Mechanics Applied to Aerospace Engineering

Lecture Notes

Version 2018.1

Aerospace Engineering Department (UC3M)

Foreword

These lecture notes are the primary learning guide for the second year course *Mechanics Applied to Aerospace Engineering* (MAAE) of the Bachelor in Aerospace Engineering (BAE) at Universidad Carlos III de Madrid (UC3M). The first version of these notes was prepared by Manuel Villalba, who tough this course from 2011 to 2015. The notes underwent an important overhaul in 2016 and 2017. The professors of the Aerospace Engineering department Rauno Cavallaro, Xin Chen, Sara Correyero, Pablo Fajardo, Alejandro Gonzalo, Manuel Moriche, Antonio Sánchez, Álvaro Sáchez, and Manuel Sanjurjo are or have been involved at some point in this course and have also contributed to the development of these notes. Moreover, many students—too many to name here—have helped us improve these notes with their comments and observations.

The MAAE course strives to fulfill a critical goal in the education of our prospective aerospace engineers. First, to become proficient in Newtonian Mechanics, a core subject in any engineering studies upon which to build more advanced physical knowledge. We cannot stress enough the importance of Newtonian Mechanics: together with a few other fundamental subjects like calculus, linear algebra, programming, and fluiid mechanics, a profound applied knowledge of Newtonian Mechanics is a defining aspect of an excellent aerospace engineer.

Experience teaches us that this is a difficult course to assimilate by the student, as it requires a profound paradigm shift in the way that we visualize, model and analyze a physical problem. This needs time, and time is short in a semester filled with other demanding courses. As the time dedicated to this subject in the exigent BAE program is limited, the situation calls for a redoubled effort from both the professor side and from the student side to get the most and best out of this course.

We have noted that poor mathematical and physical foundations in calculus and linear algebra at the beginning of the course can make things worse. The positive side of this is that, if you detect that you have knowledge gaps in these areas, this *is* an problem upon which you can act: we are certain that all students can benefit from a deep review of their calculus and linear algebra notes of first year before and while they take MAAE. Do not hesitate to ask us for help if you find yourself in trouble with this at any point in the semester! We are here to help you learn as much as possible.

These lecture notes are not intended to substitute the literature that the student must regularly consult to consolidate their understanding of the course. Various sources have been used during the preparation of these lecture notes, in particular the following books:

- Engineering Dynamics. J. Ginsberg. Cambridge Univ. Press. 2008
- Classical Mechanics. T.W.B. Kibble & F.H. Berkshire. Imperial College Press. 2004
- Curso de Mecánica Racional. Vols I & II. Manuel Prieto. ADI Madrid. 1986
- Introducción a la Ingeniería aeroespacial. S. Franchini and O. López. IDR. 2008
- Introduction to Space Dynamics. W.T. Thomson. Dover. 1986

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The authors,
Leganés, August 2018

Meaning of mathematical symbols

In this course, all scalars are written in normal typeface, e.g. x, μ. Vectors, on the contrary, will always be presented in bold face: $\boldsymbol{v}$, $\boldsymbol{F}$. Tensors will be denoted with a double bar, like $\overline{\overline{\mathcal{I}}}_0$. The following standard nomenclature is used throughout the notes and problems:

$B_0 : \{\boldsymbol{i}, \boldsymbol{j}, \boldsymbol{k}\}$: A vector basis, defined by the ordered set of three vectors $\boldsymbol{i}, \boldsymbol{j}, \boldsymbol{k}$.

$S_0 : \{O; B_0\}$: A reference frame, with origin at point O and vector basis B_0. Using the natural Cartesian coordinates in S_0, and by abuse of notation, the reference frame is often also denoted by $Oxyz$, where Ox Oy, Oz are the coordinate axis in the directions of the three vectors of B_0, and Oxy, Oxz, Oyz are the coordinated planes.

$C : \{q_1, q_2, q_3\}$: A coordinate system.

$B_0 \otimes B_0$: A tensor basis, constructed by tensor-multiplying the vectors of a vector basis $B_0 : \{\boldsymbol{i}, \boldsymbol{j}, \boldsymbol{k}\}$, i.e., $B_0 \otimes B_0 = \{\boldsymbol{i}\otimes\boldsymbol{i}, \boldsymbol{i}\otimes\boldsymbol{j}, \boldsymbol{i}\otimes\boldsymbol{k}, \boldsymbol{j}\otimes\boldsymbol{i}, \boldsymbol{j}\otimes\boldsymbol{j}, \boldsymbol{j}\otimes\boldsymbol{k}, \boldsymbol{k}\otimes\boldsymbol{i}, \boldsymbol{k}\otimes\boldsymbol{j}, \boldsymbol{k}\otimes\boldsymbol{k}\}$.

$\dot{x}, \ddot{x}$: Dots over a scalar variable denote time differentiation: $\dot{x} = \mathrm{d}x/\mathrm{d}t$, $\ddot{x} = \mathrm{d}^2x/\mathrm{d}t^2$.

$\mathrm{d}\boldsymbol{A}/\mathrm{d}t|_0$: Time derivative of some vector $\boldsymbol{A}$, taken in the reference frame S_0

∇ : Nabla operator. When applied to a scalar function f, it returns its gradient vector, ∇f. Observe that the gradient of a function is the only vector in these notes that does not appear in bold face. It is also used to express the divergence and curl (or rotor) of a vector, $\nabla \cdot \boldsymbol{A}$ and $\nabla \times \boldsymbol{A}$. In Cartesian coordinates, $\nabla = \boldsymbol{i}(\partial/\partial x) + \boldsymbol{j}(\partial/\partial y) + \boldsymbol{k}(\partial/\partial z)$.

$\boldsymbol{r}_0^P$: Position vector of a point P as seen from reference frame S_0: $\boldsymbol{r}_0^P = \boldsymbol{OP}$. Dimensions: $[L]$.

$\boldsymbol{v}_0^P$: Velocity vector of a point P as seen from reference frame S_0: $\boldsymbol{v}_0^P = \mathrm{d}\boldsymbol{r}_0^P/\mathrm{d}t|_0$. Dimensions: $[L/T]$.

$\boldsymbol{a}_0^P$: Acceleration vector of a point P as seen from reference frame S_0: $\boldsymbol{v}_0^P = \mathrm{d}\boldsymbol{v}_0^P/\mathrm{d}t|_0$. Dimensions: $[L/T^2]$.

$\boldsymbol{\omega}_{10}$: Angular velocity vector of reference frame or rigid body 1 with respect to reference frame or rigid body 0. Dimensions: $[1/T]$.

$\boldsymbol{\alpha}_{10}$: Angular acceleration vector of reference frame or rigid body 1 with respect to reference frame or rigid body 0. Dimensions: $[1/T^2]$.

$\boldsymbol{p}_0$: Linear momentum as seen from reference frame S_0. Superindices like P or B are used to indicate the particle or body to which the linear momentum refers to. Dimensions: $[ML/T]$.

$\boldsymbol{H}_{A,0}$: Angular momentum of about a point A as seen from reference frame S_0. Superindices like P or B are used to indicate the particle or body to which the angular momentum refers to. Dimensions: $[ML^2/T]$.

$\boldsymbol{F}$: Force vector. Subindices are used to label different forces. If necessary, a superindex can be used to denote the point of application of the force. Dimensions: $[ML/T^2]$.

$\boldsymbol{M}_A$: Torque moment of a force about a point A. Additional subindices are used to label the moment of different forces. Dimensions: $[ML^2/T^2]$.

$W_0, \dot{W}_0$: Work and power exerted by a force or torque as seen from reference frame S_0. In the case of work, note that the initial and final states of the motion, and in general the trajectory followed between them, must be specified. Additional subindices are used to label the work and power of different forces or torques. Dimensions: $[ML^2/T^2]$, $[ML^2/T^3]$.

V_0 : Potential energy associated to a conservative force as seen from reference frame S_0. Subindices are also used to label different forces as required. Dimensions: $[ML^2/T^2]$.

T_0, E_0 : Kinetic and mechanial energy as seen from reference frame S_0. Superindices like P or B are used to indicate the particle or body to which the variable refers to. Dimensions: $[ML^2/T^2]$.

$\mathcal{D}(\boldsymbol{r}_0^P)$: Mass distribution of a material system. It can be discrete, i.e., composed of point particles, or continuous, i.e., given as a linear, area, or volume density.

$\mathcal{M}_{O,n}^{\mathcal{D}}$: n-th moment of a mass distribution $\mathcal{D}$ about a point O. This is a n-th order tensor, i.e., $\mathcal{M}_{O,0}^{\mathcal{D}}$ is a scalar, $\mathcal{M}_{O,1}^{\mathcal{D}}$ is a vector, and $\mathcal{M}_{O,2}^{\mathcal{D}}$ is a second-order tensor. Dimensions: $[ML^n]$.

m : Mass. Superindices like P or B are used to indicate the particle or body to which the mass refers to. Dimensions: $[M]$.

I_O, I_x, I_{xy} : Moment of inertia of a mass distribution with respect to a point O, a line x, and a plane xy, respectively. Superindices like P or B are used to indicate the particle or body to which the moment of inertia refers to. Dimensions: $[ML^2]$.

$P_{xy,xz}$: Product of inertia of a mass distribution with respect to the orthogonal planes xy and xz. Superindices like P or B are used to indicate the particle or body to which the product of inertia refers to. Dimensions: $[ML^2]$.

$\overline{\overline{\mathcal{I}}}_0$: Tensor of inertia of a mass distribution about the origin of the reference frame S_0. Superindices like P or B are used to indicate the particle or body to which the tensor of inertia refers to. Dimensions: $[ML^2]$.

$\overline{\overline{\mathcal{U}}}$: Unit diagonal tensor.

ρ_∞ : Air density far upstream from the aircraft. Dimensions: $[M/L^3]$.

$\boldsymbol{u}_\infty$: Air velocity far upstream from the aircraft, relative the aircraft. Dimensions: $[L/T]$.

$\boldsymbol{L}, \boldsymbol{D}$: Lift and drag forces. Lift is always perpendicular to $\boldsymbol{u}_\infty$; drag is always parallel to $\boldsymbol{u}_\infty$. Dimensions: $[ML/T^2]$.

c_L, c_D : Dimensionless lift and drag coefficients. Dimensions: $[-]$.

α : Angle of attack. Dimensions: $[-]$.

γ : Flight path angle. Dimensions: $[-]$.

When necessary to avoid ambiguity, the to state that a point A belongs to a rigid body or reference frame B (i.e., to indicate that the points moves jointly with the body or reference frame, so that the relative velocity of the point in a body-fixed reference frame is zero) will be indicated in parenthesis, e.g. $A(B)$.

For time-dependent variables, additional subindices can be used to denote the value of some quantity at a particular instant of time. For instance, $\boldsymbol{r}_{0,0}^P$ represents $\boldsymbol{r}_0^P(t_0)$. Note that a comma "," is used to separate from other subindices if they exist. Subindices are also used to denote the vector or tensor basis used when expressing a vector or a tensor in the form of a matrix of components. For example, a vector $\boldsymbol{A}$ can be expressed through its components in a vector basis $B_0 : \{\boldsymbol{i}, \boldsymbol{j}, \boldsymbol{k}\}$ as:

$$\boldsymbol{A} = x\boldsymbol{i} + y\boldsymbol{j} + z\boldsymbol{k} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}_0,$$

and tensor $\boldsymbol{A}\otimes\boldsymbol{A}$ can be expressed in $B_0\otimes B_0 = \{\boldsymbol{i}\otimes\boldsymbol{i}, \boldsymbol{i}\otimes\boldsymbol{j}, \boldsymbol{i}\otimes\boldsymbol{k}, \boldsymbol{j}\otimes\boldsymbol{i}, \boldsymbol{j}\otimes\boldsymbol{j}, \boldsymbol{j}\otimes\boldsymbol{k}, \boldsymbol{k}\otimes\boldsymbol{i}, \boldsymbol{k}\otimes\boldsymbol{j}, \boldsymbol{k}\otimes\boldsymbol{k}\}$ as:

$$\boldsymbol{A}\otimes\boldsymbol{A} = \begin{array}{r} xx\boldsymbol{i}\otimes\boldsymbol{i} + xy\boldsymbol{i}\otimes\boldsymbol{j} + xz\boldsymbol{i}\otimes\boldsymbol{k} \\ +yx\boldsymbol{j}\otimes\boldsymbol{i} + yy\boldsymbol{j}\otimes\boldsymbol{j} + yz\boldsymbol{j}\otimes\boldsymbol{k} \\ +zx\boldsymbol{k}\otimes\boldsymbol{i} + zy\boldsymbol{k}\otimes\boldsymbol{j} + zz\boldsymbol{k}\otimes\boldsymbol{k} \end{array} = \begin{bmatrix} xx & xy & xz \\ yx & yy & yz \\ zx & zy & zz \end{bmatrix}_{00}.$$

Contents

1 Getting started

In this course we will study Classical Mechanics and its introduce its application to aerospace systems. Classical Mechanics is governed by Newton's laws and is sometimes referred to as Newtonian Mechanics. Newtonian mechanics breaks down for systems moving at speeds comparable with the speed of light, where relativistic effects become important, and also fails for systems of dimensions comparable to the size of the atom, where quantum effects cannot be neglected. Nevertheless, for practical engineering applications, Newtonian mechanics provides a very good model to represent reality, and, in fact, it is hard to find examples in aerospace where Newtonian mechanics is not adequate. The most notable perhaps are the relativistic corrections that need to be made for modeling global navigation satellite systems (where both special *and* general relativity need to be taken into account).

In aerospace engineering we are particularly interested in the application of Newtonian mechanics to:

- The motion of aircraft: the prediction of position, velocity and acceleration under the action of forces and moments as well as aircraft stability
- The motion of spacecraft: orbits, attitude, satellite stability, launch dynamics and orbit transfers.
- The motion of systems or subsystems of these: engines, flaps, landing gear, etc.

Newtonian mechanics also constitutes the basis of many other disciplines, such as fluid dynamics and aerodynamics, and structural dynamics.

While we approach the study of the motion of aircraft and spacecraft, it is convenient to apply the methods of Newtonian Mechanics to the study of simpler mechanical devices outside of the aerospace field like ground vehicles, particles on a wire, and spring-mass systems. Not only does this illustrate and facilitate learning the methods of Newtonian Mechanics, but also quite often these simple academic contraptions, or variations of them, have a direct application in the modeling of the various aerospace engineering systems.

In this initial chapter we review some of the key concepts that we will need along our discovery of Newtonian Mechanics. The emphasis is on Linear Algebra and Vector Calculus. Most of these concepts ought to be already well-known to the student. Reviewing first-year course notes of Calculus, Linear Algebra, and Physics in detail before starting with the study of the present course is strongly advised.

1.1 Mechanical modeling

All design and analysis activities in aerospace engineering require developing a *model* of the reality that we want to study. This model allows the identification of the design parameters of the system, the computation of its performance figures, and the understanding of how each design parameter affects its operation. Models range from the most simple, which may only reproduce the basic behavior of the system, to the most complex, which try to represent reality as accurately as possible. In Newtonian mechanics, like in many other areas of engineering, mathematical models are built with the tools of linear algebra and calculus.

It is important to be permanently aware, however, that the model is *not* the reality. Our models are a simplified representation of an actual system, which regardless of their level of detail, inevitably ignore certain aspects of the reality of the different system parts and the environment where it operates. Noting this distinction is crucial, because it is impossible for our human minds to work directly with reality: we can only work with *models of reality*. Indeed, trying to directly "solve reality" without simplifying and modeling it first often leads to analysis paralysis. Only when we start to develop a *model* by clearly stating its simplifying *assumptions* can we begin to solve (a version of) the problem at hand.

Developing a mathematical model in Classical Mechanics requires having a very clear understanding of:

1. What is the **system** under analysis, what is its **environment**, and where are the **boundaries** between the two.

2. What are the interactions between out system and the environment.

3. What are the **assumptions** that we are taking at each step of the derivation of the model.

When first approaching a problem, it helps to think of these points and *write them down explicitly*, before trying to actually solve it.

In this course real objects will be idealized either as *point particles* or as *rigid bodies*. Often, this idealization is one of the first simplifying assumptions that we will take:

- A **point particle** is an object of negligible dimensions. When the dimensions of the objetc are unimportant to the description of its motion, and its orientation in space does not matter to us, we will idealize it as a particle.
- A **rigid body** is an object that has a finite size and cannot deform. This will be a useful approximation when the deformation of a body under the action of external forces is negligible compared to the overall motion. For instance, we may consider an aircraft as being a rigid body when analyzing the behavior of the aircraft along its flight path, even though under some specific conditions the deflection of the wing tips may be considerable. In describing the motion of a rigid body, we are concerned not only with its position but also with its orientation, or attitude.

An aircraft, for instance, may be modeled as a point particle for the purpose of studying its flight performance, or as one or more rigid bodies to analyze its response to a perturbation in its flight condition.

On the other hand, real objects have a finite size and are always deformable under loading. In some situations it will be required to consider the elastic or inelastic deformation of the body when considering its dynamic behavior, but those models are outside the scope of the present course. Following with the previous example, each element of an aircraft is somewhat elastic and experiences deflections (some of them large) during flight.

1.2 Scalar, vector and tensor quantities

In Mechanical models we will encounter three basic kinds of physical quantities:

1. **Scalars**. *Scalars* are physical properties that can be represented by a single number (plus its units). Mass, time and energy are examples of scalar quantities. Scalars do not have direction, they only represent a magnitude. The scalars that appear in this course are real numbers, i.e. they are elements of $\mathbb{R}$, with the exception of a few complex numbers ($\mathbb{C}$).

2. **Vectors**. The second type of physical quantity cannot be fully represented by a magnitude only, and further information (associated to a direction) is required. For instance, displacement of a particle in space requires a statement as to its starting point and its direction in addition to its numerical value. Such additional information is necessary to describe completely many physical quantities such as force, velocity and acceleration. Physical quantities possessing magnitude and direction which satisfy certain additional necessary requirements are called *vectors*. The vectors that appear in Mechanics are N-dimensional real vectors, i.e., they can be represented as lists of numbers in $\mathbb{R}^N$ (plus units), where $N = 2, 3$ depending whether we are solving a planar or three-dimensional problem[1]. In these lecture notes, a vector quantity are denoted in bold face, like $\boldsymbol{v}$, while its magnitude or norm is denoted in normal text face,[2] $v = ||\boldsymbol{v}||$.

   Based on the context and what physical magnitudes they represent, vector quantities can be further classified into free, sliding or bound vectors.

   (a) **Free vectors** are those which can be shifted around as long as their magnitude and direction are not altered, for the purpose of calculations. The application point, or starting point, of free vectors does not have any physical relevance. Examples of such free vectors are angular velocity and torque.

[1] Higher dimension vectors also find some use in certain formulations of Mechanics.

[2] In handwritten text, a small arrow is used to denote a vector. Remember that failing to distinguish between vectors and scalars (and the magnitude of a vector *is* a scalar) is considered a serious error in an exam!

(b) Vectors representing quantities whose effect remains unaltered when they are moved along their line of action are called **sliding vectors**. One example is the force acting on a rigid body as we will see in later chapters. Applying a force on a different line *does* matter.

(c) Vectors whose effects depend on the point of application are called **bound vectors**. Their magnitude, direction and point of application are fixed. Force acting on a deformable body produces different effects when applied at different points, and it is therefore considered a bound vector in deformable-body problems.

3. **Tensors**. The third kind of physical quantities are second-order *tensors*. In these lecture notes, a second-order tensor will be denoted by two bars over a letter, e.g. $\overline{\overline{\mathcal{I}}}$. The second-order tensors that appear in Mechanics are real $N \times N$-dimensional tensors, i.e., they can be represented as matrices in $\mathbb{R}^N \times \mathbb{R}^N$ (plus units), where $N = 2, 3$ for planar and spatial problems, respectively. Tensors are mathematical objects that describe linear relations between vectors, scalars, and other tensors. For example, if $\boldsymbol{A}$ is a N-vector and $\overline{\overline{\mathcal{I}}}$ a second-order $N \times N$-tensor, then $\boldsymbol{B} = \overline{\overline{\mathcal{I}}} \cdot \boldsymbol{A}$ is also a N-vector, and $p = \boldsymbol{A} \cdot \overline{\overline{\mathcal{I}}} \cdot \boldsymbol{A}$ is a scalar. The best example of a tensor in Mechanics is the tensor of inertia of a rigid body.

We can think of scalars, vectors, and tensors over the field of the real numbers $\mathbb{R}$ as mathematical objects of different *order*, or dimensionality:

- Scalars ($\mathbb{R}$) have no dimensionality, so they are objects of *order zero*.
- Vectors ($\mathbb{R}^N$) have dimensionality equal to one, so they are objects of *first order*.
- Second-order tensors ($\mathbb{R}^N \times \mathbb{R}^N$) have dimensionality equal to two, so as their name implies they are objects of *second order*[3].

Importantly, not every mathematical object in $\mathbb{R}$, $\mathbb{R}^N$ and $\mathbb{R}^N \times \mathbb{R}^N$ qualifies as a physical scalar, vector or tensor. This requires that the set of these elements be endowed with certain structure and properties. In particular, vectors must have the structure of a *vector space*. Moreover, in Mechanics we also require that they have certain *physical meaning*; for example, that they represent a velocity or the inertia properties of a rigid body. An important property of actual vectors and tensors is that their meaning and behavior remain unaltered under rotations and other simple transformations of the plane and space.

1.3 Vector spaces

For a set V to be called a vector space over the field of real numbers $\mathbb{R}$, there must exist two operations that satisfy certain properties: vector addition and multiplication by a scalar. The following must be true for any elements $\boldsymbol{A}, \boldsymbol{B} \in V$ and any scalars $\lambda, \mu \in \mathbb{R}$:

1. Vector addition must be *associative* (summands may be grouped in any order) and *commutative* (independent of the order of summation). There must exist an *identity element* of addition, which is called the null vector $\boldsymbol{0}$, and each vector $\boldsymbol{A}$ must have an *inverse* vector $-\boldsymbol{A}$ such that they result in the identity element when added together, $\boldsymbol{A} + (-\boldsymbol{A}) = \boldsymbol{0}$.

2. Multiplication of a vector by a scalar in $\mathbb{R}$ must be equipped with an *identity* or unity scalar, 1, such that $1\boldsymbol{A} = \boldsymbol{A}$. The *property* $\lambda(\mu\boldsymbol{A}) = (\lambda\mu)\boldsymbol{A}$ must be satisfied, and vector addition and multiplication by a scalar satisfy the *distributivity conditions* with each other: $(\lambda + \mu)\boldsymbol{A} = \lambda\boldsymbol{A} + \mu\boldsymbol{A}$ and $\lambda(\boldsymbol{A} + \boldsymbol{B}) = \lambda\boldsymbol{A} + \lambda\boldsymbol{B}$.

The most common vector spaces in Mechanics are $\mathbb{R}^2$ (the plane) and $\mathbb{R}^3$ (space) over $\mathbb{R}$, the field of the real numbers.[4] Observe that if $\boldsymbol{B} = \lambda\boldsymbol{A}$, the magnitude of $\boldsymbol{B}$ is $B = |\lambda|A$. The vector $\boldsymbol{B}$, is parallel to $\boldsymbol{A}$ and points in the same direction if $\lambda > 0$. For $\lambda < 0$, the vector $\boldsymbol{B}$ is parallel to $\boldsymbol{A}$ but points in the opposite direction (sometimes called antiparallel).

If we multiply an arbitrary vector $\boldsymbol{A}$ by the inverse of its magnitude, $1/A$, we obtain a **unit vector** which is parallel to $\boldsymbol{A}$: $\boldsymbol{e}_A = \boldsymbol{A}/A$.

[3] Indeed, scalars can be called "tensors of order zero", and vectors "tensors of first order". Naturally, objects of higher order (e.g. tensors of third order, which can be represented as a three-dimensional array) exist. However, these objects have little interest in Newtonian Mechanics.

[4] Incidentally, you can prove that the set of all scalars and the set of all $N \times N$ tensors also have the structure of vector spaces.

1.4 Vector bases and components of a vector

A vector basis is an ordered set of N linearly-independent vectors, where N is the dimensionality of the vector space (2 for the plane, 3 for space). With N independent vectors we can *span* the whole N-vector space by linear combination (also called superposition). Using a vector basis, we can express any vector by giving its *vector components* in that basis.

Consider the canonical right-handed vector basis B_0 of $\mathbb{R}^3$, composed by the mutually-orthogonal (i.e. perpendicular) unit vectors $\boldsymbol{i}$, $\boldsymbol{j}$ and $\boldsymbol{k}$:

$$B_0 : \{\boldsymbol{i}, \boldsymbol{j}, \boldsymbol{k}\}. \tag{1.1}$$

You can think of $\boldsymbol{i}$, $\boldsymbol{j}$ and $\boldsymbol{k}$ as the unit vectors along the coordinated axes Ox, Oy, Oz, as shown in Fig. 1.1. We say that the basis is right-handed when the order in which the vectors are specified follows the right-hand rule, i.e., if we extend the thumb, index, and middle fingers of the right hand to form 90 deg with each other, the orientation of $\boldsymbol{i}, \boldsymbol{j}, \boldsymbol{k}$ coincides with the thumb-index-middle finger orientation in that order. Observe that the vectors of a vector basis are always *free vectors*, in the sense that they can be translated anywhere without affecting their meaning.

A basis of this type is termed an *orthonormal right-handed vector basis*. Orthonormal right-handed bases are the only ones that we will consider in this course.

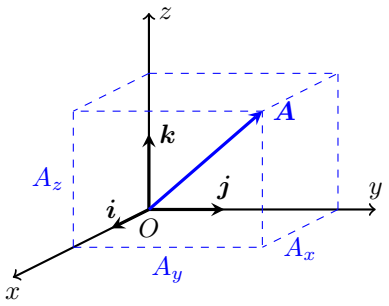

Figure 1.1: Illustration of an orthonormal right-handed vector basis $B_0 : \{\boldsymbol{i}, \boldsymbol{j}, \boldsymbol{k}\}$ and the components of a vector $\boldsymbol{A} = A_x\boldsymbol{i} + A_y\boldsymbol{j} + A_z\boldsymbol{k}$ in that basis.

A vector $\boldsymbol{A}$ can be written as a sum of the three vectors along the coordinate axes which have magnitudes A_x, A_y, and A_z and using matrix notation, as a column matrix containing the component magnitudes in that basis.

$$\boldsymbol{A} = \boldsymbol{A}_x + \boldsymbol{A}_y + \boldsymbol{A}_z = A_x\boldsymbol{i} + A_y\boldsymbol{j} + A_z\boldsymbol{k} = \begin{bmatrix} A_x \\ A_y \\ A_z \end{bmatrix}_0, \tag{1.2}$$

where the subindex 0 specifies that the components are given in basis B_0, and can be omitted only if it is evident from the context.

It is essential to remember that the components of a vector (in a given basis) are *not* the vector, but just a representation of it. The same vector will have different component representations in a different basis.

1.4.1 Components of a vector with a known angle

A recurring situation in Classical Mechanics is the need of computing the components of a vector $\boldsymbol{A}$ in a vector basis $B_0 : \{\boldsymbol{i}, \boldsymbol{j}, \boldsymbol{k}\}$, when $\boldsymbol{A}$ is contained in one of the coordinated planes, e.g. Oxy, and we know its magnitude A and the angle θ it forms with one of the coordinated axes, e.g. Ox, as depicted in Fig. 1.2. In this particular case, the solution is simply $\boldsymbol{A} = A_x\boldsymbol{i} + A_y\boldsymbol{j} + A_z\boldsymbol{k}$, where:

$$A_x = A\cos\theta, \tag{1.3}$$
$$A_y = A\sin\theta, \tag{1.4}$$
$$A_z = 0. \tag{1.5}$$

In many cases, we do not know the value of A and θ, but we can define them as unknowns and work with these variables until we solve the problem.

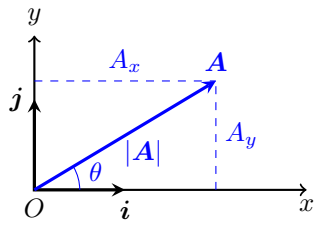

Figure 1.2: Definition of a vector in the plane Oxy in terms of its amplitude $A = |\boldsymbol{A}|$ and the angle it forms with the Ox axis, θ.

Oftentimes, the plane in which the vector is contained is not a coordinated plane. In these cases, it helps to define a new vector basis so that $\boldsymbol{A}$ is contained in one of the new coordinated planes. The new vector basis needs to be related to the former with a *basis rotation*, as explained in next sections. Usually, we will decompose this change of basis into a chain of *simple rotations* to simplify our work.

To avoid common mistakes while projecting a vector into the basis in this way, it is extremely helpful to draw more than one **sketch of the situation**: a general "view" of the projection problem, plus separate auxiliary diagrams where we draw only the pertinent coordinated plane.

1.5 Working with multiple vector bases

We often need to use different vector basis in order to describe different vector quantities as easily as possible. When we have more than one vector basis, a procedure for *transforming* the components of a vector in one basis to another becomes necessary.

Consider two different orthogonal, right-handed vector bases $B_0 : \{\boldsymbol{i}, \boldsymbol{j}, \boldsymbol{k}\}$ and $B_1 : \{\boldsymbol{i}_1, \boldsymbol{j}_1, \boldsymbol{k}_1\}$. We can say that one basis is *rotated* with respect to the other. Obviously, we can express any vector $\boldsymbol{A}$ in either of the two bases, but the physical vector itself remains the same entity:

$$\boldsymbol{A} = A_x\boldsymbol{i} + A_y\boldsymbol{j} + A_z\boldsymbol{k} = A_{x1}\boldsymbol{i}_1 + A_{y1}\boldsymbol{j}_1 + A_{z1}\boldsymbol{k}_1. \tag{1.6}$$

Knowing the components A_{x1}, A_{y1}, A_{z1} in the B_1 basis, we want to compute the components of $\boldsymbol{A}$ in the B_0 basis (i.e., the numbers A_x, A_y, A_z), and vice-versa.

To do so in a systematic way, we first express the vectors of the B_1 basis as components in the B_0 basis. For example, we can denote the components of $\boldsymbol{i}_1$ in the B_0 basis as:

$$\boldsymbol{i}_1 = \ell_{x1,x}\boldsymbol{i} + \ell_{x1,y}\boldsymbol{j} + \ell_{x1,z}\boldsymbol{k}. \tag{1.7}$$

Similarly, $\boldsymbol{j}_1$ and $\boldsymbol{k}_1$ can be expressed in B_0. The three expressions together are known as the *change of basis* from B_1 to B_0:

$$\begin{cases} \boldsymbol{i}_1 = \ell_{x1,x}\boldsymbol{i} + \ell_{x1,y}\boldsymbol{j} + \ell_{x1,z}\boldsymbol{k}, \\ \boldsymbol{j}_1 = \ell_{y1,x}\boldsymbol{i} + \ell_{y1,y}\boldsymbol{j} + \ell_{y1,z}\boldsymbol{k}, \\ \boldsymbol{k}_1 = \ell_{z1,x}\boldsymbol{i} + \ell_{z1,y}\boldsymbol{j} + \ell_{z1,z}\boldsymbol{k}. \end{cases} \tag{1.8}$$

This expression is extremely useful, since we can use it to *transform the components* of *any* vector $\boldsymbol{A}$ from basis B_1 to basis B_0. To do so, we first substitute $\boldsymbol{i}_1$, $\boldsymbol{j}_1$ and $\boldsymbol{k}_1$ from these expressions into the second equation of (1.6). After rearranging the terms that multiply $\boldsymbol{i}$, $\boldsymbol{j}$ and $\boldsymbol{k}$, we get three independent equations for A_x, A_y and A_z as a function of A_{x1}, A_{y1}, A_{z1}:

$$A_x = \ell_{x1,x}A_{x1} + \ell_{y1,x}A_{y1} + \ell_{z1,x}A_{z1}, \tag{1.9}$$
$$A_y = \ell_{x1,y}A_{x1} + \ell_{y1,y}A_{y1} + \ell_{z1,y}A_{z1}, \tag{1.10}$$
$$A_z = \ell_{x1,z}A_{x1} + \ell_{y1,z}A_{y1} + \ell_{z1,z}A_{z1}. \tag{1.11}$$

or, in matrix form:

$$\begin{bmatrix} A_x \\ A_y \\ A_z \end{bmatrix}_0 = \begin{bmatrix} \ell_{x1,x} & \ell_{y1,x} & \ell_{z1,x} \\ \ell_{x1,y} & \ell_{y1,y} & \ell_{z1,y} \\ \ell_{x1,z} & \ell_{y1,z} & \ell_{z1,z} \end{bmatrix} \cdot \begin{bmatrix} A_{x1} \\ A_{y1} \\ A_{z1} \end{bmatrix}_1 , \tag{1.12}$$

where the matrix $[{}_0R_1]$

$$[{}_0R_1] = \begin{bmatrix} \ell_{x1,x} & \ell_{y1,x} & \ell_{z1,x} \\ \ell_{x1,y} & \ell_{y1,y} & \ell_{z1,y} \\ \ell_{x1,z} & \ell_{y1,z} & \ell_{z1,z} \end{bmatrix} \tag{1.13}$$

is called the *rotation matrix*[5] from B_1 to B_0 (i.e., if we right-multiply by the component column matrix of a vector in B_1, we get the components in basis B_0 on the left hand side).

Several important properties of $[{}_0R_1]$ follow from the fact that B_0 and B_1 are both orthonormal vector bases, which makes matrix $[{}_0R_1]$ an *orthogonal matrix*. Here we enumerate some of these properties without demonstration. If we define $[{}_1R_0]$ as the rotation matrix converting components in B_0 to components in B_1 (i.e., the inverse operation above), it can be shown that:

$$[{}_1R_0] = [{}_0R_1]^{-1} = [{}_0R_1]^T , \tag{1.14}$$

and obviously then $[{}_0R_1]^T \cdot [{}_0R_1] = [\mathcal{U}]$, where $[\mathcal{U}]$ is the *identity matrix*. Also, it can be shown that:

$$|[{}_0R_1]| = 1. \tag{1.15}$$

Moreover, $[{}_0R_1]$ has one eigenvalue equal to 1. The other two eigenvalues are conjugated complex numbers of unit absolute value in general. Observe that the columns of $[{}_0R_1]$ are the components of $\boldsymbol{i}_1, \boldsymbol{j}_1, \boldsymbol{k}_1$ in B_0. This is an interesting way to view the meaning of the rotation matrix. From the properties above, we can also infer that the rows of $[{}_0R_1]$ are the components of $\boldsymbol{i}, \boldsymbol{j}, \boldsymbol{k}$ in B_1.

Sequences of rotations applied one after another are common in Mechanics. Imagine, for instance, a first rotation that goes from a basis B_2 to B_1, and a second rotation that goes from B_1 to B_0. It is possible to compute the matrix for the change of basis to go directly from B_2 to B_0 if we know the matrices of the individual rotations, by multiplying them right-to-left:

$$[{}_0R_2] = [{}_0R_1] \cdot [{}_1R_2] . \tag{1.16}$$

In general, for n sequential rotations, the resulting composite change of basis matrix is the right-to-left product of the matrices for each simple rotation,

$$[R] = [R_n] \cdots [R_2][R_1]. \tag{1.17}$$

Observe that rotation sequences like the ones above are not commutative, e.g. in general $[R_i][R_j] \neq [R_j][R_i]$. An important result due to Euler is that any rotation, or sequence of rotations, regardless of how complicated, is equivalent to **a single rotation about an axis**. That axis coincides with the eigenvector with eigenvalue 1 of the rotation matrix $[R]$ (i.e., the direction of space that remains the same after rotation).

To conclude, remember that whenever we work with more than one vector basis, it is essential to state clearly the relation between them, or the rotation that allows to change from one basis to the other. This can be done either by writing the expressions in (1.8) or by giving the rotation matrix $[{}_0R_1]$ in (1.13).

1.5.1 Simple rotations

When the axis of rotation to change from B_0 to B_1 coincides with one of the vectors of B_0, we call the rotation a *simple rotation*.

In a simple rotation, the unit vector of B_0 that acts as the axis of rotation remains stationary, while the rest of the vector space rotates an angle θ about it, positive according to the right hand rule. This rule states that, if one curls the fingers of the right hand in the sense of the rotation, the extended thumb of that hand points in the positive sense of the rotation axis. As a consequence of

[5] Note that the rotation matrix is *not* a tensor, and for several reasons. For example, the *addition* of rotation matrices is not defined (and does not make physical sense).

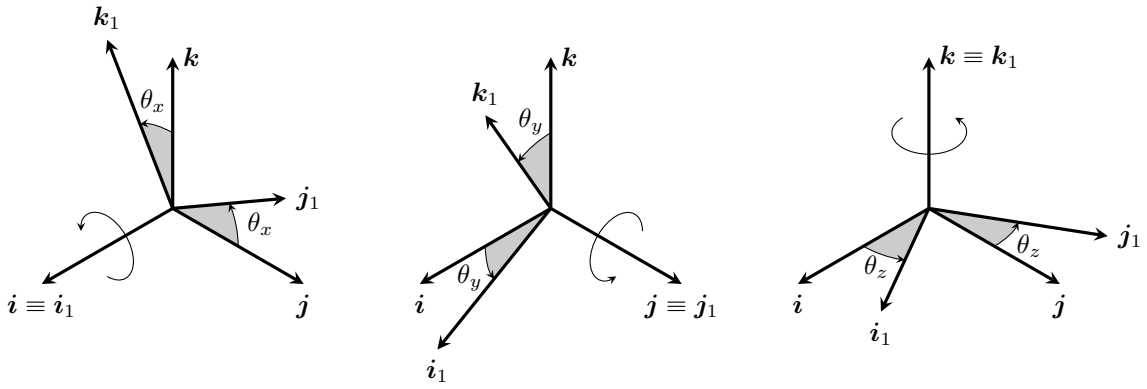

Figure 1.3: Simple rotations about each of the coordinate axes.

that unit vector of B_0 being stationary, the corresponding unit vector of B_1 coincides with it. For example, if $\boldsymbol{i}$ gives the axis and direction of rotation, then $\boldsymbol{i}_1 \equiv \boldsymbol{i}$.

The three possibilities, involving positive rotation about the $\boldsymbol{i}$, $\boldsymbol{j}$ and $\boldsymbol{k}$ are depicted in Fig. 1.3. For each case, the rotation matrices can be written directly from inspection of this figure, which leads to:

$$[{}_0R_1(x,\theta_x)] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta_x & -\sin\theta_x \\ 0 & \sin\theta_x & \cos\theta_x \end{bmatrix}, \tag{1.18}$$

$$[{}_0R_1(y,\theta_y)] = \begin{bmatrix} \cos\theta_y & 0 & \sin\theta_y \\ 0 & 1 & 0 \\ -\sin\theta_y & 0 & \cos\theta_y \end{bmatrix}, \tag{1.19}$$

$$[{}_0R_1(z,\theta_z)] = \begin{bmatrix} \cos\theta_z & -\sin\theta_z & 0 \\ \sin\theta_z & \cos\theta_z & 0 \\ 0 & 0 & 1 \end{bmatrix}. \tag{1.20}$$

Observe that the inverse matrices that transform vector components from B_0 to B_1 are equivalent to the rotations but with opposite angle sign, i.e., $\theta_x \to -\theta_x$, etc. As such, the only change in the previous expressions is the position of the minus sign before the sin operators. To avoid common sign errors while writing down the rotation matrices (or, for that matter, the expressions of Eq. (1.8)), it is recommended to **always draw a sketch** picturing the rotation by looking down from the stationary direction. For example, if the rotation is about the $\boldsymbol{i}$ direction, we would draw the Oyz plane and study the rotation there.

1.5.2 Sequences of simple rotations

Obviously, the situations that arise in Mechanics cannot be always studied with a mere simple rotation. Let us call B_0 the canonical, initial basis of our space, and B_3 a new basis where we want to work to simplify our computations. In general, none of the vectors in B_3 and B_0 coincide, and moreover, their relative orientation may be a function of time.

Notwithstanding this, it is always possible to decompose a general rotation as the product of three sequential simple rotations. The choice of which simple rotations and their order is not unique, and therefore we have some freedom as how to define it. However, once defined, the value of the angle of each rotation is unique (except in some pathological singular situations).

To illustrate the procedure of working with a sequence of concatenated simple rotations, consider that we can reach B_0 from B_3 by performing first a simple z-rotation, then a y-rotation, and finally a x-rotation. We begin by defining an *intermediate vector basis* B_2 which is a simple z-rotation of certain angle θ_z away from B_3, along the $\boldsymbol{k}_2 \equiv \boldsymbol{k}_3$ vector. We can express the components of any

vector $\boldsymbol{A}$ in B_2 as a function of its components in B_3 through the corresponding rotation matrix,

$$\begin{bmatrix} A_{x_2} \\ A_{y_2} \\ A_{z_2} \end{bmatrix} = [{}_2R_3(z,\theta_z)] \cdot \begin{bmatrix} A_{x_3} \\ A_{y_3} \\ A_{z_3} \end{bmatrix}. \tag{1.21}$$

Then, we define another intermediate basis B_1, which is a y-rotation away from B_2. Finally, if we did our choice of simple rotations correctly, B_0 and B_1 will also be a mere simple rotation away from each other. In this example, it is a x-rotation:

$$\begin{bmatrix} A_{x_1} \\ A_{y_1} \\ A_{z_1} \end{bmatrix} = [{}_1R_2(y,\theta_y)] \cdot \begin{bmatrix} A_{x_2} \\ A_{y_2} \\ A_{z_2} \end{bmatrix}; \quad \begin{bmatrix} A_{x_0} \\ A_{y_0} \\ A_{z_0} \end{bmatrix} = [{}_0R_1(x,\theta_x)] \cdot \begin{bmatrix} A_{x_1} \\ A_{y_1} \\ A_{z_1} \end{bmatrix}. \tag{1.22}$$

Plugging each expression into the former we can find a direct link between the components of $\boldsymbol{A}$ in B_3 and in B_0,

$$\begin{bmatrix} A_{x_0} \\ A_{y_0} \\ A_{z_0} \end{bmatrix} = [{}_0R_1(x,\theta_x)] \cdot [{}_1R_2(y,\theta_y)] \cdot [{}_2R_3(z,\theta_z)] \cdot \begin{bmatrix} A_{x_3} \\ A_{y_3} \\ A_{z_3} \end{bmatrix} = [{}_0R_3] \cdot \begin{bmatrix} A_{x_3} \\ A_{y_3} \\ A_{z_3} \end{bmatrix}, \tag{1.23}$$

where $[{}_0R_3] = [{}_0R_1(x,\theta_x)] \cdot [{}_1R_2(y,\theta_y)] \cdot [{}_2R_3(z,\theta_z)]$ is naturally the rotation matrix of the basis change from B_3 to B_0.

