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Ex Post Risk and Risk-Adjusted Return Measures: Foundations and Formulae

BarraOne Performance Analytics

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October 2013

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Overview

This white paper supports the BarraOne Performance Analytics Ex Post Risk and Risk-Adjusted Return Measures which have been updated for the release of BarraOne 3.8. These measures are provided in addition to attribution and contribution results provided by BarraOne Performance Analytics.

The BarraOne Performance Analytics Ex Post Risk and Risk-Adjusted Return Measures fulfill an important role in the appraisal component of the performance evaluation process. They can be used to understand whether the portfolio manager achieves a sufficient level of return for the amount of risk taken and if the portfolio's performance is likely to be sustained.

This paper presents the exact specification of how the BarraOne Performance Analytics Ex Post Risk and Risk-Adjusted Return Measures are calculated. This is not straightforward task and we will see that the precise calculation of seemingly straightforward measures is particularly subtle. The following questions apply: What is the frequency of single-period returns? How many days per year should be used when the return frequency is daily? When are multi-period or annualized returns appropriate? How is excess and active return defined? How is risk defined? Are arithmetic mean or geometric mean returns appropriate? Should sample or population-based variance be used? How is the horizon of a measure matched with the horizon of the reporting period? The framework for BarraOne Performance Analytics Ex Post Risk and Risk-Adjusted Return Measures is built on top of rigorous, analytical foundations and addresses all of these questions. The analytical foundations are discussed in Section 1. Sections 2 and 3 present the formulae for BarraOne Performance Analytics Ex Post Risk and Risk-Adjusted Return Measures. Section 4 presents the formulae for BarraOne Performance Analytics Ex Post Return Measures. The analytical foundations are completely consistent with those used to produce the attribution and contribution results by BarraOne Performance Analytics.

1 Foundations

This section presents the analytical foundations for the BarraOne Performance Analytics Ex Post Risk and Risk-Adjusted Return Measures.

1.1 Frequency of Single-Period Returns

When carrying out a performance analysis, a daily, weekly or monthly return frequency must be specified to determine the periodicity of each single-period return used in the calculation of the various BarraOne Performance Analytics Ex Post Risk and Risk-Adjusted Return Measures. A minimum number of single-period returns is required, and the BarraOne Performance Analytics Ex Post Risk and Risk-Adjusted Return Measures are not applicable if the reporting period is so short that there is insufficient data for their accurate calculation:

- If a monthly return frequency is specified, BarraOne Performance Analytics Ex Post Risk and Risk-Adjusted Return Measures will not be calculated if the reporting period is shorter than six months.
- If a weekly return frequency is specified, BarraOne Performance Analytics Ex Post Risk and Risk-Adjusted Return Measures will not be calculated if the reporting period is shorter than six weeks.
- If a daily return frequency is specified, BarraOne Performance Analytics Ex Post Risk and Risk-Adjusted Return Measures will not be calculated if the reporting period is shorter than 20 business days; for daily return frequency, non-business days are not included.

The minimum number of single-period returns required is numerically and not financially motivated. Although there are no formal standards on a minimum number of data points to calculate ex post risk and risk-adjusted return measures, GIPS requires 36 months of data for ex posts statistics on composites, UCITS rules require far more than 6 weekly observations.

1.2 Annualization of Returns

The goal of annualization is to arrive at a standard measure of return that can be used to compare performance across reporting periods of differing lengths. The caveat is that annualization should only ever be carried out if the reporting period is 1 year or more. If annualization were to be carried out for reporting periods of less than 1 year then we would be attempting to project what the performance would be over the complete year. This would be misleading as annualization is to enable comparability and not to forecast returns.

The multi-period (or cumulative) return R_T is calculated by compounding single-period returns R_t over the number of single periods as follows:

$$R_T = \prod_{t=1}^{T}(1 + R_t) - 1 \quad (1)$$

where:

R_T	Multi-period (or cumulative) return over the reporting period
R_t	Single-period return
T	Number of single periods in the reporting period, for the daily return frequency BarraOne uses the convention that T is the number of business days from the start of the reporting period to the end

Note 1: Single-period return R_t is the official base return in each period. This top-level return in the base currency of the portfolio can be provided by the user. If this is not available then the base return is calculated bottom-up from the constituents. This also applies to the benchmark, where the top-level return is the vendor’s published index return.

Note 2: For the daily return frequency, the multi-period return R_T includes non-zero returns on any calendar day during the reporting period since this includes the possibility of there being a return on a non-business day.

Note 3: BarraOne Performance Analytics Ex Post Risk and Risk-Adjusted Return Measures, as well as returns used in their calculation are always annualized for reporting periods of 1 year or more.

The annualized multi-period return R_{ann} is calculated as the multi-period return R_T annualized according to the number of single periods in one year T_{ann}, which is determined by the frequency of the single period returns:

$$R_{ann} = (1 + R_T)^{\frac{T_{ann}}{T}} - 1 \tag{1}$$

where:

R_{ann}	Annualized multi-period return
R_T	Multi-period return over reporting period T calculated by compounding single-period returns, see – (1)
T_{ann}	Number of single periods in one year $= \{12, 52, 252\}$ and corresponds to a single-period t being monthly, weekly or daily
T	Number of single periods in the reporting period

1.2.1 Number of Periods per Year for the Daily Return Frequency

When the frequency of returns is set to daily the convention for the number of periods per year T is 252 for annualizing both returns and risk measures.

Although it is possible that a portfolio could have more than 252 trading days in a year we use 252 days always. A possible counter example is for a portfolio containing securities simultaneously from US, EMEA and GCC regions which could have a return on 6 days per week since certain GCC markets trade on Sundays. In reality though emerging markets strategies are managed separately from the developed market strategies, so a particular portfolio is unlikely to have securities from both GCC and non-GCC regions. Using 252 means the BarraOne Ex Post Risk and Risk-Adjusted Return Measures are annualized consistently with how returns and attribution model results are annualized in BarraOne Performance Analytics.

1.2.2 Excess and Active Returns

It is important to specify the order in which relative return, be it an active or excess relative return, and annualization calculations are carried out. Annualized active return is required in the calculation of information ratio (Section 3.8), and it is calculated by annualizing the portfolio and the benchmark multi-period return separately, using Equation (2), before conducting the active return calculation. Annualized portfolio return in excess of the risk free rate of return is required in the calculation of Sharpe ratio (Section 3.1), Sortino ratio (Section 3.2), Treynor ratio (Section 3.3), and Jensen's Alpha (Section 3.5), and it is calculated by annualizing the portfolio multi-period return and the risk-free interest rate multi-period return separately, using Equation (2), before conducting the excess return calculation.

2 Ex Post Risk Measures

The most commonly used ex post risk measures are volatility and beta, the former is a measure of the degree of dispersion around the mean level of return, whereas the latter is a measure of the systematic risk that the portfolio is subjected to. This section provides the specification of the BarraOne Ex Post Risk Measures, namely volatility and beta, as well as the building blocks required in the calculation of these measures, which are variance, covariance, and correlation. The specification of R-squared is also provided.

2.1 Variance

The single-period variance of the return R_t^P of portfolio P over T single periods is:

$$\mathrm{Var}(R^P) = \frac{1}{T-1}\sum_{t=1}^{T}(R_t^P - \bar{R}^P)^2 \tag{1}$$

where:

$\mathrm{Var}(R^P)$	Variance of the return of portfolio P
R_t^P	Portfolio return over single-period t
T	Number of single periods in the reporting period

and where the mean portfolio return $\bar{R}^P$ over T single periods is:

$$\bar{R}^P = \frac{1}{T}\sum_{t=1}^{T} R_t^P. \tag{1}$$

The single-period variance of the return R_t^B of benchmark B is given by Equation (3) with B substituted for P.

A geometric mean could be used rather than an arithmetic mean in the calculation of variance (1). This would lead to a geometric standard deviation with an implicit assumption that the returns were drawn from a log-normally distributed population. An arithmetic mean return leads to an arithmetic standard deviation with an implicit assumption that the returns were drawn from a normally distributed population. The merits of one over the other is outside the scope of this white paper but it suffices to state that the mean return used in BarraOne Ex Post Risk and Risk-Adjusted Return Measures is the arithmetic mean return and this is used consistently in standard deviation / volatility, covariance and all other measures involving these quantities.

