HW 6

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① $P(X|Y) \sim N(\mu_y, \Sigma)$, $\forall j \neq i$ to GDA

$$P(c \mid x) = \frac{P(X|c)P(c)}{P(X)}$$

$$P(X_i \mid y, X_j)$$

$$= \frac{P(X_i, X_j \mid y)}{P(X_j \mid y)}$$ Let X_i and X_j are conditionally independent

$$= \frac{P(X_i|y)\,\cancel{P(X_j|y)}}{\cancel{P(X_j|y)}} = P(X_i|y)$$

$P(X_i \mid y, X_j) = P(X_i \mid y)$ → since X_i does not depend on X_j

Σ is $d \times d$
$$\begin{bmatrix} Var(X_1) & Cov(X_1,X_2) & \cdots & Cov(X_1,X_d) \\ Cov(X_2,X_1) & Var(X_2) & \cdots & \\ \vdots & & \ddots & \\ \vdots & & & var(X_d) \end{bmatrix}$$

$$cov(X_1, X_2) = E[(X_1-\mu_1)(X_2-\mu_2)]$$

When $i \neq j$ in $cov(X_i, X_j)$ and X_i, X_j are independent

So it the value of $cov(X_i, X_j)$ is zero. Since all elements in Σ when $i \neq j$, are zero, it shows that all off diagonal elements of Σ equals to 0, $\forall\ i \neq j$

②
$$P(C_0|X) = \frac{P(X|C_0)P(C_0)}{P(X|C_0)P(C_0) + P(X|C_1)P(C_1)}$$

ⓐ
$$\sigma(a) = \frac{1}{1+\exp(-a)} = P(C_0|X)$$

$$\frac{1}{P(C_0|X)} = 1 + \exp(-a) \qquad \frac{1-P(C_0|X)}{P(C_0|X)} = \exp(-a)$$

$$\ln\left(\frac{1-P(C_0|X)}{P(C_0|X)}\right) = -a \qquad a = \ln\left(\frac{P(C_0|X)}{1-P(C_0|X)}\right)$$

$$= \ln\left(\frac{\frac{P(X|C_0)P(C_0)}{P(X|C_0)P(C_0)+P(X|C_1)P(C_1)}}{\frac{P(X|C_1)P(C_1)}{P(X|C_0)P(C_0)+P(X|C_1)P(C_1)}}\right) = \ln\left(\frac{P(X|C_0)P(C_0)}{P(X|C_1)P(C_1)}\right)$$

$$\therefore \boxed{a = \ln\left(\frac{P(X|C_0)P(C_0)}{P(X|C_1)P(C_1)}\right)}$$

ⓑ Find ⓦ and ⓑ in an equation $a = w^Tx + b$

$$a = w^Tx + b = \ln\left(\frac{P(X|C_0)P(C_0)}{P(X|C_1)P(C_1)}\right)$$

$$* \ \frac{P(X|C_0)P(C_0)}{P(X|C_1)P(C_1)} = \frac{\cancel{\frac{1}{(2\pi)^{d/2}|\Sigma|^{1/2}}}\exp(-\frac{1}{2}(x-\mu_0)^T\Sigma^{-1}(x-\mu_0))\,P(C_0)}{\cancel{\frac{1}{(2\pi)^{d/2}|\Sigma|^{1/2}}}\exp(-\frac{1}{2}(x-\mu_1)^T\Sigma^{-1}(x-\mu_1))\,P(C_1)}$$

$$= \exp\left[-\tfrac{1}{2}(x-\mu_0)^T\Sigma^{-1}(x-\mu_0) + \tfrac{1}{2}(x-\mu_1)^T\Sigma^{-1}(x-\mu_1)\right]\frac{P(C_0)}{P(C_1)}$$

$$a = \ln(\,''\,) = w^Tx + b$$

$$= -\tfrac{1}{2}(x-\mu_0)^T\Sigma^{-1}(x-\mu_0) + \tfrac{1}{2}(x-\mu_1)^T\Sigma^{-1}(x-\mu_1) + \ln\left(\frac{P(C_0)}{P(C_1)}\right)$$