As we shall see in Chapter 9, the angles $\theta_x, \theta_y, \theta_z$ of the sequential rotations here are termed *Euler angles*. As the choice of what simple rotations to use and their order is not unique, there are multiple definitions of Euler angles.

1.6 Vector properties and operations

Besides addition and multiplication by a scalar, it is useful to define an inner product (also known as dot product or scalar product) between vectors of $\mathbb{R}^N$, which endows the vector space with an Euclidean structure (i.e., it allows to define and measure distances, norms, and angles). Similarly, it is often useful to define the cross product (also known as vector product) and the triple product (or mixed product). Finally, we also introduce here the concept of tensor product (or outer product).

1.6.1 Scalar or "dot" product

This product takes two vectors and results in a scalar quantity. The scalar product between two vectors $\boldsymbol{A}$ and $\boldsymbol{B}$ returns a scalar, is denoted by $\boldsymbol{A} \cdot \boldsymbol{B}$ and is defined as

$$p = \boldsymbol{A} \cdot \boldsymbol{B} = AB\cos\theta, \tag{1.24}$$

where θ is the *angle* between the vectors $\boldsymbol{A}$ and $\boldsymbol{B}$ when they are drawn with a common origin. The scalar or dot product is also termed *inner product*. The dot product is *commutative*, *linear* and *positive-definite*. In other words, for all $\boldsymbol{A}, \boldsymbol{B}, \boldsymbol{C} \in V$ and all scalars $\lambda, \mu \in \mathbb{R}$:

$$\boldsymbol{A} \cdot \boldsymbol{B} = \boldsymbol{B} \cdot \boldsymbol{A}, \tag{1.25}$$

$$(\lambda\boldsymbol{A} + \mu\boldsymbol{B}) \cdot \boldsymbol{C} = \lambda\boldsymbol{A} \cdot \boldsymbol{C} + \mu\boldsymbol{B} \cdot \boldsymbol{C}, \tag{1.26}$$

$$\boldsymbol{A} \cdot \boldsymbol{A} \geqslant 0; \quad \boldsymbol{A} \cdot \boldsymbol{A} = 0 \Leftrightarrow \boldsymbol{A} = \boldsymbol{0}. \tag{1.27}$$

As soon as we have defined a dot product, we say that our vector space is *Euclidean*. We can also define a *norm* based on this scalar product, $||\boldsymbol{v}|| = \sqrt{\boldsymbol{v} \cdot \boldsymbol{v}} = v$, so it is also a *normed* vector space.

We note that, since $\cos\theta = \cos(-\theta)$, it makes no difference which vector is considered first when measuring the angle θ (consistent with the commutative property). If $\boldsymbol{A} \cdot \boldsymbol{B} = 0$, then either $\boldsymbol{A} = \boldsymbol{0}$ and/or $\boldsymbol{B} = \boldsymbol{0}$, and/or, they are *orthogonal* (i.e., perpendicular), that is, $\cos\theta = 0$. If one of the vectors is a unit vector, say $B = 1$, then $\boldsymbol{A} \cdot \boldsymbol{B} = A\cos\theta$, is the *projection* of vector $\boldsymbol{A}$ along the direction of $\boldsymbol{B}$.

The projections of a vector $\boldsymbol{A}$ along the $\boldsymbol{i}$, $\boldsymbol{j}$ and $\boldsymbol{k}$ vectors of a vector basis B_0 are the components[6] of $\boldsymbol{A}$ in the that basis, i.e., $\boldsymbol{A} = A_x\boldsymbol{i} + A_y\boldsymbol{j} + A_z\boldsymbol{k}$. Indeed, in order to determine the components of $\boldsymbol{A}$, we can use the scalar product and write:

$$A_x = \boldsymbol{A}\cdot\boldsymbol{i}, \quad A_y = \boldsymbol{A}\cdot\boldsymbol{j}, \quad A_z = \boldsymbol{A}\cdot\boldsymbol{k}. \tag{1.28}$$

When projecting a unit vector, e.g. $\boldsymbol{e}$ onto the three unit vectors of B_0, the resulting scalar products are called the *director cosines* of $\boldsymbol{e}$ in that basis,

$$\boldsymbol{e} = (\boldsymbol{e}\cdot\boldsymbol{i})\boldsymbol{i} + (\boldsymbol{e}\cdot\boldsymbol{j})\boldsymbol{j} + (\boldsymbol{e}\cdot\boldsymbol{k})\boldsymbol{k} = \cos\alpha\boldsymbol{i} + \cos\beta\boldsymbol{j} + \cos\gamma\boldsymbol{k}. \tag{1.29}$$

Examination of Fig. 1.4 reveals that the values of the director angles α, β, γ should be limited to the range $0 \leqslant \alpha, \beta, \gamma \leqslant \pi$ to avoid ambiguity. If instead of $\boldsymbol{e}$ we input the unit vectors of a new basis B_1, i.e., $\boldsymbol{i}_1, \boldsymbol{j}_1, \boldsymbol{k}_1$, it is easy to prove that the corresponding director cosines are the components of the rotation matrix $[{}_0R_1]$.

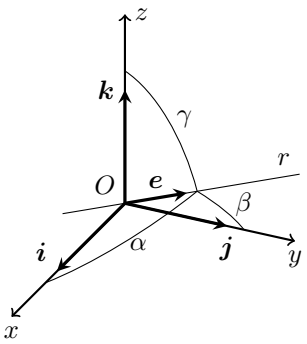

Figure 1.4: Direction angles α, β, and γ for a line r whose direction is given by the unit vector $\boldsymbol{e}$.

Finally, observe that the dot "$\cdot$" is also used to indicate the *contraction product* between a vector and a tensor, i.e. $\boldsymbol{A}\cdot\overline{\overline{M}}$. Note that the result of $\boldsymbol{A}\cdot\overline{\overline{M}}$ is a vector, and that the operation is not commutative: $\boldsymbol{A}\cdot\overline{\overline{M}} \neq \overline{\overline{M}}\cdot\boldsymbol{A}$; in fact, these products to make sense only if the dimensions of the vector and tensor that we are multiplying (i.e., contracting) match each other.

1.6.2 Vector or "cross" product

If $V = \mathbb{R}^3$, as in the case of most interest in Classical Mechanics, we define the vector or cross product as an operation that involves two vectors $\boldsymbol{A}$ and $\boldsymbol{B}$, and results in a new vector $\boldsymbol{C} = \boldsymbol{A}\times\boldsymbol{B}$. The *magnitude* of $\boldsymbol{C}$ is given by:

$$C = AB|\sin\theta|, \tag{1.30}$$

where θ is the angle between the vectors $\boldsymbol{A}$ and $\boldsymbol{B}$ when drawn with a common origin. We can easily show that C is equal to the area enclosed by the parallelogram defined by $\boldsymbol{A}$ and $\boldsymbol{B}$. The vector $\boldsymbol{C}$ is orthogonal to both $\boldsymbol{A}$ and $\boldsymbol{B}$, i.e. it is normal to the plane defined by $\boldsymbol{A}$ and $\boldsymbol{B}$. The direction of $\boldsymbol{C}$ is determined by the right-hand rule as shown in Fig. 1.5.

The vector product is *anticommutative*, as follows from this definition, and *linear*. In other words, for all $\boldsymbol{A}, \boldsymbol{B}, \boldsymbol{C} \in V$ and all scalars $\lambda, \mu \in \mathbb{R}$:

$$\boldsymbol{A}\times\boldsymbol{B} = -\boldsymbol{B}\times\boldsymbol{A}, \tag{1.31}$$

$$(\lambda\boldsymbol{A} + \mu\boldsymbol{B})\times\boldsymbol{C} = \lambda\boldsymbol{A}\times\boldsymbol{C} + \mu\boldsymbol{B}\times\boldsymbol{C}. \tag{1.32}$$

Observe that the vector product is non-associative (i.e., $\boldsymbol{A}\times(\boldsymbol{B}\times\boldsymbol{C}) \neq (\boldsymbol{A}\times\boldsymbol{B})\times\boldsymbol{C}$). We also note that if $\boldsymbol{A}\times\boldsymbol{B} = \boldsymbol{0}$, then, either $\boldsymbol{A}$ and/or $\boldsymbol{B}$ are zero, or, $\boldsymbol{A}$ and $\boldsymbol{B}$ are *collinear*, i.e. parallel or antiparallel. Thus, we also have the identity $\boldsymbol{A}\times\boldsymbol{A} = \boldsymbol{0}$.

[6] More precisely, the *covariant* components, to distinguish them from the *contravariant* components. If the basis is orthonormal, the two types of components coincide, so there is no need of making such distinction. The interested student is invited to learn more about these concepts, which are outside of the present course.

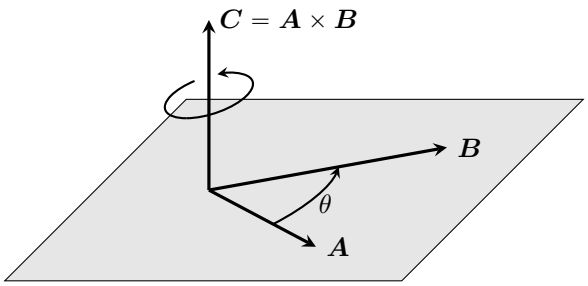

Figure 1.5: Right hand rule for the definition of the cross product.

1.6.3 Perpendicular and parallel vectors

When working with vectors, sometimes one needs to impose the condition that two vectors be perpendicular or collinear to each other. As seen above, this can be done very easily using the dot and cross products:

- In order for two vectors to be perpendicular, it must be $\boldsymbol{A} \cdot \boldsymbol{B} = 0$
- In order for two vectors to be collinear (parallel/antiparallel), it must be $\boldsymbol{A} \times \boldsymbol{B} = \boldsymbol{0}$.

1.6.4 Triple product

Given three vectors $\boldsymbol{A}$, $\boldsymbol{B}$, and $\boldsymbol{C}$, the triple product is a scalar given by $[\boldsymbol{A}, \boldsymbol{B}, \boldsymbol{C}] = \boldsymbol{A} \cdot (\boldsymbol{B} \times \boldsymbol{C})$. Geometrically, the triple product can be interpreted as the volume of the three dimensional parallelepiped defined by the three vectors $\boldsymbol{A}$, $\boldsymbol{B}$, and $\boldsymbol{C}$ as shown in Fig. 1.6. It can be easily verified that

$$[\boldsymbol{A}, \boldsymbol{B}, \boldsymbol{C}] = \boldsymbol{A} \cdot (\boldsymbol{B} \times \boldsymbol{C}) = \boldsymbol{C} \cdot (\boldsymbol{A} \times \boldsymbol{B}) = \boldsymbol{B} \cdot (\boldsymbol{C} \times \boldsymbol{A}). \tag{1.33}$$

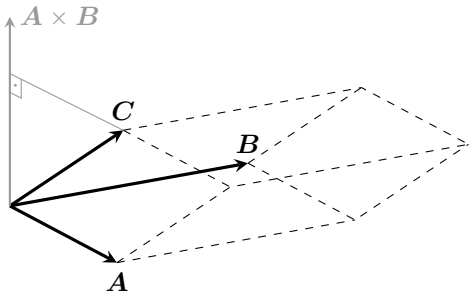

Figure 1.6: Definition of the triple product of three vectors, $[\boldsymbol{A}, \boldsymbol{B}, \boldsymbol{C}]$. The triple product is the volume of the dashed box when $\boldsymbol{A}$, $\boldsymbol{B}$, $\boldsymbol{C}$ are given in a right-hand-rule order, and minus the volume otherwise. The triple product is equal to the dot product $\boldsymbol{C} \cdot (\boldsymbol{A} \times \boldsymbol{B})$.

1.6.5 Tensor product

The tensor product (or outer or dyadic product) between two vectors $\boldsymbol{A}$ and $\boldsymbol{B}$ is an operation that returns a second-order tensor, or dyadic, $\overline{\overline{\mathcal{M}}} = \boldsymbol{A} \otimes \boldsymbol{B}$.

The tensor product is *bilinear*. In other words, for all $\boldsymbol{A}, \boldsymbol{B}, \boldsymbol{C} \in V$ and all scalars $\lambda, \mu \in \mathbb{R}$:

$$(\lambda \boldsymbol{A} + \mu \boldsymbol{B}) \otimes \boldsymbol{C} = \lambda \boldsymbol{A} \otimes \boldsymbol{C} + \mu \boldsymbol{B} \otimes \boldsymbol{C}, \tag{1.34}$$

$$\boldsymbol{A} \otimes (\lambda \boldsymbol{B} + \mu \boldsymbol{C}) = \lambda \boldsymbol{A} \otimes \boldsymbol{B} + \mu \boldsymbol{A} \otimes \boldsymbol{C}. \tag{1.35}$$

Note that the tensor or outer product is not commutative ($\boldsymbol{A} \otimes \boldsymbol{B} \neq \boldsymbol{B} \otimes \boldsymbol{A}$). However, it presents an interesting associative property when dotted with a vector:

$$\boldsymbol{A} \cdot (\boldsymbol{B} \otimes \boldsymbol{C}) = (\boldsymbol{A} \cdot \boldsymbol{B})\boldsymbol{C}; \quad (\boldsymbol{A} \otimes \boldsymbol{B}) \cdot \boldsymbol{C} = \boldsymbol{A}(\boldsymbol{B} \cdot C). \tag{1.36}$$

The *rank* of the resulting tensor $\boldsymbol{A}\otimes\boldsymbol{B}$ is equal to two if $\boldsymbol{A}$ and $\boldsymbol{B}$ are non-zero and not collinear; equal to one if $\boldsymbol{A}$ and $\boldsymbol{B}$ are non-zero and collinear; and equal to zero if $\boldsymbol{A}=\boldsymbol{0}$ and/or $\boldsymbol{B}=\boldsymbol{0}$.

Observe that while the inner product returns an object of one order lower than the inputs (i.e., it takes two vectors and returns a scalar), the outer product returns an object of one order higher than the inputs (i.e., it takes two vectors and returns a second-order tensor).

1.6.6 Useful vector relations

The double vector product results from repetition of the cross product operation. The student can check that:

$$\boldsymbol{A}\times(\boldsymbol{B}\times\boldsymbol{C})=(\boldsymbol{A}\cdot\boldsymbol{C})\boldsymbol{B}-(\boldsymbol{A}\cdot\boldsymbol{B})\boldsymbol{C}. \tag{1.37}$$

Another useful relation is:

$$(\boldsymbol{A}\times\boldsymbol{B})\cdot(\boldsymbol{C}\times\boldsymbol{D})=(\boldsymbol{A}\cdot\boldsymbol{C})(\boldsymbol{B}\cdot\boldsymbol{D})-(\boldsymbol{A}\cdot\boldsymbol{D})(\boldsymbol{B}\cdot\boldsymbol{C}). \tag{1.38}$$

1.6.7 Vector and tensor operations in component form

We have seen above that it is possible to define several operations involving vectors and tensors without ever introducing a vector basis or vector components. This is a rather important concept which explains why vectors, tensors, and vector equations are so useful to express physical entities and laws, since these **must be obviously independent of the particular vector basis used to study them**. In practice however, bases need to be introduced at some point in order to perform computations, or measure the direction and magnitude of vectors. For example, we can easily measure the direction of the velocity vector of an airplane by defining a vector basis and measuring the angle that the vector makes with the local vertical and the geographic north. Similarly, when operating with tensors, a tensor basis is defined and we work with the tensor components in that basis.

The vector operations introduced above can be expressed in terms of the vector components in a given basis in a rather straightforward manner. For instance, when we say that $\boldsymbol{A}=\boldsymbol{B}$, this implies that the projections of $\boldsymbol{A}$ and $\boldsymbol{B}$ along the axes xyz are the same, and therefore, this is equivalent to three scalar equations e.g. $A_x=B_x$, $A_y=B_y$, and $A_z=B_z$. Regarding vector addition, subtraction and multiplication by a scalar, we have that, if $\boldsymbol{C}=\lambda\boldsymbol{A}+\mu\boldsymbol{B}$, then,

$$C_x=\lambda A_x+\mu B_x,\quad C_y=\lambda A_y+\mu B_y,\quad C_z=\lambda A_z+\mu B_z. \tag{1.39}$$

Scalar product: Since $\boldsymbol{i}\cdot\boldsymbol{i}=\boldsymbol{j}\cdot\boldsymbol{j}=\boldsymbol{k}\cdot\boldsymbol{k}=1$ and $\boldsymbol{i}\cdot\boldsymbol{j}=\boldsymbol{j}\cdot\boldsymbol{k}=\boldsymbol{k}\cdot\boldsymbol{i}=0$, the scalar product of two vectors can be written as,

$$\boldsymbol{A}\cdot\boldsymbol{B}=(A_x\boldsymbol{i}+A_y\boldsymbol{j}+A_z\boldsymbol{k})\cdot(B_x\boldsymbol{i}+B_y\boldsymbol{j}+B_z\boldsymbol{k})=A_xB_x+A_yB_y+A_zB_z, \tag{1.40}$$

or, using *matrix notation*,

$$\boldsymbol{A}\cdot\boldsymbol{B}=\begin{bmatrix} A_x & A_y & A_z \end{bmatrix}_0\cdot\begin{bmatrix} B_x \\ B_y \\ B_z \end{bmatrix}_0. \tag{1.41}$$

Observe that in a matrix context, the notation $\boldsymbol{A}\cdot\boldsymbol{B}$ used here is a shorthand of $A^T\cdot B$, where A^T is the row matrix with the components of $\boldsymbol{A}$ and B the column matrix with the components of $\boldsymbol{B}$. In a vector context, the transpose symbol "T" can be dropped as the meaning of the dot product is unambiguous.

Vector product: Now, $\boldsymbol{i}\times\boldsymbol{i}=\boldsymbol{j}\times\boldsymbol{j}=\boldsymbol{k}\times\boldsymbol{k}=0$ and $\boldsymbol{i}\times\boldsymbol{j}=\boldsymbol{k}$, $\boldsymbol{j}\times\boldsymbol{k}=\boldsymbol{i}$ and $\boldsymbol{k}\times\boldsymbol{i}=\boldsymbol{j}$. Thus, we can express the tensor product as a determinant:

$$\begin{aligned}\boldsymbol{A}\times\boldsymbol{B}&=(A_x\boldsymbol{i}+A_y\boldsymbol{j}+A_z\boldsymbol{k})\times(B_x\boldsymbol{i}+B_y\boldsymbol{j}+B_z\boldsymbol{k})\\&=(A_yB_z-A_zB_y)\boldsymbol{i}+(A_zB_x-A_xB_z)\boldsymbol{j}+(A_xB_y-A_yB_x)\boldsymbol{k}=\begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix}\end{aligned} \tag{1.42}$$

Triple product: The triple product can be expressed as the following determinant

$$\boldsymbol{A}\cdot(\boldsymbol{B}\times\boldsymbol{C}) = \begin{vmatrix} A_x & A_y & A_z \\ B_x & B_y & B_z \\ C_x & C_y & C_z \end{vmatrix}, \tag{1.43}$$

which is clearly equal to zero whenever the vectors are not fully linearly independent (if two of the three vectors are linearly dependent, they must be co-planar, and therefore the parallelepiped defined by the three vectors has zero volume).

Tensor product: Using vector components in a basis B_0, the tensor product between $\boldsymbol{A}$ and $\boldsymbol{B}$ can be expressed as:

$$\begin{aligned} \boldsymbol{A}\otimes\boldsymbol{B} &= (A_x\boldsymbol{i}+A_y\boldsymbol{j}+A_z\boldsymbol{k})\otimes(B_x\boldsymbol{i}+B_y\boldsymbol{j}+b_z\boldsymbol{k}) && (1.44)\\ &= A_xB_x\boldsymbol{i}\otimes\boldsymbol{i}+A_xB_y\boldsymbol{i}\otimes\boldsymbol{j}+A_xB_z\boldsymbol{i}\otimes\boldsymbol{k} && (1.45)\\ &+ A_yB_x\boldsymbol{j}\otimes\boldsymbol{i}+A_yB_y\boldsymbol{j}\otimes\boldsymbol{j}+A_yB_z\boldsymbol{j}\otimes\boldsymbol{k} && (1.46)\\ &+ A_zB_x\boldsymbol{k}\otimes\boldsymbol{i}+A_zB_y\boldsymbol{k}\otimes\boldsymbol{j}+A_zB_z\boldsymbol{k}\otimes\boldsymbol{k}, && (1.47) \end{aligned}$$

where $B_0\otimes B_0 = \{\boldsymbol{i}\otimes\boldsymbol{i}, \boldsymbol{i}\otimes\boldsymbol{j}, \boldsymbol{i}\otimes\boldsymbol{k}, \boldsymbol{j}\otimes\boldsymbol{i}, \boldsymbol{j}\otimes\boldsymbol{j}\ldots\}$ is the *tensor basis* constructed by tensor-multiplying the vectors of the vector basis B_0. Making use of the usual *matrix notation* to express the components of the resulting tensor,

$$\boldsymbol{A}\otimes\boldsymbol{B} = \begin{bmatrix} A_xB_x & A_xB_y & A_xB_z \\ A_yB_x & A_yB_y & A_yB_z \\ A_zB_x & A_zB_y & A_zB_z \end{bmatrix}_{00}. \tag{1.48}$$

As with vector components in matrix form, the subindices 00 indicate in which basis the components of $\boldsymbol{A}\otimes\boldsymbol{B}$ are expressed, and can be omitted if there is no ambiguity as to the bases used.

To conclude, remember that vector (and tensor) operations can only be executed using components if **all the participating vectors are first expressed in the same vector basis**. Likewise, to compute vector-tensor and tensor-tensor operations using components, all vectors *and* tensors must be expressed using consistent bases.

1.6.8 Moment of a vector

Consider a sliding or bound vector $\boldsymbol{A}$ and a point O in space. If we draw a vector $\boldsymbol{OP}$ from O to any point P on the line of action of $\boldsymbol{A}$ as shown in Fig. 1.7, and compute the cross product $\boldsymbol{OP}\times\boldsymbol{A}$, then the result will be the *moment vector* $\boldsymbol{M}$ of $\boldsymbol{A}$ about O,

$$\boldsymbol{M} = \boldsymbol{OP}\times\boldsymbol{A}. \tag{1.49}$$

Note that $\boldsymbol{M}$ always goes in a direction perpendicular to the plane containing $\boldsymbol{OP}$ and $\boldsymbol{A}$.

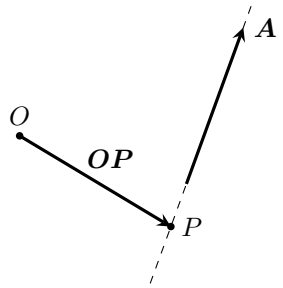

Figure 1.7: Computation of moment of the sliding or bound vector $\boldsymbol{A}$ about point O.

Observe that the moment $\boldsymbol{M}$ is independent of the point P on the line of action of $\boldsymbol{A}$ that we choose for the computation. To show this, we decompose $\boldsymbol{OP}$ into two components, parallel and perpendicular to the line of action of $\boldsymbol{A}$: $\boldsymbol{OP} = \boldsymbol{OP}_{\parallel} + \boldsymbol{OP}_{\perp}$. Then,

$$\boldsymbol{M} = \boldsymbol{OP}\times\boldsymbol{A} = \cancel{\boldsymbol{OP}_{\parallel}\times\boldsymbol{A}} + \boldsymbol{OP}_{\perp}\times\boldsymbol{A} = \ell A\boldsymbol{e}_M, \tag{1.50}$$

where $\ell = |\boldsymbol{OP}_\perp|$ is the distance from O to the line of action of $\boldsymbol{A}$, A is the module (i.e. magnitude) of $\boldsymbol{A}$, and $\boldsymbol{e}_M$ is the unit vector in the direction of $\boldsymbol{M}$.

1.7 Vector differential calculus

The differential change of a vector *is* a vector, meaning that it also has magnitude *and* direction. As usual, we denote the differential change of a vector $\boldsymbol{A}$ as $\mathrm{d}\boldsymbol{A}$. Thus, if a vector is a function of time t, then its derivative with respect to time is also a vector. Analogously, the differential and the time derivative of a tensor are tensors.

1.7.1 Time derivative of a vector

Consider a vector $\boldsymbol{A}(t)$ which is a function of time, and that there is only one observer. The derivative of $\boldsymbol{A}$ with respect to time is defined as:

$$\frac{\mathrm{d}\boldsymbol{A}}{\mathrm{d}t} = \lim_{\Delta t \to 0} \frac{\boldsymbol{A}(t+\Delta t) - \boldsymbol{A}(t)}{\Delta t}. \tag{1.51}$$

A vector has magnitude and direction, and it changes whenever either of them, or both of them, do. The rate of change of a vector will be equal to the sum of the changes due to magnitude and direction.

Rate of change due to magnitude changes: When a vector only changes in magnitude from $\boldsymbol{A}$ to $\boldsymbol{A} + \mathrm{d}\boldsymbol{A}$ during a differential time $\mathrm{d}t$, the rate of change vector $\mathrm{d}\boldsymbol{A}$ is clearly parallel to the original vector $\boldsymbol{A}$. We can then write:

$$\frac{\mathrm{d}\boldsymbol{A}}{\mathrm{d}t} = \frac{\mathrm{d}A}{\mathrm{d}t}\boldsymbol{e}_A, \tag{1.52}$$

where $\boldsymbol{e}_A = \boldsymbol{A}/A$ is the unitary vector in the direction of $\boldsymbol{A}$.

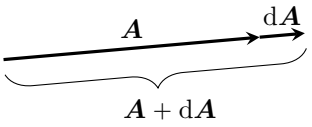

Figure 1.8: Differential vector $\mathrm{d}\boldsymbol{A}$ due to a change in the magnitude of $\boldsymbol{A}$.

Rate of change due to direction changes: Consider now the situation where only the direction of the vector changes, e.g. by a differential angle $\mathrm{d}\theta$ in a time $\mathrm{d}t$, while the magnitude stays constant. Then, the dot product of $\boldsymbol{A}$ with itself is constant in time. Hence, its differential is zero,

$$\boldsymbol{A}\cdot\boldsymbol{A} = \text{constant} \;\rightarrow\; \mathrm{d}(\boldsymbol{A}\cdot\boldsymbol{A}) = 0 = 2\boldsymbol{A}\cdot\mathrm{d}\boldsymbol{A} \tag{1.53}$$

which shows that $\boldsymbol{A}$, and $\mathrm{d}\boldsymbol{A}$, must be orthogonal. Note that in Fig. 1.9 the vector $\mathrm{d}\boldsymbol{A}$ has been drawn with a finite magnitude and therefore, $\boldsymbol{A}$ and $\mathrm{d}\boldsymbol{A}$ are not exactly perpendicular in the sketch. In the limit $\mathrm{d}\theta$, also $\mathrm{d}\boldsymbol{A} \to \boldsymbol{0}$ and then it is true that $\boldsymbol{A} \perp \mathrm{d}\boldsymbol{A}$.

Suppose that $\boldsymbol{A}$ is rotating in the plane of the paper at a rate[7] $\dot{\theta} = \mathrm{d}\theta/\mathrm{d}t$ in the counterclockwise direction, as shown in Fig. 1.9, with no change in magnitude. We say that the vector is rotating with respect to an axis of rotation that is perpendicular to the paper, which we may denote with the direction of the unit vector $\boldsymbol{k}$ coming out of the page so that the rotation is right-handed. In a differential time interval $\mathrm{d}t$, $\boldsymbol{A}$ will rotate an amount $\mathrm{d}\theta = \dot{\theta}\mathrm{d}t$ and the magnitude of $\mathrm{d}\boldsymbol{A}$ will be[8]

$$|\mathrm{d}\boldsymbol{A}| = A\mathrm{d}\theta = A\dot{\theta}\mathrm{d}t. \tag{1.54}$$

[7] The dot over a scalar variable always denotes differentiation with respect to time.

[8] Observe that whenever the direction of $\boldsymbol{A}$ changes, the differential change of the magnitude of $\boldsymbol{A}$ is *not* the same as the magnitude of the differential of $\boldsymbol{A}$, i.e., $\mathrm{d}A = \mathrm{d}|\boldsymbol{A}| \neq |\mathrm{d}\boldsymbol{A}|$. This is evident in this example, where $A = \text{const}$ but $|\mathrm{d}\boldsymbol{A}| \neq 0$.

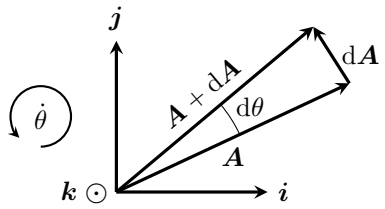

Figure 1.9: Differential vector $\mathrm{d}\boldsymbol{A}$ due to a change in the direction of $\boldsymbol{A}$, in the case where the rotation is along the $\boldsymbol{k}$ direction.

Hence, the magnitude of the vector derivative is

$$\left|\frac{\mathrm{d}\boldsymbol{A}}{\mathrm{d}t}\right| = A\dot{\theta}. \tag{1.55}$$

Then, noting that the direction of $\mathrm{d}\boldsymbol{A}$ is the same as that of $\boldsymbol{k} \times \boldsymbol{A}$, we can finally write:

$$\frac{\mathrm{d}\boldsymbol{A}}{\mathrm{d}t} = \dot{\theta}\boldsymbol{k} \times \boldsymbol{A}. \tag{1.56}$$

Suppose now that the vector $\boldsymbol{A}$ of constant magnitude is rotating with respect to a general axis of rotation, not necessarily perpendicular to the paper, whose direction is given by a unit vector $\boldsymbol{e}_\omega$, as shown in Fig. 1.10. Note that the direction of $\boldsymbol{e}_\omega$ is also chosen so that the direction of rotation obeys the right-hand rule.

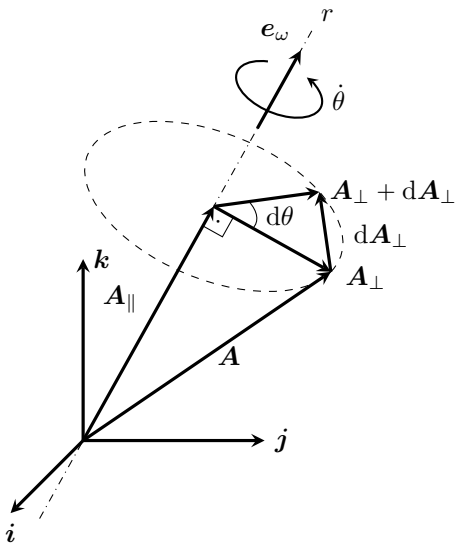

Figure 1.10: Differential vector $\mathrm{d}\boldsymbol{A}$ due to a change in the direction of $\boldsymbol{A}$, in a general three-dimensional rotation. The direction of the rotation is given by the unit vector $\boldsymbol{e}_\omega$ according to the right-hand rule. The vector $\boldsymbol{A}$ is decomposed in its parallel and perpendicular components to $\boldsymbol{e}_\omega$.

Let us decompose $\boldsymbol{A}$ as the sum of two vectors, $\boldsymbol{A}_\parallel$, which is parallel to $\boldsymbol{e}_\omega$, and $\boldsymbol{A}_\perp$, perpendicular to it:

$$\boldsymbol{A} = \boldsymbol{A}_\parallel + \boldsymbol{A}_\perp, \tag{1.57}$$

where the magnitude of each part is:

$$A_\parallel = \boldsymbol{e}_\omega \cdot \boldsymbol{A}, \tag{1.58}$$

$$A_\perp = |\boldsymbol{e}_\omega \times \boldsymbol{A}|. \tag{1.59}$$

Clearly, $\boldsymbol{A}_{\parallel}$ remains constant through the rotation, as it is not affected by it, whereas $\boldsymbol{A}_{\perp}$ describes a circle in a plane perpendicular to $\boldsymbol{e}_{\omega}$. Hence, $\mathrm{d}\boldsymbol{A} \equiv \mathrm{d}\boldsymbol{A}_{\perp}$. If the angle rotated about the axis in a differential time interval $\mathrm{d}t$ is $\mathrm{d}\theta$, the magnitude of the differential change of $\boldsymbol{A}$ is then:

$$|\mathrm{d}\boldsymbol{A}| = |\mathrm{d}\boldsymbol{A}_{\perp}| = A_{\perp}\mathrm{d}\theta = \dot{\theta}|\boldsymbol{e}_{\omega} \times \boldsymbol{A}|\mathrm{d}t, \tag{1.60}$$

so that the magnitude of the derivative of $\boldsymbol{A}$ is

$$\left|\frac{\mathrm{d}\boldsymbol{A}}{\mathrm{d}t}\right| = \dot{\theta}|\boldsymbol{e}_{\omega} \times \boldsymbol{A}|. \tag{1.61}$$

Finally, it is easy to see that the direction of $\mathrm{d}\boldsymbol{A}_{\perp}$ the same as the direction of the vector $\boldsymbol{e}_{\omega} \times \boldsymbol{A}$. Therefore, in the general three-dimensional case of a vector $\boldsymbol{A}$ of constant magnitude rotating with respect to a general axis given by $\boldsymbol{e}_{\omega}$, we can write the derivative of $\boldsymbol{A}$ as:

$$\frac{\mathrm{d}\boldsymbol{A}}{\mathrm{d}t} = \dot{\theta}\boldsymbol{e}_{\omega} \times \boldsymbol{A} = \boldsymbol{\omega} \times \boldsymbol{A}, \tag{1.62}$$

where $\boldsymbol{\omega} = \dot{\theta}\boldsymbol{e}_{\omega}$ is a vector whose direction encodes the direction of the axis of rotation, and whose magnitude encodes the rate of rotation about the axis. We call $\boldsymbol{\omega}$ the *angular velocity vector* of the rotation of $\boldsymbol{A}$.

This is indeed an important result: Rotations of vectors in time can be expressed in general by an angular velocity vector $\boldsymbol{\omega}$ that is aligned with the axis of rotation, points in the direction of the rotation according to the right-hand rule, and has a magnitude equal to the instantaneous angular rate of the rotation ($\dot{\theta}$ in the previous example).

Equation (1.62) is also valid in the more general case where $\boldsymbol{A}$ is rotating about an axis which does not pass through the origin of $\boldsymbol{A}$, since a rotation about an arbitrary axis can always be written as a rotation about a parallel axis plus a translation, and translations do not affect the magnitude nor the direction of a vector.

Rate of change in the general case: We can now go back to the general expression for the derivative of a vector when it changes in magnitude *and* direction due to a rotation (1.51). We can always write $\boldsymbol{A}(t) = A(t)\boldsymbol{e}_A(t)$, where $\boldsymbol{e}_A$ is a unit vector in the direction of $\boldsymbol{A}$, and decompose the differential of the vector, $\mathrm{d}\boldsymbol{A}$, into a differential associated to the change of magnitude plus a differential associated to the change of direction. In other words, applying the chain rule

$$\frac{\mathrm{d}\boldsymbol{A}}{\mathrm{d}t} = \frac{\mathrm{d}A}{\mathrm{d}t}\boldsymbol{e}_A + A\frac{\mathrm{d}\boldsymbol{e}_A}{\mathrm{d}t} = \frac{\mathrm{d}A}{\mathrm{d}t}\boldsymbol{e}_A + A\boldsymbol{\omega} \times \boldsymbol{e}_A = \frac{\mathrm{d}A}{\mathrm{d}t}\boldsymbol{e}_A + \boldsymbol{\omega} \times \boldsymbol{A}. \tag{1.63}$$

Note that the first term (due to the change of magnitude only) is parallel to $\boldsymbol{A}$ and the second term (due to the change of direction only) is orthogonal to $\boldsymbol{A}$ and $\boldsymbol{\omega}$.

Rules for vector differentiation: Vector differentiation follows similar rules to scalar differentiation regarding addition, multiplication by a scalar, and the rest of products. In particular we have that, for any vectors $\boldsymbol{A}$, $\boldsymbol{B}$, and any scalar λ,

$$\mathrm{d}(\lambda\boldsymbol{A}) = \mathrm{d}\lambda\boldsymbol{A} + \lambda\mathrm{d}\boldsymbol{A} \tag{1.64}$$

$$\mathrm{d}(\boldsymbol{A} + \boldsymbol{B}) = \mathrm{d}\boldsymbol{A} + \mathrm{d}\boldsymbol{B} \tag{1.65}$$

$$\mathrm{d}(\boldsymbol{A} \cdot \boldsymbol{B}) = \mathrm{d}\boldsymbol{A} \cdot \boldsymbol{B} + \boldsymbol{A} \cdot \mathrm{d}\boldsymbol{B} \tag{1.66}$$

$$\mathrm{d}(\boldsymbol{A} \times \boldsymbol{B}) = \mathrm{d}\boldsymbol{A} \times \boldsymbol{B} + \boldsymbol{A} \times \mathrm{d}\boldsymbol{B} \tag{1.67}$$

1.7.2 Infinitesimal rotations

As we showed in §1.5, the product of rotation matrices is in general not commutative. For any two small simple rotations orthogonal to each other, however, the difference between $[R_2][R_1]$ and $[R_1][R_2]$ becomes negligible. In the limiting case of *infinitesimal* rotations (i.e. differential rotations),

the product rotation is independent of the order in which the rotations are applied. When rotations are infinitesimal ($\Delta\theta \to \mathrm{d}\theta$), we can approximate

$$\cos(\mathrm{d}\theta) = 1 + \mathcal{O}(\mathrm{d}\theta^2), \tag{1.68}$$

$$\sin(\mathrm{d}\theta) = \mathrm{d}\theta + \mathcal{O}(\mathrm{d}\theta^3). \tag{1.69}$$

If we keep only the first-order terms, the simple rotation matrices in Eq. (1.20) may be written

$$[R_x] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -\mathrm{d}\theta_x \\ 0 & \mathrm{d}\theta_x & 1 \end{bmatrix}, \; [R_y] = \begin{bmatrix} 1 & 0 & \mathrm{d}\theta_y \\ 0 & 1 & 0 \\ -\mathrm{d}\theta_y & 0 & 1 \end{bmatrix}, \; [R_z] = \begin{bmatrix} 1 & -\mathrm{d}\theta_z & 0 \\ \mathrm{d}\theta_z & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}. \tag{1.70}$$

The product of three simple rotations, one about each coordinated axis, is

$$[R] = [R_x][R_y][R_z] = \begin{bmatrix} 1 & -\mathrm{d}\theta_z & \mathrm{d}\theta_y \\ \mathrm{d}\theta_z & 1 & -\mathrm{d}\theta_x \\ -\mathrm{d}\theta_y & \mathrm{d}\theta_x & 1 \end{bmatrix}, \tag{1.71}$$

where, again, we have kept only the first-order terms. As it can be easily checked, this result is independent of the order in which the matrices are multiplied. Indeed, it can be shown that for infinitesimal rotations are commutative in general, in contrast to finite rotations. Furthermore, observe that to fully define the resulting matrix $[R]$, we only need to specify three numbers, $\mathrm{d}\theta_x$, $\mathrm{d}\theta_y$, $\mathrm{d}\theta_z$.

We discuss now the time derivative of a vector $\boldsymbol{A}$ of constant magnitude, as we did in §1.7.1, when $\boldsymbol{A}$ undergoes a differential rotation as given by the matrix $[R]$ of Eq. (1.71) between a time t and $t + \mathrm{d}t$. Then, we can write the time derivative of $\boldsymbol{A}$ as:

$$\frac{\mathrm{d}\boldsymbol{A}}{\mathrm{d}t} = \frac{[R]\cdot\boldsymbol{A} - \boldsymbol{A}}{\mathrm{d}t} = \frac{([R]-[U])\cdot\boldsymbol{A}}{\mathrm{d}t} = [\Omega]\cdot\boldsymbol{A} = \begin{bmatrix} 0 & -\mathrm{d}\theta_z/\mathrm{d}t & \mathrm{d}\theta_y/\mathrm{d}t \\ \mathrm{d}\theta_z/\mathrm{d}t & 0 & -\mathrm{d}\theta_x/\mathrm{d}t \\ -\mathrm{d}\theta_y/\mathrm{d}t & \mathrm{d}\theta_x/\mathrm{d}t & 0 \end{bmatrix} \cdot \begin{bmatrix} A_x \\ A_y \\ A_z \end{bmatrix}.$$

Matrix $[\Omega]$ can be termed the *angular velocity matrix*, and as it can be seen, it is antisymmetric. It can be easily checked that the result of the product $[\Omega]\cdot\boldsymbol{A}$ is equivalent to the cross product $\boldsymbol{\omega}\times\boldsymbol{A}$,

$$[\Omega]\cdot\boldsymbol{A} \equiv \boldsymbol{\omega}\times\boldsymbol{A} \tag{1.72}$$

where $\boldsymbol{\omega}$ is the so-called *angular velocity vector*, which condenses the same information as $[\Omega]$, and is defined as

$$\boldsymbol{\omega} = \frac{\mathrm{d}\theta_x}{\mathrm{d}t}\boldsymbol{i} + \frac{\mathrm{d}\theta_y}{\mathrm{d}t}\boldsymbol{j} + \frac{\mathrm{d}\theta_z}{\mathrm{d}t}\boldsymbol{k}. \tag{1.73}$$

In mechanics, the angular velocity vector $\boldsymbol{\omega}$ is customarily used instead of matrix $[\Omega]$. Vector $\boldsymbol{\omega}$ has the direction of the axis of rotation, and is oriented according to the right-hand rule. The magnitude of $\boldsymbol{\omega}$ equals the angular rate of rotation.

