Exploring the Fokas Method for Heat Equation IBVPs: Analytical Solutions and Numerical Illustrations

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Abstract. We investigate the solutions to Initial-Boundary Value Problems (IBVPs) for the heat equation via the Fokas (Unified transform) method. Analytical solutions are derived for problems on the half-line with general Robin boundary conditions and source terms, as well as on a finite interval with general Dirichlet boundary conditions. Illustrative examples are provided to demonstrate the method's effectiveness and potential applications, including scenarios involving periodic heating and thermal insulation. We also compare this method with traditional approaches, such as the image method, and highlight the advantages of this unified framework.

1. Introduction.

Solving Initial-Boundary Value Problems (IBVPs) for evolution partial differential equations (PDEs) has been one of the tent poles of applied mathematics. Its applications run into physics, engineering, and more. Among the prominent methods, the Fokas method, or unified transform, stands out as a method that can draw uniformly and analytically convergent solutions even under very challenging boundary conditions. Originally designed for integrable nonlinear PDEs, the method was adapted later to be relevant for linear evolution PDEs, such as heat equations, which the study focuses on.

It unifies and extends traditional techniques through the application of complex analysis, as opposed to other classical approaches such as the separation of variables or Fourier transform. This yields not only new integral representations but also enhanced computational efficiency and accuracy, particularly for problems involving inhomogeneous boundary conditions or complex geometries.

Here, we present the Fokas application to the heat equation on the half-line and to finite domains. The study seeks to give a theoretical framework and numerical illustrations by developing solutions for general Robin and Dirichlet boundary conditions and also with source terms.

2. Solving the Heat Equation on the Half-Line via the Fokas Method .

The Fokas method for the Half-Line Heat Equation involves three steps. The first step is to derive the Global Relation using the Fourier Transform. The second step requires shifting the integration contour into the complex plane using Cauchy's Theorem and Jordan's Lemma. The third step aims to eliminate the unknown terms by manipulating the Global Relation. This general strategy is outlined in [12, 14], and other literature, as well as online resources such as Fokas Method (unified transform) and Lectures on the Unified Transform Method.

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$$\frac{\partial u}{\partial t}-\frac{\partial^2 u}{\partial x^2}=0,\quad 0<t<T,\quad 0<x<\infty; \tag{2.1a}$$

$$u(x,0)=f(x),\quad 0<x<\infty; \tag{2.1b}$$

$$-\frac{\partial u}{\partial x}+hu=g(t),\quad 0<t<T,\quad x=0; \tag{2.1c}$$

$$u(x,t)\to 0,\quad 0<t<T,\quad x\to\infty, \tag{2.1d}$$

where $h \geq 0$, $g \in C^1([0,T])$, $f \in L^1(\mathbb{R}^+)\cap C^0(\mathbb{R}^+)$.

2.1. Step 1—Derive the Global Relation. Take Eq.(2.1a) and apply the Half-Fourier Transform on both sides

$$\begin{aligned}\frac{\partial \hat{u}}{\partial t}(\lambda,t) &= \int_0^\infty \frac{\partial u}{\partial t}e^{-i\lambda x}dx = \int_0^\infty \frac{\partial^2 u}{\partial x^2}e^{-i\lambda x}dx \\ &= -\frac{\partial u}{\partial x}(0,t)+i\lambda\int_0^\infty \frac{\partial u}{\partial x}e^{-i\lambda x}dx \\ &= -\frac{\partial u}{\partial x}(0,t)-i\lambda u(0,t)-\lambda^2\int_0^\infty u(x,t)e^{-i\lambda x}dx \\ &= -\frac{\partial u}{\partial x}(0,t)-i\lambda u(0,t)-\lambda^2\hat{u}(\lambda,t).\end{aligned}$$

Hence, We obtained a first-order ODE

$$\frac{\partial \hat{u}}{\partial t}(\lambda,t)+\lambda^2\hat{u}(\lambda,t)=-g_1(t)-i\lambda g_0(t),\quad \operatorname{Im}(\lambda)\leqslant 0. \tag{2.2}$$

where $g_0(t)=u(0,t)$, $g_1(t)=u_x(x,t)|_{x=0}$. The solution to Eq.(2.2) is

$$\hat{u}(\lambda,t)=e^{-\lambda^2 t}\left(\hat{f}(\lambda)-\tilde{g_1}(\lambda^2,t)-i\lambda\tilde{g_0}(\lambda^2,t)\right),\quad \operatorname{Im}(\lambda)\leqslant 0. \tag{2.3}$$

Definition 2.1. *we adopt $\hat{}$ and $\tilde{}$ to denote half-line Fourier transform and Laplace transform, respectively. i.e.*

$$\begin{aligned}\tilde{g_j}(\lambda,t) &\equiv \int_0^t e^{\lambda\tau}g_0(\tau)d\tau,\quad j=0,1.\\ \hat{f}(\lambda) &\equiv \int_0^\infty f(x)e^{-i\lambda x}dx.\end{aligned}$$

Note that $\tilde{g_0}(\lambda,t)$, and $\tilde{g_1}(\lambda,t)$ are entire functions with respect to $\lambda\in\mathbb{C}$. $\hat{u}(\lambda,t)$, $\hat{f}(\lambda)$ are well-defined within $\{\lambda\in\mathbb{C}:\operatorname{Im}(\lambda)\leqslant 0\}$. Hence, Eq.(2.3) extends to $\{\lambda\in\mathbb{C}:\operatorname{Im}(\lambda)\leqslant 0\}$, which is the desired Global Relation. This statement can be interpreted as implying that $\hat{u}(\lambda,t)$ extends to become a holomorphic function in the lower-half λ plane.

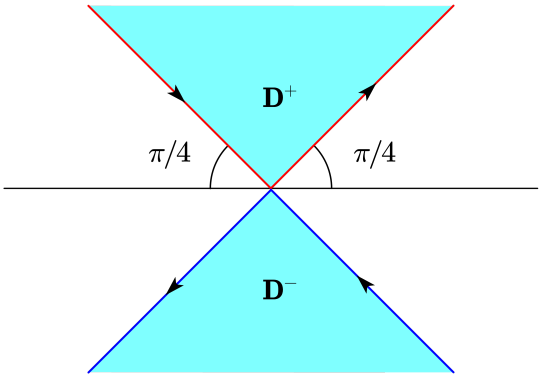


Figure 1. *The regions D^+ and D^- in the complex λ plane, with $D = D^+ \cup D^-$. The boundaries ∂D^+ and ∂D^- are both equipped with counterclockwise orientation.*

2.2. Step 2—Shift the Contour of Integration. Consider the Fourier Inverse Transform of Eq.(2.3)

$$\begin{aligned} u(x,t) &= \frac{1}{2\pi}\int_{\mathbb{R}} e^{i\lambda x-\lambda^2 t}\hat{f}(\lambda)d\lambda \\ &- \frac{1}{2\pi}\int_{\mathbb{R}} e^{i\lambda x-\lambda^2 t}[\tilde{g}_1(\lambda^2,t)+i\lambda\tilde{g}_0(\lambda^2,t)]d\lambda. \end{aligned} \tag{2.4}$$

We claim Eq.(2.4) is equivalent to

$$\begin{aligned} u(x,t) &= \frac{1}{2\pi}\int_{\mathbb{R}} e^{i\lambda x-\lambda^2 t}\hat{f}(\lambda)d\lambda \\ &- \frac{1}{2\pi}\int_{\partial D^+} e^{i\lambda x-\lambda^2 t}[\tilde{g}_1(\lambda^2,t)+i\lambda\tilde{g}_0(\lambda^2,t)]d\lambda. \end{aligned} \tag{2.5}$$

where ∂D^+ is specified in (Fig.1). In order to deform the contour of integration from the real line to ∂D^+, we use the Cauchy's Theorem and Jordan's Lemma. Consider the integrands

$$e^{i\lambda x-\lambda^2 t}\tilde{g}_1(\lambda^2,t) = e^{i\lambda x}\int_0^t e^{-\lambda^2(t-\tau)}g_1(\tau)d\tau \tag{2.6}$$

$$i\lambda e^{i\lambda x-\lambda^2 t}\tilde{g}_0(\lambda^2,t) = i\lambda e^{i\lambda x}\int_0^t e^{-\lambda^2(t-\tau)}g_0(\tau)d\tau. \tag{2.7}$$

$D^+ = \{\lambda \in \mathbb{C} : \mathrm{Im}(\lambda) \geqslant 0, \mathrm{Re}(\lambda^2) < 0\}$ (See Fig.2). On $\{\lambda \in \mathbb{C} : \mathrm{Im}(\lambda) \geqslant 0\} \setminus D^+$, $|e^{i\lambda x}| \leqslant 1$, because $x \geqslant 0$; and $|e^{-\lambda^2(t-\tau)}| \leqslant 1$, since $t-\tau \geqslant 0$. Apply integration by parts to Eq.(2.6)

$$e^{i\lambda x-\lambda^2 t}\tilde{g}_1(\lambda^2,t) = \frac{e^{i\lambda x}}{\lambda^2}\left(g_1(t) - e^{-\lambda^2 t}g_1(0) - \int_0^t e^{-\lambda^2(t-\tau)}g_1'(\tau)d\tau\right).$$

Hence, $e^{-\lambda^2 t}\tilde{g}_1(\lambda^2,t) = O\left(\frac{1}{\lambda^2}\right)$ as $\lambda \to \infty$ in $\{\lambda \in \mathbb{C} : \mathrm{Im}(\lambda) \geqslant 0\} - D^+$. Similarly, for Eq.(2.7),

$$i\lambda e^{i\lambda x-\lambda^2 t}\tilde{g}_0(\lambda^2,t) = \frac{ie^{i\lambda x}}{\lambda}\left(g_0(t) - e^{-\lambda^2 t}g_0(0) - \int_0^t e^{-\lambda^2(t-\tau)}g_0'(\tau)d\tau\right).$$

Then, $ie^{-\lambda^2 t}\tilde{g}_0(\lambda^2,t) = O\left(\frac{1}{\lambda}\right)$ as $\lambda \to \infty$ in $\{\lambda \in \mathbb{C} : \mathrm{Im}(\lambda) \geqslant 0\} - D^+$.

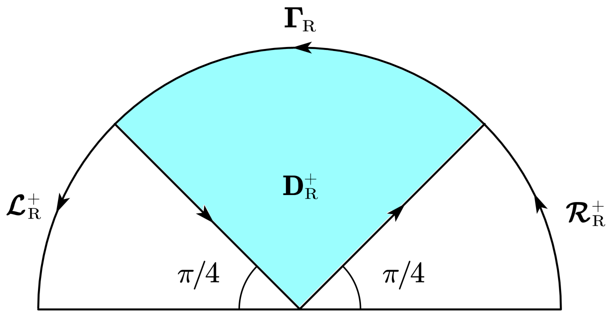


Figure 2. *The contours $\mathcal{L}^+(R)$ and $\mathcal{R}^+(R)$*

By Jordan's Lemma, for any $x > 0$:

$$\lim_{R\to\infty}\int_{\mathcal{L}_R^+} e^{-\lambda^2 t}[\tilde{g}_1(\lambda^2,t) + i\lambda\tilde{g}_0(\lambda^2,t)]e^{i\lambda x}d\lambda = 0;$$
$$\lim_{R\to\infty}\int_{\mathcal{R}_R^+} e^{-\lambda^2 t}[\tilde{g}_1(\lambda^2,t) + i\lambda\tilde{g}_0(\lambda^2,t)]e^{i\lambda x}d\lambda = 0.$$

Apply Cauchy's Theorem to obtain

$$\begin{aligned}&\int_{\mathbb{R}} e^{i\lambda x-\lambda^2 t}[\tilde{g}_1(\lambda^2,t) + i\lambda\tilde{g}_0(\lambda^2,t)]d\lambda \\ =&\int_{\partial D^+} e^{i\lambda x-\lambda^2 t}[\tilde{g}_1(\lambda^2,t) + i\lambda\tilde{g}_0(\lambda^2,t)]d\lambda.\end{aligned}$$

Therefore, Eq.(2.5) holds. But it is not a solution of the Ehrenpreis form, namely of the form

$$u(x,t) = \int U(\lambda)e^{i\lambda x-\lambda^2 t}d\lambda. \tag{2.8}$$

Moreover, $\tilde{g}_0(\lambda^2,t)$ and $\tilde{g}_1(\lambda^2,t)$ remain unsolved.

