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An immersed body in a semi-infinite nematic liquid crystal

Abstract

In this article, we investigate the configuration of a single rigid body immersed in a semi-infinite two-dimensional liquid crystal domain and explore asymptotic in the far field.

Keywords: liquid crystals

Contents

§ 1 Introduction

We consider a body D_1 on a semi-infinite liquid crystal domain. We assume the liquid crystal is subject to tangential-anchoring on D_1, with large dimensionless anchoring strength ω_1. The half plane is denoted D_2, with large anhoring ω_2, the domain $D := \mathbb{C}\backslash(D_1 \cup D_2)$. We write the effective domains as D_1^ω and D_2^ω respectievly.

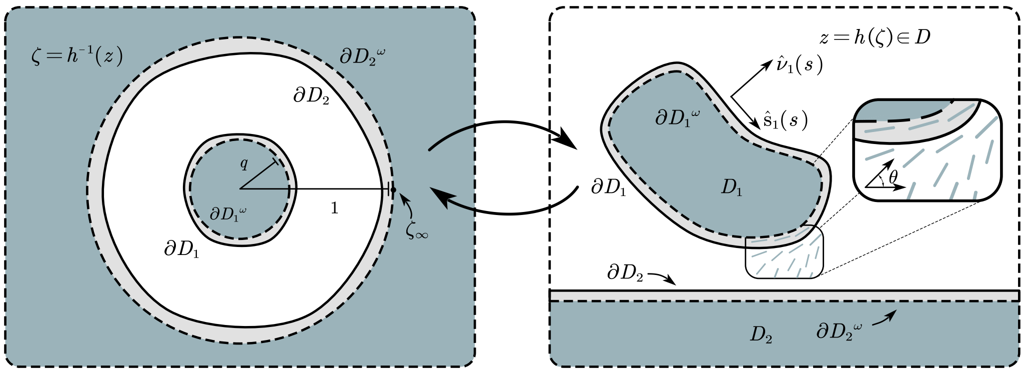

Figure 1: Right: the physical domain with one rigid body immersed in a two-dimensional nematic liquid crystal with an infinite wall, where $z = x + iy$. The orientation of the liquid crystal is portrayed by a director field $\boldsymbol{n} = (\cos\theta, \sin\theta)$ with director angle $\theta(z) \in (0, \pi]$ for $z \in D$. The boundary of the body and the wall are plotted as solid curves, ∂D_1 and ∂D_2. ∂D_1 is equipped with unit normal and tangent vector fields $\hat{\boldsymbol{\nu}}_1(s)$ and $\hat{\boldsymbol{s}}_1(s)$, respectively. The effective boundaries are shown as dotted curves, namely ∂D_1^ω and $\partial D_2(\omega)$. Left: the conformally-equivalent-domain of D. The pole at $\zeta_\infty = 1$ corresponds to $z \to \infty$ in the physical domain D.

Although we usually work with a wedge shaped domain, we assert that restricting to a half-plane geometry is of more generality. The analysis presented here can immediately be extended to any doubly connected semi-infinite field, *i.e.*, a dobuly connected liquid crystal domain with an unbounded complement, such as liquid crystal in the first quadrant with an immersed body. This is due to [1], which states that any doubly connected domain is conformally equivalent to an annular region.

§ 2 Construction of analytical solution

In this section, we will derive anlytical solutions for any single immersed body in a wedge shaped liquid crystal domain of turning angle $\gamma \in (0, 2\pi)$, which can be adapted to any doubly connected semi-infinite domain with minor modifications. We write the complex director as Ω and define the periods of Ω around D_1 and D_2 as

$$\oint_{\partial D_1} d\Omega \equiv \frac{1}{i} \int_{\partial D_1} \theta_x - i\theta_y dz = \Upsilon_1 - 2\pi i M_1 \tag{2.1a}$$

$$\int_{\partial D_2} d\Omega \equiv \frac{1}{i} \int_{\partial D_2} \theta_x - i\theta_y dz = \Upsilon_2 - 2\pi i M_2. \tag{2.1b}$$

We assume $\Omega(z) \to 0$ as $|z| \to \infty$ in D, then $M_2 = 0$ by definition. Then it follows that $M_1 = 0$ and $\Upsilon_1 = -\Upsilon_2 = -\Upsilon$ because Ω is locally analytic in D, and we might deform the contour integral defined in (2.1) from ∂D_1 to ∂D_2 joined with an infinite semicircle and use the fact that $\Omega \to 0$ at infinity. Due to [2, 3], At large anchoring strengths, the boundary conditions are given by

$$\begin{aligned} \mathrm{Im}[e^{\Omega(z)} z_s] &= \mathcal{O}(1/\omega_1^3) \quad \text{on } \partial D_1^\omega, \\ \mathrm{Im}[e^{\Omega(z)} z_s] &= \mathcal{O}(1/\omega_2^3) \quad \text{on } \partial D_2^\omega. \end{aligned} \tag{2.2}$$

With all periods vanishing in (2.1), we write the complex director as

$$\Omega(z) \equiv \log f'(z) + g(z), \quad \mathrm{Im}(z) > 0, \tag{2.3}$$

where $g(z)$ is a locally analytic function that accounts for the periods in (2.1). These yield the problem:

$$g(z) \text{ locally holomorphic} \quad \text{in } D^\omega, \tag{2.4a}$$
$$\mathrm{Im}\ g(z) = \alpha \quad \text{on } \partial D_1^\omega, \tag{2.4b}$$
$$\mathrm{Im}\ g(z) = 0 \quad \text{on } \partial D_2^\omega, \tag{2.4c}$$
$$g(z) \sim e^{i\beta} \quad \text{as } |z| \to \infty, \tag{2.4d}$$
$$\oint_{\partial D_1} dg = -\Upsilon \quad \text{and} \quad \int_{\partial D_2} dg = \Upsilon, \tag{2.4e}$$

for some real constants α and β that are to be determined, and

$$f(z) \text{ locally holomorphic} \quad \text{in } D^\omega, \tag{2.5a}$$
$$\mathrm{Im}\ [e^{i\alpha} f(z)] = C \quad \text{on } \partial D_1^\omega, \tag{2.5b}$$
$$\mathrm{Im}\ f(z) = 0 \quad \text{on } \partial D_2^\omega, \tag{2.5c}$$
$$f(z) \sim e^{-i\beta} z \quad \text{as } |z| \to \infty, \tag{2.5d}$$

for some unkonwn constant C, where (2.5b) and (2.5c) follows from integrating (2.2) with respect to arc-length. It is worth mentioning that $e^{\Omega(z)} \to 1$ as $z \to \infty$, which induces a real singularity when integrating on ∂D_2^ω, but this does no harm since $z_s = 1$ along ∂D_2^ω and we are taking imaginary parts in (2.2). Here, we have fixed the gauge of f such that the constant in (2.5b) vanishes. A unique solution is specified by fixing the period of f around D_1^ω, *i.e.*

$$\oint_{\partial D_1^\omega} df = \Gamma. \quad \int_{\partial D_2^\omega} df = \hat{\Gamma}, \quad \Gamma \in \mathbb{R}, \tag{2.6}$$

where $\hat{\Gamma}$ is unspecified. We don't have the freedom to choose $\hat{\Gamma}$ because it is implied by the far-field condition. We shall see why this yields a unique solution in appendix A.4. An extension of Riemann's mapping theorem due to Koebe[4], guarantees that there exists a conformal map $z = h(\zeta)$, from $z \in D^\omega$ to the annulus $q \leqslant |\zeta| \leqslant 1$, with ∂D_1^ω mapped onto $|\zeta| = q$, ∂D_2^ω onto $|\zeta| = 1$, and $z = \infty$ onto $\zeta = \zeta_\infty \equiv 1$

on the unit circle, such that $z = h(\zeta) \sim C_\infty/(\zeta - \zeta_\infty)$ as $\zeta \to \zeta_\infty$. In the ζ-plane, $G(\zeta) := g(z(\zeta))$ and $F(\zeta) := f(h(\zeta))$ satisfies

$$\begin{aligned}
G(\zeta) \text{ locally holomorphic} &\quad \text{in } q < |\zeta| < 1, &\quad (2.7a)\\
\operatorname{Im} G(\zeta) = 0 &\quad \text{on } |\zeta| = 1, &\quad (2.7b)\\
\operatorname{Im} [G(\zeta)] = \alpha &\quad \text{on } |\zeta| = q, &\quad (2.7c)\\
G(\zeta) = i\beta &\quad \text{at } \zeta = \zeta_\infty, &\quad (2.7d)\\
\oint_{|\zeta|=q} dG = -\Upsilon, &\quad \text{and } \oint_{|\zeta|=1} dG = -\Upsilon, &\quad (2.7e)
\end{aligned}$$

and

$$\begin{aligned}
F(\zeta) \text{ locally holomorphic} &\quad \text{in } q < |\zeta| < 1, &\quad (2.8a)\\
\operatorname{Im} F(\zeta) = 0 &\quad \text{on } |\zeta| = 1, &\quad (2.8b)\\
\operatorname{Im} [e^{i\alpha} F(\zeta)] = C &\quad \text{on } |\zeta| = q, &\quad (2.8c)\\
F(\zeta) \sim C_\infty e^{-i\beta}/(\zeta - \zeta_\infty) &\quad \text{as } \zeta \to \zeta_\infty, &\quad (2.8d)\\
\oint_{|\zeta|=q} dF = \Gamma, &\quad \text{and } \oint_{|\zeta|=1} dF = \hat{\Gamma}. &\quad (2.8e)
\end{aligned}$$

With details provided in the appendix A, we find that

$$G(\zeta) = -\frac{\Upsilon}{2\pi} \log \zeta, \quad \alpha = \frac{\Upsilon}{2\pi} \log q, \quad \beta = \frac{\Upsilon}{2\pi} \log \zeta_\infty = 0. \tag{2.9}$$

We also know from the construction of the modified Green's functions and their derivatives that, up to an additive constant, the corresponding potential of (2.8) is given by

$$F(\zeta) = C_\infty K(\zeta; \alpha, q) + \frac{\Gamma}{2\pi i} \log \zeta, \tag{2.10}$$

where

$$P(\zeta; \alpha, q) := (1 - \zeta) \prod_{k=1}^{\infty} (1 - q^{2k}\zeta)^{e^{2ki\alpha}} (1 - q^{2k}/\zeta)^{e^{-2ki\alpha}}, \tag{2.11a}$$

$$K(\zeta; \alpha, q) := \frac{\zeta P'(\zeta, q)}{P(\zeta, q)} = \frac{\zeta}{\zeta - 1} + \sum_{k=1}^{\infty} \left(\frac{e^{-2ki\alpha} q^{2k}}{\zeta - q^{2k}} - \frac{e^{2ki\alpha} q^{2k}}{1/\zeta - q^{2k}} \right). \tag{2.11b}$$

Since $z = \infty$ and the infinite line ∂D_2^ω can't be seperated by any conformal map, ζ_∞ is inevitably on $|\zeta| = 1$, and we have the freedom to choose $\zeta_\infty = 1$. But given that $q < 1$, (2.7) converge to a holomolphic function in $q < |\zeta| < 1$. Refer to appendix A.4 on how to make sense of admitting a period of F around $|\zeta| = 1$ that goes to infinity, while passing $\zeta_\infty \to 1$.

§ 3 Example: A cylinder on a half plane liquid crystal

We assume the cylinder $D_1 := \{z \in \mathbb{C} : |z - de^{i\chi}| \leqslant b\}$ has vanishing topological charge, and naturally $\mathrm{Im}(de^{i\chi}) > b - 2/\omega_2$. Let $D_1^\omega := \{z \in \mathbb{C} : |z - de^{i\chi}| \leqslant b\rho(b\omega_1)\}$, where

$$\rho(\omega) := \left(\sqrt{1 + \frac{4}{\omega_1^2}} - \frac{2}{\omega_1}\right)^{1/2} \tag{3.1}$$

We set $D_2 := \{z \in \mathbb{C} : \mathrm{Im}(z) \leqslant 2/\omega_2\}$, so that $\partial D_2^\omega = \{z \in \mathbb{C} : \mathrm{Im}(z) = 0\}$.