On a separate note we could have chosen to use T rather than $T-1$ to calculate a population variance rather than a sample variance. To choose population over sample variance we should know what the population is that we are drawing observations from. However, knowing the population is subjective since if we fix the reporting period then we know how many single-period returns there will be but since the user is free to set the reporting period to any length from the available time series of portfolio returns we prefer to use sample variance.

2.1.1 Matching the Horizon of Variance to the Horizon of the Reporting Period

The calculation of variance (3) determines a single period variance which depends on the return frequency of the single-period returns, which may be daily, weekly or monthly returns. We ensure that the horizon of variance matches the horizon of the reporting period using the additive property of variance.

The horizon variance is the single-period variance of returns $\mathrm{Var}(R^P)$ multiplied by either the:

1. Number of single periods in the reporting period, T, if the reporting period is less than one year, such that the horizon variance is

$$\mathrm{Var}_T(R^P) = \mathrm{Var}(R^P)T \quad \text{if} \quad T < 1 \text{ year} \tag{2}$$

2. Number of single periods in one year, T_{ann}, if the reporting period is 1 year or more, such that the horizon variance is

$$\mathrm{Var}_{ann}(R^P) = \mathrm{Var}(R^P)T_{ann} \qquad \text{if} \quad T \geq 1 \text{ year} \tag{3}$$

where $T_{ann} = \{12, 52, 252\}$ and corresponds to a single-period t being monthly, weekly or daily.

For a reporting period of 1 year or more variance is annualized, here the horizon of the reporting period is always assumed to be 1 year.

2.2 Volatility

BarraOne Performance Analytics calculates the ex post risk measure of volatility as the standard deviation of the single-period returns R_t^P of portfolio P over T single periods, which is the square root of variance (3):

$$\sigma(R^P) = \sqrt{\frac{1}{T-1}\sum_{t=1}^{T}(R_t^P - \bar{R}^P)^2} \tag{3}$$

where:

$\sigma(R^P)$	Volatility of the return of portfolio P
R_t^P	Portfolio return over single-period t

T	Number of single periods in the reporting period
$\bar{R}^P$	Mean portfolio return over T single periods — see (4)

The volatility of benchmark returns R_t^B is given by Equation (3) with B substituted for P.

2.2.1 Matching the Horizon of the Volatility to the Horizon of the Reporting Period

The calculation of volatility (3) provides a single period volatility which depends on the return frequency of the single period returns, which may be daily, weekly or monthly returns. We ensure that the horizon of the volatility matches the horizon of the reporting period by using what is known as the "square root of T rule". It is a logical result of the additive property of variance.

The horizon volatility is the single-period volatility of returns $\sigma(R^P)$ multiplied by either the:

1. Square root of the number of single periods in the reporting period, T, if the reporting period is less than one year, such that the horizon volatility is

$$\sigma_T(R^P) = \sigma(R^P)\sqrt{T} \quad \text{if} \quad T < 1 \text{ year} \tag{4}$$

2. Square root of the number of single periods in one year, T_{ann}, if the reporting period is 1 year or more, such that the horizon volatility is

$$\sigma_{ann}(R^P) = \sigma(R^P)\sqrt{T_{ann}} \quad \text{if} \quad T \geq 1 \text{ year} \tag{5}$$

where $T_{ann} = \{12, 52, 252\}$ and corresponds to a single-period t being monthly, weekly, or daily

For a reporting period of 1 year or more volatility is annualized, here the horizon of the reporting period is always assumed to be 1 year.

Volatility is reported as a percentage.

Horizon volatility is required in the calculation of the following BarraOne Performance Analytics Ex Post Risk-Adjusted Return Measures: Sharpe ratio (Section 3.1); Sortino ratio (Section 3.2), in which a similar horizon matching technique is used, but in which Sortino ratio involves semideviation rather than standard deviation; M-Squared (Section 3.3); tracking error (Section 3.7); and information ratio (Section 3.8).

2.2.2 Attributing Volatility

The standalone horizon volatility $\sigma_T(Q_m)$ of the multi-period contribution Q_m for an attribute computed via multi-period linking is given by:

$$\sigma_T(Q_m) = \sigma(Q_m)\sqrt{T} \quad \text{if} \quad T < 1 \text{ year} \tag{6}$$

where

T	Number of single periods in the reporting period

or

$$\sigma_{ann}(Q_m) = \sigma(Q_m)\sqrt{T_{ann}} \quad \text{if} \quad T \geq 1 \text{ year} \tag{7}$$

where

T_{ann} Number of single periods in a year

and where the standalone volatility $\sigma(Q_m)$ of attribution effect m is:

$$\sigma(Q_m) = \sqrt{\frac{1}{T-1}\sum_{t=1}^{T}\left(Q_{m,t} - \bar{Q}_m\right)^2}, \tag{7}$$

where:

$t = 1, \dots, T$ Range of dates specifying a time series of single periods

$Q_{m,t}$ Attribution effect over a single-period t,

where the mean attribution effect over T single periods is:

$$\bar{Q}_m = \frac{1}{T}\sum_{t=1}^{T} Q_{m,t}. \tag{7}$$

Volatility is attributed to attribution effects only in the Equity Factor Attribution model.

2.3 Covariance

Covariance measures the tendency of portfolio and benchmark returns to move together:

$$\mathrm{Cov}(R^P, R^B) = \frac{1}{T-1}\sum_{\mathrm{t}=1}^{\mathrm{T}}(R_t^P - \bar{R}^P)(R_t^B - \bar{R}^B) \tag{7}$$

where:

$\mathrm{Cov}(R^P, R^B)$	Covariance between the time series of portfolio and benchmark returns
R^P	Time series of portfolio P returns
R^B	Time series of benchmark B returns
T	Number of single periods in the reporting period
R_t^P	Portfolio return over single-period t
R_t^B	Benchmark return over single-period t
$\bar{R}^P$	Mean portfolio return over T single periods — see (4)
$\bar{R}^B$	Mean benchmark return over T single periods — see (4)

2.3.1 Matching the Horizon of Covariance to the Horizon of the Reporting Period

The calculation of covariance (7) provides a single-period covariance which depends on the return frequency of the single-period returns, which may be daily, weekly or monthly returns. We ensure that the horizon of covariance matches the horizon of the reporting period, again using the additive property of variance which extends naturally to covariance.

The horizon covariance is the single-period covariance of returns $\mathrm{Cov}(R^P)$ is multiplied by either the:

1. Number of single periods in the reporting period, T, if the reporting period is less than one year, such that the horizon covariance is

$$\mathrm{Cov}_T(R^P) = \mathrm{Cov}(R^P)T \quad \text{if} \quad T < 1 \text{ year} \tag{8}$$

2. Number of single periods in one year, T_{ann}, if the reporting period is 1 year or more, such that the horizon covariance is

$$\mathrm{Cov}_{ann}(R^P) = \mathrm{Cov}(R^P)T_{ann} \qquad \text{if} \quad T \geq 1 \text{ year} \tag{9}$$

where $T_{ann} = \{12, 52, 252\}$ and corresponds to a single-period t being monthly, weekly or daily.

For a reporting period of 1 year or more covariance is annualized, here the horizon of the reporting period is always assumed to be 1 year period is always assumed to be 1 year. The Appendix illustrates the importance of horizon matching, particularly for reporting periods less than 1 year.

2.4 Correlation

Correlation is especially important in modern portfolio theory, since the most effective way of reducing risk via diversification is by selecting assets for which the returns are as uncorrelated as possible. When two sets of returns are perfectly positively correlated, their correlation is 1. When they fluctuate independently of one another, their correlation is zero. When two sets of returns are perfectly negatively correlated, their correlation is -1.