2.3. Step 3—Eliminate the Unknowns. To complete our solution, we need to use the boundary conditions and the Global Relation to solve for $\tilde{g}_0$ and $\tilde{g}_1$. Multiply $e^{\lambda^2\tau}$ on both sides of (2.1c) and integrate with respect to τ:

$$-\tilde{g}_1(\lambda^2,t) + h\tilde{g}_0(\lambda^2,t) = \int_0^t e^{\lambda^2\tau}g(\tau)d\tau = \tilde{g}(\lambda^2,t). \tag{2.9}$$

Eq.(2.9) provides us with one equation to solve for $\tilde{g}_0(\lambda^2,t)$ and $\tilde{g}_1(\lambda^2,t)$, we need another equation. It turns out a clever manipulation of the Global Relation(2.3) offers the equation we need.

Note that Eq.(2.3) is valid in the lower-half complex λ plane, but the range of integration of Eq.(2.5) is located in the upper-half plane. It is natural to consider replacing λ by $-\lambda$, noticing that this substitution does affect $\tilde{g}_0(\lambda^2,t)$ and $\tilde{g}_1(\lambda^2,t)$:

$$\hat{u}(-\lambda,t)e^{\lambda^2 t} = \hat{f}(-\lambda) - \tilde{g}_1(\lambda^2,t) + i\lambda\tilde{g}_0(\lambda^2,t), \quad \mathrm{Im}(\lambda) \geqslant 0. \tag{2.10}$$

Combine Eq.(2.9) and Eq.(2.10) to obtain the matrix equation

$$\mathbb{A}\boldsymbol{v} = \boldsymbol{b}, \quad \text{where} \quad \mathbb{A} = \begin{pmatrix} h & -1 \\ -i\lambda & 1 \end{pmatrix},$$
$$\boldsymbol{v} = \begin{pmatrix} \tilde{g}_0(\lambda^2,t) \\ \tilde{g}_1(\lambda^2,t) \end{pmatrix}, \quad \text{and} \quad \boldsymbol{b} = \begin{pmatrix} \tilde{g}(\lambda^2,t) \\ \hat{f}(-\lambda) - \hat{u}(-\lambda,t)e^{\lambda^2 t} \end{pmatrix}.$$

Then the solution for $\boldsymbol{v}$ is

$$\begin{aligned} \boldsymbol{v} = \mathbb{A}^{-1}b &= \frac{1}{h-i\lambda}\begin{pmatrix} 1 & 1 \\ i\lambda & h \end{pmatrix}\begin{pmatrix} \tilde{g}(\lambda^2,t) \\ \hat{f}(-\lambda) - \hat{u}(-\lambda,t)e^{\lambda^2 t} \end{pmatrix} \\ &= \frac{1}{h-i\lambda}\begin{pmatrix} \tilde{g}(\lambda^2,t) + \hat{f}(-\lambda) - \hat{u}(-\lambda,t)e^{\lambda^2 t} \\ i\lambda\tilde{g}(\lambda^2,t) + h\hat{f}(-\lambda) - h\hat{u}(-\lambda,t)e^{\lambda^2 t} \end{pmatrix}. \end{aligned}$$

Thus,

$$\begin{aligned} &\tilde{g}_1(\lambda^2,t) + i\lambda\tilde{g}_0(\lambda^2,t) = \langle (i\lambda,1), \boldsymbol{v}\rangle \\ &= \frac{1}{h-i\lambda}\left(2i\lambda\tilde{g}(\lambda^2,t) + (h+i\lambda)[\hat{f}(-\lambda) - \hat{u}(-\lambda,t)e^{\lambda^2 t}]\right), \end{aligned} \tag{2.11}$$

which can be inserted into Eq.(2.5) to obtain:

$$\begin{aligned} u(x,t) &= \frac{1}{2\pi}\int_{\mathbb{R}} e^{i\lambda x - \lambda^2 t}\hat{f}(\lambda)d\lambda \\ &- \frac{1}{2\pi}\int_{\partial D^+} \frac{e^{i\lambda x-\lambda^2 t}}{h-i\lambda}\left(2i\lambda\tilde{g}(\lambda^2,t) + (h+i\lambda)[\hat{f}(-\lambda) - \hat{u}(-\lambda,t)e^{\lambda^2 t}]\right)d\lambda. \end{aligned} \tag{2.12}$$

But we are not done with such an expression, since $\hat{u}(-\lambda,t)$ is an unknown. The situation seems a bit frustrating, because we have run out of equations. Fortunately, it turns out that the contribution of $\hat{u}(-\lambda,t)$ vanishes. This is again due to Cauchy's Theorem. The term $\hat{u}(-\lambda,t)$ gives rise to a contribution of

$$\frac{1}{2\pi}\int_{\partial D^+} \frac{h+i\lambda}{h-i\lambda}\hat{u}(-\lambda,t)e^{i\lambda x}d\lambda,$$

where $\hat{u}(-\lambda,t)$ is bounded in the upper-half λ plane. In fact,

$$|\hat{u}(-\lambda,t)| = \left|\int_0^\infty u(x,t)e^{i\lambda x}dx\right| = \int_0^\infty |u(x,t)e^{-\mathrm{Im}(\lambda)x}|dx < \infty, \quad \mathrm{Im}(\lambda) > 0.$$

Since $\lim_{x\to\infty} u(x,t) = 0 \Rightarrow u(x,t)e^{-\mathrm{Im}(\lambda)x} \sim o(e^{-\mathrm{Im}(\lambda)x})$, as $x \to \infty$. Let's denote $D^+ \cap \{\lambda \in \mathbb{C} : |\lambda| = R\}$ as Γ_R (See Fig.[2]), then

$$\left|\frac{h+i\lambda}{h-i\lambda}\right| \leqslant \frac{R+h}{R-h} \leqslant 3, \quad \lambda \in \Gamma_R, \quad R \geqslant 2h.$$

For all $x > 0$:

$$\begin{aligned}&\left|\int_{\Gamma_R} \frac{h+i\lambda}{h-i\lambda}\hat{u}(-\lambda,t)e^{i\lambda x}d\lambda\right| \lesssim R\int_{\pi/4}^{3\pi/4} e^{-xR\sin\theta}d\theta = 2R\int_{\pi/4}^{\pi/2} e^{-xR\sin\theta}d\theta\\ &\leqslant 2R\int_{\pi/4}^{\pi/2} e^{-2xR\theta/\pi}d\theta = \frac{\pi}{x}\left(e^{-xR/2} - e^{-xR/4}\right) \longrightarrow 0, \quad \text{as} \quad R \to \infty.\end{aligned}$$

Note that if $h = 0$, $\lambda = 0$ would be a removable singularity of $\frac{h+i\lambda}{h-i\lambda}$. Thus, $\frac{h+i\lambda}{h-i\lambda}\hat{u}(-\lambda,t)e^{i\lambda x}$ is holomorphic in $\{\lambda \in \mathbb{C} : \mathrm{Im}(\lambda) \geqslant 0\}$. By Cauchy's Theorem,

$$\int_{\partial D^+} \frac{h+i\lambda}{h-i\lambda}\hat{u}(-\lambda,t)e^{i\lambda x}d\lambda = -\lim_{R\to\infty}\int_{\Gamma_R} \frac{h+i\lambda}{h-i\lambda}\hat{u}(-\lambda,t)e^{i\lambda x}d\lambda = 0. \tag{2.13}$$

Insert Eq.(2.13) into Eq.(2.14) to acquire

$$\begin{aligned}u(x,t) = &\frac{1}{2\pi}\int_{\mathbb{R}} e^{i\lambda x-\lambda^2 t}\hat{f}(\lambda)d\lambda\\ &-\frac{1}{2\pi}\int_{\partial D^+} e^{i\lambda x-\lambda^2 t}\left(\frac{2i\lambda}{h-i\lambda}\tilde{g}(\lambda^2,t) + \frac{h+i\lambda}{h-i\lambda}\hat{f}(-\lambda)\right)d\lambda.\end{aligned} \tag{2.14}$$

However, Eq.(2.14) is still not of the form of Eq.(2.8). We can refine our solution by tackling with the boundary conditions even more.

2.4. Refine the Solution to the Ehrenpreis Form. By Eq.(2.1c), the heat equation is valid for $0 < t < T$. Let

$$\begin{aligned}\tilde{g}_0(\lambda) &= \tilde{g}_0(\lambda,T) = \int_0^T e^{\lambda\tau}g_0(\tau)d\tau;\\ \tilde{g}_1(\lambda) &= \tilde{g}_1(\lambda,T) = \int_0^T e^{\lambda\tau}g_0(\tau)d\tau;\\ \tilde{g}(\lambda) &= \tilde{g}(\lambda,T) = \int_0^T e^{\lambda\tau}g(\tau)d\tau.\end{aligned} \tag{2.15}$$

Then, Eq.(2.5) is equivalent to the equation

$$\begin{aligned}u(x,t) = &\frac{1}{2\pi}\int_{\mathbb{R}} e^{i\lambda x-\lambda^2 t}\hat{f}(\lambda)d\lambda\\ &-\frac{1}{2\pi}\int_{\partial D^+} e^{i\lambda x-\lambda^2 t}[\tilde{g}_1(\lambda^2) + i\lambda\tilde{g}_0(\lambda^2)]d\lambda.\end{aligned} \tag{2.16}$$

Although Eq.(2.5) and Eq.(2.16) differ by

$$\frac{1}{2\pi}\int_{\partial D^+} e^{i\lambda x}\left[\int_t^T \left(e^{-\lambda^2(t-\tau)}g_1(\tau)+i\lambda e^{-\lambda^2(t-\tau)}g_0(\tau)\right)d\tau\right]d\lambda. \tag{2.17}$$

Again, integration by parts, Cauchy's Theorem supplemented with Jordan's Lemma imply that Eq.(2.17) vanishes. Note that $\tau \geqslant t$ in Eq.(2.17) and $\mathrm{Re}(\lambda^2)<0$ in D^+ are crucial here, as we deform the contour of integration from ∂D^+ to Γ_R.

Analogously, Eq.(2.14) is equivalent to the equation

$$\begin{aligned} u(x,t) &= \frac{1}{2\pi}\int_{\mathbb{R}} e^{i\lambda x-\lambda^2 t}\hat{f}(\lambda)d\lambda \\ &-\frac{1}{2\pi}\int_{\partial D^+} e^{i\lambda x-\lambda^2 t}\left(\frac{2i\lambda}{h-i\lambda}\tilde{g}(\lambda^2)+\frac{h+i\lambda}{h-i\lambda}\hat{f}(-\lambda)\right)d\lambda. \end{aligned} \tag{2.18}$$

In fact, Eq.(2.14) and Eq.(2.18) differ by

$$\frac{1}{2\pi}\int_{\partial D^+}\left(\frac{2i\lambda}{h-i\lambda}\int_t^T e^{-\lambda^2(t-\tau)}g(\tau)d\tau\right)e^{i\lambda x}d\lambda. \tag{2.19}$$

Note that

$$\begin{aligned} &\left|\frac{2i\lambda}{h-i\lambda}\right| \leqslant \frac{2R}{R-h} \leqslant 4, \quad \lambda\in\Gamma_R, \quad R\geqslant 2h; \\ &\left|\int_t^T e^{-\lambda^2(t-\tau)}g(\tau)d\tau\right| \leqslant \int_0^T |g(\tau)|d\tau<\infty. \end{aligned}$$

Similar to the estimate made in the previous subsection, we have

$$\lim_{R\to\infty}\int_{\Gamma_R}\left(\frac{2i\lambda}{h-i\lambda}\int_t^T e^{-\lambda^2(t-\tau)}g(\tau)d\tau\right)e^{i\lambda x}d\lambda=0.$$

It follows from Cauchy's Theorem that Eq.(2.18) vanishes. Hence, we obtain a solution of the Ehrenpreis form, which is given by Eq.(2.18). It remains to verify that Eq.(2.18) is the solution to Eq.(2.1).