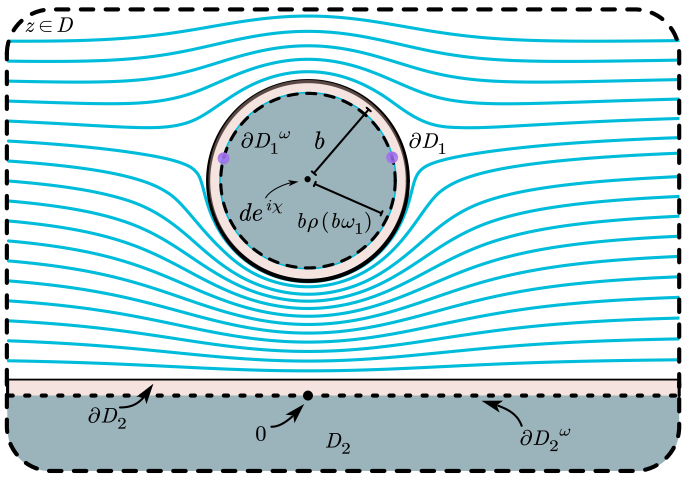

Figure 2: Two-dimensional liquid crystal outside a cylinder centred at $z = de^{i\chi}$ with radius b on the upper-half plane. The effective domain boundary consists of a shrunken cylinder of radius $b\rho(b\omega)$ and the real line. Integral curves of the director field are shown in blue for $\omega_1 = \omega_2 = 1000$, $d = 1$, $b = 0.5$ and $\chi = \pi/2$, and using numerically determined energy-minimizing period: $\Gamma \approx 1.8833$. The boundaries of the effective domain are intensionally plotted away from the boundaries of the physical domain for visibility.

§ 3.1 Deriving the complex director

The next step is to find a conformal map, $z = h(\zeta)$, which maps the effective domain D^ω to the annulus $q \leqslant |\zeta| \leqslant 1$. Firstly, we consider the Cayley transform:

$$\hat{z} = \frac{z - de^{i\chi}}{z - de^{-i\chi}}, \tag{3.2}$$

which maps ∂D_2^ω onto $|\hat{z}| = 1$, $z = \infty$ to $\hat{z} = 1$ and ∂D_1^ω onto

$$\left|\frac{\hat{z}}{1 - \hat{z}}\right| = \frac{b\rho(b\omega_1)}{2d \sin\chi} := r \Leftrightarrow |\hat{z} + \lambda| = \sqrt{\lambda + \lambda^2}, \tag{3.3}$$

where $\lambda = r^2/(1 - r^2)$. We denote $\mathcal{C} := \{\hat{z} \in \mathbb{C} : |\hat{z} + \lambda| = \sqrt{\lambda + \lambda^2}\}$. Since D_1 is on the upper half-plane, $\mathrm{Im}(de^{i\chi}) = d \sin\chi > b$, and $\rho(b\omega_1) < 1$, we have that

$0 < r < 1/2$, which implies $0 < \lambda < 1/3$, hence $\mathcal{C}$ is contained in the unit disc. Then we construct a Möbius transformation, which is an automorphism of the unit disc that centres $\mathcal{C}$ and $|\hat{z}| = 1$ at the origin. Consider $|a| < 1$ such that

$$\zeta = \frac{\hat{z} - a}{1 - \overline{a}\hat{z}}, \quad \text{where } a = \frac{\lambda - 1 + \sqrt{1 - 2\lambda - 3\lambda^2}}{2\lambda}. \tag{3.4}$$

Note that a is on the real line and $-1 < a < 0$ as λ ranges from 0 to 1/3. $\mathcal{C}$ is mapped to

$$\{\zeta \in \mathbb{C} : |\zeta| = q\}, \quad \text{where } q^2 = \frac{\lambda - 2a\lambda - a^2}{1 + 2a\lambda - \lambda a^2}, \tag{3.5}$$

which satisfies $0 < q < 1$. Combine (3.2) (3.3) and (3.4), we find that

$$\zeta = h^{-1}(z) = \frac{(1 - a)z + d(ae^{-i\chi} - e^{i\chi})}{(1 - a)z + \overline{d(ae^{-i\chi} - e^{i\chi})}}, \tag{3.6a}$$

$$a = \frac{\lambda - 1 + \sqrt{1 - 2\lambda - 3\lambda^2}}{2\lambda}, \quad \lambda = \frac{b^2\rho^2(b\omega_1)}{4d^2 \sin^2 \chi - b^2\rho^2(b\omega_1)}. \tag{3.6b}$$

which is a single Cayley transformation that maps ∂D_1^ω to $|\zeta| = q$ and ∂D_2^ω to $|\zeta| = 1$ and $z = \infty$ to $\zeta_\infty = 1$, with q prescribed in (3.5). Hence, the complex director angle is thus simplified to

$$\Omega(z) = \log\left(\sum_{k=-\infty}^{\infty}\left[\frac{q^{2k}}{(\zeta - q^{2k})^2}\right] - \frac{\Gamma(1 - a)}{4\pi d \sin \chi (1 + a)\zeta}\right) + 2\log(\zeta - 1), \tag{3.7}$$

where $\zeta(z)$, λ and a are given in (3.6). Refer to the appendix B.1 for full derivation.

§ 4 Far-field asymptotics of the immersed body

For the general configuration of an immersed body in a half-plane liquid crystal, as discussed in section 2, there is no period around the body. Hence, the director field decays as a quadrupole, $\theta \sim \mathcal{O}(1/|z|^2)$[2], which is evidently reflected in (B.46), where we derived the asymptotic director field for an immersed cylinder in the half plane liquid crystal domain. We know from (2.3) that $\Omega(z) = \log F'(\zeta) - \log h'(\zeta)$, where $z = h(\zeta)$ is the conformal map that takes $q < |\zeta| < 1$ to the liquid crystal domain D. This suggests us to find an asymptotic conformal map, and analyze the asymptotic potential using (2.7), to compute the resulting body force and torque.

§ 4.1 Asymptotic conformal map

The conformal map $h(\zeta)$, which maps $|\zeta| = q$ to ∂D_1^ω, $|\zeta| = 1$ to ∂D_2^ω, and $\zeta_\infty = 1$ to $z = \infty$, is analytic in $q < |\zeta| < 1$ and singular at $\zeta_\infty = 1$ by construction. Thus, we may decompose it as its singular part at $\zeta_\infty = 1$ plus the Laurent series of the rest at $\zeta = 0$, namely

$$z = h(\zeta) = \frac{C_\infty}{\zeta - 1} + \sum_{j=0}^{\infty} h_j \zeta^j + \sum_{j=1}^{\infty} \frac{H_j}{\zeta^j}. \tag{4.1}$$

For instance, for the cylinder introduced in section 3, the corresponding expansion for the conformal map $z = \tilde{h}(\zeta)$ is given by

$$z = \tilde{h}(\zeta) = -\frac{2id\sin\chi(1+a)}{(1-a)(\zeta-1)} - \frac{d(ae^{i\chi} - e^{-i\chi})}{1-a}. \tag{4.2}$$

Given the decay of the director field, it is natural to consider D_1 and D_2 as isolated in the far-field, then there exists two seperate conformal maps, $z = \mu(\xi)$ and $z = \nu(\xi)$ such that $z =$, that map the exterior of ∂D_1^{ω} and the exterior of ∂D_2^{ω} to a unit disk $|\xi| < 1$ in the $\xi-$plane. This follows directly from the usual Riemann mapping theorem for simply connected domains. On mapping an unbounded preimage to the bounded unit disk $|\xi| < 1$, we introduced a singularity at $z = \infty$, we may always find such mapping $z = \mu(\xi)$ such that $z = \infty$ is mapped to $\xi = 0$, and $z = \nu(\xi)$ such that $z = \infty$ is mapped to $\xi = 1$. These mappings are analytic elsewhere, so we write their series representation

$$z = \mu(\xi) = \frac{\mu_{-1}}{\xi} + \sum_{j=0}^{\infty} \mu_j \xi^j, \quad z = \nu(\xi) = \frac{\nu_{-1}}{\xi - 1} + \sum_{j=0}^{\infty} \nu_j \xi^j, \tag{4.3}$$

for complex coefficients μ_j and ν_j, $j \geqslant -1$. Since there is a rotational degree of freedom for $z = \mu(\xi)$, which we shall fix by choosing the map such that $\mu_{-1} \in \mathbb{R}^+$. Since $z = \mu(\xi)$ has no signularity at $\xi = 1$, we choose a mapping $z = \nu(\xi)$ such that $\nu_0 - \nu_{-1} = \mu_0$. For example, for the cylinder considered in Section 3, the corresponding mappings $\tilde{\mu}$ and $\tilde{\nu}$ are given by

$$\tilde{\mu}(\xi) = \frac{b\rho(b\omega_1)}{\xi} + de^{i\chi}, \quad \tilde{\nu}(\xi) = \frac{-2id\sin\chi}{\xi - 1} + de^{-i\chi}. \tag{4.4}$$

This inspires us to define the parameters d and $e^{i\chi}$ for the general case, which measure the far-field separation distance and relative argument repectively.

$$d := |\mu_0|, \quad e^{i\chi} = \frac{\mu_0}{|\mu_0|}. \tag{4.5}$$

As the separation distance increases (*i.e.* $d \to \infty$), the continuity of conformal maps naturally implies $q \to 0$, while the rate of decay depends on the mapping itself, $q \sim \mathcal{O}(1/d)$ in the cylinder case. Moreover, the conformal maps from $q < |\zeta| < 1$ to the doubly connected semi infinite liquid crystal domain, $h(\zeta)$, and from the inverted annulus (the same annulus with swapped boundary circles) to the liquid crystal domain, $h(q/\zeta)$, should recover $\mu(\xi)$ up to a rotational degree of freedom. *i.e.*, $h(q/\zeta) \sim \mu(e^{i\alpha}\zeta)$ and $h(\zeta) \sim \nu(e^{i\beta}\zeta)$ as $d \to \infty$ for some $\alpha, \beta \in (-\pi, \pi]$ to be determined. On substituting (4.1) and (4.3) into the the relation under $q \ll |\zeta| \ll 1$, we find the asymptotic balance

$$\sum_{j=0}^{\infty} (h_j - C_\infty)\zeta^j + \sum_{j=1}^{\infty} \frac{H_j}{\zeta^j} \sim \sum_{j=0}^{\infty} (\nu_j - \nu_{-1})(e^{i\beta}\zeta)^j, \tag{4.6a}$$

$$\sum_{j=0}^{\infty} \frac{(h_j - C_\infty)q^j}{\zeta^j} + \sum_{j=1}^{\infty} \frac{H_j \zeta^j}{q^j} \sim \frac{\mu_{-1}}{e^{i\alpha}\zeta} + \sum_{j=0}^{\infty} \mu_j (e^{i\alpha}\zeta)^j, \tag{4.6b}$$

as $d \to \infty$. Recall that $\mu_{-1} \in \mathbb{R}^+$, $\chi \in (0, \pi)$ and $q > 0$, these restrictions yield $\beta = 0$ and $\alpha = -\pi/2$. Thus, on matching the coefficients of (4.6a) and (4.6b), we may write the asymptotic expressions for the coefficients of the mapping $z = h(\zeta)$,

$$h_j \sim \nu_j, \quad C_\infty \sim \nu_1, \quad q \sim -i\mu_{-1}/\nu_{-1}, \quad H_j \sim \mu_j(-iq_j)^j. \tag{4.7}$$

§ A Analytic potentials for doubly-connected circular domain

We shall derive analytical solution to the potential problem (2.8) in this appendix. We construct the modified Green's functions for the doubly connected ciruclar domain $q < |\zeta| < 1$, which can be represented using Schottky-Klein prime functions[5]. These modified Green's functions and their derivatives can be adopted to represent the analytic solution to the potential. In particular, we will make sense of passing the singular point of these functions to the boundary.