Correlation is calculated as covariance divided by the product of the volatilities of portfolio return and benchmark return,

$$\rho(R^P, R^B) = \frac{\mathrm{Cov}(R^P, R^B)}{\sigma(R^P)\sigma(R^B)} \tag{9}$$

where:

$\rho(R^P, R^B)$	Correlation between the time series of portfolio and benchmark returns
$\mathrm{Cov}(R^P, R^B)$	Covariance between the time series of portfolio and benchmark returns — see (7)
$\sigma(R^P)$	Volatility of the portfolio return of portfolio P — see (3)

$\sigma(R^B)$	Volatility of the benchmark return of benchmark B — see (3) with B substituted for P

2.5 R-Squared

R-squared is the proportion of the variance in portfolio returns that is related to the variance (volatility squared) of benchmark returns; it is a measure of portfolio diversification.

R-squared $R^2(R^P, R^B)$ is the square of correlation (9),

$$R^2(R^P, R^B) = \left(\rho(R^P, R^B)\right)^2 \tag{9}$$

where:

$\rho(R^P, R^B)$	Correlation between the time series of portfolio and benchmark returns (9)
R_t^P	Portfolio return over single-period t
R_t^B	Benchmark return over single-period t.

The closer $R^2(R^P, R^B)$ is to 1, the more the portfolio's variance is explained by the benchmark's variance.

2.6 Beta

Beta is used to measure the sensitivity of the portfolio to its benchmark. For example, a portfolio with beta less than 1 would be expected to underperform its benchmark in rising markets, and to outperform its benchmark in falling markets. Portfolio beta captures the correlated variance of the portfolio return series in relation to the variance of the benchmark return series. It is a measure of the systematic risk relative to the benchmark that the portfolio is subjected to.

BarraOne Performance Analytics calculates the ex post risk measure of portfolio beta as follows:

$$\beta(R^P, R^B) = \frac{\mathrm{Cov}(R^P, R^B)}{\mathrm{Var}(R^B)} \tag{9}$$

where:

$\beta(R^P, R^B)$	Beta of portfolio return R^P relative to benchmark return R^B
$\mathrm{Cov}(R^P, R^B)$	Covariance between the time series of portfolio and benchmark returns — see (7)
$\mathrm{Var}(R^B)$	Variance of the time series of benchmark returns, which is given by Equation (3) but with B substitued for P.

The following measures do not need to be annualized or horizon matched for daily, weekly, or monthly return series: correlation, R-squared, and beta.

3 Ex Post Risk-Adjusted Return Measures

This section provides the specification of the BarraOne Ex Post Risk-Adjusted Return Measures, of which there are many.

3.1 Sharpe Ratio

We begin with the most commonly known ex post risk-adjusted return measure, the Sharpe ratio , which is a measure of the excess return per unit of risk, where excess return is the return beyond the risk free rate of return and risk is defined as the volatility of the portfolio returns during the reporting period.

For reporting periods of less than 1 year the portfolio's Sharpe ratio SR_T^P is

$$SR_T^P = \frac{R_T^P - c_T^{base}}{\sigma(R^P)\sqrt{T}} \tag{10}$$

where:

R_T^P	Multi-period portfolio return over reporting period T calculated by compounding single-period returns, see – (1)
c_T^{base}	Multi-period base currency risk-free return over reporting period T calculated by compounding single-period returns, see (1)
$\sigma(R^P)$	Single-period volatility of the portfolio return, see (3)

Notice that the horizon of the denominator in (10) matches the horizon of the numerator, i.e. returns are multi-period returns over the reporting period and the denominator has a horizon equal to the length of the reporting period.

For reporting periods of 1 year or more the annualized Sharpe ratio SR_{ann}^P is calculated as

$$SR_{ann}^P = \frac{R_{ann}^P - c_{ann}^{base}}{\sigma(R^P)\sqrt{T_{ann}}} \tag{10}$$

where:

R_{ann}^P	Annualized portfolio return given by (2) using the portfolio multi-period return
c_{ann}^{base}	Annualized base currency risk-free return given by (2) using the base currency multi-period risk-free return
$\sigma(R^P)$	Single-period volatility of the portfolio return, see (3)
T_{ann}	Number of single periods in one year $= \{12, 52, 252\}$ and corresponds to a single-period t being monthly, weekly or daily

For a reporting period of 1 year or more Sharpe ratio is annualized, here the horizon of the reporting period is always assumed to be 1 year.

Rather than using multi-period returns, arithmetic mean returns could have been used in the numerator of the Sharpe ratio calculation. In fact the original article by Sharpe uses arithmetic mean returns and states that multi-period returns may make the relationship more complicated. However, we chose to use multi-period returns as they are exactly the same returns which form the basis of the BarraOne Performance Analytics attribution and contribution results, and we wish to be consistent in the analytical foundations used throughout the framework.

3.2 Sortino Ratio

The Sortino ratio is a modification of the Sharpe ratio, but it penalizes only those returns falling below a user-specified target, or required rate of return, whereas the Sharpe ratio penalizes both upside and downside volatility equally.

For reporting periods of less than 1 year the portfolio's Sortino ratio S_T^P is:

$$S_T^P = \frac{R_T^P - TA_T}{DR\sqrt{T}} \tag{11}$$

where:

R_T^P	Multi-period portfolio return over reporting period T calculated by compounding single-period returns, see (1)
TA_T	Multi-period target rate of return for the investment strategy under consideration (currently as it is not possible to specify the target rate of return, instead TA_T is assumed be the multi-period base currency risk-free rate of return c_T^{base}), see (1)
T	Number of single periods in the reporting period

where DR is the target semideviation or downside risk that measures the variability of underperformance below a minimum target rate and is calculated as follows:

$$DR = \left(\sum_{t=1}^{T} \frac{\min[(R_t^P - TA_t), 0]^2}{T-1} \right)^{1/2} \tag{12}$$

where:

R_t^P	Portfolio return over single-period t
TA_t	Single-period target rate of return for the investment strategy under consideration, currently it is not possible to specify the target rate of return in BarraOne Performance Analytics, instead TA_t is assumed be the single-period base currency risk-free rate of return c_t^{base}

T Number of single periods in the reporting period

Notice that the horizon of the denominator in (11) matches the horizon of the numerator, i.e. the returns are multi-period returns over the reporting period and the downside risk has a horizon equal to the length of the reporting period.

For reporting periods of 1 year or more the annualized Sortino ratio S_{ann}^{P} is calculated as

$$S_{ann}^{P} = \frac{R_{ann}^{P} - TA_{ann}}{DR\sqrt{T_{ann}}} \tag{12}$$

where:

R_{ann}^{P} Annualized portfolio return given by (2) using the portfolio multi-period return

TA_{ann} Annualized target rate of return for the investment strategy under consideration, currently it is not possible to specify the target rate of return in BarraOne Performance Analytics, instead TA_{ann} it is assumed be the annualized base currency risk-free rate of return

DR Target semideviation or downside risk, see (12)

T_{ann} Number of single periods in a year

For a reporting period of 1 year or more Sortino ratio is annualized, here the horizon of the reporting period is always assumed to be 1 year.

3.3 Treynor Ratio

The Treynor ratio is a measurement of the return earned in excess of that which could have been earned on an investment that has no diversifiable risk (e.g. Treasury Bills or a completely diversified portfolio), per unit of market risk assumed. Hence, it is determined as the portfolio return in excess of the risk free rate of return per unit of systematic risk.

For reporting periods of less than 1 year the portfolio's Treynor ratio TR_{T}^{P} is

$$TR_{T}^{P} = \frac{R_{T}^{P} - c_{T}^{base}}{\beta(R^{P}, R^{B})} \tag{13}$$

where:

R_{T}^{P} Multi-period portfolio return over reporting period T calculated by compounding single-period returns, see (1)

c_{T}^{base} Multi-period base currency risk-free return over reporting period T calculated by compounding single-period returns, see (1)

$\beta(R^{P}, R^{B})$ Beta of portfolio return R^{P} relative to benchmark return R^{B}, see Equation (9)

For reporting periods of 1 year or more the annualized portfolio Treynor ratio TR_{ann}^{P} is calculated as

$$TR_{ann}^P = \frac{R_{ann}^P - c_{ann}^{base}}{\beta(R^P, R^B)} \quad (13)$$

where:

R_{ann}^P	Annualized portfolio return given by (2) using the portfolio multi-period return
c_{ann}^{base}	Annualized base currency risk-free return given by (2) using the base currency multi-period risk-free return
$\beta(R^P, R^M)$	Beta of portfolio return R^P relative to benchmark return R^B, see Equation (9)

3.4 Alpha

Alpha is an ex post risk-adjusted measure of the portfolio return, where the portfolio return is adjusted for systematic risk.

