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Orthogonalized factors and systematic risk decomposition

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ABSTRACT

In the context of linear multi-factor models, this study proposes an egalitarian, optimal and unique procedure to find orthogonalized factors, which also facilitates the decomposition of the coefficient of determination. Importantly, the new risk factors may diverge significantly from the original ones. The decomposition of risk allows one to explicitly examine the impact of individual factors on the return variation of risky assets, which provides discriminative power for factor selection. The procedure is experimentally robust even for small samples. Empirically we find that even though, on average, approximately eighty (sixty-five) percent of style (industry) portfolios' volatility is explained by the *market* and *size* factors, other factors such as *value*, *momentum* and *contrarian* still play an important role for certain portfolios. The components of systematic risk, while dynamic over time, generally exhibit negative correlation between *market*, on one side, and *size*, *value*, *momentum* and *contrarian*, on the other side.

1. Introduction

Under the traditional single-factor Sharpe (1964) and Lintner (1965) Capital Asset Pricing Model (CAPM), the market beta captures a stock's systematic risk for all rational, risk-averse investors. Therefore, a decomposition of the market beta is sufficient to break down the systematic risk of a stock.[2] For example, Campbell and Vuolteenaho (2004) break the market beta of a stock into a 'bad' component, that reflects news about the market's future cash flows, and a 'good' component, that reflects news about the market's discount rates. In an earlier paper, Campbell and Mei (1993) show that the market beta can be decomposed into three sub-betas that reflect news about future cash flows, future real interest rates and a stock's future excess returns, respectively. Acharya and Pedersen (2005) develop a CAPM with liquidity risk by dividing the market beta of a stock into four sub-betas that reflect the impact of illiquidity costs on the systematic risk of an asset. Researchers frequently apply decompositions of the market beta to examine the size and/or book-to-market anomalies. Although beta-decompositions are useful to describe the structure and source of systematic variation of returns on risky assets, they are complicated under multi-factor frameworks. For instance, Campbell and Mei (1993) show that one complication is due to the possible covariance between the risk price of one factor and the other factors, which prevents identifying a neat linear relationship between the overall beta of an asset and its beta of news about future cash flows.

The purpose of this paper is to develop an optimal procedure to identify the underlying uncorrelated components of common factors, by a simultaneous and symmetric orthogonal transformation of sample data, such that the linear dependence is removed and the systematic variation of stock returns becomes decomposable. We empirically compare our approach with two popular orthogonalization methods, Principal Component Analysis (PCA) and the Gram-Schmidt (GS) process, and unsurprisingly find that our technique has the essential advantage of maintaining maximum resemblance with the original factors.[3]

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2 According to the CAPM, the systematic risk is measured as $(\beta_j \sigma_{RM})^2$. Since the market factor is the only priced risk factor faced by all investors, β_j is sufficient to determine the systematic risk.

3 For instance, Baker and Wurgler (2006, 2007) employ PCA to develop measures of investor sentiment, shown to have significant effects on the cross-section of stock

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In the past two decades, one of the most extensively researched areas in finance has concentrated on alternative common risk factors, in addition to market risk, that could characterize the cross-section of expected stock returns. Fama and French (1992, 1993, 1996, 1998) document that a company's market capitalization, *size*, and the company's *value*, which is assessed by ratios of book-to-market (B/M), earnings to price (E/P) or cash flows to price (C/P), together predict the return on a portfolio of stocks with much higher accuracy than the market beta alone, or the traditional CAPM.[4] In addition to the *size* and *value* effects, Jegadeesh and Titman (1993), Jegadeesh and Titman (2001), Rouwenhorst (1998), and Chan, Jegadeesh, and Lakonishok (1996) report that short-term past returns or past earnings predict future returns. Average returns on the best prior performing stocks (i.e., the *winners*) exceed those of the worst prior performing stocks (i.e., the *losers*), attesting the existence of *momentum* in stock prices. Conversely, De Bondt and Thaler (1985, 1987) detect a *contrarian* effect by which stocks exhibiting low long-term past returns outperform stocks with high long-term past returns. De Bondt and Thaler (1985, 1987), Chopra, Lakonishok, and Ritter (1992), and Balvers, Wu, and Gilliland (2000) suggest a profitable contrarian strategy of buying the *losers* and shorting the *winners*.

Consequently, for the determination of the return generating process for risky assets, one needs to consider more than just the market risk factor. For this reason, multi-factor market models have been widely employed by both academics and practitioners. Under the multi-factor framework, the expected excess return on a risky asset is specified as a linear combination of beta coefficients and expected premia of individual factors. Fama and French (1993) emphasize that, if there are multiple common factors in stock returns, they must be in the market return, as well as in other well-diversified portfolios that contain these stocks. This indicates that returns on common factors must be, to some degree, correlated with the market and with each other. Consequently, in a multiple linear regression setting, although the beta coefficient corresponding to an individual factor provides a sensitivity measure of an asset's return to the factor's variation, it is not sufficient to assess the systematic variation of the asset's return with respect to that factor. The volatility of an asset's return is determined jointly not only by the betas, but also by the variances and covariances of the factors' premia. Therefore, determining the factors' underlying uncorrelated components helps us achieve a clearer identification of the separate roles of common factors in stock returns.

This paper proposes an optimal simultaneous orthogonal transformation of factor returns. The data transformation allows us to identify the underlying uncorrelated components of common factors. Specifically, the inherent components of factors retain their variances, but their cross-sectional covariances are equal to zero. Moreover, a multi-factor regression using the orthogonalized factors has the same coefficient of determination, *R*-square, as that using the original, non-orthogonalized factors. Importantly, the coefficient of determination (the ratio of systematic variation to the overall volatility of a risky asset) is a measure of the systematic risk of an asset. Therefore, disentangling the *R*-square based on factors' volatilities and their corresponding betas enables us to decompose the systematic risk. For that, we need to extract the core, standalone components of common factors. Fama and French (1993) clearly demonstrate that since the market return is a mixture of the multiple common factors, an orthogonalization of the market factor is necessary so that it can capture common variation in returns left from other factors such as *size* or *value*. We argue that not only the market factor, but all factors need to be orthogonally transformed to eliminate any dependence among them. Although Fama and French's (1993) orthogonalization procedure for the market factor is straightforward, it cannot be extended to eliminate the correlations between all variables in a model, without generating two related biases. Firstly, similarly to GS, it leaves one factor (call it *leader*) unchanged. Secondly, it is a sequential (i.e., order-dependent) procedure. Therefore, a different selection of the leader or a different orthogonalization sequence generates different transformation results. Our method avoids these two biases by construction.

Using Monte Carlo simulations, we demonstrate that our orthogonal transformation is robust, in that it produces precise estimates of the population systematic risk even for small samples. By applying our methodology to some of the Kenneth French's style and industry portfolios, we show empirically that the systematic return variation can now be unequivocally allocated to the common factors.[5] We find that, over a time period from January 1931 to December 2008, the *market* and *size* factors are the largest sources of systematic risk, while other factors such as *value*, *momentum* and *contrarian* play relatively small roles in stock volatility.

The paper is organized as follows. In Section 2, after explaining why the systematic risk decomposition is problematic under multi-factor models, we present our procedure of symmetric orthogonal transformation and risk decomposition. In Section 3, we illustrate the procedure empirically, using monthly U.S. Research Returns Data obtained from Kenneth French's Data Library, for the time interval January 1931–December 2008. The final section of the paper provides concluding remarks.

2. Orthogonalization procedure

Suppose a risky asset *j*'s return generating process is linearly determined by a set of *K* common factors (f^k), such as *market* (*RM*), *size* (*SMB*), *value* (*HML*), *momentum* (*Mom*), and *long-term reversal* (*Rev*), as shown in the following general linear factor model.

$$r_t^j = \alpha_j + \sum_{k=1}^{K} \beta_{k_j} f_t^k + \varepsilon_t^j, \tag{1}$$

where f^k are assumed to be uncorrelated with the residual term (ε_j), but not with each other. For instance, the *market* factor is a mixture of the multiple common factors, while the factor-mimicking portfolios of *size*, *value*, *momentum* and *contrarian* are all formed using securities in the same market, and thus their returns are not uncorrelated.

The systematic return variation (σ_s^2) of asset *j* can then be measured as

$$\sigma_{s_j}^2 = \sum_{l=1}^{K} \sum_{k=1}^{K} \beta_{k_j} \beta_{l_j} Cov(f^k, f^l), \tag{2}$$

while the coefficient of determination, *R*-square, is the ratio of systematic variation to total return variation ($\sigma_{s_j}^2/\sigma_j^2$).

It is important to note that under the multi-factor framework, systematic risk depends not only on the beta coefficients but also on the factors' variance–covariance. Thus, beta coefficients alone are

returns. Boubakri and Ghouma (2010) remove the multicollinearity between their variables using the Gram-Schmidt algorithm.

[4] Fama and French (1992, 1996, 1998) show that the investment strategy of buying the *Small – Value* stocks and shorting the *Big – Growth* stocks produces positive returns.

[5] Kenneth French's Data Library is located at http://mba.tuck.dartmouth.edu/pages/faculty/ken.french/data_library.html.

inappropriate measures of systematic variation. One of the goals of this paper is to develop a decomposable systematic risk measure.

As shown in (2), $\sigma^2_{s_j}$ is not decomposable into *individual* systematic risk components, due to the covariance between factors.[6] Kennedy (2008, p. 46), among others, points out that the total *R*-square cannot be allocated unequivocally to each explanatory variable, unless we have zero multicollinearity between the variables. To be able to achieve zero multicollinearity, thus eliminating the impact of covariances, we employ an orthogonal transformation which helps us identify the underlying components of all factors. We argue that, even though numerous orthogonalization techniques are available, the optimal alternative for finding the proper orthogonal proxies of the original factors is the so-called *symmetric* procedure of Schweinler and Wigner (1970) and Löwdin (1970), as adapted in this paper (henceforth denoted as SW/L).[7] For instance, the popular Principal Component Analysis, often used for dimensionality reduction, though similar to our procedure in some aspects, cannot offer by itself a meaningful one-to-one and onto correspondence from the original to the orthogonalized set of factors, if the number of variables is larger than two. And even in the cases where we have only two explanatory variables, if they are significantly correlated, it is difficult to maintain a strong resemblance to the original variables once the transformation is performed. Table 1 considers all the possible combinations of pairs out of the five factors mentioned above and, using two measures of similitude, compares SW/L, PCA and the classical GS process. A major deficiency of this last procedure is that it requires a choice of the initial starting factor, which will not be transformed, thus failing to give all factors equal footing. We are nevertheless interested to see whether the deviations in the other factor are greater or less than those generated by the first two methods.

