Space Vehicles and Orbital Dynamics

Exercises and problems

Version 2020.2

Aerospace Engineering Department (UC3M)

Foreword

This is a collection of exercises and problems for the course *Space Vehicles and Orbital Dynamics*. The document is currently a work in progress, and as such they may include typos and errors. We require all your help and feedback to improve it and make it more useful for future students. Please contact us if you see any areas of improvement: mario.merino@uc3m.es.

Several problems are taken or adapted from other references, in particular, from "Orbital Mechanics for Engineering Students" by H.D. Curtis, 3rd edition, Elsevier (2010).

The authors,
Leganés, January 2019

Contents

Basic exercises

1. A ground vehicle P is traveling Northward along the Greenwich meridian with a constant speed of $v_1^P = 100$ m/s with respect to the Earth-centered, Earth-fixed (ECEF) reference frame S_1. At time $t = t_0$, its latitude is 45 deg North. Compute: (a) Its position, velocity and acceleration in ECEF at that instant. (b) Its velocity and acceleration in a Earth-cencerted Inertial (ECI) reference frame S_0 at that instant.

2. The Cartesian coordinates $\{x, y, z\}$ of a point P are

$$
\begin{aligned}
x &= 1, \\
y &= 1, \\
z &= \sqrt{2}.
\end{aligned}
$$

   Calculate its cylindrical and spherical coordinates, and find the normalized local tangent vector basis for each coordinate system at that point

Two-body problem

3. At two points on a geocentric orbit the altitude and true anomaly are $\zeta_1 = 1538$ km, $\theta_1 = 126$ deg and $\zeta_2 = 845$ km, $\theta_2 = 58$ deg. Find: (a) the orbit eccentricity; (b) the altitude at perigee; (c) the semimajor axis; and (d) the orbital period.

4. Ida is an asteroid, the first found to have a natural satellite, the tiny Dactyl, whose mass is negligible with respect to that of Ida. This fact (the existence of binary asteroids, or asteroids with moons) was discovered by the Galileo probe in its way to Jupiter. Knowing that the semi-major axis of the orbit of Dactyl about Ida is 60 km and that the sidereal period is 15.3 h, estimate the mass of Ida.

5. A satellite is orbiting the Earth with a perigee altitude of $\zeta_p = 393$ km and an eccentricity $e = 0.6$.
   (a) What is the velocity at perigee and apogee?
   (b) What velocity *should* the satellite have at perigee, to be in a circular orbit?
   (c) Find out the escape velocity at perigee and apogee. Which one is smaller? At what point is it cheaper to give an impulse to the satellite in the elliptic orbit, so that it escapes to infinity?

6. The *hodograph* is the plot of the velocity of a point particle in velocity space as it moves along its trajectory. When using polar coordinates, the hodograph represents the evolution of the velocity in the $v_r = \dot{r}$, $v_\theta = r\dot{\theta}$ plane. Represent the hodograph in the two-body problem in the case of (a) an elliptic orbit (b) a parabolic orbit and (c) an hyperbolic orbit. Identify the flight path angle γ and the velocity at pericenter and apocenter in each diagram, and represent qualitatively in a separate plot γ as a function of θ.

7. A satellite on an elliptic Earth orbit has $r_p = 10000$ km and $r_a = 20000$ km. Determine: (a) the semi-major axis a, (b) the eccentricity e, (c) the mechanical energy ξ, (d) the time to coast from periapsis to apoapsis, (e) the radius and true anomaly at which the orbital velocity equals the local circular velocity at that radius, $v = v_c(r)$. Do you identify these points in the ellipse?

8. We have set a satellite into a parabolic escape trajectory from Earth. The radius of perigee is 7000 km. Compute the distance d between the two points of the parabolic trajectory after perigee that have radius $r_1 = 8000$ and $r_2 = 16000$ km, respectively.

9. At a given instant of time, a spacecraft is in a geocentric trajectory with radius $r = 14600$ km, velocity $v = 8.6$ km/s, and flight path angle $\gamma = 50$ deg. You are asked to (a) prove that the trajectory is hyperbolic, and then to compute: (b) angular momentum (c) eccentricity, (d) true anomaly at that point, (e) radius of the periapsis of the orbit, (f) semi-major axis, (g) characteristic energy c_3, (h) turning angle δ, (i) impact parameter Δ.

10. A satellite, described in an ECI-equatorial reference frame S_0, has classical orbital elements $\Omega = 25.4$ deg, $i = 12.5$ deg, $\omega = 232.6$ deg $a = 20000$ km, $e = 0.05$, and $\theta = 245.1$ deg. Compute the position and velocity vectors $\boldsymbol{r}$, $\boldsymbol{v}$ at that instant in S_0, expressed in the basis of S_0. Then, calculate the velocity vector that an observer on an ECEF reference frame S_1 would observe, $\boldsymbol{v}_{\text{rel}}$.

11. A spacecraft is known to have

$$\boldsymbol{r} = -6045\boldsymbol{i} - 3490\boldsymbol{j} + 2500\boldsymbol{k} \text{ km}$$
$$\boldsymbol{v} = -3.457\boldsymbol{i} + 6.618\boldsymbol{j} + 2.533\boldsymbol{k} \text{ km/s}$$

in an ECI-equatorial reference frame. Find the classical orbital elements of the spacecraft.

12. We are on the equator of the Moon an we throw a rock toward the East, at a 45 deg angle with respect to the local horizontal. The rock reaches an apoapsis of 400 km altitude before falling back to the Moon. Compute (a) the velocity magnitude v_0 in a Luna-centered inertial reference frame right after the impulse (b) the Δv we have imparted to the rock, taking into account Luna's rotation.