For reporting periods of less than 1 year the portfolio alpha α_T^P is

$$\alpha_T^P = R_T^P - \beta(R^P, R^B) R_T^B \quad (14)$$

where:

R_T^P	Multi-period portfolio return over reporting period T calculated by compounding single-period returns, see (1)
$\beta(R^P, R^B)$	Beta of portfolio return R^P relative to benchmark return R^B, see Equation (9)
R_T^B	Multi-period benchmark return over reporting period T calculated by compounding single-period returns, see (1)

For reporting periods of 1 year or more the annualized portfolio alpha α_{ann}^P is calculated as

$$\alpha_{ann}^P = R_{ann}^P - \beta(R^P, R^B) R_{ann}^B \quad (15)$$

where:

R_{ann}^P	Annualized portfolio return given by (2) using the portfolio multi-period return
$\beta(R^P, R^B)$	Beta of portfolio return R^P relative to benchmark return R^B, see Equation (9)
R_{ann}^B	Annualized benchmark return given by (2) using the benchmark multi-period return

Alpha is expressed as a percentage.

3.5 Jensen's Alpha

Jensen's alpha is used to determine the abnormal return of the portfolio beyond the theoretical expected return by quantifying the portfolio return in excess of the security market line in the capital asset pricing model.

For reporting periods of less than 1 year the portfolio Jensen's alpha $J\alpha_T^P$ is

$$J\alpha_T^P = \left(R_T^P - c_T^{base}\right) - \beta(R^P, R^B)\left(R_T^B - c_T^{base}\right) \tag{16}$$

where:

R_T^P	Multi-period portfolio return over reporting period T calculated by compounding single-period returns, see (1)
$\beta(R^P, R^B)$	Beta of portfolio return R^P relative to benchmark return R^B, see Equation (9)
R_T^B	Multi-period benchmark return over reporting period T calculated by compounding single-period returns, see (1)
c_T^{base}	Multi-period base currency risk-free return over reporting period T calculated by compounding single-period returns, see (1)

For reporting periods of 1 year or more the annualized portfolio Jensen's alpha $J\alpha_{ann}^P$ is calculated as

$$J\alpha_{ann}^P = \left(R_{ann}^P - c_{ann}^{base}\right) - \beta(R^P, R^B)\left(R_{ann}^B - c_{ann}^{base}\right) \tag{17}$$

where:

R_{ann}^P	Annualized portfolio return given by (2) using the portfolio multi-period return
$\beta(R^P, R^B)$	Beta of portfolio return R^P relative to benchmark return R^B, see Equation (9)
R_{ann}^B	Annualized benchmark return given by (2) using the benchmark multi-period return
c_T^{base}	Annualized base currency risk-free return given by (2) using the base currency risk-free multi-period return

Jensen's Alpha is expressed as a percentage.

Note 4: Return in excess of the risk-free rate is calculated using an arithmetic difference for both an arithmetic and geometric analysis, and as such, Sharpe Ratio, Sortino Ratio, Treynor Ratio, and Jensen's Alpha are always calculated using arithmetic excess returns.

3.6 M-Squared

M-squared or the Modigliani-Modigliani measure quantifies the return of the portfolio adjusted for the risk of the portfolio relative to the benchmark. It is particularly useful for comparing portfolios with different levels of risk.

For reporting periods of less than 1 year M-squared can be expressed in terms of the Sharpe ratio as

$$M_T^2 = R_T^P + SR_T^P\left(\sigma(R^B) - \sigma(R^P)\right)\sqrt{T} \tag{18}$$

where:

R_T^P	Multi-period portfolio return over reporting period T calculated by compounding single-

period returns, see (1)

SR_T^P	Portfolio Sharpe ratio over the reporting period T, see (10)
$\sigma(R^P)$	Volatility of the portfolio return, see (3)
$\sigma(R^B)$	Volatility of the benchmark return, see (3) but with R^B instead of R^P

Alternatively, Equation (18) can be rearranged and calculated in a more straightforward manner where the risk of the portfolio relative to the benchmark is used explicitly as follows:

$$\begin{aligned} M_T^2 &= R_T^P + SR_T^P\big(\sigma(R^B) - \sigma(R^P)\big)\sqrt{T} \\ &= R_T^P + \frac{R_T^P - c_T^{base}}{\sigma(R^P)}\big(\sigma(R^B) - \sigma(R^P)\big) \\ &= \big(R_T^P - c_T^{base}\big)\frac{\sigma(R^B)}{\sigma(R^P)} + c_T^{base} \end{aligned} \tag{19}$$

For reporting periods of 1 year or more annualized M-squared can be expressed in terms of the Sharpe ratio as

$$M_{ann}^2 = R_{ann}^P + SR_{ann}^P\big(\sigma(R^B) - \sigma(R^P)\big)\sqrt{T_{ann}} \tag{20}$$

where:

R_{ann}^P	Annualized portfolio return given by (2) using the portfolio multi-period return
SR_{ann}^P	Annualized portfolio Sharpe ratio over the reporting period T, see (10)
$\sigma(R^P)$	Volatility of the portfolio return, see (3)
$\sigma(R^B)$	Volatility of the benchmark return, see (3) but with R^B instead of R^P

Equation (20) can be rearranged as is calculated in a more straightforward where the risk of the portfolio relative to the benchmark is used explicitly manner as follows:

$$\begin{aligned} M_{ann}^2 &= R_{ann}^P + SR_{ann}^P\big(\sigma(R^B) - \sigma(R^P)\big)\sqrt{T_{ann}} \\ &= R_{ann}^P + \frac{R_{ann}^P - c_{ann}^{base}}{\sigma(R^P)\sqrt{T_{ann}}}\big(\sigma(R^B) - \sigma(R^P)\big)\sqrt{T_{ann}} \\ &= \big(R_{ann}^P - c_{ann}^{base}\big)\frac{\sigma(R^B)}{\sigma(R^P)} + c_{ann}^{base} \end{aligned} \tag{21}$$

For a reporting period of 1 year or more M-squared is annualized, here the horizon of the reporting period is always assumed to be 1 year.

M-Squared is expressed as a percentage.

As was the case for the Sharpe ratio, rather than using multi-period returns, arithmetic mean returns could have been used to calculate M-squared in Equation (19). The original article by Modigliani and Modigliani used arithmetic mean returns but with the motivation to follow a consistent analytical framework throughout BarraOne Performance Analytics we prefer to use multi-period returns.

3.7 Tracking Error

Tracking error is the standard deviation of a portfolio's active return.

Only ex post risk-adjusted return measures involving active returns should have geometric variants; this applies to tracking error, information ratio and t-statistic.

3.7.1 Arithmetic Tracking Error

BarraOne Performance Analytics calculates arithmetic tracking error as follows:

$$\sigma\left(R^{ari,A}\right) = \sqrt{\frac{1}{T-1}\sum_{t=1}^{T}\left(R_t^{ari,A} - \bar{R}^{ari,A}\right)^2} \tag{21}$$

where:

$\sigma\left(R^{ari,A}\right)$ Arithmetic tracking error

$R_t^{ari,A}$ Arithmetic active return for single period t calculated as the arithmetic difference of the single-period portfolio and benchmark return

where:

$\bar{R}^{ari,A}$ Mean arithmetic active return over period T

$$\bar{R}^{ari,A} = \frac{1}{T}\sum_{t=1}^{T} R_t^{ari,A}. \tag{21}$$

3.7.2 Geometric Tracking Error

Geometric ex post risk and risk-adjusted return measures that are a function of active (benchmark relative) return use a geometric difference in the calculation of the active return.

