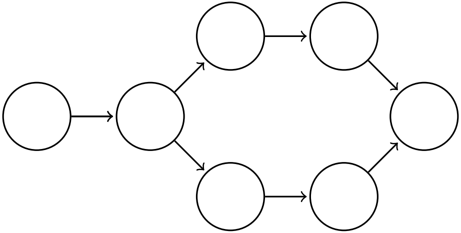
COMP 526

- Conditional independence
- Bayesian Networks
- Learning parameters of networks

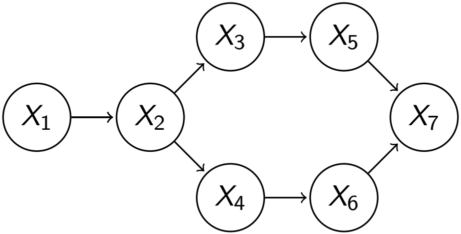
Specifying a graphical model

The starting point is a directed acyclic graph (DAG) over n nodes.



Specifying a graphical model

Each of these nodes corresponds to a random variable.



and each node has a conditional probability associated with it

$$p(X_j | X_{\mathbf{pa}(j)})$$

where $\mathbf{pa}\,(j)$ is a list of parent nodes of node j, e.g.

$$p(X_7 | X_5, X_6)$$

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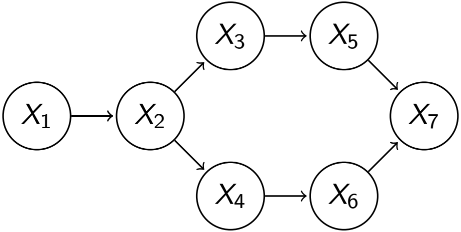

This graphical model specifies a joint distribution

$$
\begin{aligned}
p(X_1, X_2, X_3, X_4, X_5, X_6, X_7) &= \prod_i p(X_i | X_{\mathbf{pa}(i)}) \\
&= p(X_1)
\end{aligned}
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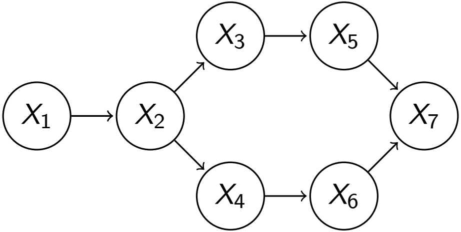

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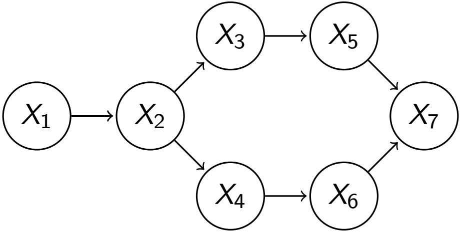

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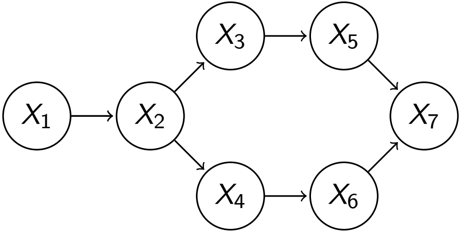

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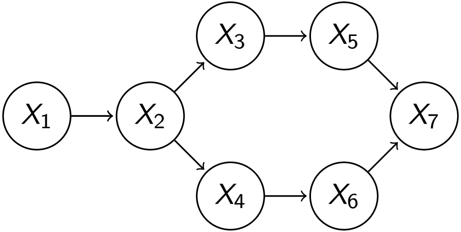

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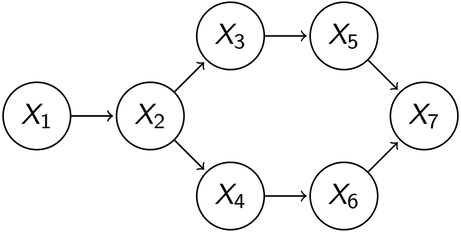

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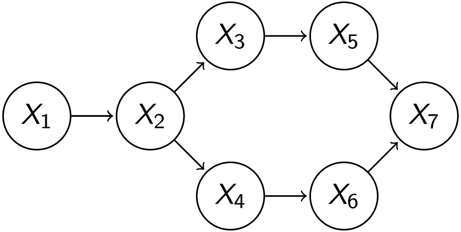

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Determining conditional independencies from graphs

In the topological order of nodes of a DAG, parent nodes precede child nodes. There can be many topological orders.

