To Represent a Certain Function Value by Integrals

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Abstract

I am taking a complex analysis course at Nanjing University. Recently, we learned something about Cauchy Integral Formula. It occurred to me that we have learned something like this in Mathematical Analysis courses, in regard to how to represent the function value at a certain spot by integrals, just like what is shown by Cauchy Integral Formula, but not in the field of complex Analysis.

Keywords: Represent the function value at a certain spot by integrals

1 Some Examples

Lets go through some examples and see what we can figure out.

1.1 Problem

$$u \in \mathbb{C}^2(\mathbb{R}^3), supp(u) := \left\{ x \in \mathbb{R}^3 \mid \quad u(x) \neq 0 \right\}$$
$$supp(u) \subset \mathbb{B}^3, \mathbb{B}^3 := \left\{ x \in \mathbb{R}^3 \mid \quad \|x\| < 1 \right\}, y \in \mathbb{B}^3, x \in \mathbb{R}^3$$

prove that

$$x \mapsto \frac{1}{\|x - y\|}$$

is a harmonic function and

$$u(y) = \int_{\mathbb{B}^3} -\frac{\Delta u(x) dx}{4\pi \|x - y\|}$$

Proof

$$v : \mathbb{R}^3 \to \mathbb{R}, x \mapsto \|x-y\|, B_y(\varepsilon) := \left\{x \in \mathbb{R}^3 \,|\|x-y\| < \varepsilon\right\},$$
$$\Omega_\varepsilon := \mathbb{B}^3 - B_y(\varepsilon), x = (x_1, x_2, x_3)^T, y = (y_1, y_2, y_3)^T$$
$$r = \|x-y\| = \sqrt{(x_1-y_1)^2 + (x_2-y_2)^2 + (x_3-y_3)^2}$$

Then we have

$$\nabla v(x) = \left(\frac{x_1-y_1}{r^3}, \frac{x_2-y_2}{r^3}, \frac{x_3-y_3}{r^3}\right)$$

then

$$\frac{\partial^2 v}{\partial x_1^2} = -\frac{r^3 - 3r(x_1-y_1)^2}{r^6}$$

similarly,we have

$$\frac{\partial^2 v}{\partial x_2^2} = -\frac{r^3 - 3r(x_2-y_2)^2}{r^6}, \frac{\partial^2 v}{\partial x_3^2} = -\frac{r^3 - 3r(x_3-y_3)^2}{r^6}$$

then we could prove

$$\Delta v(x) = -\frac{3r^3 - 3r\cdot r^2}{r^6} = 0$$

By

$$\int_{\mathbb{B}^3} -\frac{\Delta u(x)dx}{4\pi\|x-y\|} = -\frac{1}{4\pi}\int_{\mathbb{B}^3} v(x)\Delta u(x)dx$$

We have

$$\begin{aligned} -\frac{1}{4\pi}\int_{\mathbb{B}^3} v(x)\Delta u(x)dx = -\frac{1}{4\pi}\left(\int_{\Omega_\varepsilon} + \int_{B_y(\varepsilon)} v(x)\Delta u(x)dx\right), \\ I = -\frac{1}{4\pi}\int_{\Omega_\varepsilon} v(x)\Delta u(x)dx, \qquad J = \frac{1}{4\pi}\int_{B_y(\varepsilon)} v(x)\Delta u(x)dx \end{aligned} \tag{1}$$

By Green Equality

$$\int_{\Omega_\varepsilon}(u\Delta v - v\Delta u) = -\int_{\Omega_\varepsilon}(v\Delta u) = \int_{\partial\Omega_\varepsilon}\left(u\frac{\partial v}{\partial \boldsymbol{n}} - v\frac{\partial u}{\partial \boldsymbol{n}}\right)$$

$$= \frac{1}{4\pi}\left(\int_{\partial\mathbb{B}^3} u\frac{\partial v}{\partial \boldsymbol{n}} - \int_{\partial B_y(\varepsilon)} u\frac{\partial v}{\partial \boldsymbol{n}} - \int_{\partial\mathbb{B}^3} v\frac{\partial u}{\partial \boldsymbol{n}} + \int_{\partial B_y(\varepsilon)} v\frac{\partial u}{\partial \boldsymbol{n}}\right)$$

Now that

$$supp(u) \subset \mathbb{B}^3$$

instantly,we have

$$u(x)=0, \frac{\partial u}{\partial \boldsymbol{n}}=0, \forall x \in \partial \mathbb{B}^3$$