Panel A presents the correlation coefficients between the original and their orthogonalized counterparts. It is easily noticeable that SW/L outperforms both PCA and GS. For instance, its lowest correlation of 0.945 (between *Rev* and orthogonalized *Rev*) is more than double the minimum correlation for PCA, 0.450, and about 20 percent larger than that of GS, 0.787, all for *HML* & *Rev*.[8] Also, given that GS does not modify the first factor, which can be interpreted as perfect correlation, while the correlations corresponding to the second factor are greater than the lower coefficients for PCA, we can conclude that GS performs better than PCA in all ten cases. Panel B reports, on a comparative scale, the Frobenius norm of the $T \times K$ matrix whose elements are the deviations of the orthogonalized from the original sets of data where, in these cases, $T = 936$ months (from January 1931 to December 2008) and $K = 2$ factors.[9] Since we want the original factors to be modified as little as possible, the results confirm the superiority of our *symmetric* procedure, which has the lowest (square root of the) sum of squared deviations, in all ten cases (i.e., the lowest Frobenius norm). Similar to Panel A, the principal component scores are further from the values of the original variables, compared to the GS transformation. This is very severe especially when the original variables show significant correlation. That is why some researchers, for ease of interpretation of the factor loading pattern, prefer to perform rotations, once the PCA is completed.

Table 1
Method comparison with respect to resemblance criteria.

Panel A: Correlation coefficients between the original and the orthogonalized factors

	SW/L		PCA		GS	
(1)	(2)	(3)	(4)	(5)	(6)	(7)
RM & *SMB*	0.992	0.978	0.986	0.874	1.000	0.941
RM & *HML*	0.996	0.990	0.987	0.922	1.000	0.970
RM & *Mom*	0.987	0.982	0.909	0.709	1.000	0.946
RM & *Rev*	0.995	0.989	0.988	0.920	1.000	0.968
SMB & *HML*	0.999	0.999	0.869	0.913	1.000	0.994
SMB & *Mom*	0.996	0.998	0.957	0.990	1.000	0.991
SMB & *Rev*	0.976	0.979	0.590	0.871	1.000	0.908
HML & *Mom*	0.972	0.984	0.746	0.952	1.000	0.925
HML & *Rev*	0.947	0.945	0.904	0.450	1.000	0.787
Mom & *Rev*	0.994	0.990	0.970	0.882	1.000	0.975

Panel B: Frobenius norm values of the deviation matrix

	SW/L	PCA	GS
RM & *SMB*	30.518	57.877	35.994
RM & *HML*	22.171	51.239	29.428
RM & *Mom*	38.748	128.797	54.975
RM & *Rev*	22.824	50.529	28.768
SMB & *HML*	7.465	70.009	17.464
SMB & *Mom*	12.920	36.675	30.671
SMB & *Rev*	31.769	123.197	47.165
HML & *Mom*	37.327	88.974	63.458
HML & *Rev*	51.105	116.381	71.404
Mom & *Rev*	21.661	63.158	28.691

This table considers all the possible combinations of pairs out of five stock-market factor portfolios: *RM*, *SMB*, *HML*, *Mom* and *Rev*. For each pair of original factors, their orthogonalized counterparts are computed. In Panel A, each of the columns (2)–(7) reports the correlation coefficient between the original and the orthogonalized factors, in the same order in which they are specified in column (1). Panel B reports the Frobenius norm values of the deviations between the non-orthogonalized and the orthogonalized forms. The orthogonalized factors are obtained using each of the following methods: The Schweinler–Wigner/Löwdin (1970) symmetric procedure, as adapted in this paper (SW/L), Principal Component Analysis (PCA), and the Gram-Schmidt process (GS). *RM*, *SMB* and *HML* are the Fama/French factors: *RM* is the market risk premium; *SMB* is Small Minus Big – size, while *HML* is High Minus Low – book-to-market. *Mom* is the momentum factor, while *Rev* is the long-term reversal factor. All five factors follow the description and are obtained from Kenneth French's data library at http://mba.tuck.dartmouth.edu/pages/faculty/ken.french/. The sample period is January 1931 to December 2008. The Frobenius norm of a real matrix is defined as the square root of the sum of the squares of its elements. Note that GS leaves the first factor unchanged. Hence, the correlation for the first factor indicated in column (1) is always equal to one.

A number of papers (see, for instance, Garrett, Hyde, & Lozano, 2011) use the Pearson system to generate data with the same first four moments as those of the original factors. By construction, between these simulated factors there is zero or near zero correlation. While useful for certain applications, even if the possible mismatching with the data is avoided, the Pearson system does not maintain maximum resemblance in the least squares sense. For example, as can be seen in Table 1, the worst case that involves *RM* (i.e., *RM & Mom*) results in a Frobenius norm value of 38.748 for SW/L. The same metric for Pearson simulated data (not reported in this paper) takes a value roughly eight times larger than that (308.095).

In the following sub-section we describe our procedure in detail and we emphasize its essential properties.

2.1. Methodology

For convenience purposes, we present the transformation procedure in a matrix format. Let $f^k_{T\times 1} = [f^k_1, f^k_2, \ldots, f^k_T]'$ be the sample returns of the kth factor, for $k = 1, 2, \ldots, K$, and $F_{T\times K} = [f^k_t]^{k=1,\ldots,K}_{t=1,\ldots,T}$ their corresponding T by K matrix, where T is the number of periods,

[6] For simplicity and without loss of generality, henceforth we will refer to common factors only as *factor portfolio returns*, but the model can also be applied to other systematic factors (e.g., macro-economic or fundamental variables).

[7] The actual procedure is presented in Section 2.1.

[8] *HML* and *Rev*, in their original form, exhibit the highest correlation among all pairs of factors, over the period January 1931–December 2008 (see Table 3).

[9] The Frobenius norm of a real matrix is defined as the square root of the sum of the squares of its elements.

and K is the number of factors.[10] Our purpose is to derive from $F_{T\times K}$ a matrix of mutually uncorrelated and variance-preserving vectors, denoted by $F^{\perp}_{T\times K} = [f_t^{k\perp}]_{t=1,\dots,T}^{k=1,\dots,K}$, so that the systematic return variation can be estimated by the following decomposable form

$$\hat{\sigma}^2_{s_j} = \sum_{k=1}^{K} \left(\hat{\beta}^{\perp}_{k_j}\hat{\sigma}^{\perp}_{f^k}\right)^2 = \hat{\sigma}^2_j - \hat{\sigma}^2_{\varepsilon_j}, \tag{3}$$

where $\hat{\sigma}^2_{s_j}$ is the estimate of $\sigma^2_{s_j}$, $\hat{\beta}^{\perp}$ and $\hat{\sigma}^{\perp}(=\hat{\sigma})$ are the estimates of beta and standard deviation from sample data after the orthogonal transformation, while $\hat{\sigma}^2_j$ and $\hat{\sigma}^2_{\varepsilon_j}$ are the estimated variance of asset j's returns and its residual variance, respectively. Importantly, both the intercept and the error term in Eq. (1) remain unchanged after transformation (please see Appendix A for a proof of this result).

We propose a procedure that encompasses the following three important characteristics:

Property 1: The *symmetric* orthogonalization of matrix $F_{T\times K}$ is *democratic* (or *egalitarian*), as opposed to sequential.
Property 2: The *symmetric* orthogonalization of matrix $F_{T\times K}$ is *optimal*, in the sense that $F^{\perp}_{T\times K}$ maximally resembles $F_{T\times K}$ with respect to all the von Neumann–Schatten p-norms.[11]
Property 3: The *symmetric* orthogonalization of matrix $F_{T\times K}$ is *unique*.

To obtain $F^{\perp}_{T\times K}$, we employ a methodology attributed to Schweinler and Wigner (1970) in the wavelet literature and to Löwdin (1970) in the quantum chemistry literature. This is a *democratic* (or *egalitarian*) procedure by construction, as opposed to sequential approaches that are sensitive to the order in which vectors are selected.[12] This distinctive characteristic is essential for a proper decomposition, as we need to treat all the factors on an equal footing. We need to avoid the bias towards factors that enter early in an order-dependent orthogonalization, as is the case with the Gram-Schmidt process (see Frank, Paulsen, & Tiballi, 2001). Thus, the orthogonal transformation of all factors has to be conducted jointly and simultaneously.[13]

As emphasized in Löwdin (1970) and Srivastava (2000), the *symmetric* form of orthogonalization minimizes the overall difference between the original and the orthogonal vectors, thus maximizing the resemblance between the two sets of data. Moreover, as explained in Aiken, Erdos, and Goldstein (1980), the *symmetric* procedure is optimal with respect to all the commonly used norms (i.e., for all the von Neumann–Schatten norms $||\cdot||_p$, $1 \le p \le \infty$). In particular, SW/L minimizes the Frobenius norm ($p = 2$ in this case).

Closely related to Property 2 is the *uniqueness* characteristic of SW/L. Aiken et al. (1980) prove that the minimum distance between the original and the orthogonalized vectors, in a least squares sense, is attained only for *symmetric* orthogonalizations. They also prove that this minimum is *unique*.

We apply the orthogonalization to the demeaned original factors, which ensures that the resulting vectors are not only mathematically orthogonal, but also uncorrelated. Let $\tilde{F}_{T\times K} = [\tilde{f}_t^k]_{t=1,\dots,T}^{k=1,\dots,K} = [f_t^k - \bar{f}^k]_{t=1,\dots,T}^{k=1,\dots,K}$ be the demeaned matrix of $F_{T\times K}$. We define a linear transformation $S_{S\times K}$ of the set $F_{T\times K}$ to $\tilde{F}^{\perp}_{T\times K}$, as follows

$$\tilde{F}^{\perp}_{T\times K} = \tilde{F}_{T\times K} S_{K\times K}. \tag{4}$$

To obtain $S_{K\times K}$ (and then $\tilde{F}^{\perp}_{T\times K}$), the first step is to calculate the variance–covariance matrix of the factors' returns ($\Sigma_{K\times K}$), and take $M_{K\times K} = (T-1)\Sigma_{K\times K}$, that is

$$M_{K\times K} = \begin{bmatrix} (\tilde{f}^1)'(\tilde{f}^1) & (\tilde{f}^1)'(\tilde{f}^2) & \cdots & (\tilde{f}^1)'(\tilde{f}^K) \\ (\tilde{f}^2)'(\tilde{f}^1) & (\tilde{f}^2)'(\tilde{f}^2) & \cdots & (\tilde{f}^2)'(\tilde{f}^K) \\ \vdots & \vdots & \vdots & \vdots \\ (\tilde{f}^K)'(\tilde{f}^1) & (\tilde{f}^K)'(\tilde{f}^2) & \cdots & (\tilde{f}^K)'(\tilde{f}^K) \end{bmatrix}. \tag{5}$$