1.7.3 Gradient of a scalar function

Consider a scalar function of space, $f = f(x, y, z)$, where x, y, z are the Cartesian coordinates of space. Consider also the canonical vector basis $B_0 : \{\boldsymbol{i}, \boldsymbol{j}, \boldsymbol{k}\}$. We define its *gradient*, ∇f, as the vector given by the following component expression in B_0:

$$\nabla f = \frac{\partial f}{\partial x}\boldsymbol{i} + \frac{\partial f}{\partial y}\boldsymbol{j} + \frac{\partial f}{\partial z}\boldsymbol{k}, \tag{1.74}$$

where the nabla symbol (∇) is a shorthand for the differential operator $\nabla = (\partial/\partial x)\boldsymbol{i} + (\partial/\partial y)\boldsymbol{j} + (\partial/\partial z)\boldsymbol{k}$. The gradient of a function can be interpreted as the vector that indicates the direction of growth of f. The magnitude of ∇f, i.e. $|\nabla f|$, indicates the rate of growth of f in that direction.

1.7.4 Divergence and curl of a vector function

Given a vector function $\boldsymbol{A} = \boldsymbol{A}(x, y, z)$, where x, y, z are the Cartesian coordinates of space. Like before, consider also the canonical vector basis $B_0 : \{\boldsymbol{i}, \boldsymbol{j}, \boldsymbol{k}\}$. We define the *divergence* and *curl* (or *rotational*) of $\boldsymbol{A}$ as

$$\mathrm{div}(\boldsymbol{A}) = \nabla \cdot \boldsymbol{A} = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}; \tag{1.75}$$

$$\mathrm{rot}(\boldsymbol{A}) = \nabla \times \boldsymbol{A} = \left(\frac{\partial A_y}{\partial z} - \frac{\partial A_z}{\partial y}\right)\boldsymbol{i} + \left(\frac{\partial A_z}{\partial x} - \frac{\partial A_x}{\partial z}\right)\boldsymbol{j} + \left(\frac{\partial A_x}{\partial y} - \frac{\partial A_y}{\partial x}\right)\boldsymbol{k}. \tag{1.76}$$

Observe the symmetry of these operations with the dot and cross product between two vectors. The divergence of a vector is a scalar, whereas the rotational of a vector is a vector. It is easy to check that

$$\nabla \times (\nabla f) = \boldsymbol{0} \tag{1.77}$$

$$\nabla \cdot (\nabla \times \boldsymbol{A}) = 0 \tag{1.78}$$

A vector function $\boldsymbol{A}$ that has zero divergence ($\nabla \cdot \boldsymbol{A} = 0$) is called *solenoidal*; a vector function $\boldsymbol{A}$ that has zero curl ($\nabla \times \boldsymbol{A} = \boldsymbol{0}$) is called *irrotational*.

Helmholtz's theorem states that if a vector field (i.e. a vector function) $\boldsymbol{A} = \boldsymbol{A}(x, y, z)$ is twice continuously differentiable, then we can find a scalar field ϕ and a vector field $\boldsymbol{B}$ such that we can write $\boldsymbol{A}$ as

$$\boldsymbol{A} = \nabla\phi + \nabla \times \boldsymbol{B}. \tag{1.79}$$

It is easy to check from the corresponding definitions and Eqs. (1.77) and (1.78) that if $\boldsymbol{A}$ is solenoidal, then we can write $\boldsymbol{A} = \nabla \times \boldsymbol{B}$; if $\boldsymbol{A}$ is irrotational, then we can write $\boldsymbol{A} = \nabla\phi$.

Part I
Mechanics of the point particle

2 Kinematics of point particles

Kinematics is the branch of Classical Mechanics that describes the motion of bodies without consideration of the forces that cause the motion. In practice, it is concerned with *expressing positions, velocities and accelerations of particles and bodies as functions of useful variables, or coordinates, identifying relations among them in different reference frames, and the constraints of the system.* Solving the kinematics of a system is always the first step toward solving a problem in mechanics.

2.1 Reference frames

In Newtonian Mechanics, we are interested in the change in the position of point particles and bodies in space with time. The motion of an object can only be described relative to *something else*—other bodies, or a set of space-time coordinates. In other words, the description of motion only makes sense relative to a *reference frame*.

A **reference frame** represents an *observer*, i.e., the viewpoint of a virtual person localized in space, with a certain orientation, and with a certain state of linear motion and rotation (collectively referred to simply as "motion"). While it may seem a simple concept, the idea of reference frame is a profound one that deserves some reflection. For all practical purposes, we can define a reference frame S_0 as the combination of a point O in space, which we will call *origin*, and an orthonormal, right-handed vector basis $B_0 : \{\boldsymbol{i}, \boldsymbol{j}, \boldsymbol{k}\}$. We will usually write

$$S_0 : \{O; B_0\}, \tag{2.1}$$

or, equivalently, $S_0 : Oxyz$, where x, y, z denote the coordinated directions defined by the vectors of B_0. When we have more than one reference frame, e.g. $S_1 : \{A; B_1\}$ and $S_0 : \{O; B_0\}$, the linear motion of the observer mounted on S_1 is given by the evolution in time of the position of point A as seen from S_0, while its rotation is given by the evolution in time of the vectors of the basis B_1 with respect to B_0. Note that we need to specify both aspects of the motion of the observer to fully define S_1 with respect to S_0.

Sometimes it is useful to think of a reference frame as the whole Euclidean space that is moving with the observer[9] Later in this course, we will see that we can identify a reference frame with a rigid body, and vice-versa. Indeed, reference frames are often defined using rigid bodies to which we say the reference frame is *attached*: a reference frame attached to a rotating platform, to the chassis of an aircraft, to the Earth, to the Sun, etc. This means that the observer is moving with that body and has the same rotation as the body.

2.2 Position, velocity and acceleration

As soon as we have defined a reference frame $S_0\{O; B_0\}$, we can apply vector calculus to describe the kinematics of point particles.

The **position vector** of a point particle P in S_0 is defined as the vector from O to P,

$$\boldsymbol{r}_0^P = \boldsymbol{OP}. \tag{2.2}$$

Here, the subindex 0 indicates that this is a position vector with respect to the origin of reference frame S_0. The subindex and/or the superindex can dropped only if there is no ambiguity as to what position vector we are referring.

When a point particle P moves with respect to S_0, its position vector is a function of time, $\boldsymbol{r}_0^P = \boldsymbol{r}_0^P(t)$, and its magnitude, its direction, or both can change in time. The **velocity vector** of

[9] A reference frame in Relativistic Mechanics, where space-time is not an absolute for all observers, is far more complicated, and is not covered in this course.

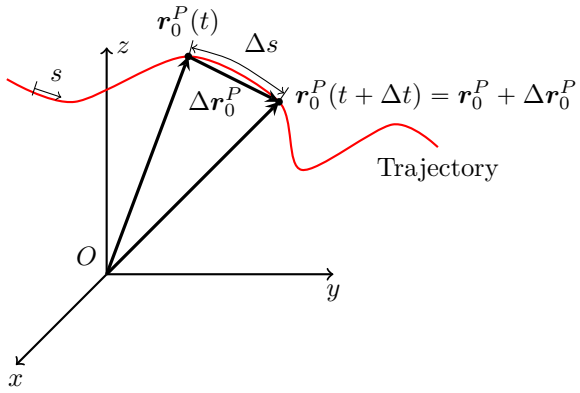

Figure 2.1: Position vector $\boldsymbol{r}_0^P$ at two consecutive instants of time. The arc length coordinate s has been defined with an arbitrary origin on the trajectory.

point particle P relative to S_0 is given by the following time derivative taken in S_0, i.e., as seen by an observer mounted on that reference frame (see Fig. 2.1):

$$\boldsymbol{v}_0^P = \lim_{\Delta t \to 0} \frac{(\boldsymbol{r}_0^P + \Delta \boldsymbol{r}_0^P) - \boldsymbol{r}_0^P}{\Delta t} = \lim_{\Delta t \to 0} \frac{\Delta \boldsymbol{r}_0^P}{\Delta t} = \left. \frac{\mathrm{d}\boldsymbol{r}_0^P}{\mathrm{d}t} \right|_0, \tag{2.3}$$

where the subindex 0 on the derivative term, $\mathrm{d}\boldsymbol{r}_0^P/\mathrm{d}t|_0$, indicates that it *has been taken in* the S_0 reference frame[10]. Again, as long as there is no ambiguity (as far as we work with a single reference frame), this symbol may be dropped for brevity. The magnitude of $\boldsymbol{v}_0^P$ is often called *speed* or *celerity*. Note that it is incorrect to call the velocity vector just "speed."

Lastly, the **acceleration vector** of point particle P relative to S_0 is the rate of change in time of the velocity vector $\boldsymbol{v}_0^P$, which can also involve a change of magnitude, direction or both:

$$\boldsymbol{a}_0^P = \lim_{\Delta t \to 0} \frac{(\boldsymbol{v}_0^P + \Delta \boldsymbol{v}_0^P) - \boldsymbol{v}_0^P}{\Delta t} = \lim_{\Delta t \to 0} \frac{\Delta \boldsymbol{v}_0^P}{\Delta t} = \left. \frac{\mathrm{d}\boldsymbol{v}_0^P}{\mathrm{d}t} \right|_0 = \left. \frac{\mathrm{d}^2\boldsymbol{r}_0^P}{\mathrm{d}t^2} \right|_0. \tag{2.4}$$

If we express $\boldsymbol{r}_0^P$ in basis B_0,

$$\boldsymbol{r}_0^P = x\boldsymbol{i} + y\boldsymbol{j} + z\boldsymbol{k}, \tag{2.5}$$

taking time derivatives in S_0, we obtain:

$$\boldsymbol{v}_0^P = \left. \frac{\mathrm{d}\boldsymbol{r}_0^P}{\mathrm{d}t} \right|_0 = \dot{x}\boldsymbol{i} + \dot{y}\boldsymbol{j} + \dot{z}\boldsymbol{k} \tag{2.6}$$

where $\boldsymbol{i}$, $\boldsymbol{j}$ and $\boldsymbol{k}$ are treated as constants since these vectors do not change in time with respect to S_0. Taking time derivatives again[11] we may write:

$$\boldsymbol{a}_0^P = \left. \frac{\mathrm{d}\boldsymbol{v}_0^P}{\mathrm{d}t} \right|_0 = \ddot{x}\boldsymbol{i} + \ddot{y}\boldsymbol{j} + \ddot{z}\boldsymbol{k}. \tag{2.7}$$

Observe that, while the value of the position vector depends on the point O chosen as the origin, the velocity and acceleration vectors do not depend on it.

2.3 Trajectory and arc length coordinate

The curve traced by point P is the *trajectory* or *path* of P as seen from S_0. The *path coordinate* or *arc length coordinate*, s, measures the distance traveled by the particle along the trajectory, as

[10] The reason why it is crucial to specify this will become apparent in next Chapter.

[11] In case you are wondering, we *could* continue and take higher derivatives of $\boldsymbol{r}_0^P$. However, higher derivatives have little interest in mechanics, as the laws of Newton involve only the second derivative.

defined in Fig. 2.1. If we know the geometric shape of the trajectory, we know $\boldsymbol{r}_0^P = \boldsymbol{r}_0^P(s)$; then, if we also know the arc length coordinate as a function of time, $s = s(t)$, we unambiguously know $\boldsymbol{r}_0^P$ at all instants of time: $\boldsymbol{r}_0^P = \boldsymbol{r}_0^P(s(t))$. Therefore, having complete knowledge of the trajectory $\boldsymbol{r}_0^P(s)$ *and* the time law $s(t)$ is equivalent to solving the motion of the particle. We can then derive $\boldsymbol{v}_0^P$, $\boldsymbol{a}_0^P$, or any other quantity associated to the motion of P.

Observe that the trajectory of a particle is specific to the reference frame from where it is observed: another observer, on a different reference frame, will perceive a different curve for the motion of P.

2.4 Intrinsic vector basis and reference frame

Consider the difference between two position vectors on the trajectory for a small increment of s,

$$\Delta\boldsymbol{r}_0^P = \boldsymbol{r}_0^P(s+\Delta s) - \boldsymbol{r}_0^P(s). \tag{2.8}$$

As can be inferred from inspecting Fig. 2.1, in the limit $\Delta s \to \mathrm{d}s$, the direction of $\Delta\boldsymbol{r}_0^P$ approaches the *tangent direction* to the curve at that point. We can also see that the magnitude of $\Delta\boldsymbol{r}_0^P$, approaches $\left|\mathrm{d}\boldsymbol{r}_0^P\right| = \mathrm{d}s$. Hence, the vector $\mathrm{d}\boldsymbol{r}_0^P/\mathrm{d}s$ is the *unit tangent vector* $\boldsymbol{e}_t$ (i.e., $|\boldsymbol{e}_t| = 1$):

$$\frac{\mathrm{d}\boldsymbol{r}_0^P}{\mathrm{d}s} = \lim_{\Delta s\to 0}\frac{\Delta\boldsymbol{r}_0^P}{\Delta s} \equiv \boldsymbol{e}_t. \tag{2.9}$$

The tangent vector $\boldsymbol{e}_t$ changes as the point P travels along the trajectory, i.e., $\boldsymbol{e}_t = \boldsymbol{e}_t(s)$. We can compute the derivative of $\boldsymbol{e}_t$ along s just as we did with $\boldsymbol{r}_0^P$. As $\boldsymbol{e}_t$ is a unit vector, $\boldsymbol{e}_t \cdot \boldsymbol{e}_t = 1$, and differentiating this expression:

$$2\frac{\mathrm{d}\boldsymbol{e}_t}{\mathrm{d}s}\cdot\boldsymbol{e}_t = 0. \tag{2.10}$$

Therefore, the vector $\mathrm{d}\boldsymbol{e}_t/\mathrm{d}s$ is always perpendicular to $\boldsymbol{e}_t$. We define the *normal unit vector* at each point of the trajectory, $\boldsymbol{e}_n$, as the normalized value of this derivative:

$$\boldsymbol{e}_n = \frac{\mathrm{d}\boldsymbol{e}_t}{\mathrm{d}s}\Big/\left|\frac{\mathrm{d}\boldsymbol{e}_t}{\mathrm{d}s}\right|. \tag{2.11}$$

At each point of the trajectory, we may zoom in and approximate the trajectory *locally* as a circular arc as shown on Fig. 2.2. The approximating circle is known as the *osculating circle*. The normal vector $\boldsymbol{e}_n$ always points toward the center of the osculating circle C (i.e., the local *center of curvature*). Lastly, the plane defined by vectors $\boldsymbol{e}_t$ and $\boldsymbol{e}_n$ is known as the local *osculating plane*.

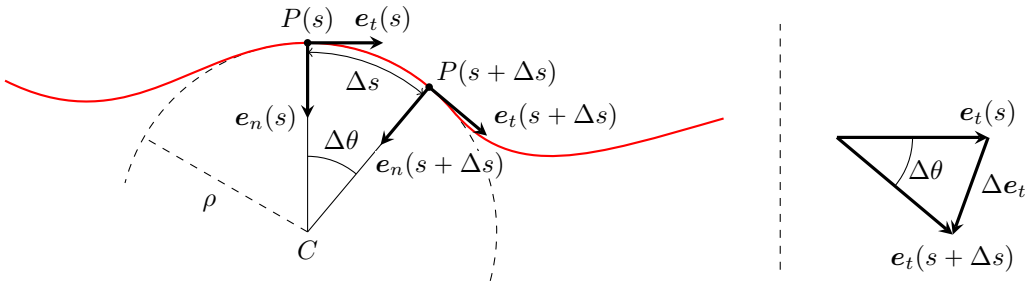

Figure 2.2: Osculating circle at one point of the trajectory of a point particle P, and unit vectors $\boldsymbol{e}_t$ and $\boldsymbol{e}_n$ at two consecutive positions along the trajectory. The auxiliary diagram on the right provides a detail view of $\Delta\boldsymbol{e}_t$.

In Fig. 2.2, the vectors $\boldsymbol{e}_t(s)$ and $\boldsymbol{e}_t(s+\Delta s)$ form an isosceles triangle, in which the angle $\Delta\theta$ is the same as the one subtended between $P(s)$ and $P(s+\Delta s)$ as seen from the center C of the osculating circle. In the limit $\Delta s \to \mathrm{d}s$, since $\boldsymbol{e}_t$ is a unit vector, $|\mathrm{d}\boldsymbol{e}_t| \to \mathrm{d}\theta$. Hence,

$$\left|\frac{\mathrm{d}\boldsymbol{e}_t}{\mathrm{d}s}\right| = \frac{\mathrm{d}\theta}{\mathrm{d}s} = \frac{1}{\rho} = \kappa. \tag{2.12}$$

It turns out that the magnitude of $\mathrm{d}\boldsymbol{e}_t/\mathrm{d}s$, which has units of L^{-1}, has a clear physical meaning: it is the inverse of the local *radius of curvature* ρ at that point of the trajectory, i.e., the radius of the osculating circle. The local *curvature*, $\kappa = 1/\rho$, can be used instead. In a purely circular trajectory, ρ is constant with s and is simply the radius of the trajectory at all points. In a general case, the radius of curvature and the osculating circle change with s.

Lastly, the unit vector along the perpendicular direction to the osculating plane, defined as $\boldsymbol{e}_b = \boldsymbol{e}_t \times \boldsymbol{e}_n$, is called the *binormal vector*. Importantly, the vectors $\boldsymbol{e}_t, \boldsymbol{e}_n, \boldsymbol{e}_b$ depend only on the trajectory $\boldsymbol{r}_0^P(s)$. They do not depend on how quickly or slowly the particle travels along this trajectory, i.e., the time function $s(t)$. As such, they are considered an intrinsic property of the trajectory, and for this reason they are sometimes referred to as *path variables*.

The set of these three vectors at a point P in the trajectory,

$$B_F(P) : \{\boldsymbol{e}_t, \boldsymbol{e}_n, \boldsymbol{e}_b\}, \tag{2.13}$$

constitutes a right-handed orthonormal vector basis, named the *intrinsic vector basis*. The intrinsic vector basis at P can be used to define a *intrinsic reference frame* (or *Frenet-Serret reference frame*), using P as the origin:

$$S_F(P) : \{P; B_F(P)\}. \tag{2.14}$$

Observe that this reference frame changes as the particle travels along the trajectory (both its origin and its orientation, given by $B_F(P)$, may change). The intrinsic reference frame is useful for describing vectorial quantities associated to the motion of the point particle.[12]

We may apply the same differentiation procedure as above to the unit vectors $\boldsymbol{e}_n(s)$ and $\boldsymbol{e}_b(s)$. In a planar trajectory, it is easy to conclude that $\mathrm{d}\boldsymbol{e}_n/\mathrm{d}s = -\kappa\boldsymbol{e}_t = -\boldsymbol{e}_t/\rho$, and that $\boldsymbol{e}_b = \text{const}$, so that $\mathrm{d}\boldsymbol{e}_b/\mathrm{d}s = 0$ (see Fig. 2.2). However, the trajectory of a particle can also be a three-dimensional curve in space. In this case, $\mathrm{d}\boldsymbol{e}_n/\mathrm{d}s$ can have an additional component along $\boldsymbol{e}_b$, so in general, $\mathrm{d}e_n/\mathrm{d}s = -\kappa\boldsymbol{e}_t + \tau\boldsymbol{e}_b$. The magnitude of this new component, τ, is known as the local *torsion* of the curve at that point, which also has units of L^{-1}. Similarly, observing the geometry of this situation we find $\mathrm{d}\boldsymbol{e}_b/\mathrm{d}s = -\tau\boldsymbol{e}_n$. The relations between the unit vectors of the intrinsic vector basis and their derivatives along s in the three-dimensional case are termed the *Frenet-Serret formulas*. To summarize,

$$\frac{\mathrm{d}\boldsymbol{e}_t}{\mathrm{d}s} = \kappa\boldsymbol{e}_n; \quad \frac{\mathrm{d}\boldsymbol{e}_n}{\mathrm{d}s} = -\kappa\boldsymbol{e}_t + \tau\boldsymbol{e}_b; \quad \frac{\mathrm{d}\boldsymbol{e}_b}{\mathrm{d}s} = -\tau\boldsymbol{e}_n. \tag{2.15}$$

As a final comment, note that just as the trajectory is specific of the reference frame S_0 from where we observe the motion of the particle P, since $B_F(P)$ and S_F are *intrinsic* to the trajectory, they also depend on the reference frame of observation. This would warrant the use of a subindex 0 to denote in which reference frame the trajectory $\boldsymbol{r}_0^P(s)$, the arc length s, and the unit vectors $\boldsymbol{e}_t, \boldsymbol{e}_n, \boldsymbol{e}_b$ are described. Notwithstanding this, simultaneous use of these concepts in two different reference frame is unusual, and therefore, there is seldom any ambiguity, so the reference frame subindices have been dropped here. This is the approach taken in the rest of these lecture notes when using s, $\boldsymbol{r}_0^P(s)$, and the vectors $\boldsymbol{e}_t, \boldsymbol{e}_n, \boldsymbol{e}_b$.

2.5 Velocity and acceleration vectors in the intrinsic vector basis

Applying the chain rule, and dropping the 0 subindex for simplicity when we work with a single reference frame S_0, we can express the velocity vector as

$$\boldsymbol{v}^P = \frac{\mathrm{d}\boldsymbol{r}^P}{\mathrm{d}t} = \frac{\mathrm{d}\boldsymbol{r}^P}{\mathrm{d}s}\frac{\mathrm{d}s}{\mathrm{d}t} = \dot{s}\boldsymbol{e}_t. \tag{2.16}$$

In the previous expression, $\dot{s} \equiv v$ is the speed or celerity at which the particle travels the trajectory. Again, observe that $\boldsymbol{e}_t$ is a purely geometric entity which only depends on the shape of the trajectory, and not on how fast this trajectory is traveled by the particle.

[12] Observe, however, that for a straight trajectory, or in straight segments of a general trajectory, $\rho \to \infty$, and the direction of $\boldsymbol{e}_n$ and $\boldsymbol{e}_b$ is not defined. In these cases, the intrinsic reference frame must be redefined taking this into account (i.e., $\boldsymbol{e}_n$ can be chosen along any direction $\perp \boldsymbol{e}_t$).

In a similar way, we can obtain a useful expression for the acceleration of the particle:

$$\boldsymbol{a}^P = \frac{\mathrm{d}\boldsymbol{v}^P}{\mathrm{d}t} = \ddot{s}\boldsymbol{e}_t + \dot{s}\frac{\mathrm{d}\boldsymbol{e}_t}{\mathrm{d}s}\frac{\mathrm{d}s}{\mathrm{d}t} = \ddot{s}\boldsymbol{e}_t + \frac{\dot{s}^2}{\rho}\boldsymbol{e}_n. \tag{2.17}$$

From this expression we define the *tangential acceleration* and the *normal acceleration* respectively as:

$$a_t = \ddot{s}; \quad a_n = \frac{\dot{s}^2}{\rho} \equiv \frac{v^2}{\rho}. \tag{2.18}$$

These relations reveal many important facts about the motion of a particle along its trajectory:

- $\boldsymbol{v}^P$ is always tangent to the path (i.e., it only has a tangential component parallel to $\boldsymbol{e}_t$. The normal and binormal velocity components are always zero!)
- $\boldsymbol{a}^P$ can have a tangent and a normal component (but never a binormal component!).
- $\boldsymbol{v}^P$ and $\boldsymbol{a}^P$ always lie in the osculating plane defined by $\boldsymbol{e}_t$ and $\boldsymbol{e}_n$.
- The osculating plane is constant only if the path is planar. Otherwise, the osculating plane twists around as the particle moves along its path, due to its torsion.
- The tangential acceleration component is related to changes in speed (i.e., changes in the velocity magnitude).
- The normal acceleration component is related to changes in the direction of the velocity vector.
- Both the tangential and normal acceleration components exist and are non-zero unless v is constant ($a_t = 0$) or the path is straight ($a_n = 0$ since $\rho \to \infty$) or $v = 0$ ($a_n = 0$).

2.5.1 Solving planar problems using path variables

Planar problems are common in Mechanics. Sometimes we are given certain conditions that the velocity, acceleration, normal acceleration, tangential acceleration or radius of curvature must fulfill, and we are asked to obtain the trajectory $\boldsymbol{r}^P(s)$ and the time law $s(t)$; i.e., to solve the motion of the particle. If the problem is planar, we can find the components x, y of $\boldsymbol{r}^P$ in the B_0 basis ($\boldsymbol{r}^P = x\boldsymbol{i}+y\boldsymbol{j}$) as a function of s if we can integrate (see Fig. 2.3):

$$\begin{aligned} \mathrm{d}x &= \mathrm{d}s\cos\theta, \\ \mathrm{d}y &= \mathrm{d}s\sin\theta, \end{aligned} \tag{2.19}$$

where $\mathrm{d}s$ is the differential of the arc length coordinate and θ is the angle that the tangent to the curve forms with the Ox axis. It is then necessary to integrate θ as a function of s using the conditions given in the problem. To do so, the relations between the variables below can prove useful:

$$v = \dot{s}, \; a_t = \dot{v} = \ddot{s}, \; a_n = \frac{v^2}{\rho}, \; \frac{\mathrm{d}\theta}{\mathrm{d}s} = \kappa = \frac{1}{\rho}. \tag{2.20}$$

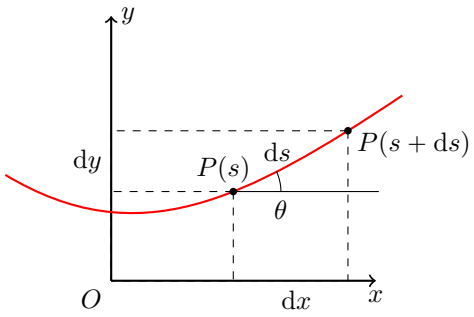

Figure 2.3: Expressing $\mathrm{d}x$ and $\mathrm{d}y$ as functions of $\mathrm{d}s$ and θ in a planar problem

2.6 Coordinate systems

We have introduced so far two closely-related concepts, namely vector bases like B_0, which allow us to describe any vector in terms of components, and reference frames like $S_0 : \{O; B_0\}$, which allow defining the position vector of a point particle P as $\boldsymbol{r}_0^P = \boldsymbol{OP}$ and introduce the important notion of the observer (his position, orientation, linear motion and rotational motion, which affect his perception of the motion of the point particle).

It turns out that the position of a point P can be given not only by its position vector $\boldsymbol{r}_0^P$, but also as its *coordinates* in a given *coordinate system*. There are many possible coordinate systems that we may define. In the plane, two coordinates are needed to locate a point: $C : \{q_1, q_2\}$, whereas in space three coordinates are required: $C : \{q_1, q_2, q_3\}$. A list of common examples of spatial coordinates is the following:

1. **Cartesian coordinates** $\{x, y, z\}$. These is the most familiar coordinate system to us, and we have already referred to it implicitly when we mentioned the coordinated axes Ox, Oy, Oz, the coordinated planes Oxy, Oxz, Oyz and the whole space $Oxyz$. We can think of Cartesian coordinates as existing by default for any reference frame $S_0 : \{O; B_0 : \{\boldsymbol{i}, \boldsymbol{j}, \boldsymbol{k}\}\}$ that we define: given a point P, its x coordinate is the rectilinear distance from O to P in the direction of the first vector of the basis $\boldsymbol{i}$, y is the distance along $\boldsymbol{j}$ and z the distance along $\boldsymbol{k}$.

   The Cartesian coordinates of P have the special property that they coincide numerically with the vector components of $\boldsymbol{r}_0^P$ in the B_0 basis: $P = \{x, y, z\}$ and $\boldsymbol{r}_0^P = x\boldsymbol{i} + y\boldsymbol{j} + z\boldsymbol{k}$. This is not true of other coordinate systems in general.

2. **Cylindrical coordinates** $\{R, \theta, z\}$: This is a particular case of *orthogonal curvilinear coordinates*. The Cartesian x, y are substituted by R, θ (cylindrical radius and azimuthal angle), while z has the same meaning as before. When working only on the Oxy plane, cylindrical coordinates are often called *polar coordinates*. In this case, θ is also called polar angle. Cylindrical coordinates are described in more detail below.

3. **Spherical coordinates** $\{r, \theta, \phi\}$: This is also a set of orthogonal curvilinear coordinates. While θ has the same meaning as in cylindrical coordinates, R and z are substituted for r and ϕ (spherical radius and latitude). In some contexts, however, the complementary angle to ϕ is used instead (measured from the pole and termed co-latitude). Spherical coordinates have the special property that $|\boldsymbol{r}_0^P| = r$. Spherical coordinates are also described in more detail below.

2.6.1 Local vector basis in orthogonal curvilinear coordinates

The existence of a coordinate system $C : \{q_1, q_2, q_3\}$ enables us to define a local vector basis,

$$B_C(P) : \{\boldsymbol{e}_{q_1}, \boldsymbol{e}_{q_2}, \boldsymbol{e}_{q_3}\}, \tag{2.21}$$

associated to the *tangent directions* of the coordinate system at each point P. This is particularly interesting in the case of orthogonal curvilinear coordinates, since this basis is orthogonal.

The direction of the vectors of the basis at P is the direction along which each of the coordinates grows when keeping the other two constant. In the trivial example of Cartesian coordinates, the direction of growth of x while keeping y, z constant is $\boldsymbol{e}_x = \boldsymbol{i}$ direction. Similarly, the growth direction of y is $\boldsymbol{e}_y = \boldsymbol{j}$, and for z it is $\boldsymbol{e}_z = \boldsymbol{k}$. Indeed, the locally-defined basis at P in Cartesian coordinates coincides with the initial vector basis that we had to start with: $B_0 : \{\boldsymbol{i}, \boldsymbol{j}, \boldsymbol{k}\}$. This is not so in other coordinate systems, as will become evident in the following examples.

The basis vectors are normalized if necessary to be unit vectors, and $\{q_1, q_2, q_3\}$ are always ordered so that $\{\boldsymbol{e}_{q_1}, \boldsymbol{e}_{q_2}, \boldsymbol{e}_{q_3}\}$ is an orthonormal, right-handed basis. Lastly, observe that the local vector basis that we define at P from the coordinate system can be used to establish a *local reference frame*,

$$S_C : \{P; B_C(P)\}, \tag{2.22}$$

which changes from point to point in general.

2.6.2 Cylindrical coordinates

Cylindrical coordinates are particularly useful to study motions with some form of cylindrical symmetry. The position of a point P is given by the three coordinates $\{R, \theta, z\}$ as shown in Fig. 2.4:

1. R is the so-called cylindrical radius, i.e., the distance from the Oz axis to P.
2. θ is the azimuthal angle between the Oxz plane and the plane defined by the Oz and P. It becomes undefined when P is on the Oz axis.
3. z is the vertical height, and coincides with the Cartesian z.

We can immediately write the following coordinate transformation relations:

$$\begin{cases} x = R\cos\theta \\ y = R\sin\theta \end{cases} \qquad \begin{cases} R = \sqrt{x^2 + y^2} \\ \theta = \arctan(y/x) \end{cases} \tag{2.23}$$

As can be inferred from Fig. 2.4, the unit vectors of $B_C(P)$ in cylindrical coordinates are

$$\begin{cases} \boldsymbol{e}_R = \cos\theta\boldsymbol{i} + \sin\theta\boldsymbol{j}, \\ \boldsymbol{e}_\theta = -\sin\theta\boldsymbol{i} + \cos\theta\boldsymbol{j}, \\ \boldsymbol{e}_z = \boldsymbol{k}, \end{cases} \tag{2.24}$$

and their time derivatives are

$$\frac{\mathrm{d}\boldsymbol{e}_R}{\mathrm{d}t} = \dot\theta(-\sin\theta\boldsymbol{i} + \cos\theta\boldsymbol{j}) = \dot\theta\boldsymbol{e}_\theta, \tag{2.25}$$

$$\frac{\mathrm{d}\boldsymbol{e}_\theta}{\mathrm{d}t} = -\dot\theta(\cos\theta\boldsymbol{i} + \sin\theta\boldsymbol{j}) = -\dot\theta\boldsymbol{e}_R, \tag{2.26}$$

$$\frac{\mathrm{d}\boldsymbol{e}_z}{\mathrm{d}t} = \boldsymbol{0}. \tag{2.27}$$

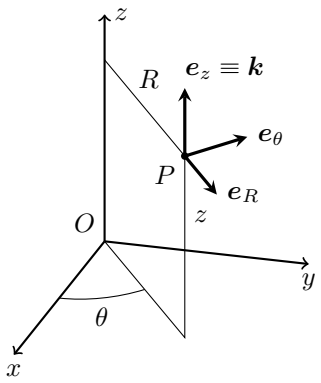

Figure 2.4: Definition of cylindrical coordinates.

Using these relations it is easy to obtain the expressions for position, velocity and acceleration in cylindrical coordinates:

$$\boldsymbol{r}_0^P = R\boldsymbol{e}_R + z\boldsymbol{e}_z, \tag{2.28}$$

$$\boldsymbol{v}_0^P = \dot R\boldsymbol{e}_R + R\dot\theta\boldsymbol{e}_\theta + \dot z, \boldsymbol{e}_z \tag{2.29}$$

$$\boldsymbol{a}_0^P = (\ddot R - R\dot\theta^2)\boldsymbol{e}_R + (R\ddot\theta + 2\dot R\dot\theta)\boldsymbol{e}_\theta + \ddot z\boldsymbol{e}_z. \tag{2.30}$$

Observe that $(\ddot R - R\dot\theta^2)$ and $(R\ddot\theta + 2\dot R\dot\theta)$ cannot be identified with the tangential and normal acceleration components, in general.

2.6.3 Spherical coordinates

Spherical coordinates are often used in some three-dimensional problems. The position of a point P is given by the three coordinates $\{r, \theta, \phi\}$ as shown in Fig. 2.5:

1. r is the radius. It coincides with the distance from O to P: $r = |\boldsymbol{r}_0^P|$.
2. θ is the azimuthal angle as in cylindrical coordinates.
3. ϕ is the latitude angle, formed between the Oxy plane and the $\boldsymbol{r}_0^P$ vector.

Note that spherical coordinates are not well defined if $r = 0$ or if $\phi = \pi/2$. The coordinate transformation relations for spherical coordinates are:

$$\begin{cases} x = r\cos\phi\cos\theta \\ y = r\cos\phi\sin\theta \\ z = r\sin\phi \end{cases} \qquad \begin{cases} r = \sqrt{x^2+y^2+z^2} \\ \theta = \arctan(y/x) \\ \phi = \arctan(z/\sqrt{x^2+y^2}) \end{cases} \tag{2.31}$$

With the help of Fig. 2.5, the unit vectors of $B_C(P)$ in spherical coordinates are:

$$\begin{cases} \boldsymbol{e}_r = \cos\phi\cos\theta\boldsymbol{i} + \cos\phi\sin\theta\boldsymbol{j} + \sin\phi\boldsymbol{k}, \\ \boldsymbol{e}_\theta = -\sin\theta\boldsymbol{i} + \cos\theta\boldsymbol{j}, \\ \boldsymbol{e}_\phi = -\sin\phi\cos\theta\boldsymbol{i} - \sin\phi\sin\theta\boldsymbol{j} + \cos\phi\boldsymbol{k}, \end{cases} \tag{2.32}$$

and their time derivatives are:

$$\frac{\mathrm{d}\boldsymbol{e}_r}{\mathrm{d}t} = \dot\theta\cos\phi(-\sin\theta\boldsymbol{i} + \cos\theta\boldsymbol{j}) + \dot\phi(-\sin\phi\cos\theta\boldsymbol{i} - \sin\phi\sin\theta\boldsymbol{j} + \cos\phi\boldsymbol{k}) = \dot\theta\cos\phi\boldsymbol{e}_\theta + \dot\phi\boldsymbol{e}_\phi, \tag{2.33}$$

$$\frac{\mathrm{d}\boldsymbol{e}_\theta}{\mathrm{d}t} = -\dot\theta(\cos\theta\boldsymbol{i} + \sin\theta\boldsymbol{j}) = -\dot\theta\cos\phi\boldsymbol{e}_r + \dot\theta\sin\phi\boldsymbol{e}_\phi, \tag{2.34}$$

$$\frac{\mathrm{d}\boldsymbol{e}_\phi}{\mathrm{d}t} = \dot\theta\sin\phi(\sin\theta\boldsymbol{i} - \cos\theta\boldsymbol{j}) + \dot\phi(-\cos\phi\cos\theta\boldsymbol{i} - \cos\phi\sin\theta\boldsymbol{j} - \sin\phi\boldsymbol{k}) = -\dot\theta\sin\phi\boldsymbol{e}_\theta - \dot\phi\boldsymbol{e}_r. \tag{2.35}$$

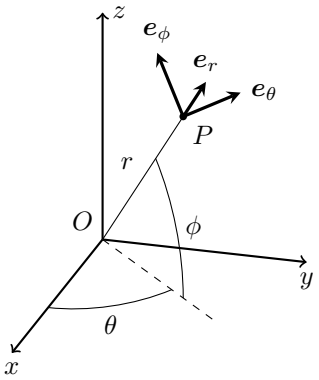

Figure 2.5: Definition of spherical coordinates.

Using these relations it is easy to obtain the expressions for position, velocity and acceleration in spherical coordinates:

$$\boldsymbol{r}_0^P = r\boldsymbol{e}_r, \tag{2.36}$$

$$\boldsymbol{v}_0^P = \dot r\boldsymbol{e}_r + r\dot\theta\cos\phi\boldsymbol{e}_\theta + r\dot\phi\boldsymbol{e}_\phi, \tag{2.37}$$

$$\boldsymbol{a}_0^P = (\ddot r - r\dot\theta^2\cos^2\phi - r\dot\phi^2)\boldsymbol{e}_r \tag{2.38}$$

$$+ (2\dot r\dot\theta\cos\phi + r\ddot\theta\cos\phi - 2r\dot\theta\dot\phi\sin\phi)\boldsymbol{e}_\theta \tag{2.39}$$

$$+ (2\dot r\dot\phi + r\ddot\phi + r\dot\theta^2\cos\phi\sin\phi)\boldsymbol{e}_\phi. \tag{2.40}$$

2.7 Degrees of freedom and constraints

The term *coordinate* is also used in a broader sense. The effective degrees of freedom of a mechanical system refer to the number of independent parameters, or *generalized coordinates*, that must be specified to fully determine the position and orientation of all its parts. If the object under study is a point particle, only three coordinates are required to determine its position, so it has 3 degrees of freedom. On the other hand, if the object is a rigid body, such as an aircraft, three position coordinates of one of its points and three angular coordinates are required to completely specify its position and orientation in space, as we will see in Chapter 9.

In many situations the number of independent coordinates will be reduced below this number, either because the number of spatial dimensions is reduced or because there are relationships specified among the spatial coordinates. When setting up problems for solution it is useful to think of these relationships as *constraints*. For example, if a point mass is constrained to move in a plane (two dimensions) the number of spatial coordinates necessary to describe its motion goes down to two. If instead of being a point mass, this body has extended dimensions, such as a flat plate confined to a plane, it requires three coordinates to specify its position and orientation: two position coordinates for one of its points and one angular coordinate. If a point particle is confined to move along a curve in either two or three dimensions, such as a bead sliding on a wire, the number of independent coordinates necessary to describe its motion is one. Another source of constraints on the motion of particles and bodies are the connections that may exist between them. For example, two particles connected by a cable or string passing over a pulley are constrained to move in equal and opposite directions. More complex arrangements are possible and can be analyzed using these ideas.

In general, we do not need to restrict ourselves to the classical Cartesian, cylindrical or spherical coordinates to define the position of a point and angular coordinates to define orientation of a body. It is often possible to use other valid sets of parameters $\{q_i\}$, such as distances, areas, etc. to fully determine the position (and orientation) of the system. When we choose this path to solve the problem, we refer to these parameters as generalized coordinates[13].

[13]The advanced student might want to read about Analyitical Mechanics, which plays with generalized coordinates to reduce the equations of motion to the simplest form in a systematic way. The concepts of Analytical Mechanics are beyond the scope of this course.

3 Relative motion

When we drive a car or travel on an airplane, the reference frame for our observations is moving with respect to the reference frame of an observer on the ground. Clearly, the velocities and accelerations that we measure in our moving reference frame are different from those seen from the ground. Quite often one can easily calculate velocities and accelerations in one of this two reference frames but needs them in the other one. Indeed, as we shall see in Chapter 4, Newton laws only apply in inertial reference frames, and any accelerating or rotating reference frame are non-inertial. If we wish to use such observations to formulate Newton's Laws, we need to first "translate" our problem from the moving, non-inertial reference frame to the fixed, inertial reference frame (or include so-called *inertial or fictitious forces* to take this "non-inertiality" into account, as we shall see in the next Chapters). In this Chapter we develop a procedure to correlate observations of position, velocity, and acceleration from fixed and moving reference frames (or, more generally, between any two reference frames).