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Given an order O, let $\mathbf{pnp}_O(i)$ denote a set of nodes that precede node i in a topological order but are not its parents.

We can show that

$$X_i \perp X_{\mathbf{pnp}_O(i)} | X_{\mathbf{pa}(i)}$$

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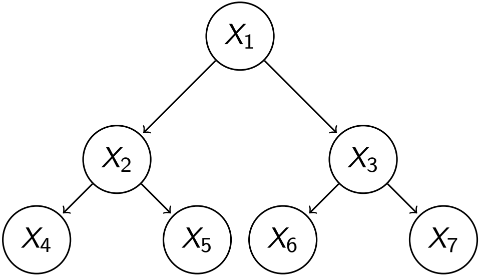
These are basic conditional independence relationships.

Obtaining basic conditional independencies



A topological order:
$X_1, X_2, X_3, X_4, X_5, X_6, X_7$

$$X_1 \perp \emptyset \quad | \quad \emptyset$$

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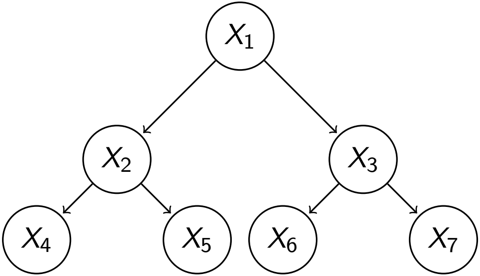

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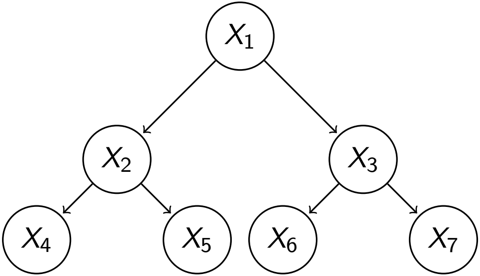

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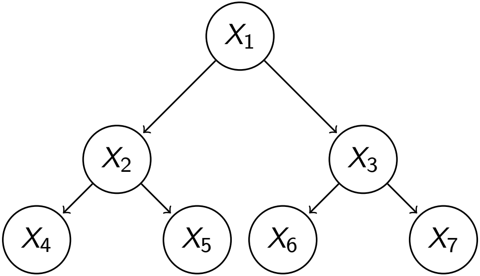

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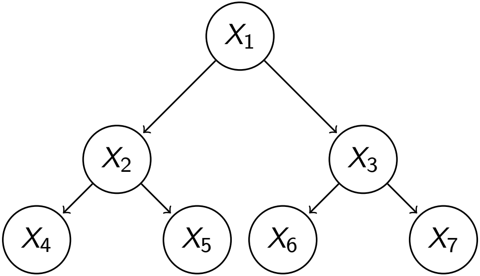

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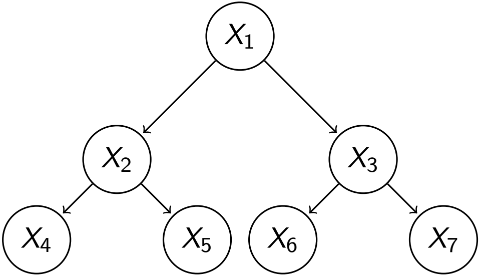

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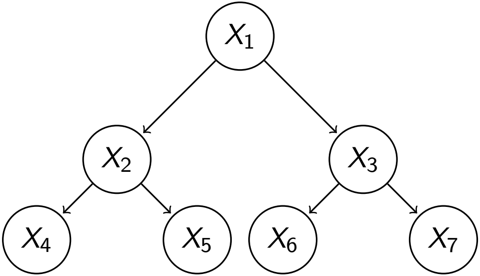