Theorem 2.2. (2.18) *is the solution of the Ehrenpreis form, to the initial-boundary value problem* (2.1).

Proof. Consider Eq.(2.18), the (x,t) dependence lies only within the term $e^{i\lambda x-\lambda^2 t}$, which clearly satisfies the heat equation. It is intuitively obvious that Eq.(2.18) should satisfy the heat equation. Let's verify that.

$$\int_{\mathbb{R}}\frac{\partial}{\partial t}e^{i\lambda x-\lambda^2 t}f(\hat{\lambda})d\lambda=-\int_{\mathbb{R}}\lambda^2 e^{i\lambda x-\lambda^2 t}\hat{f}(\lambda)d\lambda. \tag{2.20}$$

Since $\hat{f}(\lambda)$ is bounded by $\|f\|_{L^1(\mathbb{R}^+)}$, $\lambda^2 e^{i\lambda x-\lambda^2 t}\in L^1(\mathbb{R})$, it follows from the Dominated Convergence Theorem that

$$\frac{\partial}{\partial t}\int_{\mathbb{R}} e^{i\lambda x-\lambda^2 t}f(\hat{\lambda})d\lambda=\int_{\mathbb{R}}\frac{\partial}{\partial t}e^{i\lambda x-\lambda^2 t}f(\hat{\lambda})d\lambda. \tag{2.21}$$

Combine Eq.(2.20) and Eq.(2.21) to write

$$\frac{\partial}{\partial t}\int_{\mathbb{R}} e^{i\lambda x-\lambda^2 t} f(\hat{\lambda})d\lambda = -\int_{\mathbb{R}} \lambda^2 e^{i\lambda x-\lambda^2 t}\hat{f}(\lambda)d\lambda. \tag{2.22}$$

Similary, we have the the x derivative

$$\frac{\partial^2}{\partial x^2}\int_{\mathbb{R}} e^{i\lambda x-\lambda^2 t} f(\hat{\lambda})d\lambda = -\int_{\mathbb{R}} \lambda^2 e^{i\lambda x-\lambda^2 t}\hat{f}(\lambda)d\lambda. \tag{2.23}$$

Join Eq.(2.22) and Eq.(2.23) to obtained

$$\left(\frac{\partial}{\partial t}-\frac{\partial^2}{\partial x^2}\right)\int_{\mathbb{R}} e^{i\lambda x-\lambda^2 t} f(\hat{\lambda})d\lambda = 0. \tag{2.24}$$

Analogously, It is easy to verify that

$$\begin{aligned}&\left(\frac{\partial}{\partial t}-\frac{\partial^2}{\partial x^2}\right)\int_{\partial D^+} e^{i\lambda x-\lambda^2 t}\left(\frac{2i\lambda}{h-i\lambda}\tilde{g}(\lambda^2)+\frac{h+i\lambda}{h-i\lambda}\hat{f}(-\lambda)\right)d\lambda\\ &=\int_{\partial D^+}\left(\frac{\partial}{\partial t}-\frac{\partial^2}{\partial x^2}\right)e^{i\lambda x-\lambda^2 t}\left(\frac{2i\lambda}{h-i\lambda}\tilde{g}(\lambda^2)+\frac{h+i\lambda}{h-i\lambda}\hat{f}(-\lambda)\right)d\lambda = 0.\end{aligned} \tag{2.25}$$

Eq.(2.24) with Eq.(2.25) suggest that Eq.(2.18) satisfies Eq.(2.1a). Next, we shall show that Eq.(2.18) causes no conflict with the boundary conditions Eq.(2.1b) and Eq.(2.1c). Evaluate Eq.(2.18) at $t = 0$, $x > 0$, we find

$$\begin{aligned}u(x,0) &= \frac{1}{2\pi}\int_{\mathbb{R}} e^{i\lambda x}\hat{f}(\lambda)d\lambda\\ &-\frac{1}{2\pi}\int_{\partial D^+} e^{i\lambda x}\left(\frac{2i\lambda}{h-i\lambda}\tilde{g}(\lambda^2)+\frac{h+i\lambda}{h-i\lambda}\hat{f}(-\lambda)\right)d\lambda.\end{aligned} \tag{2.26}$$

Using the boundedness of $\tilde{g}(\lambda^2)$, $\hat{f}(-\lambda)$, $\dfrac{2i\lambda}{h-i\lambda}$ and $\dfrac{h+i\lambda}{h-i\lambda}$ on Γ_R, Similar to the arguments made to obtain Eq.(2.13), we have

$$\begin{aligned}&\int_{\partial D^+} e^{i\lambda x}\left(\frac{2i\lambda}{h-i\lambda}\tilde{g}(\lambda^2)+\frac{h+i\lambda}{h-i\lambda}\hat{f}(-\lambda)\right)d\lambda\\ &=-\lim_{R\to\infty}\int_{\Gamma_R} e^{i\lambda x}\left(\frac{2i\lambda}{h-i\lambda}\tilde{g}(\lambda^2)+\frac{h+i\lambda}{h-i\lambda}\hat{f}(-\lambda)\right)d\lambda = 0.\end{aligned} \tag{2.27}$$

Thus, by the Fourier Inversion theorem, Eq.(2.26) becomes

$$u(x,0) = \frac{1}{2\pi}\int_{\mathbb{R}} e^{i\lambda x}\hat{f}(\lambda)d\lambda = f(x), \tag{2.28}$$

which is Eq.(2.1b). Note that if we drop the condition that $f \in C^0(\mathbb{R}^+)$, we would have $\lim_{t\to 0} u(x,t) = f(x)$ almost everywhere. Now evaluate Eq.(2.18) at $x = 0$, $t > 0$, we find

$$\begin{aligned}u(0,t) &= \frac{1}{2\pi}\int_{\mathbb{R}} e^{-\lambda^2 t}\hat{f}(\lambda)d\lambda\\ &-\frac{1}{2\pi}\int_{\partial D^+} e^{-\lambda^2 t}\left(\frac{2i\lambda}{h-i\lambda}\tilde{g}(\lambda^2)+\frac{h+i\lambda}{h-i\lambda}\hat{f}(-\lambda)\right)d\lambda.\end{aligned} \tag{2.29}$$

Take the x derivative of Eq.(2.18) and evaluate at $x=0$, $t>0$, we find

$$\begin{aligned} u_x(x,t)|_{x=0} &= \frac{1}{2\pi}\int_{\mathbb{R}} i\lambda e^{-\lambda^2 t}\hat{f}(\lambda)d\lambda \\ &- \frac{1}{2\pi}\int_{\partial D^+} i\lambda e^{-\lambda^2 t}\left(\frac{2i\lambda}{h-i\lambda}\tilde{g}(\lambda^2)+\frac{h+i\lambda}{h-i\lambda}\hat{f}(-\lambda)\right)d\lambda. \end{aligned} \tag{2.30}$$

h times Eq.(2.29) and minus Eq.(2.30), we obtain

$$\begin{aligned} -u_x(x,t)|_{x=0}+hu(0,t) &= \frac{1}{2\pi}\int_{\mathbb{R}}(h-i\lambda)e^{-\lambda^2 t}\hat{f}(\lambda)d\lambda \\ &- \frac{1}{2\pi}\int_{\partial D^+}(h+i\lambda)e^{-\lambda^2 t}\hat{f}(-\lambda)d\lambda - \frac{1}{2\pi}\int_{\partial D^+} 2i\lambda e^{-\lambda^2 t}\tilde{g}(\lambda^2)d\lambda. \end{aligned} \tag{2.31}$$

Deform the second integral of Eq.(2.31) to the real line by Cauchy's Theorem and Jordan's Lemma:

$$\begin{aligned} &\frac{1}{2\pi}\int_{\partial D^+}(h+i\lambda)e^{-\lambda^2 t}\hat{f}(-\lambda)d\lambda \\ =&\frac{1}{2\pi}\int_{\mathbb{R}}(h+i\lambda)e^{-\lambda^2 t}\hat{f}(-\lambda)d\lambda = \frac{1}{2\pi}\int_{\mathbb{R}}(h-i\lambda)e^{-\lambda^2 t}\hat{f}(\lambda)d\lambda. \end{aligned} \tag{2.32}$$

Plug Eq.(2.32) into Eq.(2.31) to find Eq.(2.1c) is satisfied:

$$\begin{aligned} &-u_x(x,t)|_{x=0}+hu(0,t) = -\frac{1}{2\pi}\int_{\partial D^+} 2i\lambda e^{-\lambda^2 t}\tilde{g}(\lambda^2)d\lambda \\ &= \frac{1}{2\pi}\int_{\mathbb{R}} e^{i\mu t}\tilde{g}(-i\mu)d\mu = \frac{1}{2\pi}\int_{\mathbb{R}} e^{i\mu t}\left(\int_0^T e^{-i\mu\tau}g(\tau)d\tau\right)d\mu \\ &= g(t), \qquad \text{where} \quad \mu = i\lambda^2\in\mathbb{R}. \end{aligned} \tag{2.33}$$

The last equality of Eq.(2.33) is due to the Fourier Inversion theorem. It remains to check that Eq.(2.18) satisfies Eq.(2.1d). Since $\hat{f}(\lambda)$, $\tilde{g}(\lambda^2)$, $\hat{f}(-\lambda)$, $\dfrac{2i\lambda}{h-i\lambda}$ and $\dfrac{h+i\lambda}{h-i\lambda}$ are bounded $\left(\text{it is easy to verify that } \left|\dfrac{2i\lambda}{h-i\lambda}\right|\leqslant 2, \left|\dfrac{h+i\lambda}{h-i\lambda}\right|\leqslant 1, \lambda\in\partial D^+\right)$, and $e^{-\lambda^2 t}$ is simply oscillatory on ∂D^+. Hence,

$$\int_{\partial D^+} e^{i\lambda x-\lambda^2 t}\left(\frac{2i\lambda}{h-i\lambda}\tilde{g}(\lambda^2)+\frac{h+i\lambda}{h-i\lambda}\hat{f}(-\lambda)\right)d\lambda = O(\frac{1}{x}), \quad x\to\infty.$$

Besides, the Riemann-Lebesgue Lemma implies $\mathcal{F}^{-1}(L^1(\mathbb{R}))\subset C_0(\mathbb{R})$. That is to say

$$\hat{f}(\lambda)\leqslant \|f\|_{L^1(\mathbb{R}^+)} \Rightarrow e^{-\lambda^2 t}\hat{f}(\lambda)\in L^1(\mathbb{R}) \Rightarrow \lim_{x\to\infty}\int_{\mathbb{R}} e^{i\lambda x-\lambda^2 t}\hat{f}(\lambda)d\lambda = 0.$$

This confirms the regularity condition Eq.(2.1d). Therefore, the solution to Eq.(2.1) is indeed

Eq.(2.18):