Then

$$I=-\frac{1}{4\pi}\int_{\Omega_\varepsilon} v\Delta u=\frac{1}{4\pi}\left(-\int_{\partial B_y(\varepsilon)} u\frac{\partial v}{\partial \boldsymbol{n}}+\int_{\partial B_y(\varepsilon)} v\frac{\partial u}{\partial \boldsymbol{n}}\right)$$

While

$$\left|\int_{\partial B_y(\varepsilon)} v\frac{\partial u}{\partial \boldsymbol{n}}\right| \leq \max_{x\in\partial B_y(\varepsilon)} |\nabla u(x)| \cdot \left|\int_{\partial B_y(\varepsilon)} v\right|$$

$$=\max_{x\in\partial B_y(\varepsilon)} |\nabla u(x)| \cdot \int_{\partial B_y(\varepsilon)} \frac{1}{\varepsilon}=\max_{x\in\partial B_y(\varepsilon)} |\nabla u(x)| \cdot 4\pi\varepsilon=O(\varepsilon)$$

and

$$-\frac{1}{4\pi}\int_{\partial B_y(\varepsilon)} u\frac{\partial v}{\partial \boldsymbol{n}}=-\frac{1}{4\pi}\int_{\partial B_y(\varepsilon)} u\langle\nabla v,\vec{n}\rangle=\frac{1}{4\pi}\int_{\partial B_y(\varepsilon)} u\frac{r}{r^3}=\frac{1}{4\pi}\int_{\partial B_y(\varepsilon)} \frac{u}{\varepsilon^2}$$

By Integral Mean Value Theorem

$$\lim_{\varepsilon\to 0^+}\frac{1}{4\pi}\int_{\partial B_y(\varepsilon)} \frac{u}{\varepsilon^2}=\frac{1}{4\pi}u(y)\frac{4\pi\varepsilon^2}{\varepsilon^2}=u(y)$$

Then we consider J,Through Spherical coordinate transformation

$$\lim_{\varepsilon\to 0^+}\int_{B_y(\varepsilon)} v\Delta u=\lim_{\varepsilon\to 0^+}\int_{B_y(\varepsilon)} \frac{\Delta u}{r}$$

$$=\lim_{\varepsilon\to 0^+}\int_0^\pi d\theta\int_0^{2\pi} d\phi\int_0^\varepsilon \frac{r^2 sin\theta}{r}\Delta u dr=lim_{\varepsilon\to 0^+}O(\varepsilon^2)=0$$

Consider the equation 1,let $\varepsilon \to 0^+$, then we have

$$u(y)=\int_{\mathbb{B}^3} -\frac{\Delta u(x)dx}{4\pi\|x-y\|}$$

Q.E.D

1.2 Problem

Suppose $\Omega \subset \mathbb{R}^3$,bounded, $\partial\Omega$ is of $\mathbb{C}^1.u \in \mathbb{C}^2(\Omega)$ and $\Delta u = 0$. Prove that

$$u(\boldsymbol{y}) = \frac{1}{4\pi}\int_{\partial\Omega}\left(\frac{X\cdot\vec{n}}{r^2}u + \frac{1}{r}\frac{\partial u}{\partial\boldsymbol{n}}\right)d\sigma \tag{2}$$

where

$$r = \sqrt{(x-x_0)^2+(y-y_0)^2+(z-z_0)^2}, \quad X = \frac{1}{r}(x-x_0, y-y_0, z-z_0)$$

$\boldsymbol{n}$ is the outward unit normal vector of $\partial\Omega$.

Proof

$$\boldsymbol{x} = (x,y,z)^T, \boldsymbol{y} = (x_0,y_0,z_0)^T, \varepsilon < dist(\partial\Omega, \boldsymbol{y})$$

$$B_y(\varepsilon) := \left\{x \in \mathbb{R}^3 \mid \quad \|x-y\| < \varepsilon\right\}, \Omega_\varepsilon := \Omega - B_y(\varepsilon)$$

$$v : \mathbb{R}^3 \to \mathbb{R}, x \mapsto \frac{1}{\|\boldsymbol{x}-\boldsymbol{y}\|}, \Delta v(x) = 0, x \neq y$$

Then

$$\int_{\partial\Omega}\left(\frac{X\cdot\vec{n}}{r^2}u + \frac{1}{r}\frac{\partial u}{\partial\boldsymbol{n}}\right)d\sigma = \int_{\partial\Omega_\varepsilon} + \int_{\partial B_y(\varepsilon)}\left(\frac{X\cdot\vec{n}}{r^2}u + \frac{1}{r}\frac{\partial u}{\partial\boldsymbol{n}}\right)d\sigma$$