The matrix $\tilde{F}^{\perp}_{T\times K}$ will be orthonormal if

$$\begin{aligned}(\tilde{F}^{\perp}_{T\times K})'\tilde{F}^{\perp}_{T\times K} &= (\tilde{F}_{T\times K}S_{K\times K})'\tilde{F}_{T\times K}S_{K\times K} = S'_{K\times K}(\tilde{F}'_{T\times K}\tilde{F}_{T\times K})S_{K\times K} \\ &= S'_{K\times K}M_{K\times K}S_{K\times K} = I_{K\times K}\end{aligned} \tag{6}$$

or equivalently,

$$S_{K\times K}S'_{K\times K} = M^{-1}_{K\times K}. \tag{7}$$

The general solution of Eq. (7) is $S_{K\times K} = M^{-1/2}_{K\times K}C_{K\times K}$, where C is an arbitrary orthogonal matrix. For $C_{K\times K} = I_{K\times K}$, where $I_{K\times K}$ is the identity matrix, the orthogonalization procedure is called *symmetric*. To be able to calculate $S_{K\times K}$, we identify an orthogonal matrix $O_{K\times K}$ (i.e., $O'_{K\times K} = O^{-1}_{K\times K}$) that brings $M_{K\times K}$ to a diagonal form $D_{K\times K}$ (i.e., $O^{-1}_{K\times K}M_{K\times K}O_{K\times K} = D_{K\times K}$).[14] Thus, $M_{K\times K}$ can be factorized as

$$M_{K\times K} = O_{K\times K}D_{K\times K}O^{-1}_{K\times K}, \tag{8}$$

where the kth column of $O_{K\times K}$ is the kth eigenvector of the matrix $M_{K\times K}$, and $D_{K\times K}$ is the diagonal matrix whose diagonal elements are the corresponding eigenvalues (λ), that is, $D_{kk} = \lambda_k$, where k goes from 1 to K. Note that Eq. (7) could also be solved using the Cholesky factorization (often employed, for instance, in the Generalized Least Squares (GLS) estimation), but the procedure would produce orthogonal factors whose values depend on their sequence (i.e., the algorithm would not be democratic).[15] Also note that the GLS estimation, different from our procedure, transforms not only the explanatory variables, but also the response variable and the error term.

Solving for $S_{K\times K}$ from Eqs. (7) and (8), we obtain the symmetric matrix

$$S_{K\times K} = O_{K\times K}D^{-1/2}_{K\times K}O'_{K\times K}, \tag{9}$$

where

$$D^{-1/2}_{K\times K} = \begin{bmatrix} 1/\sqrt{\lambda_1} & 0 & \cdots & 0 \\ 0 & 1/\sqrt{\lambda_2} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1/\sqrt{\lambda_K} \end{bmatrix}. \tag{10}$$

[10] The factors are assumed to be linearly independent (i.e., none of them can be written as a linear combination of the other factors, meaning that the matrix $F_{T\times K}$ is full rank).
[11] Given a real number $p \ge 1$, the von Neumann–Schatten p-norm of a matrix X is defined as $||X||_p = \left(\sum_i s_i^p\right)^{1/p}$, where s_i denote the singular values of X.
[12] For a comparison between the two approaches, see for instance Chaturvedi, Kapoor, and Srinivasan (1998) and Löwdin (1970).
[13] Compared to orthogonal rotation techniques employed in factor analysis or principal component analysis (e.g., *varimax*, *quartimax* or *equamax*), our procedure has the advantage of guaranteeing a bijective transformation of the original variables (i.e. a one-to-one and onto correspondence between the original and the orthogonalized sets).
[14] If $C_{K\times K} = O_{K\times K}$ instead, the orthogonalization is called *canonical*. This form is not appropriate in our case, as it does not maintain the resemblance with the original data.
[15] For example, Dewachter and Lyrio (2006) use the Cholesky factorization to compute the orthogonalized components of four macroeconomic factors aimed to provide a description of the yield curve.

Finally, we rescale the factors to their original variances, using the following transformation:

$$S_{K\times K} \mapsto S_{K\times K}\sqrt{T-1}\begin{bmatrix} \sigma_1 & 0 & \cdots & 0 \\ 0 & \sigma_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \sigma_K \end{bmatrix}, \tag{11}$$

where σ_k represents the standard deviation of factor k, with $k = 1, 2, \ldots, K$.

Hence, the matrix $S_{K\times K}$, as transformed in Eq. (11), when substituted into Eq. (4), gives the *symmetric* orthogonal transformation of the demeaned factor-matrix $\tilde{F}_{T\times K}$.

To obtain $F^{\perp}_{T\times K}$, we perform the following straight-forward transformation:

$$\begin{aligned}\tilde{F}^{\perp}_{T\times K} \quad &+ \mathbf{1}_{T\times 1}\bar{F}_{1\times K}S_{K\times K} = \tilde{F}_{T\times K}S_{K\times K} + \mathbf{1}_{T\times 1}\bar{F}_{1\times K}S_{K\times K} \\ &= (\tilde{F}_{T\times K} + \mathbf{1}_{T\times 1}\bar{F}_{1\times K})S_{K\times K} = F_{T\times K}S_{K\times K} = F^{\perp}_{T\times K},\end{aligned} \tag{12}$$

where $\mathbf{1}_{T\times 1}$ is a vector of ones and $\bar{F}_{1\times K}$ is the mean of $F_{T\times K}$.

Hence, the matrix $F^{\perp}_{T\times K}$ is a conversion of matrix $\tilde{F}^{\perp}_{T\times K}$, not an orthogonalized matrix per se. However, considering that adding constant terms to orthogonal vectors results in uncorrelated vectors, we can refer to $F^{\perp}_{T\times K}$, for simplicity, as the orthogonalized matrix of $F_{T\times K}$. The orthogonalization procedure described above can be easily replicated to suit any researcher's needs by implementing Eqs. (4)–(12).

To be able to understand what matrix $S_{K\times K}$ (or its inverse) represents, we can write each factor f^k where $k = 1, 2, \ldots, K$, as

$$f^k = \psi_{1k}f^{1^{\perp}} + \psi_{2k}f^{2^{\perp}} + \cdots + \psi_{Kk}f^{K^{\perp}}, \tag{13}$$

where the coefficients ψ_{lk} are obtained through the orthogonalization procedure described above (i.e., they are the elements of the inverse of matrix $S_{K\times K}$, in its final form).

If we calculate the covariance between f^k and $f^{k^{\perp}}$, we obtain that

$$\begin{aligned}cov(f^k, f^{k^{\perp}}) = \psi_{kk} \times var(f^{k^{\perp}}) \Rightarrow \psi_{kk} &= \frac{cov(f^k, f^{k^{\perp}})}{var(f^{k^{\perp}})} \\ &= \frac{cov(f^k, f^{k^{\perp}})}{\sigma_k\sigma_k^{\perp}} = \mathrm{corr}(f^k, f^{k^{\perp}})\end{aligned} \tag{14}$$

Moreover, it can be shown (see Appendix B) that for any k and l, $\psi_{kl} = \mathrm{corr}(f^k, f^{l^{\perp}})$. Thus, the inverse of matrix $S_{K\times K}$ is the correlation matrix between the original and the orthogonal factors. So, if we consider $S_{K\times K} = \Psi^{-1}_{K\times K}$, where $\Psi_{K\times K} = [\mathrm{Corr}(f^k, f^{l^{\perp}})]_{\substack{k=1,\ldots,K \\ l=1,\ldots,K}}$, the last equality in Eq. (12) can be rewritten as

$$F^{\perp}_{T\times K} = F_{T\times K}\Psi^{-1}_{K\times K} \tag{15}$$

That is, the orthogonal factors are linear combinations of the original factors, with the coefficients taken from the inverse correlation matrix between the original and the uncorrelated factors. Each orthogonal factor deviates from its original counterpart in such a way that the common variation is partitioned symmetrically and $F^{\perp}_{T\times K}$ optimally resembles $F_{T\times K}$.

To demonstrate the consistency between the decomposable systematic risk estimate, $\sum_{k=1}^{K}(\hat{\beta}^{\perp}_{k_j}\hat{\sigma}^{\perp}_{f^k})^2$, and the systematic risk estimate from regression, $\hat{\sigma}^2_{s_j} = \hat{\sigma}^2_j - \hat{\sigma}^2_{\varepsilon_j}$ as shown in Eq. (3), we note that the orthogonal transformation retains the original sum of squared errors (*SSE*) of Eq. (1), that is, $min\{\varepsilon'\varepsilon\}$. Mathematically, the space generated by $F^{\perp}$ is the same, by definition, as the one generated by F.[16] That is, $\{F\dot{\beta}|\dot{\beta} \in \Re^5\} = \{F^{\perp}\dot{\beta}^{\perp}|\dot{\beta}^{\perp} \in \Re^5\}$, meaning that all linear combinations of F span the same space as all linear combinations of $F^{\perp}$. Therefore, the range of the function of F, defined as $(r - F\dot{\beta})'(r - F\dot{\beta})$, is identical to that of the function of $F^{\perp}$, defined as $(r - F^{\perp}\dot{\beta}^{\perp})'(r - F^{\perp}\dot{\beta}^{\perp})$. Since the lower boundary of the two ranges is the same, $min\ \{\varepsilon'\varepsilon\}$ for F is identical to that for $F^{\perp}$.[17]

Since the orthogonal transformation is a numerical data process, it is important to examine the sampling errors, in order to assess the robustness of our estimation.

2.2. Monte Carlo simulations

We generate data for a five-factor linear model that has the following structure

$$r = 0.02 + 1.2f_1 + 0.1f_2 + 0.3f_3 - 0.2f_4 + 0.7f_5 + e \tag{16}$$

Table 2 reports the mean squared errors (*MSE*) for a set of Monte Carlo simulations. In Panel A we report the correlation coefficients between the five right-hand side variables. We choose those coefficients to range in magnitude from very low correlation (0.08, between f_2 and f_3) to moderately high (0.64, between f_3 and f_5). This covers the more common cases where individual variables are not depleted of unique or specific information. The loadings in Eq. (16) are arbitrarily chosen so that their magnitudes make sense economically.[18] Different values in loadings and/or correlations will not cause significant changes in the robustness of our estimators.

In Panel B, f_1 through f_5 follow two hypothetical forms of distribution, multivariate-normal and multivariate-lognormal, respectively, while the zero-mean residuals (e) follow by turns, one of the two processes, homoscedastic (white noise) and heteroscedastic (GARCH(1,1)). We calculate the *MSE* of the *aggregate* decomposed systematic risk estimates for 10,000 trials. The random samples have, by turns, five different sizes: 50, 150, 300, 500, and 1000. The results show that the *MSE* consistently and roughly proportionally decreases as the sample size increases. Specifically, the *MSE* drops by more than 80 percent when the sample size is increased from 50 to 300, and by more than 90 percent when increased from 50 to 500. This suggests that significant gains in precision are attained for larger sample sizes.