13. Implement a Newton-Rhapson solver `x=newton(f,df,x,tol,maxiter)` in Matlab to find a root of a function $f(x) = 0$ based on the iteration formula

$$x_n = x_{n-1} - \frac{f(x_{n-1})}{f'(x_{n-1})}. \tag{0.1}$$

Use it to find the root of $f(x) = \cos(x) - x$ around $x = 0.75$ with a tolerance of $\varepsilon = 10^{-8}$.

14. At an instant of time t_0, a satellite was seen to be at $\theta = 45$ deg in an Earth orbit that has $a = 20000$ km and $e = 0.4$. (a) Find its radius, radial rate, and angular rate at that instant of time. (b) What are the eccentric anomaly and the true anomaly after 40 min? (c) Compute the radius, radial rate, and angular rate after 40 min.

15. A satellite describes an elliptic orbit with $a = 18000$ km and $e = 0.1$. At $t = 0$, it is at $\theta = 20$ deg. Calculate the time t when the satellite is at $\theta = 40, 60, 80, 100, 120, 140, 160, 180$ and 200 deg. Create a small table to report your results:

θ	E	M_e	t
...	...	...	...

16. An interplanetary probe flies by Jupiter in an hyperbolic orbit with $e = 1.1$ and $p = 50000$ km. At time $t_1 = 0$, the probe is found at $\theta_1 = -85.2$ deg. What is the time of flight $\Delta t_{\text{tof}} = t_2 - t_1$ from that instant until the probe is found at $\theta_2 = 46.5$ deg?

Orbital Maneuvering

17. A spacecraft is in a 480 km by 800 km earth orbit (orbit 1 in Figure 0.1). Find (a) the Δv required at perigee A to place the spacecraft in a 480 km by 16 000 km transfer orbit (orbit 2); and (b) the Δv (apogee kick) required at B of the transfer orbit to establish a circular orbit of 16 000 km altitude (orbit 3). Data: $\mu_{Earth} = 0.39860 \cdot 10^6$ km^3/s^2 . $R_{Earth} = 6378.137$ km

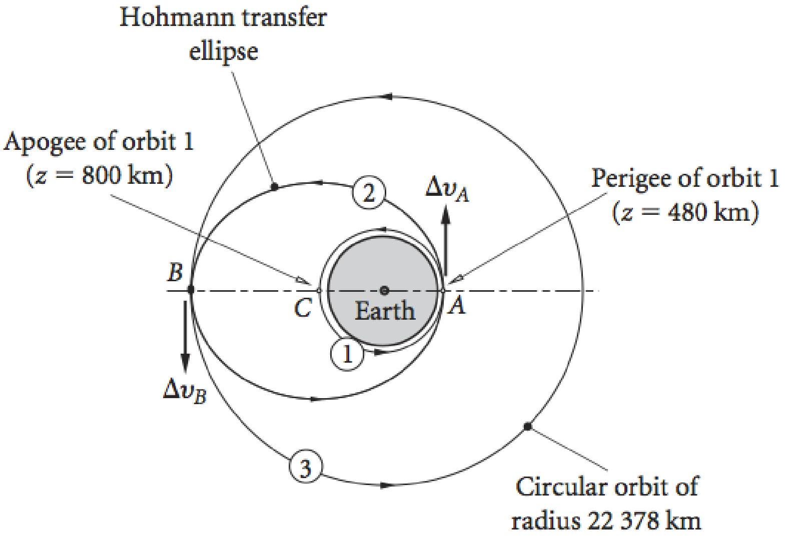

Figure 0.1: Hohmann transfer. Figure exercise 17

18. Find the total Δv requirement for a bi-elliptical Hohmann transfer from a geocentric circular orbit of 7000 km radius to one of 105 000 km radius. Let the apogee of the first ellipse be 210 000 km. Compare the Δv schedule and total flight time with that for an ordinary single Hohmann transfer ellipse. Data: $\mu_{Earth} = 0.39860 \cdot 10^6$ km^3/s^2 . $R_{Earth} = 6378.137$ km

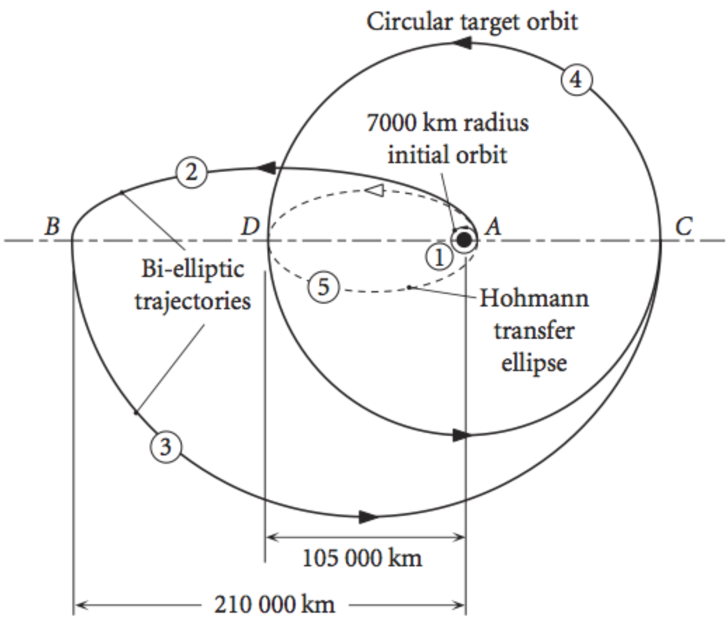

Figure 0.2: Bi-elliptical Hohmann Transfer. Figure exercise 18

19. The space station and spacecraft A and B are all in the same circular Earth orbit of 350 km altitude. Spacecraft A is 600 km behind the space station and spacecraft B is 600 km ahead of the space station. At the same instant, both spacecraft apply a $\Delta v_{\perp}$ so as to arrive at the space station in one revolution of their phasing orbits.