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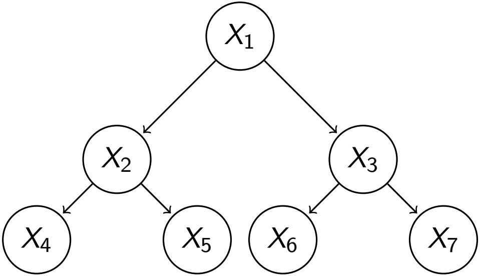

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And we are going to verify one of these because it highlights the distributive property that is crucial for message passing algorithm derivations.

Verifying a conditional independence

We want to show

$$p(X_3|X_2, X_1) = \frac{p(X_3, X_2, X_1)}{p(X_2, X_1)} = p(X_3|X_1)$$

Marginal $p(X_3, X_2, X_1)$

$$p(X_1, X_2, X_3) =$$
$$\sum_{X_4} \sum_{X_5} \sum_{X_6} \sum_{X_7} p(X_1)p(X_2|X_1)p(X_3|X_1)p(X_4|X_2)p(X_5|X_2)p(X_6|X_3)p(X_7|X_3)$$

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Marginals are

$$\begin{aligned} p(X_3, X_2, X_1) &= p(X_1)p(X_2|X_1)p(X_3|X_1) \\ p(X_2, X_1) &= p(X_1)p(X_2|X_1) \end{aligned}$$

plugging them in we get

$$p(X_3|X_2, X_1) = \frac{p(X_3, X_2, X_1)}{p(X_2, X_1)} = \frac{p(X_1)p(X_2|X_1)p(X_3|X_1)}{p(X_1)p(X_2|X_1)} = p(X_3|X_1)$$

and this confirms $X_3 \perp X_2 | X_1$

Bayes ball (http://uai.sis.pitt.edu/papers/98/p480-shachter.pdf)

You want to check if $X \perp Y | \mathcal{Z}$.

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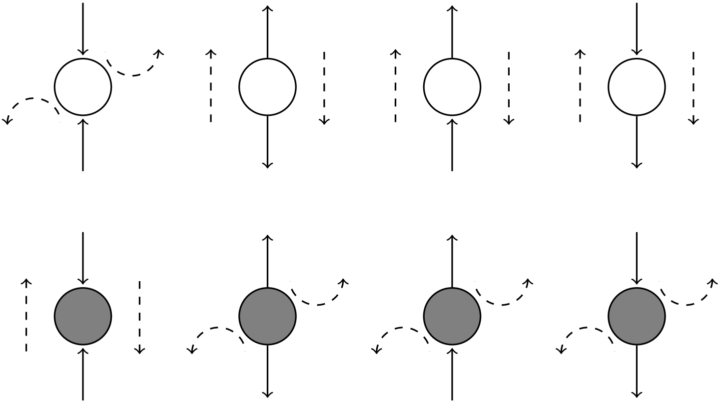

V-structure and Bayes ball

$X \perp Y | \mathcal{Z}$ is equivalent to no path from X to Y using these rules (nodes in $\mathcal{Z}$ are gray)

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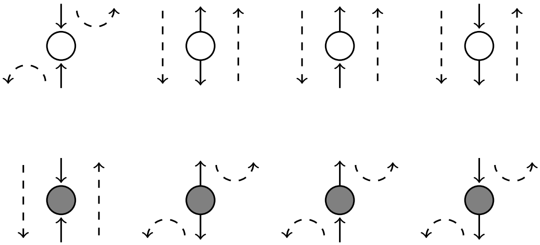
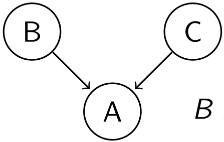
$X \perp Y | \mathcal{Z}$ is equivalent to no path from X to Y using these rules (nodes in $\mathcal{Z}$ are gray)