$$= \underbrace{-\tfrac{1}{2}(\mu_1-\mu_0)^T\Sigma^{-1}}\,x + \underbrace{\tfrac{1}{2}(\mu_0^T\Sigma^{-1}\mu_0 - \mu_1^T\Sigma^{-1}\mu_1) + \ln\left(\frac{P(C_0)}{P(C_1)}\right)}$$

$$= (W^T \quad)\,x + b$$

$$W^T = -\tfrac{1}{2}(\mu_1-\mu_0)^T\Sigma^{-1} \ // \ W = -\tfrac{1}{2}(\Sigma^T)^{-1}(\mu_1-\mu_0)$$

$$b = \tfrac{1}{2}(\mu_0^T\Sigma^{-1}\mu_0 - \mu_1^T\Sigma^{-1}\mu_1) + \ln\left(\frac{P(C_0)}{P(C_1)}\right)$$

ⓒ
$$\frac{P(X|C_0)P(C_0)}{P(X|C_1)P(C_1)} = \frac{\frac{1}{\cancel{(2\pi)^{d/2}}|\Sigma_1|^{1/2}}\exp(-\frac{1}{2}(x-\mu_0)^T\Sigma_1^{-1}(x-\mu_0))\,P(C_0)}{\frac{1}{\cancel{(2\pi)^{d/2}}|\Sigma_2|^{1/2}}\exp(-\frac{1}{2}(x-\mu_1)^T\Sigma_2^{-1}(x-\mu_1))\,P(C_1)}$$

$$= \left(\frac{\Sigma_2}{\Sigma_1}\right)^{1/2}\exp\left(-\tfrac{1}{2}(x-\mu_0)^T\Sigma_1^{-1}(x-\mu_0) + \tfrac{1}{2}(x-\mu_1)^T\Sigma_2^{-1}(x-\mu_1)\right)\cdot\frac{P(C_0)}{P(C_1)}$$

$$\ln\left(\frac{P(X|C_0)P(C_0)}{P(X|C_1)P(C_1)}\right) = \tfrac{1}{2}\ln\left(\frac{\Sigma_2}{\Sigma_1}\right) - \tfrac{1}{2}(x-\mu_0)^T\Sigma_1^{-1}(x-\mu_0) + \tfrac{1}{2}(x-\mu_1)^T\Sigma_2^{-1}(x-\mu_1) + \ln\left(\frac{P(C_0)}{P(C_1)}\right)$$

$$= \left[\underbrace{-\tfrac{1}{2}(x-\mu_0)^T\Sigma_1^{-1}(x-\mu_0)}_{①} + \underbrace{\tfrac{1}{2}(x-\mu_1)^T\Sigma_2^{-1}(x-\mu_1)}_{②}\right] + \left[\tfrac{1}{2}\ln\left(\frac{\Sigma_2}{\Sigma_1}\right) + \ln\left(\frac{P(C_0)}{P(C_1)}\right)\right]$$

$$①: (x-\mu_0)^T\Sigma_1^{-1}(x-\mu_0) = (x^T\Sigma_1^{-1} - \mu_0^T\Sigma_1^{-1})(x-\mu_0)$$
$$= x^T\Sigma_1^{-1}x - \mu_0^T\Sigma_1^{-1}x - x^T\Sigma_1^{-1}\mu_0 + \mu_0^T\Sigma_1^{-1}\mu_0$$

$$②: \quad = x^T\Sigma_2^{-1}x - \mu_1^T\Sigma_2^{-1}x - x^T\Sigma_2^{-1}\mu_1 + \mu_1^T\Sigma_2^{-1}\mu_1$$

① : $(x-\mu_0)^T\Sigma_1^{-1}(x-\mu_0) = (x^T\Sigma_1^{-1} - \mu_0^T\Sigma_1^{-1})(x-\mu_0)$

$$= x^T\Sigma_1^{-1}x - \mu_0^T\Sigma_1^{-1}x - x^T\Sigma_1^{-1}\mu_0 + \mu_0^T\Sigma_1^{-1}\mu_0$$