BarraOne Performance Analytics calculates geometric tracking error in the following way:

$$\sigma(R^{geo,A}) = \sqrt{\frac{1}{T-1}\sum_{t=1}^{T}\left(R_t^{geo,A} - \bar{R}^{geo,A}\right)^2} \tag{21}$$

where:

$\sigma(R^{geo,A})$ Geometric tracking error

$R_t^{geo,A}$ Geometric active return for single period t calculated as the geometric difference of the single-period portfolio and benchmark return

where:

$\bar{R}^{geo,A}$ Mean geometric active return over period T

$$\bar{R}^{geo,A} = \frac{1}{T}\sum_{t=1}^{T} R_t^{geo,A}. \tag{21}$$

BarraOne Performance Analytics matches the horizon of tracking error to the horizon of the reporting period using the same approach as for volatility (see Section 2.2.1), such that the horizon tracking error for a reporting period of less than 1 year is

$$\sigma_T(R^A) = \sigma(R^A)\sqrt{T} \quad \text{if} \quad T < 1 \text{ year} \tag{22}$$

where

T Number of single periods in the reporting period

and the horizon tracking error for a reporting period of 1 year or more is

$$\sigma_{ann}(R^A) = \sigma(R^A)\sqrt{T_{ann}} \quad \text{if} \quad T \geq 1 \text{ year} \tag{23}$$

where

T_{ann} Number of single periods in a year

For a reporting period of 1 year or more tracking error is annualized, here the horizon of the reporting period is always assumed to be 1 year.

Tracking error is expressed as a percentage.

3.8 Information Ratio

The information ratio measures the active return achieved per unit of active risk, in which active risk is calculated as the standard deviation of active return. Information ratio measures the information content of an active investment process and is frequently interpreted as a measurement of the skill the portfolio manager possesses. Being a standardized measure, it can provide an “apples-to-apples” comparison between different periods and between different active managers or different active strategies.

3.8.1 Arithmetic Information Ratio

For reporting periods of less than 1 year BarraOne Performance Analytics calculates the arithmetic information ratio IR_T^{ari} as portfolio return less the benchmark return relative to the horizon tracking error using the following formula:

$$IR_T^{ari} = \frac{R_T^P - R_T^B}{\sigma(R^{ari,A})\sqrt{T}} \tag{23}$$

where:

R_T^P Multi-period portfolio return over reporting period T calculated by compounding single-period returns, see (1)

R_T^B	Multi-period benchmark return over reporting period T calculated by compounding single-period returns, see (1)
$\sigma\left(R^{ari,A}\right)$	Arithmetic tracking error — see Equation (21).
T	Number of single periods in the reporting period

Notice that the horizon of the denominator in (23) matches the horizon of the numerator, i.e. the returns are multi-period returns over the reporting period and the arithmetic tracking error has a horizon equal to the length of the reporting period.

Annualized arithmetic information ratio IR_{ann}^{ari} is calculated as annualized portfolio return less the annualized benchmark return relative to the annualized tracking error:

$$IR_{ann}^{ari} = \frac{R_{ann}^P - R_{ann}^B}{\sigma(R^{ari,A})\sqrt{T_{ann}}} \tag{23}$$

where:

R_{ann}^P	Annualized portfolio return given by (2) using the portfolio multi-period return
R_{ann}^B	Annualized benchmark return given by (2) using the benchmark multi-period return
$\sigma\left(R^{ari,A}\right)$	Arithmetic tracking error — see Equation (21)
T_{ann}	Number of single periods in a year.

For a reporting period of 1 year or more arithmetic information ratio is annualized, here the horizon of the reporting period is always assumed to be 1 year.

3.8.2 Geometric Information Ratio

For reporting periods of less than 1 year BarraOne Performance Analytics calculates the geometric information ratio IR_T^{geo} as the geometric difference of portfolio return and the benchmark return relative to the horizon tracking error using the following formula:

$$IR_T^{geo} = \frac{\frac{1 + R_T^P}{1 + R_T^B} - 1}{\sigma(R^{geo,A})\sqrt{T}} \tag{23}$$

where:

R_T^P	Multi-period portfolio return over reporting period T calculated by compounding single-period returns, see (1)
R_T^B	Multi-period benchmark return over reporting period T calculated by compounding single-period returns, see (1)
$\sigma(R^{geo,A})$	Geometric tracking error, see Equation (21)
T	Number of single periods in the reporting period

Notice that the horizon of the denominator in (23) matches the horizon of the numerator, i.e. the returns are multi-period returns over the reporting period and the geometric tracking error has a horizon equal to the length of the reporting period.

Annualized geometric information ratio IR_{ann}^{geo} is calculated as the geometric difference of annualized portfolio return and the annualized benchmark relative to the annualized tracking error:

$$IR_{ann}^{geo} = \frac{\dfrac{1 + R_{ann}^{P}}{1 + R_{ann}^{B}} - 1}{\sigma(R^{geo,A})\sqrt{T_{ann}}} \tag{23}$$

where:

R_{ann}^{P}	Annualized portfolio return given by (2) using the portfolio multi-period return
R_{ann}^{B}	Annualized benchmark return given by (2) using the portfolio multi-period return
$\sigma(R^{geo,A})$	Geometric tracking error — see Equation (21)
T_{ann}	Number of single periods in a year

For a reporting period of 1 year or more geometric information ratio is annualized, here the horizon of the reporting period is always assumed to be 1 year.

3.8.3 Attributing Information Ratio

The annualized standalone information ratio $IR_{ann}(Q_m)$ of an attribution effect is estimated by:

$$IR_{ann}(Q_m) \approx \frac{Q_{m,ann}}{\sigma_{ann}(Q_m)} \tag{23}$$

where:

$Q_{m,ann}$	Annualized multi-period contribution for attribution effect m
$\sigma_{ann}(Q_m)$	Annualized standalone volatility of the multi-period contribution for an attribute is given by Equation (7).

Information ratio is attributed to attribution effects only in the Equity Factor Attribution model.

3.9 t-Statistic

The t-statistic indicates the statistical significance of a non-zero active return. If the t-statistic is greater than 2 then we can say the mean active return is statistically significant.

By definition the t-statistic is a measure of the statistical significance of a sample-based estimate $\hat{\bar{X}}$ of the mean:

$$t - \text{stat} = \left| \frac{\bar{X}}{SE\left(\hat{\bar{X}}\right)} \right| \tag{23}$$

where:

X Normally distributed random variable

$\bar{X}$ Mean of X

$\hat{\bar{X}}$ Sample-based estimate of $\bar{X}$

SE Standard error function.

In particular, it measures the distance from the sample-based estimate of $\bar{X}$ to 0 as a number of estimated standard deviations of the sample-based estimate of $\bar{X}$. Thus, if the *t*-statistic of the estimate is 2 or more, then the mean is 0 with probability less than 5%.[1]

When the variable is observable, then the standard error of the estimate is itself estimated by the realized standard deviation of the variable over the square root of the number of samples:

$$SE\left(\hat{\bar{X}}\right) = \left|\frac{\hat{\sigma}(X)}{\sqrt{T}}\right| \tag{23}$$

where:

T Number of samples.

The *t*-statistic formula above is used to estimate the standard error of the estimate, even when the random variable is not observable. Combing the two equations above, the *t*-statistic is estimated by:

$$t - \mathrm{stat} = \left|\sqrt{T}\,\frac{\hat{\bar{X}}}{\hat{\sigma}(X)}\right|. \tag{23}$$

A *t*-statistic can be quoted as a means of gauging the statistical significance of nonzero active return. In this case, the random variable in question is the single-period active return, and so Equation (23) becomes:

$$t - \mathrm{stat} \approx \left|\sqrt{T} \cdot IR_{period}\right| \tag{24}$$

where the information ratio is quoted on a single-period basis.

