Hardy-Littlewood Maximal Inequalities and Their Applications in Interpolation Problems

Junhao Yin

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Abstract

This project delves into the Hardy-Littlewood maximal inequalities, both strong and weak forms, in the context of real analysis. By systematically deriving and analyzing these inequalities, we investigate their applications, including their extension to function spaces such as $L^p(\mathbb{R}^n)$ and the characterization of singular integral operators. Furthermore, the study provides insights into related inequalities and their significance in the broader framework of harmonic analysis and PDEs.

§ 1 Introduction

The Hardy-Littlewood maximal inequalities represent foundational results in real analysis and harmonic analysis, offering powerful tools for understanding the behavior of functions and their transformations under maximal operators. These inequalities are instrumental in establishing bounds for integrals and serve as a bridge to advanced results in harmonic analysis.

The strong form of the Hardy-Littlewood inequality ensures control over maximal functions in L^p spaces, while the weak form extends applicability to broader settings. Together, they provide a comprehensive toolkit for analyzing functions in real and functional analysis.

This project focuses on deriving these inequalities, emphasizing their geometric and analytical underpinnings. The study begins with the Hardy-Littlewood maximal operator, progresses to proving its weak and strong L^p -type bounds, and explores applications to problems in harmonic analysis and PDEs, including interpolation theorems and singular integral operator bounds.

This exploration showcases the enduring relevance of the Hardy-Littlewood maximal inequalities in mathematical analysis, revealing their critical role in understanding function spaces and integral operators.

§ 2 Hardy-Littlewood inequality for uncentered balls

Suppose $f:\mathbb{R}^n \to \mathbb{R} \in L^1(\mathbb{R}^n), \mathcal{M}f(x) := \sup\limits_{r>0} \dfrac{1}{|B(x,r)|}\displaystyle\int_{B(x,r)} |f|dm.$
$\mathcal{N}f(x) := \sup\limits_{t>0}\{\dfrac{1}{t}\displaystyle\int_{B_t} |f|dm, B_t = B(y,r) \subseteq \mathbb{R}^n, m(B_t) = t, x \in B_t\}$
show that $\lambda \cdot m(\{\mathcal{N}f(x) > \lambda\}) \leqslant 6^n \|f\|_{L^1(\mathbb{R}^n)}, \forall \lambda > 0.$

Proof. $\forall t > 0, x \in B_t = B(y,r), B_t \subset B(x, |y-x| + r) \subset B(x, 2r)$. Thus we have the estimate:

$$\int_{B_t} |f|dm \leqslant \int_{B(x,2r)} |f|dm$$

Therefore

$$\frac{1}{t}\int_{B_t} |f|dm \leqslant 2^n \frac{1}{|B(x,2r)|}\int_{B(x,2r)} |f|dm \leqslant 2^n \mathcal{M}f(x)$$

Take the sup with respect to t on the left hand side, then we get the inequality $\mathcal{N}f(x) \leqslant 2^n \mathcal{M}f(x)$. Then observe that $\{\mathcal{N}f(x) > \lambda\} \subset \{\mathcal{M}f(x) > 2^{-n}\lambda\}$. So, $m(\{\mathcal{N}f(x) > \lambda\}) \leqslant m(\{\mathcal{M}f(x) > 2^{-n}\lambda\})$. Notice that $\lambda \cdot m(\{\mathcal{M}f(x) > \lambda\}) \leqslant 3^n \|f\|_{L^1(\mathbb{R}^n)}$. Finally, we conclude $\lambda \cdot m(\{\mathcal{N}f(x) > \lambda\}) \leqslant \lambda \cdot m(\{\mathcal{M}f(x) > 2^{-n}\lambda\}) \leqslant 6^n \|f\|_{L^1(\mathbb{R}^n)}, \forall \lambda > 0.$

§ 3 (Strong) Hardy-Littlewood maximal inequality

Suppose $f:\mathbb{R}^n \to \mathbb{R}, f \in L^p(\mathbb{R}^n) \cap L^1(\mathbb{R}^n), 1 < p \leqslant \infty$. Show that

exists $C_{p,n} > 0, s.t. \|\mathcal{M}f\|_{L^p(\mathbb{R}^n)} \leqslant C_{p,n}\|f\|_{L^p(\mathbb{R}^n)}$.