    (a) Calculate the times required for each spacecraft to reach the space station.

    (b) Calculate the total Δv requirement for each spacecraft.

    Data: $\mu_{Earth} = 0.39860 \cdot 10^6$ km^3/s^2 . $R_{Earth} = 6378.137$ km

20. Assuming the orbits of Earth and Mars are circular and coplanar, calculate

    (a) the time required for a Hohmann transfer from Earth to Mars, and

    (b) the initial position of Mars (α) in its orbit relative to earth for interception to occur.

    Data: Radius of earth orbit= $1.496 \cdot 10^8$ km. Radius of Mars orbit= $2.279 \cdot 10^8$ km. $\mu_{Sun} = 1.327 \cdot 10^{11}$ km^3/s^2.

21. The shuttle orbiter has a mass of 125 000 kg. The two orbital manoeuvring engines produce a combined (non-throttleable) thrust of 53.4 kN. The orbiter is in a 300 km circular orbit. A delta-v manoeuvre transfers the spacecraft to a coplanar 250 km by 300 km elliptical orbit. Neglecting propellant loss and using elementary physics (linear impulse equals change in linear momentum, distance equals speed times time), estimate

    (a) the time required for the Δv burn, and

    (b) the distance traveled by the orbiter during the burn.

    (c) Calculate the ratio of your answer for (b) to the circumference of the initial circular orbit.

    Data: $\mu_{Earth} = 0.39860 \cdot 10^6$ km^3/s^2 . $R_{Earth} = 6378.137$ km

22. A spacecraft in circular orbit of radius r leaves for infinity on parabolic trajectory and returns from infinity on a parabolic trajectory to a circular orbit of radius 12 r. Find the total Δv required for this non-Hohmann orbit change manoeuvre.

23. An earth satellite is in an elliptical orbit of eccentricity 0.3 and angular momentum 60 000 km^2/s. Find the Δv required for a 90^o change in inclination at apogee (no change in speed).

24. The goal of this exercise is to compare numerically the performance of the Hohmann and bi-elliptical Hohmann transfers in terms of the required Δv. Firstly, a function `dv_Hohmann` is to be implemented, providing the total Δv of the manoeuvre as output, as a function of the initial, r_A, and the final radius, , r_C, (and the gravitational parameter of the central body). Secondly, a function `dv_Bielliptical` is to be implemented, providing the total Δv of the manoeuvre as a function of the initial, the intermediate and the final radius (and the gravitational parameter of the central body). Finally, for the comparison, we will consider ratios of r_C/r_A from 1 to 30, and ratios of r_B/r_A from r_C/r_A to 100.

25. This exercise is a propagation problem from given initial conditions taking into account the gravitational attraction of the main body and the thrust acceleration of the propulsion system of an spacecraft. The equations of motion are written in polar coordinates and non-dimensional form as [1]:

$$\dot{r} = u$$
$$\dot{\alpha} = \frac{v}{r}$$
$$\dot{u} = \frac{v^2}{r} - \frac{1}{r^2} + a \sin\theta$$
$$\dot{v} = -\frac{uv}{r} + a\cos\theta$$

[1] The characteristic length is the initial radius, r_0, the characteristic time is $\sqrt{\mu/r_0^3}$

where θ is the thrust path angle ($\theta = 0$, tangential thrust, $\theta = \pi/2$ radial thrust). It is considered that the thrust is on during the whole trajectory, with constant thrust, T. The propellant consumption rate ($\dot{m}$) is also constant. Therefore, $a = T/(1 - |\dot{m}|t)$. The initial conditions are (departing from a circular orbit): $r(0) = 1$, $\alpha(0) = 0$, $u(0) = 0$, $v(0) = 1$. For performing the propagation, follow the steps:

(a) Implement a function `derPolar` that computes the derivatives of the state $[r, \, , \alpha, \, u, \, v]^T$.

(b) With $a = 0$, integrate the problem and check that the resulting orbit is circular.

(c) Include $a = T/(1 - |\dot{m}|t)$ with a constant value of θ.

(d) Pass an array of times and values of θ to the function `derPolar`, to compute the steering law $\theta(t)$ as the interpolation among the given values (you can use the built-in matlab function `interp1`).

Numerical values: $T = 0.1405$, $|\dot{m}| = 0.07489$, $t_f = 3.3155$ (time of flight).

26. In the previous problem, it is not possible to establish a priori, the final radius or velocity. In this exercise, we will use a Direct Method to optimise the control law $\theta(t)$ to achieve the maximum possible radius of a circular orbit. The problem to solve is, therefore,

$$\max_{\theta(t)\in\Theta} r(t_f)$$

Subject to: $u(t_f) = 0$ $v(t_f) = 1/\sqrt{r_f}$ and the equations of motion from the previous exercise.