The relation between a single-period information ratio IR_{period} and the horizon information ratio, for periods less than 1 year, is

$$IR_{period} = \frac{1}{\sqrt{T}} IR_T \qquad \text{if} \quad T < 1 \text{ year} \tag{24}$$

and for reporting periods of 1 year or more is:

[1] This is only an approximation to the true probability, because the standard error is only a sample-based estimate of the standard deviation of the sample mean. See the refinement (7-11) on p. 372 of Johnson and Wichern that accounts for this use of the standard error. This shows that the t-stat test is correct for practical purposes.

$$IR_{period} = \frac{1}{\sqrt{T_{ann}}} IR_{ann} \qquad \text{if } \; T \geq 1 \text{ year} \tag{24}$$

Since the information ratio quoted on a single-period basis is calculated differently depending on the number of single periods in the reporting period, it follows that the estimated t-statistic is also calculated differently.

Substituting Equation (24) into (24) produces the estimated t-statistic for reporting periods of less than 1 year:

$$t - \text{stat} = |IR_T| \qquad \text{if } \; T < 1 \text{ year} \tag{24}$$

Substituting Equation (24) into (24) produces the estimated t-statistic for reporting periods of 1 year or more:

$$t - \text{stat} = \left| \sqrt{\frac{T}{T_{ann}}} IR_{ann} \right| \qquad \text{if } \; T \geq 1 \text{ year} \tag{24}$$

where:

IR_T	Horizon information ratio for periods less than 1 year, which is Equation (23) within an arithmetic analysis, and Equation (23) within a geometric analysis.
IR_{ann}	Annualized information ratio, which is Equation (23) within an arithmetic analysis, and Equation (23) within a geometric analysis.
T	Number of single periods in the reporting period
T_{ann}	Number of single periods in a year

3.9.1 Attributing t-Statistic

A t-statistic $t - \text{stat}(Q_m)$ for each return contribution Q_m can be quoted as a means of gauging the statistical significance of a nonzero attribution effect. In this case, the random variable in question is the single-period attribution effect, and so Equation (24) becomes:

$$t - \text{stat}(Q_m) \approx \left| \sqrt{T} \cdot IR_{period}(Q_m) \right| \tag{24}$$

where the information ratio is quoted on a single-period basis.

The relation between a single-period information ratio $IR_{period}(Q_m)$ and the horizon information ratio for reporting periods of less than 1 year is:

$$IR_{period}(Q_m) = \frac{1}{\sqrt{T}} IR_T(Q_m) \qquad \text{if } \; T < 1 \text{ year} \tag{24}$$

and for reporting periods of 1 year or more is:

$$IR_{period}(Q_m) = \frac{1}{\sqrt{T_{ann}}} IR_{ann}(Q_m) \qquad \text{if } \; T \geq 1 \text{ year} \tag{24}$$

Substituting Equation (24) into (24) produces the estimated t-statistic for an attribution effect m for reporting periods of less than 1 year:

$$t - \mathrm{stat}(Q_m) = \left| \frac{Q_{m,T}}{\sigma_T(Q_m)} \right| \qquad \text{if } \; T < 1 \text{ year} \tag{24}$$

Substituting Equation (24) into (24) produces the estimated t-statistic for an attribution effect m for reporting periods of less than 1 year:

$$t - \mathrm{stat}(Q_m) = \left| \sqrt{\frac{T}{T_{ann}}} \frac{Q_{m,ann}}{\sigma_{ann}(Q_m)} \right| \qquad \text{if } \; T \geq 1 \text{ year} \tag{24}$$

where:

$Q_{m,T}$	Multi-period contribution for attribution effect m computed via multi-period linking
$Q_{m,ann}$	Annualized multi-period contribution for attribution effect m
$\sigma_T(Q_m)$	Standalone volatility of the multi-period contribution for an attribution effect m, see (6)
$\sigma_{ann}(Q_m)$	Annualized standalone volatility of the multi-period annualized contribution for an attribution effect m, see (7)
T	Number of single periods in the reporting period
T_{ann}	Number of single periods in a year

t-statistic is attributed to attribution effects only in the Equity Factor Attribution model.

4 Ex Post Return Measures

The final part of this white paper covers 3 ex post measures which are neither risk measures nor risk-adjusted return measures but are measures calculated directly from portfolio and benchmark returns.

This section provides the exact specification of how the BarraOne Ex Post Return Measures are calculated.

4.1 Upside Capture Ratio

The upside capture ratio, also called up-market capture ratio, shows whether a portfolio has outperformed by gaining more than the benchmark during periods of market strength, and if so, by how much. An upside capture ratio over 100% indicates a portfolio has generally outperformed the benchmark during periods of positive returns for the benchmark.

The upside capture ratio UC_T for reporting periods of less than 1 year is calculated as the multi-period portfolio return during periods in which the benchmark has positive return divided by the multi-period benchmark return during periods of positive return:

$$UC_T = \frac{\prod_{t=1}^{T}\left(1 + R_t^{P|B^+}\right) - 1}{\prod_{t=1}^{T}\left(1 + R_t^{B^+}\right) - 1} \tag{24}$$

where:

$R_t^{P|B^+}$ Portfolio return given that the benchmark return is greater than zero for period t, i.e. $R_t^{P|B^+} = R_t^P$, if $R_t^B > 0$, otherwise $R_t^{P|B^+} = 0$

$R_t^{B^+}$ Benchmark return given that the benchmark return is greater than zero for period t, i.e. $R_t^{B^+} = R_t^B$, if $R_t^B > 0$, otherwise $R_t^{B^+} = 0$

T Number of single periods in the reporting period

For reporting periods of 1 year or more the annualized upside capture ratio UC_{ann} is calculated using annualized returns as follows

$$UC_{ann} = \frac{\left(\prod_{t=1}^{T}\left(1 + R_t^{P|B^+}\right)\right)^{\frac{T_{ann}}{T}} - 1}{\left(\prod_{t=1}^{T}\left(1 + R_t^{B^+}\right)\right)^{\frac{T_{ann}}{T}} - 1} \tag{24}$$

where:

$R_t^{P|B^+}$ Portfolio return given that the benchmark return is greater than zero for period t, i.e. $R_t^{P|B^+} = R_t^P$, if $R_t^B > 0$, otherwise $R_t^{P|B^+} = 0$

$R_t^{B^+}$ Benchmark return given that the benchmark return is greater than zero for period t, i.e. $R_t^{B^+} = R_t^B$, if $R_t^B > 0$, otherwise $R_t^{B^+} = 0$

T_{ann} Number of single periods in a year

T Number of single periods in the reporting period

For a reporting period of 1 year or more upside capture ratio is annualized, here the horizon of the reporting period is always assumed to be 1 year.

Note 5: Upside capture ratio is not applicable in certain scenarios

The upside capture ratio quantifies portfolio outperformance during periods of market strength, i.e. when the benchmark has positive returns. If there are no periods of “market strength”, then Equations (24) and (24) are not well defined as the denominators are zero and in this scenario the upside capture ratio is not applicable.

Upside capture ratio is expressed as a percentage.

4.2 Downside Capture Ratio

The downside capture ratio, also called down-market capture ratio, shows whether a portfolio has outperformed by losing less than a benchmark during periods of market weakness, and if so, by how much. A downside capture ratio less than 100% indicates a portfolio has lost less than its benchmark during periods of when the benchmark provides a negative return. However, if the portfolio generates positive returns while the benchmark declines, the portfolio's downside capture ratio will be negative, which means the portfolio has moved in the opposite direction to the benchmark.