Proof. Set $g_t(x) := f(x)\chi_{\{|f(x)|>t/2\}}(x), \varphi_t(x) := f(x)\chi_{\{|f(x)\leqslant t/2\}}(x)$, $\forall 0 < \varepsilon < 1$. $f(x) = g_t(x)+\varphi_t(x)$ implies $\mathcal{M}f(x) \leqslant \mathcal{M}g_t(x)+\mathcal{M}\varphi_t(x)$. Apparently, $\mathcal{M}\varphi_t(x) \leqslant t/2$. for any $x \in \{\mathcal{M}f(x) > t\}, \mathcal{M}g_t(x) \geqslant \mathcal{M}f(x)-\mathcal{M}\varphi_t(x) > t/2$. So $m(\{\mathcal{M}f(x) > t\}) \leqslant m(\{\mathcal{M}g_t(x) > t/2\})$. Apply the weak estimate to $m(\{\mathcal{M}g_t(x) > t/2\})$ to get:

$$m(\{\mathcal{M}f(x) > t\}) \leqslant \frac{2\cdot 3^n}{t}\|g_t\|_{L^1(\mathbb{R}^n)} = \frac{2\cdot 3^n}{t}\int_{\{|f(x)|>t/2\}} |f|dm$$

Consider shifting the integrand with distribution function uisng the Fubini-Tonelli theorem:

$$\begin{aligned}
LHS &= \|\mathcal{M}f\|^p_{L^p(\mathbb{R}^n)} = \int_{\mathbb{R}^n} |\mathcal{M}f(x)|^p dx = \int_{\mathbb{R}^n}\int_0^{\mathcal{M}f(x)} pt^{p-1}dtdx \\
&= \int_{\mathbb{R}^n}\int_0^\infty pt^{p-1}\chi_{\{0\leqslant t\leqslant \mathcal{M}f(x)\}}(t)dtdx \\
&= \int_0^\infty\int_{\mathbb{R}^n} pt^{p-1}\chi_{\{\mathcal{M}f(x)>t\}}(x)dxdt \\
&= \int_0^\infty pt^{p-1}m(\{\mathcal{M}f(x) > t\})dt \\
&\leqslant \int_0^\infty pt^{p-1}\left(\frac{2\cdot 3^n}{t}\int_{\{|f(x)|>t/2\}} |f(x)|dx\right)dt \\
&= 2\cdot 3^n p\int_0^\infty t^{p-2}\int_{\mathbb{R}^n} \chi_{\{|f(x)|>t/2\}}(x)|f(x)dxdt \\
&= 2\cdot 3^n p\int_{\mathbb{R}^n}\int_0^\infty t^{p-2}\chi_{\{t<2|f(x)|\}}(t)|f(x)|dtdx \\
&= \frac{3^n 2^p p}{p-1}\int_{\mathbb{R}^n} |f(x)|^{p-1}|f(x)|dx := C^p_{p,n}\|f\|^p_{L^p(\mathbb{R}^n)} = RHS
\end{aligned}$$

Therefore, $\|\mathcal{M}f\|_{L^p(\mathbb{R}^n)} \leqslant C_{p,n}\|f\|_{L^p(\mathbb{R}^n)}, \quad \text{where } C_{p,n} = \left(\frac{3^n 2^p p}{p-1}\right)^{1/p}.$

§ 4 (Weak) Hardy-Littlewood maximal inequality

Suppose $f : \mathbb{R}^n \to \mathbb{R} \in L^1(\mathbb{R}^n), E \subset \mathbb{R}^n, m(E) < \infty$. Then for any

$0<q<1$, show that $\|\mathcal{M}f(x)\|_{L^q(E)}^q \leqslant C_q m(E)^{1-q}\|f\|_{L^1(\mathbb{R}^n)}^q$, $C_q>0$.

Proof. Without loss of generality, we assume $m(E)>0$ and $\|f\|_{L^1(\mathbb{R}^n)}>0$. Using the same trick as above,we have:

$$
\begin{aligned}
& \|\mathcal{M}f(x)\|_{L^q(E)}^q = \int_E |\mathcal{M}f(x)|^q dx = \int_E\int_0^{\mathcal{M}f(x)} qt^{q-1}dtdx \\
= & \int_E\int_0^\infty qt^{q-1}\chi_{\{0<t<\mathcal{M}f(x)\}}(t)dtdx = \int_0^\infty\int_E qt^{q-1}\chi_{\{\mathcal{M}f(x)>t\}}(x)dxdt \\
= & \int_0^\infty qt^{q-1}m(\{E\cap\mathcal{M}f(x)>t\})dt \\
\leqslant & \int_0^\infty qt^{q-1}min\{m(E), m(\{\mathcal{M}f(x)>t\})\}dt \\
= & \int_0^A + \int_A^\infty qt^{q-1}min\{m(E), \frac{3^n}{t}\|f\|_{L^1(\mathbb{R}^n)}\}dt \\
= & \int_0^A q3^{n(q-\delta)}t^{\delta-1}m(E)^{1-q+\delta}\|f\|_{L^1(\mathbb{R}^n)}^{q-\delta}dt \\
+ & \int_A^\infty q3^{n(q+\delta)}t^{-\delta-1}m(E)^{1-q-\delta}\|f\|_{L^1(\mathbb{R}^n)}^{q+\delta}dt \\
= & \frac{q}{\delta}A^\delta 3^{n(q-\delta)}\left(\frac{m(E)}{\|f\|_{L^1(\mathbb{R}^n)}}\right)^\delta m(E)^{1-q}\|f\|_{L^1(\mathbb{R}^n)}^q \\
+ & \frac{q}{\delta}A^{-\delta}3^{n(q+\delta)}\left(\frac{\|f\|_{L^1(\mathbb{R}^n)}}{m(E)}\right)^\delta m(E)^{1-q}\|f\|_{L^1(\mathbb{R}^n)}^q \\
= & \frac{2q}{\delta}3^{nq}m(E)^{1-q}\|f\|_{L^1(\mathbb{R}^n)}^q, \quad A=\frac{3^n\|f\|_{L^1(\mathbb{R}^n)}}{m(E)}, \quad \delta<min\{q,1-q\}.
\end{aligned}
$$

Let $\delta\to min\{q,1-q\} \Rightarrow \|\mathcal{M}f(x)\|_{L^q(E)}^q \leqslant \dfrac{2q\cdot 3^{nq}}{min\{q,1-q\}}m(E)^{1-q}\|f\|_{L^1(\mathbb{R}^n)}^q$

§ 5 Problem

Suppose $f\in L^q(\mathbb{R}^n)$, $0<a<n$, $p>q>1$, $\dfrac{1}{p}-\dfrac{1}{q}+1=\dfrac{a}{n}$, $\mathcal{C}f(x) := f*(id)^{-a} = \displaystyle\int_{\mathbb{R}^n}\frac{f(y)}{|x-y|^a}dy$. Show that exists $C_{n,a,q}>0$, s.t. $\|\mathcal{C}f\|_{L^p(\mathbb{R}^n)} \leqslant C_{n,a,q}\|f\|_{L^q(\mathbb{R}^n)}$.

§ 5.1 Direct Proof using Dyadic Decomposition

Without loss to generality, assume $m(\{|f(x)| > 0\}) > 0$. For any $x \in \mathbb{R}^n$,$r > 0$:

$$\begin{aligned}
&\left|\int_{B(x,r)} \frac{f(y)}{|y-x|^a} dy\right| \leqslant \int_{B(x,r)} \frac{|f(y)|}{|y-x|^a} dy \\
=\ & \sum_{j=0}^{\infty} \int_{B(x,\frac{r}{2^j})\setminus B(x,\frac{r}{2^{j+1}})} \frac{|f(y)|}{|y-x|^a} dy \\
\leqslant\ & \sum_{j=0}^{\infty} \int_{B(x,\frac{r}{2^j})\setminus B(x,\frac{r}{2^{j+1}})} \left(\frac{2^{j+1}}{r}\right)^a |f(y)| dy \\
\leqslant\ & \sum_{j=0}^{\infty} \int_{B(x,\frac{r}{2^j})} \left(\frac{2^{j+1}}{r}\right)^a |f(y)| dy \\
=\ & \sum_{j=0}^{\infty} \left(\frac{2^{j+1}}{r}\right)^a \alpha(n) \left(\frac{r}{2^j}\right)^n \frac{1}{|B(x,\frac{r}{2^j})|} \int_{B(x,\frac{r}{2^j})} |f(y)| dy \\
\leqslant\ & \left(\alpha(n) 2^a \sum_{j=0}^{\infty} 2^{j(a-n)}\right) r^{n-a} \mathcal{M}f(x) := C_{n,a} r^{n-a} \mathcal{M}f(x). \quad C_{n,a} < \infty.
\end{aligned}$$

Consider $\dfrac{1}{q} + \dfrac{1}{q'} = 1$, then $\dfrac{1}{p} + \dfrac{1}{q'} = \dfrac{a}{n}$.