Additionally, in Panel C, we examine the robustness of *individual* decomposed systematic risk measures with respect to the set of five correlated factors, in a hypothetical population with a finite number of outcomes (25,000). The uncorrelated components of the factors in the population are determined numerically by the orthogonal transformation. The population decomposed systematic risk for each factor k, $(\beta^{\perp}_k\sigma^{\perp}_{f_k})^2$, is then calculated. Again, we generate 10,000 random samples for each sample size (50, 150, 300, 500, and 1000 observations). Similar to the results reported in Panel B, the *MSE* decreases, roughly proportionally, as the sample size increases. It drops by approximately two-thirds when the sample size increases from 50 to 150 observations.

Next, we turn our attention towards determining the contribution of each individual factor to the systematic risk of an asset.

[16] Moreover, by adding a column vector of ones to F and to $F^{\perp}$, to account for intercepts, the two resulting spaces will still be identical.

[17] In a less elegant manner, it can be shown that $min\{\varepsilon'\varepsilon\}$ remains the same using the observation that ε remains unchanged, as demonstrated in Appendix A.

[18] For instance, the coefficients in Table 6 of Fama and French (1993), which considers the 25 style portfolios formed on *size* and *book-to-market*, are estimated to range from −0.52 to 1.46.

Table 2
The mean squared errors (*MSE*) of the decomposed systematic risk estimates.

Panel A: Correlation matrix

	f_1	f_2	f_3	f_4	f_5
f_1	1.00	0.34	0.21	−0.36	0.27
f_2	0.34	1.00	0.08	−0.19	0.44
f_3	0.21	0.08	1.00	−0.41	0.64
f_4	−0.36	−0.19	−0.41	1.00	−0.25
f_5	0.27	0.44	0.64	−0.25	1.00

Panel B: Aggregate decomposed systematic risk estimates

$$MSE = E\left(\sum_{k=1}^{5}(\beta^{\perp}_{k_j}\sigma^{\perp}_{fk})^2 - \sigma^2_{sr}\right)^2$$

Sample size	Normal f_k, $k = 1, 2, \ldots, 5$		Log-normal f_k, $k = 1, 2, \ldots, 5$	
	Homoscedastic (e)	Heteroscedastic (e)	Homoscedastic (e)	Heteroscedastic (e)
50	0.02373	0.02198	0.00119	0.00013
150	0.00817	0.00720	0.00038	0.00004
300	0.00404	0.00354	0.00019	0.00002
500	0.00232	0.00217	0.00011	0.00001
1000	0.00121	0.00110	0.00006	0.00001

Panel C: Individual decomposed systematic risk estimates

$MSE = E[(\beta^{\perp}_{k}\sigma^{\perp}_{f_k})^2 - (\beta^{\perp}_{k}\sigma^{\perp}_{f_k})^2]^2$, for $k = 1, 2, \ldots, 5$

Sample size	f_1	f_2	f_3	f_4	f_5
50	0.01268	0.00035	0.00070	0.00090	0.00134
150	0.00398	0.00011	0.00022	0.00031	0.00043
300	0.00202	0.00005	0.00011	0.00015	0.00021
500	0.00123	0.00003	0.00006	0.00009	0.00012
1000	0.00058	0.00001	0.00003	0.00005	0.00006

This table presents the Monte Carlo simulation results (10,000 trials) for the *MSE* of our *aggregate* (Panel B) and *individual* (Panel C) decomposed systematic risk estimates. We generate data for a five-factor linear model that has the following structure: $r = 0.02 + 1.2f_1 + 0.1f_2 + 0.3f_3 - 0.2f_4 + 0.7f_5 + e$. Panel A reports the correlation coefficients between the five variables. In Panel B, f_1 through f_5 follow two hypothetical forms of distribution, multivariate-normal and multivariate-lognormal, respectively. The residuals (e) follow by turns, one of the two processes: homoscedastic [white noise with a mean of zero and a standard deviation of 0.21] and heteroscedastic [GARCH(1,1), with coefficients 0.001, 0.5, 0.3]. In Panel C, we define a population with a finite number of observations (25,000), where f_1 through f_5 follow a multivariate-normal distribution. The distribution of the residuals (e) is normal, with a mean of zero and a standard deviation of 0.21.

2.3. R-square decomposition

We derive an important extension of Eq. (3), dividing it by the estimated variance of asset j's returns ($\hat{\sigma}^2_j$). We are then able to decompose the estimate of the coefficient of determination, as follows

$$R^2_j = \sum_{k=1}^{K} DR^2_{j,k}, \quad \text{where } DR^2_{j,k} = \left(\hat{\beta}^{\perp}_{k_j}\frac{\hat{\sigma}_{fk}}{\hat{\sigma}_j}\right)^2. \tag{17}$$

Note that since the idiosyncratic risk can be measured as $(1 - R^2)$, the sum of the individual decomposed systematic risk measures and the idiosyncratic risk equals one. Moreover, from a statistical viewpoint, the decomposition of *R*-square characterizes the segments of goodness-of-fit. Parts of the total *R*-square can now be allocated unequivocally to each orthogonalized factor, indicating their relative contribution to the variation in the dependent variable (in our case, the return on the risky asset j).

3. Empirical illustration

We apply the orthogonal transformation procedure described in Section 2, to monthly returns on five well known equity pricing factors obtained from Kenneth R. French's Data Library: *RM*, *SMB*, *HML*, *Mom* and *Rev*. Historical observations suggest that market equity, book-to-market ratio, past short- or long-term returns may be proxies for exposures to various sources of systematic risk, not captured by the CAPM beta, and hence generating return premiums. Risk-based explanations for the return premiums to these factors might consider, for instance, that the returns on the *HML* and *SMB* portfolios seem to predict GDP growth, and thus they may be proxies for business cycle risk (see, for instance, Liew & Vassalou, 2000).

Once the transformation is performed, the decomposed systematic risk and decomposed *R*-square (henceforth denoted as DR^2) are calculated for style and industry portfolios, obtained from the same data library.

Table 3 reports the moments of the distribution and the correlation matrix of factors' monthly returns over a sample period from January 1931 to December 2008 for both the original and the orthogonally-transformed data.

As expected (see Panel A), sample variances are identical before and after the orthogonal transformation. Although the other distributional moments are different between non-orthogonalized and orthogonalized data, that does not affect the effectiveness of our decomposed systematic risk measures. Importantly, after transformation, the premia for all factors will change. In our case, the mean returns of *RM*, *HML* and *Mom* increase, while for *SMB* and *Rev*, they decrease.

Note that in Panel B of Table 3, the original *RM*, *SMB*, *HML*, *Mom*, and *Rev* are correlated with each other to various degrees (in absolute value, from 0.10 to 0.61), while after the orthogonal transformation they become uncorrelated. Importantly, the orthogonal factors maintain a high resemblance to their original counterparts; the correlation coefficients between the original and the orthogonally-transformed

Table 3
Distribution properties of factors' sample returns: original vs. orthogonal.

Panel A: Distribution parameters

	Original returns					Orthogonal returns				
	RM	*SMB*	*HML*	*Mom*	*Rev*	$RM^{\perp}$	$SMB^{\perp}$	$HML^{\perp}$	$Mom^{\perp}$	$Rev^{\perp}$
Mean	0.61	0.29	0.44	0.70	0.35	0.67	0.23	0.55	0.99	0.17
Std. Dev.	5.40	3.36	3.61	4.71	3.54	5.40	3.36	3.61	4.71	3.54
Skewness	0.30	2.29	1.91	−3.04	2.95	−0.24	1.44	0.63	−2.04	1.97
Kurtosis	8.34	22.99	16.11	28.33	24.22	4.82	14.89	5.54	15.45	17.40

Panel B: Correlation coefficients

Factor	*Original returns*					Factor	*Orthogonal Returns*				
	RM	*SMB*	*HML*	*Mom*	*Rev*		$RM^{\perp}$	$SMB^{\perp}$	$HML^{\perp}$	$Mom^{\perp}$	$Rev^{\perp}$
RM	1	0.33	0.23	−0.34	0.24	$RM^{\perp}$	1	0.00	0.00	0.00	0.00
SMB		1	0.10	−0.15	0.41	$SMB^{\perp}$	0.00	1	0.00	0.00	0.00
HML			1	−0.40	0.61	$HML^{\perp}$	0.00	0.00	1	0.00	0.00
Mom				1	−0.24	$Mom^{\perp}$	0.00	0.00	0.00	1	0.00
Rev					1	$Rev^{\perp}$	0.00	0.00	0.00	0.00	1

This table reports the distribution parameters (Panel A) and correlation coefficients (Panel B) for the monthly returns on five stock-market factor portfolios: *RM*, *SMB*, *HML*, *Mom* and *Rev*, both non-orthogonalized and orthogonalized. *RM*, *SMB* and *HML* are the Fama-French factors: *RM* is the market risk premium; *SMB* is Small Minus Big – size, while *HML* is High Minus Low – book-to-market. *Mom* is the momentum factor, while *Rev* is the long-term reversal factor. All five factors follow the description and are obtained from Kenneth French's data library at http://mba.tuck.dartmouth.edu/pages/faculty/ken.french/. The sample period is January 1931 to December 2008. The orthogonalized measures, denoted by the symbol "⊥", are obtained using the Schweinler–Wigner/Löwdin (1970) procedure.

returns are very high (i.e., 0.97, 0.96, 0.92, 0.97 and 0.91, respectively).

Next, we estimate the "orthogonal" beta coefficients using the orthogonally-transformed data, which avoids the loss of information incurred by the collinearity between factors.[19] Since more information is employed to obtain these coefficients, thus reducing their variances, they can be considered more stable or precise compared to their "non-orthogonal" equivalents (see, for instance, Kennedy, 2008, pp. 45, 46 and 194). We use the five-factor regression model in Eq. (1), applied to eight style portfolios, characterized by *size*, *value/growth*, *momentum*, and *contrarian*, respectively (*Small/Big – Growth*, *Value*, *Down Mom*, and *Low Rev*).[20] Table 4 presents the results for the equally-weighted portfolios.