To do that, a function `performanceIndex` should be implemented. This function has as input an array of values of θ at equally space times, and bring back as output the value of $r(t_f)$. In addition, a function `constraintFcn` is to be implemented, accepting as input an array of equally spaced in time values of θ, and bringing back as output the value of the constraints that should be equal to 0 ($u(t_f)$ and $v(t_f) - 1/\sqrt{r_f}$). Once both functions are implemented, the built-in matlab routine `fmincon` can be used to solve the optimisation problem. In the matlab help you can find more information about the direct method implemented in this function. Try to call it as: `thSol = fmincon(@performanceIndex, thIG, [], [], [], [], zeros(length(thIG)),2*pi*ones(length(thIG)), ... @constraintFcn);` . The array `thIG` contains an initial guess of the solution, the first intent to get the array of values of θ that maximises the final radius. The other element in the `fmincon` window are set to give boundaries to the elements of the array of θs. The array `thSol` is the solution of the problem. You can test the final value of r introducing this array in the propagation problem of the previous exercise.

27. The previous problem is intended to be solved using the Indirect Approach of the Optimal Control Problem. For that, it is necessary to derive the Euler-Lagrange equations of the system.

(a) Thus, taking $\mathbf{x}^T = [r, \, \alpha, \, u, \, v]$ and $\lambda^T = [\lambda_r, \, \lambda_\alpha, \lambda_u, \, \lambda_v]$, obtain the explicit form of:

$$\begin{aligned}
\dot{\mathbf{x}}^T &= \frac{\partial \mathcal{H}(\mathbf{x}, \mathbf{u}, \lambda, t)}{\partial \lambda} \\
\dot{\lambda}^T &= -\frac{\partial \mathcal{H}(\mathbf{x}, \mathbf{u}, \lambda, t)}{\partial \mathbf{x}} \\
\mathbf{0} &= \frac{\partial \mathcal{H}(\mathbf{x}, \mathbf{u}, \lambda, t)}{\partial \mathbf{u}}
\end{aligned}$$

From the last equation, get an explicit expression of θ as a function of λ.

(b) Implement the set of six ordinary differential equations for the evolution of the system in a matlab function `derOCP`.

(c) The solution of the problem should also fulfil the initial conditions: $\mathbf{x}(0) = [1, 0, 0, 1]$, the final conditions $u(t_f) = 0; v(t_f) = 1/\sqrt{r_f}$ and the transversality conditions $\lambda_r(t_f) =$

$1 + \lambda_v(t_f)/(2r(t_f)^{3/2})$. Implement a function `fpOCP` that given as input the values of the co-states at t_0, provide the vector function $\mathbf{F}(\mathbf{x}) = \mathbf{0}$.

$$\mathbf{F} = \begin{cases} u(t_f) \\ v(t_f) - 1/\sqrt{r_f} \\ \lambda_r(t_f) - 1 - \lambda_v(t_f)/(2r(t_f)^{3/2}) \end{cases}$$

(d) Use the built-in matlab function `fsolve` to obtain the values of the co-states at t_0 that provide the solution to the optimal control problem.

Initial Orbit Determination and Lambert's Problem

28. A ground station located at 40.3320 deg N, 3.7687 deg W, 600 m above sea level, measures the position and velocity of a spacecraft in its topocentric (SEZ) reference frame. The results of the measurements are

$$\rho = 2793.1 \text{ km}; \qquad \beta = 78.20 \text{ deg}; \qquad \varepsilon = 25.80 \text{ deg};$$
$$\dot{\rho} = 2.9453 \text{ km/s}; \qquad \dot{\beta} = -0.0984 \text{ deg/s}; \qquad \dot{\varepsilon} = -0.1075 \text{ deg/s};$$

Find the position and velocity vector of the spacecraft in ECI, if two sidereal hours have passed since Greenwich meridian crossed the $\boldsymbol{i}_0$ direction of ECI. From there, compute the COEs using the functions you developed in previous classes.

29. Three sequential observations are made in ECI of a spacecraft in low Earth orbit,

$$\begin{aligned} \boldsymbol{r}_1 &= -2014.52\boldsymbol{i}_0 + 4254.09\boldsymbol{j}_0 + 5409.70\boldsymbol{k}_0 \\ \boldsymbol{r}_2 &= -9550.16\boldsymbol{i}_0 - 1683.95\boldsymbol{j}_0 + 9097.54\boldsymbol{k}_0 \\ \boldsymbol{r}_3 &= 3596.23\boldsymbol{i}_0 - 7594.21\boldsymbol{j}_0 - 9657.16\boldsymbol{k}_0 \end{aligned}$$

where all the data is given in km. Armed with this information:

(a) Check that the three position vectors are indeed coplanar.

(b) Find out the COEs of the spacecraft.

(c) Compute the velocity that the spacecraft had when it was at $\boldsymbol{r}_2$.

30. We want to determine the orbit of a S/C knowing two position vectors in ECI:

$$\begin{aligned} \boldsymbol{r}_1 &= 15945.34\boldsymbol{i}_0 \text{ km} \\ \boldsymbol{r}_2 &= (12214.83899\boldsymbol{i}_0 + 10249.64731\boldsymbol{k}_0)\text{km} \end{aligned}$$

and the time span between the two observations, $\Delta t = 76$ minutes. Do the following analyses:

(a) Compute the two possible solutions (short arc, long arc). Check whether one of them hits the Earth and discard it.

(b) Explore how the long arc and short arc solution change, if the time span between observations was different, between a minimum of 30 minutes to a maximum of 240 minutes.

Orbital Perturbations

31. Compute and plot the time history of the semi-major axis, a, and eccentricity, e, of an Earth orbit with the following initial conditions:

$$
\begin{aligned}
a_0 &= 7200 \text{ km} & e_0 &= 0.05 \\
i_0 &= 30 \text{ deg} & \Omega_0 &= 45 \text{ deg} \\
\omega_0 &= 50 \text{ deg} & \nu_0 &= 100 \text{ deg}
\end{aligned}
$$

Integrate the equations of motion using the Cowell's method, based on the following set of ordinary differential equations:

$$\ddot{\mathbf{r}} = -\frac{\mu}{r^3}\mathbf{r} + \mathbf{a}_p$$

In this exercise, do not consider any perturbing acceleration $\mathbf{a}_p = \mathbf{0}$. Use as final time 1 orbital period, 10 orbital periods and 100 orbital periods.