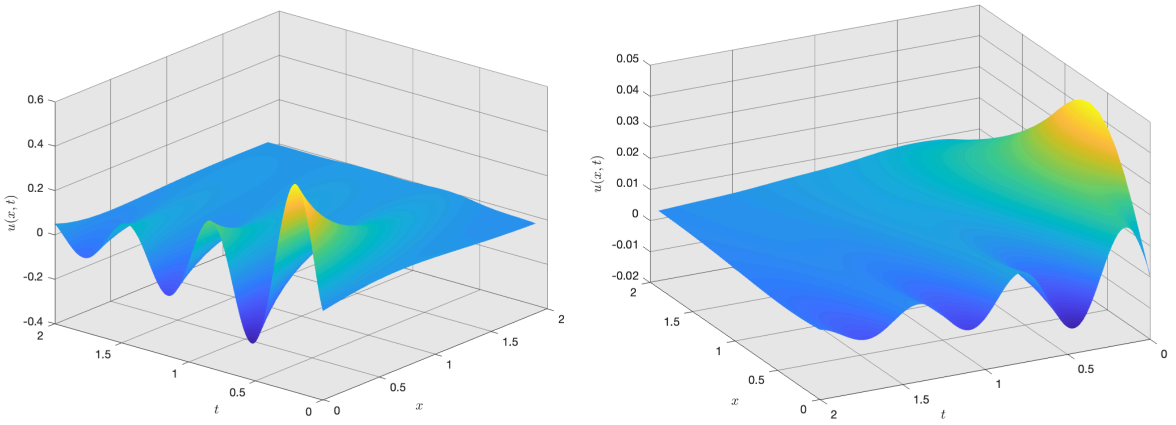
$$u(x,t) = \frac{1}{2\pi}\int_{\mathbb{R}} e^{i\lambda x-\lambda^2 t}\hat{f}(\lambda)d\lambda$$
$$-\frac{1}{2\pi}\int_{\partial D^+} e^{i\lambda x-\lambda^2 t}\left(\frac{2i\lambda}{h-i\lambda}\tilde{g}(\lambda^2)+\frac{h+i\lambda}{h-i\lambda}\hat{f}(-\lambda)\right)d\lambda.$$

■

2.5. Numerical visualizations and Applications. We shall adopt (2.18) to solve (5.1) under physical configurations.

Example 2.3. *Periodic heating with decay on the boundary:*

$$g(t) = A\sin(\omega t)e^{-\alpha t}, \quad \alpha > 0.$$



(a) $g(t) = 2\sin(10t)e^{-t}$, $f(x) = \sin(\frac{x}{10})e^{-x^2}$, $h = 1$ (b) $g(t) = 2\sin(10t)e^{-t}$, $f(x) = \sin(\frac{x}{10})e^{-x^2}$, $h = 100$

Figure 3. *Temperature evolution with periodic heating with decay on the boundary*

This setup is relevant in several real-world applications:

1. Pulsed laser heating: Models scenarios where a material is periodically heated by a laser, with the intensity of heating decreasing over time due to energy dissipation or laser pulse attenuation.

2. Thermal stress analysis: Simulates systems where periodic thermal loads are applied, such as in engine components or electronic devices, and the intensity reduces as the system stabilizes or cools over time.

3. Material fatigue testing: Represents experimental setups to assess how materials respond to oscillatory thermal loads that decay, providing insights into their durability under cyclic thermal conditions.

4. Environmental modeling: Mimics natural scenarios such as temperature oscillations due to solar radiation, where heat input diminishes during nighttime or over longer periods.

Example 2.4. *Insulated boundary condition:*

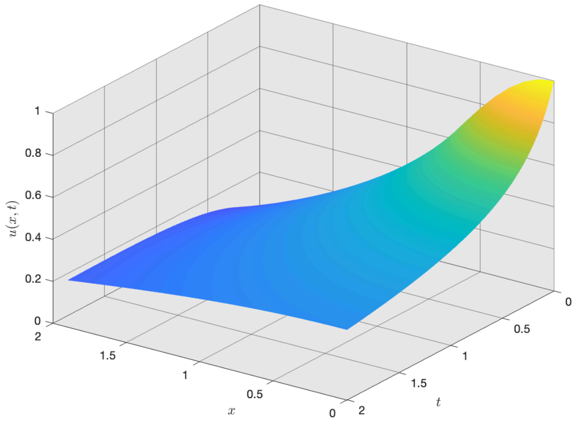
$$\left.\frac{\partial u}{\partial x}\right|_{x=0} = 0.$$

This setup has practical applications in various fields:

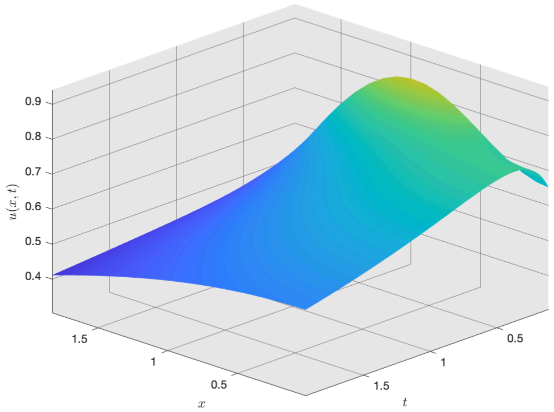
1. Thermal insulation: Models a metal rod or other objects wrapped in a perfect insulator, preventing any heat exchange with the environment. This scenario is critical for understanding thermal stability and designing energy-efficient systems.

2. Chemical reactions: Represents systems where heat is retained within the domain to ensure uniform temperature conditions, facilitating consistent reaction rates and optimizing reaction efficiency.

3. Cryogenics: Simulates cryogenic storage tanks or systems with insulated boundaries to minimize heat transfer, ensuring the preservation of low-temperature conditions required for certain materials and processes.



(a) $g(t) = 0$, $f(x) = e^{-x^2}$, $h = 0$



(b) $g(t) = 0$, $e^{-(x-1)^2}$, $h = 0$

Figure 4. *Temperature evolution with no-flux condition on the boundary*

3. The Inhomogeneous Problem.

Consider the inhomogeneous version of Eq.(2.1)

$$\frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial x^2} = \phi(x,t), \quad 0 < t < T, \quad 0 < x < \infty; \tag{3.1a}$$

$$u(x,0) = f(x), \quad 0 < x < \infty; \tag{3.1b}$$

$$-\frac{\partial u}{\partial x} + hu = g(t), \quad 0 < t < T, \quad x = 0; \tag{3.1c}$$

$$u(x,t) \to 0, \quad 0 < t < T, \quad x \to \infty, \tag{3.1d}$$

where $h \geqslant 0$, and ϕ is continuous and integrable.

Theorem 3.1. *The soluiton to (3.1) is given by*

$$
\begin{aligned}
u(x,t) = & \frac{1}{2\pi}\int_{\mathbb{R}} e^{i\lambda x-\lambda^2 t}\left(\hat{f}(\lambda)+\tilde{\hat{\phi}}(\lambda,\omega(\lambda),t)\right)d\lambda \\
& -\frac{1}{2\pi}\int_{\partial D^+} e^{i\lambda x-\lambda^2 t}\frac{h+i\lambda}{h-i\lambda}\left(\hat{f}(-\lambda)+\tilde{\hat{\phi}}(-\lambda,\omega(\lambda))\right) \\
& -\frac{1}{2\pi}\int_{\partial D^+} e^{i\lambda x-\lambda^2 t}\frac{2i\lambda}{h-i\lambda}\tilde{g}(\lambda^2)d\lambda.
\end{aligned}
$$

Proof. The Duhamel's Principle yields that the solution to Eq.(3.1) is given by

$$
\begin{aligned}
& u(x,t) = \frac{1}{2\pi}\int_{\mathbb{R}} e^{i\lambda x-\lambda^2 t}\hat{f}(\lambda)d\lambda - \frac{1}{2\pi}\int_{\partial D^+} e^{i\lambda x-\lambda^2 t}\frac{h+i\lambda}{h-i\lambda}\hat{f}(-\lambda)d\lambda \\
& +\frac{1}{2\pi}\int_0^t\int_{\mathbb{R}} e^{i\lambda x-\lambda^2(t-s)}\hat{\phi}(\lambda,s)d\lambda ds - \frac{1}{2\pi}\int_0^t\int_{\partial D^+} e^{i\lambda x-\lambda^2(t-s)}\frac{h+i\lambda}{h-i\lambda}\hat{\phi}(-\lambda,s)d\lambda ds \\
& -\frac{1}{2\pi}\int_{\partial D^+} e^{i\lambda x-\lambda^2 t}\frac{2i\lambda}{h-i\lambda}\tilde{g}(\lambda^2)d\lambda,
\end{aligned}
\tag{3.2}
$$

where

$$
\hat{\phi}(\lambda,s) = \int_0^\infty e^{-i\lambda y}\phi(y,s)dy.
$$

By the Fubini's Theorem,

$$
\begin{aligned}
& \frac{1}{2\pi}\int_0^t\int_{\mathbb{R}} e^{i\lambda x-\lambda^2(t-s)}\hat{\phi}(\lambda,s)d\lambda ds = \frac{1}{2\pi}\int_{\mathbb{R}} e^{i\lambda x-\lambda^2 t}\int_0^t e^{\lambda^2 s}\hat{\phi}(\lambda,s)ds d\lambda \\
= & \frac{1}{2\pi}\int_{\mathbb{R}} e^{i\lambda x-\lambda^2 t}\tilde{\hat{\phi}}(\lambda,\omega(\lambda),t)d\lambda; \\
& \frac{1}{2\pi}\int_0^t\int_{\partial D^+} e^{i\lambda x-\lambda^2(t-s)}\frac{h+i\lambda}{h-i\lambda}\hat{\phi}(-\lambda,s)d\lambda ds \\
= & \frac{1}{2\pi}\int_{\partial D^+} e^{i\lambda x-\lambda^2 t}\frac{h+i\lambda}{h-i\lambda}\tilde{\hat{\phi}}(-\lambda,\omega(\lambda),t)d\lambda,
\end{aligned}
$$

where

$$
\tilde{\hat{\phi}}(\lambda,\omega,t) = \int_0^t e^{\omega\tau}\hat{\phi}(\lambda,\tau)d\tau, \quad \omega(\lambda)=\lambda^2.
$$

Therefore, Eq(3.2) is simplified to

$$
\begin{aligned}
u(x,t) = & \frac{1}{2\pi}\int_{\mathbb{R}} e^{i\lambda x-\lambda^2 t}\left(\hat{f}(\lambda)+\tilde{\hat{\phi}}(\lambda,\omega(\lambda),t)\right)d\lambda \\
& -\frac{1}{2\pi}\int_{\partial D^+} e^{i\lambda x-\lambda^2 t}\frac{h+i\lambda}{h-i\lambda}\left(\hat{f}(-\lambda)+\tilde{\hat{\phi}}(-\lambda,\omega(\lambda),t)\right) \\
& -\frac{1}{2\pi}\int_{\partial D^+} e^{i\lambda x-\lambda^2 t}\frac{2i\lambda}{h-i\lambda}\tilde{g}(\lambda^2)d\lambda.
\end{aligned}
\tag{3.3}
$$

But, analogous to how we refine the solution to Eq.(2.1), we claim Eq.(3.3) is equivalent to

$$
\begin{aligned}
u(x,t) = {} & \frac{1}{2\pi}\int_{\mathbb{R}} e^{i\lambda x-\lambda^2 t}\left(\hat{f}(\lambda)+\tilde{\hat{\phi}}(\lambda,\omega(\lambda),t)\right)d\lambda \\
& -\frac{1}{2\pi}\int_{\partial D^+} e^{i\lambda x-\lambda^2 t}\frac{h+i\lambda}{h-i\lambda}\left(\hat{f}(-\lambda)+\tilde{\hat{\phi}}(-\lambda,\omega(\lambda))\right) \\
& -\frac{1}{2\pi}\int_{\partial D^+} e^{i\lambda x-\lambda^2 t}\frac{2i\lambda}{h-i\lambda}\tilde{g}(\lambda^2)d\lambda.
\end{aligned} \tag{3.4}
$$

where

$$
\tilde{\hat{\phi}}(\lambda,\omega(\lambda)) = \int_0^T e^{\omega\tau}\hat{\phi}(\lambda,\tau)d\tau, \quad \omega(\lambda)=\lambda^2.
$$

