J. By Spherical coordinate transformation,

$$\lim_{\varepsilon\to0^+}\int_{B_y(\varepsilon)}v\Delta u=\lim_{\varepsilon\to0^+}\int_{B_y(\varepsilon)}\frac{\Delta u}{r}$$

$$=\lim_{\varepsilon\to0^+}\int_0^\pi d\theta\int_0^{2\pi}d\phi\int_0^\varepsilon\frac{r^2\sin\theta}{r}\Delta u dr=\lim_{\varepsilon\to0^+}O(\varepsilon^2)=0$$

Consider the equation 1, let $\varepsilon\to0^+$, then we have

$$u(y)=\int_{\mathbb{B}^3}-\frac{\Delta u(x)dx}{4\pi\|x-y\|}$$

Q.E.D

3 Problem 2

Suppose $\Omega\subset\mathbb{R}^3$,bounded, $\partial\Omega$ is of $\mathbb{C}^1$.$u\in\mathbb{C}^2(\Omega)$ and $\Delta u=0$. Prove that

$$u(\boldsymbol{y})=\frac{1}{4\pi}\int_{\partial\Omega}\left(\frac{X\cdot\vec{n}}{r^2}u+\frac{1}{r}\frac{\partial u}{\partial\boldsymbol{n}}\right)d\sigma \tag{2}$$

where

$$r=\sqrt{(x-x_0)^2+(y-y_0)^2+(z-z_0)^2},\quad X=\frac{1}{r}(x-x_0,y-y_0,z-z_0)$$

$\boldsymbol{n}$ is the outward unit normal vector of $\partial\Omega$.

Proof

$$\boldsymbol{x}=(x,y,z)^T,\boldsymbol{y}=(x_0,y_0,z_0)^T,\varepsilon<dist(\partial\Omega,\boldsymbol{y})$$
$$B_y(\varepsilon):=\left\{x\in\mathbb{R}^3\mid\quad\|x-y\|<\varepsilon\right\},\Omega_\varepsilon:=\Omega-B_y(\varepsilon)$$
$$v:\mathbb{R}^3\to\mathbb{R},x\mapsto\frac{1}{\|\boldsymbol{x}-\boldsymbol{y}\|},\Delta v(x)=0,x\neq y$$

Then

$$\int_{\partial\Omega}\left(\frac{X\cdot\vec{n}}{r^2}u+\frac{1}{r}\frac{\partial u}{\partial \boldsymbol{n}}\right)d\sigma=\int_{\partial\Omega_\varepsilon}+\int_{\partial B_y(\varepsilon)}\left(\frac{X\cdot\vec{n}}{r^2}u+\frac{1}{r}\frac{\partial u}{\partial \boldsymbol{n}}\right)d\sigma$$

Let

$$I=\int_{\partial\Omega_\varepsilon}\left(\frac{X\cdot\vec{n}}{r^2}u+\frac{1}{r}\frac{\partial u}{\partial \boldsymbol{n}}\right)d\sigma,\quad J=\int_{\partial B_y(\varepsilon)}\left(\frac{X\cdot\vec{n}}{r^2}u+\frac{1}{r}\frac{\partial u}{\partial \boldsymbol{n}}\right)d\sigma$$

Because

$$\int_{\partial B_y(\varepsilon)}\frac{X\cdot\vec{n}}{r^2}u=\int_{\partial B_y(\varepsilon)}\frac{\vec{n}^2}{r^2}\longrightarrow u(\boldsymbol{y})\cdot\frac{4\pi\varepsilon^2}{\varepsilon^2}=4\pi u(\boldsymbol{y}),\varepsilon\to 0^+$$

and

$$\left|\int_{\partial B_y(\varepsilon)}v\frac{\partial u}{\partial \boldsymbol{n}}\right|\le\max_{x\in\partial B_y(\varepsilon)}|\nabla u(\boldsymbol{x})|\left|\int_{\partial B_y(\varepsilon)}\frac{1}{r}\right|=\max_{x\in\partial B_y(\varepsilon)}|\nabla u(\boldsymbol{x})|\cdot 4\pi\varepsilon=O(\varepsilon)$$