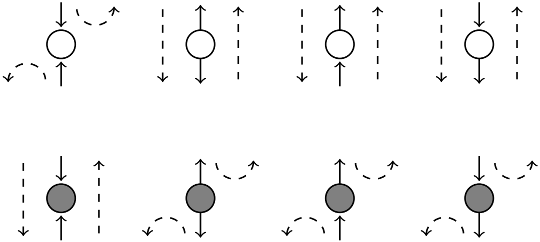


V-structure and Bayes ball



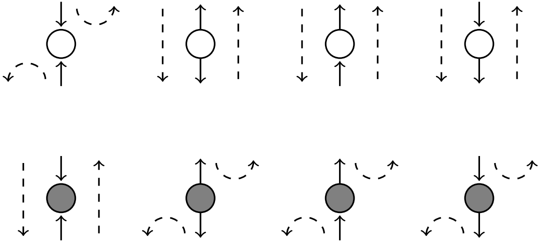
$B \perp C | \emptyset$

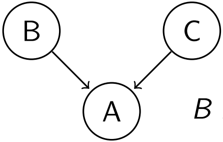
$X \perp Y | \mathcal{Z}$ is equivalent to no path from X to Y using these rules (nodes in $\mathcal{Z}$ are gray)



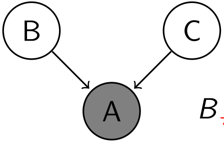
V-structure and Bayes ball

$X \perp Y|\mathcal{Z}$ is equivalent to no path from X to Y using these rules (nodes in $\mathcal{Z}$ are gray)





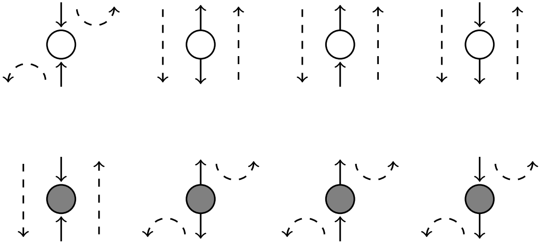
$B \perp C|\emptyset$

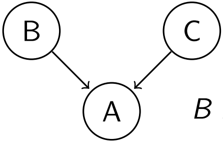


$B \not\perp C|A$

V-structure and Bayes ball

$X \perp Y | \mathcal{Z}$ is equivalent to no path from X to Y using these rules (nodes in $\mathcal{Z}$ are gray)





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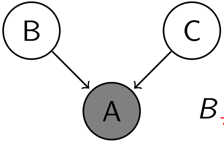
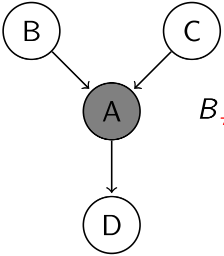
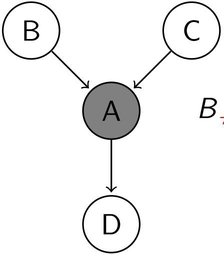


$B \not\perp C | A$



$B \not\perp C | A$

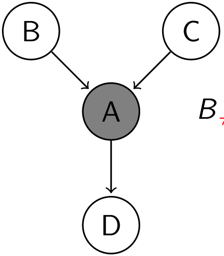
V-structure with a hanging unobserved variable



$B \not\perp C | A$

$$p(B, C|A) = \frac{p(A, B, C)}{p(A)} \quad =$$

V-structure with a hanging unobserved variable

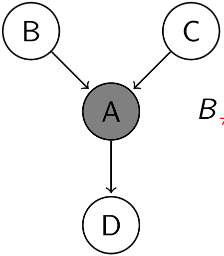


$B \not\perp C | A$

$$p(B, C|A) = \frac{p(A, B, C)}{p(A)} = \frac{\sum_D p(B)p(C)p(A|B, C)p(D|A)}{\sum_B \sum_C \sum_D p(B)p(C)p(A|B, C)p(D|A)}$$

$$=$$

V-structure with a hanging unobserved variable

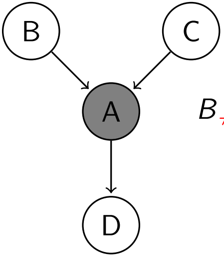


$B \not\perp C | A$

$$p(B, C|A) = \frac{p(A, B, C)}{p(A)} = \frac{\sum_D p(B)p(C)p(A|B, C)p(D|A)}{\sum_B \sum_C \sum_D p(B)p(C)p(A|B, C)p(D|A)}$$

$$= \frac{p(B)p(C)p(A|B, C)\sum_D p(D|A)}{\sum_B \sum_C p(B)p(C)p(A|B, C)\sum_D p(D|A)}$$