In fact, Eq.(3.3) and Eq.(3.4) differ by

$$
\frac{1}{2\pi}\int_{\partial D^+} e^{i\lambda x}\left(\frac{h+i\lambda}{h-i\lambda}\int_t^T e^{-\lambda^2(t-\tau)}\hat{\phi}(-\lambda,\tau)d\tau\right)d\lambda. \tag{3.5}
$$

But $t \leqslant \tau$ when $t \leqslant \tau \leqslant T$, $\mathrm{Re}(\lambda^2) < 0$ on D^+, and $\hat{\phi}(-\lambda,t)$ is a bounded holomorphic function in $\{\lambda \in \mathbb{C} : \mathrm{Im}(\lambda) \geqslant 0\}$. Again, for all $x > 0$:

$$
\begin{aligned}
& \int_{\partial D^+} e^{i\lambda x}\left(\frac{h+i\lambda}{h-i\lambda}\int_t^T e^{-\lambda^2(t-\tau)}\hat{\phi}(-\lambda,\tau)d\tau\right)d\lambda \\
= & -\lim_{R\to\infty}\int_{\Gamma_R} e^{i\lambda x}\left(\frac{h+i\lambda}{h-i\lambda}\int_t^T e^{-\lambda^2(t-\tau)}\hat{\phi}(-\lambda,\tau)d\tau\right)d\lambda = 0.
\end{aligned} \tag{3.6}
$$

It is straightforward to verify that Eq.(3.4) satisfies the initial value Eq.(3.1b), the boundary condition Eq.(3.1c) and the regularity condition Eq.(3.1d). Now it remains to verify that Eq.(3.4) agrees with the inhomogeneous heat equation Eq.(3.1a). But

$$
\begin{aligned}
& \frac{1}{2\pi}\left(\frac{\partial}{\partial t}-\frac{\partial^2}{\partial x^2}\right)\int_{\mathbb{R}} e^{i\lambda x-\lambda^2 t}\tilde{\hat{\phi}}(\lambda,\omega(\lambda),t)d\lambda = -\frac{1}{2\pi}\int_{\mathbb{R}}\lambda^2 e^{i\lambda x-\lambda^2 t}\tilde{\hat{\phi}}(\lambda,\omega(\lambda),t)d\lambda \\
+ & \frac{1}{2\pi}\int_{\mathbb{R}} e^{i\lambda x-\lambda^2 t}e^{\lambda^2 t}\hat{\phi}(\lambda,t)d\lambda + \frac{1}{2\pi}\int_{\mathbb{R}}\lambda^2 e^{i\lambda x-\lambda^2 t}\tilde{\hat{\phi}}(\lambda,\omega(\lambda),t)d\lambda \\
= & \frac{1}{2\pi}\int_{\mathbb{R}} e^{i\lambda x}\hat{\phi}(\lambda,t)d\lambda = \phi(x,t), \qquad \text{by the Fourier Inversion theorem.}
\end{aligned}
$$

Moreover, for all $\lambda \in \partial D^+$, $\mathrm{Re}(\lambda^2) = 0$. Hence,

$$
\begin{aligned}
& \left| e^{-\lambda^2 t}\frac{h+i\lambda}{h-i\lambda}\tilde{\hat{\phi}}(\lambda,\omega(\lambda))\right| \leqslant \int_0^T |e^{\lambda^2(t-\tau)}|\cdot|\hat{\phi}(-\lambda,\tau)|d\tau \\
= & \int_0^T |\hat{\phi}(-\lambda,\tau)|d\tau \leqslant \int_0^T\int_0^\infty |\phi(x,t)|dxdt,
\end{aligned}
$$

Since $\lambda^n e^{i\lambda x}$ are integrable along ∂D^+, for all $n \in \mathbb{N}$ and $x > 0$. It follows from the dominated convergence theorem that

$$\begin{aligned}
&\frac{1}{2\pi}\left(\frac{\partial}{\partial t}-\frac{\partial^2}{\partial x^2}\right)\int_{\partial D^+} e^{i\lambda x-\lambda^2 t}\frac{h+i\lambda}{h-i\lambda}\tilde{\hat{\phi}}(-\lambda,\omega(\lambda))d\lambda \\
=&\frac{1}{2\pi}\int_{\partial D^+}\left(\frac{\partial}{\partial t}-\frac{\partial^2}{\partial x^2}\right) e^{i\lambda x-\lambda^2 t}\frac{h+i\lambda}{h-i\lambda}\tilde{\hat{\phi}}(-\lambda,\omega(\lambda))d\lambda = 0.
\end{aligned} \tag{3.7}$$

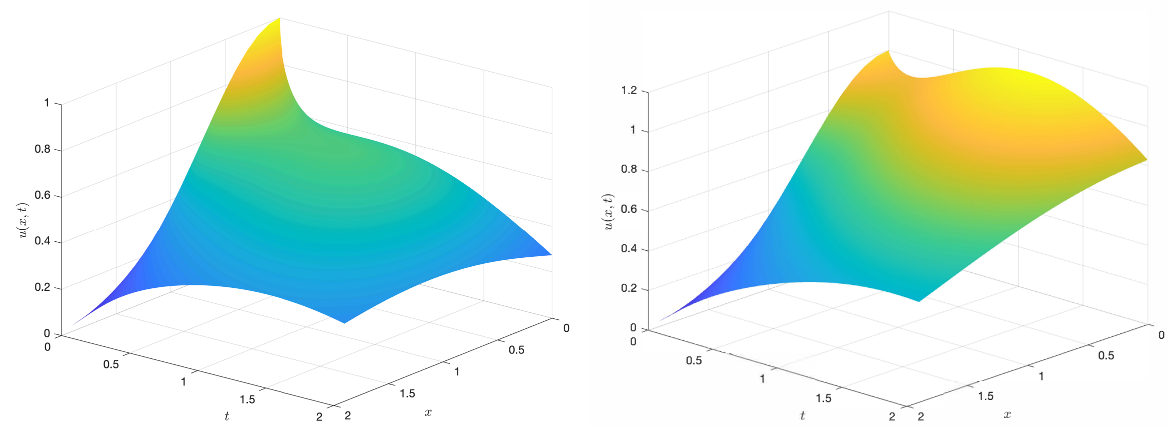
Combine Eq.(2.24), Eq.(2.25), Eq.(3.7), by linearity, we find that the inhomogeneous heat equation Eq.(3.1a) holds. Thus, Eq.(3.4) is indeed a solution to the inhomogeneous IBVP Eq.(3.1). ■

3.1. Numerical illustrations and applications.

Example 3.2. *Thermal Energy Distribution in Manufacturing Processes, with a source term:*

$$\phi(x,t) = H_0 e^{-\alpha x}\cos(\omega t), \quad \alpha > 0,$$

where H_0 is the heat intensity, α models heat dissipation, and ω is the oscillation frequency.



(a) $g(t) = \sin(t)e^{-t}$, $f(x) = e^{-x^2}$, $h = 1$, $\phi(x,t) = e^{-x}\cos(t)$.

(b) $g(t) = \sin(t)e^{-t}$, e^{-x^2}, $h = 0.1$, $\phi(x,t) = e^{-x}\cos(t)$

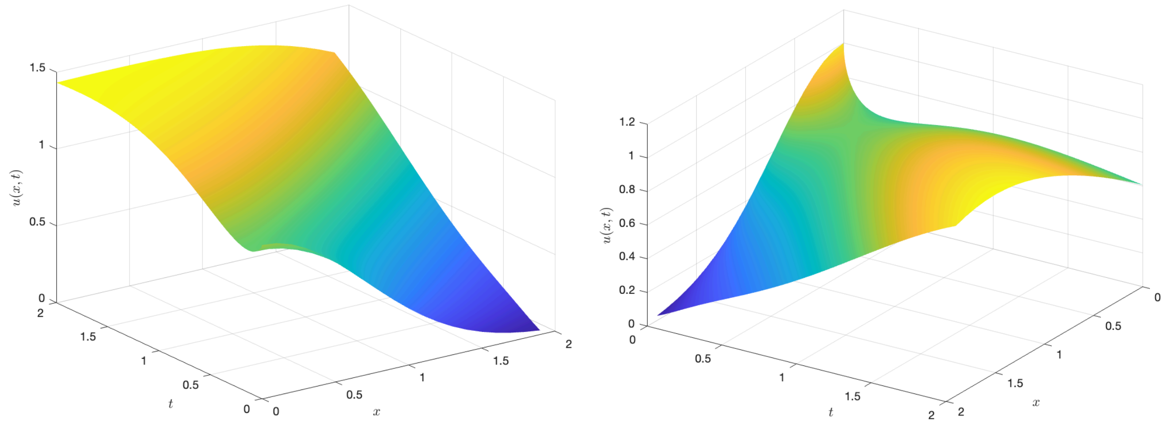
Figure 5. *Thermal Energy Distribution in Manufacturing Processes*

Physical Application: This models a situation where a laser or external heater applies a periodic heat source along a rod. For instance, in additive manufacturing, heat is applied to selectively melt material layers, with the intensity decaying further along the rod due to energy loss.

Example 3.3. *Combustion and Energy Release, with a source term:*

$$\phi(x,t) = Qe^{-\alpha(x-vt)^2/\sigma^2},$$

where Q is the heat release rate, v the velocity of the combustion front, and σ the width of the heat source. Physical Applications:



(a) $g(t) = \sin(t)e^{-t}$, $f(x) = e^{-x^2}$, $h = 0.1$, $\phi(x,t) = e^{-(x-t)^2}$.

(b) $g(t) = \sin(t)e^{-t}$, e^{-x^2}, $h = 1$, $\phi(x,t) = e^{-(x-t)^2}$

Figure 6. *Combustion and Energy Release*

• Rocket propulsion: Describes heat propagation from burning fuel in pipes or combustion chambers, aiding propulsion analysis.

• Wildfires: Simulates heat transfer along a moving burn line, offering insights into wildfire behavior and spread.