It is standard terminology to refer to any quantity that is measured relative to a fixed reference frame as *absolute*, whereas quantities measured with respect to any moving reference frame are *relative*. Of course, these naming is not universal, as it depend on what we choose to call our "fixed reference frame."

3.1 Derivative of a vector in a moving reference frame

Consider a vector $\boldsymbol{A}(t)$ that changes in magnitude and direction with time, a fixed reference S_0 and a moving reference frame S_1, with the same origin as S_0. Initially, S_1 and S_0 coincide, i.e., their coordinated axes are aligned and the vector bases coincide. The reference frame S_1 rotates with an angular velocity $\boldsymbol{\omega}_{10}$ with respect to S_0. As a consequence, the vectors $\boldsymbol{i}_1$, $\boldsymbol{j}_1$ and $\boldsymbol{k}_1$ of the basis of S_1 change in direction with time as seen from S_0. A sketch of this is provided in Fig. 3.1.

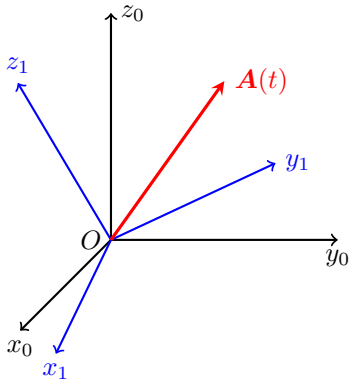

Figure 3.1: Two reference frames S_0 and S_1 rotating with respect to each other, and a time-varying vector $\boldsymbol{A}(t)$.

Vector $\boldsymbol{A}$ may be described in terms of its components with respect to either set of unit vectors,

$$\boldsymbol{A} = A_{x0}\boldsymbol{i}_0 + A_{y0}\boldsymbol{j}_0 + A_{z0}\boldsymbol{k}_0 = A_{x1}\boldsymbol{i}_1 + A_{y1}\boldsymbol{j}_1 + A_{z1}\boldsymbol{k}_1. \tag{3.1}$$

Taking the time derivative of this expression **in the fixed reference frame** S_0, we obtain

$$\begin{aligned}\left.\frac{\mathrm{d}\boldsymbol{A}}{\mathrm{d}t}\right|_0 &= \frac{\mathrm{d}A_{x0}}{\mathrm{d}t}\boldsymbol{i}_0 + \frac{\mathrm{d}A_{y0}}{\mathrm{d}t}\boldsymbol{j}_0 + \frac{\mathrm{d}A_{z0}}{\mathrm{d}t}\boldsymbol{k}_0 \\ &= \frac{\mathrm{d}A_{x1}}{\mathrm{d}t}\boldsymbol{i}_1 + \frac{\mathrm{d}A_{y1}}{\mathrm{d}t}\boldsymbol{j}_1 + \frac{\mathrm{d}A_{z1}}{\mathrm{d}t}\boldsymbol{k}_1 + A_{x1}\left.\frac{\mathrm{d}\boldsymbol{i}_1}{\mathrm{d}t}\right|_0 + A_{y1}\left.\frac{\mathrm{d}\boldsymbol{j}_1}{\mathrm{d}t}\right|_0 + A_{z1}\left.\frac{\mathrm{d}\boldsymbol{k}_1}{\mathrm{d}t}\right|_0,\end{aligned} \tag{3.2}$$

where we have taken into account that, as mentioned above, the vectors of the basis of S_1 are not constant in time *as seen from* S_0. Consequently, when taking the derivative of the right hand side of Eq. (3.1), we must apply the chain rule and keep the time derivatives of $\boldsymbol{i}_1$, $\boldsymbol{j}_1$, $\boldsymbol{k}_1$. Since these

vectors are unit vectors, they do not change its magnitude, only in direction. Following §1.7.2, we may write

$$\left.\frac{\mathrm{d}\boldsymbol{i}_1}{\mathrm{d}t}\right|_0 = \boldsymbol{\omega}_{10} \times \boldsymbol{i}_1, \tag{3.3}$$

$$\left.\frac{\mathrm{d}\boldsymbol{j}_1}{\mathrm{d}t}\right|_0 = \boldsymbol{\omega}_{10} \times \boldsymbol{j}_1, \tag{3.4}$$

$$\left.\frac{\mathrm{d}\boldsymbol{k}_1}{\mathrm{d}t}\right|_0 = \boldsymbol{\omega}_{10} \times \boldsymbol{k}_1. \tag{3.5}$$

We can now write the last three terms in the right hand side of Eq. (3.2) in a more compact form:

$$A_{x1}\left.\frac{\mathrm{d}\boldsymbol{i}_1}{\mathrm{d}t}\right|_0 + A_{y1}\left.\frac{\mathrm{d}\boldsymbol{j}_1}{\mathrm{d}t}\right|_0 + A_{z1}\left.\frac{\mathrm{d}\boldsymbol{k}_1}{\mathrm{d}t}\right|_0 = \boldsymbol{\omega}_{10} \times (A_{x1}\boldsymbol{i}_1 + A_{y1}\boldsymbol{j}_1 + A_{z1}\boldsymbol{k}_1) = \boldsymbol{\omega}_{10} \times \boldsymbol{A}. \tag{3.6}$$

On the other hand, the first three terms in the right hand side of Eq. (3.2) can be seem to correspond to the time derivative of $\boldsymbol{A}$, *as seen from* S_1: with respect to their own reference frame, the vectors $\boldsymbol{i}_1$, $\boldsymbol{j}_1$, $\boldsymbol{k}_1$ are constant in time. Thus, we may write

$$\frac{\mathrm{d}A_{x1}}{\mathrm{d}t}\boldsymbol{i}_1 + \frac{\mathrm{d}A_{y1}}{\mathrm{d}t}\boldsymbol{j}_1 + \frac{\mathrm{d}A_{z1}}{\mathrm{d}t}\boldsymbol{k}_1 = \left.\frac{\mathrm{d}\boldsymbol{A}}{\mathrm{d}t}\right|_1 \tag{3.7}$$

Thus, we can rewrite Eq. (3.2) as

$$\left.\frac{\mathrm{d}\boldsymbol{A}}{\mathrm{d}t}\right|_0 = \left.\frac{\mathrm{d}\boldsymbol{A}}{\mathrm{d}t}\right|_1 + \boldsymbol{\omega}_{10} \times \boldsymbol{A}. \tag{3.8}$$

Equation (3.8), which is known as *Coriolis theorem*, states that the time derivative of a vector in an absolute reference frame can be computed as the time derivative of the vector in a moving reference frame plus an additional term that takes into account the rotation of the moving frame with respect to the fixed one.

Coriolis theorem shows that when taking time derivatives of *vectors*, the reference frame from which we look at the problem matters. That is why denoting the reference frame in which the time derivative of a vector is computed is essential to avoid ambiguity. Derivatives of scalar quantities, on the contrary, are the same regardless of the reference frame used when differentiating.

While we have derived this result for the derivative of a generic a physical vector $\boldsymbol{A}$ in a reference frame S_1 that rotates—but does not translate—with respect to a reference frame S_0, the general case of rotation plus translation leads to exactly the same expression. The reason is that the translation of reference frame S_1 does not alter the orientation of its vector basis $\{\boldsymbol{i}_1, \boldsymbol{j}_1, \boldsymbol{k}_1\}$ with respect to the fixed reference frame S_0.

As a particular case, the time derivative of the angular velocity vector $\boldsymbol{\omega}_{10}$ (i.e., the *angular acceleration vector* $\boldsymbol{\alpha}_{10}$) coincides in S_0 and S_1:

$$\boldsymbol{\alpha}_{10} \equiv \left.\frac{\mathrm{d}\boldsymbol{\omega}_{10}}{\mathrm{d}t}\right|_0 = \left.\frac{\mathrm{d}\boldsymbol{\omega}_{10}}{\mathrm{d}t}\right|_1 + \cancel{\boldsymbol{\omega}_{10} \times \boldsymbol{\omega}_{10}} = \left.\frac{\mathrm{d}\boldsymbol{\omega}_{10}}{\mathrm{d}t}\right|_1. \tag{3.9}$$

Observe that the angular acceleration vector may be, in general, not parallel to the angular velocity vector.

Whenever we work with more than two reference frames, more subindices will be used to clarify what angular velocity and acceleration vectors we refer to: in general, $\boldsymbol{\omega}_{lm}$ denotes the angular velocity vector of reference frame l as seen from reference frame m. Observe that $\boldsymbol{\omega}_{lm} = -\boldsymbol{\omega}_{ml}$ and that $\boldsymbol{\alpha}_{lm} = -\boldsymbol{\alpha}_{ml}$. Also, note that the composition angular velocities is additive. For example, if we consider three reference frames,

$$\boldsymbol{\omega}_{20} = \boldsymbol{\omega}_{21} + \boldsymbol{\omega}_{10} \tag{3.10}$$

but composition of angular accelerations is *not*. Indeed, if we take the time derivative of the equation above we obtain

$$\boldsymbol{\alpha}_{20} = \left.\frac{\mathrm{d}\boldsymbol{\omega}_{20}}{\mathrm{d}t}\right|_0 = \left.\frac{\mathrm{d}\boldsymbol{\omega}_{21}}{\mathrm{d}t}\right|_0 + \left.\frac{\mathrm{d}\boldsymbol{\omega}_{10}}{\mathrm{d}t}\right|_0 = \left.\frac{\mathrm{d}\boldsymbol{\omega}_{21}}{\mathrm{d}t}\right|_1 + \boldsymbol{\omega}_{10} \times \boldsymbol{\omega}_{21} + \boldsymbol{\alpha}_{10} = \tag{3.11}$$

$$= \boldsymbol{\alpha}_{21} + \boldsymbol{\omega}_{10} \times \boldsymbol{\omega}_{21} + \boldsymbol{\alpha}_{10} \tag{3.12}$$

In summary, we obtain the following composition formulas:

$$\boldsymbol{\omega}_{10} = -\boldsymbol{\omega}_{01}; \quad \boldsymbol{\alpha}_{10} = -\boldsymbol{\alpha}_{01}, \tag{3.13}$$

$$\boldsymbol{\omega}_{20} = \boldsymbol{\omega}_{21} + \boldsymbol{\omega}_{10}, \tag{3.14}$$

$$\boldsymbol{\alpha}_{20} = \boldsymbol{\alpha}_{21} + \boldsymbol{\omega}_{10} \times \boldsymbol{\omega}_{21} + \boldsymbol{\alpha}_{10}. \tag{3.15}$$

It is important to remember that, when operating with the equations above in component form, one must first express each intervening vector in the same vector basis.

3.2 Velocity and acceleration in a moving reference frame

Figure 3.2 depicts a general situation in which a point P is being observed from a moving reference frame $S_1 : \{A; B_1\}$ and from a fixed reference frame $S_0 : \{O; B_0\}$. In the following, it is assumed that the position, orientation and motion of S_1 with respect to S_0 are fully known. This is equivalent to knowing:

1. The linear motion of the origin A, i.e., its position vector $\boldsymbol{r}_0^A$, velocity vector $\boldsymbol{v}_0^A$ and acceleration vector $\boldsymbol{a}_0^A$.
2. The angular motion of the basis B_1, i.e., the rotation matrix $[{}_0R_1]$, the angular velocity vector $\boldsymbol{\omega}_{10}$ and acceleration vector $\boldsymbol{\alpha}_{10}$.

It is apparent from Fig. 3.2 that one can express the *absolute position vector* $\boldsymbol{r}_0^P$ as a vector sum, adding together the absolute position vector of A and the *relative position vector* $\boldsymbol{r}_1^P$:

$$\boldsymbol{r}_0^P = \boldsymbol{OP} = \boldsymbol{OA} + \boldsymbol{AP} = \boldsymbol{r}_0^A + \boldsymbol{r}_1^P. \tag{3.16}$$

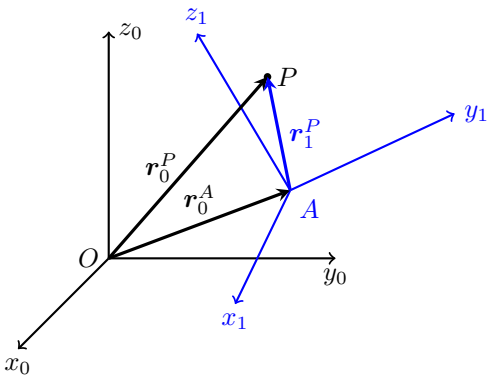

Figure 3.2: Fixed and moving reference frames for observing the position of a point P.

Both $\boldsymbol{r}_0^P$ and $\boldsymbol{r}_1^P$ are the vectors that describe position as seen from a specific reference frame.

The velocity of P with respect to the fixed reference frame S_0 (called the *absolute velocity vector*) is, by definition, the time derivative of $\boldsymbol{r}_0^P$ in S_0. Taking the time derivative of Eq. (3.16):

$$\boldsymbol{v}_0^P \equiv \left.\frac{\mathrm{d}\boldsymbol{r}_0^P}{\mathrm{d}t}\right|_0 = \left.\frac{\mathrm{d}\boldsymbol{r}_0^A}{\mathrm{d}t}\right|_0 + \left.\frac{\mathrm{d}\boldsymbol{r}_1^P}{\mathrm{d}t}\right|_0 \tag{3.17}$$

Using Coriolis theorem, Eq. (3.8), the derivative of $\boldsymbol{r}_1^P$ can be written as

$$\left.\frac{\mathrm{d}\boldsymbol{r}_1^P}{\mathrm{d}t}\right|_0 = \left.\frac{\mathrm{d}\boldsymbol{r}_1^P}{\mathrm{d}t}\right|_1 + \boldsymbol{\omega}_{10} \times \boldsymbol{r}_1^P, \tag{3.18}$$

where we can identify the time derivative of the relative position vector $\boldsymbol{r}_1^P$ as seen from the moving frame S_1 as the *relative velocity vector*:

$$\left.\frac{\mathrm{d}\boldsymbol{r}_1^P}{\mathrm{d}t}\right|_1 = \boldsymbol{v}_1^P \tag{3.19}$$

Using this result we may write $\boldsymbol{v}_0^P$ as

$$\boldsymbol{v}_0^P = \boldsymbol{v}_0^A + \boldsymbol{v}_1^P + \boldsymbol{\omega}_{10} \times \boldsymbol{r}_1^P, \tag{3.20}$$

i.e., the velocity of the origin A, plus the relative velocity $\boldsymbol{v}_1^P$, plus the term $\boldsymbol{\omega}_{10} \times \boldsymbol{r}_1^P$ associated to the rotation of S_1.

Taking time derivatives once again we obtain an expression of the absolute acceleration of point P, $\boldsymbol{a}_0^P$:

$$\boldsymbol{a}_0^P \equiv \left.\frac{\mathrm{d}\boldsymbol{v}_0^P}{\mathrm{d}t}\right|_0 = \left.\frac{\mathrm{d}\boldsymbol{v}_0^A}{\mathrm{d}t}\right|_0 + \left.\frac{\mathrm{d}\boldsymbol{v}_1^P}{\mathrm{d}t}\right|_0 + \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_0 \left(\boldsymbol{\omega}_{10} \times \boldsymbol{r}_1^P\right). \tag{3.21}$$

Using again Coriolis theorem (3.8), we may write:

$$\left.\frac{\mathrm{d}\boldsymbol{v}_1^P}{\mathrm{d}t}\right|_0 = \left.\frac{\mathrm{d}\boldsymbol{v}_1^P}{\mathrm{d}t}\right|_1 + \boldsymbol{\omega}_{10} \times \boldsymbol{v}_1^P, \tag{3.22}$$

$$\left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_0 \left(\boldsymbol{\omega}_{10} \times \boldsymbol{r}_1^P\right) = \boldsymbol{\alpha}_{10} \times \boldsymbol{r}_1^P + \boldsymbol{\omega}_{10} \times \boldsymbol{v}_1^P + \boldsymbol{\omega}_{10} \times \left(\boldsymbol{\omega}_{10} \times \boldsymbol{r}_1^P\right). \tag{3.23}$$

We define the *relative acceleration vector* as the time derivative of the relative velocity as seen from the moving frame S_1:

$$\boldsymbol{a}_1^P = \left.\frac{\mathrm{d}\boldsymbol{v}_1^P}{\mathrm{d}t}\right|_1, \tag{3.24}$$

Finally we may rewrite the absolute acceleration $\boldsymbol{a}_0^P$ as

$$\boldsymbol{a}_0^P = \boldsymbol{a}_0^A + \boldsymbol{a}_1^P + \boldsymbol{\alpha}_{10} \times \boldsymbol{r}_1^P + \boldsymbol{\omega}_{10} \times (\boldsymbol{\omega}_{10} \times \boldsymbol{r}_1^P) + 2\boldsymbol{\omega}_{10} \times \boldsymbol{v}_1^P, \tag{3.25}$$

i.e., the acceleration of the origin A, plus the relative acceleration $\boldsymbol{a}_1^P$, plus a term $\boldsymbol{\alpha}_{10} \times \boldsymbol{r}_1^P$ proportional to the angular acceleration of S_1 (sometimes called the *Euler term*), plus a centripetal term $\boldsymbol{\omega}_{10} \times (\boldsymbol{\omega}_{10} \times \boldsymbol{r}_1^P)$, plus the Coriolis term $2\boldsymbol{\omega}_{10} \times \boldsymbol{v}_1^P$.

In summary, we have found the following formulas for the velocity and acceleration of a point particle

$$\boldsymbol{v}_0^P = \boldsymbol{v}_0^A + \boldsymbol{v}_1^P + \boldsymbol{\omega}_{10} \times \boldsymbol{r}_1^P, \tag{3.26}$$

$$\boldsymbol{a}_0^P = \boldsymbol{a}_0^A + \boldsymbol{a}_1^P + \boldsymbol{\alpha}_{10} \times \boldsymbol{r}_1^P + \boldsymbol{\omega}_{10} \times (\boldsymbol{\omega}_{10} \times \boldsymbol{r}_1^P) + 2\boldsymbol{\omega}_{10} \times \boldsymbol{v}_1^P, \tag{3.27}$$

These expresions deserve some additional remarks:

1. The effect of the translation of the origin A appears only through the terms $\boldsymbol{v}_0^A$ and $\boldsymbol{a}_0^A$. All other terms are due to the relative motion of P with respect to S_1 and/or the angular motion of S_1 with respect to S_0.

2. The direction of $\boldsymbol{\omega}_{10}$ is the direction of the axis of rotation of S_1 with respect to S_0.

3. The term $\boldsymbol{\omega}_{10} \times \boldsymbol{r}_1^P$ is always perpendicular to $\boldsymbol{\omega}_{10}$ and $\boldsymbol{r}_1^P$. Its magnitude is $\omega_1 0 r_{1,\perp}^P$, where $r_{0,\perp}^P$ is the component of $\boldsymbol{r}_1^P$ perpendicular to $\boldsymbol{\omega}_{10}$.

4. A similar statement can be made of the term $\boldsymbol{\alpha}_{10} \times \boldsymbol{r}_1^P$. Since $\boldsymbol{\omega}_{10}$ and $\boldsymbol{\alpha}_{10}$ are, in general, not parallel, $\boldsymbol{\alpha}_{10} \times \boldsymbol{r}_1^P$ can have a component along $\boldsymbol{\omega}_{10}$.

5. The centripetal term has a magnitude $r_{0,\perp}^P \omega_{10}^2$, and always points toward the line defined by the origin and $\boldsymbol{\omega}_{10}$.

6. The factor 2 in the Coriolis term arises from two different effects, as can be seen from the derivation above, which have equal importance. Coriolis acceleration is only present in a rotating reference frame when the particle has relative velocity.

Finally, we close this discussion with the trivial case in which the moving reference frame S_1 is traslating without rotating. In this case, $\boldsymbol{\omega}_{10} = 0$ and $\boldsymbol{\alpha}_{10} = 0$, so the equations above reduce to

$$\boldsymbol{v}_0^P = \boldsymbol{v}_0^A + \boldsymbol{v}_1^P, \tag{3.28}$$

$$\boldsymbol{a}_0^P = \boldsymbol{a}_0^A + \boldsymbol{a}_1^P. \tag{3.29}$$

The motion of the origin and of the relative motion of the point are additive—there are no corrections for direction changes due rotations of the bases. If the moving reference frame is translating at a constant velocity $\boldsymbol{v}_0^A$ then $\boldsymbol{a}_0^A = 0$ (note that this condition requires that the origin A follows a straight path). The second equation then shows that $\boldsymbol{a}_0^P = \boldsymbol{a}_1^P$. so the absolute and relative accelerations coincide in this case.

4 Dynamics of the point particle

If kinematics is the study of the general relations between velocity and acceleration with the coordinates used to represent the problem, dynamics deals with the causes of motion and provides the differential equations that must be integrated to obtain the actual trajectory of the particle. In this Chapter we introduce the key concepts of the dynamics of a point particle, including force, linear momentum, kinetic energy and potential energy. The basic equations of dynamics are the mathematical formulation of Newton's second law. This Chapter also discusses mechanical energy, angular momentum, and the conditions that must be satisfied for either or both of them to be conserved quantities.

4.1 Force, linear momentum, and kinetic energy

A *force* represents the action of one body on another, exerted by contact or through a distance, as in the case of gravitational, electric or magnetic forces. In the case of particles and rigid bodies, to describe a force it is necessary to specify its magnitude, its direction and its line of application. In other words, forces are *sliding vector quantities*. In the international system of units, forces have units of Newtons, N, or kg·m/s^2. In these notes, force vectors are represented as $\boldsymbol{F}$, with a subindex to label the force if necessary.

The *linear momentum* of a particle is also a vector quantity, defined as the product of the mass and the velocity of the particle:

$$\boldsymbol{p}_0 = m\boldsymbol{v}_0^P. \tag{4.1}$$

Force and linear momentum are the basic quantities that appear in the *dynamics* of point particles.

The *kinetic energy* of a point particle is a scalar quantity defined as

$$T_0 = \frac{1}{2}m(v_0^P)^2. \tag{4.2}$$

Observe that $\boldsymbol{p}_0$ and T_0 depend on the reference frame on which they are defined, whereas $\boldsymbol{F}$ does not. When working with several particles, a superindex is used to indicate which particle the magnitudes refer to: $\boldsymbol{p}_0^P$, T_0^P.

4.2 Newton's laws

The laws of Newton are the hallmark of classical mechanics. They can be regarded as the axioms that condense the description of the dynamics of all particles and bodies. Newton laws only apply in so-called *inertial* reference frames, whose existence is accepted as an axiom. If we have one inertial reference frame, any other reference frame moving at a constant velocity and not rotating with respect to the former is also inertial.

The first Newton's law states:

> *"A body shall continue in a state of rest or of uniform motion in a straight line unless impressed upon by a force."*

Thus, an isolated body in the absence of forces will move in a straight line at constant speed relative to an inertial frame of reference. In fact, this first law can be turned around and be regarded as the definition of inertial reference frames (i.e., a reference frame is inertial if a particle free of forces stays at rest or in uniform motion).

The second Newton's law reads:

> *"The rate of change of momentum is proportional to the impressed force and takes place in the same direction as the force."*

This law defines the magnitude of a force in terms of the rate of change of the product of mass and velocity of the body on which it is applied, i.e., the linear momentum. In this course we only consider systems with constant mass, so we can write

$$\boldsymbol{F} = \left.\frac{\mathrm{d}\boldsymbol{p}_0}{\mathrm{d}t}\right|_0 = m\left.\frac{\mathrm{d}\boldsymbol{v}_0^P}{\mathrm{d}t}\right|_0, \tag{4.3}$$

where $\boldsymbol{F}$ denotes the resultant sum of all the forces acting on P. Newton's first law can also be regarded a special case of the second law when $\boldsymbol{F} = \boldsymbol{0}$. If the sum of the forces acting on a particle gives a null resultant force, the particle remains at rest ($\boldsymbol{p}_0 = \boldsymbol{0}$) or continues to move in a straight line with constant linear momentum ($\boldsymbol{p}_0 = \text{const}$).

The third Newton's law which states:

> *"To every action (force) there is an equal and opposite reaction (force)."*

This last law embodies the conservation of the total momentum of two or more interacting bodies and forbids the existence of unpaired forces in the universe. Incidentally, the third law lies at the foundations of aerospace propulsion, and explains why space vehicles need to expel mass (and momentum with it) to accelerate and change their motion.

4.3 Equations of motion

Consider a particle P of mass m moving freely in an inertial reference frame S_0 under a known resultant force $\boldsymbol{F}$. In general, $\boldsymbol{F}$ may be a vector function of the position of the particle, its velocity and time: $\boldsymbol{F} = \boldsymbol{F}(\boldsymbol{r}_0^P, \boldsymbol{v}_0^P, t)$. A particle moving freely in 3D space has **three degrees of freedom**, which means that three independent variables or coordinates are required to fully specify its position (e.g., the Cartesian coordinates x, y, z). Applying the second law of Newton in the inertial reference frame S_0 frame provides a differential equation of motion:

$$m \left. \frac{\mathrm{d}^2 \boldsymbol{r}_0^P}{\mathrm{d}t^2} \right|_0 = \boldsymbol{F}. \tag{4.4}$$

This is a *vector equation*, which can be expressed as three independent scalar equations by projecting it into three independent directions (i.e., by dot-multiplying it with 3 linearly independent unit vectors). The most obvious choice for these directions is to take the vectors of the basis $\{\boldsymbol{i}, \boldsymbol{j}, \boldsymbol{k}\}$ of the S_0 reference frame. However, this is not the only possibility, and indeed many problems are easier to solve if Eq. (4.4) is projected along other directions. Calling F_x, F_y, F_z the components of $\boldsymbol{F}$ in that basis, Eq. (4.4) can be expressed in two equivalent ways:

1. A system of 3 second-order ordinary differential equations,

$$\frac{\mathrm{d}^2 x}{\mathrm{d}t^2} = F_x/m, \quad \frac{\mathrm{d}^2 y}{\mathrm{d}t^2} = F_y/m, \quad \frac{\mathrm{d}^2 z}{\mathrm{d}t^2} = F_z/m, \tag{4.5}$$

   for the three unknowns $x(t)$, $y(t)$ and $z(t)$.

2. A system of 6 first-order ordinary differential equations

$$\begin{aligned} &\frac{\mathrm{d}\dot{x}}{\mathrm{d}t} = F_x/m; \quad \frac{\mathrm{d}\dot{y}}{\mathrm{d}t} = F_y/m; \quad \frac{\mathrm{d}\dot{z}}{\mathrm{d}t} = F_z/m; \\ &\frac{\mathrm{d}x}{\mathrm{d}t} = \dot{x}; \qquad \frac{\mathrm{d}y}{\mathrm{d}t} = \dot{y}; \qquad \frac{\mathrm{d}z}{\mathrm{d}t} = \dot{z}, \end{aligned} \tag{4.6}$$

   for the six unknowns $x(t)$, $y(t)$, $z(t)$, $\dot{x}(t)$, $\dot{y}(t)$ and $\dot{z}(t)$.

In both cases, the solution of the system gives us the position of the particle as a function of time and *six* arbitrary constants:

$$x = x(t; C_1, C_2, C_3, C_4, C_5, C_6), \tag{4.7}$$
$$y = y(t; C_1, C_2, C_3, C_4, C_5, C_6), \tag{4.8}$$
$$z = z(t; C_1, C_2, C_3, C_4, C_5, C_6). \tag{4.9}$$

The six arbitrary constants must be obtained once the initial position and velocity of the particle are specified, $\boldsymbol{r}_{0,0}^P$ and $\boldsymbol{v}_{0,0}^P$. Using the relations of the kinematics of the problem, it is possible to use $\boldsymbol{r}_{0,0}^P$ and $\boldsymbol{v}_{0,0}^P$ to compute the initial value of the coordinates and their derivatives (in this example, $x_0, y_0, z_0, \dot{x}_0, \dot{y}_0, \dot{z}_0$).

4.4 Power, work, and potential energy

The power exerted by a force $\boldsymbol{F}$ on a particle P as seen from a reference frame S_0 is defined as

$$\dot{W}_0 = \boldsymbol{F} \cdot \boldsymbol{v}_0^P. \tag{4.10}$$

Note that $\dot{W}_0$ can be positive or negative depending on the relative direction of $\boldsymbol{F}$ and $\boldsymbol{v}_0^P$, or even 0 if the force and the velocity are perpendicular (or, obviously, if one of them is zero). Power has units of Watts, W, or kg·m^2/t^3, in the international system. Observe that the time derivative of the kinetic energy of the particle is equal to the power exerted by the resultant force acting on it,

$$\frac{\mathrm{d}T_0}{\mathrm{d}t} = \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{2}m\boldsymbol{v}_0^P \cdot \boldsymbol{v}_0^P\right) = m\boldsymbol{a}_0^P \cdot \boldsymbol{v}_0^P = \boldsymbol{F} \cdot \boldsymbol{v}_0^P = \dot{W}_0, \tag{4.11}$$

where we have used the second law of Newton to relate $m\boldsymbol{a}_0^P$ with $\boldsymbol{F}$.

The work exerted by $\boldsymbol{F}$ from t_1 to t_2 is the time integral of the power,

$$W_0(t_1, t_2) = \int_{t_1}^{t_2} \dot{W}_0 \mathrm{d}t = \int_{t_1}^{t_2} \boldsymbol{F} \cdot \boldsymbol{v}_0^P \mathrm{d}t = \int_{t_1}^{t_2} \boldsymbol{F} \cdot \left.\frac{\mathrm{d}\boldsymbol{r}_0^P}{\mathrm{d}t}\right|_0 = \int_{\boldsymbol{r}_{0,1}^P}^{\boldsymbol{r}_{0,2}^P} \boldsymbol{F} \cdot \mathrm{d}\boldsymbol{r}_0^P. \tag{4.12}$$

Work has energy units, and in the international system the Joule, J, is used, or kg·m^2/t^2. The quantity

$$\delta W_0 = \boldsymbol{F} \cdot \mathrm{d}\boldsymbol{r}_0^P \tag{4.13}$$

is the work done on the particle in a differential time interval dt. The symbol δ has been used instead of d to denote that, in general, δW_0 is *not* an exact differential form: indeed, the integral in Eq. (4.12) is in general a function of the exact path followed by the particle in its trajectory between $\boldsymbol{r}_{0,1}^P$ and $\boldsymbol{r}_{0,2}^P$, and not just of the initial and final points only.

When δW_0 is an exact differential form, the integral of Eq. (4.12) becomes independent of the trajectory followed by the particle and depends *only* on the initial and final points. Then, by fixing $\boldsymbol{r}_{0,1}^P$ at any arbitrary position of choice, we can define a work-related function that depends only on $\boldsymbol{r}_{0,2}^P$. Dropping for brevity the subindex 2, we define the *potential energy* associated to $\boldsymbol{F}$ as

$$V_0(\boldsymbol{r}_0^P) = -\int_{\boldsymbol{r}_{0,1}^P}^{\boldsymbol{r}_0^P} \boldsymbol{F}(\boldsymbol{r}_0^P) \cdot \mathrm{d}\boldsymbol{r}_0^P, \tag{4.14}$$

where the minus sign is customary. We may regard $V_0(\boldsymbol{r}_0^P)$ as a scalar field defined on every point in space. Our choice of $\boldsymbol{r}_{0,1}^P$ is irrelevant, and is associated with the integration constant in the equation above. Whenever $V_0(\boldsymbol{r}_0^P)$ can be defined for a force $\boldsymbol{F}$ we say that $\boldsymbol{F}$ is *conservative*.

In one-dimenaional problems, for δW_0 to be an exact differential form and for $V_0(\boldsymbol{r}_0^P)$ to exist, it is necessary and sufficient that $\boldsymbol{F}$ be only a function of position, $\boldsymbol{F} = \boldsymbol{F}(\boldsymbol{r}_0^P)$. Indeed, if $\boldsymbol{r}_0^P = x\boldsymbol{i}$ and $\boldsymbol{F} = F_x(x)\boldsymbol{i}$, we may write:

$$V_0(x) = -\int_{x_1}^{x} F_x(x)\mathrm{d}x. \tag{4.15}$$

Note that Eq. (4.15) can be inverted to give F_x in terms of the potential energy:

$$F_x(x) = -\frac{\mathrm{d}V_0}{\mathrm{d}x}. \tag{4.16}$$

Plotting $V_0(x)$ is useful to visualize and understand the problem. Thanks to the minus sign in the definition of $V_0(x)$, we may think of the peaks and troughs of this curve as the analogies of mountains and valleys in some terrain. If we dropped our particle P somewhere on this figured terrain, the potential V_0 would exert a force F_x directed downslope, with a magnitude proportional to how steep the curve is. An example visualization is provided in Fig. 4.1.

In two- and three-dimensional problems, however, the condition that $\boldsymbol{F}$ be a function only of position, $\boldsymbol{F} = \boldsymbol{F}(\boldsymbol{r}_0^P)$, is necessary but not sufficient to guarantee that δW_0 is an exact differential

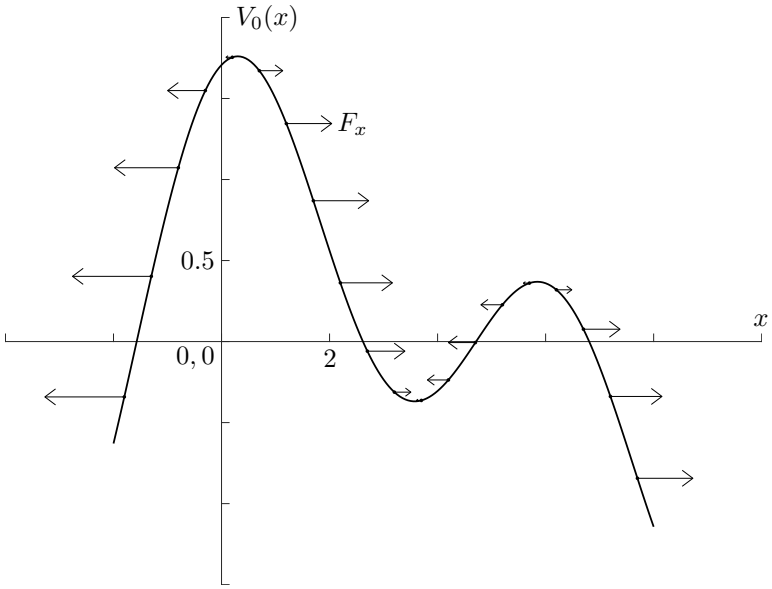

Figure 4.1: Plot of an example one-dimensional potential $V_0(x) = \cos(x) + \cos(x/2 - \pi/4)$, with arrows to indicate the direction and the relative magnitude of the corresponding force at selected points.

form and that $V_0(\boldsymbol{r}_0^P)$ exists (i.e. that $\boldsymbol{F}$ is conservative). In these cases, the additional condition must be satisfied:

$$\oint_\Gamma \boldsymbol{F}(\boldsymbol{r}_0^P) \cdot \mathrm{d}\boldsymbol{r}_0^P = 0; \quad \forall \Gamma, \tag{4.17}$$

i.e., the integral of $\boldsymbol{F} \cdot \mathrm{d}\boldsymbol{r}_0^P$ on *any* closed path Γ must be identically zero, or equivalently, the force $\boldsymbol{F}$ must be *irrotational* (see Eq. (1.76)):

$$\nabla \times \boldsymbol{F} = \boldsymbol{0}. \tag{4.18}$$

When these conditions are satisfied, $\boldsymbol{F}$ is conservative and the inversion of Eq. (4.15) gives

$$F_x = -\frac{\partial V_0}{\partial x}; \quad F_y = -\frac{\partial V_0}{\partial y}; \quad F_z = -\frac{\partial V_0}{\partial z}, \tag{4.19}$$

or, recalling the definition of the gradient of a scalar function, Eq. (1.74),

$$\boldsymbol{F} = -\nabla V_0 = -\frac{\partial V_0}{\partial x}\boldsymbol{i} - \frac{\partial V_0}{\partial y}\boldsymbol{j} - \frac{\partial V_0}{\partial z}\boldsymbol{k}. \tag{4.20}$$

4.5 Conservation of mechanical energy

We have now introduced the necessary ingredients to state one of the fundamental principles of mechanics, namely the conservation of *mechanical energy* E_0 of a particle P, which is defined as the sum of its kinetic energy T_0 plus the potential energy V_0. This principle states that

> *"If all the forces that exert non-zero work on a particle are conservative, then the mechanical energy of the particle is a conserved quantity of motion."*

To derive and discuss this result, consider first the one-dimensional motion of a particle moving along the Ox axis of an inertial reference frame S_0 under a conservative force $\boldsymbol{F} = F(x)\boldsymbol{i}$. Since $m\boldsymbol{a}_0^P = \boldsymbol{F}$ and $\boldsymbol{a}_0^P = (\mathrm{d}^2x/\mathrm{d}t^2)\boldsymbol{i} = \ddot{x}\boldsymbol{i}$, the equation of motion along the direction of $\boldsymbol{i}$ is simply

$$m\ddot{x} = F(x). \tag{4.21}$$

Since this is an ordinary differential equation of second order, we need to integrate it twice to find x as a function of t. Thus the solution will contain two arbitrary constants that may be fixed by specifying the initial values of x and $\dot{x}$, i.e., the *initial conditions* of the problem, x_0 and $\dot{x}_0$.

If we multiply Eq. (4.21) by $\dot{x}$ and integrate between two instants of time t_1 and t_2, we can write

$$\int_{t_1}^{t_2} m\dot{x}\frac{\mathrm{d}\dot{x}}{\mathrm{d}t}\mathrm{d}t - \int_{t_1}^{t_2} F(x)\frac{\mathrm{d}x}{\mathrm{d}t}\mathrm{d}t = 0. \tag{4.22}$$

The first term this equation can be integrated right away and expressed in terms of the kinetic energy T_0 at t_1 and t_2:

$$\int_{t_1}^{t_2} m\dot{x}\frac{\mathrm{d}\dot{x}}{\mathrm{d}t}\mathrm{d}t = T_{0,2} - T_{0,1}. \tag{4.23}$$

The second term equals the difference in potential energy associated to the force field $F(x)$,

$$V_{0,2} - V_{0,1} = -\int_{x_1}^{x_2} F(x)\mathrm{d}x, \tag{4.24}$$

Thus, Eq. (4.22) can be written

$$T_{0,2} + V_{0,2} - T_{0,1} - V_{0,1} = 0, \tag{4.25}$$

or, defining the mechanical energy of the particle as $E_0 = T_{0,1} + V_{0,1}$, and dropping the subindex 2,

$$T_0 + V_0 = E_0 = \text{const}, \tag{4.26}$$

i.e., the mechanical energy of P is conserved. As with T_0, a superindex P is used whenever necessary to clarify that we refer to the mechanical energy of particle P, i.e., E_0^P.

While this equation does not provide the position $\boldsymbol{r}_0^P$ of the particle as a function of time, a great deal of information about the motion of the particle can be obtained from this conservation law alone. If the initial position and velocity are given, we can calculate the value of the constant E_0. Then Eq. (4.26) in the form

$$\frac{1}{2}m\dot{x}^2 = E_0 - V_0(x) \tag{4.27}$$

gives the absolute value of the velocity of the particle as a function of its position x. Since the kinetic energy must obviously be $T_0 \geqslant 0$, the motion is confined to the *allowed regions* where

$$V_0(x) \leqslant E_0, \tag{4.28}$$

and thus the particle cannot enter any *forbidden regions* where $V_0(x) > E_0$. Increasing its mechanical energy may enable the particle to access parts of space previously unavailable. In the example of Fig. 4.2, a particle with mechanical energy $E_0 = E_{0,1}$ would be confined to the valley of the potential (or would exist in one of the external regions of the graph). For $E_0 = E_{0,2}$, the valley becomes connected with the region on the right and the previously confined particle may escape there. For $E_0 = E_{0,3}$, the mechanical energy of the particle is so high that it may access all regions of space. Note, however, that the fact that the particle *may* access a given region does not necessarily mean that it *will* travel there: finding the actual trajectory of the the particle requires solving the equation of dynamics, Eq. (4.21).

4.5.1 Two- and three-dimensional motion

Like in the case of 1D motion, the mechanical energy of a particle is sometimes conserved in 3D space. The condition that must be satisfied is the following: if all the forces *that exert work* on the particle derive from a potential, then the mechanical energy of the particle, kinetic plus potential, is conserved. Note that this statement allows mechanical energy to be conserved even if non-conservative forces act on the particle, *as long as they do no work on it*, e.g. if they are always perpendicular to the velocity.

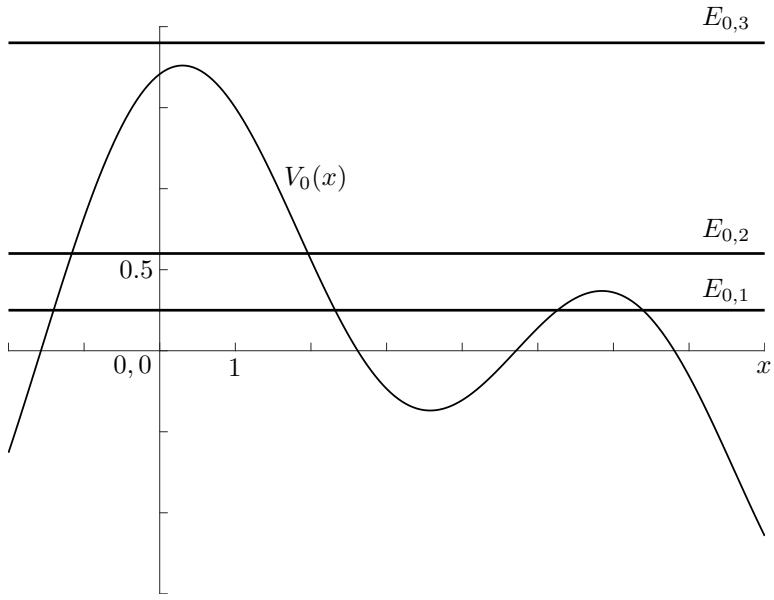

Figure 4.2: Illustration of allowed and forbidden regions for three values of the mechanical energy E_0 in the same one-dimensional potential V_0 as in Fig. 4.1.