32. Compute and plot the time history of the semi-major axis, a, and eccentricity, e, of an Earth orbit with the following initial conditions:

$$
\begin{aligned}
a_0 &= 7200 \text{ km} & e_0 &= 0.05 \\
i_0 &= 30 \text{ deg} & \Omega_0 &= 45 \text{ deg} \\
\omega_0 &= 50 \text{ deg} & \nu_0 &= 100 \text{ deg}
\end{aligned}
$$

Integrate the equations of motion using the Gauss' Planetary equations, based on the following set of ordinary differential equations:

$$
\begin{aligned}
\frac{\mathrm{d}a}{\mathrm{d}t} &= \frac{2a^2}{h}\left(e\sin\nu a_r + \frac{p}{r}a_\theta\right) \\
\frac{\mathrm{d}e}{\mathrm{d}t} &= \frac{1}{h}\left(p\sin\nu a_r + \left((p+r)\cos\nu + r\,e\right)a_\theta\right) \\
\frac{\mathrm{d}i}{\mathrm{d}t} &= \frac{r\cos(\omega+\nu)}{h}a_h \\
\frac{\mathrm{d}\Omega}{\mathrm{d}t} &= \frac{r\sin(\omega+\nu)}{h\sin i}a_h \\
\frac{\mathrm{d}\omega}{\mathrm{d}t} &= -\frac{1}{e\,h}\left[\frac{h^2}{\mu}\cos\nu a_r - \left(r + \frac{h^2}{\mu}\right)\sin\nu a_\theta\right] - \frac{r\sin(\omega+\nu)}{h\tan i}a_h \\
\frac{\mathrm{d}\nu}{\mathrm{d}t} &= \frac{h}{r^2} + \frac{1}{e\,h}\left[\frac{h^2}{\mu}\cos\nu a_r - \left(r + \frac{h^2}{\mu}\right)\sin\nu a_\theta\right]
\end{aligned}
$$

In this exercise, do not consider any perturbing acceleration $a_r = a_\theta = a_h = 0$. Use as final time 1 orbital period, 10 orbital periods and 100 orbital periods. Assess if this implementation can be used in case $e_0 = 0$.

33. Use the Cowell's method and the Gauss' Planetary equations to determine the effect of the J_2 perturbation on the variation of orbital elements a, e, i, ω and Ω over 48 hours on an Earth orbit with the following initial conditions:

$$
\begin{aligned}
a_0 &= 7200 \text{ km} & e_0 &= 0.05 \\
i_0 &= 30 \text{ deg} & \Omega_0 &= 45 \text{ deg} \\
\omega_0 &= 50 \text{ deg} & \nu_0 &= 100 \text{ deg}
\end{aligned}
$$

Take into account that the potential function of the J_2 in Cartesian coordinates is:

$$R_{J_2} = \frac{\mu J_2 R_E^2}{2r^3}\left[1 - 3\left(\frac{z}{r}\right)^2\right]$$

with $J_2 = 0.00108263$. In addition, the perturbing acceleration as a function of the orbital elements in the local-vertical local-horizontal reference frame is:

$$\begin{aligned}
a_r &= -\frac{3}{2}\frac{J_2\mu R_E^2}{r^4}\left[1 - 3\sin^2 i \sin^2(\omega+\nu)\right] \\
a_\theta &= -\frac{3}{2}\frac{J_2\mu R_E^2}{r^4}\sin^2 i \sin 2(\omega+\nu) \\
a_h &= -\frac{3}{2}\frac{J_2\mu R_E^2}{r^4}\sin(2i)\sin(\omega+\nu)
\end{aligned}$$

If the initial inclination was $i = 63.5$ deg, indicate how the results would have changed.

34. Use the Cowell's method and the Gauss' Planetary equations to determine the effect of the atmospheric drag on the variation of orbital elements a, e, i, ω and Ω over 20 orbital periods on an Earth orbit with the following initial conditions:

$$\begin{aligned}
a_0 &= 7200 \text{ km} & e_0 &= 0.05 \\
i_0 &= 30 \text{ deg} & \Omega_0 &= 45 \text{ deg} \\
\omega_0 &= 50 \text{ deg} & \nu_0 &= 100 \text{ deg}
\end{aligned}$$

The area-to-mass ratio A/m, of the spacecraft is 2 m^2/kg, and the ballistic coefficient is equal to 2. The drag model to be considered is as follows:

$$\mathbf{a}_D = \frac{1}{2}\rho C_D \frac{A}{m} v\mathbf{v}$$

Take into account that the projection of the velocity in the local-horizontal local-vertical reference frame is:

$$\begin{aligned}
v_r &= \frac{\mu}{h}\left(-\sin\nu\cos\nu + \sin\nu(e+\cos\nu)\right) \\
v_\theta &= \frac{\mu}{h}\left(\sin^2\nu + \cos\nu(e+\cos\nu)\right)
\end{aligned}$$

Interplanetary flight

35. ESA is planning to send a new orbiter to Mars and you are in charge of the preliminary mission analysis. Compute, in the framework of the restricted patched conics method, and extending exercise 20, (a) the synodic period of Mars as seen from Earth; (b) the semi-major axis and eccentricity of the transfer orbit (Hohmann transfer); (c) the velocity at perihelion and aphelion of the spacecraft in the heliocentric reference frame S_0; (d) the transfer time; (e) the initial relative phase angle for the mission to begin; (f) the minimum waiting time at Mars before returning to Earth.