4. Fokas method vs Image method.

The heat equation on the half-line is also solved using the image meathod [17]. Since one may always subtract the inhomogeneous term in the boundary condition and apply Duhamel's principle to the differential term, for argument's sake, let's consider the homogeneous version:

$$\begin{cases} \dfrac{\partial u}{\partial t} - a^2 \dfrac{\partial^2 u}{\partial x^2} = 0, & 0 < t < T, \quad 0 < x < \infty; \\ u(x,0) = \varphi(x), & 0 < x < \infty; \\ \dfrac{\partial u}{\partial x} + \alpha u = 0, & 0 < t < T, \quad x = 0; \\ u(x,t) \to 0, & 0 < t < T, \quad x \to \infty. \end{cases} \tag{4.1}$$

where $\alpha \leqslant 0$, $\varphi \in L^1(\mathbb{R}^+) \cap C^0(\mathbb{R}^+)$. In order to apply the image method, let's first consider the following IVP:

$$\begin{cases} \dfrac{\partial W}{\partial t} - a^2 \dfrac{\partial^2 W}{\partial t^2} = 0, & x \in \mathbb{R}, \quad t > 0; \\ W(x,0) = \Phi(x), & x \in \mathbb{R}, \end{cases} \quad \text{where} \quad \Phi(x) = \begin{cases} \varphi(x), & x \geqslant 0; \\ \psi(x), & x < 0, \end{cases} \tag{4.2}$$

where $\psi(x)$ will be specified later, such that $W_x(0,t) + \alpha W(0,t) = 0$. By the uniqueness of the solution, we have that $u(x,t) = W(x,t)|_{x \geqslant 0}$. But the solution to Eq.(4.2) is given by

$$W(x,t) = \int_{\mathbb{R}} \Gamma(x-\xi,t)\Phi(\xi)d\xi, \quad \Gamma(x,t) = \frac{1}{2a\sqrt{\pi t}} \exp\left(-\frac{x^2}{4a^2 t}\right), \tag{4.3}$$

where Γ is called the fundamental solution of the heat equation or the heat kernel. We deduce from Eq.(4.3) that

$$\begin{aligned}\frac{\partial W}{\partial x}(x,t) &= \int_{\mathbb{R}} \frac{\partial \Gamma}{\partial x}(x-\xi,t)\Phi(\xi)d\xi = -\int_{\mathbb{R}} \frac{\partial \Gamma}{\partial \xi}(x-\xi,t)\Phi(\xi)d\xi \\ &= -\Gamma(x,t)\psi(0) + \Gamma(x,t)\Phi(0) + \int_{\mathbb{R}} \Gamma(x-\xi,t)\Phi'(\xi)d\xi.\end{aligned} \tag{4.4}$$

But $W_x(0,t) + \alpha W(0,t) = 0$ implies that

$$-\Gamma(0,t)\psi(0) + \Gamma(0,t)\varphi(0) + \int_{\mathbb{R}} \Gamma(-\xi,t)\Phi'(\xi)d\xi + \alpha \int_{\mathbb{R}} \Gamma(-\xi,t)\Phi(\xi)d\xi = 0. \tag{4.5}$$

Now set $\psi(0) = \varphi(0)$, then it follows from Eq.(4.5) that

$$\begin{aligned}0 &= \int_{-\infty}^{0} \Gamma(\xi,t)[\Phi'(\xi) + \alpha\Phi(\xi)]d\xi + \int_{0}^{\infty} \Gamma(\xi,t)[\Phi'(\xi) + \alpha\Phi(\xi)]d\xi \\ &= \int_{-\infty}^{0} \Gamma(\xi,t)[\Phi'(\xi) + \alpha\Phi(\xi)]d\xi + \int_{0}^{-\infty} \Gamma(\xi,t)[\Phi'(-\xi) + \alpha\Phi(-\xi)](-d\xi) \\ &= \int_{-\infty}^{0} \Gamma(\xi,t)[\Phi'(\xi) + \alpha\Phi(\xi) + \Phi'(-\xi) + \alpha\Phi(-\xi)]d\xi \\ &= \int_{-\infty}^{0} \Gamma(\xi,t)[\Psi'(\xi) + \alpha\Psi(\xi) + \varphi'(-\xi) + \alpha\varphi(-\xi)]d\xi.\end{aligned} \tag{4.6}$$

Hence, we can require ψ to satisfy the following ordinary differential equation:

$$\begin{cases}\psi'(\xi) + \alpha\psi(\xi) = -\varphi'(-\xi) - \alpha\varphi(-\xi), & \xi < 0; \\ \psi(0) = \varphi(0),\end{cases} \tag{4.7}$$

which has the solution

$$\psi(\xi) = \varphi(-\xi) + 2\alpha \int_{0}^{-\xi} e^{-\alpha(\eta+\xi)}\varphi(\eta)d\eta, \quad \xi < 0. \tag{4.8}$$

Hence, we obtain Φ, which in turn give the solution to Eq.(4.2):

$$\begin{aligned}W(x,t) &= \int_{-\infty}^{0} \Gamma(x-\xi,t)\psi(\xi)d\xi + \int_{0}^{\infty} \Gamma(x-\xi,t)\varphi(\xi)d\xi \\ &= \int_{0}^{\infty} \Gamma(x+\xi,t)\psi(-\xi) + \Gamma(x-\xi,t)\varphi(\xi)d\xi \\ &= \int_{0}^{\infty} [\Gamma(x+\xi,t) + \Gamma(x-\xi,t)]\varphi(\xi)d\xi \\ &+ 2\alpha \int_{0}^{\infty} \Gamma(x+\xi,t) \int_{0}^{\xi} e^{-\alpha(\eta-\xi)}\varphi(\eta)d\eta d\xi.\end{aligned} \tag{4.9}$$

Therefore, $u(x,t) = W(x,t)|_{x\geqslant 0}$ gives the solution to Eq.(4.1).

Remark 4.1. The image method works owing to the simple geometry of the half-line. The next section, we shall discover the solution the heat equation via the Fokas method on a finite interval. In this case, the advantages of Fokas method prevails, because applying the image method to complicated geometry settings such as a finite interval would be impractical. Solving an IBVP using the image method in a convex polygon domain would be catastrophic, while the Fokas method performs rather well.

5. Solving the heat equation on a finite interval via the Fokas method. For argument's sake, let's consider the following Dirichlet IBVP:

$$\begin{cases} \dfrac{\partial u}{\partial t} - \dfrac{\partial^2 u}{\partial x^2} = 0, & 0 < t < T, \quad 0 < x < L; \\ u(x,0) = f(x), & 0 \leqslant x \leqslant L; \\ u(0,t) = g_0(t), \quad u(L,t) = h_0(t); & 0 < t < T; \end{cases} \tag{5.1}$$

where $g_0 \in C^1([0,T])$, $h_0 \in C^1([0,T])$, $f \in C^0([0,L])$.

5.1. Step 1—Derive the Global Relation. The finite Fourier transform of $u_t - u_{xx} = 0$ gives

$$\begin{aligned} \frac{\partial \hat{u}}{\partial t} &= \int_0^L \frac{\partial u}{\partial t} e^{-i\lambda x} dx = \int_0^L \frac{\partial^2 u}{\partial x^2} e^{-i\lambda x} dx \\ &= \left(\frac{\partial u}{\partial x} e^{-i\lambda x} \right)\Bigg|_0^L + \left(i\lambda u e^{-i\lambda x} \right)\Big|_0^L - \lambda^2 \hat{u}. \\ &\Rightarrow \frac{\partial \hat{u}}{\partial t} + \lambda^2 \hat{u} = -g_1(t) - i\lambda g_0(t) + e^{-i\lambda L}(h_1(t) + i\lambda h_0(t)), \\ \text{where} \quad h_1(t) &= \frac{\partial u}{\partial x}\bigg|_{x=L}, \quad g_1(t) = \frac{\partial u}{\partial x}\bigg|_{x=0}, \end{aligned} \tag{5.2}$$

the solution of which is given by

$$e^{\lambda^2 t} \hat{u}(\lambda, t) = \hat{f}(\lambda) - \tilde{g}_1(\lambda^2, t) - i\lambda \tilde{g}_0(\lambda^2, t) + e^{-i\lambda L}[\tilde{h}_1(\lambda^2, t) + i\lambda \tilde{h}_0(\lambda^2, t)], \tag{5.3}$$

where $\lambda \in \mathbb{C}$, $\tilde{g}_0(\lambda, t)$, $\tilde{g}_1(\lambda, t)$ are defined in Eq.(2.3), and

$$\begin{aligned} \hat{u}(\lambda, t) &= \int_0^L e^{-i\lambda x} u(x,t) dx, \quad \hat{f}(\lambda) = \int_0^L e^{-i\lambda x} f(x) dx; \\ \tilde{h}_0(\lambda, t) &= \int_0^t e^{\lambda \tau} h_0(\tau) d\tau, \quad \tilde{h}_1(\lambda, t) = \int_0^t e^{\lambda \tau} h_1(\tau) d\tau, \quad t > 0 \, , \, \lambda \in \mathbb{C}. \end{aligned} \tag{5.4}$$

The desired Global Relation of Eq.(5.1) is given by Eq.(5.3).

5.2. Step 2—Shift the Contour of Integration. Again, we apply the Inverse Fourier transform to Eq.(5.3), shift the contour of integration by the Jordan's Lemma and Cauchy's

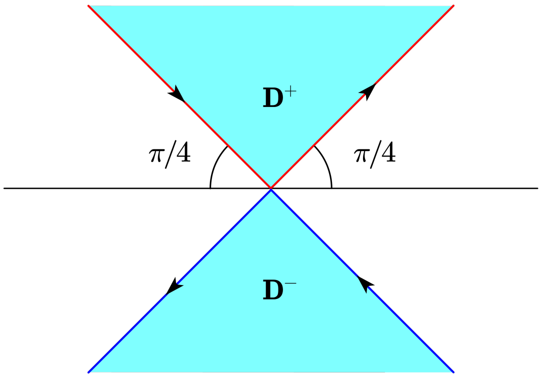


Figure 7. *The regions D^+ and D^- in the complex λ plane, with $D = D^+ \cup D^-$. The boundaries ∂D^+ and ∂D^- are both equipped with counterclockwise orientation.*

theorem, we obtain

$$
\begin{aligned}
u(x,t) = {} & \frac{1}{2\pi}\int_{\mathbb{R}} e^{i\lambda x-\lambda^2 t}\hat{f}(\lambda)d\lambda - \frac{1}{2\pi}\int_{\partial D^+} e^{i\lambda x-\lambda^2 t}[\tilde{g}_1(\lambda^2,t)+i\lambda\tilde{g}_0(\lambda^2,t)]d\lambda \\
& - \frac{1}{2\pi}\int_{\partial D^-} e^{-i\lambda(L-x)-\lambda^2 t}[\tilde{h}_1(\lambda^2,t)+i\lambda\tilde{h}_0(\lambda^2,t)]d\lambda,
\end{aligned} \tag{5.5}
$$

where ∂D^+ and ∂D^- are specified in figure(7).

5.3. Step 3—Eliminate the Unknowns. Let's write the Global Relation Eq.(5.3) in terms of the knowns and unknowns,

$$
e^{\lambda^2 t}\hat{u}(\lambda,t) = \Phi(\lambda,t) - \tilde{g}_1(\lambda^2,t) + e^{-i\lambda L}\tilde{h}_1(\lambda^2,t), \tag{5.6}
$$

where

$$
\Phi(\lambda,t) = \hat{f}(\lambda) - i\lambda\tilde{g}_0(\lambda^2,t) + i\lambda e^{-i\lambda L}\tilde{h}_0(\lambda^2,t) \tag{5.7}
$$

is given. Similarly, let $\lambda \mapsto -\lambda$ in Eq.(5.6), and recall $\tilde{g}_j$, $\tilde{h}_j$ $(j=0,1)$ are invariant under $\lambda \mapsto -\lambda$, we have that

$$
e^{\lambda^2 t}\hat{u}(-\lambda,t) = \Phi(-\lambda,t) - \tilde{g}_1(\lambda^2,t) + e^{i\lambda L}\tilde{h}_1(\lambda^2,t) \tag{5.8}
$$

Solving Eq.(5.6) and Eq.(5.8) for $\tilde{g}_1$ and $\tilde{h}_1$, we obtain

$$
\begin{aligned}
\tilde{g}_1(\lambda^2,t) = {} & \frac{e^{i\lambda L}\Phi(\lambda,t) - e^{-i\lambda L}\Phi(-\lambda,t)}{e^{i\lambda L}-e^{-i\lambda L}} \\
& + \frac{e^{\lambda^2 t}[e^{-i\lambda L}\hat{u}(-\lambda,t) - e^{i\lambda t}\hat{u}(\lambda,t)]}{e^{i\lambda L}-e^{-i\lambda L}} \\
\tilde{h}_1(\lambda^2,t) = {} & \frac{\Phi(\lambda,t)-\Phi(-\lambda,t)+e^{\lambda^2 t}[\hat{u}(-\lambda,t)-\hat{u}(\lambda,t)]}{e^{i\lambda L}-e^{-i\lambda L}}.
\end{aligned} \tag{5.9}
$$