Multiplying the second Newton law by $\boldsymbol{v}_0^P$ and integrating between two instants of time we may write

$$\int_{t_1}^{t_2} m\boldsymbol{v}_0^P \cdot \left.\frac{\mathrm{d}\boldsymbol{v}_0^P}{\mathrm{d}t}\right|_0 \mathrm{d}t - \int_{t_1}^{t_2} \boldsymbol{F}\cdot\boldsymbol{v}_0^P \mathrm{d}t = 0. \tag{4.29}$$

The integrand in the first term side equals the differential of the kinetic energy, $\mathrm{d}T_0$, so this integral is again $T_{0,2} - T_{0,1}$. Regarding the right hand side, when $\boldsymbol{F}$ is conservative, i.e., it derives from a potential as $\boldsymbol{F} = -\nabla V_0$, and we can write the second integral as

$$\begin{aligned} -\int_{t_1}^{t_2} \boldsymbol{F}\cdot\boldsymbol{v}_0^P \mathrm{d}t = \int_{\boldsymbol{r}_{0,1}^P}^{\boldsymbol{r}_{0,2}^P} \nabla V_0 \cdot \mathrm{d}\boldsymbol{r}_0^P = \int_{\boldsymbol{r}_{0,1}^P}^{\boldsymbol{r}_{0,2}^P} \left(\frac{\partial V_0}{\partial x}\boldsymbol{i} + \frac{\partial V_0}{\partial y}\boldsymbol{j} + \frac{\partial V_0}{\partial z}\boldsymbol{k}\right)\cdot(\mathrm{d}x\boldsymbol{i} + \mathrm{d}y\boldsymbol{j} + \mathrm{d}z\boldsymbol{k}) = \\ = \int_{\boldsymbol{r}_{0,1}^P}^{\boldsymbol{r}_{0,2}^P} \frac{\partial V_0}{\partial x}\mathrm{d}x + \frac{\partial V_0}{\partial y}\mathrm{d}y + \frac{\partial V_0}{\partial z}\mathrm{d}z = V_{0,2} - V_{0,1}, \end{aligned} \tag{4.30}$$

and we reach the same result as in the one-dimensional case,

$$T_0 + V_0 = E_0. \tag{4.31}$$

4.5.2 Final comments

The conservation of energy was originally derived from Newton's second law for the case where the resultant force $\boldsymbol{F}$ is only a function of position. As we have seen, then the mechanical energy is conserved. It has, however, a much wider range of application. By introducing additional forms of energy (thermal, chemical, electromagnetic, etc), it has been extended far beyond the field of Mechanics, to the point where it is now recognized as one of the most fundamental principles of Physics.

Many so-called *non-conservative* (or dissipative) forces in the macroscopic description of a problem in Mechanics are really conservative on a microscopic scale. For example, when a body penetrates a retarding medium, such as the atmosphere, it experiences a force which in the macroscopic description is velocity-dependent, and therefore non-conservative. However, if we look at the situation on a microscopic scale, we see that what happens is that the body makes a series of collisions with the molecules of the medium. In each collision, energy is conserved, and some of the kinetic energy of the incoming body is transferred to the molecules with which it collides. By means of further

collisions, this energy is gradually distributed among the surrounding molecules. The net result is a drag force on he motion of the incoming body, and to increase the average energy of the molecules in the air. This increased energy appears macroscopically as dissipated heat, and results in a rise in the temperature of the medium.

4.6 Equilibrium

When a particle or system of particles is and remains at rest, we say that it is in an *equilibrium point* or configuration. The study of equilibrium points is known as *statics*, which is a branch of mechanics.

To find the equilibrium configurations of a particle or system of particles, we just need to look in the equations of motion, Eq. (4.4), and impose that all velocities and accelerations be zero. Naturally, this is equivalent to imposing that the sum of all forces acting on each particle be null:

$$\sum \boldsymbol{F} = 0. \tag{4.32}$$

This results in an algebraic system of equations that can be solved for the value of the coordinates of the problem in the equilibrium configurations.

Equilibrium points can be classified into *stable* and *unstable*. Stable equilibrium points are those for which, if a small perturbation takes the system out of equilibrium, it would oscillate about the equilibrium point. This is the case, e.g., of a marble lying on the bottom of a bowl: a small perturbation will set it into motion, but the particle will not go very far: it will oscillate about this equilibrium point. If the oscillations are small enough, it is always possible to linearize the motion about a stable equilibrium point, reducing it to a harmonic oscillator. If the system is dissipative, these oscillations will die with time.

Unstable equilibrium points are those for which, if a small perturbation takes the system out of equilibrium, it will move farther and farther away from it with time. This is the case, e.g., of a ball lying on the peak of a mountain. It is an equilibrium point, since in the absence of perturbations it will remain there indefinitely; however, any small perturbation will topple the ball down the slope of the mountain.

In order to discuss the stability of an equilibrium point, it is necessary to study the behavior of the forces in the neighborhood of it. If the forces tend to restore the position of the system back to the equilibrium configuration, then it is stable. If the forces pull the system further away from it, then it is unstable.

If the system is conservative, then the condition in Eq. (4.32) can also be written $\nabla V_0 = 0$: i.e., the equilibrium configurations are critical points of the potential energy. Then, an equilibrium configuration is stable if and only if it is a local minimum of the potential energy, which can be checked with the second derivatives of V_0.

4.7 The simple pendulum

To illustrate the conservation of mechanical energy and the type of analysis that is possible to carry out with it, we consider a simple pendulum, consisting of a bob of mass m supported by a light rigid rod of length ℓ of negligible mass from a fixed point O of the inertial reference frame S_0 as shown in Fig. 4.3. We can easily solve the problem in polar coordinates (R, θ), choosing the $-Oy$ axis as the direction for which $\theta = 0$. We define the local basis $B_P : \{\boldsymbol{e}_R, \boldsymbol{e}_\theta\}$ as explained in §2.6.2. Since $R = \ell = \text{const}$, the problem has a single degree of freedom, represented by the coordinate θ.

The position, velocity and acceleration vectors of the particle expressed in terms of θ and its derivatives are:

$$\boldsymbol{r}_0^P = \ell \boldsymbol{e}_R, \tag{4.33}$$

$$\boldsymbol{v}_0^P = \ell\dot{\theta}\boldsymbol{e}_\theta, \tag{4.34}$$

$$\boldsymbol{a}_0^P = -\ell\dot{\theta}^2\boldsymbol{e}_R + \ell\ddot{\theta}\boldsymbol{e}_\theta. \tag{4.35}$$

The particle is subject to two forces: the constant force of gravity $\boldsymbol{F}_g = m\boldsymbol{g}$, pointing downward, and the unknown tension force from the rod $\boldsymbol{F}_t$, which is a reaction force (see next Chapters). The force of gravity expressed in the polar vector basis $\{\boldsymbol{e}_R, \boldsymbol{e}_\theta\}$ is

$$\boldsymbol{F}_g = mg\cos\theta\boldsymbol{e}_R - mg\sin\theta\boldsymbol{e}_\theta. \tag{4.36}$$

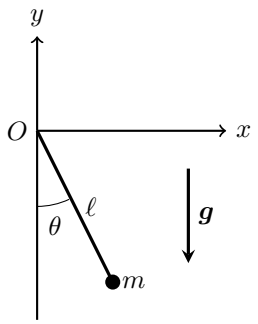

Figure 4.3: A simple pendulum of length ℓ and mass m.

The magnitude of the tension force is unknown until we fully solve the problem, but we know that it must act along the direction of the rod: $\boldsymbol{F}_t = -F_t \boldsymbol{e}_R$.

Projecting the vector equation of Newton's second law in the direction of $\boldsymbol{e}_R$ and $\boldsymbol{e}_\theta$ results in:

$$\boldsymbol{e}_R : -m\ell\dot{\theta}^2 = mg\cos\theta + F_t, \tag{4.37}$$

$$\boldsymbol{e}_\theta : \quad m\ell\ddot{\theta} = -mg\sin\theta \quad \Rightarrow \quad \ddot{\theta} + \frac{g}{\ell}\sin\theta = 0. \tag{4.38}$$

We observe that the component of the force along $\boldsymbol{e}_\theta$ is a restoring force that tries to bring the pendulum back to its equilibrium position, defined by $\theta = 0$.

Multiplying this last expression by $\dot{\theta}$ and integrating we obtain

$$\frac{1}{2}\dot{\theta}^2 + \frac{g}{\ell}(1 - \cos\theta) = C, \tag{4.39}$$

where we have chosen the integration constants as 1 and C. We note that the mass m does not appear in this expression. The same result can be obtained directly by noting that the tension force exerts no work on the particle, as it is always perpendicular to its velocity ($\boldsymbol{F}_T \perp \boldsymbol{v}$), and that the gravity force is uniform everywhere and derives from a potential form. Dropping the indices P and 0 in the following since there is only one particle and one reference frame S_0 we can write:

$$V_0(\theta) = mg(y + \ell) = mg\ell(1 - \cos\theta), \tag{4.40}$$

where we have taken $y = -\ell$ as our origin for the potential. Thus, all the forces that exert any work on the particle (i.e. gravity in this case) are conservative, and *the system conserves its mechanical energy*. Noting that the kinetic energy of the particle is

$$T_0 = \frac{1}{2}m(\ell\dot{\theta})^2, \tag{4.41}$$

we may write this conservation law as

$$T_0 + V_0 = E_0 \Rightarrow \frac{1}{2}m(\ell\dot{\theta})^2 + mg\ell(1 - \cos\theta) = E_0, \tag{4.42}$$

which is identical to Eq. (4.39) except for a constant multiplying factor $m\ell^2$.

The potential $V_0(\theta)$ has been plotted in Fig. 4.4. Notice that the points $\theta = \pi + 2\pi k$ can be easily identified with the maxima in this diagram, where $V_0 = 2mg\ell$, and the points $\theta = 0 + 2\pi k$ with the minima. To discuss the classes of motion that the pendulum can have, consider that the bob starts at its the equilibrium position ($\theta = 0$) with a velocity $v^P_{0,0}$ toward the right. We may wonder what kinds of motion are possible for different values of $v^P_{0,0}$. Since we have chosen the arbitrary constant in the potential V_0 so that $V_0(0) = 0$, the total energy is $E_0 = (1/2)m(v^P_{0,0})^2 = T_0$. Now if $E_0 < 2mg\ell$ (e.g. $E_0 = E_{0,1}$ in the figure), the motion will be confined between the two angles $\pm\theta^*$, for which $V_0(\theta^*) = E_0$. These are the *turning points* where the kinetic energy vanishes ($T_0 = 0$), so that the

pendulum bob is momentarily at rest before turning back. The motion is an oscillation of amplitude θ^*. The equation of energy provides an expression for θ^*:

$$\cos\theta^* = 1 - \frac{(v^P_{0,0})^2}{2g\ell}. \tag{4.43}$$

We can compute the period τ of the oscillations by integration during one cycle:

$$\left|\frac{\mathrm{d}\theta}{\mathrm{d}t}\right| = \sqrt{\frac{2E_0 - 2V_0(\theta)}{m\ell^2}} \quad \Rightarrow \quad \tau = 2\int_{-t^*}^{t^*} \mathrm{d}t = 2\int_{-\theta^*}^{\theta^*} \sqrt{\frac{m\ell^2}{2E_0 - 2V_0(\theta)}}\mathrm{d}\theta, \tag{4.44}$$

where we have taken into account the sign of $\dot\theta$ in the two halves of the cycle, from $-\theta^*$ to θ^* and from θ^* to $-\theta^*$.

On the other hand, if the bob is pushed so hard that $E_0 > 2mg\ell$ (e.g. $E_0 = E_{0,2}$ in the figure), then the kinetic energy never vanishes. When the bob reaches the upward vertical, it still has a non-zero velocity $v^P_{0,1}$, and a kinetic energy

$$T_{0,1} = \frac{1}{2}m(v^P_{0,1})^2 = E_0 - 2mg\ell \tag{4.45}$$

In this case, the motion is a continuous rotation rather than an oscillation.

This energy analysis is useful in general whenever the mechanical energy is conserved in a problem, but it is specially interesting, as we have seen, in problems with only one degree of freedom. In general, the equation of conservation of mechanical energy, when available, may be used to replace one of the scalar equations of motion, as it is a linear combination of them.

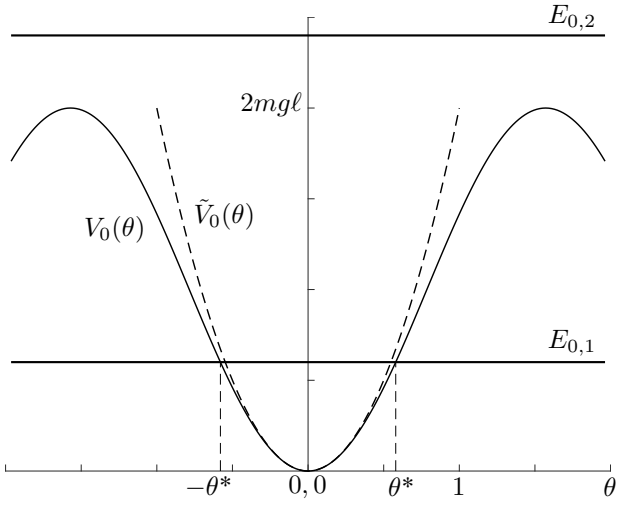

Figure 4.4: Potential energy of the simple pendulum, $V_0(\theta) = mg\ell(1-\cos\theta)$. The potential $\tilde V_0(\theta) = mg\ell\theta^2/2$ of the linearized problem is shown as a dashed line.

Lastly, it is worth noting that if the mechanical energy E is small compared to $2mg\ell$, the oscillations of the pendulum will be small, so that $\theta \ll 1$. Under these circumstances, it is possible to linearize the equations of motion, obtaining an approximated but simplified model of the pendulum. Approximating $\sin\theta \sim \theta$ and keeping only first-order terms in θ, Eq. (4.38) reads

$$\ddot\theta + \frac{g}{\ell}\theta = 0, \tag{4.46}$$

which is the equation of motion of a *harmonic oscillator* as discussed in the next Chapter. The potential energy $\tilde V_0$ associated to this linearized problem is

$$\tilde V_0(\theta) = mg\ell\theta^2/2, \tag{4.47}$$

which is shown in Fig. 4.4 for comparison with the potential energy of the full problem, V_0.

4.8 Angular momentum and torque

The moment of the (linear) momentum $\boldsymbol{p}_0 = m\boldsymbol{v}_0^P$ of a particle P of mass m about an arbitrary point A as seen from a reference frame S_0 is known as the *angular momentum* (about A):

$$\boldsymbol{H}_{A,0} = \boldsymbol{r}_A^P \times m\boldsymbol{v}_0^P, \tag{4.48}$$

where $\boldsymbol{r}_A^P = \boldsymbol{AP} = \boldsymbol{r}_0^P - \boldsymbol{r}_0^A$, as shown in Fig. 4.5. As before, a superindex P is used when we need to disambiguate what particle are we talking about: $\boldsymbol{H}_{A,0}^P$.

Observe that the angular momentum is a **vector quantity** that points in the direction perpendicular to both $\boldsymbol{r}_A^P$ and $\boldsymbol{v}_0^P$. The units in the international system are kg·m^2/s.

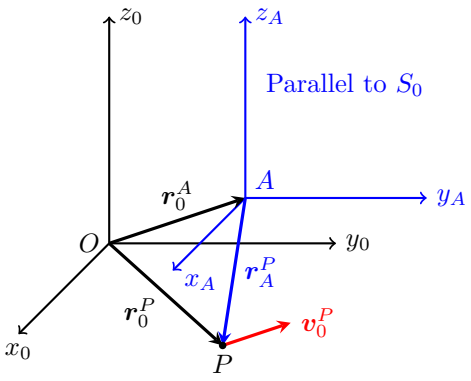

Figure 4.5: Sketch for the discussion of the momentum of a particle P about an arbitrary point A.

Whereas $\boldsymbol{p}_0$ is the fundamental quantity in linear motion, $\boldsymbol{H}_{A,0}$ is the fundamental quantity in the rotational motion about A. Just like Newton's second law gives the evolution of $\boldsymbol{p}_0$ with time, we can find the corresponding differential vector equation for the evolution of the evolution of $\boldsymbol{H}_{A,0}$. Taking derivatives in the definition above we obtain

$$\left.\frac{\mathrm{d}\boldsymbol{H}_{A,0}}{\mathrm{d}t}\right|_0 = (\boldsymbol{r}_0^P - \boldsymbol{r}_0^A) \times m\boldsymbol{a}_0^P + (\boldsymbol{v}_0^P - \boldsymbol{v}_0^A) \times m\boldsymbol{v}_0^P, \tag{4.49}$$

and since the cross product of a vector with itself vanishes we can write:

$$\left.\frac{\mathrm{d}\boldsymbol{H}_{A,0}}{\mathrm{d}t}\right|_0 = (\boldsymbol{r}_0^P - \boldsymbol{r}_0^A) \times m\boldsymbol{a}_0^P - \boldsymbol{v}_0^A \times m\boldsymbol{v}_0^P. \tag{4.50}$$

If the resultant force acting on P is $\boldsymbol{F}$, we define the *moment* of this force about A, or *torque*, as

$$\boldsymbol{M}_A = \boldsymbol{r}_A^P \times \boldsymbol{F} = (\boldsymbol{r}_0^P - \boldsymbol{r}_0^A) \times m\boldsymbol{a}_0^P, \tag{4.51}$$

where the last equation has used the second law of Newton. $\boldsymbol{M}_A$ has units of N·m, or kg·m^2/s^2. We can finally write:

$$\left.\frac{\mathrm{d}\boldsymbol{H}_{A,0}}{\mathrm{d}t}\right|_0 = \boldsymbol{M}_A - \boldsymbol{v}_0^A \times m\boldsymbol{v}_0^P. \tag{4.52}$$

An interesting conclusion can be drawn in the particular case that A is a fixed point in space ($\boldsymbol{v}_0^A = 0$), or if $\boldsymbol{v}_0^A$ and $\boldsymbol{v}_0^P$ are parallel, for which

$$\left.\frac{\mathrm{d}\boldsymbol{H}_{A,0}}{\mathrm{d}t}\right|_0 = \boldsymbol{M}_A, \tag{4.53}$$

which states that the rate of change of angular momentum about a fixed point A is equal to the moment of the resultant force about A acting on the particle. This result is analogous to Newton's second law for linear momentum. In fact, it is a direct consequence of it, and therefore *the angular momentum equation does not provide any additional information about the motion of the particle*;

however, in many circumstances working with angular momentum is easier or more enlightening on the nature of the motion.

Since this is a vector equation, its projection along three independent directions (i.e. the three vectors of *any* basis) provides three scalar equations that we can use. Like with Newton's second law, an obvious choice is the basis $B_0 : \{\boldsymbol{i}, \boldsymbol{j}, \boldsymbol{k}\}$ of the S_0 reference frame, but it is not the only possibility. If the moment of forces along a given direction is always zero, then the component of the angular momentum along that direction will remain *constant*. Again, these equations do not add extra information with respect to Newton's equations.

When A is a moving point, it is sometimes useful to define an auxiliary **parallel, non-rotating** reference frame $S_A : \{A; B_0\}$ with origin on A, as shown in Fig. 4.5. Then, since the transformation from S_0 to S_A does not involve any rotation, $\boldsymbol{v}_0^P = \boldsymbol{v}_0^A + \boldsymbol{v}_A^P$ and $\boldsymbol{a}_0^P = \boldsymbol{a}_0^A + \boldsymbol{a}_A^P$, and we may write

$$\left.\frac{\mathrm{d}\boldsymbol{H}_{A,0}}{\mathrm{d}t}\right|_0 = \boldsymbol{r}_A^P \times m(\boldsymbol{a}_0^A + \boldsymbol{a}_A^P) - \boldsymbol{v}_0^A \times m\boldsymbol{v}_A^P. \tag{4.54}$$

If A is moving with *constant velocity* (i.e., with $\boldsymbol{a}_0^A = 0$), then we can write Eq. (4.52) as

$$\boldsymbol{r}_A^P \times m\boldsymbol{a}_A^P = \left.\frac{\mathrm{d}\boldsymbol{H}_{A,A}}{\mathrm{d}t}\right|_A = \boldsymbol{M_A}, \tag{4.55}$$

where we have introduced $\boldsymbol{H}_{A,A}$, the angular momentum of P about A as seen from S_A. This equation states that the moment of the resultant force is equal to the rate of change of the *relative* angular momentum, if S_A is defined as described above.

As a comment on the quantities introduced in this chapter, observe that the quantities $\boldsymbol{p}_0$, T_0, $\dot{W}_0$, W_0, V_0, E_0 and $\boldsymbol{H}_{A,0}$ depend, in general, on the reference frame S_0 used for the analysis. Moreover, it is possible that V_0 can be defined in one some reference frames and not in others.

4.8.1 Central Force Motion

If the line of action of the resultant force $\boldsymbol{F}$ applied on a particle P is always passes by a fixed point O, the force is called a *central force*, and it can be written as

$$\boldsymbol{F} = F\boldsymbol{e}_r, \tag{4.56}$$

where $\boldsymbol{e}_r = \boldsymbol{r}_0^P / r_0^P$ is a radial unit vector. In this case, the moment of $\boldsymbol{F}$ with respect to O is always zero,

$$\boldsymbol{M}_O = \boldsymbol{r}_0^P \times \boldsymbol{F} = r_0^P \boldsymbol{e}_r \times F\boldsymbol{e}_r \equiv 0. \tag{4.57}$$

and as a consequence, the angular momentum of P about O is a *constant of motion*:

$$\boldsymbol{H}_{O,0} = \boldsymbol{r}_0^P \times m\boldsymbol{v}_0^P = \text{const}. \tag{4.58}$$

Central force problems are an important class in classical mechanics. For instance, the orbital movement of planets about the sun and satellites about the Earth due to gravity is a central force problem, known as the *Kepler's problem* or the *two body problem* (see Chapter 8). The laws that govern the two body problem were first postulated by Johannes Kepler and deduced from observation. Later, it was proven that these laws can be derived naturally from Newton's laws.

Central force problems have the following interesting properties:

1. There is a vector constant of motion $\boldsymbol{H}_{O,0} = \text{const}$, which translates into three scalar conservation laws.
2. Since the constant $\boldsymbol{H}_{O,0}$ is perpendicular to $\boldsymbol{r}_0^P$ and $\boldsymbol{v}_0^P$ by definition, the particle P always stays in the plane perpendicular to $\boldsymbol{H}_{O,0}$ that passes through O. In other words, *the motion is planar*.
3. The motion of central force problems is therefore conveniently described in polar coordinates in the plane of motion, (R, θ). In some cases, a rotating reference frame centered in O that tracks the position of the particle is useful for the analysis.
4. Since $H_{O,0} = R^2\dot{\theta} = \text{const}$, given the initial conditions of the motion of P, the angular rate $\dot{\theta}$ increases as R decreases.
5. The particle P can never reach O unless $\boldsymbol{H}_{O,0} = 0$.

4.9 Dynamics of relative motion

At the beginning of this chapter we established the equations of motion of a free point particle with respect to an inertial reference frame. In these equations, the *absolute* acceleration of the particle with respect to the inertial reference frame has to be used. Sometimes, we need to determine the motion of a particle with respect to a non-inertial reference frame, whose motion is known with respect to an inertial reference frame, e.g. when we study the motion of a spacecraft with respect to a reference frame attached to the Earth, which rotates about itself and about the Sun. In other cases, the solution of a problem may be much easier if we use some auxiliary non-inertial reference frame. In this section we will study the modifications we need to apply to the equations of motion to be able to use them in non-inertial reference frames, using the *relative* acceleration instead of the absolute one.

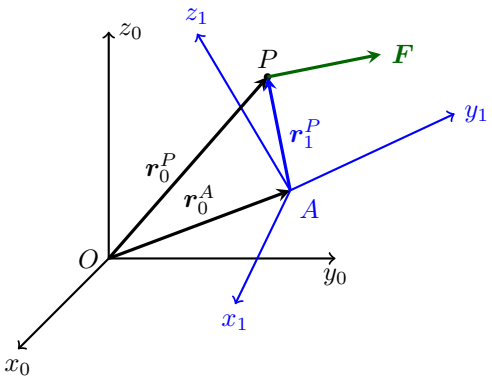

Figure 4.6: Dynamics of a point particle with respect to an inertial reference frame S_0 and a non-inertial reference frame S_1.

We consider an inertial reference frame S_0 and a non inertial reference frame S_1 as shown in Fig. 4.6. We assume that the motion of S_1 with respect to S_0 is known. We also know all the forces acting on the point particle P of mass m, with a resultant force $\boldsymbol{F}$. With respect to the inertial reference frame, we know that the equations of motion are

$$m\boldsymbol{a}_0^P = \boldsymbol{F}, \tag{4.59}$$

where $\boldsymbol{a}_0^P$ is the absolute acceleration of P. Using Eq. (3.27) we may write

$$m\left[\boldsymbol{a}_0^A + \boldsymbol{a}_1^P + \boldsymbol{\alpha}_{10} \times \boldsymbol{r}_1^P + \boldsymbol{\omega}_{10} \times (\boldsymbol{\omega}_{10} \times \boldsymbol{r}_1^P) + 2\boldsymbol{\omega}_{10} \times \boldsymbol{v}_1^P\right] = \boldsymbol{F}, \tag{4.60}$$

and moving all terms except the relative acceleration one to the right hand side,

$$m\boldsymbol{a}_1^P = \boldsymbol{F} - m\boldsymbol{a}_0^A - m\boldsymbol{\alpha}_{10} \times \boldsymbol{r}_1^P - m\boldsymbol{\omega}_{10} \times (\boldsymbol{\omega}_{10} \times \boldsymbol{r}_1^P) - 2m\boldsymbol{\omega}_{10} \times \boldsymbol{v}_1^P. \tag{4.61}$$

If we define the *fictitious or inertia force* as

$$\boldsymbol{F}_i = -m\boldsymbol{a}_0^A - m\boldsymbol{\alpha}_{10} \times \boldsymbol{r}_1^P - m\boldsymbol{\omega}_{10} \times (\boldsymbol{\omega}_{10} \times \boldsymbol{r}_1^P) - 2m\boldsymbol{\omega}_{10} \times \boldsymbol{v}_1^P, \tag{4.62}$$

we may rewrite Eq. (4.61) simply as

$$m\boldsymbol{a}_1^P = \boldsymbol{F} + \boldsymbol{F}_i. \tag{4.63}$$

Obviously Eq. (4.63) provides the same information as Eq. (4.60), since we have merely moved some terms from the right hand side to the left hand side. However, Eq. (4.63) can be given a different *interpretation*. The equation says that an observer standing on S_1 might apply Newton's second law if only he adds some additional fictitious force $\boldsymbol{F}_i$ given by Eq. (4.62) to the force $\boldsymbol{F}$. Recall that all the terms in Eq. (4.62) might be computed as a function of the relative position and velocity of P if we know the motion of S_1 with respect to S_0, i.e. if we know $\boldsymbol{\omega}_{10}$, $\boldsymbol{\alpha}_{10}$ and $\boldsymbol{a}_0^A$.

Each of the terms in $\boldsymbol{F}_i$ receives its own name:

1. The first term, $-m\boldsymbol{a}_0^A$, is the inertial force due to the *acceleration of the origin.*
2. The term $-m\boldsymbol{\alpha}_{10} \times \boldsymbol{r}_1^P$ is sometimes called *Euler's acceleration.*
3. We call *centrifugal force* to the term $-m\boldsymbol{\omega}_{10} \times (\boldsymbol{\omega}_{10} \times \boldsymbol{r}_1^P)$.
4. The *Coriolis force* is $-2m\boldsymbol{\omega}_{10} \times \boldsymbol{v}^P 1$.

In spite of their fictitious character (they result from the derivation above), an observer standing on a non-inertial reference frame might *feel* such forces as if they are very real. However, keep in mind that inertia forces are due solely to the relative motion (with respect to an inertial reference frame) of the reference frame in which we apply the (modified) Newton's second law.

We have all experienced the tendency to go forward when a vehicle suddenly brakes. It is very common to say that the inertia force (i.e., the term $-m\boldsymbol{a}_0^A$ in Eq. (4.62)) pushes us forward. Such force does not exist, and for an observer on the ground it is clear that as the vehicle decelerates due to the friction of the wheels with the ground, the passenger mass tends to continue its forward motion, without any new force coming into existence.

A similar phenomenon takes place when a vehicle turns abruptly. In this case, the vehicle receives a transverse friction force from the road that accelerates it towards the center of curvature of its trajectory (i.e., it is a normal acceleration). As the mass of the passenger tries to continue in rectilinear uniform motion, the passenger then feels being 'pushed outward.' Only when the vehicle transmits the necessary force to the passenger through his seat and his feet does he adapt his trajectory to the trajectory of the car.

In addition to the outward centrifugal force, a kid in an accelerating merry-go-round would feel a radial force pointing backward from the direction of travel due to the term $\boldsymbol{\alpha}_{10} \times \boldsymbol{r}_1^P$.

Finally, if a kid in the outer part of that merry-go-round wanted to pass a ball to a kid in the inner part, he would be surprised to see the ball turning away from its expected rectilinear trajectory, toward the direction of travel, due to the Coriolis force. The same force is used to explain the rotation direction of hurricanes in the Northern and Southern hemispheres, or why a cannonball shot toward the East lands farther than a cannonball shot otherwise equally toward the West.

In summary, we can always analyze the motion of a particle in two ways. Firstly, using the original Newton's second law in an inertial reference frame. Second, modifying Newton's second law by adding fictitious forces of inertia and working in a non-inertial reference frame.

5 The harmonic oscillator

From the classical spring-mass-damper system to the linerized aeroelastic oscillations of an airfoil or the vibrations of an aircraft structure, many mechanical systems have an equation of motion that can be reduced to the form $\ddot{\xi} + 2\gamma\dot{\xi} + \omega_0^2\xi = f(t)$, where ξ is a generalized coordinate, γ and ω_0 some constants, and $f(t)$ a forcing term. This is the most general expression of the equation of a *harmonic oscillator*. Due to their relevance in aerospace engineering, it is worth studying and understanding them in detail. As we will be studying the motion of a single particle in a single reference frame S_0, the super- and subindices P and 0 used in the previous chapter are dropped for many of the quantities to simplify the notation.

5.1 Undamped motion

Consider a point particle of a mass m attached to the origin by a spring of elastic constant k and zero length at rest. The force on the particle in this system is

$$F(x) = -kx. \tag{5.1}$$

It is immediate to see that this force derives from a potential

$$V(x) = \frac{1}{2}kx^2, \tag{5.2}$$

which corresponds to a parabola, shown in Fig. 5.1. Thus for $k > 0$ and *any* energy $E > 0$, there are two points for which $V(x) = E$, namely

$$x = \pm a = \pm\sqrt{2E/k}. \tag{5.3}$$

The motion is a *harmonic oscillation* between these two points.

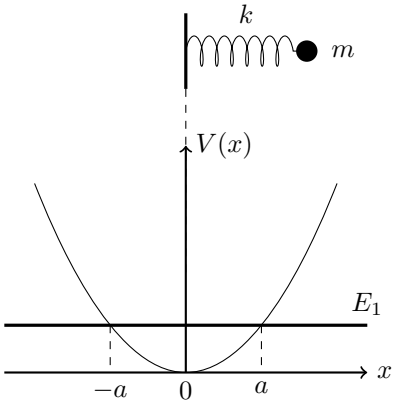

Figure 5.1: Potential energy of the harmonic oscillator.

The equation of motion of the harmonic oscillator is

$$m\ddot{x} + kx = 0. \tag{5.4}$$

In this particular case, full integration of this differential equation is straight forward. Equation (5.4) is a homogeneous second order linear differential equation with constant coefficients. Such equations have the important property that their solutions satisfy the superposition principle: if $x_1(t)$ and $x_2(t)$ are solutions, then so is any linear combination of them, i.e.,

$$x(t) = a_1x_1(t) + a_2x_2(t), \tag{5.5}$$

for any two a_1 and a_2. Moreover, if x_1 and x_2 are *linearly independent solutions* (that is, unless one is a multiple of the other), then Eq. (5.5) is actually *the* general solution, as a linear second order

equation must contain just two arbitrary constants of integration. So, to find the general solution of the problem, all we have to do is to find *any two independent solutions* $x_1(t)$ and $x_2(t)$.

In terms of the *angular frequency* $\omega_0 = \sqrt{k/m}$, Eq. (5.4) may be written

$$\ddot{x} + \omega_0^2 x = 0. \tag{5.6}$$

It is easy to check that the functions $x = \cos(\omega_0 t)$ and $x = \sin(\omega_0 t)$ are two independent solutions, and the general solution is therefore

$$x = C_1 \cos(\omega_0 t) + C_2 \sin(\omega_0 t). \tag{5.7}$$

The arbitrary constants C_1 and C_2 are to be determined by the initial conditions. If at $t = 0$ the particle is at x_0 with velocity $\dot{x}_0$, then we easily find

$$C_1 = x_0, \tag{5.8}$$
$$C_2 = \dot{x}_0/\omega_0. \tag{5.9}$$

An alternative form of Eq. (5.7) is

$$x = a \cos(\omega_0 t + \phi), \tag{5.10}$$

where the constants a, ϕ are related to C_1, C_2 through

$$C_1 = a \cos\phi, \tag{5.11}$$
$$C_2 = -a \sin\phi. \tag{5.12}$$

The constant a is called the amplitude, and ϕ the initial phase of the oscillator at $t = 0$. Constant a is the same one as the homonymous constant introduced in Eq. (5.3), and defines the limits between which the particle oscillates, $x = \pm a$. The motion is a periodic oscillation of period τ given by $\tau = 2\pi/\omega_0$. Observe that the period is independent of the amplitude a. The frequency f (not to be confused with the *angular* frequency) is the number of oscillations per unit time, or, in other words,

$$f = \frac{1}{\tau} = \frac{\omega_0}{2\pi}. \tag{5.13}$$

Incidentally, the linearized pendulum of §4.7 for small osillations is also an undamped harmonic oscillator.

5.2 Complex representation

It is often convenient to describe periodic phenomena using exponential *complex numbers*. The solution to Eq. (5.6) may also be written

$$x = \frac{1}{2} A \exp(i\omega_0 t) + \frac{1}{2} A^* \exp(-i\omega_0 t) = \frac{1}{2}(z + z^*) = \Re(z), \tag{5.14}$$

where "$*$" denotes the complex conjugate and $z = A \exp(i\omega_0 t)$. Of course, the result of this operation results in x being a real number. The relation of C_1 and C_2 with the *complex amplitude* A, which includes the information of the initial phase, is

$$A = C_1 - iC_2. \tag{5.15}$$

Observe that the complex number A contains in itself the two independent constants of the solution. Using Euler's relation

$$\exp(i\theta) = \cos(\theta) + i \sin(\theta), \tag{5.16}$$

it is straightforward to recover the solution as written in Eq. (5.7). Similarly, if we use Euler's relation and the polar form of a A,

$$A = a \exp(i\phi), \tag{5.17}$$

we obtain the solution in the form Eq. (5.10).

It is also useful to note that if $z = A\exp(i\omega_0 t) = x + iy$ is a *complex* solution of Eq. (5.4), i.e.,

$$m\ddot{z} + kz = (m\ddot{x} + kx) + i(m\ddot{y} + ky) = 0, \tag{5.18}$$

then its real and imaginary parts, $x = \Re(z)$ and $y = \Im(z)$, must each satisfy the equation separately.

If we plot the complex solution z in the complex plane, we see that z as a phasor (i.e., the arrow in the complex plane that goes from 0 to z) moves in a circle around the origin with angular velocity ω_0 (see Fig. 5.2). Incidentally, this is why ω_0 is called 'angular frequency.' The absolute value a of the complex amplitude A is the radius of the circle, and the phase ϕ of A is the initial phase of z at $t = 0$. The real part of the solution, x, is the projection of this circular motion on the real axis.

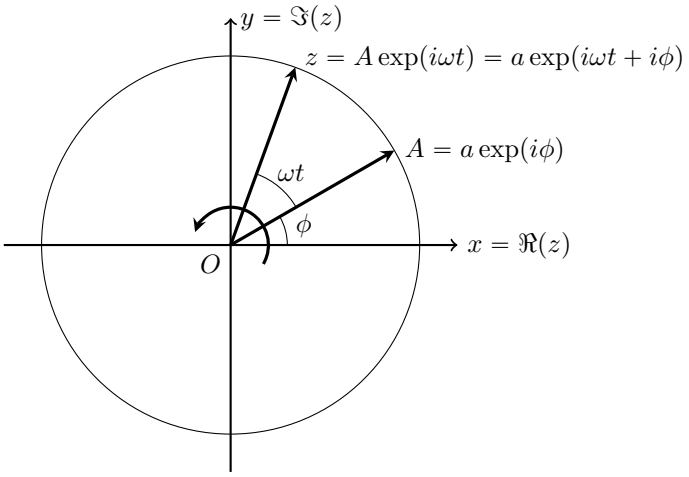

Figure 5.2: Illustration of the initial condition A and the solution z in complex plane.

5.3 The damped harmonic oscillator

If we add a viscous damper to the spring-mass system such that the damping force is proportional and opposite to the velocity with a proportionality constant λ, the total force acting in the system (with $k > 0$ and $\lambda > 0$) is now

$$F(x) = -kx - \lambda\dot{x}. \tag{5.19}$$

The equation of motion now becomes

$$m\ddot{x} + \lambda\dot{x} + kx = 0. \tag{5.20}$$

We may look for solutions of the form $x = \exp(pt)$. Differentiating to obtain $\dot{x}$ and $\ddot{x}$ and substituting in Eq. (5.20), we obtain the characteristic equation for p,

$$mp^2 + \lambda p + k = 0, \tag{5.21}$$

whose roots, which are generally complex, are given by the expression:

$$p = -\gamma \pm \sqrt{\gamma^2 - \omega_0^2}, \tag{5.22}$$

where

$$\gamma = \frac{\lambda}{2m}; \quad \omega_0 = \sqrt{\frac{k}{m}}. \tag{5.23}$$

Note that ω_0 is the angular frequency of the *undamped* oscillator.

5.3.1 Small damping

If λ is so small that $\gamma < \omega_0$, then both roots of p are complex conjugates. In this case, we say that the system is *underdamped*:

$$p = -\gamma \pm i\omega, \quad \omega = \sqrt{\omega_0^2 - \gamma^2}. \tag{5.24}$$

The general solution may be written in several equivalent forms:

$$x = \frac{1}{2}A\exp(i\omega t - \gamma t) + \frac{1}{2}A^*\exp(-i\omega t - \gamma t) \tag{5.25}$$

$$= \Re\left[A\exp(i\omega t - \gamma t)\right] = a\exp(-\gamma t)\exp(i\phi)\exp(i\omega t). \tag{5.26}$$

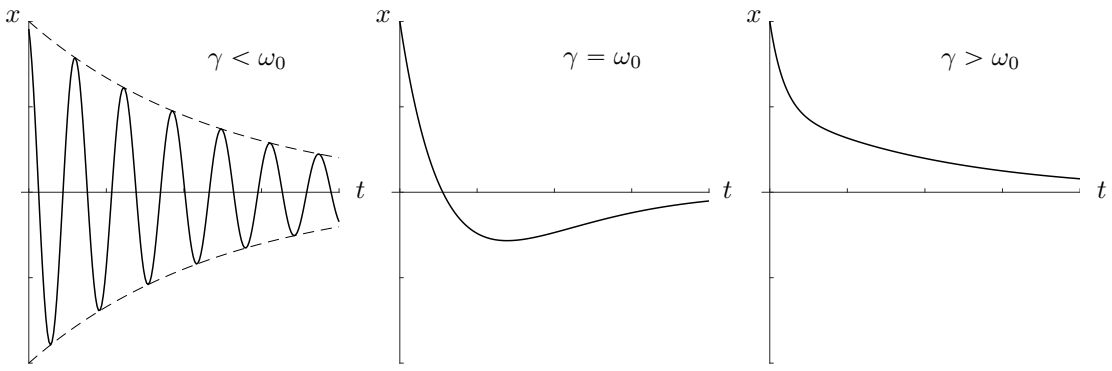

Figure 5.3: Illustration of the harmonic oscillator solution for small damping (left), critical damping (center) and large damping (right).

From Eq. (5.26) we see that this solution represents an oscillation with exponentially decreasing amplitude $a\exp(-\gamma t)$, and angular frequency ω (see Fig. 5.3). Note that ω is always less than the frequency ω_0 of the undamped oscillator. The time in which the amplitude is reduced by a factor $1/e$, called the relaxation time of the oscillator, is

$$\frac{1}{\gamma} = \frac{2m}{\lambda}. \tag{5.27}$$

It is often convenient to introduce the *quality factor*, or simply Q, of the oscillator, defined as the dimensionless number

$$Q = \frac{m\omega_0}{\lambda} = \frac{\omega_0}{2\gamma} \tag{5.28}$$

If the damping is small then Q is large. In a single oscillation period, the amplitude is reduced by the factor $\exp(-2\pi\gamma/\omega)$, or approximately $\exp(-\pi/Q)$. The number of periods in a relaxation time is roughly Q/π.