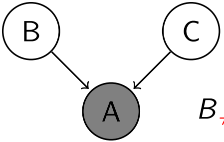
V-structure with a hanging unobserved variable



$B \not\perp C | A$

$$p(B, C|A) = \frac{p(A, B, C)}{p(A)} = \frac{\sum_D p(B)p(C)p(A|B, C)p(D|A)}{\sum_B \sum_C \sum_D p(B)p(C)p(A|B, C)p(D|A)}$$

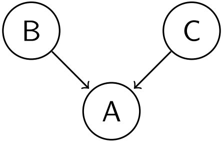
$$= \frac{p(B)p(C)p(A|B, C)\sum_D p(D|A)}{\sum_B \sum_C p(B)p(C)p(A|B, C)\sum_D p(D|A)}$$



$B \not\perp C | A$

V-structures and explaining away

Suppose we know of two competing explanations of an outcome.

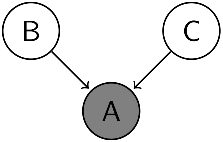


variables B and C are independent

$$\begin{aligned} p(B,C) &= \sum_A p(A|B,C)p(B)p(C) = p(B)p(C)\sum_A p(A|B,C) \\ &= p(B)p(C) \end{aligned}$$

V-structures and explaining away

But as soon as we observe A the variables B and C become dependent



$$
\begin{aligned}
p(B, C|A = a) &= \frac{p(A = a|B, C)p(B)p(C)}{\sum_B \sum_C p(A = a|B, C)p(B)p(C)} \\
&\neq p(B)p(C)
\end{aligned}
$$

Examples of explaining away

Outcome	Explanation 1	Explanation 2
wet grass	sprinkler	rain

Examples of explaining away

Outcome	Explanation 1	Explanation 2
wet grass	sprinkler	rain
student admitted	student brainy	student athletic

Examples of explaining away

Outcome	Explanation 1	Explanation 2
wet grass	sprinkler	rain
student admitted	student brainy	student athletic
house jumps	truck hits house	earthquake

Examples of explaining away

Outcome	Explanation 1	Explanation 2
wet grass	sprinkler	rain
student admitted	student brainy	student athletic
house jumps	truck hits house	earthquake

Explaining away: Given outcome two causes are not independent.

This is also called Berkson's paradox.

Worked out example of explaining away

E - earthquake, T - truck hits the house, H - house moves

$$p(E = 1) = 0.01 \quad p(T = 1) = 0.01$$

[1] So, proverbial perfect storms are unlikely; no, it does not pour when it rains, and country songs are a bit overdramatic ... in case you wondered

Worked out example of explaining away

E - earthquake, T - truck hits the house, H - house moves

$$p(E=1)=0.01 \quad p(T=1)=0.01$$
$$p(H=1|E,T)=\begin{cases}1, & T=1 \text{ or } E=1\\ 0, & \text{otherwise.}\end{cases}$$

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Worked out example of explaining away

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$$p(H=1|E,T)=\begin{cases}1, & T=1 \text{ or } E=1\\ 0, & \text{otherwise.}\end{cases}$$

Using Bayes rule we can compute $p(T=1|H=1)$ and $p(T=1|H=1,E=1)$

$$\begin{aligned} p(T=1|H=1) &= 0.5025 \\ p(T=1|H=1,E=1) &= 0.01 \end{aligned}$$

If the house moved and we know that there was an earthquake, the probability that a truck hit the house decreases [1].