Notably, correcting for correlations, the magnitude and/or the sign of betas can change. The absolute values of the orthogonal betas ($\hat{\beta}^{\perp}$) are generally higher than those of the non-orthogonal betas ($\hat{\beta}$). For instance, the non-orthogonal contrarian beta ($\hat{\beta}_{Rev}$) of the small-cap and contrarian-sensitive portfolio (*Small – Low Rev*) is 0.42, but its orthogonal beta $\hat{\beta}^{\perp}_{Rev}$ equals 0.95. Since the volatility estimates, $\hat{\sigma}$, are identical before and after orthogonalization, a lower $\hat{\beta}$ (as compared to $\hat{\beta}^{\perp}$) indicates that the systematic risk is underestimated, if the dependence between factors is ignored. As in Fama and French (1993), $\hat{\beta}_{HML}$ is negative for *Small-Growth* and *Big-Growth* portfolios. Also, it increases in magnitude from *Small-Growth* to *Big-Growth*. Interestingly, for *Small-Growth* it becomes positive, though small after orthogonalization. This alerts us to the possibility of drawing wrong conclusions if we do not adjust the model for correlations between variables. The results presented in Tables 3 and 4 highlight the importance of the orthogonal transformation in determining the proper premia and beta coefficients, and suggest that multi-factor market models should be used cautiously. Paired *t*-tests (not reported) indicate that, for the entire period (January 1931 to December 2008), the differences between the premia on the orthogonal and the original factors are statistically significant for *Mom* and *Rev* (at 1 percent level), *HML* (at 5 percent level), *SMB* (at 10 percent level), and barely significant for *RM*.[21]

We note that although the orthogonal beta ($\hat{\beta}^{\perp}$) assesses the sensitivity of an asset's return to the variation in the underlying component of a factor's premium, it alone cannot be used as a decomposed risk measurement. The appropriate approach is to take the product ${\hat{\beta}^{\perp}_{f_j}}^2\hat{\sigma}^2_f$. Table 5 shows the empirical results of systematic risk decomposition. Again, our methodology is applied to the eight style portfolios characterized by *size*, *value/growth*, *momentum*, and *contrarian*, respectively. We calculate the risk estimates for both the equally-weighted (Panel A) and value-weighted portfolios (Panel B). We use an Ordinary Least Squares (OLS) regression model and we calculate the systematic variation ($\hat{\sigma}^2_{s_j}$), taking the difference between the variance of a portfolio's returns ($\hat{\sigma}^2_j$) and its residual variance ($\hat{\sigma}^2_{\varepsilon_j}$) from the regression. From both Panel A and Panel B it is clear that the sum of the decomposed systematic risk values after the orthogonal transformation is exactly equal to the overall systematic variation ($\hat{\sigma}^2_{s_j}$), proving that the risk measure can be decomposed through orthogonalization. Conversely, due to the correlations between the original factors, as reported in Table 3, the sum of the non-orthogonal measures, $\hat{\beta}^2_{f_j}\hat{\sigma}^2_f$, is clearly different from $\hat{\sigma}^2_{s_j}$, and generally much smaller. A notable exception is, in our case, the *Big-Growth* value-weighted portfolio, due to the higher magnitudes of its original

[19] Again, the "orthogonal" betas are the coefficients obtained from the regression on the orthogonally-transformed factors, or simply by multiplying the correlation matrix Ψ by the original beta estimates. So, they are not orthogonal per se, but for ease of understanding, we prefer to call them such.

[20] The breakpoints and the resulting style portfolios (in parentheses), as described in Kenneth French's Data Library, are as follows: the size breakpoint is the median NYSE market equity at the end of June of each year (*Small/Big*); the book-to-market breakpoints are the 30th and 70th NYSE percentiles (*Growth/Neutral/Value*); the monthly prior (2–12) return breakpoints are the 30th and 70th NYSE percentiles (*Down/Medium/Up*); the monthly prior (13–60) return breakpoints are the 30th and 70th NYSE percentiles (*Low/Medium/High*).

[21] A popular way to estimate factor risk premia is the Fama–MacBeth (1973) two-pass approach. In the first pass, the beta coefficients are determined (in our case, both for the original and the orthogonalized factors). In the second pass, the risk premium for each factor is estimated, regressing the mean portfolio returns against the previously estimated betas. Our results, not reported in this paper (but available upon request), suggest insignificant differences between the original and the orthogonalized risk premia.

Table 4
Non-orthogonal vs. orthogonal beta estimates.

		Original factor returns					Orthogonal factor returns				
		$\hat{\beta}_{RM}$	$\hat{\beta}_{SMB}$	$\hat{\beta}_{HML}$	$\hat{\beta}_{Mom}$	$\hat{\beta}_{Rev}$	$\hat{\beta}^{\perp}_{RM}$	$\hat{\beta}^{\perp}_{SMB}$	$\hat{\beta}^{\perp}_{HML}$	$\hat{\beta}^{\perp}_{Mom}$	$\hat{\beta}^{\perp}_{rev}$
Small	Growth	1.0574	1.2421	−0.0920	−0.1910	−0.0497	1.2014	1.3960	0.0570	−0.4066	0.3140
		(71.27)***	(49.58)***	(−3.35)***	(−11.04)***	(−1.74)*	(91.01)***	(65.74)***	(2.89)***	(−26.83)***	(15.60)***
	Value	0.9608	1.1961	0.7280	−0.1635	0.1746	1.1754	1.3774	0.8635	−0.5134	0.7587
		(72.90)***	(53.74)***	(29.81)***	(−10.64)***	(6.87)***	(100.22)***	(73.01)***	(49.23)***	(−38.13)***	(42.42)***
	Down Mom	0.9905	1.2869	0.3427	−0.5859	0.1473	1.2496	1.4907	0.5957	−0.8649	0.6730
		(54.10)***	(41.63)***	(10.10)***	(−27.43)***	(4.17)***	(76.70)***	(56.88)***	(24.45)***	(−46.25)***	(27.09)***
	Low Rev	0.9855	1.3605	0.4973	−0.2119	0.4186	1.2281	1.5944	0.7413	−0.5512	0.9482
		(53.96)***	(44.12)***	(14.69)***	(−9.95)***	(11.88)***	(75.57)***	(60.99)***	(30.50)***	(−29.55)***	(38.26)***
Big	Growth	1.0596	0.1975	−0.2226	−0.0915	−0.0556	1.0504	0.3913	−0.0842	−0.2433	0.0476
		(142.42)***	(15.72)***	(−16.15)***	(−10.54)***	(−3.87)***	(158.65)***	(36.74)***	(−8.51)***	(−32.01)***	(4.72)***
	Value	1.1188	0.2258	0.8116	−0.1189	−0.0012	1.1968	0.4413	0.8944	−0.4569	0.4438
		(115.98)***	(13.86)***	(45.40)***	(−10.56)***	(−0.07)	(139.43)***	(31.97)***	(69.68)***	(−46.37)***	(33.90)***
	Down Mom	1.1379	0.2065	0.1419	−0.6348	−0.0227	1.2410	0.4547	0.3867	−0.8455	0.2580
		(128.65)***	(13.82)***	(8.66)***	(−61.53)***	(−1.33)	(157.68)***	(35.92)***	(32.86)***	(−93.59)***	(21.49)***
	Low Rev	1.1225	0.1000	0.2132	−0.0957	0.6474	1.1856	0.4589	0.5447	−0.3778	0.8175
		(107.02)***	(5.65)***	(10.97)***	(−7.82)***	(31.98)***	(127.02)***	(30.57)***	(39.03)***	(−35.27)***	(57.44)***

This table presents the estimated beta coefficients with and without an orthogonal transformation. We use monthly equally-weighted excess returns on eight style portfolios, obtained from Kenneth French's Data Library, for the time interval January 1931–December 2008 (936 observations). For variety we choose: four of the 6 Portfolios Formed on Size and Book-to-Market, two of the 6 Portfolios Formed on Size and Momentum, and two of the 6 Portfolios Formed on Size and Long-Term Reversal. By employing an OLS regression model, we estimate the beta coefficients with respect to five factors: *RM*, *SMB*, *HML*, *Mom* and *Rev*, respectively. *RM*, *SMB* and *HML* are the Fama-French factors: *RM* is the market risk premium; *SMB* is Small Minus Big – size, while *HML* is High Minus Low – book-to-market. *Mom* is the momentum factor, while *Rev* is the long-term reversal factor. The orthogonal measures are denoted by the symbol"⊥". The values of *t*-statistics are reported in parentheses. *, **, and *** denote significance levels of 10%, 5% and 1%, respectively.

Market and *Value* betas, relative to their orthogonal counterparts.

To examine the magnitude of individually-decomposed risk measures in relation to systematic and idiosyncratic risk, we further compute the decomposed *R*-squares, simply dividing $\hat{\beta}^{\perp 2}_{f_j}\hat{\sigma}^2_f$ by $\hat{\sigma}^2_j$. The sum of the DR^2 equals the overall systematic risk, and one minus the *R*-square becomes an assessment of the idiosyncratic risk. Moreover, the decomposition of the *R*-square can be used to discriminate against unimportant factors in model specification. Given our special interest in these important applications of our orthogonalization procedure, we extend the area of analysis to a more comprehensive list of style and industry portfolios. Firstly, Table 6 presents the risk decomposition for eighteen style portfolios (both equally-weighted and value-weighted): the six portfolios formed on *Size* and *Book-to-Market*, the six portfolios formed on *Size* and *Momentum*, and the six portfolios formed on *Size* and *Long-Term Reversal*.

Table 5
Orthogonal transformation and systematic risk decomposition.

		Original factor returns						$\hat{\sigma}^2_{sr_j}$	Orthogonal factor returns					
		$\hat{\beta}^2_{RM_j}\hat{\sigma}^2_{RM}$	$\hat{\beta}^2_{SMB_j}\hat{\sigma}^2_{SMB}$	$\hat{\beta}^2_{HML_j}\hat{\sigma}^2_{HML}$	$\hat{\beta}^2_{Mom_j}\hat{\sigma}^2_{Mom}$	$\hat{\beta}^2_{Rev_j}\hat{\sigma}^2_{Rev}$	Sum		Sum	$\hat{\beta}^{\perp 2}_{RM_j}\hat{\sigma}^2_{RM}$	$\hat{\beta}^{\perp 2}_{SMB_j}\hat{\sigma}^2_{SMB}$	$\hat{\beta}^{\perp 2}_{HML_j}\hat{\sigma}^2_{HML}$	$\hat{\beta}^{\perp 2}_{Mom_j}\hat{\sigma}^2_{Mom}$	$\hat{\beta}^{\perp 2}_{Rev_j}\hat{\sigma}^2_{Rev}$
Panel A: Equally-weighted portfolios														
Small	Growth	32.635	17.408	0.111	0.808	0.031	50.993	69.060	69.060	42.131	21.987	0.042	3.663	1.237
	Value	26.948	16.140	6.919	0.592	0.383	50.982	84.527	84.527	40.326	21.404	9.732	5.839	7.226
	Down Mom	28.638	18.685	1.533	7.605	0.273	56.734	97.544	97.544	45.584	25.070	4.632	16.572	5.685
	Low Rev	28.352	20.884	3.228	0.995	2.199	55.659	97.899	97.899	44.028	28.681	7.174	6.732	11.285
Big	Growth	32.775	0.440	0.647	0.186	0.039	34.086	35.364	35.364	32.205	1.727	0.093	1.311	0.028
	Value	36.535	0.575	8.597	0.313	0.000	46.020	61.546	61.546	41.811	2.198	10.441	4.624	2.472
	Down Mom	37.795	0.481	0.263	8.930	0.006	47.475	65.914	65.914	44.956	2.333	1.952	15.837	0.835
	Low Rev	36.777	0.113	0.593	0.203	5.261	42.947	58.829	58.829	41.028	2.376	3.873	3.163	8.389
Panel B: Value-weighted portfolios														
Small	Growth	35.031	12.247	0.573	0.077	0.005	47.934	60.757	60.757	41.353	16.771	0.036	1.527	1.070
	Value	30.189	9.482	7.277	0.043	0.020	47.011	69.520	69.520	39.879	13.301	8.759	3.351	4.231
	Down Mom	33.885	10.951	0.820	6.991	0.003	52.649	84.143	84.143	47.599	15.982	2.940	14.854	2.768
	Low Rev	32.547	12.922	2.102	0.179	1.692	49.442	80.699	80.699	44.267	19.441	5.001	3.788	8.202
Big	Growth	29.899	0.119	0.732	0.010	0.001	30.760	27.709	27.709	26.914	0.124	0.123	0.538	0.011
	Value	34.718	0.006	8.500	0.028	0.001	43.253	52.456	52.456	37.316	0.621	9.800	2.887	1.832
	Down Mom	33.913	0.063	0.017	10.154	0.000	44.146	56.307	56.307	38.473	0.370	1.157	15.895	0.412
	Low Rev	33.221	0.237	0.026	0.046	8.436	41.966	49.210	49.210	34.934	0.691	2.384	1.907	9.294