5.3.2 Large damping

If λ is so large that $\gamma > \omega_0$, then both roots of p are real and negative, and we say that the oscillator is dominated by the damping force, or that it is *overdamped*:

$$p = -\gamma_\pm; \quad \gamma_\pm = \gamma \pm \sqrt{\gamma^2 - \omega_0^2}. \tag{5.29}$$

The general solution is then

$$x = \frac{1}{2}A\exp(-\gamma_+ t) + \frac{1}{2}A^*\exp(-\gamma_- t). \tag{5.30}$$

Since both terms are multiplied by real negative exponentials, the displacement tends exponentially to zero. For large times, the dominant term is the one associated to the smaller exponent (in absolute value), γ_-. Thus the characteristic time in which the displacement x is reduced by a factor $1/e$ is of order $1/\gamma_-$.

5.3.3 Critical damping

The limit case $\gamma = \omega_0$ is the case of *critical damping*, in which $\omega = 0$ and the two roots of p coincide and are real negative ($p = -\gamma$). Then the solution in Eq. (5.30) or Eq. (5.26) involves only one arbitrary constant, and therefore it cannot be the general solution. We need to find a second independent solution in this case. In fact, it is easy to verify by direct substitution that the equation Eq. (5.20) in this special case is also satisfied by the function $t\exp(pt)$. Thus the general solution is

$$x = (a + bt)\exp(-\gamma t), \tag{5.31}$$

where a and b are the arbitrary constants in this case. Critical damping is the ideal for many applications. For example, in a measuring instrument, we want to damp out the oscillations of the pointer about its correct position as quickly as possible, but too much damping would lead to a very slow response. Let us assume that k is fixed and the amount of damping can be varied. When the damping is less than critical ($\gamma < \omega_0$), the characteristic time of response is the relaxation time $1/\gamma$, which of course decreases as γ is increased. However, when $\gamma > \omega_0$, the characteristic time is $1/\gamma_-$, as we noted above. It is not hard to verify that as γ increases, γ_- decreases, so that the response time $1/\gamma_-$ increases. Thus the shortest possible response time is obtained by choosing $\gamma = \omega_0$, that is, the critical damping.

5.4 Damped oscillator under a periodic force

We are often interested in the response of a one-dimensional oscillatory system subject to an external applied force that is a function of the time, $F(t)$. The equation of motion in this case is the *complete* version of the Eq. (5.20),

$$m\ddot{x} + \lambda\dot{x} + kx = F(t). \tag{5.32}$$

Now, if $x_1(t)$ is any solution of this equation, and $x_0(t)$ is a solution of the corresponding *homogeneous* Eq. (5.20), then, as one can easily check by direct substitution, $x_1(t) + x_0(t)$ will be another solution of Eq. (5.32). Hence we only need to find a *particular solution* of the complete equation. Its general solution is then obtained by adding to this particular solution the general solution of the homogeneous problem, Eq. (5.20), which in turn must be constructed from two linearly independent solutions.

Let us consider first the case where the applied force is periodic in time, with the simple form

$$F(t) = F_1\cos(\omega_1 t), \tag{5.33}$$

where F_1 and ω_1 are real constants. It is convenient to write this in the complex form

$$F(t) = \Re\left[F_1\exp(i\omega_1 t)\right], \tag{5.34}$$

and to solve first the complex equation with the complex force,

$$m\ddot{z} + \lambda\dot{z} + kz = F_1\exp(i\omega_1 t). \tag{5.35}$$

Then the real part $x = \Re(z)$ of this solution will be a solution of Eq. (5.32) with the force of Eq. (5.33). We shall look for a particular solution of Eq. (5.35) which is periodic in time with the same period as the applied force,

$$z = A_1\exp(i\omega_1 t) = a_1\exp[i(\omega_1 t + \phi_1)], \tag{5.36}$$

where $A_1 = a_1\exp(i\phi_1)$ is a complex constant to be computed. Substituting in (5.35), we obtain an equation for A_1:

$$\left(-m\omega_1^2 + i\lambda\omega_1 + k\right)A_1 = F_1. \tag{5.37}$$

Rearranging the terms gives

$$\left(\omega_0^2 - \omega_1^2 + 2i\gamma\omega_1\right)a_1 = \frac{F_1}{m}\exp(-i\phi_1), \tag{5.38}$$

where γ and ω_0 are defined as before by Eq. (5.23), respectively. Hence, equating real and imaginary parts,

$$\left(\omega_0^2 - \omega_1^2\right) a_1 = \frac{F_1}{m} \cos\phi_1, \tag{5.39}$$

$$2\gamma\omega_1 a_1 = -\frac{F_1}{m} \sin\phi_1. \tag{5.40}$$

The amplitude is found by squaring these equations and adding them, to give

$$a_1 = \frac{F_1/m}{\sqrt{(\omega_0^2 - \omega_1^2)^2 + 4\gamma^2\omega_1^2}}. \tag{5.41}$$

Dividing one equation by the other yields the phase angle difference between the applied force and the motion of the oscillator:

$$\tan\phi_1 = -\frac{2\gamma\omega_1}{\omega_0^2 - \omega_1^2}. \tag{5.42}$$

From the two possible solutions for the tangent, the meaningful one is for $-\pi < \phi_1 < 0$. A negative value of the phase angle ϕ_1 means that the oscillator lags behind the force, i.e., it moves after the force phasor in the complex circle.

We have found a particular solution of Eq. (5.35). Its real part, the corresponding particular solution of our original equation Eq. (5.32), is simply

$$x = a_1 \cos(\omega_1 t + \phi_1), \tag{5.43}$$

The general solution, as we discussed above, is obtained by adding to this particular solution the general solution of the corresponding homogeneous equation, namely Eq. (5.26), Eq. (5.30), or Eq. (5.31). For example, in the case where the damping is less than critical ($\gamma < \omega_0$), we obtain

$$x = a_1 \cos(\omega_1 t + \phi_1) + a \exp(-\gamma t) \cos(\omega t + \phi). \tag{5.44}$$

Here, of course, a_1 and ϕ_1 are given by Eqs. (5.41) and (5.42), but a and ϕ are arbitrary constants to be fixed by the initial conditions. The second term in the general solution Eq. (5.44), which represents a free damped oscillation, dies away exponentially with time. It is therefore related to the initial transient. After a long time, the displacement x will be given by the first term of Eq. (5.44) alone, that is, by Eq. (5.43). Thus, no matter what initial conditions we choose, the oscillations are ultimately governed solely by the external force. Note that their angular frequency is that of the applied force, ω_1, not the angular frequency of the unforced oscillator, ω_0.

5.4.1 Resonance

The amplitude a_1 and phase ϕ_1 of the forced oscillations are strongly dependent on the angular frequency of the applied force, ω_1. In particular, if the damping of the oscillator is small, the amplitude can become very large when ω_1 and ω_0 are almost equal. If we fix γ and the forcing frequency ω_1, and vary the oscillator frequency ω_0, the amplitude is a maximum when $\omega_0 = \omega_1$. In this case, we say that the system is at *resonance*. At resonance, the amplitude is

$$a_1 = \frac{F_1}{2m\gamma\omega_1} = \frac{F_1}{\lambda\omega_1}, \tag{5.45}$$

which can be very large if the damping constant λ is small. If, on the other hand, we fix the parameters γ and ω_0 of the oscillator, and vary the forcing frequency ω_1, the maximum amplitude actually occurs for a frequency slightly lower than ω_0, namely

$$\omega_1 = \sqrt{\omega_0^2 - 2\gamma^2}. \tag{5.46}$$

The maximum disappears for $\gamma > \omega_0\sqrt{1/2}$. However, if γ is small, the position of the maximum does not differ much from ω_0. Note that the natural frequency ω of the damped oscillator lies between this resonant frequency and the frequency ω_0 of the undamped oscillator.

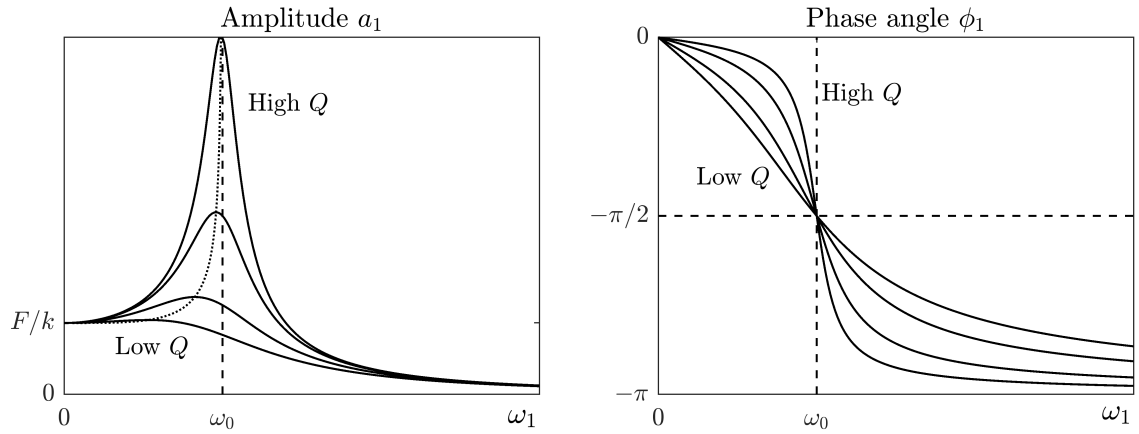

Figure 5.4: Amplitude a_1 and phase ϕ_1 against the forcing frequency ω_1 of the forced harmonic oscillator, for various values of Q (related to γ).

The dependence of the amplitude a_1 on the forcing frequency ω_1 is of the form sketched in Fig. 5.4. The width of the resonance peak, that is the range of frequencies over which the amplitude is large, is determined by γ. Indeed, the amplitude is reduced to $1/\sqrt{2}$ of its maximum value when the two terms in the denominator of Eq. (5.41) become roughly equal, and for small γ this occurs when $\omega_1 \simeq \omega_0 \pm \gamma$. Therefore, γ is called the approximate half-width of the resonance. Notice the inverse relationship between the width and height of the peak: the narrower the resonance, the higher its peak.

The quality factor $Q = \omega_0/(2\gamma)$ provides a quantitative measure of the sharpness of the resonance peak. Indeed, the ratio of the amplitude at resonance, given by Eq. (5.45), to the amplitude at $\omega_1 = 0$ (which describes the response to a constant, time-independent force) is precisely Q. Figure 5.4 also displays the line of maxima of the amplitude a_1 with different values of γ as a dotted line. This phenomenon of resonance occurs in any oscillatory system, and is of great practical importance. Since quite small forces can bring about large oscillations near resonance, great care must be taken in the design of any mechanical structure, like any part of an aircraft, to avoid this possibility.

The constant ϕ_1 specifies the phase relation between the applied force and the induced oscillations. The variation of ϕ_1 with ω_1 is shown in Fig. 5.4. If the force is slowly oscillating, i.e. if ω_1 is small, then $\phi_1 \simeq 0$, so that the induced oscillations are nearly in phase with the force. In that case, the amplitude in Eq. (5.41) is

$$a_1 \approx \frac{F_1}{m\omega_0^2} = \frac{F_1}{k}. \tag{5.47}$$

Thus the position x at any time t, given by Eq. (5.43), is approximately the equilibrium position under the force $F_1 \cos(\omega_1 t) - kx$. At resonance, the phase shift is $\phi_1 = -\pi/2$, and the induced oscillations lag behind by a quarter period. For very rapidly oscillating forces, $\phi_1 \simeq -\pi$, and the oscillations are almost exactly out of phase. In this limiting case, $a_1 \simeq F_1/m\omega_1^2$ and the oscillations correspond to those of a free particle under the applied oscillatory force (i.e., the restoring force $-kx$ and the damping force $\lambda\dot{x}$ become irrelevant in this regime). Note that the value of the damping constant is important for the forced amplitude only in the region near the resonance.

6 Constraints and reaction forces

In this Chapter we consider the kinematics and dynamics of a particle that is constrained to move along a curve or on surface. All these constraints are *integrable*, meaning that they establish a mathematical relation between coordinates of the problem. Whenever there is one or more such *constraints* on the motion of the particle, the number of *effective degrees of freedom* is reduced. In other words, a lower number of independent parameters (or *generalized coordinates*) is required to fully determine its position. At the same time, unknown *reaction forces* will exist to ensure that the motion of the particle fulfills the constraints.

6.1 Curves and surfaces

Before we approach the analysis of constrained motion, we briefly review different ways of describing curves and surfaces in three-dimensional space.

1. Curve in parametric form. A curve might be expressed by a vector function of one parameter, $\boldsymbol{r} = \boldsymbol{r}(\lambda)$, or, in Cartesian coordinates:

$$\begin{cases} x = f_1(\lambda), \\ y = f_2(\lambda), \\ z = f_3(\lambda). \end{cases} \tag{6.1}$$

   So, for example, the equation of a circumference of radius a, which is located on a horizontal plane with center in the point $C = (C_x, C_y, C_z)$ may be written as a function of a parameter θ as

$$\begin{cases} x = C_x + a\cos\theta, \\ y = C_y + a\sin\theta, \\ z = C_z. \end{cases} \tag{6.2}$$

2. Surface in parametric form. A surface might be expressed by a vector function of two independent parameters, $\boldsymbol{r} = \boldsymbol{r}(\alpha, \beta)$, or

$$\begin{cases} x = g_1(\alpha, \beta), \\ y = g_2(\alpha, \beta), \\ z = g_3(\alpha, \beta). \end{cases} \tag{6.3}$$

   For example, the equation of a sphere of radius R with center in the origin may be written as

$$\begin{cases} x = R\cos\alpha\sin\beta, \\ y = R\sin\alpha\sin\beta, \\ z = R\cos\beta. \end{cases} \tag{6.4}$$

3. Surface in implicit form. The parametric form of the equations of a surface is a set of three equations with two parameters. Eliminating the two parameters, we arrive at the equation of the surface in implicit form,

$$f(x, y, z) = 0. \tag{6.5}$$

   For example, the equation of the sphere of radius R and center at the origin is

$$x^2 + y^2 + z^2 - R^2 = 0. \tag{6.6}$$

4. Curve in implicit form. In a similar way, it is possible to eliminate the parameter of the curve and obtain two equations,

$$\begin{cases} f(x, y, z) = 0, \\ g(x, y, z) = 0. \end{cases} \tag{6.7}$$

Note that a curve can be interpreted as the intersection between two surfaces. For example the implicit equations of the circumference discussed above are

$$\begin{cases} (x - C_x)^2 + (y - C_y)^2 - R^2 = 0, \\ z - C_z = 0, \end{cases} \tag{6.8}$$

which correspond to the intersection between a cylinder and a plane (of course, this way of writing the curve is not unique, since we could give the same circumference as the intersection of a different pair of surfaces, like a plane and a sphere).

6.2 Choosing coordinates

While we have complete freedom over what coordinates we choose to solve a problem, an intelligent choice can facilitate enormously our work. The constraints on the motion of a particle are the main aspect to consider when selecting our coordinates. We need as many coordinates as effective degrees of freedom in the problem. The only exception to this rule is when we have *non-integrable* constraints, an example of which will be discussed in the second part of this course, in §13.2. In general, it is desirable that the coordinates reflect the geometry of the constraints as much as possible, so that it is easy to parametrize the constraint in terms of them, and the vector basis associated to the coordinates at each point is aligned with the tangent and normal subspaces of the constraint. To illustrate these points, some orientative examples are given below:

1. If a particle P is attached by a string or a rod of length ℓ to a fixed point A, that means that it can move on the surface of a sphere of radius ℓ and center A. This motion has two effective degrees of freedom. A good choice of coordinates could be the spherical angles θ, ϕ.

2. If the particle must remain on a cone, the azimuth angle θ and the distance to the vertex ℓ (or alternatively the height of the particle along its axis z) could be good choices of coordinates.

3. If the particle can move along a wire of fixed shape, the arc length coordinate s, or one of the Cartesian coordiantes x, y or z that can be used to parametrize the curve, would be good choices.

4. In general, if a particle P must remain on a surface, the two parameters α and β that are used to define the surface could be a good choice of coordinates.

Once the coordinates have been selected, the procedures of Kinematics enable us to express its position, velocity and acceleration vectors as a function of them and their time derivatives: this is the main output of Kinematics and the information that we need to proceed to solve the Dynamics of motion.

6.3 Tangent and normal vectors

When working with curve or surface constraints, it is useful to have a local vector basis that separates the tangent and normal directions to it:

1. In the case of a curve, the tangent subspace has dimension one, and the normal subspace dimension two.

2. In the case of a surface, the tangent subspace has dimension two, and the normal subspace dimension one.

When a constraint is defined in parametric form, partial differentiation with respect to each of the parameters will yield a vector basis of the tangent subspace. The resultant vectors may not be unitary and orthogonal in general, so we would need to operate to obtain a orthonormal vector basis if required.

This was the approach we followed in Chapter 2 to obtain the tangent vector $\boldsymbol{e}_t$ of a curve. We then obtained two normal unit vectors, $\boldsymbol{e}_n$ and $\boldsymbol{e}_b$, to define a orthonormal right-handed vector basis of space. In the case of a surface defined parametrically with two parameters α and β, taking derivatives with respect to each of them and normalizing yields two distinct unit tangent vectors, $\boldsymbol{e}_\alpha$

and $\boldsymbol{e}_\beta$, which may not be orthogonal. A normal unit vector to the surface can be found with the cross product $\boldsymbol{e}_\alpha \times \boldsymbol{e}_\beta$ and normalizing.

When a constraint is given by implicit equations like $f(x, y, z) = 0$, it is easier to first obtain a vector basis of the normal subspace. Note that, as discussed above, each of these equations represents a surface in space. By construction, the *gradient vector* points in the direction perpendicular to the surface. In Cartesian coordinates:

$$\nabla f = \frac{\partial f}{\partial x}\boldsymbol{i} + \frac{\partial f}{\partial y}\boldsymbol{j} + \frac{\partial f}{\partial z}\boldsymbol{k}. \tag{6.9}$$

Normalizing, a unit vector normal to the surface is given by;

$$\frac{\nabla f}{|\nabla f|}. \tag{6.10}$$

In the case of a surface, we can always find two directions perpendicular to this one to complete a vector basis of space. In the case of a curve, defined by two such equations, each of them has its own gradient vector, and thus we can compute two normal vectors to the curve. Note that in general the two vectors will not be orthogonal. Then, using the cross product and normalizing gives us a tangent vector to the curve to complete a vector basis of space.

Of course, we can always operate in the parametric equations of a constraint to obtain an implicit representation of it, and viceversa. Sometimes this offers a faster route to the vectors that we want to compute.

6.4 Motion of a particle on a smooth surface

We consider now the motion of a particle P which is constrained to move without friction on a given surface $f(x, y, z) = 0$, under the influence of an applied force $\boldsymbol{F}(\boldsymbol{r}_0^P, \boldsymbol{v}_0^P, t)$. The particle can move freely on the surface, which has a *tangent plane* at each point (i.e., a vector space of dimension 2), and therefore the motion has two degrees of freedom. The particle cannot move in the direction perpendicular to the surface.

The mechanism that compels the particle to stay on the surface is an additional unknown *reaction force* $\boldsymbol{N}$ acting in the normal direction to it (which is the reason behind its usual name, *normal force*). The magnitude of $\boldsymbol{N}$ will be as large as necessary to make the motion of the particle satisfy the constraint, and is unkown a priori. In summary, we may write $\boldsymbol{N} = \lambda \nabla f$, where $\lambda(t)$ is an unknown parameter that we need to determine as part of the solution.

Recall that our goal is to obtain the position of the particle as a function of time, that is, to find $x(t)$, $y(t)$ and $z(t)$. As discussed above, the equation of the constraint ($f(x, y, z) = 0$) already removes one degree of freedom of the motion of the particle, i.e., it can be used to write *locally* one coordinate as a function of the other two (for example, $z = z(x, y)$). Therefore, the set of equations to be solved is

$$\begin{cases} m\dfrac{\mathrm{d}^2\boldsymbol{r}}{\mathrm{d}t^2} = \boldsymbol{F}(\boldsymbol{r}, \dot{\boldsymbol{r}}, t) + \boldsymbol{N} = \boldsymbol{F}(\boldsymbol{r}, \dot{\boldsymbol{r}}, t) + \lambda \nabla f, \\ \\ f(x, y, z) = 0, \end{cases} \tag{6.11}$$

for the unknowns $x(t)$, $y(t)$, $z(t)$ **and** $\lambda(t)$ (4 scalar equations for 4 unknowns).

Clearly, $\boldsymbol{N}$ does not derive from a potential. Notwithstanding this, we can check that the work performed by this reaction force in a differential displacement of the particle is given by

$$\delta W = \boldsymbol{N} \cdot \mathrm{d}\boldsymbol{r} = \lambda \nabla f \cdot \mathrm{d}\boldsymbol{r} = \lambda \left(\frac{\partial f}{\partial x}\mathrm{d}x + \frac{\partial f}{\partial y}\mathrm{d}y + \frac{\partial f}{\partial z}\mathrm{d}z \right) = \lambda \mathrm{d}f, \tag{6.12}$$

but, since the particle stays on $f(x, y, z) = 0$, then $\mathrm{d}f = 0$ along the motion of the particle. This is obvious since $\boldsymbol{N}$ is always perpendicular to the surface, and the velocity $\boldsymbol{v}_0^P$ is always tangent to it, so the power $P = \boldsymbol{N} \cdot \boldsymbol{v}_0^P$ that $\boldsymbol{N}$ exerts on the particle is zero, and hence also the work performed. Therefore a normal reaction force *never performs work*. This means, in particular, that its existence does not decide whether the law of conservation of mechanical energy is applicable or not.

This conclusion about the forces of a surface is only true under the assumptions above. if the surface moves in time, or if it is not smooth so that there is friction, then:

1. If the surface changes with time, then $f(x, y, z, t) = 0$, there may be a component of $\boldsymbol{v}_0^P$ along ∇f, and the normal force may perform work.

2. If there is friction and the particle moves against it, the friction force will dissipate energy.

Surface constraints come in two different types: *unilateral* and *bilateral*. In unilateral constraints, the particle can leave the surface on one side of it. A good example is an airplane rolling on the ground, the moment it takes off. In this type of problems, it is necessary to subdivide the solution into two or more parts: when the particle is attached to the constraint, so that the constraint is in effect, and when it is not, so that the constraint is not in effect. The point at which a particle separates from a surface constraint is when the magnitude normal force becomes zero, $N = 0$. In a bilateral constraint the particle cannot leave the surface in either direction, so it is effectively trapped in it: the normal force may point to either direction as required.

6.5 Motion of a particle along a smooth curve

We consider now the motion of a particle P which is constrained to move without friction along a curve given implicitly by $f(x, y, z) = 0$, $g(x, y, z) = 0$, under the influence of a force $\boldsymbol{F}(\boldsymbol{r}_0^P, \boldsymbol{v}_0^P, t)$.

Now the normal force that compels the particle to stay on the curve can have two components, one along the normal vector of each surface that defines the curve: $\boldsymbol{N} = \lambda_1 \nabla f + \lambda_2 \nabla g$. The parameters λ_1 and λ_2 are unknown functions of time to be determined as part of the solution.

The analysis is very similar to the analysis of the motion of a particle over a surface. The set of equations to be solved are Newton's second law plus the two implicit equations of the curve:

$$\begin{cases} m\dfrac{\mathrm{d}^2\boldsymbol{r}}{\mathrm{d}t^2} = \boldsymbol{F}(\boldsymbol{r}, \dot{\boldsymbol{r}}, t) + \lambda_1 \nabla f + \lambda_2 \nabla g \\ \\ f(x, y, z) = 0, \\ \\ g(x, y, z) = 0, \end{cases} \tag{6.13}$$

for the unknowns $x(t)$, $y(t)$, $z(t)$, $\lambda_1(t)$ and $\lambda_2(t)$ (5 unknowns for 5 scalar equations).

As before, the normal reaction force does not perform work for a stationary and smooth curve. The same discussion to the applicability of the equation of conservation of mechanical energy as in the case of a surface applies here.

When studying the motion of particles along curves it is sometimes useful to use path variables, so that the acceleration, the external force and the normal reaction may be written as

$$\boldsymbol{a}_0^P = \ddot{s}\boldsymbol{e}_t + \frac{\dot{s}^2}{\rho}\boldsymbol{e}_n \tag{6.14}$$

$$\boldsymbol{F} = F_t\boldsymbol{e}_t + F_n\boldsymbol{e}_n + F_b\boldsymbol{e}_b, \tag{6.15}$$

$$\boldsymbol{N} = N_n\boldsymbol{e}_n + N_b\boldsymbol{e}_b. \tag{6.16}$$

It is particularly useful to project Newton's second law along $\boldsymbol{e}_t$, $\boldsymbol{e}_n$ and $\boldsymbol{e}_b$ using these expressions:

$$F_t = m\ddot{s}, \tag{6.17}$$

$$F_n + N_n = m\frac{\dot{s}^2}{\rho}, \tag{6.18}$$

$$F_b + N_b = 0. \tag{6.19}$$

6.6 Friction forces

In the case of a non-smooth surface or curve, a friction reaction force $\boldsymbol{R}$ must be considered in addition to the normal reaction. There exist several models for the friction force ranging in complexity. For our purposes, it will suffice to consider a simple friction model with a single friction coefficient μ,

which depends on the surface and on the particle (e.g. it is not the same a glass particle sliding over ice than a sand particle sliding over wood)[14]. The magnitude of the friction force is given by

$$|\boldsymbol{R}| \leqslant \mu |\boldsymbol{N}|. \tag{6.20}$$

Let us assume that the constraint is stationary in some reference frame S_1:

1. Whenever the velocity of the particle with respect to the constraint is zero, $\boldsymbol{v}_1^P = 0$, friction must be considered as an unknown reaction force, since then its magnitude is unknown as given by the expression above. The direction of the friction force is any direction of the tangent subspace (dimension 2 for a surface, dimension 1 for a curve, so we will have 2 new unknowns in the case of a surface and 1 for a curve). Observe that in this case friction exerts no work on the particle, since it is not moving.

2. Whenever the particle starts to move with respect to the constraint so its velocity is non-zero, $\boldsymbol{v}_1^P \neq 0$, then the magnitude of the friction force is known: it will be saturated at the maximum possible value, $|\boldsymbol{R}| = \mu |\boldsymbol{N}|$. Moreover, its direction will be antiparallel to $\boldsymbol{v}_1^P$,

$$\boldsymbol{R} = -\mu |\boldsymbol{N}| \frac{\boldsymbol{v}_1^P}{v_1^P}. \tag{6.21}$$

   Thus, having friction in this case does not introduce any new unknowns. Observe that if the particle moves, friction does negative work and dissipates energy.

Each case must be regarded as a different Mechanics problem, as the equations to be solved differ. In general, we do not know *a priori* whether friction will be large enough to keep the particle stationary or whether the particle will start to move. The particle may even begin moving and then stop, then move again, etc. Solving such a problem requires monitoring the condition $R \leqslant \mu N$ and $\boldsymbol{v}_1^P = 0$ continuously, and switch from one formulation of the problem to another as necessary.

[14] In more advanced models, one distinguishes between a *static* friction coefficient μ_S when the particle is at rest and a *dynamic* friction coefficient μ_D when the particle is in relative motion with the surface. In this course, such distinction will not be made.

7 The Newtonian mechanics approach I

We now have all the basic tools to solve *any* problem of particle motion, an essential skill for the study of flight mechanics: so far, we have learned about *vector bases*, *reference systems*, and *coordinate systems*. We have studied how to *rotate*, *project* and *differentiate vectors*. We have analyzed the *kinematics* of a particle, and learned several ways to express its velocity and acceleration. We now know about *relative motion*, and how to relate the velocities, accelerations, angular velocities, and angular accelerations with respect to different reference frames.

Additionally, we have introduced the concepts of *force* and *linear momentum*, and presented the evolution equation for the position of the particle (*Newton's second law*). We have learned how to approach particle problems with *constraints*, and the concepts of *degrees of freedom* and *reaction forces*. Also, we have brought into the discussion the concepts of *moment of a force* and *angular momentum*, together with the corresponding evolution equation. Finally, we have studied the equation of *conservation of mechanical energy*, when can it be applied, the meaning of the potential function and how to discuss *regions of existence of the solution*.

In this chapter we propose a systematic approach to solve problems in classical mechanics involving one or more particles (i.e., systems of particles). While it may not be the fastest route to the solution in some cases, this step-by-step process is recommended, especially for beginners.

7.1 Understanding the problem

It goes without saying, the first and most important step is to *undertand what we are looking for*. In this course, the statement of the problem is given to the student, whereas in real life the engineer is generally not so lucky: the problem is there in an amorphous form, and it first needs to be defined in a crystal-clear form and modeled in a way that it can then be analyzed.

When the statement of the problem is given, the mere reading of the text is not sufficient to fully understand what needs to be done and how. Usually, the text contains many conditions or constraints on the motion (the particle is attached to a string, or it moves on a plane, etc) that *need to be translated into mathematical conditions*. An essential question to answer during this step is: **how many (effective) degrees of freedom does my problem have?**

The capability to visualize the mechanical system under study is an important skill to develop. For this reason, **drawing sketches** of the problem is a great idea, as this can help us see the type of motion that we are about to analyze. Do not hesitate to do as many drawings as you need, from as many angles as you need, as simple or complex as you need, in order to understand the problem. At the very least, a drawing of the position of the paricle in a *generic configuration* of the problem is advisable. Auxiliar sketches, e.g. to analyze simple rotations, can also be of help.

Do not jump to the next step until you are certain that you understand all the parts and details of what you need to do, and have a general idea of how you want to solve the problem.

7.2 Selecting the coordinates

At this point one should think of **which variables are going to be used** to solve the problem. The number of variables, or coordinates, should match the number of effective degrees of freedom of the problem. The only exception to this rule is if we have any non-integrable constraints, where an excess of coordinates is required; this type of constraints will only make their appearance in the second part of this course, in §13.2.

The choice of coordinates is not unique, but an intelligent choice can facilitate substantially the solution of the problem. A poor choice can result in mathematical expressions that are too difficult to deal with. To choose an adequate set of independent coordinates, one must look at the problem and pay attention, in particular, to the constraints of motion. One may consider distances, arc lengths, angles, etc. that can be used to express the velocity and acceleration of the particle(s). Common choices are Cartesian coordinates x, y, z, the distance d of a particle with respect to a fixed plane, the polar angle θ of a moving line with respect to a reference, and combinations thereof. The ability to make a good choice comes from experience; hence the need for practice!

A common error when choosing coordinates is to select redundant or dependent parameters, e.g. the distance l_1 of P with respect to the plane $z = 0$ and the distance l_2 of P with respect to the

plane $z = L$ are dependent on each other, because one may write $l_2 = L - l_1$. Sometimes, however, introducing temporarily one or more redundant coordinates will facilitate posing the problem, but they must then be eliminated using the equations of the constraints.

7.3 Defining reference frames

In Newtonian mechanics there is always a predefined inertial reference frame S_0 in which the answer must be expressed. If not given explicitly, at least the conditions for a frame to be inertial will be implicit in the problem. In that case, we must define this essential reference frame ourselves.

While we could solve the whole problem in just this reference frame, the solution of the problem may become easier if one defines **auxiliary reference frames** to decompose the motion of the particle(s) into simple rotations and simple translations.

What we seek in this step is to define as many sets of axes as needed to solve the problem in a simple way. Again, the choice is not unique, and experience teaches what are in general good choices. Then again, sometimes the problem statement guides us on what auxiliary reference frames to choose. Each set of axes must be clearly defined in writing, identified in the drawing, and labeled (e.g. S_1, S_2, S_3).

Naturally, together with each reference frame that we define there is an associated *vector basis*. Additionally, other vector bases may exist which are of interest for us, such as the intrinsic vector basis or the vector basis associated to our coordinates. A second task to do in this step is to **express all the new vector bases in terms of known vectors** in a generic position of our system, i.e., making use of the coordinates that we have chosen. As an example, if a reference frame S_2 is rotated an angle θ with respect to reference frame S_1 along the $\boldsymbol{k}_1$ direction, then we should write

$$\begin{cases} \boldsymbol{i}_2 = \cos\theta \boldsymbol{i}_1 + \sin\theta \boldsymbol{j}_1, \\ \boldsymbol{j}_2 = -\sin\theta \boldsymbol{i}_1 + \cos\theta \boldsymbol{j}_1, \\ \boldsymbol{k}_2 = \boldsymbol{k}_1, \end{cases} \tag{7.1}$$

or, alternatively, write the vectors of S_1 as a function of the vectors of S_2. Once we do so, we can use any of these vectors in our calculations and in the solution of the problem. Failing to do so forbids us from using those useful vectors and to write the solution of the problem in terms of them.

The vector bases are usually chosen to be just a simple rotation away from each other, so defining each basis in terms of the previous one (i.e., before or after the simple rotation) is easy: the familiar cos and sin expression will result (see e.g. Eq. (7.1)). Rather than memorizing expressions, it is preferable to think of the vector basis change relations each time, to avoid e.g. misplacing a minus sign.

7.4 Kinematics of the particles in the problem

Armed with our choice of coordinates and reference frames, we can move on to the next step in the process, which consists of **obtaining the expressions of the position, velocity and acceleration vectors of each particle** in a generic state of the system *as a function of only the chosen coordinates and their derivatives*. In some cases we are only interested in absolute velocities and accelerations, but in some problems we are also asked about relative quantities (or we need them for any particular reason). We will also need to write down in the same manner the **angular velocity and angular acceleration vectors of all defined reference frames** with respect to the fixed one. These expressions will be essential for the rest of the process.

We should first start by obtaining the $\boldsymbol{\omega}$ and $\boldsymbol{\alpha}$ vectors of any auxiliary reference frame that we have defined with respect to the fixed one. If we have chosen our auxiliary frames to be only a simple rotation away from each other, obtaining the relative $\boldsymbol{\omega}$ vectors of one frame to the next in the chain of rotations is straightforward (e.g. if reference frame S_2 results from rotating reference frame S_1 an angle θ along the positive $\boldsymbol{k}_1$ direction, then $\boldsymbol{\omega}_{21} = \dot{\theta}\boldsymbol{k}_1$). Using the laws of composition of angular velocities, we can then obtain the absolute $\boldsymbol{\omega}$ vector of each auxiliary reference frame. Lastly, by differentiation we can get any $\boldsymbol{\alpha}$ vector that we need.

To solve the kinematics of positions, linear velocities and linear accelerations there are several equivalent ways:

1. Write the absolute position vector of each particle in the fixed reference frame using vectors of different vector bases, e.g. $\boldsymbol{r} = a\boldsymbol{i}_1 + b\boldsymbol{i}_2 + c\boldsymbol{j}_2$, and then taking time derivatives in the fixed reference frame to obtain the velocity and acceleration vectors. This is a valid vector expression, but then we have to be careful to use the Coriolis theorem when taking time derivatives to obtain the velocity and acceleration vectors in the fixed reference frame if any of the unit vectors used is not constant in that frame.

2. Write down the position vector of each particle in the fixed reference frame using *only* the vector basis of this frame, by transforming the previous expression with the relations of the type shown in Eq. (7.1). Then, take time derivatives in the fixed frame to obtain the absolute velocity and acceleration of each particle. Working directly in the fixed reference frame vector basis may result in slightly more complicated intermediate expressions in some cases, but this way we do not need to apply the Coriolis theorem for the derivatives.

3. First write down the *relative* position vector of the particle with respect to a useful auxiliary reference frame. Then, find the *relative* velocity and acceleration vectors by taking time derivatives *in that frame*. If the position vector is first expressed only in the basis of that reference frame, then the Coriolis theorem will not be needed. Finally, connect these variables with the *absolute* velocity and acceleration using the expressions of Chapter 3. This way of solving the particle kinematics may be somewhat less cumbersome in some cases, and may be necessary in some problems where we actually need the *relative* vectors for some reason.

Obviously, any of these ways to compute the vectors must yield exactly the same results in the end. As you can see, in any of these possibilities we always begin with a position vector and then take time derivatives. It may be useful to decompose the position vector in terms of simpler vectors. For example, it may be difficult to write directly the position of a point particle P, but vectors $\boldsymbol{OA}$, $\boldsymbol{AB}$, $\boldsymbol{BC}$ and $\boldsymbol{CP}$ be easy to express in our chosen coordinates. Then, by vector summation, $\boldsymbol{r}_P = \boldsymbol{OP} = \boldsymbol{OA} + \boldsymbol{AB} + \boldsymbol{BC} + \boldsymbol{CP}$.

As a general warning, do not confuse the concepts of a position, velocity or acceleration vector *relative* to some reference frame and *expressed* in the vector basis of some reference frame.

7.5 Defining the forces of the problem

The next step is to identify the magnitude and direction of all forces acting on the particle(s) and expressing them in a useful mathematical form. The forces can be classified into three groups as follows:

Applied forces: Gravity (if the particle is "heavy"), drag, buoyancy, the force of a spring, and any force exerted directly on the particle. The magnitude of applied forces is always known a priori (although it may depend on the position and velocity of the particle and also on time in a prescribed way).

Reaction (or constraint) forces: These are forces of unknown magnitude a priori, which result from the existence of a physical constraint. When the particle must remain on a plane or on a curve, or two particles are joined by a string or cable of constant length, reaction forces exist. The normal force $\boldsymbol{N}$ in the case of a surface, or the two normal force components $\boldsymbol{N}_1$ and $\boldsymbol{N}_2$ in the case of a curve, the friction force $\boldsymbol{R}$ when its magnitude and direction are unknown, and the tension in a cable $\boldsymbol{T}$ are all examples of reaction forces.

Inertia forces: If *and only if* we intend to write Newton equation in a non-inertial reference frame in the next step, then we also need to take into account the relevant forces of inertia. If we are going to apply Newton's second law in an inertial reference frame there are no forces of inertia to consider. It is a common mistake of beginners to include forces of inertia when they should not.

For each force in the problem, we must express its **magnitude and direction** as a function of only the chosen coordinates and their derivatives (and perhaps also time). In the case of reaction forces, since the magnitude of each force is unkown, we must also introduce an extra parameter for it: e.g., $\boldsymbol{N} = \lambda \nabla f$, $\boldsymbol{T} = T\boldsymbol{e}_T$, etc.

As an example of applied force, if a spring of stiffness k (and zero length at rest) connects a particle P with the origin O, the force exerted on the particle by the spring can be written (with magnitude and direction) as $\boldsymbol{F}_S = -k\boldsymbol{OA} = -k\boldsymbol{r}_0^P = -kx\boldsymbol{i} - ky\boldsymbol{j} - kz\boldsymbol{k}$. Like in the previous step, constructing the vector $\boldsymbol{OA}$ may be easier if approached as the sum of partial vectors. The direction of normal reaction forces is easy to obtain from the gradient of the surface (or surfaces) that defines the constraint.

7.6 Obtaining the equations of motion

The unknowns of the problem are the coordinates that we have chosen as a function of time, plus any additional parameter that we may have introduced if there exist reaction forces in the problem (like λ in the previous Section). To close the problem, we must find an equal number of independent scalar equations.

Newton's second law provides a vector equation that can be projected into any three independent directions to obtain three independent scalar equations per particle. Before we do it, we must decide *in which reference frame* to apply Newton's second law. If we choose an inertial reference frame, no inertial forces appear; otherwise, we must consider them (return to the previous step if necessary).

In previous steps we have obtained the accelerations and the forces in terms of only the unknowns of the problem and their derivatives, so we only need to "plug in" those values into Newton's second law to do so. The accelerations that we need to use are, of course, with respect to the reference frame in which we want to express the equation.

The projection of Newton's second law equation into three directions also requires some thought. If the directions are chosen wisely, it may simplify the integration of the problem. Typically, it is interesting to consider the directions of the tangent subspace of the constraints, so the normal reaction forces do not appear in those projection. For example, if we project along a tangent direction to a surface, the normal force will not appear in that projection. Projecting along the normal directions provides equations for the reaction forces.

Besides Newton's second law we must use the equation(s) of the constraints of the problem, if any. At this point we must perform a sanity check, and see whether we have an equal number of unknowns and independent scalar equations so the problem is closed. If we do not, we have missed something in this or in previous steps.

Alternatively to Newton's second law equation, or some of its scalar projections, we may use the *mechanical energy conservation equation* (if the applicability conditions are met) and/or the *angular momentum equation*. Importantly, these equations do *not* add additional information, i.e., they are dependent upon Newton's second law. However, in many circumnstances they can be a useful substitute for some of them.

7.7 Integration of the problem and answering particular questions

The last step is to integrate the problem. The equations we have obtained in the previous step are a mix of differential and algebraic equations; the differential equations need to be integrated to obtain the equation of the trajectory of each particle as a function of time (and the initial conditions) only (see, as an example, the problem of the harmonic oscillator). It should be kept in mind that except for very particular problems, a fully analytical integration is impossible, and one needs to resort to numerical integration.

In many situations, it is possible to work on the equations to eliminate some of the unknowns from a subset of them: in this case, we say that those equations (and the unknowns still in them) are *uncoupled* from the rest of the problem. This is highly beneficial, as we can integrate each subproblem separately (divide and win). A typical case where this is possible is with normal reaction forces: we can often produce some equations in which the normal reaction forces do not appear, as discussed in the previous Section. This allows us to solve the rest of the problem first, and only then compute the magnitude of this reaction force.

In some problems, we do not seek to fully integrate the problem however. Instead, we are only interested in certain aspects of the motion, such as the regions of existence of solution, or a discussion of the shape of the possible trajectories as a function of the initial conditions: the particular questions in each problem may vary. In these cases it may suffice to work with the original differential equations

to obtain the information that we need, or we may only need to integrate them once instead of twice. This last possibility, to reduce our problem to a system of equations that depends only on time, the unknowns and, at most, *their first derivatives*, is sometimes referred to as *reducing the problem to quadratures*. The equation of conservation of mechanical energy, is already a first-integral (or quadrature) of the motion and may be used for this purpose, if applicable.