Let

$$I = \int_{\partial\Omega_\varepsilon}\left(\frac{X\cdot\vec{n}}{r^2}u + \frac{1}{r}\frac{\partial u}{\partial\boldsymbol{n}}\right)d\sigma, \quad J = \int_{\partial B_y(\varepsilon)}\left(\frac{X\cdot\vec{n}}{r^2}u + \frac{1}{r}\frac{\partial u}{\partial\boldsymbol{n}}\right)d\sigma$$

Because

$$\int_{\partial B_y(\varepsilon)}\frac{X\cdot\vec{n}}{r^2}u = \int_{\partial B_y(\varepsilon)}\frac{\vec{n}^2}{r^2} \longrightarrow u(\boldsymbol{y})\cdot\frac{4\pi\varepsilon^2}{\varepsilon^2} = 4\pi u(\boldsymbol{y}), \varepsilon \to 0^+$$

and

$$\left|\int_{\partial B_y(\varepsilon)} v\frac{\partial u}{\partial\boldsymbol{n}}\right| \leq \max_{x\in\partial B_y(\varepsilon)}|\nabla u(\boldsymbol{x})|\left|\int_{\partial B_y(\varepsilon)}\frac{1}{r}\right| = \max_{x\in\partial B_y(\varepsilon)}|\nabla u(\boldsymbol{x})|\cdot 4\pi\varepsilon = O(\varepsilon)$$

Therefore

$$J \to 4\pi u(\boldsymbol{y}), \varepsilon \to 0^+$$

By Green Equality

$$\int_{\Omega_\varepsilon}(u\Delta v - v\Delta u)dx = \int_{\partial\Omega_\varepsilon}\left(u\frac{\partial v}{\partial\boldsymbol{n}} - v\frac{\partial u}{\partial\boldsymbol{n}}\right)d\sigma$$

We have

$$\int_{\partial\Omega_\varepsilon} \frac{1}{r}\frac{\partial u}{\partial \boldsymbol{n}} = \int_{\partial\Omega_\varepsilon} v\frac{\partial u}{\partial \boldsymbol{n}} = \int_{\partial\Omega_\varepsilon} u\frac{\partial v}{\partial \boldsymbol{n}} = \int_{\partial\Omega_\varepsilon} u\langle \nabla v, \vec{n}\rangle$$

$$= \int_{\partial\Omega_\varepsilon} u\vec{n}\cdot(-\frac{x-x_0}{r^3}, -\frac{y-y_0}{r^3}, -\frac{z-z_0}{r^3}) = \int_{\partial\Omega_\varepsilon} u\vec{n}\left(-\frac{X}{r^2}\right)$$

Thus

$$I = \int_{\partial\Omega_\varepsilon} \left(\frac{X\cdot\vec{n}}{r^2}u + \frac{1}{r}\frac{\partial u}{\partial \boldsymbol{n}}\right) d\sigma = 0, \quad \forall \varepsilon \in (0, dist(\partial\Omega, \boldsymbol{y}))$$

Consider the equation 2

$$\textbf{RHS} = \lim_{\varepsilon\to 0^+} \frac{1}{4\pi}\int_{\partial\Omega} \left(\frac{X\cdot\vec{n}}{r^2}u + \frac{1}{r}\frac{\partial u}{\partial \boldsymbol{n}}\right) d\sigma = lim_{\varepsilon\to 0^+}\frac{1}{4\pi}(I+J) = u(\boldsymbol{y})$$

Q.E.D

1.3 Problem

Suppose $\Omega \subset \mathbb{R}^2$,bounded, $\partial\Omega$ is of $\mathbb{C}^1$.$u \in \mathbb{C}^2(\Omega)$ and $\Delta u = 0$. Prove that

$$u(\boldsymbol{y}) = \frac{1}{2\pi}\int_{\partial\Omega}\left(\frac{X\cdot\vec{n}}{r}u - \frac{\partial u}{\partial \boldsymbol{n}}\log r\right) ds$$

where

$$r = \sqrt{(x-x_0)^2 + (y-y_0)^2}, X = \frac{1}{r}(x-x_0, y-y_0)$$

$\boldsymbol{n}$ is the outward unit normal vector of $\partial\Omega$.