36. The spacecraft of previous exercise is in a 300 km circular parking orbit around the Earth, waiting before the injection in a escape trajectory toward Mars. Calculate: (a) the parameters of the escape hyperbola, and (b) the Δv required to enter the escape trajectory.

37. After a long interplanetary transfer, the spacecraft of previous exercise is approaching Mars. Along the way, several trajectory control maneuvers were applied to tailor its entry into Mars space. If our goal is to insert the spacecraft into a circular science orbit with an orbital period of 4 hours, compute: (a) the radius of the science orbit, (b) the parameters of the arrival hyperbola, (c) the capture Δv.

38. We have sent a spacecraft to Jupiter (♃) using a Hohmann transfer from Earth (⊕). We want to perform a gravity assist maneuver at Jupiter to catapult the spacecraft into a higher energy heliocentric orbit. You are asked to: (a) compute the excess hyperbolic velocity of the arrival hyperbola with respect to Jupiter (b) express the turn angle δ and the velocity change vector $\Delta\boldsymbol{v}$ as a function of the impact parameter (c) the semi-major axis and eccentricity of the new heliocentric orbit after leaving Jupiterspace.

39. We are at the Summer solstice and we want to send a probe from Earth (⊕) to Mercury (☿). In order to do so, we will launch from latitude 40 deg North toward the East into a circular parking orbit at 400 km height. Find out the local time at the launchpad that enables us to acquire the necessary parking orbit, and the true anomaly where the escape maneuver shall be performed. Remember that the inclination angle between the equatorial plane and the ecliptic is 23.4 deg.

40. The probe of previous exercise is on its way to Mercury (☿). We want to insert the spacecraft into a science orbit with radius at pericenter $r_{sp} = 14000$ km and inclination 30 deg. If the declination of the spacecraft far away from Mercury is 5 deg North with respect to the equator, compute the required impact vector $\boldsymbol{B}$ in the B-plane.

41. Consider an interplanetary transfer from Earth to Mars with the following characteristics:

    - Launch dates between 1st Jan 2019 and 1st Jan 2029.
    - Transfer duration between 60 and 680 days.

    Provide the following:

    - Pork chop plot of the characteristic energy c_3 for the departure from Earth
    - Pork chop plot of the characteristic energy c_3 for the arrival at Mars
    - Pork chop plot of the total c_3.

    Finally, identify the most economic transfer in terms of c_3. This is a figure of merit proportional to the sum of the excess energy at departure and at arrival (and hence it is a good index of the total propulsive cost of the mission). Once you have identified the best transfer, plot the trajectory of the S/C, Mars and Earth. You can use the ephemerides of the Horizons database continuously or only for the initial condition of the planets at some earlier time in the problem, as you prefer.

Three-body problem

42. Consider the Sun-Earth system $(\odot, \oplus)$ as a circular restricted problem. Find the location of Lagrange points L_1, L_2, L_3 (collinear Lagrange solutions).

43. We have three point particles P_1, P_2, P_3 with nondimensional masses $m^{P_1} = 0.90$, $m^{P_2} = 0.09$, and $m^{P_3} = 0.01$. The nondimensional gravity constant is $G = 1$ The particles are known to be in one of the collinear Lagrange solutions, ordered as P_1, P_2, P_3 on the line of masses. If each mass performs a elliptic orbit with eccentricity $e = 0.1$ about the center of mass of the system, and the nondimensional period of the system is $\tau = 1$, find the nondimensional trajectory of each mass with respect to the center of mass of the system.

44. Consider the Sun-Jupiter CR3BP $(\odot, ♃)$. Write a short Matlab code that performs the following tasks, in non-dimensional units: (a) Identify the values of the energy E for which the Lagrange points L_1 through L_5 first become accessible; (b) Plot the contour levels of the potential $U(\boldsymbol{r})$ in the invariant plane of the problem using the `contour` function. (c) Plot the 3D Hill's surface using the `isosurface` function for values of energy E in between the bounds identified in the first question. You may want to add some transparency to the surface to visualize it better.

45. Consider the motion of a point particle P near the L_4 point in the Sun-Earth system $(\odot, \oplus)$. (a) Compute the derivatives U_{xx0}, U_{xy0}, U_{yy0} and U_{zz0}. Obtain the eigenvalues and eigenvectors of the $x'y'$ problem, and the angular frequency in the z' problem. Plot the analytical solution of the linearized problem for each mode, giving a magnitude of 10^{-5} to the eigenvectors. (b) Use `ode45` to integrate the full (i.e., not linearized) equations of motion of P. Plot the full solution for each mode of the linearized problem and compare the with the linearized solution. How long does it take the solutions to depart substantially? (c) Repeat the analysis for L_1.

46. Roche's limit defines the radius from the center of a planet beyond which loose matter can clump together due its own gravity to form a moon, and below which tidal forces will tear it apart to form a ring.

    Consider a planet P_1 of mass m_1 and a spherical rock P_2 of mass m_2 and radius R_2 in a circular orbit about P_1. The distance between them is L. There is a pebble P_3 sitting on P_2, looking toward P_1. (a) Write down the equation of motion of P_3 in the x direction of the synodic reference frame S_s, ignoring the contact force. (b) Linearize the equation of motion keeping only first order terms in $R_2 \ll x_2$. (b) The Roche limit occurs when $\ddot{x} = 0$. With this condition, find the Roche radius L^* as a function of the parameters of the problem.