$(\Sigma_1^{-1}\mu_0)^T = \mu_0^T(\Sigma_1^T)^{-1}$

$$= x^T\Sigma_1^{-1}x - \mu_0^T\Sigma_1^{-1}x - (\Sigma_1^{-1}\mu_0)^T x + \mu_0^T\Sigma_1^{-1}\mu_0$$

$$= x^T\Sigma_1^{-1}x + [-\mu_0^T\Sigma_1^{-1} - \mu_0^T(\Sigma_1^T)^{-1}]x + \mu_0^T\Sigma_1^{-1}\mu_0$$

② : $= x^T\Sigma_2^{-1}x - \mu_1^T\Sigma_2^{-1}x - x^T\Sigma_2^{-1}\mu_1 + \mu_1^T\Sigma_2^{-1}\mu_1$

$$= x^T\Sigma_2^{-1}x + [-\mu_1^T\Sigma_2^{-1} - \mu_1^T(\Sigma_2^T)^{-1}]x + \mu_1^T\Sigma_2^{-1}\mu_1$$

$$\ln\left(\frac{P(x|C_0)P(C_0)}{P(x|C_1)P(C_1)}\right) = -\frac{1}{2}\Big[\underbrace{(x-\mu_0)^T\Sigma_1^{-1}(x-\mu_0)}_{①} - \underbrace{(x-\mu_1)^T\Sigma_2^{-1}(x-\mu_1)}_{②}\Big] + \left[\frac{1}{2}\ln\left(\frac{\Sigma_2}{\Sigma_1}\right) + \ln\left(\frac{P(C_0)}{P(C_1)}\right)\right]$$

$$-\frac{1}{2}\cdot x^T(\Sigma_1^{-1}-\Sigma_2^{-1})x - \frac{1}{2}[-\mu_0^T\Sigma_1^{-1} - \mu_0^T(\Sigma_1^T)^{-1} + \mu_1^T\Sigma_2^{-1} + \mu_1^T(\Sigma_2^T)^{-1}]x$$
$$-\frac{1}{2}(\mu_0^T\Sigma_1^{-1}\mu_0 - \mu_1^T\Sigma_2^{-1}\mu_1) + \frac{1}{2}\ln\left(\frac{\Sigma_2}{\Sigma_1}\right) + \ln\left(\frac{P(C_0)}{P(C_1)}\right)$$

$$A = -\frac{1}{2}(\Sigma_1^{-1}-\Sigma_2^{-1})$$

$$W^T = -\frac{1}{2}[-\mu_0^T\Sigma_1^{-1} - \mu_0^T(\Sigma_1^T)^{-1} + \mu_1^T\Sigma_2^{-1} + \mu_1^T(\Sigma_2^T)^{-1}]$$

$$b = -\frac{1}{2}(\mu_0^T\Sigma_1^{-1}\mu_0 - \mu_1^T\Sigma_2^{-1}\mu_1) + \frac{1}{2}\ln\left(\frac{\Sigma_2}{\Sigma_1}\right) + \ln\left(\frac{P(C_0)}{P(C_1)}\right)$$

③ $\{(x^{(i)}, y^{(i)}); i = \{1 \dots m\}\}$

ⓐ $P(y=1) = \phi = 1 - P(y=0)$ // $P(y=0) = 1-\phi$

$$P(x^{(i)}, \dots, x^{(m)}, y^{(1)} \dots, y^{(m)}) = \prod_{i=1}^{m} P(x^{(i)}|y^{(i)})P(y^{(i)})$$

$$\boxed{= \prod_{i=1}^{m}\left[\frac{\phi}{(2\pi)^{n/2}|\Sigma|^{1/2}}\exp\left(-\frac{1}{2}(x-\mu_0)^T\Sigma^{-1}(x-\mu_0)\right)\right]^{1-y^{(i)}} \times \left[\frac{1-\phi}{(2\pi)^{n/2}|\Sigma|^{1/2}}\exp\left(-\frac{1}{2}(x-\mu_1)^T\Sigma^{-1}(x-\mu_1)\right)\right]^{y^{(i)}}}$$

$$L(\phi,\mu_0,\mu_1,\Sigma) = \ln\left(\prod_{i=1}^{m} P(x^{(i)}|y^{(i)})P(y^{(i)})\right)$$