---

[1] So, proverbial perfect storms are unlikely; no, it does not pour when it rains, and country songs are a bit overdramatic ... in case you wondered

Learning Bayesian Networks

Learning a Bayesian Network requires us to determine

- network's structure
- network's parameters

You will implement both of these in your HW2.

Learning a BayesNet

Probability of a state of all n variables $\mathbf{x}$

$$p(X_1 = x_1, ..., X_n = x_n) = \prod_{j=1}^{n} p(X_j = x_j | X_{\mathbf{pa}(j)} = x_{\mathbf{pa}(j)}, \theta_j)$$

Learning a BayesNet

Probability of a state of all n variables $\mathbf{x}$

$$p(X_1 = x_1, ..., X_n = x_n) = \prod_{j=1}^{n} p(X_j = x_j | X_{\mathbf{pa}(j)} = x_{\mathbf{pa}(j)}, \theta_j)$$

Our goal is to learn parameters $\Theta = (\theta_1, ...\theta_n)$ from multiple samples of the state of the Bayes net.

Example

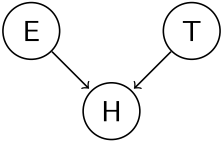
Data:

Variable / Sample	Earthquake	Truck	House moved
1	1	0	1
2	0	0	0
$\vdots$	$\vdots$	$\vdots$	$\vdots$
S	0	1	0

Example

Data:

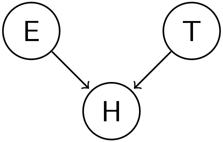
Variable / Sample	Earthquake	Truck	House moved
1	1	0	1
2	0	0	0
⋮	⋮	⋮	⋮
S	0	1	0



Example

Data:

Sample \ Variable	Earthquake	Truck	House moved
1	1	0	1
2	0	0	0
⋮	⋮	⋮	⋮
S	0	1	0



Conditional distributions and their parameters

- $p(E = 1|\theta_E) = \theta_E$, where $\theta_E \in [0, 1]$
- $p(T = 1|\theta_T) = \theta_T$, where $\theta_R \in [0, 1]$
- $p(H = 1|E = e, T = t, \theta_H) = \theta_{H,e,t}$, where $\theta_{H,e,t} \in [0, 1]$

Learning a BayesNet – likelihood

Data S instances of Bayes net's state; $x_{i,j}$ state of variable X_j in i^{th} sample.

$$L(\Theta) = \underbrace{\prod_{i=1}^{S}}_{\text{samples}} \underbrace{\prod_{j=1}^{n}}_{\text{variables}} p(X_j = x_{i,j} | X_{\mathbf{pa}(j)} = x_{i,\mathbf{pa}(j)}, {\color{red}\theta_j})$$

Likelihood Example

Data:

Sample \ Variable	Earthquake	Truck	House moved
1	1	0	1
2	0	0	0
$\vdots$	$\vdots$	$\vdots$	$\vdots$
S	0	1	0

$$
\begin{aligned}
L(\Theta) \quad = \quad & p(E=1|\theta_E)p(T=0|\theta_T)p(H=1|E=1,T=0,\theta_H) \\
& p(E=0|\theta_E)p(T=0|\theta_T)p(H=0|E=0,T=0,\theta_H) \\
& \vdots \\
& p(E=0|\theta_E)p(T=1|\theta_T)p(H=0|E=0,T=1,\theta_H)
\end{aligned}
$$

Likelihood Example – continued

$$
\begin{aligned}
L(\Theta) &= p(E=1|\theta_E)p(T=0|\theta_T)p(H=1|E=1,T=0,\theta_H) \\
&\times p(E=0|\theta_E)p(T=0|\theta_T)p(H=0|E=0,T=0,\theta_H) \\
&\times \vdots \\
&\times p(E=0|\theta_E)p(T=1|\theta_T)p(H=0|E=0,T=1,\theta_H)
\end{aligned}
$$

In terms of parameters

$$
\begin{aligned}
L(\Theta) &= \theta_E \quad (1-\theta_T) \quad \theta_{H,1,0} \\
&\times (1-\theta_E) \quad (1-\theta_T) \quad (1-\theta_{H,0,0}) \\
&\times \vdots \\
&\times (1-\theta_E) \quad \theta_T \quad (1-\theta_{H,1,0})
\end{aligned}
$$