$$\begin{aligned}
&\left|\int_{\mathbb{R}^n\setminus B(x,r)} \frac{f(y)}{|y-x|^a} dy\right| \leqslant \left(\int_{\mathbb{R}^n\setminus B(x,r)} |f|^q dm\right)^{\frac{1}{q}} \\
\cdot\ & \left(\int_{\mathbb{R}^n\setminus B(x,r)} |y-x|^{-q'a} dy\right)^{\frac{1}{q'}} \\
\leqslant\ & \|f\|_{L^q(\mathbb{R}^n)} \cdot \left(\int_r^{\infty} \int_{\partial B(0,\rho)} \rho^{-q'a} dS d\rho\right)^{\frac{1}{q'}} \\
=\ & \|f\|_{L^q(\mathbb{R}^n)} \cdot \left(\int_r^{\infty} \omega_n \rho^{n-1-q'a} d\rho\right)^{\frac{1}{q'}} \\
=\ & \|f\|_{L^q(\mathbb{R}^n)} \left(\frac{\omega_n}{q'a-n}\right)^{\frac{1}{q'}} \cdot r^{\frac{n}{q'}-a} := \tilde{C}_{n,a,q} \cdot r^{-\frac{n}{p}} \|f\|_{L^q(\mathbb{R}^n)}.
\end{aligned}$$

$C := \max\{C_{n,a}, \tilde{C}_{n,a,q}\}$, We have the following estimate for $|\mathcal{C}f(x)|$:

$$|\mathcal{C}f(x)| = \left|\int_{\mathbb{R}^n} \frac{f(y)}{|x-y|^a} dy\right| \leqslant C\left(r^{n-a}\mathcal{M}f(x) + r^{-\frac{n}{p}}\|f\|_{L^q(\mathbb{R}^n)}\right) \quad (*)$$

By the Strong Hardy-Littlewood maximal inequality: $\|\mathcal{M}f\|_{L^q(\mathbb{R}^n)} \leqslant C_{q,n}\|f\|_{L^q(\mathbb{R}^n)}$, where $C_{p,n} = \left(\dfrac{3^n 2^p p}{p-1}\right)^{1/p}$. Besides, by our assumption, $m(\{|f(x)| > 0\}) > 0 \Rightarrow \mathcal{M}f(x) > 0$. Let $r^{n-a}\mathcal{M}f(x) = r^{-\frac{n}{p}}\|f\|_{L^q(\mathbb{R}^n)}$. Then $r = (\|f\|_{L^q(\mathbb{R}^n)})^{\frac{q}{n}}(\mathcal{M}f(x))^{-\frac{q}{n}}$.

So $(*)$ yields that $|\mathcal{C}f(x)|^p \leqslant (2C)^p(\mathcal{M}f(x))^q(\|f\|_{L^q(\mathbb{R}^n)})^{p-q}$.

Then integrates on both sides:

$$\|\mathcal{C}f\|_{L^p(\mathbb{R}^n)} = \left(\int_{\mathbb{R}^n} |\mathcal{C}f(x)|^p dx\right)^{\frac{1}{p}} \leqslant 2C(\|f\|_{L^q(\mathbb{R}^n)})^{1-\frac{q}{p}}\left(\int_{\mathbb{R}^n} (\mathcal{M}f(x))^q dx\right)^{\frac{1}{p}}$$

$$\leqslant 2C(\|f\|_{L^q(\mathbb{R}^n)})^{1-\frac{q}{p}}(C_{q,n})^{\frac{q}{p}} \cdot \|f\|^{\frac{q}{p}}_{L^q(\mathbb{R}^n)} = 2C \cdot C^{\frac{q}{p}}_{q,n} \cdot \|f\|_{L^q(\mathbb{R}^n)} := C_{n,a,q} \cdot \|f\|_{L^q(\mathbb{R}^n)}.$$

where $C_{n,a,q} = 2\max\left\{\alpha(n)2^a \sum_{j=0}^{\infty} 2^{j(a-n)}, \left(\dfrac{\omega_n}{q'a-n}\right)^{\frac{1}{q'}}\right\} \cdot \left(\dfrac{3^n 2^p p}{p-1}\right)^{q/p^2}$.