We can then substitude $\tilde{g}_1$ and $\tilde{h}_1$ in Eq.(5.5). We claim the terms involving $\hat{u}(\pm\lambda, t)$ yield a zero contribution. Consider the contributions

$$\left\{\begin{aligned} &\frac{1}{2\pi}\int_{\partial D^+}\frac{e^{-i\lambda L}\hat{u}(-\lambda,t)-e^{i\lambda L}\hat{u}(\lambda,t)}{e^{i\lambda L}-e^{-i\lambda L}}e^{i\lambda x}d\lambda;\\ &\frac{1}{2\pi}\int_{\partial D^-}\frac{\hat{u}(-\lambda,t)-\hat{u}(\lambda,t)}{e^{i\lambda L}-e^{-i\lambda L}}e^{-i\lambda(L-x)}d\lambda.\end{aligned}\right. \tag{5.10}$$

On D^+, $e^{-i\lambda L}$ grows exponentially, then

$$\lim_{\lambda\to\infty}\frac{e^{-i\lambda L}\hat{u}(-\lambda,t)-e^{i\lambda L}\hat{u}(\lambda,t)}{e^{i\lambda L}-e^{-i\lambda L}} = -\hat{u}(-\lambda,t)+e^{i\lambda L}\int_0^L e^{i\lambda(L-x)}u(x,t)dx, \tag{5.11}$$

which is bounded. On D^-, $e^{i\lambda L}$ grows exponentially, so

$$\lim_{\lambda\to\infty}\frac{\hat{u}(-\lambda,t)-\hat{u}(\lambda,t)}{e^{i\lambda L}-e^{-i\lambda L}} = -e^{-i\lambda L}\hat{u}(\lambda,t)+\int_0^L e^{-i\lambda(L-x)}u(x,t)dx, \tag{5.12}$$

which is also bounded. Moreover, observe that $\lambda = 0$ is a removable singularity of both

$$\frac{e^{-i\lambda L}\hat{u}(-\lambda,t)-e^{i\lambda L}\hat{u}(\lambda,t)}{e^{i\lambda L}-e^{-i\lambda L}} \quad \text{and} \quad \frac{\hat{u}(-\lambda,t)-\hat{u}(\lambda,t)}{e^{i\lambda L}-e^{-i\lambda L}},$$

since

$$e^{-i\lambda L}\hat{u}(-\lambda,t)-e^{i\lambda L}\hat{u}(\lambda,t)\Big|_{\lambda=0} = \hat{u}(-\lambda,t)-\hat{u}(\lambda,t)|_{\lambda=0} = 0, \tag{5.13}$$

and $\lambda = 0$ is a zero of order one of $e^{i\lambda L} - e^{-i\lambda L}$. Then, with the two functions holomorphic in the upper-half plane and the lower-half plane respectively, we apply the same arguments in subsection(2.3) to verify that both terms in Eq.(5.10) vanish. Therefore, Eq.(5.5) becomes

$$\begin{aligned} u(x,t) = &\frac{1}{2\pi}\int_{\mathbb{R}} e^{i\lambda x-\lambda^2 t}\hat{f}(\lambda)d\lambda\\ &-\frac{1}{2\pi}\int_{\partial D^+} e^{i\lambda x-\lambda^2 t}\left(i\lambda\tilde{g}_0(\lambda^2,t)+\frac{e^{i\lambda L}\Phi(\lambda,t)-e^{-i\lambda L}\Phi(-\lambda,t)}{e^{i\lambda L}-e^{-i\lambda L}}\right)d\lambda\\ &-\frac{1}{2\pi}\int_{\partial D^-} e^{-i\lambda(L-x)-\lambda^2 t}\left(i\lambda\tilde{h}_0(\lambda^2,t)+\frac{\Phi(\lambda,t)-\Phi(-\lambda,t)}{e^{i\lambda L}-e^{-i\lambda L}}\right)d\lambda.\end{aligned} \tag{5.14}$$

5.4. Improving the Solution to the Ehrenpreis's Form. Eq.(5.1) is valid only when $t \in [0, T]$, let's introduce

$$\begin{aligned} \tilde{g}_0(\lambda) &= \int_0^T e^{\lambda\tau}g_0(\tau)d\tau, \quad \tilde{h}_0(\lambda) = \int_0^T e^{\lambda\tau}h_0(\tau)d\tau;\\ \tilde{g}_1(\lambda) &= \int_0^T e^{\lambda\tau}g_1(\tau)d\tau, \quad \tilde{h}_1(\lambda) = \int_0^T e^{\lambda\tau}h_1(\tau)d\tau.\end{aligned} \tag{5.15}$$

Then, Eq.(5.5) is equivalent to

$$u(x,t) = \frac{1}{2\pi}\int_{\mathbb{R}} e^{i\lambda x-\lambda^2 t}\hat{f}(\lambda)d\lambda - \frac{1}{2\pi}\int_{\partial D^+} e^{i\lambda x-\lambda^2 t}[\tilde{g}_1(\lambda^2)+i\lambda\tilde{g}_0(\lambda^2)]d\lambda - \frac{1}{2\pi}\int_{\partial D^-} e^{-i\lambda(L-x)-\lambda^2 t}[\tilde{h}_1(\lambda^2)+i\lambda\tilde{h}_0(\lambda^2)]d\lambda. \tag{5.16}$$

In fact, Eq.(5.16) and Eq.(5.5) differ by

$$\frac{1}{2\pi}\int_{\partial D^+} e^{i\lambda x}\left[\int_t^T\left(e^{-\lambda^2(t-\tau)}g_1(\tau)+i\lambda e^{-\lambda^2(t-\tau)}g_0(\tau)\right)d\tau\right]d\lambda + \frac{1}{2\pi}\int_{\partial D^+} e^{-i\lambda(L-x)}\left[\int_t^T\left(e^{-\lambda^2(t-\tau)}h_1(\tau)+i\lambda e^{-\lambda^2(t-\tau)}h_0(\tau)\right)d\tau\right]d\lambda, \tag{5.17}$$

which vanishes according to the arguments in subsection(2.4). Then, by an exact same approach in subsection(2.4), we find that Eq.(5.14) is equivalent to

$$\begin{aligned} u(x,t) = {} & \frac{1}{2\pi}\int_{\mathbb{R}} e^{i\lambda x-\lambda^2 t}\hat{f}(\lambda)d\lambda \\ & - \frac{1}{2\pi}\int_{\partial D^+} e^{i\lambda x-\lambda^2 t}\left(i\lambda\tilde{g}_0(\lambda^2)+\frac{e^{i\lambda L}\Phi(\lambda)-e^{-i\lambda L}\Phi(-\lambda)}{e^{i\lambda L}-e^{-i\lambda L}}\right)d\lambda \\ & - \frac{1}{2\pi}\int_{\partial D^-} e^{-i\lambda(L-x)-\lambda^2 t}\left(i\lambda\tilde{h}_0(\lambda^2)+\frac{\Phi(\lambda)-\Phi(-\lambda)}{e^{i\lambda L}-e^{-i\lambda L}}\right)d\lambda, \end{aligned} \tag{5.18}$$

where

$$\Phi(\lambda) = \hat{f}(\lambda) - i\lambda\tilde{g}_0(\lambda^2) + i\lambda e^{-i\lambda L}\tilde{h}_0(\lambda^2). \tag{5.19}$$

Now, one can easily verify that Eq.(5.18) is the solution to Eq.(5.1), because Eq.(5.18) is of the Ehrenpreis's form and one can pass the derivatives into the integrals using the Lebesgue dominated convergence theorem.

5.5. Rederiving the Series Solution. It is possible to deform ∂D^+ and ∂D^- back to the real axis and then apply the Residue theorem to recover the usual trigonometric solution. For simplicity, we assume homogeneous Dirichlet boundary conditions, that is, $g_0(t) = h_0(t) = 0$, then the solution is now given by

$$\begin{aligned} u(x,t) = {} & \frac{1}{2\pi}\int_{\mathbb{R}} e^{i\lambda x-\lambda^2 t}\hat{f}(\lambda)d\lambda - \frac{1}{2\pi}\int_{\partial D^-} e^{-i\lambda(L-x)-\lambda^2 t}\left(\frac{\hat{f}(\lambda)-\hat{f}(-\lambda)}{e^{i\lambda L}-e^{-i\lambda L}}\right)d\lambda \\ & - \frac{1}{2\pi}\int_{\partial D^+} e^{i\lambda x-\lambda^2 t}\left(\frac{e^{i\lambda L}\hat{f}(\lambda)-e^{-i\lambda L}\hat{f}(-\lambda)}{e^{i\lambda L}-e^{-i\lambda L}}\right)d\lambda. \end{aligned} \tag{5.20}$$

To simplify the notation, let's define

$$\phi^+(x,t,\lambda) = e^{i\lambda x-\lambda^2 t}\left(\frac{e^{i\lambda L}\hat{f}(\lambda)-e^{-i\lambda L}\hat{f}(-\lambda)}{e^{i\lambda L}-e^{-i\lambda L}}\right), \quad \mathrm{Im}(\lambda)\geqslant 0, \mathrm{Re}(\lambda)\notin \bigcup_{n\in\mathbb{Z}^*}\{\frac{n\pi}{L}\};$$

$$\phi^-(x,t,\lambda) = e^{-i\lambda(L-x)-\lambda^2 t}\left(\frac{\hat{f}(\lambda)-\hat{f}(-\lambda)}{e^{i\lambda L}-e^{-i\lambda L}}\right), \quad \mathrm{Im}(\lambda)\leqslant 0, \mathrm{Re}(\lambda)\notin \bigcup_{n\in\mathbb{Z}^*}\{\frac{n\pi}{L}\}.$$

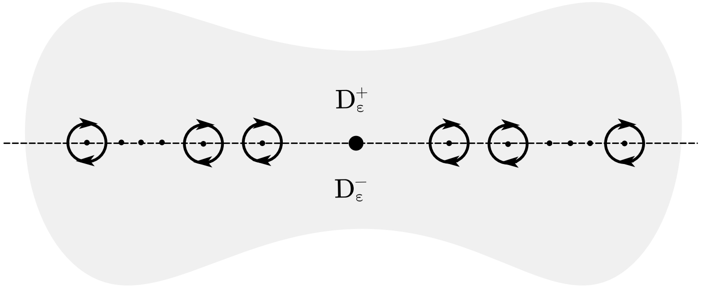


Figure 8. $D_\varepsilon^+ = \{\lambda \in \mathbb{C} : Im(\lambda \geqslant 0)\} \setminus (\bigcup_{n\in\mathbb{Z}^*} D(n\pi/L, \varepsilon))$, *with* ∂D_ε^+ *equipped with clockwise orientation;* $D_\varepsilon^- = \{\lambda \in \mathbb{C} : Im(\lambda \leqslant 0)\} \setminus (\bigcup_{n\in\mathbb{Z}^*} D(n\pi/L, \varepsilon))$, *with* ∂D_ε^- *equipped with clockwise orientation, where* $\varepsilon \ll \frac{\pi}{2L}$.