Therefore

$$J\to 4\pi u(\boldsymbol{y}),\varepsilon\to 0^+$$

By Green Equality

$$\int_{\Omega_\varepsilon}(u\Delta v-v\Delta u)dx=\int_{\partial\Omega_\varepsilon}\left(u\frac{\partial v}{\partial \boldsymbol{n}}-v\frac{\partial u}{\partial \boldsymbol{n}}\right)d\sigma$$

We have

$$\int_{\partial\Omega_\varepsilon}\frac{1}{r}\frac{\partial u}{\partial \boldsymbol{n}}=\int_{\partial\Omega_\varepsilon}v\frac{\partial u}{\partial \boldsymbol{n}}=\int_{\partial\Omega_\varepsilon}u\frac{\partial v}{\partial \boldsymbol{n}}=\int_{\partial\Omega_\varepsilon}u\langle\nabla v,\vec{n}\rangle$$

$$=\int_{\partial\Omega_\varepsilon}u\vec{n}\cdot(-\frac{x-x_0}{r^3},-\frac{y-y_0}{r^3},-\frac{z-z_0}{r^3})=\int_{\partial\Omega_\varepsilon}u\vec{n}\left(-\frac{X}{r^2}\right)$$

Thus

$$I=\int_{\partial\Omega_\varepsilon}\left(\frac{X\cdot\vec{n}}{r^2}u+\frac{1}{r}\frac{\partial u}{\partial \boldsymbol{n}}\right)d\sigma=0,\quad\forall\varepsilon\in(0,dist(\partial\Omega,\boldsymbol{y}))$$

Consider the equation 2

$$\textbf{RHS}=\lim_{\varepsilon\to 0^+}\frac{1}{4\pi}\int_{\partial\Omega}\left(\frac{X\cdot\vec{n}}{r^2}u+\frac{1}{r}\frac{\partial u}{\partial \boldsymbol{n}}\right)d\sigma=\lim_{\varepsilon\to 0^+}\frac{1}{4\pi}(I+J)=u(\boldsymbol{y})$$

Q.E.D

4 Problem 3

Suppose $\Omega \subset \mathbb{R}^2$,bounded, $\partial\Omega$ is of $\mathbb{C}^1$.$u \in \mathbb{C}^2(\Omega)$ and $\Delta u = 0$. Prove that

$$u(\boldsymbol{y}) = \frac{1}{2\pi} \int_{\partial\Omega} \left(\frac{X \cdot \vec{n}}{r} u - \frac{\partial u}{\partial \boldsymbol{n}} \log r \right) ds$$

where

$$r = \sqrt{(x - x_0)^2 + (y - y_0)^2}, X = \frac{1}{r}(x - x_0, y - y_0)$$

$\boldsymbol{n}$ is the outward unit normal vector of $\partial\Omega$.

Proof

$$\boldsymbol{x} = (x, y)^T, \boldsymbol{y} = (x_0, y_0)^T, \varepsilon < dist(\partial\Omega, \boldsymbol{y})$$

$$B_y(\varepsilon) := \left\{ x \in \mathbb{R}^2 \,|\quad \|x - y\| < \varepsilon \right\}, \Omega_\varepsilon := \Omega - B_y(\varepsilon)$$

$$v : \mathbb{R}^2 \to \mathbb{R}, x \mapsto \log \|\boldsymbol{x} - \boldsymbol{y}\|, \boldsymbol{x} \neq \boldsymbol{y}$$