$$= \sum_{i=1}^{N}\left\{\log\left[\frac{1-\phi}{(2\pi)^{n/2}|\Sigma|^{1/2}}\exp\left(-\frac{1}{2}(x-\mu_0)^T\Sigma^{-1}(x-\mu_0)\right)\right]^{1-y_i} + \left[\frac{\phi}{(2\pi)^{n/2}|\Sigma|^{1/2}}\exp\left(-\frac{1}{2}(x-\mu_1)^T\Sigma^{-1}(x-\mu_1)\right)\right]^{y_i}\right\}$$

$$\boxed{= \sum_{i=1}^{N}\left\{(1-y_i)\ln\left[\frac{1-\phi}{(2\pi)^{n/2}|\Sigma|^{1/2}}\exp\left(-\frac{1}{2}(x-\mu_0)^T\Sigma^{-1}(x-\mu_0)\right)\right] + y_i\ln\left[\frac{\phi}{(2\pi)^{n/2}|\Sigma|^{1/2}}\exp\left(-\frac{1}{2}(x-\mu_1)^T\Sigma^{-1}(x-\mu_1)\right)\right]\right\}}$$

ⓑ
$$L = \sum_{i=1}^{N}\left\{(1-y_i)\ln\left[\frac{1-\phi}{(2\pi)^{n/2}|\Sigma|^{1/2}}\exp\left(-\frac{1}{2}(x-\mu_0)^T\Sigma^{-1}(x-\mu_0)\right)\right] + y_i\ln\left[\frac{\phi}{(2\pi)^{n/2}|\Sigma|^{1/2}}\exp\left(-\frac{1}{2}(x-\mu_1)^T\Sigma^{-1}(x-\mu_1)\right)\right]\right\}$$

$$\frac{\partial L}{\partial\phi} = \sum_{i=1}^{N}\left\{(1-y_i)\frac{\frac{-1}{(2\pi)^{n/2}|\Sigma|^{1/2}}}{\frac{1-\phi}{(2\pi)^{n/2}|\Sigma|^{1/2}}} + y_i\frac{\frac{1}{(2\pi)^{n/2}|\Sigma|^{1/2}}}{\frac{\phi}{(2\pi)^{n/2}|\Sigma|^{1/2}}}\right\}$$

$$= \sum_{i=1}^{N}\left\{\frac{(1-y_i)}{\phi-1} + \frac{y_i}{\phi}\right\}$$

$$= \sum_{i=1}^{N}\left\{\frac{(1-y_i)\phi + y_i(\phi-1)}{\phi(\phi-1)}\right\}$$

$$= \sum_{i=1}^{N}\left\{\frac{(1-2y_i)\phi - y_i}{\phi(\phi-1)}\right\}$$

$\phi = \dfrac{y_i}{1-2y_i}$ for the max

$$\frac{\partial}{\partial\phi}\left(\frac{\partial L}{\partial\phi}\right) = \sum_{i=1}^{N}\frac{-(1-y_i)}{(\phi-1)^2} + \frac{-y_i}{(\phi)^2}$$

For the second derivative of L is all negative. This is because $\frac{-(1-y_i)}{(\phi-1)^2}$ and $\frac{-y_i}{(\phi)^2}$ are both not positive value. It means that it is the best but not the worst.

ⓒ
$$L = \sum_{i=1}^{N}\left\{(1-y_i)\ln\left[\frac{1-\phi}{(2\pi)^{n/2}|\Sigma|^{1/2}}\exp\left(-\frac{1}{2}(x-\mu_0)^T\Sigma^{-1}(x-\mu_0)\right)\right] + y_i\ln\left[\frac{\phi}{(2\pi)^{n/2}|\Sigma|^{1/2}}\exp\left(-\frac{1}{2}(x-\mu_1)^T\Sigma^{-1}(x-\mu_1)\right)\right]\right\}$$

$$= \sum_{i=1}^{N}\left\{(1-y_i)\left(\ln(1-\phi) - \ln\left((2\pi)^{n/2}|\Sigma|^{1/2}\right) - \frac{1}{2}(x-\mu_0)^T\Sigma^{-1}(x-\mu_0)\right) + y_i\left(\ln(\phi) - \ln\left((2\pi)^{n/2}|\Sigma|^{1/2}\right) - \frac{1}{2}(x-\mu_1)^T\Sigma^{-1}(x-\mu_1)\right)\right\}$$