A common case in mechanics is when we can take all our quadratures and algebraic equations and produce an expression of the form $f(\dot{\theta}^2, \theta) = 0$, where θ here is any single coordinate of our problem. This function f will of course depend also on the initial conditions. This situation allows us to discuss the *regions of existence* of the solution in terms of the variable θ and the initial conditions. To do so, we observe that for a real solution to exist, $\dot{\theta}^2 \geqslant 0$. Then, the limits of the regions of existence are given by the single variable equation $f(0, \theta) = 0$. We can visualize them by plotting an "effective potential" function against θ.

7.8 Final comments

To paraphrase the famous quote by Newton "to stand on the shoulders of giants," new knowledge and abilities must build on top of previous ones. The student may find this course difficult and confusing if he is not already proficient with the necessary mathematical tools to solve the problems.

A list of essential mathematical knowledge and skills is given below in no particular order. Students are urged to review their notes from other courses until they are *totally confident* with all of the following matters:

- Linear algebra, vector operations, and vector projections: being able to express vectors in any basis, transforming from one basis to another, projecting any vector in a 3D orthogonal basis with sin and cos of the relevant angles.
- Equations: general skills for manipulating and rearranging systems of equations. Trigonometric relations, hyperbolic functions relations, complex exponentials.
- Calculus: being able to differentiate any mathematical expression without difficulty. Being able to apply changes of variable and integration by parts to solve immediate integrals. Solving linear ordinary differential equations with constant coefficients.
- Curve sketching: given any single-variable function, being able to plot it in a faithfull way, clearly identifying maxima and minima, zeros, asymptotes, and the correct behavior at infinity.

8 Orbital mechanics: Kepler's problem

To illustrate the Newtonian mechanics approach of the previous Chapter, we consider here the motion of a spacecraft of mass m^P around the Earth of mass $m^\oplus$ as shown in Fig. 8.1. The spacecraft is modeled as a point particle P, and the Earth as a point particle O. We assume there is nothing else in the universe that can perturb the motion of P, and consider an inertial reference frame S_0 with the origin on the barycenter of P and O. Since $m^P \ll m^\oplus$, we can neglect the acceleration of O and take O as the barycenter of the system. This problem is known as Kepler's problem or the two-body problem in celestial mechanics.

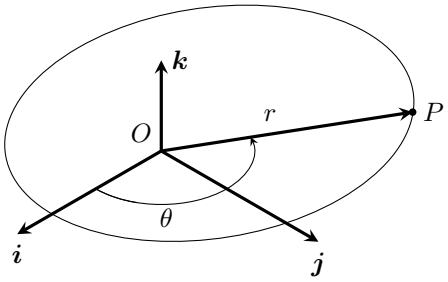

Figure 8.1: Definition of Kepler's problem.

The particle P is subject to the gravitational force of the Earth, which is inversely proportional to the square of the distance between the particle and the origin,

$$\boldsymbol{F} = -\frac{\mu}{r^2} m^P \boldsymbol{e}_r, \tag{8.1}$$

where $\mu = Gm^\oplus$ is the gravitational parameter of the Earth, r is the modulus of $\boldsymbol{r}_0^P$, and $\boldsymbol{e}_r = \boldsymbol{r}_0^P/r$ is the unit vector in the direction of $\boldsymbol{r}_0^P$.

8.1 Kepler's laws

The two body problem was studied by Kepler (1571–1630) who lived before Newton was born. He described the motion of planets around the sun based only in observations, and postulated the following *Kepler's laws*:

1. The orbits of the planets are ellipses, with the Sun at one of the foci.
2. The line joining a planet to the Sun sweeps out equal areas in equal intervals of time
3. The square of the period of a planet is proportional to the cube of the major axis of its elliptical orbit

However, he provided no explanation as to why the observations matched those results. We had to wait until Newton established his three laws of Mechanics and his law of gravitation to explain these results as a consequence of a physicomathematical framework. Naturally, in our problem, we have to substitute "planet" by spacecraft, and "Sun" by Earth.

In the following, we will start from Newton's laws and verify that the above three laws of Kepler can indeed be derived from Newtonian mechanics.

8.2 Kinematics

Particle P is not subject to any constraint so it has three effective degrees of freedom. We note that the line of action of the only force in the problem, $\boldsymbol{F}$, always passes by O, and thus $\boldsymbol{F}$ is a *central force*. This already tells us that the motion is planar. thus, we choose our inertial reference frame S_0 so that the plane of motion coincides with the S_0 reference plane. Moreover, we choose cylindrical

coordinates $\{r, \theta, z\}$ to describe the motion of P about O. The expressions of the vectors of the local basis associated to the cylindrical coordinates in terms of the vector basis of S_0 are:

$$\begin{cases} \boldsymbol{e}_r = \cos\theta \boldsymbol{i} + \sin\theta \boldsymbol{j}, \\ \boldsymbol{e}_\theta = -\sin\theta \boldsymbol{i} + \cos\theta \boldsymbol{j}, \\ \boldsymbol{e}_z = \boldsymbol{k}. \end{cases} . \tag{8.2}$$

From our choice of reference frame, $z = 0$ throughout the motion. The position, velocity and acceleration vectors of P can be written in terms of r, θ and their derivatives as:

$$\begin{cases} \boldsymbol{r}_0^P = r\boldsymbol{e}_r \\ \boldsymbol{v}_0^P = \dot{r}\boldsymbol{e}_r + r\dot{\theta}\boldsymbol{e}_\theta \\ \boldsymbol{a}_0^P = (\ddot{r} - r\dot{\theta}^2)\boldsymbol{e}_r + (2\dot{r}\dot{\theta} + r\ddot{\theta})\boldsymbol{e}_\theta \end{cases} \tag{8.3}$$

8.3 Dynamics

Newton's second law applied to P in the inertial reference frame S_0 states that

$$\boldsymbol{F} = -\frac{\mu}{r^2} m^P \boldsymbol{e}_r = m^P \boldsymbol{a}_0^P . \tag{8.4}$$

Projecting this equation along $\boldsymbol{e}_r$ and $\boldsymbol{e}_\theta$ yields

$$\boldsymbol{e}_r : -\frac{\mu}{r^2} = \ddot{r} - r\dot{\theta}^2, \tag{8.5}$$

$$\boldsymbol{e}_\theta : \quad 0 = r\ddot{\theta} + 2\dot{r}\dot{\theta}. \tag{8.6}$$

After multiplying Eq. (8.6) by r, we can integrate this expression directly to yield

$$h = r^2\dot{\theta} = \text{const.} \tag{8.7}$$

The quantity h is the magnitude of the *mass-specific* angular momentum vector about the origin O as seen from S_0, $\boldsymbol{h} = \boldsymbol{r}_0^P \times \boldsymbol{v}_0^P$, and it is constant as expected in a central force problem. The vector $\boldsymbol{h}$ points in the direction perpendicular to the plane of motion. The value of $\boldsymbol{h}$ must be determined from the initial conditions of the problem.

8.4 Proof of Kepler's laws from Newtonian mechanics

The area swept by $\boldsymbol{r}$ in an infinitesimal time interval $\mathrm{d}t$ is $\mathrm{d}A = r^2\mathrm{d}\theta/2$ (the area of the triangle is one half its base times its height). Dividing this expression by $\mathrm{d}t$ it is clear that

$$\frac{\mathrm{d}A}{\mathrm{d}t} = \frac{1}{2} r^2\dot{\theta} = \frac{h}{2} \tag{8.8}$$

which proves Kepler's second law: the line joining the spacecraft to the Earth sweeps out equal areas in equal intervals of time.

To integrate Eq. (8.5) we first note that

$$-r^2 \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{r}\right) = \dot{r} \tag{8.9}$$

and from Eq. (8.7), $r^2 = h/\dot{\theta}$, we can write

$$\dot{r} = -\frac{h}{\dot{\theta}} \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{r}\right) = -h \frac{\mathrm{d}}{\mathrm{d}\theta}\left(\frac{1}{r}\right) . \tag{8.10}$$

Differentiating with respect to time:

$$\ddot{r} = -h \frac{\mathrm{d}^2}{\mathrm{d}\theta^2}\left(\frac{1}{r}\right)\dot{\theta} = -\frac{h^2}{r^2}\frac{\mathrm{d}^2}{\mathrm{d}\theta^2}\left(\frac{1}{r}\right) , \tag{8.11}$$

where again Eq. (8.7) has been used to replace $\dot{\theta}$ in the last expression. Inserting this expression into Eq. (8.5), and using once more Eq. (8.7), we obtain the following differential equation as a function of θ:

$$\frac{\mathrm{d}^2}{\mathrm{d}\theta^2}\left(\frac{1}{r}\right)+\frac{1}{r}=\frac{\mu}{h^2}. \tag{8.12}$$

This is the linear second order ordinary differential equation of a forced harmonic oscillator in the variable $(1/r)$, and its general solution reads:

$$\frac{1}{r}=\frac{\mu}{h^2}(1+e\cos(\theta+\psi)), \tag{8.13}$$

where e and ψ are two integration constants. If we redefine our cylindrical coordinates so that $\theta=0$ when r is minimum, then e will be positive, and $\psi=0$. The equation describing the trajectory will be

$$r=\frac{h^2/\mu}{1+e\cos\theta}. \tag{8.14}$$

This is the equation of a conic section in cylindrical coordinates about one of its foci, with eccentricity e and semi-parameter $p=h^2/\mu$. Depending on the value of e, the conic section will be either an open or closed curve. In particular, we have that if

1. $e=0$, the curve is a circle;
2. $e<1$, the curve is an ellipse;
3. $e=1$, the curve is a parabola;
4. $e>1$, the curve is a hyperbola.

When $e<1$, the trajectory is an ellipse, thus proving Kepler's first law: the orbit of the spacecraft is an ellipse with the Earth at one focus. Actually, our result shows that other types of orbits are also possible; these other orbits are used in interplanetary flight. The point in the trajectory which is closest to the focus is called the *periapsis* or *pericenter*, r_p. For elliptic orbits, the point in the trajectory which is farthest away from the focus is called the *apoapsis* or*apocenter*, r_a. When considering orbits around the Earth, these points are called the *perigee* and *apogee*, whereas for orbits around the Sun, these points are called the *perihelion aphelion*, respectively.

Two parameters, p and e, suffice to determine the geometry of a conic section orbit. however, it is common to substitute p with the semi-major axis a of the orbit, given by

$$2a=r_p+r_a. \tag{8.15}$$

Using Eq. (8.14) at $\theta=0$ and π we can find $r_p=a(1-e)$ and $r_a=a(1+e)$ (the second expression makes sense only for an ellipse). The semi-minor axis b and the semi-major axis are related through $b=a\sqrt{1-e^2}$. Lastly, the semi-parameter can be written as $p=a(1-e^2)$. In the case of circles and ellipses, $a>0$, for parabolas $a=\infty$, and for hyperbolas $a<0$.

The area of the ellipse is πab. Since $\mathrm{d}A/\mathrm{d}t=h/2$ is a constant, we have $A_e=h\tau/2$ where τ is the period of the orbit. Equating these two expressions and expressing h in terms of the semi-major axis as

$$h^2=\mu p=\mu a(1-e^2), \tag{8.16}$$

we have

$$\mu=\left(\frac{2\pi}{\tau}\right)^2 a^3, \tag{8.17}$$

which proves Kepler's third law: the square of the spacecraft is proportional to the cube of the major axis of its elliptical orbit. This can be rewritten to obtain the time of flight or period of the orbit

$$\tau=\frac{2\pi}{\sqrt{\mu}}a^{3/2}. \tag{8.18}$$

8.5 Vis-viva equation

To conclude, we note that the mechanical energy of P is conserved, since the only force of the problem (gravity) is conservative. We could have used the energy conservation equation to substitute one of the projections of the equation in Newton's second law. If we define the *mass-specific* mechanical energy ξ, this equation reads:

$$\xi = \frac{v^2}{2} - \frac{\mu}{r} = \text{const} = -\frac{\mu}{2a}, \tag{8.19}$$

where the value of the constant has been computed particularizing the expression at pericenter and apocenter, and using the rest of relations obtained so far. This equation is known as the *vis-viva* equation.

This expression is very useful to obtain various characteristic velocities of the orbital motion. For example, if the spacecraft is in a circular orbit, then $r = a$ and it has *circular velocity* of magnitude v_c:

$$v_c = \sqrt{\frac{\mu}{r}}. \tag{8.20}$$

The minimum velocity that a spacecraft needs to leave the Earth and reach infinity (As $r \to \infty$ the second term in Eq. (8.19) goes to zero) is the *escape velocity*, which can be easily found to be $\sqrt{2}$ times the circular velocity:

$$v_e = \sqrt{\frac{2\mu}{r}}. \tag{8.21}$$

Part II
Mechanics of the rigid body

9 Kinematics of rigid bodies

Modeling a physical object as a point particle has its limitations: it ignores the object dimensions and mass distribution, and moreover, it cannot describe the orientation (or attitude) of the object in space. The rigid body model, i.e., a body with a finite, indeformable geometry, overcomes these limitations. Observe that the concept of a purely rigid body is an idealized one, just like the concept of a point particle, because all materials deform to some extent when forces are applied onto them. Nevertheless, there is a wide range of problems in Flight Mechanics where the small deformations of bodies can be ignored in the study of their motion.

In a rigid body, the distance between any pair of points is invariant. If we select any three non-collinear points P, Q, R of the body, we can define the *body-fixed* reference frame $S_B : \{P; B_B\}$ attached to the body as shown in Fig. 9.1, with $B_B = \{\boldsymbol{PQ}, \boldsymbol{PR}, \boldsymbol{PQ} \times \boldsymbol{PR}\}$. Note that the choice of the origin point P and the basis B_B are arbitrary, as long as P is a fixed point of the body, and the vectors of B_B are attached to the body. Hence, we commonly choose a right-handed orthonormal vector basis. We can make two important observations about rigid bodies and body-fixed reference frames:

1. Since the reference frame S_B is attached to the rigid body, it will maintain its orientation relative to the body when the latter moves, i.e., they will rotate with the same angular velocity vector $\boldsymbol{\omega}_{B0}$. Thus, knowing the orientation of of the body and knowing the orientation of S_B is equivalent.

2. For an observer on S_B, all points of the rigid body are at rest relative to him, i.e., their relative velocity vector is zero. Indeed, we can think of each point of the Euclidean space attached to S_B as a point of the rigid body, even if this point is outside the physical body itself, and we can assign it a rigid body velocity and acceleration vector like any other point of the body.

These two observations allow us to use the concept of rigid body and reference frame interchangeably: to study the motion of any rigid body, we can replace it by a body-fixed reference frame and study the motion of that reference frame instead. Likewise, we can think of any reference frame as a rigid body, with an Euclidean space attached to it as it moves in space.

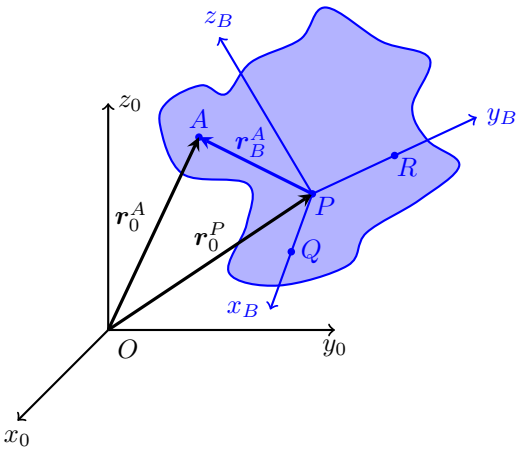

Figure 9.1: A rigid body with angular velocity $\boldsymbol{\omega}$ and a body-fixed reference frame S_B with origin in a body point P.

9.1 Velocity and acceleration fields of a rigid body

Consider points A and P of the rigid body as shown in Fig. 9.1. Using Eqs. (3.26)–(3.27) for the composition of velocities and accelerations of A, taking into account that the relative velocity and acceleration of A in $S_B : \{P; B_B\}$ is zero, we may write

$$\boldsymbol{v}_0^A = \cancel{\boldsymbol{v}_B^A} + \boldsymbol{v}_0^P + \boldsymbol{\omega}_{B0} \times \boldsymbol{r}_B^A, \tag{9.1}$$

$$\boldsymbol{a}_0^A = \cancel{\boldsymbol{a}_B^A} + \boldsymbol{a}_0^P + \boldsymbol{\alpha} \times \boldsymbol{r}_B^A + \boldsymbol{\omega}_{B0} \times (\boldsymbol{\omega}_{B0} \times \boldsymbol{r}_B^A) + \cancel{2\boldsymbol{\omega} \times \boldsymbol{v}_B^A}. \tag{9.2}$$

The velocity and acceleration vectors of any point A of the rigid body are the superposition of the movement of an arbitrarily selected origin point P and a rotation about P:

$$\boldsymbol{v}_0^A = \boldsymbol{v}_0^P + \boldsymbol{\omega}_{B0} \times \boldsymbol{PA}, \tag{9.3}$$

$$\boldsymbol{a}_0^A = \boldsymbol{a}_0^P + \boldsymbol{\alpha}_{B0} \times \boldsymbol{PA} + \boldsymbol{\omega}_{B0} \times (\boldsymbol{\omega}_{B0} \times \boldsymbol{PA}). \tag{9.4}$$

Equations (9.3) and (9.4) are called the *velocity and acceleration fields* of the rigid body.

9.1.1 Properties of the velocity field of a rigid body

It is evident that to fully determine the velocity field of a rigid body at a given instant of time t, we just need to know the velocity of one of its point and its angular velocity. Some conclusions can be drawn from Eq. (9.3):

1. If the angular velocity of the body is $\boldsymbol{\omega}_{B0} = 0$, then $\boldsymbol{v}_0^A = \boldsymbol{v}_0^P$. All the points of the body have the same velocity and the motion is a *pure translation*.

2. Defining the unit vector in the direction of the angular velocity $\boldsymbol{e}_\omega = \boldsymbol{\omega}_{B0}/\omega_{B0}$, if we dot-multiply Eq. (9.3) by $\boldsymbol{e}_\omega$, we find in a similar way that

   $$\boldsymbol{v}_0^A \cdot \boldsymbol{e}_\omega = \boldsymbol{v}_0^P \cdot \boldsymbol{e}_\omega = v_{0\parallel}, \tag{9.5}$$

   which states that all the points in the rigid body have the same velocity component $v_{0\parallel}$ along the direction of the angular velocity vector. If $v_{0\parallel} = 0$, then the motion is termed a *pure rotation*.

   Observe that it is always possible to decompose the instantaneous velocity of any body point A as the sum of a component in the parallel direction to the angular velocity, $\boldsymbol{v}_{0\parallel} = (\boldsymbol{v}_0^A \cdot \boldsymbol{e}_\omega)\boldsymbol{e}_\omega$, which is identical for all points in the rigid body, and a component which is contained in the plane perpendicular to the angular velocity, $\boldsymbol{v}_{0\perp}^A$, particular for each point in the rigid body:

   $$\boldsymbol{v}_0^A = \boldsymbol{v}_{0\parallel} + \boldsymbol{v}_{0\perp}^A. \tag{9.6}$$

   The parallel component $\boldsymbol{v}_{0\parallel}$ is often termed *slip velocity*.

3. Likewise, if we dot-multiply Eq. (9.3) by $\boldsymbol{PA}$, we find that

   $$\boldsymbol{v}_0^A \cdot \boldsymbol{PA} = \boldsymbol{v}_0^P \cdot \boldsymbol{PA}, \tag{9.7}$$

   i.e., any two points in a rigid body have the same velocity projection along the line that joins the two points, as shown in Fig. 9.2.

9.1.2 Instantaneous axis of rotation (and slip)

In a pure rotation ($\boldsymbol{v}_{0\parallel} = \boldsymbol{0}$) we define the *instantaneous axis of rotation* as the locus of rigid body points with zero velocity. To locate the instantaneous axis of rotation, we need to find one such body point I whose velocity is $\boldsymbol{v}_0^I = \boldsymbol{0}$. The instantaneous axis of rotation is then the line parallel to $\boldsymbol{\omega}_{B0}$ passing through I. To find I given a point A of the rigid body and its velocity $\boldsymbol{v}_0^A$, and knowing $\boldsymbol{\omega}_{B0}$, we use the expression of the velocity field of the rigid body to impose the following condition:

$$\boldsymbol{v}_0^I = \boldsymbol{v}_0^A + \boldsymbol{\omega}_{B0} \times \boldsymbol{AI} = \boldsymbol{0}. \tag{9.8}$$

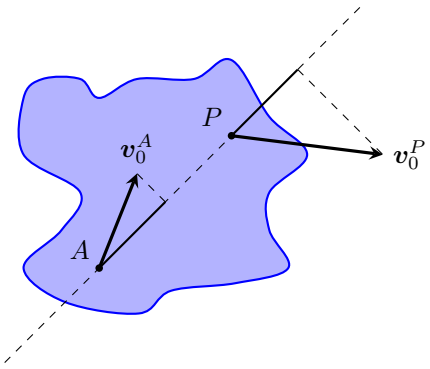

Figure 9.2: Illustration of the rigid body velocity field property $\boldsymbol{v}_0^A \cdot \boldsymbol{PA} = \boldsymbol{v}_0^P \cdot \boldsymbol{PA}$.

Cross-multiplying by $\boldsymbol{\omega}_{B0}$ and using the vector identity Eq. (1.37) we obtain

$$\boldsymbol{\omega}_{B0} \times \boldsymbol{v}_0^A + (\boldsymbol{\omega}_{B0} \cdot \boldsymbol{AI})\boldsymbol{\omega}_{B0} - \omega_{B0}^2 \boldsymbol{AI} = \boldsymbol{0} \tag{9.9}$$

Now, from all the possible points I that satisfy Eq. (9.8), if we choose to find the perpendicular foot from A to the instantaneous axis of rotation (see Fig. 9.3), then the dot product in the second term of the last equation vanishes, as $\boldsymbol{AI}$ is perpendicular to $\boldsymbol{\omega}$ in that case. Then, we can find I from the position of A simply as

$$\boldsymbol{AI} = \frac{\boldsymbol{\omega}_{B0} \times \boldsymbol{v}_0^A}{\omega_{B0}^2}, \tag{9.10}$$

The instantaneous axis of rotation is then defined by point I and the vector $\boldsymbol{\omega}_{B0}$. Note that:

1. The axis of rotation may change in time (hence the adjective, instantaneous). The ruled surfaces that results when the instantaneous axis of rotation varies in time, as seen from S_0 and from S_B, are sometimes useful to visualize the motion of rigid bodies.

2. Point I (or even the totality of the instantaneous axis of rotation) may lie outside the physical boundaries of the rigid body. Nonetheless, it is always a point of the Euclidean space attached to S_B, and is therefore still considered a body point, as described in the last part of §9.1.4.

3. The instantaneous axis of rotation depends on the reference frame from where it is computed. An observer on another reference frame may find that the instantaneous axis of rotation is different.

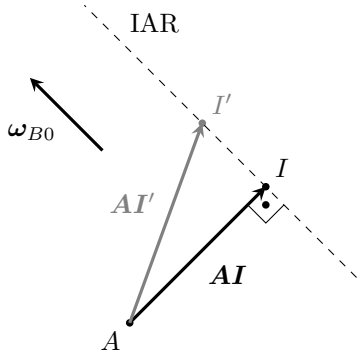

Figure 9.3: Finding point I as the perpendicular foot from A to the instantaneous axis of rotation (IAR). Any other possible choice (e.g. I') would result in a more complicated calculation, since the second term in Eq. (9.9) would not vanish.

If $\boldsymbol{v}_{0\parallel} \neq \boldsymbol{0}$, the most general case, we can define instead the *instantaneous axis of rotation and slip*. This is the locus of points which have minimal velocity. Since the velocity component $\boldsymbol{v}_{0\parallel}$ is identical for all points in the body, we want to locate a point I whose velocity is $\boldsymbol{v}_0^I = \boldsymbol{v}_{0\parallel}$, i.e., a point that does not have any component in the perpendicular direction, $\boldsymbol{v}_{0\perp} = \boldsymbol{0}$. The equation for $\boldsymbol{v}_0^I$ then reads

$$\boldsymbol{v}_0^I = \boldsymbol{v}_0^A + \boldsymbol{\omega}_{B0} \times \boldsymbol{AI} = \boldsymbol{v}_{0\parallel}. \tag{9.11}$$

Observe that if we cross-multiply by $\boldsymbol{\omega}_{B0}$ we can find $\boldsymbol{AI}$ as before with Eq. (9.10), since $\boldsymbol{\omega}_{B0} \times \boldsymbol{v}_{0\parallel} = \boldsymbol{0}$.

9.1.3 General rigid body motion

In general, the instantaneous motion of a rigid body can be decomposed as the superposition of a pure rotation about an instantaneous axis, plus a pure translation in the direction of $\boldsymbol{\omega}_{B0}$ with slip velocity $\boldsymbol{v}_{0\parallel}$. Therefore, the more general instantaneous velocity field of a rigid body is that of a screw, as depicted in Fig. 9.4. Remember that the instantaneous axis of rotation and slip may lie outside of the physical body. Of course, both the position of the axis of rotation, its orientation, and the magnitude and direction of the slip velocity $\boldsymbol{v}_{0\parallel}$, may change in time.

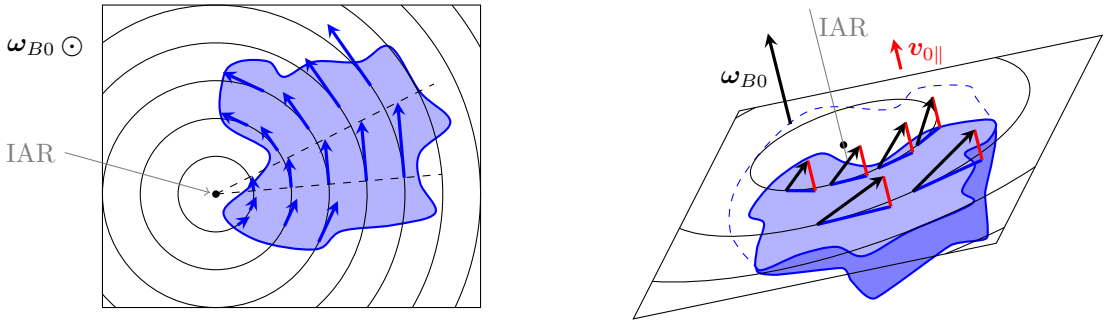

Figure 9.4: Screw motion velocity field, the most general case for a rigid body. The sketch on the left shows the velocity field in one plane perpendicular to $\boldsymbol{\omega}_{B0}$. The vector component $\boldsymbol{v}_{0\perp}$ (in blue) of the velocity of each point is arranged in concentric circles about the instantaneous axis of rotation and slip (IAR); its magnitude increases linearly with the distance from this axis. The sketch on the right shows a 3D view of the velocity field in this plane, including the slip velocity $\boldsymbol{v}_{0\parallel}$ (in red), which is identical for all points in the rigid body.

Observe that there are multiple ways of determining completely the velocity field of a body. For example:

1. Giving the velocity $\boldsymbol{v}_0^A$ of a known point A of the body and the angular velocity vector $\boldsymbol{\omega}_{B0}$.
2. Specifying the position of a point I of the instantaneous axis, the angular velocity vector $\boldsymbol{\omega}_{B_0}$ and the magnitude $v_{0\parallel}$.
3. Providing the velocities of any three non-collinear points A, C, D, and using the velocity field expression between them:

$$\boldsymbol{v}_0^C = \boldsymbol{v}_0^A + \boldsymbol{\omega}_{B0} \times \boldsymbol{AC}, \tag{9.12}$$

$$\boldsymbol{v}_0^D = \boldsymbol{v}_0^A + \boldsymbol{\omega}_{B0} \times \boldsymbol{AD}. \tag{9.13}$$

4. Giving the velocity vectors of any two points and an extra independent condition that completes the description.

9.1.4 Points of a rigid body

It is important to note that Eqs. (9.3) and (9.4) assume that points P and A are *points of the rigid body*, i.e., points that move with the rigid body, so that they have zero relative velocity in any body-fixed reference frame like S_B.

In most cases, it is possible to define any point of the body with a simple geometric condition. For example, if our rigid body is a rectangle, "the body point M which lies on the center of the rectangle" is clearly and unambiguously defined. These cases do not represent a conceptual difficulty and are easy to understand.

Sometimes, however, we need to refer to a body point using an advanced spatial condition. For example, consider a rigid body B moving in space, and a geometrical point A which is not part of the body, whose position vector in S_0 is $\boldsymbol{r}_0^A$. We may need to work with "the body point whose position vector instantaneously coincides with $\boldsymbol{r}_0^A$." To distinguish this point from A itself, in these notes we denote it as $A(B)$. Observe that, while the position vectors of A and $A(B)$ are equal, their velocity vectors (and acceleration vectors) can be different:

$$\boldsymbol{r}_0^A = \boldsymbol{r}_0^{A(B)}, \tag{9.14}$$

$$\boldsymbol{v}_0^A \neq \boldsymbol{v}_0^{A(B)}. \tag{9.15}$$

While $\boldsymbol{v}_0^{A(B)}$ is given by Eq. (9.3) as $A(B)$ is a rigid body point, it would be incorrect to plug $\boldsymbol{v}_0^A$ in this equation, as A is not a point of the rigid body. The same can be said, mutatis mutandis, of $\boldsymbol{a}_0^{A(B)}$ and $\boldsymbol{a}_0^A$. Note that point $A(B)$ represents the point of the rigid body that lies on top of A at a particular instant: if the rigid body moves, the actual body point that satisfies this condition will change in time. The nomenclature $A(B)$ is only used when strictly necessary to avoid ambiguity between a body point and a point which does not belong to the body. Needless to say, when working with rigid bodies it is essential to define with precision each point used, stating clearly if it is a rigid body point or not.

Lastly, observe also that expressions (9.3) and (9.4) assign a velocity vector and an acceleration vector to *any* point of the Euclidean space attached to the S_B reference frame, even if it lies "outside" of the physical boundaries of the rigid body itself. In mechanics, these external points are considered body points since they move solidarily with it.

9.2 Euler angles

As discussed above, it is possible to substitute a rigid body with a body-fixed reference frame S_B for the purpose of studying its motion in space, both translational and rotational. As advanced in §1.5.2, the orientation (or attitude) of S_B with respect to $S_0 : \{O; B_0\}$ can be defined with only three angles. When these three angles correspond to three sequential simple rotations, they are known as *Euler angles*.

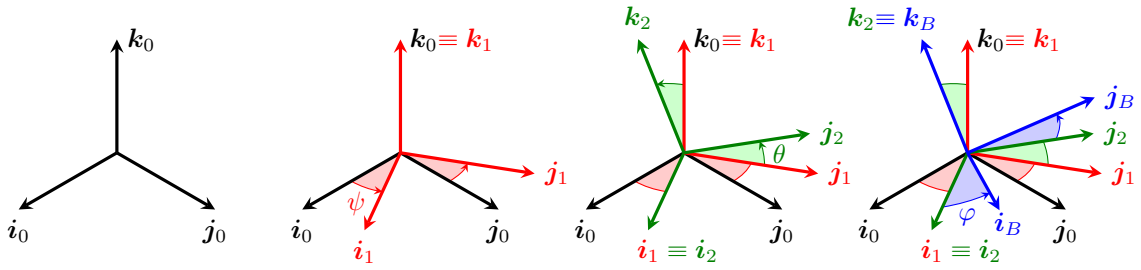

Figure 9.5: Sequence of three simple rotations of type 3-1-3 that defined the standard Euler angles.

To understand the standard Euler angles, we will follow the intermediate simple rotations as we go from S_0 to S_B, as shown in Fig. 9.5. The first rotation is about the z_0 axis, goes from S_0 to S_1, and is called *precession*. The angle of rotation in the precession is typically denoted ψ. The new vector basis $B_1 : \{\boldsymbol{i_1}, \boldsymbol{j_1}, \boldsymbol{k_1}\}$ and the rotation matrix that transforms vector components in B_1 to components in B_0 are:

$$\begin{cases} \boldsymbol{i}_1 = \cos\psi \boldsymbol{i}_0 + \sin\psi \boldsymbol{j}_0, \\ \boldsymbol{j}_1 = -\sin\psi \boldsymbol{i}_0 + \cos\psi \boldsymbol{j}_0, \\ \boldsymbol{k}_1 = \boldsymbol{k}_0. \end{cases} \qquad [{}_0R_1(z,\psi)] = \begin{bmatrix} \cos\psi & -\sin\psi & 0 \\ \sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{bmatrix}. \tag{9.16}$$

The second simple rotation is denoted *nutation*, θ, is performed about axis x_1, and goes from S_1 to S_2. The new vector basis $B_2 = \{\boldsymbol{i_2}, \boldsymbol{j_2}, \boldsymbol{k_2}\}$ and the corresponding rotation matrix are:

$$\begin{cases} \boldsymbol{i}_2 = \boldsymbol{i}_1, \\ \boldsymbol{j}_2 = \cos\theta \boldsymbol{j}_1 + \sin\theta \boldsymbol{k}_1, \\ \boldsymbol{k}_2 = -\sin\theta \boldsymbol{j}_1 + \cos\theta \boldsymbol{k}_1. \end{cases} \qquad [{}_1R_2(x,\theta)] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{bmatrix}. \tag{9.17}$$

Finally, the third and last rotation called *spin* φ, is performed about axis z_2, and goes from S_2 to S_B:

$$\begin{cases} \boldsymbol{i}_B = \cos\varphi \boldsymbol{i}_2 + \sin\varphi \boldsymbol{j}_2, \\ \boldsymbol{j}_B = -\sin\varphi \boldsymbol{i}_2 + \cos\varphi \boldsymbol{j}_2, \\ \boldsymbol{k}_B = \boldsymbol{k}_0. \end{cases} \qquad [{}_2R_B(z,\varphi)] = \begin{bmatrix} \cos\varphi & -\sin\varphi & 0 \\ \sin\varphi & \cos\varphi & 0 \\ 0 & 0 & 1 \end{bmatrix}. \tag{9.18}$$

The overall vector component transformation from B_B to B_0 can be obtained with the composed rotation matrix, i.e.,

$$[{}_0R_B] = [{}_0R_1(z,\psi)] \cdot [{}_1R_2(x,\theta)] \cdot [{}_2R_B(z,\varphi)]. \tag{9.19}$$

Thanks to the properties of orthogonal matrices, the inverse rotation matrix to transform vector components from B_0 to B_B is:

$$[{}_BR_0] = [{}_0R_B]^T = [{}_2R_B(z,\varphi)]^T \cdot [{}_1R_2(x,\theta)]^T \cdot [{}_0R_1(z,\psi)]^T. \tag{9.20}$$

The angular velocity of the rigid body is readily expressed by adding the precession, nutation, and spin rates about the respective axes using the composition formula for angular velocities, Eq. (3.14). The precession is about the z_0 axis, so the first rotation rate is $\boldsymbol{\omega}_{10} = \dot{\psi}\boldsymbol{k}_0$. The nutation occurs about the x_1 axis, so the rotation rate is $\boldsymbol{\omega}_{21} = \dot{\theta}\boldsymbol{i_1}$. The spin occurs about the z_B axis, so the rotation rate is $\boldsymbol{\omega}_{B2} = \dot{\varphi}\boldsymbol{k_B}$. The angular velocity of S_B with respect to S_0, $\boldsymbol{\omega}_{B0}$ results from their vector sum:

$$\boldsymbol{\omega}_{B0} = \boldsymbol{\omega}_{B2} + \boldsymbol{\omega}_{21} + \boldsymbol{\omega}_{10} = \dot{\varphi}\boldsymbol{k}_B + \dot{\theta}\boldsymbol{i_1} + \dot{\psi}\boldsymbol{k}_0. \tag{9.21}$$

When working on the kinematics of a rigid body, it is often necessary to make use of all the unit vectors defined above. Sometimes, a physical vector is written as a mixed sum of unit vectors of different bases. However, before using the result in vector component computations, it must be expressed in a single vector basis. For example, if we choose the B_B basis to express $\boldsymbol{\omega}_{B0}$ as $\boldsymbol{\omega}_{B0} = \omega_{B0,x_B}\boldsymbol{i}_B + \omega_{B0,y_B}\boldsymbol{j}_B + \omega_{B0,z_B}\boldsymbol{k}_B$, then using the expressions above to transform $\boldsymbol{i}_1$ and $\boldsymbol{k}_0$ in Eq. (9.21) into the B_B basis we find:

$$\omega_{B0,x_B} = \dot{\theta}\cos\varphi + \dot{\psi}\sin\theta\sin\varphi, \tag{9.22}$$

$$\omega_{B0,y_B} = -\dot{\theta}\sin\varphi + \dot{\psi}\sin\theta\cos\varphi, \tag{9.23}$$

$$\omega_{B0,z_B} = \dot{\varphi} + \dot{\psi}\cos\theta. \tag{9.24}$$

It is usually desirable to choose B_B or B_0 to express all vectors in component form, rather than any of the intermediate bases.

To compute the angular acceleration $\boldsymbol{\alpha}_{B0}$, recall from Eq. (3.9) that one of the special properties of the angular velocity vector is that $\mathrm{d}\boldsymbol{\omega}_{B0}/\mathrm{d}t|_0 = \mathrm{d}\boldsymbol{\omega}_{B0}/\mathrm{d}t|_B$, and so $\boldsymbol{\omega}_{B0}$ can be indistinctly differentiated in S_B or S_0 to obtain $\boldsymbol{\alpha}_{B0}$ (this would not be true in an arbitrary reference frame, where the application of Coriolis theorem would result in extra terms).

9.2.1 Euler angle definitions

The set of Euler angles just defined as "standard" can also be termed "3-1-3," since we are rotating first about the z axis (third axis if we label x, y, z as 1, 2, 3), then about the x_1 axis (axis 1) and then about z_2 (axis 3 again). Clearly, this is not the only possible choice: Euler angles can be defined in 12 different manners, according to the order of the simple rotations, with the only constraint that two successive rotations are never applied along the same axis:

1-2-1;	1-2-3;	1-3-1;	1-3-2;
2-1-2;	2-1-3;	2-3-1;	2-3-2;
3-1-2;	3-1-3;	3-2-1;	3-2-3.

The standard 3-1-3 Euler angles are used, for instance, in orbital mechanics to indicate the orientation of the orbit of a satellite with respect to an Earth-centered, inertial, equatorial reference frame. In this case, ψ (or Ω) is also called the *right ascension of the ascending node*, θ (or i) the *inclination angle*, and φ (or ω) the *argument of the pericenter*.

Another commonly used set of Euler angles in Flight Mechanics is the 3-2-1 type: In this case, the first angle ψ is known as *yaw*, the second angle θ as *pitch*, and the third angle φ as *roll*[15].

9.2.2 Singularities

Euler angles present certain limitations when describing arbitrary orientations, as all possible angle definitions discussed above lead to some singularity. These are particular orientations of the rigid body that arise when the second rotation aligns the first and third axes of rotation, as then the first and third rotation angles are ambiguously defined.

For instance, in the standard 3-1-3 Euler angles, the precession and spin angle are not uniquely defined when $\theta = 0$ or π. In the yaw-pitch-roll case (3-2-1 Euler angles), the singularity arises when $\theta = \pi/2$ or $3\pi/2$. In practical problems, one chooses the definition of Euler angles in order to minimize the occurrence of the singularity during the motion of the rigid body (think, for example, that an aircraft will very rarely fly completely nose-up or nose-down). The existence of a singularity in the Euler angles is related to the problem of "gimbal lock" in rotating systems composed of three gimbals.

To describe the orientation of a rigid body in a way that completely avoids singularities, other mathematical devices are used, such as the so-called *quaternions*. The study of quaternions, whose use is frequent in orbital mechanics, is beyond the scope of this course.

9.3 Degrees of freedom of a rigid body

In previous Sections we have seen that the motion of a rigid body can be regarded as the combination of the translation of an arbitrary point plus a rotation about it, Eq. (9.3). In the absence of constraints, determining the position of that arbitrary point requires three coordinates. It is quite common to choose the center of mass G of the rigid body, which will be defined in the next chapter, as that arbitrary point. Likewise, determining the angular position of the body requires three additional coordinates, such as the Euler angles introduced above.

Therefore, in general, each free rigid body in a problem adds six degrees of freedom: given these six parameters, the full configuration of the rigid body, and thus the position of any of its points, can be known without ambiguity.

[15] In Spanish: *guiñada*, *cabeceo* and *alabeo*.

10 Geometry of masses and inertia properties

While the necessary inertia properties of a point particle were just its mass m, in the case of a rigid body we need to take into account its spatial mass distribution, $\mathcal{D}(\boldsymbol{r}_0^P)$. In this chapter we define the total mass, center of mass, and moments/products/tensor of inertia of a rigid body using the zeroth, first and second moments of this mass distribution. As we shall see in the following Chapters, these are all the inertia properties of a rigid body that we need to consider to study its dynamics. These definitions are also valid for any non-rigid body or for a system of particles, so, in order to generalize, we will speak of a 'material system' rather than a rigid body. In all cases, a superindex B or similar should be used if necessary to indicate the rigid body or material system that these variables refer to.