Learning a BayesNet – log-likelihood

Log-likelihood is given by

$$\mathrm{LL}(\Theta) = \underbrace{\sum_{i=1}^{S}}_{\text{samples}} \underbrace{\sum_{j=1}^{n}}_{\text{variables}} \log p(X_j = x_{i,j} | X_{\mathbf{pa}(j)} = x_{i,\mathbf{pa}(j)}, \theta_j)$$

Crucial observation:

$$\mathrm{LL}(\Theta) = \underbrace{\sum_{j=1}^{n}}_{\text{variables}} \underbrace{\sum_{i=1}^{S}}_{\text{samples}} \log p(X_j = x_{i,j} | X_{\mathbf{pa}(j)} = x_{i,\mathbf{pa}(j)}, \theta_j)$$

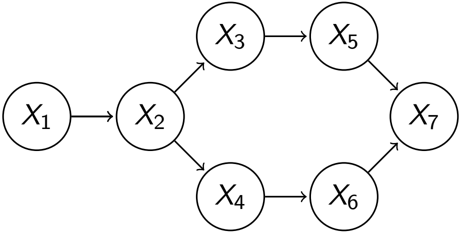
Learning a BayesNet – maximizing log-likelihood

$$
\begin{aligned}
\underset{\theta_j}{\operatorname{argmax}} \operatorname{LL}(\Theta) &= \underset{\theta_j}{\operatorname{argmax}} \sum_{j=1}^{n} \sum_{i=1}^{S} \log p(X_j = x_{i,j} | X_{\mathbf{pa}(j)} = x_{i,\mathbf{pa}(j)}, \theta_j) \\
&= \underset{\theta_j}{\operatorname{argmax}} \sum_{i=1}^{S} \log p(X_j = x_{i,j} | X_{\mathbf{pa}(j)} = x_{i,\mathbf{pa}(j)}, \theta_j)
\end{aligned}
$$

To learn parameters θ_j we only need states of node j and its parents.

Further, we do not need to concern ourselves with the rest of the graph.

For example



Learning $p(X_3|X_2)$ uses only states for X_2 and X_3

Sample \ Variable	X_1	X_2	X_3	X_4	X_5	X_6	X_7
1		0	1				
⋮		⋮	⋮				
S		1	0				

Learning BayesNets – structure learning

If we assume that BayesNet is a tree – each variable has a single parent – we can derive an algorithm to discover the optimal structure.
Given an empirical distribution of the data f we wish to find optimal distribution with factorization

$$p(X_1, \ldots, X_n) = \prod_{j=1}^{n} p(X_j | X_{\mathbf{pa}(j)})$$

where $\mathbf{pa}(j)$ has at most one element for all j.
Under this assumption, we can define edge set

$$\mathrm{Edges} = \{(j, \mathbf{pa}(j)) | j = 1, \ldots, n\}$$

Learning BayesNets – structure learning

Writing out log-likelihood

$$\mathrm{LL}(\Theta) = \sum_{i=1}^{S} \sum_{j} \log p(x_{i,j} | x_{i,\mathbf{pa}(j)})$$

Learning BayesNets – structure learning

Writing out log-likelihood

$$
\begin{aligned}
\mathrm{LL}(\Theta) &= \sum_{i=1}^{S} \sum_{j} \log p(x_{i,j} | x_{i,\mathbf{pa}(j)}) \\
&= \sum_{i=1}^{S} \sum_{j=1}^{n} \sum_{k=1}^{n} [(j,k) \in \mathrm{Edges}] \log p(x_{i,j} | x_{i,k})
\end{aligned}
$$