§ A.1 The first modified Green's function

The first modified Green's function on $q < |\zeta| < 1$, $\mathcal{G}_0(\zeta, a)$ satisfy the following

$$\mathcal{G}_0(\zeta; a, \alpha) \text{ locally analytic} \quad \text{in } q < |\zeta| < 1, \tag{A.1a}$$

$$\text{Im } [\mathcal{G}_0(\zeta; a, \alpha)] = 0 \quad \text{on } |\zeta| = 1, \tag{A.1b}$$

$$\text{Im } [e^{i\alpha}\mathcal{G}_0(\zeta; a, \alpha)] = C_1 \quad \text{on } |\zeta| = q, \tag{A.1c}$$

$$\mathcal{G}_0(\zeta; a, \alpha) \sim \frac{1}{2\pi i}\log(\zeta - a) \quad \text{as } \zeta \to a, \tag{A.1d}$$

$$\oint_{|\zeta|=1} d\mathcal{G}_0(\zeta; a, \alpha) = 1 \quad \text{and} \quad \oint_{|\zeta|=q} e^{i\alpha} d\mathcal{G}_0(\zeta, a, \alpha) = 0, \tag{A.1e}$$

for some constant C_1 that depends only on the choice of complex constant a. On representing $\mathcal{G}_0$ with Schottky-Klein prime function[5] yields

$$\mathcal{G}_0(\zeta, a) = \frac{1}{2\pi i}\log\left(\frac{\omega(\zeta, a)}{|a|\omega(\zeta, 1/\overline{a})}\right), \tag{A.2}$$

where $\omega(z, a)$ stands for the Schottky-Klein prime function in $q < |\zeta| < 1$. In the doubly connected case, the Schottky group Θ has only one generator, which is the mobius transform $\zeta \mapsto q^2\zeta$ (or $\zeta \mapsto q^2/\zeta$). It can be shown that

$$\omega(\zeta; a, \alpha) := (\zeta - a)\prod_{\theta_k \in \Theta''}\left[\frac{(\theta_k(\zeta) - a)(\theta_k(a) - \zeta)}{(\theta_k(\zeta) - \zeta)(\theta_k(a) - a)}\right]^{e^{2ki\alpha}} = -\frac{a}{C^2}P(\zeta/a; \alpha, q), \tag{A.3}$$

where the first equation is due to the construction by Baker[4], with a power of $e^{2ki\alpha}$ on each term to match up with (A.1c). Θ denotes all mappings in Θ excluding the identity and all inverses. The second equation simplifies the product with

$$P(\zeta; \alpha, q) = (1 - \zeta)\prod_{k=1}^{\infty}(1 - q^{2k}\zeta)^{e^{2ki\alpha}}(1 - q^{2k}/\zeta)^{e^{-2ki\alpha}}, \quad C = \prod_{k=1}^{\infty}(1 - q^{2k}), \tag{A.4}$$

which follows from [6]. (A.2), (A.3) and (A.4) thus yields a representation for the first modified Green's function

$$\mathcal{G}_0(\zeta; a, \alpha) = \frac{1}{2\pi i}\log\left(|a|\frac{P(\zeta/a; \alpha, q)}{P(\overline{a}\zeta; \alpha, q)}\right). \tag{A.5}$$

§ A.2 The second modified Green's function

The second modified Green's function in $q < |\zeta| < 1$, $\mathcal{G}_1(\zeta, a)$ is defined to be the solution to

$$\mathcal{G}_1(\zeta; a, \alpha) \text{ locally analytic} \quad \text{in } q < |\zeta| < 1, \tag{A.6a}$$

$$\text{Im}\,[\mathcal{G}_1(\zeta; a, \alpha)] = 0 \quad \text{on } |\zeta| = 1, \tag{A.6b}$$

$$\text{Im}\,[e^{i\alpha}\mathcal{G}_1(\zeta; a, \alpha)] = C_2 \quad \text{on } |\zeta| = q, \tag{A.6c}$$

$$\mathcal{G}_1(\zeta; a, \alpha) \sim \frac{e^{-i\alpha}}{2\pi i} \log(\zeta - a) \quad \text{as } \zeta \to a, \tag{A.6d}$$

$$\oint_{|\zeta|=1} d\mathcal{G}_1(\zeta; a, \alpha) = 0 \quad \text{and} \quad \oint_{|\zeta|=q} e^{i\alpha} d\mathcal{G}_1(\zeta, a, \alpha) = -1, \tag{A.6e}$$

for another constant C_2. It is shown by Crowdy[5] that $\mathcal{G}_1(\zeta; a, \alpha) = \mathcal{G}_0(\zeta; a, \alpha) - v(\zeta)$ for the case when $\alpha = 0$, where $v(\zeta) = \log(\zeta)/(2\pi i)$. It turns out that the case when $\alpha = 0$ is sufficient. This together with (A.5) yields the second modified Green's function

$$\mathcal{G}_1(\zeta; a, \alpha) = \frac{1}{2\pi i} \log\left(-|a| \frac{P(\zeta/a; \alpha, q)}{P(\overline{a}\zeta/q^2; \alpha, q)}\right), \quad \alpha = 0. \tag{A.7}$$

where we used the fact that $P(q^2\zeta; \alpha, q) = -\zeta^{-1} P(\zeta; \alpha, q)^{e^{-2i\alpha}}$.

§ A.3 Derivatives of the modified Green's functions

We need to obtain a potential that has a simple pole at $\zeta = \zeta_\infty$, while the Green's functions have only logarithmic singularities by construction. To obtain other types of singularities, we need to take the derivatives with respect to $a_x = \text{Re}\ a$ and $a_y = \text{Re}\ a$. Such process is legitimate because the modified Green's functions are biholomorphic by construction with the symmetry property $\mathcal{G}_j(\zeta, a) = \mathcal{G}_j(a, \zeta)$, for $j = 0, 1$[5]. With the same approach as in [3], we consider the a linear combination of derivatives

$$\hat{\mathcal{G}}(\zeta; a, \alpha, b) := 2\pi \left(b_y \frac{\partial \mathcal{G}_0}{\partial a_x} - b_x \frac{\partial \mathcal{G}_0}{\partial a_y} \right), \tag{A.8}$$

where $b = b_x + i b_y$ is a constant. We can show that (A.8) solves the following equation:

$$\hat{\mathcal{G}}(\zeta; a, \alpha, b) \text{ locally analytic} \quad \text{in } q < |\zeta| < 1, \tag{A.9a}$$

$$\text{Im}[\hat{\mathcal{G}}(\zeta; a, \alpha, b)] = 0 \quad \text{on } |\zeta| = 1, \tag{A.9b}$$

$$\text{Im}[e^{i\alpha}\hat{\mathcal{G}}(\zeta; a, \alpha, b)] = \hat{C} \quad \text{on } |\zeta| = q, \tag{A.9c}$$

$$\hat{\mathcal{G}}(\zeta; a, \alpha, b) \sim \frac{b}{\zeta - a} \quad \text{as } \zeta \to a, \tag{A.9d}$$

$$\oint_{|\zeta|=1} d\hat{\mathcal{G}}(\zeta; a, \alpha, b) = 0 \quad \text{and} \quad \oint_{|\zeta|=q} d\hat{\mathcal{G}}(\zeta; a, \alpha, b) = 0, \tag{A.9e}$$

for some constant $\hat{C}$. where

$$\hat{\mathcal{G}}(\zeta; a, \alpha, b) = 2\pi(b_y - ib_x)\frac{\partial \mathcal{G}_0}{\partial a} \sim \frac{b_y - ib_x}{i}\frac{\partial}{\partial a}\log(\zeta - a) = \frac{b}{\zeta - a}, \tag{A.10}$$

as $\zeta \to a$. On applying the Wirtinger derivatives [7],

$$\frac{\partial}{\partial a} = \frac{1}{2}\left(\frac{\partial}{\partial a_x} - i\frac{\partial}{\partial a_y}\right) \quad \text{and} \quad \frac{\partial}{\partial \overline{a}} = \frac{1}{2}\left(\frac{\partial}{\partial a_x} + i\frac{\partial}{\partial a_y}\right), \tag{A.11}$$

we can represent (A.8) as

$$\hat{\mathcal{G}}(\zeta; a, \alpha, b) = 2\pi i\left(\overline{b}\frac{\partial \mathcal{G}_0}{\partial \overline{a}} - b\frac{\partial \mathcal{G}_0}{\partial a}\right). \tag{A.12}$$

On substituting (A.5), we find that

$$\hat{\mathcal{G}}(\zeta; a, \alpha, b) = \frac{b}{a}K(\zeta/a; \alpha, q) - \frac{\overline{b}}{\overline{a}}K(\overline{a}\zeta; \alpha, q) - \frac{b}{2a} + \frac{\overline{b}}{2\overline{a}}, \tag{A.13}$$

where

$$K(\zeta; \alpha, q) := \frac{\zeta P'(\zeta; \alpha, q)}{P(\zeta; \alpha, q)} = \frac{\zeta}{\zeta - 1} + \sum_{k=1}^{\infty}\left(\frac{e^{-2ki\alpha}q^{2k}}{\zeta - q^{2k}} - \frac{e^{2ki\alpha}q^{2k}}{1/\zeta - q^{2k}}\right). \tag{A.14}$$

§ A.4 Constructing the potential

The solution to (2.7) is given by

$$G(\zeta) = -\Upsilon[\mathcal{G}_0(\zeta; \zeta_\infty, 0) - \mathcal{G}_1(\zeta; \zeta_\infty, 0)] = -\frac{\Upsilon}{2\pi i}\log(\zeta), \tag{A.15}$$

Thus, α and β in (2.7) are given by

$$\alpha = \frac{\Upsilon}{2\pi}\log q, \quad \beta = \frac{\Upsilon}{2\pi}\log \zeta_\infty \tag{A.16}$$

Neglecting the condition that the period of F around $|\zeta| = 1$ is positive infinity, then the solution of the potential to (2.8) is given by

$$\begin{aligned}\hat{F}(\zeta) &= \hat{\mathcal{G}}(\zeta; \zeta_\infty, \alpha, C_\infty) - \Gamma\mathcal{G}_1(\zeta, \zeta_\infty, \alpha)\\ &= \frac{C_\infty e^{-i\beta}}{\zeta_\infty}K(\zeta/\zeta_\infty; \alpha, q) - \frac{\overline{C_\infty}e^{i\beta}}{\zeta_\infty}K(\zeta_\infty\zeta; \alpha, q)\\ &- \frac{\Gamma}{2\pi i}\log\frac{P(\zeta/\zeta_\infty; \alpha, q)}{P(\zeta_\infty\zeta/q^2; \alpha, q)},\end{aligned} \tag{A.17}$$

up to an additive real constant, where ζ_∞ is chosen to be real. We claim the solution to the full (2.8) follows from (A.17) without much difficulties. It is worth noting that

(A.17) doesn't yield a unique solution to (2.8) if $\zeta_\infty \neq 1$. If $|\zeta_\infty| < 1$, the solution is uniquely determined by further specifying the period around $|\zeta| = 1$, meaning that

$$\begin{aligned} F(\zeta) &= \hat{\mathcal{G}}(\zeta;\zeta_\infty,\alpha,C_\infty) - \Gamma\mathcal{G}_1(\zeta;\zeta_\infty,\alpha) - \hat{\Gamma}\mathcal{G}_0(\zeta;\zeta_\infty,\alpha) \\ &= \frac{C_\infty e^{-i\beta}}{\zeta_\infty} K(\zeta/\zeta_\infty;\alpha,q) - \frac{\overline{C_\infty} e^{i\beta}}{\zeta_\infty} K(\zeta_\infty\zeta;\alpha,q) \\ &- \frac{\Gamma e^{-i\alpha}}{2\pi i}\log\frac{P(\zeta/\zeta_\infty;\alpha,q)}{P(\zeta_\infty\zeta/q^2;\alpha,q)} - \frac{\hat{\Gamma}}{2\pi i}\log\left(\frac{P(\zeta/\zeta_\infty;\alpha,q)}{P(\zeta\zeta_\infty;\alpha,q)}\right), \end{aligned} \tag{A.18}$$

up to an additive constant. where $\hat{\Gamma}$ is the the prescribed period of F around $|\zeta| = 1$. Hence, it's impiritive that we analyze the behavior of the term induced by $\hat{\Gamma}$ in the limit as $\zeta_\infty \to 1$. Consider the difference between (A.17) and (A.18):

$$\frac{1}{2\pi i}\left[\oint_{|\zeta|=1} dF(\zeta)\right]\log\left(\frac{P(\zeta/\zeta_\infty,q)}{P(\zeta_\infty\zeta,q)}\right). \tag{A.19}$$