We claim that

$$\Delta v = 0$$

In fact

$$\nabla v = \left(\frac{x - x_0}{r^2}, \frac{y - y_0}{r^2} \right)$$

$$\frac{\partial^2 v}{\partial x^2} = \frac{r^2 - 2(x - x_0^2)}{r^4}$$

Symmetrically

$$\frac{\partial^2 v}{\partial y^2} = \frac{r^2 - 2(y - y_0^2)}{r^4}$$

Therefore

$$\Delta v = \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0$$

Consider

$$\int_{\partial\Omega} \left(\frac{X \cdot \vec{n}}{r} u - \frac{\partial u}{\partial \boldsymbol{n}} \log r \right) ds = \int_{\partial\Omega_\varepsilon} + \int_{\partial B_y(\varepsilon)} \left(\frac{X \cdot \vec{n}}{r} u - \frac{\partial u}{\partial \boldsymbol{n}} \log r \right) ds$$

Let

$$I = \int_{\partial\Omega_\varepsilon} \left(\frac{X \cdot \vec{n}}{r} u - \frac{\partial u}{\partial \boldsymbol{n}} \log r \right) ds, \quad J = \int_{\partial B_y(\varepsilon)} \left(\frac{X \cdot \vec{n}}{r} u - \frac{\partial u}{\partial \boldsymbol{n}} \log r \right) ds$$

By Green Equality

$$\int_{\Omega_\varepsilon} (u\Delta v - v\Delta u)dx = \int_{\partial\Omega_\varepsilon} \left(u\frac{\partial v}{\partial \boldsymbol{n}} - v\frac{\partial u}{\partial \boldsymbol{n}} \right) ds$$

We have

$$\int_{\partial\Omega_\varepsilon} \frac{\partial u}{\partial \boldsymbol{n}} \log r = \int_{\partial\Omega_\varepsilon} v\frac{\partial u}{\partial \boldsymbol{n}} = \int_{\partial\Omega_\varepsilon} u\frac{\partial v}{\partial \boldsymbol{n}} = \int_{\partial\Omega_\varepsilon} u\langle \nabla v, \vec{n}\rangle$$

$$= \int_{\partial\Omega_\varepsilon} u\vec{n} \cdot \left(\frac{x - x_0}{r^2}, \frac{y - y_0}{r^2} \right) = \int_{\partial\Omega_\varepsilon} u\frac{X \cdot \vec{n}}{r}$$

Thus

$$I = 0, \quad \forall\varepsilon \in (0, dist(\partial\Omega, \boldsymbol{y}))$$

while

$$\int_{\partial B_y(\varepsilon)} \frac{X \cdot \vec{n}}{r} u = \int_{\partial B_y(\varepsilon)} \frac{\vec{n}^2}{r} u = \int_{\partial B_y(\varepsilon)} \frac{u}{r} \to u(\boldsymbol{y}) \cdot \frac{2\pi\varepsilon}{\varepsilon} = 2\pi u(\boldsymbol{y}), \varepsilon \to 0^+$$

and

$$\left| \int_{\partial B_y(\varepsilon)} v\frac{\partial u}{\partial \boldsymbol{n}} \right| \leq \max_{x \in \partial B_y(\varepsilon)} |\nabla u(\boldsymbol{x})| \left| \int_{\partial B_y(\varepsilon)} \log \varepsilon \right| = \max_{x \in \partial B_y(\varepsilon)} |\nabla u(\boldsymbol{x})| \cdot 2\pi\varepsilon \log \varepsilon \to 0, \varepsilon \to 0^+$$

That is

$$\lim_{\varepsilon \to 0^+} J = 2\pi u(\boldsymbol{y})$$

Thus

$$u(\boldsymbol{y}) = \lim_{\varepsilon \to 0^+} \frac{1}{2\pi} \int_{\partial\Omega} \left(\frac{X \cdot \vec{n}}{r} u - \frac{\partial u}{\partial \boldsymbol{n}} \log r \right) ds = \frac{1}{2\pi} \int_{\partial\Omega} \left(\frac{X \cdot \vec{n}}{r} u - \frac{\partial u}{\partial \boldsymbol{n}} \log r \right) ds$$

Q.E.D