Learning BayesNets – structure learning

Writing out log-likelihood

$$\begin{aligned}
\mathrm{LL}(\Theta) &= \sum_{i=1}^{S} \sum_{j} \log p(x_{i,j} | x_{i,\mathbf{pa}(j)}) \\
&= \sum_{i=1}^{S} \sum_{j=1}^{n} \sum_{k=1}^{n} [(j,k) \in \mathrm{Edges}] \log p(x_{i,j} | x_{i,k}) \\
&= \sum_{j=1}^{n} \sum_{k=1}^{n} [(j,k) \in \mathrm{Edges}] \left(\sum_{i=1}^{S} \log p(x_{i,j} | x_{i,k}) \right)
\end{aligned}$$

Learning BayesNets – structure learning

Writing out log-likelihood

$$
\begin{aligned}
\mathrm{LL}(\Theta) &= \sum_{i=1}^{S} \sum_{j} \log p(x_{i,j} | x_{i,\mathbf{pa}(j)}) \\
&= \sum_{i=1}^{S} \sum_{j=1}^{n} \sum_{k=1}^{n} [(j,k) \in \mathrm{Edges}] \log p(x_{i,j} | x_{i,k}) \\
&= \sum_{j=1}^{n} \sum_{k=1}^{n} [(j,k) \in \mathrm{Edges}] \left(\sum_{i=1}^{S} \log p(x_{i,j} | x_{i,k}) \right) \\
&= \sum_{(j,k) \in \mathrm{Edges}} \left(\sum_{i=1}^{S} \log p(x_{i,j} | x_{i,k}) \right)
\end{aligned}
$$

Learning BayesNets – structure learning

$$\mathrm{LL}(\Theta) = \sum_{(j,k)\in\mathrm{Edges}} \left(\sum_{i=1}^{S} \log p(x_{i,j} | x_{i,k}) \right)$$

We observed that we can learn optimal $p(X_j | X_k, \theta_j)$ independently of the graph structure.

$$\theta_j^* = \operatorname*{argmax}_{\theta_j} \sum_{i=1}^{S} \log p(x_{i,j} | x_{i,k}, \theta_j)$$

Learning BayesNets – structure learning

$$\mathrm{LL}(\Theta) \sum_{(j,k)\in \mathrm{Edges}} \underbrace{\sum_{i=1}^{S} \log p(x_{i,j} | x_{i,k}, \theta_j^*)}_{\text{weight of edge (j,k)}}$$

Hence, we wish to find a tree with maximum weight, where each edge's weight is

$$w_{j,k} = \sum_{i=1}^{S} \log p(X_j = x_{i,j} | X_k = x_{i,k}, \theta_j^*)$$

Chow-Liu tree learning algorithm

Observe that

$$\sum_{i=1}^{S} \log p(X_j = x_{i,j} | X_k = x_{i,k}) = I(X_j, X_k) - H(X_j)$$

and since $H(X_j)$ does not depend on the edges we can rewrite the problem as

$$\operatorname*{argmax}_{\text{Edges}} \sum_{(j,k) \in Edges} I(X_j, X_k) - H(X_j) = \operatorname*{argmax}_{\text{Edges}} \sum_{(j,k) \in Edges} I(X_j, X_k)$$

Note that this is an example of **maximum spanning tree** problem which can be solved efficiently – HW2.

Chow-Liu tree learning algorithm – details

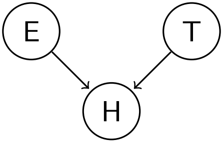
Mutual information $I(X_j, X_k)$ is computed on the empirical distribution

$$
\begin{aligned}
f_{j,k}(a,b) &= \frac{\sum_{i=1}^{S}[X_j = a, X_k = b]}{S} \\
f_l(a) &= \frac{\sum_{i=1}^{S}[X_l = a]}{S} \\
I(X_j, X_k) &= \sum_a \sum_b f_{j,k}(a,b) \log \frac{f_{j,k}(a,b)}{f_j(a) f_k(b)}
\end{aligned}
$$

Chow-Liu tree – downsides

Single parent assumptions can be restrictive.

The tree models do not permit explaining away.



Tree structured Bayes nets are step above independent models.

On the upside, they are extremely easy to learn.

Today

- Bayesian Networks
- Learning Bayesian Networks