Assume $\zeta_\infty \to 1$, it follows from (2.8d) that

$$\oint_{|\zeta|=1} dF(\zeta) \sim \frac{1}{1-\zeta_\infty}, \tag{A.20}$$

where $\sim$ refers to multiplication of a constant. In fact, we are able to determine this multilicative constant with merely the local information (2.8d), in the limit, we are free to substitude F with (2.8d) plus an analytic function $\tilde{F}$, which gives

$$\begin{aligned} &\lim_{\zeta_\infty\to 1}\oint_{|\zeta|=1}(1-\zeta_\infty)dF(\zeta) = \lim_{\zeta_\infty\to 1}\oint_{|\zeta|=1}(1-\zeta_\infty)d\left(\frac{C_\infty}{\zeta-\zeta_\infty}+\tilde{F}(\zeta)\right) \\ &= \lim_{\zeta_\infty\to 1}\oint_{|\zeta|=1}(1-\zeta_\infty)d\frac{C_\infty}{\zeta-\zeta_\infty} = \lim_{\zeta_\infty\to 1}\oint_{|\zeta|=1}\frac{C_\infty(\zeta_\infty-1)}{(\zeta-\zeta_\infty)^2}d\zeta \\ &= -\lim_{\zeta_\infty\to 1}\oint_{|\zeta|=1}\frac{C_\infty}{2\zeta-2\zeta_\infty}d\zeta = -\lim_{\zeta_\infty\to 1}\pi i C_\infty, \end{aligned} \tag{A.21}$$

where the second equality is due to L'Hôpital's rule, and we note that C_∞ depends on ζ_∞, and should be imaginary as in the limit as $\zeta_\infty \to 1$. It is clear that

$$\frac{\partial}{\partial\zeta}\log P(\zeta;\alpha,q) = \frac{P'(\zeta,q)}{P(\zeta,q)} = \frac{1}{\zeta-1} + \sum_{k=1}^{\infty}\left(\frac{e^{-2ki\alpha}q^{2k}}{\zeta^2-\zeta q^{2k}} - \frac{e^{2ki\alpha}q^{2k}}{1-\zeta q^{2k}}\right), \tag{A.22}$$

which combined with L'Hôpital's rule and (A.21) gives that

$$\begin{aligned} &\lim_{\zeta_\infty\to 1^-}\frac{1}{2\pi i}\left[\oint_{|\zeta|=1} dF(\zeta)\right]\log\left(\frac{P(\zeta/\zeta_\infty,q)}{P(\zeta_\infty\zeta,q)}\right) \\ &= -\lim_{\zeta_\infty\to 1^-}\frac{C_\infty}{2}\frac{\log P(\zeta/\zeta_\infty,q) - \log P(\zeta_\infty\zeta,q)}{\zeta_\infty - 1} \\ &= \lim_{\zeta_\infty\to 1}\frac{-C_\infty}{2}\frac{\partial\log P(\zeta/\zeta_\infty,q)}{\partial\zeta_\infty} + \frac{C_\infty}{2}\frac{\partial\log P(\zeta\zeta_\infty,q)}{\partial\zeta_\infty} = C_\infty K(\zeta,q), \end{aligned} \tag{A.23}$$

where C_∞ denotes $\lim_{\zeta_\infty \to 1} C_\infty$. On taking the limit as $\zeta_\infty \to 1$ of (A.18) and substituting (A.23), we obtain the unique potential that satisfies (2.8), namely

$$\begin{aligned} F(\zeta) &= -\overline{C_\infty} K(\zeta; \alpha, q) - \frac{\Gamma}{2\pi i} \log \frac{P(\zeta; \alpha, q)}{P(\zeta/q^2; \alpha, q)} \\ &= C_\infty K(\zeta; \alpha, q) + \frac{\Gamma}{2\pi i} \log \zeta, \end{aligned} \tag{A.24}$$

up to an additive constant, where $P(\zeta, q)$ and $K(\zeta, q)$ are specified in (A.4) and (A.14) repectively, $\alpha = \Upsilon \log q / 2\pi$.

§ B An immersed cylinder with an infinite wall

§ B.1 Derivation of the complex director

For (3.3), we write $\hat{z} = x + iy$, then the equation:

$$\begin{aligned} &\left| \frac{\hat{z}}{1 - \hat{z}} \right| = \frac{b\rho(b\omega)}{2d \sin \chi} := r \Leftrightarrow x^2 + y^2 = r^2 (x-1)^2 + r^2 y^2 \\ &\qquad \Leftrightarrow \left(x + \frac{r^2}{1 - r^2} \right)^2 + y^2 = \frac{r^2}{1 - r^2} + \left(\frac{r^2}{1 - r^2} \right)^2 \\ &\qquad := |\hat{z} + \lambda| = (\lambda + \lambda^2)^{1/2}, \quad \lambda =: \frac{r^2}{1 - r^2} = \frac{b^2 \rho^2 (b\omega_1)}{4d^2 \sin^2 \chi - b^2 \rho^2 (b\omega_1)}. \end{aligned} \tag{B.1}$$

For (3.4),

$$\zeta = \frac{\hat{z} - a}{1 - \bar{a}\hat{z}} \Leftrightarrow \hat{z} = \frac{\zeta + a}{1 + \bar{a}\zeta}. \tag{B.2}$$

Plug the right hand side of (B.2) into (3.3), we get

$$\begin{aligned} &\left| \frac{\zeta + a + \lambda + \lambda \bar{a} \zeta}{1 + \bar{a}\zeta} \right| = (\lambda + \lambda^2)^{1/2} \\ &\Leftrightarrow (\zeta + a + \lambda + \lambda \bar{a} \zeta)(\bar{\zeta} + \bar{a} + \lambda + \lambda a \bar{\zeta}) = (\lambda + \lambda^2)(1 + \bar{a}\zeta)(1 + a\bar{\zeta}) \\ &\Leftrightarrow (1 + \lambda a + \lambda \bar{a} - \lambda a \bar{a}) \zeta \bar{\zeta} + (\lambda a^2 + a - \lambda a + \lambda) \bar{\zeta} \\ &+ (\overline{\lambda a^2 + a - \lambda a + \lambda}) \zeta = \lambda - \lambda(a + \bar{a}) - a\bar{a}. \\ &\Leftrightarrow |\zeta| = q. \end{aligned} \tag{B.3}$$

Thus, a must satisfy the equation $\lambda a^2 + (1 - \lambda) a + \lambda = 0$, which yields

$$a = \frac{\lambda - 1 \pm \sqrt{1 - 2\lambda - 3\lambda^2}}{2\lambda}. \tag{B.4}$$

We select a branch to make sure $|a| < 1$, as in (3.4). Then, (B.3) gives

$$|\zeta|^2 = \frac{\lambda - 2a\lambda - a^2}{1 + 2a\lambda - \lambda a^2} := q^2. \tag{B.5}$$

Hence, we have h and h^{-1}:

$$\zeta = \zeta(\hat{z}(z)) = \frac{\hat{z} - a}{1 - a\hat{z}} = \frac{(1-a)z + d(ae^{-i\chi} - e^{i\chi})}{(1-a)z + \overline{d(ae^{-i\chi} - e^{i\chi})}}; \tag{B.6a}$$

$$z = \frac{\overline{d(ae^{-i\chi} - e^{i\chi})}\zeta - d(ae^{-i\chi} - e^{i\chi})}{(1-a)(1-\zeta)}. \tag{B.6b}$$

From (B.6b), we get

$$C_\infty = \lim_{\zeta \to \zeta_\infty} z(\zeta - \zeta_\infty) = -2id \sin\chi \left(\frac{1+a}{1-a}\right), \tag{B.7}$$

With C_∞ specified,

$$\Omega(z) = \log f'(z) = \log F'(\zeta) + \log \frac{\partial \zeta}{\partial z} \tag{B.8a}$$

$$\begin{aligned} K'(\zeta) &= -\frac{1}{(\zeta-1)^2} - \sum_{k=1}^{\infty}\left(\frac{q^{2k}}{(\zeta - q^{2k})^2} + \frac{q^{2k}}{(1 - q^{2k}\zeta)^2}\right) \\ &= -\frac{1}{(\zeta-1)^2} - \sum_{k=1}^{\infty}\left(\frac{q^{2k}}{(\zeta - q^{2k})^2} + \frac{q^{-2k}}{(\zeta - q^{-2k})^2}\right) \\ &= -\sum_{k=-\infty}^{\infty} \frac{q^{2k}}{(\zeta - q^{2k})^2} \end{aligned} \tag{B.8b}$$

$$\begin{aligned} \frac{\partial}{\partial \zeta}\left(\log \frac{P(\zeta)}{P(\zeta/q^2)}\right) &= \frac{\partial}{\partial \zeta} \log\left(\frac{1-\zeta}{1-\zeta/q^2}\frac{\prod_{k=1}^{\infty}(1-q^{2k}\zeta)(1-q^{2k}/\zeta)}{\prod_{k=1}^{\infty}(1-q^{2k-2}\zeta)(1-q^{2k+2}/\zeta)}\right) \\ &= \frac{\partial}{\partial \zeta}\left(\log\frac{1-q^2/\zeta}{1-\zeta/q^2}\right) = -\frac{1}{\zeta} \end{aligned} \tag{B.8c}$$

$$\begin{aligned} \log \frac{\partial \zeta}{\partial z} &= \log\left(\frac{\partial}{\partial z}\frac{(1-a)z + d(ae^{-i\chi} - e^{i\chi})}{(1-a)z + \overline{d(ae^{-i\chi} - e^{i\chi})}}\right) \\ &= 2\log(\zeta - 1) - \log\left(2id\sin\chi \cdot \frac{1+a}{1-a}\right). \end{aligned} \tag{B.8d}$$

Plug (2.7), (B.8b), (B.8c) and (B.8d) into (B.8a), we obtain

$$F'(\zeta) = 2id\sin\chi\frac{1+a}{1-a}\sum_{k=-\infty}^{\infty}\left[\frac{q^{2k}}{(\zeta - q^{2k})^2}\right] + \frac{\Gamma}{2\pi i\zeta}; \tag{B.9a}$$

$$\begin{aligned} \Omega(z) &= \log\left(2id\sin\chi\frac{1+a}{1-a}\sum_{k=-\infty}^{\infty}\left[\frac{q^{2k}}{(\zeta - q^{2k})^2}\right] + \frac{\Gamma}{2\pi i\zeta}\right) + 2\log(\zeta - 1) \\ &\quad - \log\left(2id\sin\chi \cdot \frac{1+a}{1-a}\right) \\ &= \log\left(\sum_{k=-\infty}^{\infty}\left[\frac{q^{2k}}{(\zeta - q^{2k})^2}\right] - \frac{\Gamma(1-a)}{4\pi d\sin\chi(1+a)\zeta}\right) + 2\log(\zeta - 1), \end{aligned} \tag{B.9b}$$

where $\zeta(z)$ is given in (B.6a), λ and a are prescribed in (B.1) and (A.4) respectively.

§ B.2 Derivation of surface tractions

According to [2], the non-dimensional surface traction $\hat{\boldsymbol{t}} \equiv (\hat{t}_x, \hat{t}_y)$ may be written as

$$\hat{t}_x - i\hat{t}_y = \frac{1}{2i}\Omega'(z)^2 \frac{\partial z}{\partial s} + \frac{\omega}{8}\frac{\partial}{\partial s}\left(\frac{\partial \overline{z}}{\partial s}[2 + e^{-2i\phi}e^{\overline{\Omega(z)}-\Omega(z)} - 3e^{i\phi}e^{\Omega(z)-\overline{\Omega(z)}}]\right). \quad \text{(B.10)}$$

For simplicity, we denote

$$\begin{aligned} \tilde{E}(\zeta) = e^{\Omega(z)} &= \left(\sum_{k=-\infty}^{\infty}\left[\frac{q^{2k}}{(\zeta - q^{2k})^2}\right] - \frac{\Gamma(1-a)}{4\pi d \sin\chi(1+a)\zeta}\right)(\zeta-1)^2, \\ E(\zeta) = \frac{\tilde{E}(\zeta)}{(\zeta-1)^2} &= \sum_{k=-\infty}^{\infty}\left[\frac{q^{2k}}{(\zeta - q^{2k})^2}\right] - \frac{\Gamma(1-a)}{4\pi d \sin\chi(1+a)\zeta}, \end{aligned} \quad \text{(B.11)}$$

and let logarithm on the ζ-plane be defined on $\mathbb{C}\setminus\mathbb{R}^-$ to avoid all the singularities of $\Omega(z)$. The argument function is hence defined on $(-\pi, \pi]$, which is compatible wit. Then, $\overline{\log(H(\zeta))} = \log \overline{H(\zeta)}$, for any analytic function H. Thus,

$$\begin{aligned} \Omega'(z) &= \frac{\partial}{\partial \zeta}\left(\log \tilde{E}(\zeta)\right)\frac{\partial \zeta}{\partial z} = \frac{\tilde{E}'(\zeta)}{\tilde{E}(\zeta)}\frac{\partial \zeta}{\partial z} \\ &= \left(\frac{2}{\zeta - 1} + \frac{E'(\zeta)}{E(\zeta)}\right)\cdot(\zeta-1)^2\left(2id\sin\chi\cdot\frac{1+a}{1-a}\right)^{-1} \\ &= c^{-1}\left(2\zeta - 2 + (\zeta-1)^2\frac{E'(\zeta)}{E(\zeta)}\right). \end{aligned} \quad \text{(B.12a)}$$

$$\begin{aligned} \text{where } E'(\zeta) &= -\sum_{k=-\infty}^{\infty}\left[\frac{2q^{2k}}{(\zeta - q^{2k})^3}\right] + \frac{\Gamma(1-a)}{4\pi d\sin\chi(1+a)\zeta^2}, \\ c &:= 2id\sin\chi\cdot\frac{1+a}{1-a}, \quad \tilde{E}'(\zeta) = E'(\zeta)(\zeta-1)^2 + 2E(\zeta)(\zeta-1). \end{aligned} \quad \text{(B.12b)}$$

B.2.1 Traction on the cylinder

On $\partial D_1 = \{z \in \mathbb{C} : |z - de^{i\chi}| = b\}$, $e^{i\phi} = z'(s) = ib^{-1}(z - de^{i\chi})$, plug in $z = z(\zeta)$ to write

$$z'(s) = \frac{2d\sin\chi}{b(1-a)}\frac{\zeta + a}{\zeta - 1} := T_1(\zeta), \quad \text{(B.13a)}$$

$$T_1'(\zeta) = \frac{ic}{b(\zeta-1)^2}. \quad \text{(B.13b)}$$

Since $z'(s) = 1$, we have that

$$\left|\frac{\zeta + a}{\zeta - 1}\right| = \frac{b(1-a)}{2d\sin\chi} := R, \quad \text{(B.14)}$$

which implies

$$\overline{\zeta} = \frac{R^2 - R^2\zeta - a\zeta - a^2}{\zeta + a - R^2\zeta + R^2} := C_1(\zeta); \tag{B.15a}$$

$$C_1'(\zeta) = -\frac{R^2 a(a+2)}{(\zeta - R^2\zeta + a + R^2)^2}; \tag{B.15b}$$

$$\left|\zeta + \frac{a+R^2}{1-R^2}\right| = \left(\frac{R^2 - a^2}{1-R^2} + \left[\frac{a+R^2}{1-R^2}\right]^2\right)^{1/2}. \tag{B.15c}$$