Relative Motion Problems

47. A chaser satellite and the target satellite are in close proximity, co-planar circular orbits. At $t = 0$, the position of the chaser relative to the target is $\delta \boldsymbol{r}_0 = a\boldsymbol{i}$, in the Hill's reference frame. Find the total delta-V required for the chaser to end up in a new circular orbit at $\delta \boldsymbol{r}_f = -a\boldsymbol{i}$ at $t = \pi/\omega$, where ω is the mean motion of the target.

48. A space station is in a 90-minute period circular earth orbit. At $t = 0$, a satellite has the following position and velocity components relative to a Clohessy-Wiltshire frame attached to the space station: $\delta \boldsymbol{r} = [1\ 0\ 0]^T$ (km), $\delta \boldsymbol{v} = [0\ 10\ 0]^T$ (m/s). How far is the satellite from the space station 15 minutes later?

49. In order to avoid some orbiting debris, a geosynchronous satellite in a circular equatorial orbit applies a ΔV_1 (radial). Some time later the satellite has the following relative position with respect to its non-manoeuvring location: $x = -120.6$ km, $y = 71.12$ km. What is the time that has passed? The satellite will now initiate a two-impulse manoeuvre to return to its original location in two hours. What are the magnitudes and directions of these ΔV_2 and ΔV_3?

50. Show that the required ΔV to perform a homing manoeuvre (i.e., displacement of distance Δx in the R-bar direction to reach the V-bar axis in half the orbital period using a tangential impulsive manoeuvre) is $\omega/2\Delta x$ (where ω is the angular velocity of the reference circular orbit).

51. Show that the required ΔV to perform a closing manoeuvre (i.e., displacement of distance Δy in the V-bar direction in half the orbital period using a radial manoeuvre (along the R-bar)) is $\omega/2\Delta y$ (where ω is the angular velocity of the reference circular orbit). If the manoeuvre is performed along the V-bar, compute the required ΔV for the same given Δy.

Attitude Dynamics Problems

52. A rigid spacecraft V has center of mass G. We define a body-fixed reference frame oriented along the principal axes of inertia of the spacecraft, $S_V : \{G; B_V\}$. The principal moments of inertia with respect to G are $I_x = 4000$ kg m^2, $I_y = 7500$ kg m^2, and $I_z = 8500$ kg m^2. The initial angular velocity of the spacecraft with respect to an inertial reference frame S_0 is $\boldsymbol{\omega}_{V0}(0) = (0.1, -0.2, 0.5)^T$ rad/s expressed in the vector basis B_V, and the initial attitude is given by the Euler angles of S_V with respect to S_0: $\psi(0) = 0, \theta(0) = \pi/2, \varphi(0) = 0$. Simulate the subsequent rotation of the spacecraft.

    Proposed task: Compare the results with an axisymmetric spacecraft.
    Proposed task: Compare the results with the approximation of small disturbance.

53. Let us study the torque-free attitude dynamics of a solid cylindrical spacecraft of mass $M = 50$ kg, radius $R = 0.25$ m, length $L = 0.1$ m and center of mass G. We will work with a body-fixed reference frame $S_b : \{G, B_b\}$, being $B_b : \{\boldsymbol{i}_b, \boldsymbol{j}_b, \boldsymbol{k}_b\}$ with the $\boldsymbol{k}_b$ direction defined along the axis of symmetry. This reference frame coincide with principal axes of inertia, having $I_z = \frac{1}{2}MR^2$ and $I_x = I_y = \frac{1}{2}I_z + \frac{1}{12}ML^2$. The cylinder is initially in an equilibrium state, rotating with $\boldsymbol{\omega}_{b0} = 30\boldsymbol{k}_b$ deg/s with respect to an inertial reference frame S_0. If this is equilibrium is perturbed such that $\boldsymbol{\omega}_{b0} = 5\boldsymbol{j}_b + 30\boldsymbol{k}_b$ deg/s at $t = 0$, compute the evolution of precession (ψ), nutation (θ) and spin (ϕ) angles and rates ($\dot{\psi}$, $\dot{\theta}$, $\dot{\phi}$) using a Simulink model. To do so:

    (a) Compute the angular momentum vector $\boldsymbol{H}^b_{G0} = \mathcal{I}^b_G \cdot \boldsymbol{\omega}_{b0}$.

    (b) Compute, at $t = 0$, θ and ϕ such that $\boldsymbol{H}^b_{G0}$ points in the direction defined by $\boldsymbol{k}_0$, i.e. $\boldsymbol{H}^b_{G0} = H^b_{G0}\boldsymbol{k}_0$. Consider that, initially, $\psi = 0$.

    (c) Express the previous Euler angles in quaternion form.

    (d) Use the initial quaternion and $\boldsymbol{\omega}_{b0} = 5\boldsymbol{j}_b + 30\boldsymbol{k}_b$ deg/s as intial conditions for integrating the attitude dynamic and kinematic equations in Simulink.

    Try to use quaternions for the kinematics part, by computing the time evolution of the quaternion q giving the required rotation to move from S_0 to S_b. The angular velocity quaternion $\omega = (0, \boldsymbol{\omega}_{b0})$, with the components of $\boldsymbol{\omega}_{b0}$ projected on B_b, relates with $\dot{q}$ through $\dot{q} = \frac{1}{2}q\omega$.

    Compare your results with the analytical solution for an axisymmetric body. What would you expect to see, from the results of the linear stability analysis? What if you try with an asymmetric geometry with $I_y = 1.8I_x$? And with $I_y = 1.9I_x$?