Proof

$$\boldsymbol{x} = (x,y)^T, \boldsymbol{y} = (x_0,y_0)^T, \varepsilon < dist(\partial\Omega, \boldsymbol{y})$$

$$B_y(\varepsilon) := \left\{x \in \mathbb{R}^2 \mid \quad \|x-y\| < \varepsilon\right\}, \Omega_\varepsilon := \Omega - B_y(\varepsilon)$$

$$v : \mathbb{R}^2 \to \mathbb{R}, x \mapsto \log\|\boldsymbol{x}-\boldsymbol{y}\|, \boldsymbol{x} \neq \boldsymbol{y}$$

We claim that

$$\Delta v = 0$$

In fact

$$\nabla v = \left(\frac{x-x_0}{r^2}, \frac{y-y_0}{r^2}\right)$$

$$\frac{\partial^2 v}{\partial x^2} = \frac{r^2 - 2(x - x_0^2)}{r^4}$$

Symmetrically

$$\frac{\partial^2 v}{\partial y^2} = \frac{r^2 - 2(y - y_0^2)}{r^4}$$

Therefore

$$\Delta v = \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0$$

Consider

$$\int_{\partial\Omega} \left(\frac{X \cdot \vec{n}}{r} u - \frac{\partial u}{\partial \boldsymbol{n}} \log r \right) ds = \int_{\partial\Omega_\varepsilon} + \int_{\partial B_y(\varepsilon)} \left(\frac{X \cdot \vec{n}}{r} u - \frac{\partial u}{\partial \boldsymbol{n}} \log r \right) ds$$

Let

$$I = \int_{\partial\Omega_\varepsilon} \left(\frac{X \cdot \vec{n}}{r} u - \frac{\partial u}{\partial \boldsymbol{n}} \log r \right) ds, \quad J = \int_{\partial B_y(\varepsilon)} \left(\frac{X \cdot \vec{n}}{r} u - \frac{\partial u}{\partial \boldsymbol{n}} \log r \right) ds$$

By Green Equality

$$\int_{\Omega_\varepsilon} (u\Delta v - v\Delta u) dx = \int_{\partial\Omega_\varepsilon} \left(u \frac{\partial v}{\partial \boldsymbol{n}} - v \frac{\partial u}{\partial \boldsymbol{n}} \right) ds$$

We have

$$\begin{aligned} \int_{\partial\Omega_\varepsilon} \frac{\partial u}{\partial \boldsymbol{n}} \log r &= \int_{\partial\Omega_\varepsilon} v \frac{\partial u}{\partial \boldsymbol{n}} = \int_{\partial\Omega_\varepsilon} u \frac{\partial v}{\partial \boldsymbol{n}} = \int_{\partial\Omega_\varepsilon} u \langle \nabla v, \vec{n} \rangle \\ &= \int_{\partial\Omega_\varepsilon} u\vec{n} \cdot \left(\frac{x - x_0}{r^2}, \frac{y - y_0}{r^2} \right) = \int_{\partial\Omega_\varepsilon} u \frac{X \cdot \vec{n}}{r} \end{aligned}$$

Thus

$$I = 0, \quad \forall \varepsilon \in (0, dist(\partial\Omega, \boldsymbol{y}))$$

while

$$\int_{\partial B_y(\varepsilon)} \frac{X \cdot \vec{n}}{r} u = \int_{\partial B_y(\varepsilon)} \frac{\vec{n}^2}{r} u = \int_{\partial B_y(\varepsilon)} \frac{u}{r} \to u(\boldsymbol{y}) \cdot \frac{2\pi\varepsilon}{\varepsilon} = 2\pi u(\boldsymbol{y}), \varepsilon \to 0^+$$

and

$$\left| \int_{\partial B_y(\varepsilon)} v \frac{\partial u}{\partial \boldsymbol{n}} \right| \leq \max_{x \in \partial B_y(\varepsilon)} |\nabla u(\boldsymbol{x})| \left| \int_{\partial B_y(\varepsilon)} \log \varepsilon \right| = \max_{x \in \partial B_y(\varepsilon)} |\nabla u(\boldsymbol{x})| \cdot 2\pi\varepsilon \log \varepsilon \to 0, \varepsilon \to 0^+$$

That is

$$\lim_{\varepsilon \to 0^+} J = 2\pi u(\boldsymbol{y})$$

Thus

$$u(\boldsymbol{y}) = \lim_{\varepsilon \to 0^+} \frac{1}{2\pi} \int_{\partial \Omega} \left(\frac{X \cdot \vec{n}}{r} u - \frac{\partial u}{\partial \boldsymbol{n}} \log r \right) ds = \frac{1}{2\pi} \int_{\partial \Omega} \left(\frac{X \cdot \vec{n}}{r} u - \frac{\partial u}{\partial \boldsymbol{n}} \log r \right) ds$$

Q.E.D