In addition, $|z - de^{i\chi}| = b$ implies

$$\overline{z} = \frac{b^2 - d^2 + de^{-i\chi}z}{z - de^{i\chi}}, \tag{B.16a}$$

$$\frac{\partial \overline{z}}{\partial s} = \frac{-ib}{z - de^{i\chi}} = \frac{1}{T_1(s)}. \tag{B.16b}$$

Now, the right hand side of (B.10) becomes

$$\begin{aligned}
&\frac{\omega_1}{8}\frac{\partial}{\partial\zeta}\left(T_1^{-1}(\zeta)\left[2 + T_1^{-2}(\zeta)\frac{\tilde{E}(C_1(\zeta))}{\tilde{E}(\zeta)} - 3T_1(\zeta)\frac{\tilde{E}(\zeta)}{\tilde{E}(C_1(\zeta))}\right]\right)\frac{\partial\zeta}{\partial z}\frac{\partial z}{\partial s} \\
&+ \frac{1}{2i}T_1(\zeta)c^{-2}\left(2\zeta - 2 + \frac{E'(\zeta)}{E(\zeta)}\right)^2 \\
&= \frac{\omega_1}{8c}(\zeta-1)^2\left(-\frac{2T_1'(\zeta)}{T_1(\zeta)} - \frac{3T_1'(\zeta)}{T_1^3(\zeta)}\frac{\tilde{E}(C_1(\zeta))}{\tilde{E}(\zeta)}\right) \\
&+ \frac{\omega_1}{8c}(\zeta-1)^2\left(\frac{\tilde{E}'(C_1(\zeta))C_1'(\zeta)\tilde{E}(\zeta) - \tilde{E}'(\zeta)\tilde{E}(C_1(\zeta))}{T_1^2(\zeta)\tilde{E}^2(\zeta)}\right) \\
&- \frac{3\omega_1}{8c}(\zeta-1)^2 T_1(\zeta)\left(\frac{\tilde{E}'(\zeta)\tilde{E}(C_1(\zeta)) - \tilde{E}'(C_1(\zeta))C_1'(\zeta)\tilde{E}(\zeta)}{\tilde{E}^2(C_1(\zeta))}\right) \\
&+ \frac{1}{2i}T_1(\zeta)c^{-2}\left(2\zeta - 2 + (\zeta-1)^2\frac{E'(\zeta)}{E(\zeta)}\right)^2 = \hat{t}_x - i\hat{t}_y.
\end{aligned} \tag{B.17}$$

B.2.2 Traction on the wall

On $\partial D_2 = \{z \in \mathbb{C} : \mathrm{Im}(z) = 2/\omega_2\}$, $e^{i\phi} = z'(s) = 1$, $\overline{z} = z - 4i/\omega_2$, $\overline{z}'(s) = 1$. Plug in $\zeta(z)$ to obatin

$$C_2(\zeta) := \overline{\zeta} = \frac{c\omega_2 + 4i(\zeta - 1)}{c\omega_2\zeta + 4i(\zeta-1)}, \tag{B.18a}$$

$$C_2'(\zeta) = \frac{-c^2\omega_2^2}{[(c\omega_2 + 4i)\zeta - 4i]^2}. \tag{B.18b}$$

where c is prescribed in (B.12b). Hence, (B.10) simplifies to

$$
\begin{aligned}
\hat{t}_x - i\hat{t}_y &= \frac{1}{2i}\Omega'(z)^2\frac{\partial z}{\partial s} + \frac{\omega_2}{8}\frac{\partial}{\partial \zeta}\left(2 + \frac{\tilde{E}(C_2(\zeta))}{\tilde{E}(\zeta)} - \frac{3\tilde{E}(\zeta)}{\tilde{E}(C_2(\zeta))}\right)\frac{\partial \zeta}{\partial z}\frac{\partial z}{\partial s} \\
&= \frac{\omega_2(\zeta-1)^2}{8c}\left(\frac{\tilde{E}'(C_2(\zeta))C_2'(\zeta)\tilde{E}(\zeta) - \tilde{E}'(\zeta)\tilde{E}(C_2(\zeta))}{\tilde{E}^2(\zeta)}\right) \\
&- \frac{3\omega_2(\zeta-1)^2}{8c}\left(\frac{\tilde{E}'(\zeta)\tilde{E}(C_2(\zeta)) - \tilde{E}'(C_2(\zeta))C_2'(\zeta)\tilde{E}(\zeta)}{\tilde{E}^2(C_2(\zeta))}\right) \\
&+ \frac{1}{2ic^2}\left(2\zeta - 2 + (\zeta-1)^2\frac{E'(\zeta)}{E(\zeta)}\right)^2.
\end{aligned}
\tag{B.19}
$$

§ B.3 Derivation of body forces

By [2], the non-dimensional body forces $\hat{\boldsymbol{F}} \equiv (\hat{F}_x, \hat{F}_y)$ can be given by integrating the traction along the surfaces. We follow the convention and view D_2 as an infinite body, which is the entire lower-half surface, or an infinite cylinder in the perspective of conformal geometry. This implies that usual orientation of $\partial D_2 : -\infty + 2i/\omega_2 \to \infty + 2i/\omega_2$, turns out to be a clockwise orientation of ∂D_2, which suggests we should put a minus sign when calculating the body force acting on D_2.

$$
\begin{aligned}
\hat{F}_{1x} - i\hat{F}_{1y} &= \oint_{\partial D_1} \hat{t}_x - i\hat{t}_y ds = \frac{1}{2i}\oint_{\partial D_1} \Omega'(z)^2 dz \\
&+ \frac{\omega}{8}\oint_{\partial D_1} \frac{\partial}{\partial s}\left(\frac{\partial \overline{z}}{\partial s}[2 + e^{-2i\phi}e^{\overline{\Omega(z)}-\Omega(z)} - 3e^{i\phi}e^{\Omega(z)-\overline{\Omega(z)}}]\right) ds \\
&= \frac{1}{2i}\oint_{\partial D_1} \Omega'(z)^2 dz.
\end{aligned}
\tag{B.20}
$$

$$
\begin{aligned}
\hat{F}_{2x} - i\hat{F}_{2y} &= -\int_{\partial D_2} \hat{t}_x - i\hat{t}_y ds = -\frac{1}{2i}\int_{\partial D_2} \Omega'(z)^2 dz \\
&+ \frac{\omega}{8}\int_{\partial D_2} \frac{\partial}{\partial s}\left(\frac{\partial \overline{z}}{\partial s}[2 + e^{-2i\phi}e^{\overline{\Omega(z)}-\Omega(z)} - 3e^{i\phi}e^{\Omega(z)-\overline{\Omega(z)}}]\right) ds \\
&= -\frac{1}{2i}\int_{\partial D_2} \Omega'(z)^2 dz,
\end{aligned}
\tag{B.21}
$$

where cancellation in (B.21) is due to the left-right symmetry of the configuration. With $E(\zeta)$ and $E'(\zeta)$ given by (B.11) and (B.12b), we write

$$
E''(\zeta) = \sum_{k=-\infty}^{\infty}\left[\frac{6q^{2k}}{(\zeta - q^{2k})^4}\right] - \frac{\Gamma(1-a)}{8\pi d \sin\chi(1+a)\zeta^3}.
\tag{B.22}
$$

B.3.1 body force on the cylinder

Not that there are two -1 defects on the ∂D_1^{ω}, where $z = de^{i\chi} + b\rho(b\omega_1)e^{i\alpha}$ and $z = de^{i\chi} - b\rho(b\omega_1)e^{-i\alpha}$, for an unknown angular parameter α. Their image under

$\zeta(z)$ is given by

$$\zeta_1 := \frac{(1-a)b\rho(b\omega_1)e^{-i\alpha} + 2aid\sin\chi}{(1-a)b\rho(b\omega_1)e^{-i\alpha} - 2id\sin\chi} \tag{B.23a}$$

$$\zeta_2 := \frac{(1-a)b\rho(b\omega_1)e^{i\alpha} - 2aid\sin\chi}{(1-a)b\rho(b\omega_1)e^{i\alpha} + 2id\sin\chi} \tag{B.23b}$$

Also note that the all the poles of $\Omega'(z)$ in $|\zeta| < 1$ are given by $\mathcal{P} := \{q^{2k} : k \in \mathbb{Z}^+\} \cup \{0, \zeta_1, \zeta_2\}$, and all of the poles are of order 1. Besides, $\mathcal{P}$ is contained in $|\zeta| < q$. However, $\zeta = 0$ is an accumulated singularity, which forbids us from applying the residue theorem.

$$\begin{aligned}
\hat{F}_{1x} - i\hat{F}_{1y} &= \frac{1}{2i}\oint_{\partial D_1} \Omega'(z)^2 dz = \frac{1}{2i}\oint_{\partial h^{-1}(D_1)} \Omega'(z)^2 \frac{\partial z}{\partial \zeta} d\zeta \\
&= \oint_{\partial h^{-1}(D_1)} \frac{1}{2ic(\zeta-1)^2}\left(2\zeta - 2 + (\zeta-1)^2\frac{E'(\zeta)}{E(\zeta)}\right)^2 d\zeta \\
&= c^{-1}\oint_{\partial h^{-1}(D_1)} 4(\zeta-1)\frac{E'(\zeta)}{E(\zeta)} + (\zeta-1)^2\frac{E'(\zeta)^2}{E^2(\zeta)} d\zeta.
\end{aligned} \tag{B.24}$$

B.3.2 body force on the wall

Although $E'(\zeta)/E(\zeta)$ has a singularity of order 1 at $\zeta = 1$, it is a removable singularity for Ω', as one might notice from (B.12a). Since Ω' has no singularities in $q < |\zeta| \leqslant 1$, we are free to deform the integration along $\partial h(D_2)$ to $\partial h^{-1}(D_1)$, as a consequence of Cauchy's integral theorem.

$$\begin{aligned}
\hat{F}_{2x} - i\hat{F}_{2y} &= -\frac{1}{2i}\int_{\partial D_2} \Omega'(z)^2 dz = -\frac{1}{2i}\oint_{\partial h^{-1}(D_2)} \Omega'(z)^2 \frac{\partial z}{\partial \zeta} d\zeta \\
&= -\frac{1}{2i}\oint_{\partial h^{-1}(D_1)} \Omega'(z)^2 \frac{\partial z}{\partial \zeta} d\zeta = -\hat{F}_{1x} + i\hat{F}_{1y}
\end{aligned} \tag{B.25}$$

§ B.4 Free energies

The net free energy is given by:

$$\hat{\mathcal{E}}_1 = -\frac{1}{4}\mathrm{Im}\left(\oint_{\partial D_1} \overline{\Omega(z)}\Omega'(z)dz\right) + \frac{\omega_1}{4}\mathrm{Re}\left(2\pi b - \oint_{\partial D_1} e^{\Omega(z)-\overline{\Omega(z)}}\frac{e^{2i\phi}}{z'(s)}dz\right). \tag{B.26a}$$

$$\begin{aligned}
\hat{\mathcal{E}}_2 &= -\frac{1}{4}\mathrm{Im}\left(\int_{\partial D_2} \overline{\Omega(z)}\Omega'(z)dz\right) + \frac{\omega_2}{4}\mathrm{Re}\left(\int_{\partial D_2} [1 - e^{\Omega(z)-\overline{\Omega(z)}}]\frac{e^{2i\phi}}{z'(s)}dz\right) \\
&= -\frac{1}{4}\mathrm{Im}\left(\int_{\partial D_2} \overline{\Omega(z)}\Omega'(z)dz\right) + \frac{\omega_2}{4}\mathrm{Re}\left(\int_{\partial D_2} [1 - e^{\Omega(z)-\overline{\Omega(z)}}]dz\right).
\end{aligned} \tag{B.26b}$$

$$\hat{\mathcal{E}} = \hat{\mathcal{E}}_1 + \hat{\mathcal{E}}_2. \tag{B.26c}$$

The energy should be minimized in order to determine the circulation constant Γ.