54. For the spacecraft with the inertia tensor and initial condition given in the first exercise, consider the addition of a wheel R, initially at rest relative to the spacecraft. The center of mass of the rotor coincides with the center of mass of the spacecraft, G. We define the body-fixed reference frame $S_R : \{G; B_R\}$ attached to the wheel, which initially coincides with S_V. The wheel axis is aligned with the $\boldsymbol{k}_V$ direction of the spacecraft and has an inertia tensor with respect to G (not considered in the inertia tensor of the spacecraft) whose components in basis B_V are:

    $$\bar{\bar{I}}^R_G = \begin{bmatrix} 50 & 0 & 0 \\ 0 & 50 & 0 \\ 0 & 0 & 100 \end{bmatrix} \qquad \text{kg m}^2 \tag{0.2}$$

    A motor applies a torque on the wheel along $\boldsymbol{k}_V$, with the following time profile:

    $$\boldsymbol{M}_G = \begin{cases} -200\boldsymbol{k}_V & \text{N m} \quad 0 \leqslant t \leqslant 5 \text{ s} \\ \boldsymbol{0} \quad \text{N m} & 5 < t \text{ s} \end{cases} \tag{0.3}$$

    Simulate the response of the spacecraft for $0 \leqslant t \leqslant 40$ s.

55. For the spacecraft with the inertia tensor and initial condition given in the first exercise, consider the addition of a control moment gyro R, initially at rest relative to the spacecraft. The center of mass of the rotor coincides with the center of mass of the spacecraft, G. We define the body-fixed reference frame $S_R : \{G; B_R\}$ attached to the rotor. The rotor has an inertia tensor

with respect to G (not considered in the inertia tensor of the spacecraft) whose components in basis B_R are:

$$\bar{\bar{I}}_G^R = \begin{bmatrix} 50 & -10 & 0 \\ -10 & 100 & 15 \\ 0 & 15 & 250 \end{bmatrix} \qquad \text{kg m}^2 \tag{0.4}$$

Note that the tensor of inertia of the control moment gyro as seen from S_V changes with time as it moves. A three-axis motion of the rotor is initiated by the application of the following motor torque profile beginning at $t = 0$:

$$\boldsymbol{M}_G = \begin{cases} 7\boldsymbol{i}_R - 10\boldsymbol{j}_R - 200\boldsymbol{k}_R \quad \text{N m} \quad 0 \leqslant t \leqslant 5 \text{ s} \\ -7\boldsymbol{i}_R + 10\boldsymbol{j}_R \quad \text{N m} \quad 5 < t \leqslant 10 \text{ s} \\ \boldsymbol{0} \quad \text{N m} \quad 10 < t \text{ s} \end{cases} \tag{0.5}$$

Simulate the response of the spacecraft for $0 \leqslant t \leqslant 40$ s.

56. Consider an axisymmetric, spin-stabilized, rigid spacecraft with principal moments of inertia $I_x = I_y = 1500$ kg m^2 and $I_z = 500$ kg m^2 and spin rate $\omega_z = 1$ rad/s. A pair of attitude thrusters mounted normal to the spin axis produces a constant torque at each one-hundredth second firing. Simulate the bang-bang response to two thruster firings spaced half a precession period apart, in order to achieve the maximum spin-axis deflection.

Spacecraft subsystems

57. We want to size the solar arrays and batteries of a 5-year LEO mission. The selected orbit is a noon-midnight sun-synchronous orbit at 800 km altitude. The power consumption is constant and equal to 400 W throughout the duration of the mission. The solar arrays have a net efficiency of 25% at beginning of life (BOL), including all losses. The solar cells degrade 3% per year. The batteries have 60 W·h/kg and a cycle life given by $N = 2 \cdot 10^6/DOD(\%)$, where N is the number of cycles and DOD the depth of discharge (in %). Compute: (a) the orbital period and the length of the eclipse, (b) the number of orbits in the duration of the mission, (c) the required solar array area (d) the required battery mass

    NOTE: you can consider worst-case scenario eclipses throughout the mission. You do not need to consider any design margins or additional losses in this exercise. The solar irradiance at 1 AU is $S = 1366$ W/m^2.

58. A polar low Earth orbit mission at 800 km altitude takes a picture of 10 MB every minute. It must download all mission data to a ground station on the North pole that it visits every orbit using the S band at 3 GHz and ensuring a signal-to-noise ratio figure $E_b/N_0 > 10$. The receiving antenna on the ground has an effective aperture area of 5 m^2. The minimum elevation angle for communication is 5 deg over the horizon, and the effective noise temperature of the system is 400 K. You are asked to (a) determine the duration of the pass and the amount of bits to download on a pass; (b) compute the EIRP; (c) discuss reasonable values of the transmitter antenna diameter and telecommunications powers on the spacecraft.

    Note: ignore any atmospheric attenuation and line losses. The speed of light is $c = 3 \cdot 10^8$ m/s. Boltzmann constant is $k = 1.38 \cdot 10^{-23}$ m^{2}kg/(s^{2}K).

59. A Sputnik-like satellite can be modelled as a 1 m diameter isothermal aluminum sphere that dissipates 50 W internally. Determine its steady-state temperature in the sunlight and in eclipse.

    Note: Ignore any thermal loads other than internal dissipation and the Sun. Aluminum has $\varepsilon_{IR} = 0.08$ and $\alpha_{SUN} = 0.24$. Stefan-Boltzmann constant is $\sigma = 5.67 \cdot 10^{-8}$ W/(m^{2}K^4). The solar irradiance at 1 AU is $S = 1366$ W/m^2.