This table presents decomposed systematic risk measures with and without an orthogonal transformation. We use monthly equally-weighted (Panel A)/value-weighted (Panel B) excess returns on eight style portfolios, obtained from Kenneth French's Data Library for the time interval January 1931–December 2008 (936 observations). For variety, we choose: four of the 6 Portfolios Formed on Size and Book-to-Market, two of the 6 Portfolios Formed on Size and Momentum, and two of the 6 Portfolios Formed on Size and Long-Term Reversal. We calculate the beta coefficients and factor variances with respect to five factors: *RM*, *SMB*, *HML*, *Mom* and *Rev*, respectively. *RM*, *SMB* and *HML* are the Fama-French factors: *RM* is the market risk premium; *SMB* is Small Minus Big – size, while *HML* is High Minus Low – book-to-market. *Mom* is the momentum factor, while *Rev* is the long-term reversal factor. The systematic variation $\hat{\sigma}^2_{sr_j}$ is also computed, based on Eqs. (1) and (2). In addition, after employing an orthogonal transformation of the aforementioned factors, we re-estimate the beta coefficients. The orthogonal measures are denoted by the symbol "⊥".

Table 6
Risk decomposition for style portfolios.

		Small						Big					
		Decomposed-R^2					$1-R^2$	Decomposed-R^2					$1-R^2$
		RM	*SMB*	*HML*	*Mom*	*Rev*		*RM*	*SMB*	*HML*	*Mom*	*Rev*	
Panel A: Equally-weighted portfolios													
	Growth	57.09%	29.80%	0.06%	4.96%	1.68%	6.41%	88.10%	4.73%	0.25%	3.59%	0.08%	3.26%
BE/ME	Neutral	58.09%	25.02%	3.82%	5.10%	4.58%	3.40%	80.12%	3.20%	5.54%	5.05%	1.94%	4.15%
	Value	45.69%	24.25%	11.03%	6.62%	8.19%	4.23%	65.80%	3.46%	16.43%	7.28%	3.89%	3.15%
	Down	43.52%	23.93%	4.42%	15.82%	5.43%	6.88%	66.51%	3.45%	2.89%	23.43%	1.24%	2.49%
Momentum	Medium	53.39%	23.26%	7.68%	5.34%	5.76%	4.58%	81.98%	3.05%	4.14%	6.04%	1.03%	3.76%
	Up	59.37%	27.14%	2.51%	0.01%	6.17%	4.79%	87.08%	6.79%	0.57%	1.68%	0.84%	3.03%
	Low	41.90%	27.30%	6.83%	6.41%	10.74%	6.82%	67.05%	3.88%	6.33%	5.17%	13.71%	3.86%
Reversal	Medium	57.99%	19.99%	7.48%	6.69%	4.68%	3.18%	80.18%	2.73%	5.87%	4.36%	2.77%	4.09%
	High	62.65%	21.41%	3.83%	7.51%	0.28%	4.32%	88.44%	4.03%	0.78%	3.38%	0.55%	2.82%
Panel B: Value-weighted portfolios													
	Growth	66.52%	26.98%	0.06%	2.46%	1.72%	2.26%	95.10%	0.44%	0.43%	1.90%	0.04%	2.09%
BE/ME	Neutral	66.86%	20.87%	3.99%	2.74%	3.41%	2.13%	83.00%	0.17%	6.04%	4.25%	1.97%	4.56%
	Value	56.91%	18.98%	12.50%	4.78%	6.04%	0.79%	69.11%	1.15%	18.15%	5.35%	3.39%	2.85%
	Down	55.48%	18.63%	3.43%	17.31%	3.23%	1.92%	66.42%	0.64%	2.00%	27.44%	0.71%	2.79%
Momentum	Medium	63.17%	19.02%	6.12%	4.96%	3.58%	3.15%	86.78%	0.19%	2.24%	6.53%	0.29%	3.98%
	Up	68.76%	23.23%	1.38%	0.29%	3.81%	2.54%	92.26%	1.32%	0.43%	2.42%	0.10%	3.46%
	Low	53.67%	23.57%	6.06%	4.59%	9.94%	2.16%	68.03%	1.35%	4.64%	3.71%	18.10%	4.17%
Reversal	Medium	64.45%	16.13%	6.89%	5.67%	3.86%	3.01%	85.23%	0.23%	3.62%	3.04%	2.69%	5.19%
	High	68.44%	19.24%	2.89%	6.25%	0.01%	3.18%	93.26%	0.84%	0.17%	1.77%	1.21%	2.76%

We use three sets of monthly return data of style portfolios from Kenneth French's Data Library, including: the 6 Portfolios Formed on Size and Book-to-Market, the 6 Portfolios Formed on Size and Momentum, and the 6 Portfolios Formed on Size and Long-Term Reversal (all for the time interval January 1931–December 2008). *RM*, *SMB* and *HML* are the Fama/French factors: *RM* is the market risk premium; *SMB* is Small Minus Big – size, while *HML* is High Minus Low – book-to-market. *Mom* is the momentum factor, while *Rev* is the long-term reversal factor. The R^2 value is calculated based on the five-factor model in Eq. (1), while $1-R^2$ is a measure of idiosyncratic risk. Further, by employing an orthogonal transformation of common risk factors and determining orthogonal beta coefficients, we calculate, according to Eq. (17), the decomposed-R^2 values with respect to each factor. The sum of all decomposed-R^2 and $1-R^2$ equals 100%. Two panels are reported in this table: (A) equally-weighted and (B) value-weighted portfolios, respectively.

The proportions of risk contributed systematically by *Market* (*RM*), *Size* (*SMB*), *Value* (*HML*), *Momentum* (*Mom*), and *Contrarian* (*Rev*) for the equally-weighted (value-weighted) small-cap portfolios are, on average, 53% (63%), 25% (21%), 5% (5%), 6% (5%) and 5% (4%), respectively. This indicates that approximately 80 percent of the systematic return-variation of the small-cap portfolios is caused by two sources: *market* and *size* factors. The idiosyncratic risk of the value-weighted small-cap portfolios (around 2 percent) seems to be lower than that of the equally-weighted funds (approximately 5 percent).

For large-cap portfolios, on the other hand, roughly 80 percent of volatility comes from the *market* factor alone, while the *size* factor is relatively unimportant. Specifically, the decomposed *R*-squares of *size* are, on average, only 4 percent for the equally-weighted portfolios and under 1 percent for the value-weighted funds. The unsystematic risk is about 3–4 percent.

These results indicate that the conventional single index market model is valid for large-cap stocks, but one needs to consider the *size* factor for small-cap stocks. Furthermore, although the *value*, *momentum* and *contrarian* factors seem to be unimportant for average portfolios of both large-cap and small-cap stocks, they do have some impact on the volatility of the style portfolios that carry their names. For instance, 16.43 (18.15) percent of the overall volatility for equally-weighted (value-weighted) large-cap –value portfolio comes from the value-factor (*HML*). At the same time, 23.43 (27.44) percent of the volatility for equally-weighted (value-weighted) large-cap – down-momentum portfolio comes from the *momentum* factor (*Mom*). The decomposed *R*-squares of low-reversal portfolios with respect to the *Rev* factor range from 9.94% to 18.10%. This indicates that the factor-specification in market models is heterogeneous and varies by different styles of portfolio formation.

When comparing Tables 4 and 6, it is even more interesting that higher original betas (in absolute value) of one factor versus another factor do not necessarily imply a relatively higher importance of the former. For example, the four small-cap portfolios in Table 4 have higher betas of *SMB* compared to *RM*; still their corresponding decomposed-R^2 values are lower.

Next, to account for the distinct risk patterns among industries, we apply our risk decomposition procedure to monthly returns on 30-industry portfolios. As shown in Table 7, the unsystematic variation of industry portfolios is much larger than that of style portfolios. It ranges from 7.76% (*Fabricated Products and Machinery*) to 56.23% (*Tobacco Products*) for the equally-weighted portfolios and from 13.69% (*Banking*, *Insurance*, *Real Estate*, *Trading*) to 71.97% (*Coal*) for the value-weighted portfolios. This suggests that other factors, specific to particular industries, are influential. The equally-weighted portfolios exhibit greater proportions of *size* risk (DR^2_{SMB}) than the value-weighted portfolios. This confirms that the *size* factor is critical for pricing small-cap stocks. Furthermore, the *value* factor has a weak impact on the return variation of equally-weighted industry portfolios. For example, the decomposed *R*-square measures for the *value* factor (DR^2_{HML}) for *Transportation*, *Utilities*, *Finance*, and *Coal* are 12.12%, 9.00%, 8.68%, and 8.51%, respectively. It appears that the influence of the *momentum* and *contrarian* factors on the industry portfolios is small and relatively insignificant.

From the overall sample analysis for the period ranging from January 1931 to December 2008, more than 85 percent, on average, of the return variation of style portfolios is attributed to the DR^2 of *RM*, *SMB* and *HML*. This indicates that the Fama-French Three-Factor Model quantifies fairly well the risk-return structure of well-diversified equity portfolios. However, it is well known that the volatility of stock portfolios changes over time. An examination of the time variation of equity risk decomposition is important. Monthly *R*-squares and decomposed *R*-squares are computed based on overlapping regression estimation for every 60-month (*t*-59 to *t*) window, over a period ranging from January 1936 to December 2008. We illustrate, in Fig. 1, the dynamic

Table 7
Risk decomposition for the 30-industry portfolios.