The downside capture ratio DC_T for reporting periods of less than 1 year is calculated as the multi-period portfolio return during periods in which the benchmark has negative return divided by the multi-period benchmark return during periods of negative return:

$$DC_T = \frac{\prod_{t=1}^{T}\left(1 + R_t^{P|B^-}\right) - 1}{\prod_{t=1}^{T}(1 + R_t^{B^-}) - 1} \tag{24}$$

where:

$R_t^{P\|B^-}$	Portfolio return given that the benchmark return is less than zero for period t, i.e. $R_t^{P\|B^-} = R_t^P$, if $R_t^B < 0$, otherwise $R_t^{P\|B^-} = 0$
$R_t^{B^-}$	Benchmark return given that the benchmark return is less than zero for period t, i.e. $R_t^{B^-} = R_t^B$, if $R_t^B < 0$, otherwise $R_t^{B^-} = 0$
T	Number of single periods in the reporting period

For reporting periods of 1 year or more annualized downside capture ratio DC_{ann} is calculated using annualized returns as follows

$$DC_{ann} = \frac{\left(\prod_{t=1}^{T}\left(1 + R_t^{P|B^-}\right)\right)^{\frac{T_{ann}}{T}} - 1}{\left(\prod_{t=1}^{T}(1 + R_t^{B^-})\right)^{\frac{T_{ann}}{T}} - 1} \tag{24}$$

where:

$R_t^{P\|B^-}$	Portfolio return given that the benchmark return is less than zero for period t, i.e. $R_t^{P\|B^-} = R_t^P$, if $R_t^B < 0$, otherwise $R_t^{P\|B^-} = 0$
$R_t^{B^-}$	Benchmark return given that the benchmark return is less than zero for period t, i.e. $R_t^{B^-} = R_t^B$, if $R_t^B < 0$, otherwise $R_t^{B^-} = 0$
T_{ann}	Number of single periods in a year
T	Number of single periods in the reporting period

For a reporting period of 1 year or more downside capture ratio is annualized, here the horizon of the reporting period is always assumed to be 1 year.

Note 6: Downside capture ratio is not applicable in certain scenarios

The downside capture ratio quantifies portfolio outperformance during periods of market weakness, i.e. when the benchmark has negative returns. If there are no periods of "market weakness", then

Equations (24) and (24) are not well defined as the denominators are zero and in this scenario the downside capture ratio is not applicable.

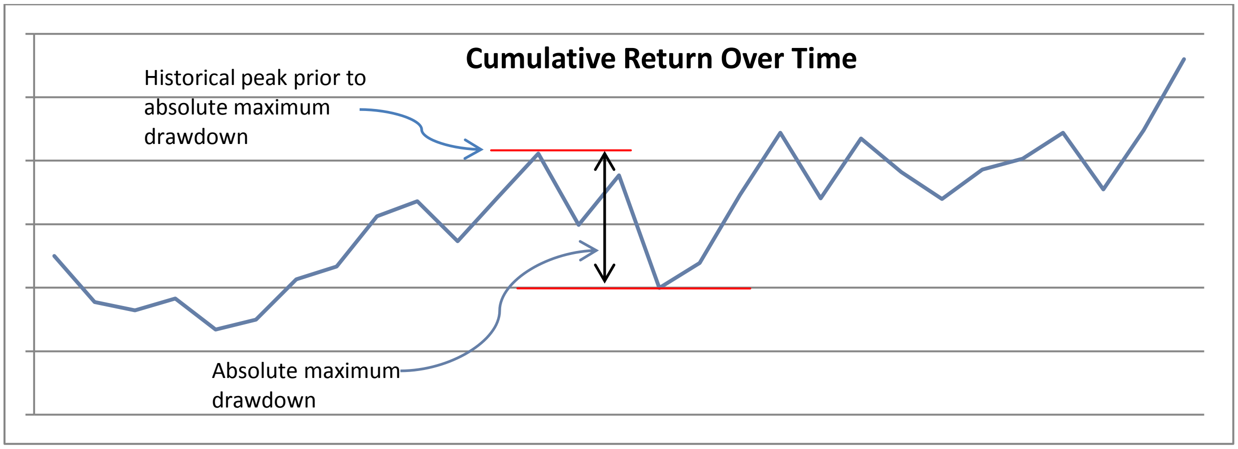
Downside capture ratio is expressed as a percentage.

4.3 Maximum Drawdown

Maximum drawdown is the maximum potential loss over a specific time period, and measures the maximum decline from a historical peak to a trough and is the loss an investor would have suffered by contributing to the fund at the highest point and withdrawing at the lowest, i.e. the largest peak-to-trough.

Figure 1 illustrates the absolute maximum drawdown and can be seen by eye to be the largest peak-to-trough in the cumulative return over the reporting period.

Figure 1: Illustration of cumulative return over time and the absolute maximum drawdown



The maximum drawdown MDD_T up to time T is the absolute maximum drawdown in the multi-period or cumulative return relative to the historical peak during the reporting period and is calculated as follows:

$$MDD_T = \frac{\min_{\tau \in (1,T)} \left[R_\tau - \max_{\bar{t} \in (1,\tau)} R_{\bar{t}} \right]}{1 + R^{max}} \tag{24}$$

where:

$R_\tau = \prod_{t=1}^{\tau} (1 + R_t) - 1$	Multi-period base currency portfolio return at time τ, which is the single-period return R_t compounded from $t = 1$ to $t = \tau$
$R_{\bar{t}} = \prod_{t=1}^{\bar{t}} (1 + R_t) - 1$	Multi-period base currency portfolio return at time $\bar{t}$, which is the single-period return R_t compounded from $t = 1$ to $t = \bar{t}$
T	Number of single periods in the reporting period

$\min_{\tau\in(1,T)}\left[R_\tau - \max_{\bar{t}\in(1,\tau)} R_{\bar{t}}\right]$	Absolute maximum drawdown from historical peak to trough during reporting period
R^{max}	Historical peak in the multi-period return prior to the absolute maximum drawdown

Note 7: Maximum drawdown always provides a negative result which reflects that it is a measure of the possible loss in value and is expressed as a percentage.

5 Conclusion

This white paper presents the exact specification of how BarraOne Performance Analytics Ex Post Risk and Risk-Adjusted Return Measures are calculated. The details on the following analytical foundations were described: the frequency of single-period returns, the number of days per year when the return frequency is daily, when multi-period or annualized returns are appropriate, the definition of excess and active return, the definition of risk, using arithmetic mean rather than geometric mean returns, using sample rather than population-based variance, the importance of matching the horizon of the measures with the horizon of the reporting period. The focus of the white paper was to present the exact specification of how the numerous BarraOne Performance Analytics Ex Post Risk and Risk-Adjusted Return Measures are calculated.

The analytical foundations of the BarraOne Performance Analytics Ex Post Risk and Risk-Adjusted Return Measures are completely consistent with analytical foundations of the attribution and contribution analytics in BarraOne Performance Analytics. This is vital as the measures are presented as complementary results to the BarraOne Performance Analytics attribution and contribution results.

In a future release the coverage of the BarraOne Ex Post Risk and Risk-Adjusted Return Measures will be extended further and measures such as different types of drawdown, Calmar ratio, Sterling ratio, etc, may be introduced.

Appendix: Horizon Matching Below 1 Year

The following results provide a time series analysis of various BarraOne Ex Post Risk and Risk-Adjusted Return Measures with and without employing horizon matching for reporting periods below 1 year. This is an interesting horizon to focus on as above 1 year, it is well known that horizon matching using annualization is appropriate.

Daily portfolio and benchmark returns were simulated with the reporting period extended by 1 day for each successive data point; hence, the horizontal axis is the length of the reporting period in days.

From Figures 2, 3, 4, 5, 6 and 7 it is clear that the horizon matched ex post measures avoid erroneous results for reporting periods below 1 year. Ex post risk and risk-adjusted return measures calculated without the horizon matching for reporting periods below 1 year would inappropriately use single-period variance, covariance, volatility / standard deviation.