§ B.5 Asymptotic energy, body force and torque in the far field

From now on, we set $\chi = \pi/2$. When $d \to \infty$, Taylor expansion with respect to $1/d$ gives

$$a(d) = \frac{-b^2\rho^2(b\omega_1)}{4d^2} + \mathcal{O}\left(\frac{1}{d^4}\right) \tag{B.27a}$$

$$q(d) = \frac{b\rho(b\omega_1)}{2d} + \frac{b^3\rho^3(b\omega_1)}{8d^3} + \mathcal{O}\left(\frac{1}{d^4}\right) \tag{B.27b}$$

$$\frac{(1-a)}{4\pi d(1+a)} = \frac{1}{4\pi d} + \frac{b^2\rho^2(b\omega_1)}{8\pi d^3} + \mathcal{O}\left(\frac{1}{d^5}\right) \tag{B.27c}$$

$$\frac{\partial\zeta}{\partial z} = \frac{(1-a)(\zeta-1)^2}{2i(1+a)\cdot d} \tag{B.27d}$$

Note that the circulation is flipped under the mapping $de^{i\chi} \mapsto -de^{i\chi}$, we may assume that $\Gamma \sim \mathcal{O}(1/d)$, denote $\Gamma^\star = \Gamma d$. Hence, when $|\zeta| \sim q \sim 1/d$, we have that

$$\begin{aligned}
E(\zeta) &= \sum_{k=1}^{\infty} \frac{q^{2k}}{\zeta^2}\left(\sum_{j=0}^{\infty}(j+1)\frac{q^{2jk}}{\zeta^j}\right) + \sum_{k=1}^{\infty} q^{2k}\left(\sum_{j=0}^{\infty}(j+1)q^{2jk}\zeta^j\right) \\
&+ \frac{1}{(\zeta-1)^2} - \frac{\Gamma_\star}{4\pi\zeta d^2} - \frac{b^2\rho^2\Gamma_\star}{8\pi\zeta d^4} \\
&= \frac{b^2\rho^2}{4\zeta^2 d^2} + \frac{b^4\rho^4}{8d^4\zeta^3} + \frac{1}{(\zeta-1)^2} - \frac{\Gamma_\star}{4\pi\zeta d^2} - \frac{b^2\rho^2\Gamma_\star}{8\pi\zeta d^4} \\
&+ \frac{3b^6\rho^6 + 12b^4d^2\zeta^2\rho^4 + 16b^2d^4\zeta^4\rho^2}{64d^6\zeta^4} + \frac{b^8\rho^8 + 8b^6d^2\zeta^2\rho^6}{64d^8\zeta^5} \\
&+ \frac{3\left(3b^8\rho^8 + 4b^6d^2\zeta^2\rho^6 + 8b^4d^4\zeta^4\rho^4\right)}{128d^8\zeta^4} + O\left(\frac{1}{d^5}\right), \quad |\zeta| \sim 1/d.
\end{aligned} \tag{B.28}$$

Hence, asymptotic director field $\Omega(z) = \log E(\zeta) + 2\log(\zeta - 1)$ is given by

$$\begin{aligned}
\Omega(z) &= \log\left(\frac{b^2\rho^2}{4d^2\zeta^2} + 1\right) - \frac{\pi\left(-b^4\right)\rho^4 + 4\pi b^2d^2\zeta^2\rho^2 + 2d^2\zeta^2\Gamma_\star}{d\left(2\pi b^2 d\zeta\rho^2 + 8\pi d^3\zeta^3\right)} \\
&+ \frac{\begin{bmatrix} -2d^4\zeta^4\left(8\pi^2b^4\rho^4 + 4\Gamma_\star^2\right) + 64\pi d^6\zeta^6\left(2\pi b^2\rho^2 + 2\Gamma_\star\right) \\ + 4\pi b^4d^2\zeta^2\rho^4\left(6\pi b^2\rho^2 + 2\Gamma_\star\right) - \pi^2b^8\rho^8 \end{bmatrix}}{16\pi^2d^4\zeta^2\left(b^2\rho^2 + 4d^2\zeta^2\right)^2} \\
&+ \frac{\begin{bmatrix} \pi b^{12}\rho^{12} + 12\pi b^{10}d^2\rho^{10}\zeta^2 + 192\pi b^8d^4\rho^8\zeta^4 - 1408\pi b^6d^6\rho^6\zeta^6 \\ + 48b^6d^3\Gamma_\star\rho^6\zeta^3 - 768\pi b^4d^8\rho^4\zeta^8 + 192b^4d^5\Gamma_\star\rho^4\zeta^5 \\ - 3072\pi b^2d^{10}\rho^2\zeta^{10} + 768b^2d^7\Gamma_\star\rho^2\zeta^7 + 3072d^9\Gamma_\star\zeta^9 \end{bmatrix}}{96\pi d^3\left(b^2d\rho^2\zeta + 4d^3\zeta^3\right)^3} \\
&+ \mathcal{O}\left(\frac{1}{d^4}\right), \quad |\zeta| \sim 1/d.
\end{aligned} \tag{B.29}$$

Differentiating (B.29) yields

$$
\begin{aligned}
\frac{\partial\Omega}{\partial\zeta} &= -\frac{2d\left(b^2\rho^2\right)}{\eta\left(b^2\rho^2+4\eta^2\right)} + \frac{-b^6\rho^6-16b^4\eta^2\rho^4+16b^2\eta^4\rho^2}{2\eta^2\left(b^2\rho^2+4\eta^2\right)^2} \\
&+ \frac{\begin{bmatrix}\pi\left(-b^{10}\right)\rho^{10}-12\pi b^8\eta^2\rho^8-192\pi b^6\eta^4\rho^6 \\ +384\pi b^4\eta^6\rho^4-8b^4\eta^3\Gamma_\star\rho^4+128\eta^7\Gamma_\star\end{bmatrix}}{8\pi d\eta^3\left(b^2\rho^2+4\eta^2\right)^3} \\
&+ \frac{\begin{bmatrix}-\pi b^{14}\rho^{14}-16\pi b^{12}\eta^2\rho^{12}-48\pi b^{10}\eta^4\rho^{10} \\ -2688\pi b^8\eta^6\rho^8+4352\pi b^6\eta^8\rho^6-256b^6\eta^5\Gamma_\star\rho^6 \\ -6144\pi b^4\eta^{10}\rho^4-4096\pi b^2\eta^{12}\rho^2+4096b^2\eta^9\Gamma_\star\rho^2\end{bmatrix}}{32\pi d^2\eta^4\left(b^2\rho^2+4\eta^2\right)^4} \\
&+ \mathcal{O}\left(\frac{1}{d^3}\right), \quad |\zeta|\sim 1/d, \quad \eta=\zeta d
\end{aligned}
\tag{B.30}
$$

B.5.1 Asymptotic energy

Since the argument function is defined on $(-\pi,\pi]$, we have that

$$
\begin{aligned}
\overline{\Omega(z)} &= \log\left(\frac{b^2\rho^2}{4d^2\overline{\zeta}^2}+1\right) - \frac{\pi\left(-b^4\right)\rho^4+4\pi b^2d^2\overline{\zeta}^2\rho^2+2d^2\overline{\zeta}^2\Gamma_\star}{d\left(2\pi b^2d\overline{\zeta}\rho^2+8\pi d^3\overline{\zeta}^3\right)} \\
&+ \frac{\begin{bmatrix}-2d^4\overline{\zeta}^4\left(8\pi^2b^4\rho^4+4\Gamma_\star^2\right)+64\pi d^6\overline{\zeta}^6\left(2\pi b^2\rho^2+2\Gamma_\star\right) \\ +4\pi b^4d^2\overline{\zeta}^2\rho^4\left(6\pi b^2\rho^2+2\Gamma_\star\right)-\pi^2b^8\rho^8\end{bmatrix}}{16\pi^2d^4\overline{\zeta}^2\left(b^2\rho^2+4d^2\overline{\zeta}^2\right)^2} \\
&+ \frac{\begin{bmatrix}\pi b^{12}\rho^{12}+12\pi b^{10}d^2\rho^{10}\overline{\zeta}^2+192\pi b^8d^4\rho^8\overline{\zeta}^4-1408\pi b^6d^6\rho^6\overline{\zeta}^6 \\ +48b^6d^3\Gamma_\star\rho^6\overline{\zeta}^3-768\pi b^4d^8\rho^4\overline{\zeta}^8+192b^4d^5\Gamma_\star\rho^4\overline{\zeta}^5 \\ -3072\pi b^2d^{10}\rho^2\overline{\zeta}^{10}+768b^2d^7\Gamma_\star\rho^2\overline{\zeta}^7+3072d^9\Gamma_\star\overline{\zeta}^9\end{bmatrix}}{96\pi d^3\left(b^2d\rho^2\overline{\zeta}+4d^3\overline{\zeta}^3\right)^3} \\
&+ \mathcal{O}\left(\frac{1}{d^4}\right), \quad |\zeta|\sim|\overline{\zeta}|\sim 1/d.
\end{aligned}
\tag{B.31}
$$

Due to (B.15a), on $\partial h^{-1}(D_1)$,

$$
\begin{aligned}
\overline{\zeta} &= \frac{R^2-R^2\zeta-a\zeta-a^2}{\zeta+a-R^2\zeta+R^2} = \frac{b^2}{4\zeta\cdot d^2} \\
&+ \frac{b^4\rho^2-b^4+4b^2d^2\rho^2\zeta^2-4b^2d^2\zeta^2}{16d^4\zeta^2} + \mathcal{O}\left(\frac{1}{d^3}\right).
\end{aligned}
\tag{B.32}
$$

Inserting into (B.31) yields

$$
\begin{aligned}
\overline{\Omega(z)} = & -\frac{2\eta(b^2\rho^2 - 4\eta^2\rho^4)}{d(b^2+4\eta^2\rho^2)} + \log\left(\frac{4\eta^2\rho^2}{b^2}+1\right) \\
& + \frac{\begin{bmatrix} \pi b^6\rho^2 - 4\pi b^4\eta^2\rho^4 + 40\pi b^2\eta^4\rho^6 - 2b^2\eta\Gamma_\star \\ + 32\pi\eta^6\rho^8 - 8\eta^3\Gamma_\star\rho^2 \end{bmatrix}}{2\pi d^2\left(b^2+4\eta^2\rho^2\right)^2} \\
& + \frac{\begin{bmatrix} -3\pi b^{10}\rho^2 - 12\pi b^8\eta^2\rho^4 - 352\pi b^6\eta^4\rho^6 + 12b^6\eta\Gamma_\star \\ + 768\pi b^4\eta^6\rho^8 + 48b^4\eta^3\Gamma_\star\rho^2 + 768\pi b^2\eta^8\rho^{10} \\ + 192b^2\eta^5\Gamma_\star\rho^4 + 1024\pi\eta^{10}\rho^{12} + 768\eta^7\Gamma_\star\rho^6 \end{bmatrix}}{24\pi d^3\eta\left(b^2+4\eta^2\rho^2\right)^3} \\
& + \mathcal{O}\left(\frac{1}{d^4}\right), \quad |\zeta| \sim 1/d, \quad \eta = \zeta d.
\end{aligned} \tag{B.33}
$$

From a simple obeservation of (B.33) and (B.30), we know that the principle part of $\overline{\Omega(z)}$ has no singularities within $h^{-1}(D_1)$, and the principle part of $\partial\Omega/\partial\zeta$ admits singularities at only $\zeta = 0$ and the two asymptotic topological defects $\zeta_1, \zeta_2 = \pm ib\rho/(2d)$. We perform the Laurent expansion of $\partial\Omega/\partial\zeta$ and $\overline{\Omega(z)}$ at a neighborhood of $\zeta = 0$, with $|\zeta| \sim 1/d$.

$$
\frac{\partial\Omega}{\partial\zeta} = -\frac{b^2\rho^2}{2d^2\zeta^2} - \frac{2}{\zeta} + \sum_{k=0}^{\infty} a_k\zeta^k + \mathcal{O}\left(\frac{1}{d^3}\right), \tag{B.34a}
$$

$$
\overline{\Omega(z)} = \frac{\pi b^2\rho^2 + \Gamma_\star}{2\pi d^2} - \frac{4\zeta^2 d^2\left(2\pi b^2\rho^2 + \Gamma_\star\right)}{\pi b^4\left(\rho^2-1\right)} + \sum_{k=3}^{\infty} b_k\zeta^k + \mathcal{O}\left(\frac{1}{d^4}\right). \tag{B.34b}
$$