Industry	Panel A: Equally-weighted portfolios						Panel B: Value-weighted portfolios					
	Decomposed-R^2					$1-R^2$	Decomposed-R^2					$1-R^2$
	RM	*SMB*	*HML*	*Mom*	*Rev*		*RM*	*SMB*	*HML*	*Mom*	*Rev*	
Food	58.46%	12.45%	5.78%	5.11%	3.61%	14.58%	67.56%	0.07%	1.22%	1.53%	0.10%	29.51%
Beer	36.86%	15.75%	3.21%	0.98%	2.42%	40.78%	44.65%	4.29%	1.63%	0.28%	1.07%	48.08%
Smoke	33.76%	3.81%	1.36%	2.39%	2.46%	56.23%	32.88%	0.04%	0.69%	1.22%	0.25%	64.92%
Games	51.26%	15.62%	2.98%	5.61%	2.73%	21.78%	60.47%	5.19%	1.01%	4.04%	1.57%	27.72%
Books	52.68%	18.54%	2.65%	4.58%	2.09%	19.47%	64.85%	5.31%	1.68%	1.91%	0.89%	25.36%
Hshld	57.45%	19.65%	2.30%	3.71%	3.84%	13.05%	66.36%	0.22%	0.08%	0.91%	1.43%	30.99%
Clths	44.65%	22.34%	5.44%	6.77%	4.11%	16.69%	44.66%	9.48%	0.06%	1.73%	0.32%	43.75%
Hlth	57.69%	17.53%	0.61%	1.49%	1.14%	21.54%	64.67%	0.16%	0.28%	0.93%	0.22%	33.73%
Chems	69.16%	11.15%	3.19%	4.13%	1.70%	10.66%	75.38%	0.08%	0.52%	2.19%	0.58%	21.25%
Txtls	47.83%	18.14%	4.32%	5.37%	2.98%	21.36%	55.98%	9.82%	4.13%	4.49%	1.88%	23.70%
Cnstr	55.62%	21.03%	4.42%	4.49%	3.54%	10.89%	77.22%	4.14%	1.28%	1.97%	0.67%	14.71%
Steel	59.94%	10.85%	7.19%	6.47%	2.86%	12.69%	65.43%	3.00%	3.45%	4.22%	1.63%	22.27%
FabPr	64.77%	17.55%	3.21%	4.02%	2.69%	7.76%	75.75%	4.23%	1.37%	3.83%	0.92%	13.90%
ElcEq	63.66%	15.84%	1.72%	3.22%	2.27%	13.28%	77.36%	0.50%	0.31%	2.74%	1.29%	17.80%
Autos	59.08%	14.86%	4.65%	5.81%	4.04%	11.55%	63.32%	0.65%	2.25%	5.26%	2.65%	25.88%
Carry	52.99%	12.68%	7.21%	3.74%	1.92%	21.46%	60.95%	3.50%	4.83%	2.63%	0.62%	27.47%
Mines	35.57%	11.98%	4.84%	2.23%	1.79%	43.60%	40.44%	4.32%	2.05%	1.01%	0.12%	52.06%
Coal	26.62%	11.52%	8.51%	3.04%	0.75%	49.55%	22.13%	3.66%	1.90%	0.00%	0.34%	71.97%
Oil	49.36%	6.96%	5.54%	0.77%	0.49%	36.87%	59.66%	0.06%	4.41%	0.26%	0.02%	35.59%
Util	46.08%	2.35%	9.00%	4.54%	5.28%	32.75%	51.66%	0.01%	6.59%	3.05%	1.42%	37.27%
Telcm	55.72%	10.63%	0.84%	4.15%	0.78%	27.88%	55.38%	0.00%	0.07%	3.70%	0.43%	40.42%
Servs	39.49%	24.60%	0.36%	1.68%	0.99%	32.88%	29.23%	4.53%	3.16%	0.14%	0.01%	62.93%
BusEq	59.54%	19.89%	0.47%	2.73%	1.68%	15.70%	70.49%	2.61%	4.56%	1.18%	0.41%	20.75%
Paper	53.88%	18.38%	3.49%	4.93%	2.51%	16.82%	71.15%	0.49%	0.91%	2.48%	0.24%	24.72%
Trans	45.72%	16.94%	12.12%	7.41%	3.98%	13.83%	63.30%	3.35%	8.74%	5.42%	0.54%	18.64%
Whlsl	40.20%	29.64%	1.55%	2.52%	3.97%	22.11%	58.90%	11.13%	0.61%	0.90%	1.11%	27.35%
Rtail	56.43%	18.29%	1.44%	5.73%	2.53%	15.58%	69.42%	1.68%	0.01%	3.11%	0.32%	25.45%
Meals	44.20%	18.99%	1.32%	2.09%	0.67%	32.73%	54.04%	5.93%	0.46%	1.24%	0.05%	38.29%
Fin	56.98%	11.13%	8.68%	7.24%	4.17%	11.79%	75.16%	0.61%	3.55%	6.13%	0.84%	13.69%
Other	49.34%	22.70%	1.31%	3.13%	3.14%	20.38%	65.77%	5.58%	0.13%	1.35%	0.83%	26.33%

We apply the R^2 decomposition approach to monthly returns on 30-industry portfolios obtained from Kenneth French's Data Library. The sample period is January 1931–December 2008. *RM*, *SMB* and *HML* are the Fama/French factors. *Mom* is the momentum factor, while *Rev* is the long-term reversal factor. The R^2 value is calculated based on the five-factor model in Eq. (1), while $1-R^2$ is a measure of idiosyncratic risk. After an orthogonal transformation of common risk factors, we calculate decomposed R^2 with respect to each factor. The sum of all decomposed R^2 and $1-R^2$ equals 100 percent. Two panels are reported in this table: (A) equally-weighted and (B) value-weighted portfolios, respectively.

risk-decomposition for the value-weighted *Small-Value* and *Big-Value* style portfolios. In this case, for ease of comparability with Fama and French (1993), they are selected from the Fama and French 25 portfolios formed on size and book-to-market. Again, the volatility is decomposed linearly into six components: *market* (DR^2_{RM}), *size* (DR^2_{SMB}), *value* (DR^2_{HML}), *momentum* (DR^2_{Mom}), *contrarian* (DR^2_{Rev}), and *idiosyncratic risk* ($1-R^2$).

In general, the largest component of return variation is captured by the *market* factor. *RM* maintains a similar importance when going from *Small* to *Big-Value* (DR^2_{RM} for *Small* and for *Big* have a high correlation of 77 percent). Conversely, *SMB* is highly significant for *Small* and insignificant for *Big*, while *HML* is moderately more significant for *Big*. These results are in line with Fama and French (1993). Additionally, as expected, the idiosyncratic component is consistently higher for *Big-Value*. In the two cases, both DR^2_{Mom} and DR^2_{Rev} values are small: with a few exceptions they fall below 20 percent. From Fig. 1, it appears that the components of risk not only are dynamic over time, but they may also exhibit significant correlations.[22] For example, DR^2_{Rev} (for *Small*) and DR^2_{HML} (for *Big*) move inversely with DR^2_{RM} (with correlation coefficients of −0.62 and −0.67, respectively).

Recently, a number of studies have used Hansen's (1982) Generalized Method of Moments (GMM) econometric procedure in

[22] Finding the possible reasons for the increases and decreases in the individual components over time is beyond the purpose of this paper, and we leave it to future research.

Table 8
GMM test results for style portfolios.

	RM	*SMB*	*HML*	*Mom*	*Rev*	J_T-Test
Panel A: Original factors						
μ	0.500	0.294	0.466	0.738	0.295	33.444
	2.739	*2.730*	*3.874*	*5.430*	*2.388*	(0.000)
b	0.039	0.050	0.072	0.068	−0.017	
	4.021	*3.393*	*3.648*	*5.989*	*−0.845*	
Panel B: Orthogonal factors						
μ	0.563	0.266	0.618	1.016	0.084	33.410
	2.980	*2.482*	*4.749*	*7.332*	*0.617*	(0.000)
b	0.037	0.047	0.050	0.046	0.015	
	4.222	*3.856*	*3.736*	*5.321*	*1.105*	

This table reports the results for Hansen's (1982) GMM procedure, considering the original factors (Panel A) and the orthogonalized factors (Panel B). *RM*, *SMB* and *HML* are the Fama/French factors: *RM* is the market risk premium; *SMB* is Small Minus Big – size, while *HML* is High Minus Low – book-to-market. *Mom* is the momentum factor, while *Rev* is the long-term reversal factor. We use monthly equally-weighted excess returns on eight style portfolios, obtained from Kenneth French's Data Library, for the time interval January 1931–December 2008 (936 observations). For variety we choose: four of the 6 Portfolios Formed on Size and Book-to-Market (*Small/Growth*, *Small/Value*, *Big/Growth* and *Big/Value*), two of the 6 Portfolios Formed on Size and Momentum (*Small/Down Mom* and *Big/Down Mom*), and two of the 6 Portfolios Formed on Size and Long-Term Reversal (*Small/Low Rev* and *Big/Low Rev*). We have the following $K+N$ moment conditions: $E(f_t)=\mu$, $E[r^j_t - r^j_t b'(f_t-\mu)]=0$, where K is the number of factors (five in this case) and N is the number of portfolios (eight in this instance). We have $2K$ parameters to estimate: K μ's and K b's. The estimates and their *t*-statistics (in italics) are provided in the table. The table also reports the J_T test (*p*-value in parenthesis), which assesses the null hypothesis that the model is valid and follows a χ^2 distribution with the number of degrees of freedom equal to the number of over-identifications, $N-K$.