Figure 2: Portfolio horizon variance compared to variance without horizon matching below 1 year

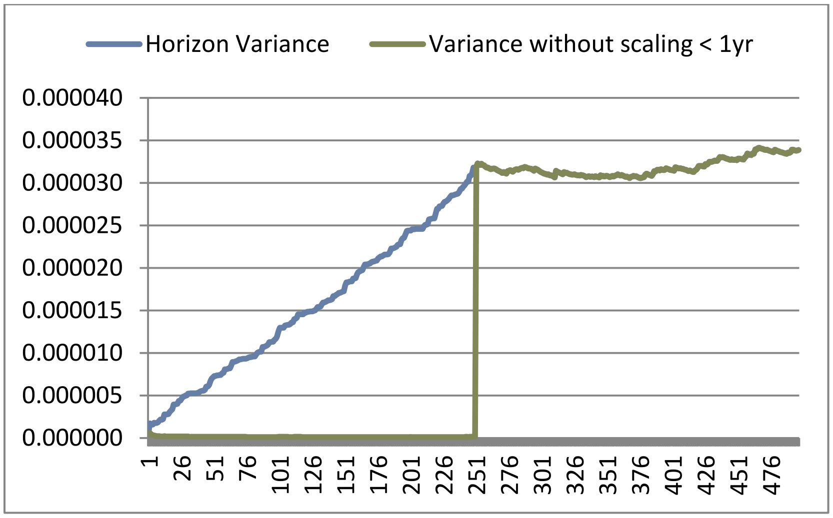


Figure 3: Portfolio horizon volatility compared to volatility without horizon matching below 1 year

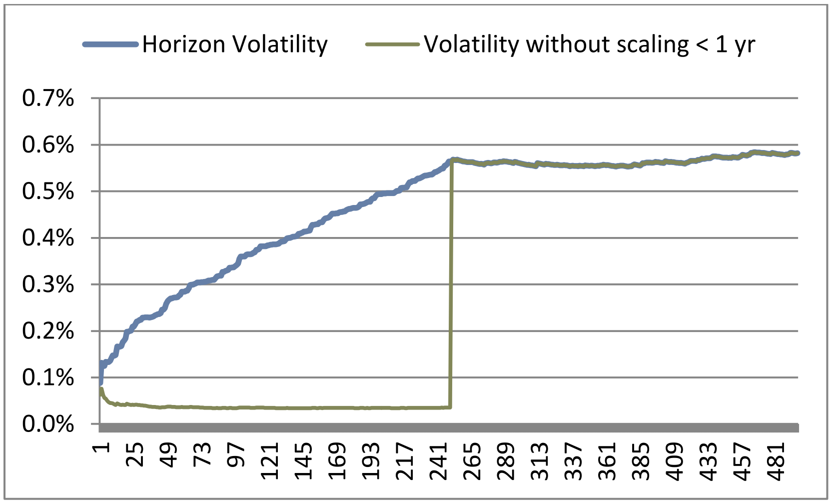
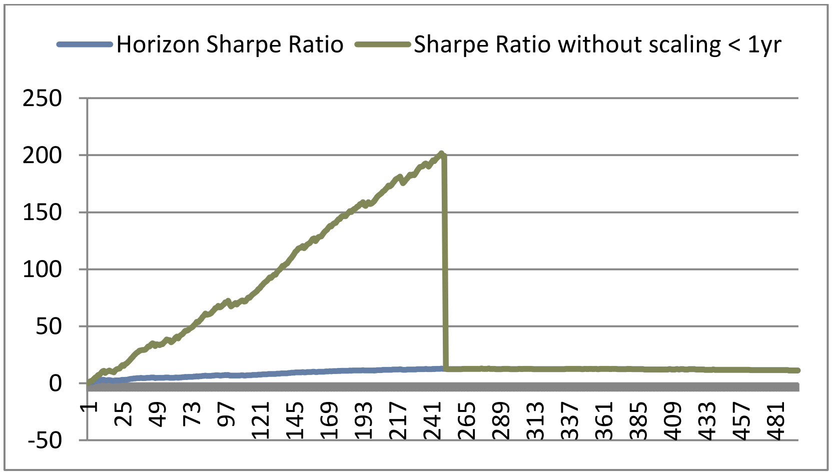


Figure 4: Horizon Sharpe ratio compared to Sharpe ratio without horizon matching below 1 year



Note: The results for the benchmark follow the same trends as for the portfolio analysis.

Figure 5: Cumulative arithmetic active return compared to the annualized arithmetic active return

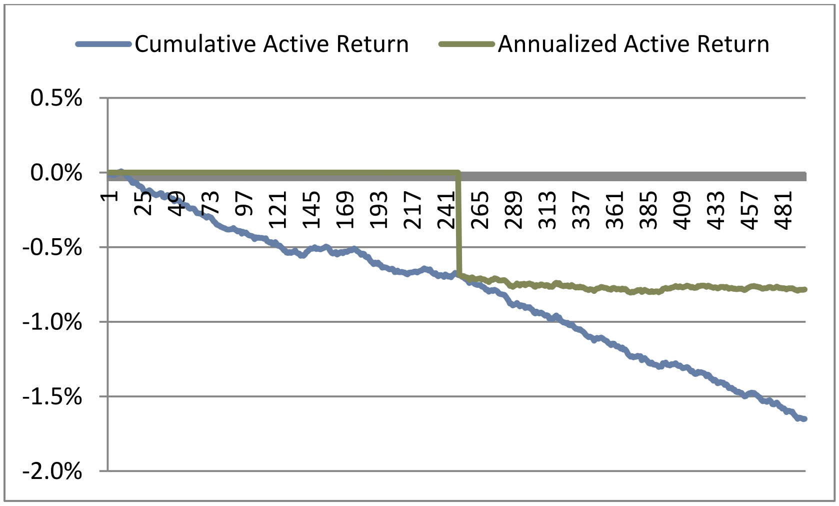


Figure 6: Horizon arithmetic tracking error vs. tracking error without horizon matching below 1 year

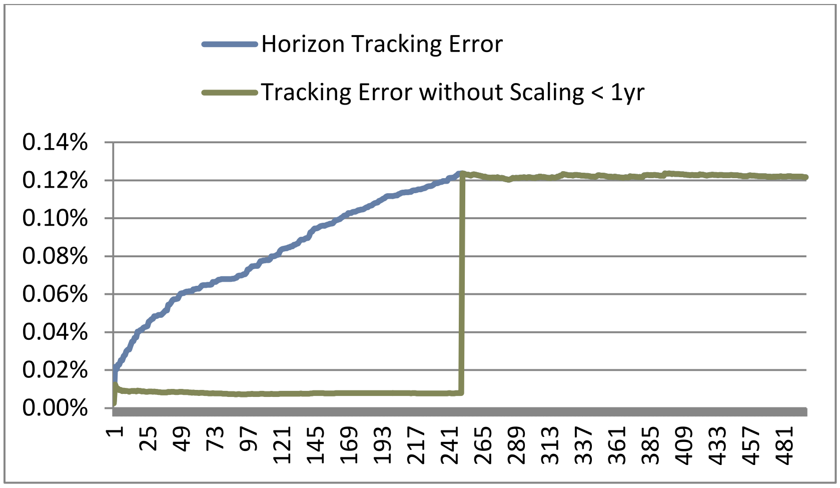
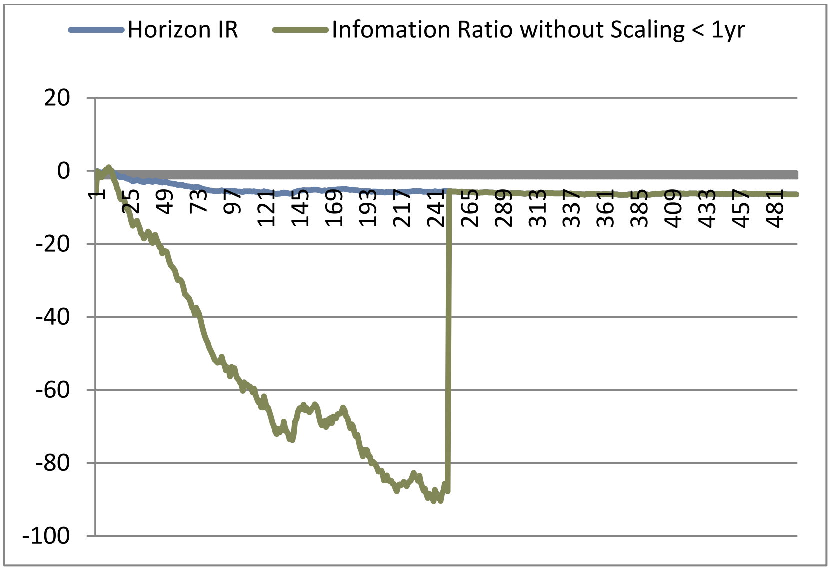


Figure 7: Horizon arithmetic information ratio vs. information ratio without the horizon matching below 1 year



Note: The geometric results follow the same trends as for the arithmetic analysis.

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[1] *As of September 30, 2012, as published by eVestment, Lipper and Bloomberg on January 31, 2013* *Apr 2013*