Similarly, Laurent expansions can be calculated at ζ_1 and ζ_2. Thus, we may compute the first term of the asymptotic energy as

$$
\begin{aligned}
& -\frac{1}{4}\mathrm{Im}\left(\oint_{\partial D_1} \overline{\Omega(z)}\Omega'(z)dz\right) = -\frac{1}{4}\mathrm{Im}\left(\oint_{\partial h^{-1}(D_1)} \overline{\Omega(h(\zeta))}\frac{\partial\Omega}{\partial\zeta}d\zeta\right) \\
& = -\frac{1}{4}\mathrm{Im}\left(2\pi i\sum_{j=0}^{2} \mathrm{Res}\left[\overline{\Omega(h(\zeta))}\frac{\partial\Omega}{\partial\zeta}, \zeta = \zeta_j\right]\right) \\
& = -\pi\log(1-\rho^4) + \frac{\pi\rho^2 b^2(7\rho^2-1)}{4(\rho^2+1)d^2} + \frac{\Gamma_\star^2 + 2b^2\pi\Gamma_\star(\rho^2+1)^2}{8b^2\pi(\rho^2+1)^2 d^2} + \mathcal{O}\left(\frac{1}{d^4}\right).
\end{aligned} \tag{B.35}
$$

For the second term of the aymptotic energy, we start by inserting (B.32) into (B.28) multiplied by $(\zeta-1)^2$ to obtain $e^{\overline{\Omega(z)}}$:

$$
\begin{aligned}
e^{\overline{\Omega(z)}} = {} & 1 - \frac{\zeta\left(2\pi b^2\rho^4 - 8\pi d^2\rho^2\zeta^2 + \Gamma_\star\right)}{\pi b^2} \\
& + \frac{4d^2\rho^2\zeta^2}{b^2} + \frac{\begin{bmatrix} 6\pi b^4\rho^6 - 8\pi b^4\rho^4 + 6\pi b^4\rho^2 - 40\pi b^2 d^2\rho^4\zeta^2 \\ + 32\pi b^2 d^2\rho^2\zeta^2 + 2b^2\Gamma_\star\rho^2 + 2b^2\Gamma_\star \\ + 96\pi d^4\rho^2\zeta^4 + 8d^2\Gamma_\star\rho^2\zeta^2 - 8d^2\Gamma_\star\zeta^2 \end{bmatrix}}{8\pi b^2 d^2} \\
& + \frac{\begin{bmatrix} -8\pi b^6\rho^8 + 20\pi b^6\rho^6 - 16\pi b^6\rho^4 + 80\pi b^4 d^2\rho^6\zeta^2 - 176\pi b^4 d^2\rho^4\zeta^2 \\ + 64\pi b^4 d^2\rho^2\zeta^2 - 2b^4\Gamma_\star\rho^4 + 4b^4\Gamma_\star\rho^2 - 4b^4\Gamma_\star + 64\pi b^2 d^4\rho^6\zeta^4 \\ - 320\pi b^2 d^4\rho^4\zeta^4 + 320\pi b^2 d^4\rho^2\zeta^4 - 16b^2 d^2\Gamma_\star\rho^4\zeta^2 + 16b^2 d^2\Gamma_\star\rho^2\zeta^2 \\ - 16b^2 d^2\Gamma_\star\zeta^2 + 512\pi d^6\rho^2\zeta^6 - 32d^4\Gamma_\star\rho^4\zeta^4 + 64d^4\Gamma_\star\rho^2\zeta^4 - 32d^4\Gamma_\star\zeta^4 \end{bmatrix}}{32\pi b^2 d^4\zeta} \\
& + \mathcal{O}\left(\frac{1}{d^4}\right), \quad |\zeta| \sim 1/d.
\end{aligned} \tag{B.36}
$$

By (B.13a) and (B.8d), we compute the aymptotics for $z'(s) = e^{i\phi}$ and $\partial z/\partial\zeta$ as the following

$$
\begin{aligned}
e^{i\phi} &= \frac{2d}{b(1-a)}\frac{\zeta + a}{\zeta - 1} = \frac{b^2\rho^2 - 4d^2\zeta^2}{2bd} + \frac{b^2 d\rho^2\zeta - 2d^3\zeta^3}{bd^2} \\
&\quad - \frac{2d\zeta}{b} + \frac{b^4\rho^4 + 8b^2d^2\rho^2\zeta^2 - 16d^4\zeta^4}{8bd^3} + \mathcal{O}\left(\frac{1}{d^4}\right),
\end{aligned} \tag{B.37a}
$$

$$
\begin{aligned}
\frac{\partial z}{\partial\zeta} &= 2id\frac{1+a}{1-a}\cdot\frac{1}{(\zeta-1)^2} \\
&= \frac{i\left(6d^2\zeta^2 - b^2\rho^2\right)}{d} + \frac{2i\left(4d^3\zeta^3 - b^2 d\rho^2\zeta\right)}{d^2} + 4id\zeta + 2id \\
&\quad + \frac{i\left(-b^4\rho^4 - 12b^2d^2\rho^2\zeta^2 + 40d^4\zeta^4\right)}{4d^3} + \mathcal{O}\left(\frac{1}{d^4}\right).
\end{aligned} \tag{B.37b}
$$

An obeservation of (B.33) implies that the principle part of $e^{-\overline{\Omega(z)}}$ admits singularities at $\zeta = 0$ and $\zeta = \pm ib/(2\rho d)$, where $\pm ib/(2\rho d) \notin h^{-1}(D_1)$. The principle part of $e^{\Omega(z)}$ admits a singularity at only $\zeta = 0$ within $h^{-1}(D_1)$, $e^{i\phi}$ and $\partial z/\partial\zeta$ are non-singular, we can apply Laurent expansion to the integrand for the second part of energy integration at $\zeta = 0$ and apply the Residue theorem to obtain

$$
\begin{aligned}
&\frac{\omega_1}{4}\mathrm{Re}\left(2\pi b - \oint_{\partial D_1} e^{\Omega(z) - \overline{\Omega(z)}}\frac{e^{2i\phi}}{z'(s)}dz\right) \\
&= \frac{\omega_1}{4}\mathrm{Re}\left(2\pi b - \oint_{\partial h^{-1}(D_1)} e^{\Omega(z) - \overline{\Omega(z)}}\frac{e^{2i\phi}}{z'(s)}\frac{\partial z}{\partial\zeta}d\zeta\right) \\
&= \frac{\omega_1}{4}\mathrm{Re}\left[2\pi b - 2\pi i\mathrm{Res}\left(e^{\Omega(z) - \overline{\Omega(z)}}\frac{e^{2i\phi}}{z'(s)}\frac{\partial z}{\partial\zeta}, \zeta = 0\right)\right] \\
&= \frac{\pi\omega_1 b(1-\rho^2)}{2} + \frac{\pi\omega_1 b^3\rho^4(\rho^2-1)^2}{4d^2} + \mathcal{O}\left(\frac{1}{d^4}\right).
\end{aligned} \tag{B.38}
$$

We proceed to calculate the energy induced by the boundary wall. Firstly, for $z \in \partial D_2$, $\overline{\zeta} = \zeta - 4i/\omega_2$. We then write

$$\begin{aligned}
\overline{\zeta} &= h^{-1}(\overline{z}) = h^{-1}(h(\zeta) - 4i/\omega_2) \\
&= \frac{1}{\zeta} + \frac{(\zeta-1)^2\left(b^2\omega_2^2\rho^2\zeta^2 + 8\zeta^2 - 16\zeta + 8\right)}{d^3\omega_2^3\zeta^4} \\
&\quad - \frac{4\left(\zeta^3 - 3\zeta^2 + 3\zeta - 1\right)}{d^2\omega_2^2\zeta^3} + \frac{2(\zeta-1)^2}{d\omega_2\zeta^2} + \mathcal{O}\left(\frac{1}{d^4}\right).
\end{aligned} \tag{B.39}$$

Hence, for $|\zeta| \sim 1$, we compute that

$$\Omega(z) = \frac{(\zeta-1)^2\left(\pi b^2\rho^2\left(\zeta^2+1\right) - \Gamma_\star\zeta\right)}{4\pi d^2\zeta^2} + \mathcal{O}\left(\frac{1}{d^4}\right), \tag{B.40a}$$

$$\frac{\partial\Omega}{\partial\zeta} = \frac{2\pi b^2\rho^2 + \Gamma_\star}{4\pi d^2\zeta^2} - \frac{2\pi b^2\rho^2 + \Gamma_\star}{4\pi d^2} - \frac{b^2\rho^2}{2d^2\zeta^3} + \frac{b^2\rho^2\zeta}{2d^2} + \mathcal{O}\left(\frac{1}{d^4}\right), \tag{B.40b}$$

$$\begin{aligned}
\overline{\Omega(z)} &= \frac{\pi b^2 d\omega_2\rho^2 - 12\pi b^2\rho^2 - 2\Gamma_\star}{4\pi d^3\omega_2\zeta^2} + \frac{-2\pi b^2 d\omega_2\rho^2 + 12\pi b^2\rho^2 - d\Gamma_\star\omega_2 + 4\Gamma_\star}{4\pi d^3\omega_2\zeta} \\
&\quad - \frac{b^2\rho^2\zeta^3}{d^3\omega_2} + \frac{b^2\rho^2}{d^3\omega_2\zeta^3} + \frac{\pi b^2\rho^2 + \Gamma_\star}{2\pi d^2} + \zeta^2\left(\frac{6\pi b^2\rho^2 + \Gamma_\star}{2\pi d^3\omega_2} + \frac{b^2\rho^2}{4d^2}\right) \\
&\quad - \zeta\left(\frac{3\pi b^2\rho^2 + \Gamma_\star}{\pi d^3\omega_2} + \frac{2\pi b^2\rho^2 + \Gamma_\star}{4\pi d^2}\right) + \mathcal{O}\left(\frac{1}{d^4}\right).
\end{aligned} \tag{B.40c}$$

(B.40d)

Together with

$$\frac{\partial z}{\partial\zeta} = -\frac{ib^4\rho^4}{4d^3(\zeta-1)^2} - \frac{ib^2\rho^2}{d(\zeta-1)^2} + \frac{2id}{(\zeta-1)^2} + \mathcal{O}\left(\frac{1}{d^5}\right), \tag{B.41a}$$

$$\begin{aligned}
e^{\Omega(z)} &= 1 + \frac{b^2\rho^2(\zeta-1)^2}{4\zeta^2 d^2} + \frac{b^2\rho^2(\zeta-1)^2}{4d^2} - \frac{\Gamma_\star(\zeta-1)^2}{4\pi\zeta d^2} \\
&\quad + \frac{(\zeta-1)^2\begin{pmatrix}2\pi b^4\rho^4\zeta^4 + 3\pi b^4\rho^4\zeta^3 + 3\pi b^4\rho^4\zeta \\ + 2\pi b^4\rho^4 - 2b^2\Gamma_\star\rho^2\zeta^2\end{pmatrix}}{16\pi d^4\zeta^3} + \mathcal{O}\left(\frac{1}{d^5}\right),
\end{aligned} \tag{B.41b}$$

$$
\begin{aligned}
e^{-\overline{\Omega(z)}} &= 1 + \frac{(\zeta-1)^3\left(2\pi b^2\rho^2\zeta^3 + 2\pi b^2\rho^2 - \Gamma_\star\zeta^2 - \Gamma_\star\zeta\right)}{2\pi d^3\omega_2\zeta^3} \\
&+ \frac{(\zeta-1)^4\left(\pi b^2\rho^2\zeta^2 + \pi b^2\rho^2 - \Gamma_\star\zeta\right)^2}{32\pi^2\zeta^4 d^4} \\
&- \frac{(\zeta-1)^2\begin{pmatrix} 2\pi b^4\omega_2^2\rho^4\zeta^5 + 3\pi b^4\omega_2^2\rho^4\zeta^4 + 3\pi b^4\omega_2^2\rho^4\zeta^2 \\ - 2b^2\Gamma_\star\omega_2^2\rho^2\zeta^3 + 48\pi b^2\rho^2\zeta^6 - 96\pi b^2\rho^2\zeta^5 \\ + 48\pi b^2\rho^2\zeta^2 - 96\pi b^2\rho^2\zeta + 48\pi b^2\rho^2 + 48\pi b^2\rho^2\zeta^4 \\ - 16\Gamma_\star\zeta^5 + 16\Gamma_\star\zeta^4 + 16\Gamma_\star\zeta^2 - 16\Gamma_\star\zeta + 2\pi b^4\omega_2^2\rho^4\zeta \end{pmatrix}}{16\pi\omega_2^2\zeta^4 d^4} \\
&- \frac{(\zeta-1)^2\left(\pi b^2\rho^2\zeta^2 + \pi b^2\rho^2 - \Gamma_\star\zeta\right)}{4\pi d^2\zeta^2} + \mathcal{O}\left(\frac{1}{d^5}\right).
\end{aligned} \tag{B.41c}
$$