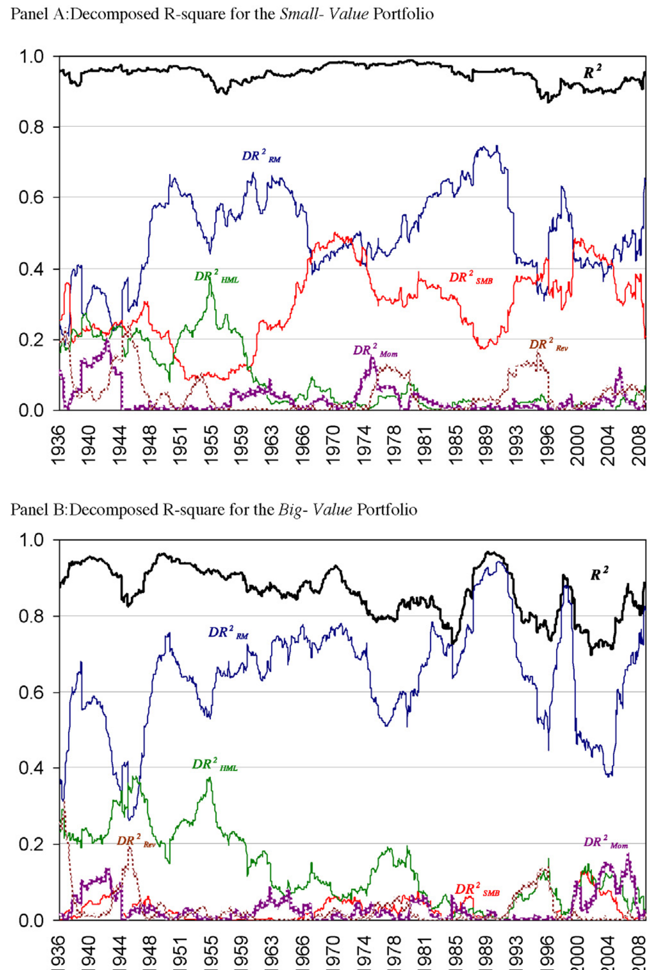

Fig. 1. Decomposed risk over time. The figure graphs the monthly variation in equity risk decomposition, from January 1936 to December 2008. The R^2 and the decomposed-R^2 (denoted as DR^2) are calculated based on overlapping regression estimation for every 60-month (t-59 to t) window. Specifically, we present our empirical results on R^2 and DR^2 for two of the 25 value-weighted portfolios formed on Size and Book-to-Market (i.e., *Small-Value* and *Big-Value*), obtained from Kenneth French's data library at http://mba.tuck.dartmouth.edu/pages/faculty/ken.french/.

order to estimate asset pricing models, adopting the pricing kernel instead of the risk premium approach (see, for instance, Dumas & Solnik, 1995; Lozano & Rubio, 2011; Nagel & Singleton, 2011; Zhou, 1994). We also use this alternative to further identify the consequences of employing our orthogonalized factors instead of the original. The pricing kernel approach does not consider betas, but rather focuses on the prices of risk, b_{kj}, as in the following equation

$$E(r_t^j) = \sum_{k=1}^{K} b_{k_j}\ cov(f_t^k, r_t^j), \tag{18}$$

where $k = 1, \ldots, K$ are the factors and $j = 1, \ldots, N$ are the portfolios.

The pricing kernel, M_t, has the following properties:

$$E[M_t] = 1, \tag{19}$$

$$E[M_t r_t^j] = 0. \tag{20}$$

In a linear asset pricing model, the stochastic discount factor (SDF) can be identified as

$$M_t = a - b'(f_t - \mu). \tag{21}$$

By normalization, a can simply be set to one. In this case, the moment conditions are

$$E(f_t) = \mu \tag{22a}$$

$$E[r_t^j - r_t^j b'(f_t - \mu)] = 0, \tag{22b}$$

We have K factors and this gives us $2K$ parameters (b and μ) to estimate. Eqs. (22a) and (22b) provide $K+N$ moment conditions. The GMM procedure reports estimates for the parameters, as well as the J_T test of over-identifying restrictions. The J_T test calculates the sum of squared pricing errors and follows a χ^2 distribution, with the number of degrees of freedom equal to the number of over-identifications ($N-K$ in our case).

In Table 8 we report the GMM results for both the original (Panel A) and the orthogonal (Panel B) factor returns, considering the same style portfolios as in Tables 4 and 5. As expected, the adequacy of the models does not change as the J_T test rejects both models. However, the coefficients and their statistical significance do change. So, either focusing on the beta loadings, or prices of risk, the orthogonalization procedure corrects for potential errors in estimates resulting from correlations between factors.

4. Conclusions

Financial researchers and professionals have extensively used multi-factor models employing additional variables to the *market* factor, such as *size*, *value*, *momentum*, and/or *reversal*. Due to the dependence among factors, decomposing the systematic variation of asset returns with respect to different factors has been a methodological challenge. This study aims to fill this gap and proposes a simple procedure for decomposing the coefficient of determination (R-square). This procedure allows us to examine the marginal contribution of individual factors to the return volatility of an asset. The key component of our procedure is a simultaneously orthogonal transformation of the data, which is able to extract jointly the underlying uncorrelated components of individual factors. The covariance between the original factors is eliminated symmetrically, to achieve a maximum overall resemblance between the original and the transformed data sets. Experimentally, it appears that the decomposition is robust even for small sample sizes.

The decomposition procedure is applied to monthly return data on U.S. style and industry portfolios, obtained from Kenneth French's Data Library, for the time interval January 1931–December 2008. In absolute value, orthogonal beta estimates are generally greater than the original betas. So, not removing the correlations, we generally underestimate the contribution of risk attributable to each factor (given that the variance of each factor is preserved after the orthogonal transformation). In general, the return variation of well-diversified equity portfolios is explained, primarily, by the *market* and *size* factors (in this order). Nevertheless, the decomposed elements of systematic risk and the systematic risk itself change over time.

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Appendix A. Intercept and error term after the orthogonal transformation

We transform matrices $F_{T\times K}$ and $F_{T\times K}^{\perp}$, defined in Section 2.1, by inserting a column vector of ones, to account for the constant term.

We name the resulting matrices $X_{T\times(K+1)}$ and $X^{\perp}_{T\times(K+1)}$, respectively. The matrix $\psi_{K\times K}$, whose inverse orthogonalizes $F_{T\times K}$, as in Eq. (15), becomes

$$\tilde{\Psi}_{(K+1)\times(K+1)} = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & \psi_{11} & \cdots & \psi_{1K} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & \psi_{K1} & \cdots & \psi_{KK} \end{bmatrix}$$

The "orthogonal" beta coefficients can be calculated as follows:

$$\begin{aligned}\hat{\beta}^{\perp}_{(K+1)\times 1} &= \left[(X\tilde{\Psi}^{-1})'X\tilde{\Psi}^{-1}\right]^{-1}(X\tilde{\Psi}^{-1})'Y = [(\tilde{\Psi}^{-1})'(X'X)\tilde{\Psi}^{-1}]^{-1} \\ &\quad \times(X\tilde{\Psi}^{-1})'Y = \tilde{\Psi}(X'X)^{-1}[(\tilde{\Psi}^{-1})']^{-1}(\tilde{\Psi}^{-1})'X'Y \\ &= \tilde{\Psi}[(X'X)^{-1}X'Y] = \tilde{\Psi}_{(K+1)\times(K+1)}\hat{\beta}_{(K+1)\times 1}\end{aligned}$$

Consequently, it is straightforward that $\hat{\alpha}^{\perp} = \hat{\alpha}$. Moreover, since $X^{\perp}\hat{\beta}^{\perp} = (X\tilde{\Psi}^{-1})(\tilde{\Psi}\hat{\beta}) = X\hat{\beta}$, we obtain that $\varepsilon^{\perp} = \varepsilon$. So, both the intercept and the error term remain unchanged after the orthogonal transformation.

Appendix B. The nature of the relationship between the original and the orthogonalized factors

Considering that each factor f^k can be written, according to Eq. (13) as $f^k = \psi_{1k}f^{1\perp} + \psi_{2k}f^{2\perp} + \cdots + \psi_{Kk}f^{K\perp}$, where $k = 1, 2, \ldots, K$ and the coefficients ψ_{lk} are the elements of the inverse of matrix $S_{K\times K}$, as transformed in Eq. (11), we want to prove that for any k and l, $\psi_{kl} = \mathrm{corr}(f^k, f^{l\perp})$.

Proof. The inverse of matrix $S_{K\times K}$ (in its final form), can be written as follows

$$\begin{aligned} S^{-1}_{K\times K} &= \frac{1}{\sqrt{T-1}}\begin{bmatrix} 1/\sigma_1 & 0 & \cdots & 0 \\ 0 & 1/\sigma_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1/\sigma_K \end{bmatrix}\begin{bmatrix} \dot{s}_{11} & \dot{s}_{12} & \cdots & \dot{s}_{1K} \\ \dot{s}_{21} & \dot{s}_{22} & \cdots & \dot{s}_{2K} \\ \vdots & \vdots & \ddots & \vdots \\ \dot{s}_{K1} & \dot{s}_{K2} & \cdots & \dot{s}_{KK} \end{bmatrix} \\ &= \frac{1}{\sqrt{T-1}}\begin{bmatrix} \dot{s}_{11}/\sigma_1 & \dot{s}_{12}/\sigma_1 & \cdots & \dot{s}_{1K}/\sigma_1 \\ \dot{s}_{21}/\sigma_2 & \dot{s}_{22}/\sigma_2 & \cdots & \dot{s}_{2K}/\sigma_2 \\ \vdots & \vdots & \ddots & \vdots \\ \dot{s}_{K1}/\sigma_K & \dot{s}_{K2}/\sigma_K & \cdots & \dot{s}_{KK}/\sigma_K \end{bmatrix},\end{aligned}$$

where $\dot{s}_{kl} = \dot{s}_{lk}$ (the inverse of a symmetric matrix is symmetric).

So, $\psi_{lk} = [1/(\sqrt{T-1})][\dot{s}_{lk}/\sigma_l]$, where k and l go from 1 to K.

Thus, $f^k = (1/(\sqrt{T-1}))((\dot{s}_{1k}/\sigma_1)f^{1\perp} + (\dot{s}_{2k}/\sigma_2)f^{2\perp} + \cdots + (\dot{s}_{lk}/\sigma_l)f^{l\perp} + \cdots + (\dot{s}_{Kk}/\sigma_K)f^{K\perp})$.

We can now calculate the covariance between f^k and $f^{l\perp}$. Note that for any $i = 1, 2, \ldots, K$, $var(f^{i\perp}) = var(f^i)$.

$$cov(f^k, f^{l\perp}) = \frac{1}{\sqrt{T-1}}\frac{\dot{s}_{lk}}{\sigma_l}var(f^{l\perp}) = \frac{1}{\sqrt{T-1}} \times \dot{s}_{lk} \times \sigma_l$$

Similarly, $cov(f^l, f^{k\perp}) = (1/\sqrt{T-1}) \times \dot{s}_{kl} \times \sigma_k$.

But

$$\dot{s}_{lk} = \dot{s}_{kl} \Rightarrow \frac{cov(f^k, f^{l\perp})}{\sigma_l} = \frac{cov(f^l, f^{k\perp})}{\sigma_k}.$$

Writing Eq. (13) for factor l, we have

$$f^l = \psi_{1l}f^{1\perp} + \psi_{2l}f^{2\perp} + \cdots + \psi_{kl}f^{k\perp} + \cdots + \psi_{Kl}f^{K\perp}.$$

Thus, $cov(f^l, f^{k\perp}) = \psi_{kl} \times var(f^{k\perp}) \Rightarrow$

$$\begin{aligned}\psi_{kl} &= \frac{cov(f^l, f^{k\perp})}{var(f^{k\perp})} \overset{\text{from}(*)}{=} \frac{cov(f^k, f^{l\perp}) \times \sigma_k}{\sigma_l \times \sigma_k^2} \\ &= \frac{cov(f^k, f^{l\perp})}{\sigma_l \times \sigma_k} = \mathrm{corr}(f^k, f^{l\perp}).\end{aligned}$$

□

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