§ 5.2 The same problem

If $0 < \alpha < n$, define an operator T_α on functions on $\mathbb{R}^n$ by

$$T_\alpha(x) = \int |x-y|^{-\alpha} f(y) dy \qquad (1)$$

Then T_α is weak type $(1, n\alpha^{-1})$ and strong type (p, r) with respect to the Lebesgue measure on $\mathbb{R}^n$, where $1 < p < \dfrac{n}{n-\alpha}$, $\dfrac{1}{r} = \dfrac{1}{p} - \dfrac{n-\alpha}{n}$[9]. (Note that r is the same as q in the last problem)

§ 5.3 Proof using Marcinkiewicz Interpolation

Given $f \in L^p(\mathbb{R}^n)$, we assume without loss to generality that $\|f\|_{L^p} = 1$. For a non-negative μ, which will be specified later, define

$$K(x) := |x|^{-\alpha} = K(x)\chi_{\{|x|>\mu\}} + K(x)\chi_{\{|x|\leqslant\mu\}} =: K_1(x) + K_2(x)$$

where

$$K_1(x) = K(x)\chi_{\{|x|>\mu\}}, \quad K_2(x) = K(x)\chi_{\{|x|\leqslant\mu\}}$$

Then

$$T_\alpha f(x) = K * f(x) = K_1 * f(x) + K_2 * f(x)$$

Note that $K_1 \in L^\infty$ implies $k_1 * f \in L^\infty$, and $K_2 \in L^1$ implies $K_2 * f \in L^p$, due to Young's Inequality. Moreover, for all $\lambda > 0$, $m(\{|K * f| > 2\lambda\}) \leqslant m(\{|K_1 * f| > \lambda\}) + m(\{|K_2 * f| > \lambda\})$. Let $1 < q < \infty$ be the conjugate exponent of p: $p^{-1} + q^{-1} = 1$, note that $\alpha q > n$.

$$\|K_1 * f\|_{L^\infty} \leqslant \|K_1\|_{L^q} \cdot \|f\|_{L^p} = \|K_1\|_{L^q} = \left(\int_{\{|x|>\mu\}} |x|^{-\alpha} dx\right)^{\frac{1}{q}}$$

$$= \left(\int_\mu^\infty \omega_n t^{n-1-\alpha q} dt\right)^{\frac{1}{q}} = \left(\frac{\omega_n}{\alpha q - n}\right)^{\frac{1}{q}} \mu^{-\frac{n}{r}} =: C_1 \mu^{-\frac{n}{r}}$$

where $\omega_n = \displaystyle\int_{\partial B(0,1)} dS$ denotes the area of $\partial B(0,1)$. Let $C_1\mu^{-\frac{n}{r}} = \lambda$, i.e. $\mu = {C_1}^{\frac{r}{n}} \lambda^{-\frac{r}{n}}$. Hence, $\|K_1 * f\|_{L^\infty} > \lambda$, and $m(\{|K_1 * f| > \lambda\}) = 0$. Besides,

$$m(\{|K_2 * f| > \lambda\}) \leqslant \lambda^{-p}\|K_2 * f\|_{L^p}^p \leqslant \lambda^{-p}\|f\|_{L^p}^p \|K_2\|_{L^1}^p$$

$$= \lambda^{-p}\|K_2\|_{L^1}^p = \lambda^{-p}\left(\int_{\{|x|\leqslant\mu\}} |x|^{-\alpha}\right) = \lambda^{-p}\left(\int_0^\mu \omega_n t^{n-1-\alpha} dt\right)^p$$

$$= \left(\frac{\omega_n}{n-\alpha}\right)^p \left(\frac{\mu^{n-\alpha}}{\lambda}\right)^p = {C_1}^{r-p}\left(\frac{\omega_n}{n-\alpha}\right)^p \lambda^{-r} =: C_2\lambda^{-r}$$

Hence,

$$(2\lambda)^r m(\{|K * f| > 2\lambda\}) \leqslant 2^r C_2 \tag{2}$$

Where C_2 does not depend on λ. Therefore, by letting $p = 1$, $r = n\alpha^{-1}$, T_α is weak type $(1, n\alpha^{-1})$. For any fixed $1 < p < \dfrac{n}{n-\alpha}$. Let $p < \tilde{p} < \dfrac{n}{n-\alpha}$, $\dfrac{1}{\tilde{r}} = \dfrac{1}{\tilde{p}} - \dfrac{n-\alpha}{n}$, then T_α is also weak type $(\tilde{p}, \tilde{r})$. Let

$$\frac{1}{p} = \frac{1-t}{1} + \frac{t}{\tilde{p}}, \quad \frac{1}{r} = \frac{1-t}{n\alpha^{-1}} + \frac{t}{\tilde{r}} \Rightarrow r = \frac{1}{p} - \frac{n-\alpha}{n}$$

It follows from the Marcinkiewicz Interpolation Theorem that T_α is strong type (p, r).

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