The principle parts of $\overline{\Omega(z)}$ and $\partial\Omega/\partial\zeta$ admit singularities within $\partial h^{-1}(D_2)$ only at $\zeta = 0$. Besides, the principle parts of $e^{-\overline{\Omega(z)}}$ and $e^{\Omega(z)}$ are singular only at $\zeta = 0$ within $\partial h^{-1}(D_2)$. Thus, we evaluate that

$$
\begin{aligned}
&-\frac{1}{4}\mathrm{Im}\left(\int_{\partial D_2}\overline{\Omega(z)}\Omega'(z)dz\right) = -\frac{1}{4}\mathrm{Im}\left(\oint_{\partial h^{-1}(D_2)}\overline{\Omega(z)}\frac{\partial\Omega}{\partial\zeta}d\zeta\right) \\
&= -\frac{1}{4}\mathrm{Im}\left[2\pi i\mathrm{Res}\left(\overline{\Omega(z)}\frac{\partial\Omega}{\partial\zeta}, \zeta = 0\right)\right] \\
&= \frac{6\pi b^4\rho^4}{2d^5\omega_2} + \frac{3b^2\Gamma_\star\rho^2}{4d^5\omega_2} + \frac{\Gamma_\star^2}{4\pi d^5\omega_2} = \mathcal{O}\left(\frac{1}{d^5}\right).
\end{aligned} \tag{B.42a}
$$

$$
\begin{aligned}
&\frac{\omega_2}{4}\mathrm{Re}\left(\int_{\partial D_2}[1 - e^{\Omega(z)-\overline{\Omega(z)}}]dz\right) = \frac{\omega_2}{4}\mathrm{Re}\left(\int_{\partial h^{-1}(D_2)}[1 - e^{\Omega(z)-\overline{\Omega(z)}}]\frac{\partial z}{\partial\zeta}d\zeta\right) \\
&= \frac{\omega_2}{4}\mathrm{Re}\left[2\pi i\mathrm{Res}\left([1 - e^{\Omega(z)-\overline{\Omega(z)}}]\frac{\partial z}{\partial\zeta}, \zeta = 0\right)\right] \\
&= -\frac{\begin{pmatrix} 32\pi^3 b^6\omega_2^2\rho^6 + 88\pi^2 b^4\Gamma_\star\omega_2^2\rho^4 - 1536\pi^3 b^4\rho^4 \\ + 64\pi b^2\Gamma_\star^2\omega_2^2\rho^2 - 768\pi^2 b^2\Gamma_\star\rho^2 + 24\Gamma_\star^3\omega_2^2 - 128\pi\Gamma_\star^2 \end{pmatrix}}{256\pi^2\omega_2 d^5} = \mathcal{O}\left(\frac{1}{d^5}\right).
\end{aligned} \tag{B.42b}
$$

By (B.26), the asymptotic energy is given by

$$
\begin{aligned}
\hat{\mathcal{E}} &= \frac{\pi\omega_1 b(1-\rho^2)}{2} - \pi\log(1-\rho^4) + \frac{\pi\omega_1 b^3\rho^4(\rho^2-1)^2}{4d^2} + \frac{\pi\rho^2 b^2(7\rho^2-1)}{4(\rho^2+1)d^2} \\
&+ \frac{\Gamma_\star^2 + 2b^2\pi\Gamma_\star(\rho^2+1)^2}{8b^2\pi(\rho^2+1)^2 d^2} + \mathcal{O}\left(\frac{1}{d^4}\right).
\end{aligned} \tag{B.43}
$$

Therefore, the minimized aymptotic energy is given by

$$
\begin{aligned}
\hat{\mathcal{E}}_{\min} &= \frac{\pi\omega_1 b(1-\rho^2)}{2} - \pi\log(1-\rho^4) + \frac{\pi\omega_1 b^3\rho^4(\rho^2-1)^2}{4d^2} \\
&+ \frac{\pi\rho^2 b^2(7\rho^2-1)}{4(\rho^2+1)d^2} - \frac{\pi b^2(\rho^2+1)^2}{8d^2} + \mathcal{O}\left(\frac{1}{d^4}\right),
\end{aligned} \tag{B.44a}
$$

$$\Gamma_{\text{min}} = \frac{\Gamma_\star}{d} = \frac{-\pi b^2(\rho^2+1)^2}{d}. \tag{B.44b}$$

Now that we secured the asymptotic for Γ_{min} which minimizes the net free energy in the far field, we substitute it into the asymptotic director field to derive asymptotic body forces and torques.

B.5.2 Asymptotic director field

For $q \ll \zeta| < 1$, The asymptotic expansion for $E(\zeta)$ is given by

$$\begin{aligned}
E(\zeta) &= \sum_{k=-\infty}^{\infty}\left[\frac{q^{2k}}{(\zeta-q^{2k})^2}\right] - \frac{\Gamma_{\text{min}}(1-a)}{4\pi d(1+a)\zeta} \\
&= \frac{1}{(\zeta-1)^2} + \sum_{k=1}^{\infty}\frac{q^{2k}}{(\zeta-q^{2k})^2} + \sum_{k=1}^{\infty}\frac{q^{2k}}{(1-q^{2k}\zeta)^2} - \frac{\Gamma_{\text{min}}(1-a)}{4\pi d(1+a)\zeta} \\
&= \sum_{k=1}^{\infty}\frac{q^{2k}}{\zeta^2}\left(\sum_{j=0}^{\infty}(j+1)\frac{q^{2jk}}{\zeta^j}\right) + \sum_{k=1}^{\infty}q^{2k}\left(\sum_{j=0}^{\infty}(j+1)q^{2jk}\zeta^j\right) \\
&+ \frac{1}{(\zeta-1)^2} - \frac{\Gamma_{\text{min}}}{4\pi\zeta d} - \frac{b^2\rho^2(b\omega_1)\Gamma_{\text{min}}}{8\pi\zeta d^3} \\
&= \frac{b^2\left(\rho^4\zeta+\rho^2(\zeta+1)^2+\zeta\right)}{4d^2\zeta^2} + \frac{1}{(\zeta-1)^2} \\
&+ \frac{b^4\rho^2\left(2\rho^4\zeta^2+\rho^2\left(2\zeta^4+3\zeta^3+4\zeta^2+3\zeta+2\right)+2\zeta^2\right)}{16d^4\zeta^3} + \mathcal{O}\left(\frac{1}{d^6}\right).
\end{aligned} \tag{B.45}$$

Then, we get the asymptotic director field

$$\begin{aligned}
\Omega(z) &= \log(E(\zeta)) + 2\log(\zeta-1) \\
&= \frac{b^2(\zeta-1)^2\left(\rho^4\zeta+\rho^2(\zeta+1)^2+\zeta\right)}{4d^2\zeta^2} + \frac{(\zeta-1)^2}{32d^4\zeta^2} \\
&- \frac{b^4(\zeta-1)^2\begin{pmatrix} 2\rho^2\left(\zeta^5-4\zeta^3+\zeta\right) \\ +\rho^8(\zeta-1)^2\zeta^2+2\rho^6\left(\zeta^5-4\zeta^3+\zeta\right) \\ +\rho^4\left(\zeta^6-2\zeta^5-5\zeta^4-16\zeta^3-5\zeta^2-2\zeta+1\right)\end{pmatrix}}{32d^4\zeta^4} \\
&+ \mathcal{O}\left(\frac{1}{d^6}\right), \quad q \ll |\zeta| < 1.
\end{aligned} \tag{B.46}$$

B.5.3 Asymptotic force and torque

We take the derivative with respect to ζ in (B.46)

$$\begin{aligned}
\frac{\partial\Omega}{\partial\zeta} &= \frac{b^2(\zeta-1)^2\left(\rho^4+2\rho^2(\zeta+1)+1\right)}{4\zeta^2d^2} + \frac{b^2(\zeta-1)\left(\rho^4\zeta+\rho^2(\zeta+1)^2+\zeta\right)}{2\zeta^2d^2} \\
&- \frac{b^2(\zeta-1)^2\left(\rho^4\zeta+\rho^2(\zeta+1)^2+\zeta\right)}{2\zeta^3d^2} + \mathcal{O}\left(\frac{1}{d^4}\right), \quad q \ll |\zeta| < 1.
\end{aligned} \tag{B.47}$$

By (B.27d), we have that

$$\frac{\partial\zeta}{\partial z}=-\frac{i(\zeta-1)^2}{2d}-\frac{ib^2\rho^2(\zeta-1)^2}{4d^3}+\mathcal{O}\left(\frac{1}{d^4}\right),\quad q\ll|\zeta|<1. \tag{B.48}$$

To evaluate the body force, we first apply Laurent expansion to $\partial\Omega/\partial\zeta$ and $\partial\zeta/\partial z$ at the origin respectively,

$$\frac{\partial\Omega}{\partial\zeta}=-\frac{b^2\left(\rho^4+1\right)}{4d^2\zeta^2}-\frac{b^2\rho^2}{2d^2\zeta^3}+\sum_{k=0}^{\infty}a_k\zeta^k+\mathcal{O}\left(\frac{1}{d^4}\right), \tag{B.49a}$$

$$\begin{aligned}\frac{\partial\zeta}{\partial z}=&-\frac{i\zeta^2\left(b^2\rho^2+2d^2\right)}{4d^3}+\frac{i\zeta\left(b^2\rho^2+2d^2\right)}{2d^3}-\frac{i\left(b^2\rho^2+2d^2\right)}{4d^3}\\&+\sum_{j=4}b_j\zeta^j+\mathcal{O}\left(\frac{1}{d^4}\right).\end{aligned} \tag{B.49b}$$

Note that $\Omega'(z)$ is analytic in $q<|\zeta|<1$, we may deform the contour of integration from $\partial h^{-1}(D_1)$ to $|\zeta|=(1+q)/2$, where we could apply all the above estimates. By (B.47), the principle part of $\partial\Omega/\partial\zeta$ only admits a singularity at $\zeta=0$ within $\{\zeta:|\zeta|\leqslant(1+q)/2\}$, while $\partial\zeta/\partial$ is analytic. The aymptotic body force $\hat{F}_{1x}-i\hat{F}_{1y}=-\hat{F}_{2x}+i\hat{F}_{2y}$ is calculated as

$$\begin{aligned}&\frac{1}{2i}\oint_{\partial D_1}\Omega'(z)^2dz=\frac{1}{2i}\oint_{\partial h^{-1}(D_1)}\left(\frac{\partial\Omega}{\partial\zeta}\right)^2\frac{\partial\zeta}{\partial z}d\zeta=\frac{1}{2i}\oint_{|\zeta|=(1+q)/2}\left(\frac{\partial\Omega}{\partial\zeta}\right)^2\frac{\partial\zeta}{\partial z}d\zeta\\&=\pi\mathrm{Res}\left[\left(\frac{\partial\Omega}{\partial\zeta}\right)^2\frac{\partial\zeta}{\partial z},\zeta=0\right]=-\frac{i\pi b^4\left[(\rho^4+1)(\rho^2-1)^2+4\rho^4\right]}{8d^5}+\mathcal{O}\left(\frac{1}{d^7}\right).\end{aligned} \tag{B.50}$$

By (B.6b),

$$\begin{aligned}&z-d=2id\frac{\zeta+a}{(1-a)(1-\zeta)}.\\&\Rightarrow\frac{\partial\zeta}{\partial z}(z-d)=\frac{(1-a)(\zeta-1)^2}{2id(1+a)}(z-d)=\frac{(\zeta-1)^2}{1+a}+\zeta-1\\&=\zeta(1-\zeta)+\frac{b^2\rho^2(\zeta-1)^2}{4\cdot d^2}+\mathcal{O}\left(\frac{1}{d^4}\right)\\&=\zeta^2\left(\frac{b^2\rho^2}{4d^2}+1\right)+\zeta\left(-\frac{b^2\rho^2}{2d^2}-1\right)+\frac{b^2\rho^2}{4d^2}+\sum_{k=4}^{\infty}c_k\zeta^k+\mathcal{O}\left(\frac{1}{d^4}\right).\end{aligned} \tag{B.51}$$

By a similar argument as above, we shift the contour of integration and apply the Residue theorem to evaluate the torque as

$$
\begin{aligned}
\hat{T}_1 &= \frac{1}{2}\mathrm{Re}\left(\oint_{\partial D_1}(z-d)\Omega'(z)^2 dz\right) = \frac{1}{2}\mathrm{Re}\left[\oint_{\partial h^{-1}(D_1)}\left(\frac{\partial\Omega}{\partial\zeta}\right)^2\frac{\partial\zeta}{\partial z}(z-d)d\zeta\right] \\
&= \frac{1}{2}\mathrm{Re}\left[\oint_{\partial|\zeta|=(1+q)/2}\left(\frac{\partial\Omega}{\partial\zeta}\right)^2\frac{\partial\zeta}{\partial z}(z-d)d\zeta\right] \\
&= \mathrm{Re}\left(\pi i\mathrm{Res}\left[\left(\frac{\partial\Omega}{\partial\zeta}\right)^2\frac{\partial\zeta}{\partial z}(z-d),\zeta=0\right]\right) = \mathcal{O}\left(\frac{1}{d^4}\right),
\end{aligned}
\tag{B.52}
$$

because all the coefficients from the principle parts of $\partial\Omega/\partial\zeta$ and $\partial\zeta/\partial z(z-d)$ are real, which yields a real residue of the integrand at $\zeta = 0$.

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