Then, Jordan's lemma implies that

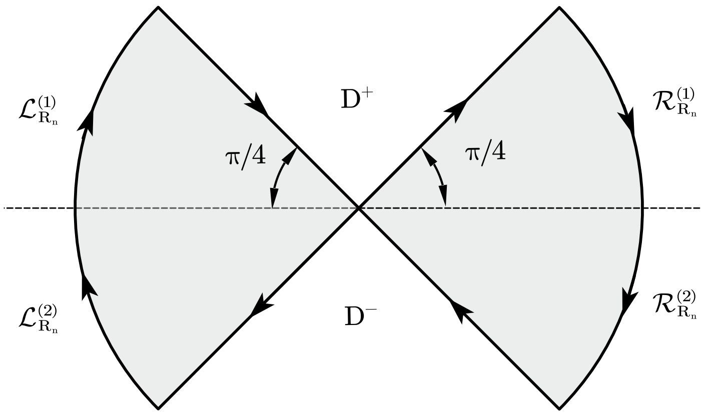


Figure 9. $\mathcal{R}^{(1)}_{R_n} = \{\lambda \in \mathbb{C} : |\lambda| = R_n = \frac{(2n+1)\pi}{2L}, 0 \leqslant \arg\lambda \leqslant \pi/4\}$, $\mathcal{L}^{(1)}_{R_n} = \{\lambda \in \mathbb{C} : |\lambda| = R_n = \frac{(2n+1)\pi}{2L}, 3\pi/4 \leqslant \arg\lambda \leqslant \pi\}$, $\mathcal{R}^{(2)}_{R_n} = \{\lambda \in \mathbb{C} : |\lambda| = R_n = \frac{(2n+1)\pi}{2L}, 7\pi/4 \leqslant \arg\lambda \leqslant 2\pi\}$, $\mathcal{L}^{(2)}_{R_n} = \{\lambda \in \mathbb{C} : |\lambda| = R_n = \frac{(2n+1)\pi}{2L}, \pi \leqslant \arg\lambda \leqslant 5\pi/4\}$, *all equipped with clockwise orientation.*

$$\begin{cases} \lim\limits_{n\to\infty} \int_{\mathcal{L}^{(1)}_{R_n}} \phi^+(x,t,\lambda)d\lambda = \lim\limits_{n\to\infty} \int_{\mathcal{R}^{(1)}_{R_n}} \phi^+(x,t,\lambda)d\lambda = 0, \\ \lim\limits_{n\to\infty} \int_{\mathcal{L}^{(2)}_{R_n}} \phi^-(x,t,\lambda)d\lambda = \lim\limits_{n\to\infty} \int_{\mathcal{R}^{(2)}_{R_n}} \phi^-(x,t,\lambda)d\lambda = 0, \end{cases} \tag{5.21}$$

where $\mathcal{L}^{(j)}_{R_n}$ and $\mathcal{R}^{(j)}_{R_n}$ are specified in Figure 9, for $j = 1, 2$. With $D_\varepsilon^\pm$ specified in figure(8), we can deform $\partial D^\pm$ to ∂D_ε^+ by Cauchy's theorem in Eq.(5.20) and write

$$u(x,t) = \frac{1}{2\pi}\int_{\mathbb{R}} e^{i\lambda x - \lambda^2 t}\hat{f}(\lambda)d\lambda - \frac{1}{2\pi}\int_{\partial D_\varepsilon^-} \phi^-(x,t,\lambda)d\lambda - \frac{1}{2\pi}\int_{\partial D_\varepsilon^+} \phi^+(x,t,\lambda)d\lambda.$$

$\phi^{\pm}(x,t,\lambda)$ have only countably many simple poles along the real axis. But the symmetricity with respect to λ in the definition for $\phi^{\pm}(x,t,\lambda)$ yields that

$$\begin{aligned}\int_{\partial D^+(n\pi/L,\varepsilon)}\phi^+(x,t,\lambda)d\lambda &= -\pi i\cdot \mathrm{Res}(\phi^+,n\pi/L),\\ \int_{\partial D^-(n\pi/L,\varepsilon)}\phi^-(x,t,\lambda)d\lambda &= -\pi i\cdot \mathrm{Res}(\phi^-,n\pi/L),\end{aligned} \tag{5.22}$$

where $\partial D^+(n\pi/L,\varepsilon)$ denotes the upper semi-circle centering at $n\pi/L$ with radius ε, $\partial D^-(n\pi/L,\varepsilon)$ denotes the lower semi-circle centering at $n\pi/L$ with radius ε, both equipped with clockwise orientation. Hence, we have that

$$\begin{aligned}u(x,t) &= \frac{1}{2\pi}\int_{\mathbb{R}} e^{i\lambda x-\lambda^2 t}\hat{f}(\lambda)d\lambda\\ &\quad -\frac{1}{2\pi}\left(\int_{\mathbb{R}\backslash(\cup_{n\in\mathbb{Z}^*}[n\pi/L-\varepsilon,n\pi/L+\varepsilon])}\phi^+(x,t,\lambda)d\lambda-\sum_{n\in\mathbb{Z}^*}\pi i\cdot\mathrm{Res}(\phi^+,n\pi/L)\right)\\ &\quad +\frac{1}{2\pi}\left(\int_{\mathbb{R}\backslash(\cup_{n\in\mathbb{Z}^*}[n\pi/L-\varepsilon,n\pi/L+\varepsilon])}\phi^-(x,t,\lambda)d\lambda+\sum_{n\in\mathbb{Z}^*}\pi i\cdot\mathrm{Res}(\phi^-,n\pi/L)\right)\\ &= \frac{1}{2\pi}\int_{\mathbb{R}} e^{i\lambda x-\lambda^2 t}\hat{f}(\lambda)d\lambda-\frac{1}{2\pi}\int_{\mathbb{R}\backslash(\cup_{n\in\mathbb{Z}^*}[n\pi/L-\varepsilon,n\pi/L+\varepsilon])}\hat{f}(\lambda)d\lambda\\ &\quad +\frac{i}{2}\sum_{n\in\mathbb{Z}^*}\{\mathrm{Res}(\phi^+,n\pi/L)+\mathrm{Res}(\phi^-,n\pi/L)\}.\end{aligned}$$

By passing $\varepsilon\to 0$, we find that the first two integrals cancel each other,

$$\begin{aligned}u(x,t) &= \frac{i}{2}\sum_{z\in\mathbb{Z}^*}\{\mathrm{Res}(\phi^+,n\pi/L)+\mathrm{Res}(\phi^-,n\pi/L)\}\\ &= \frac{i}{2}\sum_{z\in\mathbb{Z}^*}\left(\lim_{\lambda\to n\pi/L}(\lambda-n\pi/L)[\phi^+(x,t,L)+\phi^-(x,t,L)]\right)\\ &= -\frac{i}{2}\sum_{z\in\mathbb{Z}^*} e^{i\lambda_n x-\lambda_n^2 t}\left(\frac{2}{L}\int_0^L \sin(\lambda_n x)f(x)dx\right)\\ &= \sum_{n=1}^{\infty}\left(\frac{2}{L}\int_0^L\sin(\lambda_n x)f(x)dx\right)e^{-\lambda_n^2 t}\sin(\lambda_n x),\end{aligned} \tag{5.23}$$

where $\lambda_n = n\pi/L$, the last equality is due to the property that

$$f_n = \left(\frac{2}{L}\int_0^L \sin(\lambda_n x)f(x)dx\right) = -f_{-n}. \tag{5.24}$$

Therefore, we obtain the classical sine series representation of the solution to Eq.(5.1), which is given by Eq.(5.23).

5.6. Applications and numerical implementations.

Example 5.1. *Heat Flow in Building Walls with Day-Night Cycles*

In this scenario, a wall separates indoor and outdoor environments. The indoor temperature is maintained at a fixed value, while the outdoor temperature oscillates sinusoidally to model day-night temperature variations. For instance,

$$\begin{cases} u(0,t) = T_{\text{indoor}} = 22\,^\circ\text{C} \\ u(L,t) = T_{\text{outdoor}}(t) = 10 + 10\sin\left(\frac{2\pi t}{24}\right)\,^\circ\text{C}. \end{cases} \tag{5.25}$$

Applications:

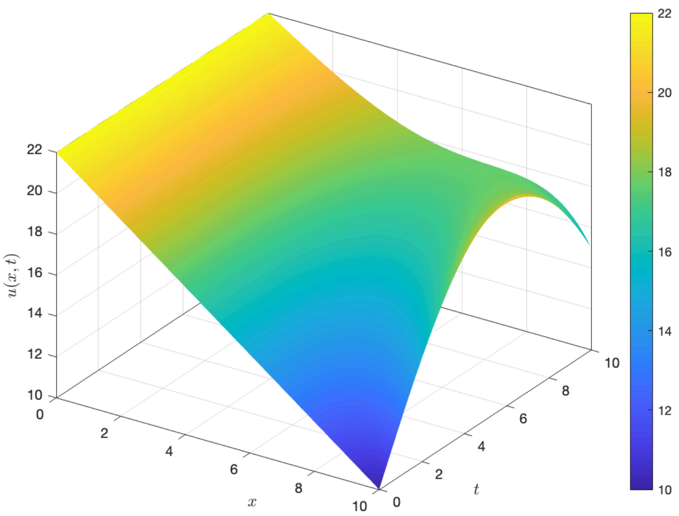


Figure 10. *Heat Flow in Building Walls with Day-Night Cycles.* $L = 10$, $f(x) = 22 - 1.2x$, $u(0,t)$ *and* $u(L,t)$ *as in* (5.25)

- Modeling heat transfer in buildings exposed to fluctuating external conditions.
- Designing insulation materials to stabilize indoor temperatures.

Example 5.2. *Thermal Testing of Electronic Components*

In this configuration, an electronic chip is tested under controlled conditions, with one end cooled and the other subjected to a time-dependent heat source.

$$\begin{cases} u(0,t) = T_{\text{cooling}}(t) = 5\sin\left(\frac{\pi t}{12}\right)\,^\circ\text{C}. \\ u(L,t) = T_{\text{heating}}(t) = 80 + 20\cos\left(\frac{\pi t}{6}\right)\,^\circ\text{C}. \end{cases} \tag{5.26}$$

Applications:
- Simulating temperature changes in microchips due to dynamic workloads.
- Optimizing thermal management strategies, such as cooling systems or heat sinks.

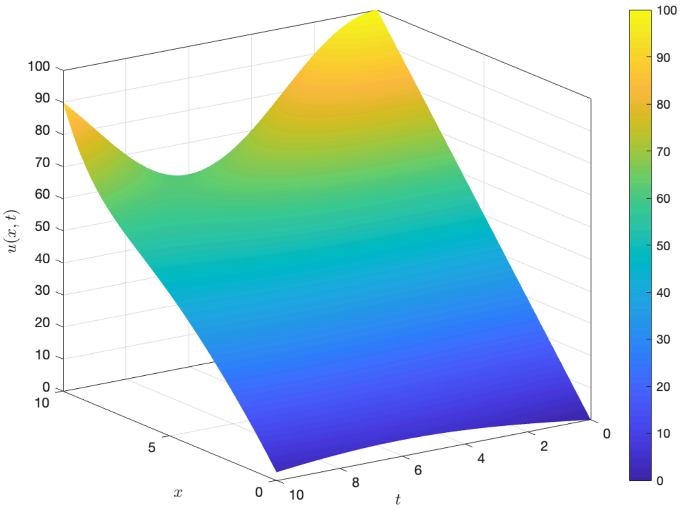


Figure 11. *Heat Flow in Building Walls with Day-Night Cycles.* $L = 10$, $f(x) = 10x$, $u(0,t)$ *and* $u(L,t)$ *as in* (5.26)

6. Conclusion.

This paper has demonstrated the versatility and power of the Fokas method in addressing Initial-Boundary Value Problems (IBVPs) for the heat equation. Analytical solutions for both the half-line and finite interval cases highlight the method's ability to handle general boundary conditions, including Robin and time-dependent Dirichlet conditions, as well as source terms. Numerical illustrations further validate the theoretical results and showcase the method's application to practical scenarios such as periodic heating, thermal insulation, and combustion.

By comparing the Fokas method with traditional approaches like the image method, this study underscores its advantages in terms of uniform convergence and computational efficiency. These features make it a compelling choice for solving IBVPs in both academic and applied settings. Future work could explore extending these techniques to multidimensional problems and investigating their integration with modern numerical solvers.

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