10.1 Moment of a mass distribution

Consider a material system B made of point particles of mass m_i and position $\boldsymbol{r}_0^{P_i}$ for $i = 1, \ldots, N$, as shown in Fig. 10.1. The n-th moment of this discrete mass distribution $\mathcal{D}(\boldsymbol{r}_0^P)$ about a point O is defined as:

$$\mathcal{M}^{\mathcal{D}}_{O,n} = \sum_{i=1}^{N} \underbrace{\boldsymbol{r}_0^{P_i} \otimes \boldsymbol{r}_0^{P_i} \otimes \ldots \otimes \boldsymbol{r}_0^{P_i}}_{n} m_i \tag{10.1}$$

where $\otimes$ denotes the tensor product as defined in §1.6.5. For $n = 0, 1, 2$, this general expression simplifies to:

$$\mathcal{M}^{\mathcal{D}}_{O,0} = \sum_{i=1}^{N} m_i; \quad \mathcal{M}^{\mathcal{D}}_{O,1} = \sum_{i=1}^{N} \boldsymbol{r}_0^{P_i} m_i; \quad \mathcal{M}^{\mathcal{D}}_{O,2} = \sum_{i=1}^{N} \boldsymbol{r}_0^{P_i} \otimes \boldsymbol{r}_0^{P_i} m_i. \tag{10.2}$$

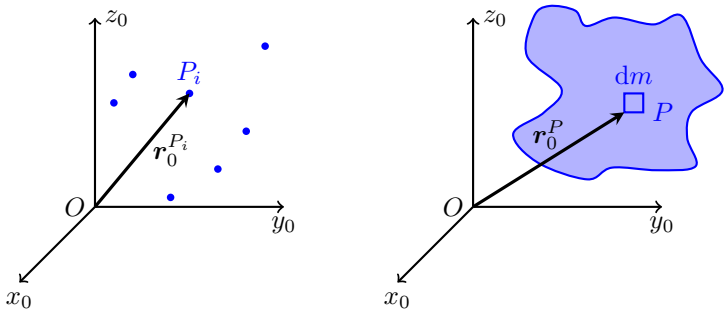

Figure 10.1: A material system made of discrete point particles (left) and a continuous mass distribution (right).

If $\mathcal{D}(\boldsymbol{r}_0^P)$ is a continuous mass distribution instead, the sums above must be replaced by integrals:

$$\mathcal{M}^{\mathcal{D}}_{O,n} = \int_{\mathcal{D}} \underbrace{\boldsymbol{r}_0^P \otimes \boldsymbol{r}_0^P \otimes \ldots \otimes \boldsymbol{r}_0^P}_{n} \mathrm{d}m, \tag{10.3}$$

where $\boldsymbol{r}_0^P$ is the position vector of the integration point P that runs over the whole mass distribution with $\mathrm{d}m$. The moments for $n = 0, 1, 2$ then read:

$$\mathcal{M}^{\mathcal{D}}_{O,0} = \int_{\mathcal{D}} \mathrm{d}m; \quad \mathcal{M}^{\mathcal{D}}_{O,1} = \int_{\mathcal{D}} \boldsymbol{r}_0^P \mathrm{d}m; \quad \mathcal{M}^{\mathcal{D}}_{O,2} = \int_{\mathcal{D}} \boldsymbol{r}_0^P \otimes \boldsymbol{r}_0^P \mathrm{d}m. \tag{10.4}$$

Depending on their geometry, continuous mass distributions can be roughly classified as 1D objects (e.g. a bar), 2D objects (e.g. a plate) and 3D objects (e.g. a sphere), as shown in Fig. 10.2(d). For each of these cases it is useful to define a corresponding mass density:

- In 1D, linear density $\mathrm{d}m = \lambda \mathrm{d}\ell$,
- In 2D, area density $\mathrm{d}m = \sigma \mathrm{d}A$,

- In 3D, volume density $\mathrm{d}m = \rho \mathrm{d}V$,

where $\mathrm{d}\ell$, $\mathrm{d}A$ and $\mathrm{d}V$ are the differential elements of length, area and volume, respectively. If the density is constant then the system or body is said to be *homogeneous*. These definitions of densities are useful to replace the mass differential $\mathrm{d}m$ in the integrals in terms of a length, area or volume differential.

One of the biggest difficulties when computing integrals like the ones in Eq. (10.4) is expressing $\mathrm{d}\ell$, $\mathrm{d}A$ or $\mathrm{d}V$ as a function of the differentials of the coordinates that are used to parametrize the mass distribution in space. For instance, in Cartesian (x, y, z), cylindrical (R, θ, z), and spherical (r, θ, ϕ) coordinates the volume differential can be written as

$$\mathrm{d}V = \mathrm{d}x\mathrm{d}y\mathrm{d}z = R\mathrm{d}R\mathrm{d}\theta\mathrm{d}z = r^2 \cos\phi \mathrm{d}r\mathrm{d}\theta\mathrm{d}\phi. \tag{10.5}$$

After this change, a multiple integral needs to be solved, for which we must write the limits of integration on each of the coordinates. For more exotic coordinate sets, it is always possible to start in Cartesian coordinates and change variables to the new coordinates by introducing the Jacobian of the transformation.

In the following, only the expressions for a continuous mass distribution will be shown for brevity, but it should be kept in mind that any discrete masses will give rise to discrete sums as in Eq. (10.1).

10.2 Total mass

The *total mass* m of a material system B is defined simply as the zeroth-moment of its mass distribution, $\mathcal{M}^{\mathcal{D}}_{n,0}$, i.e., as the integral of its mass over the whole system:

$$m = \int_{\mathcal{D}} \mathrm{d}m. \tag{10.6}$$

10.3 Center of mass

The *center of mass* of a material system B is the point G where, if we concentrated the total mass the system, the first moment of the mass distribution would remain the same. Mathematically,

$$m\boldsymbol{r}_0^G = \mathcal{M}^{\mathcal{D}}_{O,1} = \int_{\mathcal{D}} \boldsymbol{r}_0^P \mathrm{d}m. \tag{10.7}$$

The position of the center of mass $\boldsymbol{r}_0^G$ is therefore defined as

$$\boldsymbol{r}_0^G = \frac{1}{m}\int_{\mathcal{D}} \boldsymbol{r}_0^P \mathrm{d}m = \frac{\mathcal{M}^{\mathcal{D}}_{O,1}}{\mathcal{M}^{\mathcal{D}}_{O,0}}, \tag{10.8}$$

i.e., the quotient between the first and the zeroth moment of the mass distribution.

When the mass distribution has a plane or an axis of symmetry, the center of mass lies on it. This is an important property that often helps simplify the calculation of the position of the center of mass.

10.4 Moments and products of inertia

The moments and products of inertia of a material system B are related to the second moment of the mass distribution function. These inertia properties are important for the rotational dynamics of rigid bodies and, in general, any material system. They have dimensions of $[ML^2]$, and in the international system they have units of kg·m^2. The *moment of inertia* with respect to a point, a line or a plane is defined as

$$I = \int_{\mathcal{D}} d^2 \mathrm{d}m, \tag{10.9}$$

where d is the distance between the differential element of mass $\mathrm{d}m$ and the point, line or plane under consideration. A subindex on I is typically used to denote the point, line or plane with respect to

which the moment of inertia is defined. For example, the moment of inertia with respect to the origin O of a reference frame S_0 (sometimes called the *polar* moment of inertia) is

$$I_O = \int_{\mathcal{D}} (r_0^P)^2 \mathrm{d}m = \int_{\mathcal{D}} \left(x^2 + y^2 + z^2\right) \mathrm{d}m. \tag{10.10}$$

With respect to the coordinated axes Ox, Oy and Oz the moment of inertia takes the form:

$$I_x = \int_{\mathcal{D}} \left(y^2 + z^2\right) \mathrm{d}m; \quad I_y = \int_{\mathcal{D}} \left(x^2 + z^2\right) \mathrm{d}m; \quad I_z = \int_{\mathcal{D}} \left(x^2 + y^2\right) \mathrm{d}m, \tag{10.11}$$

and, with respect to the coordinated planes Oxy, Oxz, Oyz we have:

$$I_{xy} = \int_{\mathcal{D}} z^2 \mathrm{d}m; \quad I_{xz} = \int_{\mathcal{D}} y^2 \mathrm{d}m; \quad I_{yz} = \int_{\mathcal{D}} x^2 \mathrm{d}m. \tag{10.12}$$

Observe that since the integrand above is always positive, the moment of inertia (with respect to a point, line or plane) is always positive. The following relations can be established between the different moments of inertia, which follow directly from the definitions above. They are useful to simplify calculations when computing moments of inertia:

$$2I_O = 2\left(I_{xy} + I_{xz} + I_{yz}\right) = I_x + I_y + I_z \tag{10.13}$$

The *product of inertia* of a system with respect to two orthogonal planes π_1 and π_2 is defined as

$$P_{\pi_1,\pi_2} = \int_{\mathcal{D}} d_1 d_2 \mathrm{d}m, \tag{10.14}$$

where d_1 and d_2 are the signed distances between the differential element of mass $\mathrm{d}m$ and each of the planes. As before, subindices are used to indicate what planes are used in the definition. When the planes are two of the coordinated planes Oxy, Oxz, Oyz we have:

$$P_{xy,xz} = \int_{\mathcal{D}} zy \mathrm{d}m; \quad P_{xy,yz} = \int_{\mathcal{D}} zx \mathrm{d}m; \quad P_{xz,yz} = \int_{\mathcal{D}} yx \mathrm{d}m. \tag{10.15}$$

Note that the sign of the distances d_1 and d_2 is taken into account in the definition of P. Therefore, the product of inertia can be negative.

Whenever one of the planes involved in the calculation of a product of inertia is a plane of symmetry of the material system, the product of inertia is zero. Observing the symmetries of the system before starting the computation of its inertia properties can save an important amount of effort.

10.5 Examples

To illustrate the calculation of the total mass, center of mass, and moments/products of inertia, in this Section a number of examples are shown.

Half-circumference wire

We are asked to compute the inertia properties of the 1D wire $\mathcal{D}$ with linear density λ shown in Fig. 10.2(a). The equation of the wire in parametric form is:

$$\begin{cases} x = R\cos\theta, \\ y = R\sin\theta, \end{cases} \tag{10.16}$$

with θ in the range $[0, \pi]$. We begin by expressing the mass differential in terms of a length differential, and finally in terms of the differential of our parameter θ

$$\mathrm{d}m = \lambda \mathrm{d}\ell = \lambda R \mathrm{d}\theta. \tag{10.17}$$

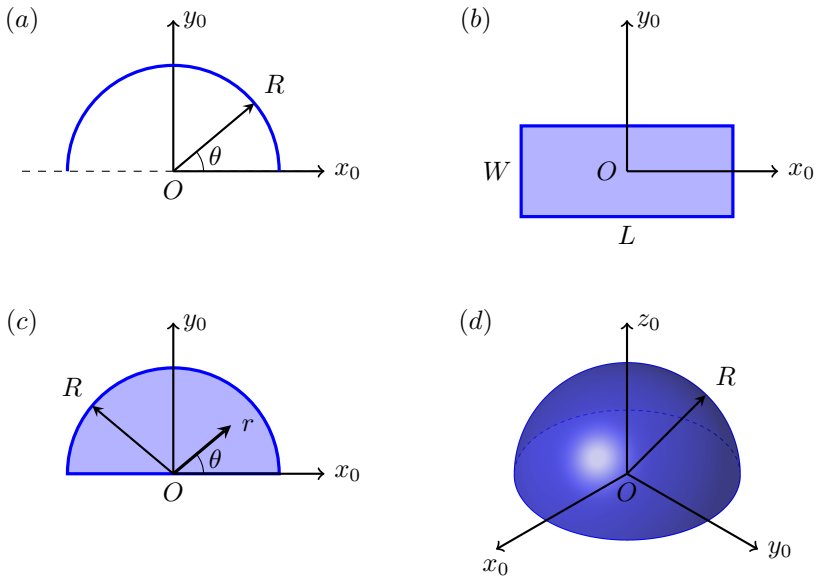

Figure 10.2: Example rigid bodies for inertia properties calculatation: (a) a half-circumference wire; (b) a rectangular plate; (c) a half disk; (d) a half sphere.

With this information, it is easy to perform the integrals for the total mass m and center of mass $\boldsymbol{r}_0^G$:

$$m = \int_0^\pi \lambda R \mathrm{d}\theta = \pi\lambda R, \tag{10.18}$$

$$\boldsymbol{r}_0^G = \frac{1}{m}\int_{\mathcal{D}} (x\boldsymbol{i} + y\boldsymbol{j})\mathrm{d}m = \frac{1}{\pi\lambda R}\int_0^\pi \left(\lambda R^2\cos\theta\boldsymbol{i} + \lambda R^2\sin\theta\boldsymbol{j}\right)\mathrm{d}\theta = \frac{2}{\pi}R\boldsymbol{j}. \tag{10.19}$$

Note that the center of mass lies on the symmetry axis, as expected.

The moments of inertia with respect to the coordinated axes are

$$I_z = \lambda R^3\int_0^\pi \left(\cos^2\theta + \sin^2\theta\right)\mathrm{d}\theta = \pi\lambda R^3 = mR^2 \equiv I_0, \tag{10.20}$$

$$I_x = \lambda R^3\int_0^\pi \sin^2\theta\mathrm{d}\theta = \frac{\pi}{2}\lambda R^3 = \frac{1}{2}mR^2, \tag{10.21}$$

and, using Eq. (10.13), which for planar bodies of negligible thickness can be written as $I_0 = I_x + I_y$, we find

$$I_y = \frac{1}{2}mR^2. \tag{10.22}$$

The products of inertia that involve the plane Oxy are zero, since the system is contained in this plane. For $P_{xz,yz}$, since Oyz is a plane of symmetry, we have:

$$P_{xz,yz} = \lambda R^2\int_0^\pi \cos\theta\sin\theta\mathrm{d}\theta = 0. \tag{10.23}$$

Homogeneous plate

Consider a 2D homogeneous plate of mass M, length L, width W and negligible thickness, as shown in Fig. 10.2. We can compute the superficial density as $\sigma = M/(LW)$, and we can express the differential mass element as $\mathrm{d}m = M/(LW)\mathrm{d}x\mathrm{d}y$. By symmetry considerations, the center of mass lies on the center of the plate, i.e., it coincides with point O, $G \equiv O$.

The polar moment of inertia with respect to O coincides with the moment of inertia with respect to the Oz axis since the body is planar,

$$I_0 = I_z = \int_{-W/2}^{W/2}\int_{-L/2}^{L/2}\frac{M}{LW}(x^2+y^2)\mathrm{d}x\mathrm{d}y = \frac{1}{12}M(L^2+W^2). \tag{10.24}$$

The moments of inertia of with respect to the axis Ox and the axis Oy are

$$I_x = \int_{-W/2}^{W/2}\int_{-L/2}^{L/2} \frac{M}{LW} y^2 \mathrm{d}x\mathrm{d}y = \frac{1}{12}MW^2. \tag{10.25}$$

$$I_y = \int_{-W/2}^{W/2}\int_{-L/2}^{L/2} \frac{M}{LW} x^2 \mathrm{d}x\mathrm{d}y = \frac{1}{12}ML^2 = I_0 - I_x. \tag{10.26}$$

As before, the products of inertia in S_0 are all zero due to symmetry. Changing to a different reference frame S_1 as shown in Fig. 10.2(b), the products of inertia that involve the plane Ax_1y_1 are still zero, since the body is planar with negligible thickness. However, the product of inertia with respect to the planes Ax_1z_1 and Ay_1z_1 is

$$P_{x_1z_1,y_1z_1} = \int_0^W \int_0^L \frac{M}{LW} xy \mathrm{d}x\mathrm{d}y = \frac{1}{4}MLW. \tag{10.27}$$

Half disk

Consider now the center of mass of a homogeneous half disk with area density σ as shown in Fig. 10.2(c). The equation of the half-disk in parametric form is:

$$\begin{cases} x = r\cos\theta, \\ y = r\sin\theta, \end{cases} \tag{10.28}$$

with r in the range $[0, R]$ and θ in $[0, \pi]$. The differential element of mass is now

$$\mathrm{d}m = \sigma \mathrm{d}A = \sigma r \mathrm{d}r\mathrm{d}\theta. \tag{10.29}$$

Naturally, we now have to deal with double integrals, as this is an integral over the surface of the half disk. The total mass of the half disk equals its density σ times its area:

$$m = \sigma \int_0^\pi \int_0^R r\mathrm{d}r\mathrm{d}\theta = \sigma\frac{\pi R^2}{2}. \tag{10.30}$$

Symmetry considerations lead to the conclusion that the center of mass lies as before in the symmetry axis, so $x_G = 0$. The vertical position of G is given by

$$y_G = \frac{1}{m}\int_{\mathcal{D}} y\mathrm{d}m = \frac{2}{\pi\sigma R^2}\sigma \int_0^\pi \int_0^R r^2 \sin\theta \mathrm{d}r\mathrm{d}\theta = \frac{4}{3\pi}R. \tag{10.31}$$

The computation of the moments of inertia with respect to the coordinated axes in this case is left as an exercise. The products of inertia are all zero for the same symmetry reasons as in the previous case.

Half sphere

Finally, as a 3D example, consider the homogeneous half-sphere of radius R and volumetric density ρ shown in Fig. 10.2(c). We write $\mathrm{d}m$ using spherical coordinates as

$$\mathrm{d}m = r^2\cos\phi \mathrm{d}r\mathrm{d}\theta\mathrm{d}\phi. \tag{10.32}$$

The total mass of the body is

$$m = \rho\int_0^R r^2\mathrm{d}r \int_0^{\pi/2} \cos\phi\mathrm{d}\phi \int_0^{2\pi} \mathrm{d}\theta = \frac{2}{3}\rho\pi R^3. \tag{10.33}$$

The vertical position of the center of mass, taking into account that $z = r\sin\phi$, is

$$z_G = \frac{1}{m}\int_{\mathcal{D}} z\mathrm{d}m = \frac{1}{m}\rho\int_0^R r^3\mathrm{d}r \int_0^{\pi/2} \sin\phi\cos\phi\mathrm{d}\phi \int_0^{2\pi} \mathrm{d}\theta = \frac{3}{8}R. \tag{10.34}$$

The computation of moments and products of inertia is again left as an exercise. Taking into account the symmetries of the body is essential to reduce the amount of calculations required.

10.6 Tensor of inertia

The moments and products of inertia introduced in the previous sections appear in the dynamics of material systems as the elements of a second order tensor, termed *tensor of inertia*. The tensor of inertia of a material system B about a the origin point O of reference frame $S_0\{O; B_0\}$ is defined as

$$\overline{\overline{\mathcal{I}}}_0 = \int_{\mathcal{D}} \left[(r_0^P)^2 \overline{\overline{\mathcal{U}}} - \boldsymbol{r}_0^P \otimes \boldsymbol{r}_0^P \right] \mathrm{d}m, \tag{10.35}$$

where the tensor $\overline{\overline{\mathcal{U}}}$ is the unit diagonal tensor. If we choose vector basis $B_0 = \{\boldsymbol{i}, \boldsymbol{j}, \boldsymbol{k}\}$ to express $\boldsymbol{r}_0^P = x\boldsymbol{i} + y\boldsymbol{j} + z\boldsymbol{k}$, then the components of the first term in the integral above in the associated tensor basis $B_0 \otimes B_0 = \{\boldsymbol{i}\otimes\boldsymbol{i}, \boldsymbol{i}\otimes\boldsymbol{j}, \boldsymbol{i}\otimes\boldsymbol{k}, \boldsymbol{j}\otimes\boldsymbol{i}, \ldots\}$ are simply:

$$(r_0^P)^2 \overline{\overline{\mathcal{U}}} = \begin{bmatrix} x^2+y^2+z^2 & 0 & 0 \\ 0 & x^2+y^2+z^2 & 0 \\ 0 & 0 & x^2+y^2+z^2 \end{bmatrix}_{00}, \tag{10.36}$$

and the components of the the tensor $\boldsymbol{r}_0^P \otimes \boldsymbol{r}_0^P$ (also called dyad) are

$$\boldsymbol{r}_0^P \otimes \boldsymbol{r}_0^P = \begin{bmatrix} x^2 & xy & xz \\ xy & y^2 & yz \\ xz & yz & z^2 \end{bmatrix}_{00}. \tag{10.37}$$

Note that this second term of the integral is the second moment of the mass distribution, $\mathcal{M}^{\mathcal{D}}_{O,2}$.

Introducing these expressions into the definition Eq. (10.35) we finally obtain:

$$\overline{\overline{\mathcal{I}}}_0 = \int_{\mathcal{D}} \begin{bmatrix} y^2+z^2 & -xy & -xz \\ -xy & x^2+z^2 & -yz \\ -xz & -yz & x^2+y^2 \end{bmatrix}_{00} \mathrm{d}m = \begin{bmatrix} I_x & -P_{xz,yz} & -P_{xy,yz} \\ -P_{xz,yz} & I_y & -P_{xy,xz} \\ -P_{xy,yz} & -P_{xy,xz} & I_z \end{bmatrix}_{00}. \tag{10.38}$$

Observe that the matrix of components of the inertia tensor is symmetric.

The tensor of inertia $\overline{\overline{\mathcal{I}}}_0$ can be used to compute the moment of inertia of the body with respect to any line ℓ that passes through the origin O. If the unit vector of such line is $\boldsymbol{u}$, then the moment of inertia of the body with respect to ℓ is given by

$$I_\ell = \boldsymbol{u} \cdot \overline{\overline{\mathcal{I}}}_0 \cdot \boldsymbol{u}. \tag{10.39}$$

Likewise, the tensor of inertia can be used to compute the product of inertia with respect to two perpendicular planes π_1 and π_2 that contain the point O and whose normal unit vectors are[16] $\boldsymbol{u}_1$ and $\boldsymbol{u}_2$:

$$P_{\pi_1,\pi_2} = -\boldsymbol{u}_1 \cdot \overline{\overline{\mathcal{I}}}_0 \cdot \boldsymbol{u}_2. \tag{10.40}$$

10.7 Change of tensor basis

Whenever we need to operate with a tensor of inertia and multiply it by a vector, be it on the left or on the right, the tensor and the vector must be expressed in consistent bases. In other words, if the tensor of inertia is expressed in the tensor basis $B_0 \otimes B_0$ (which is constructed by tensor-multiplying the vectors in the vector basis B_0), then we must express the vector in the vector basis B_0.

Assume the components of $\overline{\overline{\mathcal{I}}}_0$ in $B_0 \otimes B_0$ are known, and that we need to compute the tensor components a rotated basis $B_1 \otimes B_1$ (constructed from the vector basis B_1), If the rotation matrix from B_1 to B_0 is $[{}_0R_1]$, then the change of basis for the inertia tensor follows the standard rules for all tensors of second order:

$$\left[\overline{\overline{\mathcal{I}}}_0\right]_{11} = [{}_0R_1]^T \cdot \left[\overline{\overline{\mathcal{I}}}_0\right]_{00} \cdot [{}_0R_1]. \tag{10.41}$$

Remember that, just as a vector, a tensor remains the same physical entity, even if we change the basis we use to express it in the form of a component matrix.

[16] Note that if we choose $-\boldsymbol{u}_1$ instead of $\boldsymbol{u}_1$ (or $-\boldsymbol{u}_2$ instead of $\boldsymbol{u}_2$) the sign of P_{12} would be the opposite.

10.7.1 Principal axes of inertia

We can always find a rotated vector basis $B_1 : \{\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3\}$ such that the tensor of inertia in the new tensor basis $B_1 \otimes B_1$ becomes *diagonal*:

$$\overline{\overline{\mathcal{I}}}_0 = \begin{bmatrix} I_1 & 0 & 0 \\ 0 & I_2 & 0 \\ 0 & 0 & I_3 \end{bmatrix}_{11}. \tag{10.42}$$

The directions of the vectors $\{\boldsymbol{e}_1,\ \boldsymbol{e}_2,$ and $\boldsymbol{e}_3\}$ are known as *principal axes or directions of inertia*. Since the tensor of inertia is symmetric, the spectral theorem of linear algebra states that it is always possible to find $\{\boldsymbol{e_1}, \boldsymbol{e_2}, \boldsymbol{e_3}\}$ and I_1, I_2, I_3 as the eigenvectors and eigenvalues of the following problem

$$\overline{\overline{\mathcal{I}}}_0 \cdot \boldsymbol{e}_i = I_i \boldsymbol{e}_i. \tag{10.43}$$

10.8 Change of reference frame origin: Steiner's theorem

Consider a material system B with mass distribution $\mathcal{D}(\boldsymbol{r}_0^P)$, a reference frame $S_0 : \{O; B_0\}$ centered on O, and a parallel reference frame $S_G : \{G; B_0\}$ centered on G, the center of mass of B. Steiner's theorem establishes a relation between $\overline{\overline{\mathcal{I}}}_0$, the inertia tensor about O, and $\overline{\overline{\mathcal{I}}}_G$ the inertia tensor about the center of mass G of the material system.

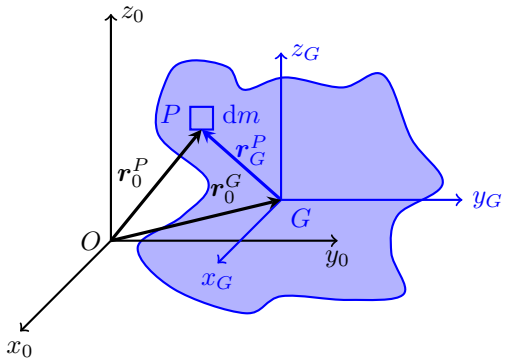

Figure 10.3: The parallel reference frame at G, S_G, used in the definition of Steiner's theorem.

Introducing $\boldsymbol{r}_0^P = \boldsymbol{r}_0^G + \boldsymbol{r}_G^P$ into the definition of $\overline{\overline{\mathcal{I}}}_0$ and taking into account the following identities:

$$\int_{\mathcal{D}} \boldsymbol{r}_G^P \mathrm{d}m = \boldsymbol{0}, \tag{10.44}$$

$$\int_{\mathcal{D}} (\boldsymbol{r}_0^G \cdot \boldsymbol{r}_G^P)\overline{\overline{\mathcal{U}}}\mathrm{d}m = \boldsymbol{r}_0^G \cdot \left(\int_{\mathcal{D}} \boldsymbol{r}_G^P \mathrm{d}m\right)\overline{\overline{\mathcal{U}}} = \overline{\overline{0}}, \tag{10.45}$$

$$\int_{\mathcal{D}} (\boldsymbol{r}_G^P \otimes \boldsymbol{r}_0^G)\mathrm{d}m = \left(\int_{\mathcal{D}} \boldsymbol{r}_G^P \mathrm{d}m\right) \otimes \boldsymbol{r}_0^G = \overline{\overline{0}}, \tag{10.46}$$

then, using the definition of the tensor of inertia about G,

$$\overline{\overline{\mathcal{I}}}_G = \int_{\mathcal{D}} \left[(r_G^P)^2 \overline{\overline{\mathcal{U}}} - \boldsymbol{r}_G^P \otimes \boldsymbol{r}_G^P\right] \mathrm{d}m, \tag{10.47}$$

we obtain Steiner's theorem

$$\overline{\overline{\mathcal{I}}}_0 = \overline{\overline{\mathcal{I}}}_G + m\left[(r_0^G)^2 \overline{\overline{\mathcal{U}}} - \boldsymbol{r}_0^G \otimes \boldsymbol{r}_0^G\right]. \tag{10.48}$$

This result states that the tensor of inertia with respect to S_0 can be calculated if the tensor of inertia with respect to S_G is known, by adding to it the tensor of a point particle with the same total mass m located at $\boldsymbol{r}_0^G$.

When using Steiner's theorem with the components of the tensors, remember that it is necessary to express all terms of Eq. (10.48) in the same tensor basis. This is the reason why we have defined S_G parallel to S_0 as shown in Fig. 10.3, i.e., with the same vector basis B_0 for both of them.

Note that in order to translate the tensor of inertia about a point A to another point B, Steiner's theorem need to be applied twice, since it is necessary to first translate the tensor to G.

10.8.1 Parallel axis theorem

Oftentimes, we know the moment of inertia with respect to an axis Gz_G that passes through the center of mass G, and we need to compute the moment of inertia with respect to a parallel axis Oz_0, as shown in Fig. 10.4. From the definition of moment of inertia, and noting that $x_0^P = x_0^G + x_G^P$, $y_0^P = y_0^G + y_G^P$,

$$
\begin{aligned}
I_{z_0} &= \int_{\mathcal{D}} \left[(x_0^P)^2 + (y_0^P)^2\right] \mathrm{d}m = \int_{\mathcal{D}} \left[(x_G^P)^2 + (x_0^G)^2 + 2x_G^P x_0^G + (y_G^P)^2 + (y_0^G)^2 + 2y_G^P y_0\right] \mathrm{d}m = \\
&= \int_{\mathcal{D}} \left[(x_G^P)^2 + (y_G^P)^2\right] \mathrm{d}m + \left[(x_0^G)^2 + (y_0^G)^2\right] \int_{\mathcal{D}} \mathrm{d}m + 2x_0^G \left(\int_{\mathcal{D}} x_G^P \mathrm{d}m\right) + 2y_0^G \left(\int_{\mathcal{D}} y_G^P \mathrm{d}m\right) = \\
&= I_{z_G} + m\left[(x_0^G)^2 + (y_0^G)^2\right],
\end{aligned}
$$

where the integrals in parentheses vanish. This compact result states that the moment of inertia about an axis can be computed as the moment of inertia about a parallel axis through G plus the moment of inertia of a particle that concentrates the total mass m of the body at G, and is sometimes termed the *parallel axis theorem*. Naturally, the parallel axis theorem is just a particularization of the more general Steiner's theorem introduced above.

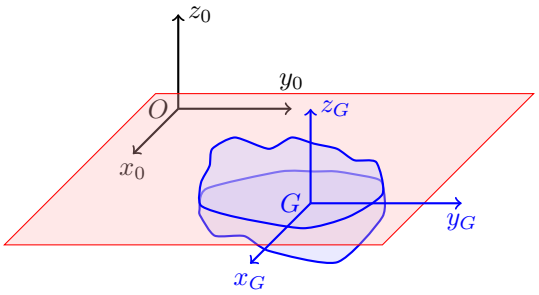

Figure 10.4: Steiner's theorem particular case: parallel axis theorem problem definition.

Since the second term in the equation is always positive, it is easy to conclude that the minimum moment of inertia with respect to an axis that passes through G is minimal, compared to the moment of inertia with respect to any other axis parallel to it.

11 Linear momentum, angular momentum and kinetic energy

The magnitudes $\boldsymbol{p}_0$, $\boldsymbol{H}_{A,0}$ and T_0 that we have defined for a point particle need to be extended to rigid bodies, and, in general, to arbitrary material systems. Our goal in this chapter is to obtain simple expressions of these quantities in terms of the kinematic variables and inertia properties of the rigid body. As in the case of a point particle, a superindex on these variables can be used to indicate the rigid body they refer to.

11.1 Linear momentum

Consider a material system B with mass distribution $\mathcal{D}(\boldsymbol{r}_0^P)$ as shown in Fig. 11.1. The linear momentum in a reference frame S_0 of a differential element of mass $\mathrm{d}m$ with position $\boldsymbol{r}_0^P$ and velocity $\boldsymbol{v}_0^P$ is $\mathrm{d}\boldsymbol{p}_0 = \boldsymbol{v}_0^P \mathrm{d}m$. Integrating over the whole system we obtain the total *linear momentum*,

$$\boldsymbol{p}_0 = \int_{\mathcal{D}} \mathrm{d}\boldsymbol{p}_0 = \int_{\mathcal{D}} \boldsymbol{v}_0^P \mathrm{d}m. \tag{11.1}$$

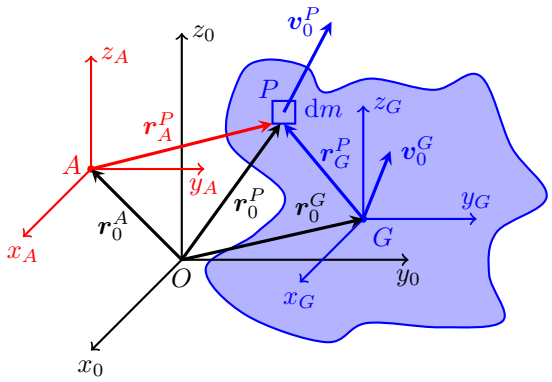

Figure 11.1: Sketch of a material system B moving with respect to a fixed reference frame $S_0\{O; B_0\}$. Parallel reference frames $S_A\{A; B_0\}$ and $S_G\{G; B_0\}$ with origin on a generic position A and the center of mass G of the system, respectively, are also shown.

Differentiating in the definition of center of mass of Eq. (10.8):

$$m\boldsymbol{r}_0^G = \int_{\mathcal{D}} \boldsymbol{r}_0^P \mathrm{d}m. \tag{11.2}$$

$$m\boldsymbol{v}_0^G = \int_{\mathcal{D}} \boldsymbol{v}_0^P \mathrm{d}m = \boldsymbol{p}_0. \tag{11.3}$$

This expression states that the linear momentum of the rigid body is equal to the momentum of an equivalent point particle with the total mass of the body, m, moving with the velocity of the center of mass, $\boldsymbol{v}_0^G$.

11.2 Angular momentum

The angular momentum of a differential element of mass $\mathrm{d}m$ about a point A with respect to S_0 is $\mathrm{d}\boldsymbol{H}_{A,0} = \boldsymbol{r}_A^P \times \boldsymbol{v}_0^P \mathrm{d}m$. Integrating over the system we obtain the total *angular momentum about* A,

$$\boldsymbol{H}_{A,0} = \int_{\mathcal{D}} \mathrm{d}\boldsymbol{H}_{A,0} = \int_{\mathcal{D}} \boldsymbol{r}_A^P \times \boldsymbol{v}_0^P \mathrm{d}m. \tag{11.4}$$

In the case of a rigid body, to obtain a simplified expression of this integral we may proceed in two different ways. Firstly, we express the velocity of any point P of the rigid body with the equation of its velocity field as

$$\boldsymbol{v}_0^P = \boldsymbol{v}_0^{A(B)} + \boldsymbol{\omega}_{B0} \times \boldsymbol{r}_A^P, \tag{11.5}$$

where the nomenclature $A(B)$ has been used to denote the point *of the rigid body* B whose position coincides with A at each instant of time, to distinguish it from the non-body point A, as explained in §9.1.4.

Note that $\boldsymbol{r}_0^A \equiv \boldsymbol{r}_0^{A(B)}$, since the position of the points A and $A(B)$ coincides by definition. With this in mind we may write:

$$\boldsymbol{H}_{A,0} = \left(\int_{\mathcal{D}} \boldsymbol{r}_A^P \mathrm{d}m\right) \times \boldsymbol{v}_0^{A(B)} + \int_{\mathcal{D}} \boldsymbol{r}_A^P \times (\boldsymbol{\omega}_{B0} \times \boldsymbol{r}_A^P)\mathrm{d}m. \tag{11.6}$$

The first term is easily integrated taking into account the definition of the center of mass,

$$\left(\int_{\mathcal{D}} \boldsymbol{r}_A^P \mathrm{d}m\right) \times \boldsymbol{v}_0^{A(B)} = m\boldsymbol{r}_A^G \times \boldsymbol{v}_0^{A(B)}. \tag{11.7}$$

The second term to be integrated is a double vector product. Recall from Eq. (1.37) that the double vector product of three vectors is

$$\boldsymbol{A} \times (\boldsymbol{B} \times \boldsymbol{C}) = (\boldsymbol{A} \cdot \boldsymbol{C})\boldsymbol{B} - (\boldsymbol{A} \cdot \boldsymbol{B})\boldsymbol{C}, \tag{11.8}$$

which in the present case leads to

$$\int_{\mathcal{D}} \boldsymbol{r}_A^P \times (\boldsymbol{\omega}_{B0} \times \boldsymbol{r}_A^P)\mathrm{d}m = \int_{\mathcal{D}} \left[(r_A^P)^2 \boldsymbol{\omega}_{B0} - (\boldsymbol{r}_A^P \cdot \boldsymbol{\omega}_{B0})\boldsymbol{r}_A^P\right] \mathrm{d}m. \tag{11.9}$$

The first term in the last expression can be written as $(r_A^P)^2\boldsymbol{\omega}_{B0} = (r_A^P)^2\overline{\overline{\mathcal{U}}} \cdot \boldsymbol{\omega}_{B0}$. The second term is[17] $\boldsymbol{r}_A^P(\boldsymbol{r}_A^P \cdot \boldsymbol{\omega}_{B0}) = (\boldsymbol{r}_A^P \otimes \boldsymbol{r}_A^P) \cdot \boldsymbol{\omega}_{B0}$. Therefore,

$$\int_{\mathcal{D}} \boldsymbol{r}_A^P \times (\boldsymbol{\omega}_{B0} \times \boldsymbol{r}_A^P)\mathrm{d}m = \int_{\mathcal{D}} \left[(r_A^P)^2\overline{\overline{\mathcal{U}}} \cdot \boldsymbol{\omega}_{B0} - (\boldsymbol{r}_A^P \otimes \boldsymbol{r}_A^P) \cdot \boldsymbol{\omega}_{B0}\right] \mathrm{d}m = \overline{\overline{\mathcal{I}}}_A \cdot \boldsymbol{\omega}_{B0}, \tag{11.10}$$

where the tensor of inertia about A defined in Eq. (10.35) has been introduced.

In summary, we obtain the following result for $\boldsymbol{H}_{A,0}$:

$$\boldsymbol{H}_{A,0} = \overline{\overline{\mathcal{I}}}_A \cdot \boldsymbol{\omega}_{B0} + m\boldsymbol{r}_A^G \times \boldsymbol{v}_0^{A(B)}. \tag{11.11}$$

This expression is notably useful when the point of analysis A is chosen to coincide with the center of mass, $A \equiv G$ so that $\boldsymbol{r}_A^G \equiv \boldsymbol{0}$, or when the point A is a fixed body point so that $\boldsymbol{v}_0^{A(B)} = \boldsymbol{v}_0^A = \boldsymbol{0}$. In these cases we can write

$$\boldsymbol{H}_{G,0} = \overline{\overline{\mathcal{I}}}_G \cdot \boldsymbol{\omega}_{B0}, \tag{11.12}$$

$$\boldsymbol{H}_{A,0} = \overline{\overline{\mathcal{I}}}_A \cdot \boldsymbol{\omega}_{B0}. \tag{11.13}$$

Secondly, there is another useful simplified expression for the angular momentum of a rigid body which makes use of the center of mass G. We begin by defining the reference frame $S_G : \{G; B_0\}$ centered at G and parallel to $S_0 : \{O; B_0\}$, as shown in Fig. 11.1. Introducing $\boldsymbol{r}_A^P = \boldsymbol{r}_A^G + \boldsymbol{r}_G^P$ in the definition of $\boldsymbol{H}_{A,0}$ in Eq. (11.4), using again the expression of the velocity field of the rigid body, this time in the form $\boldsymbol{v}_0^P = \boldsymbol{v}_0^G + \boldsymbol{\omega}_{B0} \times \boldsymbol{r}_G^P$, and noting that $\boldsymbol{v}_G^P = \boldsymbol{\omega}_{B0} \times \boldsymbol{r}_G^P$:

$$\begin{aligned}\boldsymbol{H}_{A,0} &= \int_{\mathcal{D}} (\boldsymbol{r}_A^G + \boldsymbol{r}_G^P) \times (\boldsymbol{v}_0^G + \boldsymbol{\omega}_{B0} \times \boldsymbol{r}_G^P)\mathrm{d}m \\ &= \int_{\mathcal{D}} \boldsymbol{r}_A^G \times \boldsymbol{v}_0^G \mathrm{d}m + \int_{\mathcal{D}} \boldsymbol{r}_G^P \times \boldsymbol{v}_0^G \mathrm{d}m + \int_{\mathcal{D}} \boldsymbol{r}_A^G \times (\boldsymbol{\omega}_{B0} \times \boldsymbol{r}_G^P)\mathrm{d}m + \int_{\mathcal{D}} \boldsymbol{r}_G^P \times (\boldsymbol{\omega}_{B0} \times \boldsymbol{r}_G^P)\mathrm{d}m.\end{aligned}$$

[17] As an exercise, check that this statement is true component by component.

Note that $\boldsymbol{r}_A^G$, $\boldsymbol{v}_0^G$ and $\boldsymbol{\omega}_{B0}$ are constants inside the integrals, so:

$$\int_{\mathcal{D}} \boldsymbol{r}_A^G \times \boldsymbol{v}_0^G \mathrm{d}m = m\boldsymbol{r}_A^G \times \boldsymbol{v}_0^G, \tag{11.14}$$

$$\int_{\mathcal{D}} \boldsymbol{r}_G^P \times \boldsymbol{v}_0^G \mathrm{d}m = \left(\int_{\mathcal{D}} \boldsymbol{r}_G^P \mathrm{d}m\right) \times \boldsymbol{v}_0^G = \boldsymbol{0}, \tag{11.15}$$

$$\int_{\mathcal{D}} \boldsymbol{r}_A^G \times (\boldsymbol{\omega}_{B0} \times \boldsymbol{r}_G^P)\mathrm{d}m = \boldsymbol{r}_A^G \times \left(\boldsymbol{\omega}_{B0} \times \int_{\mathcal{D}} \boldsymbol{r}_G^P \mathrm{d}m\right) = \boldsymbol{0}, \tag{11.16}$$

$$\int_{\mathcal{D}} \boldsymbol{r}_G^P \times (\boldsymbol{\omega}_{B0} \times \boldsymbol{r}_G^P)\mathrm{d}m = \int_{\mathcal{D}} \boldsymbol{r}_G^P \times \boldsymbol{v}_G^P \mathrm{d}m = \boldsymbol{H}_{G,G}, \tag{11.17}$$

where Eq. (10.44) has been used, and we have introduced the angular momentum of the rigid body with respect to the center of mass G as seen from the reference frame S_G, i.e. $\boldsymbol{H}_{G,G}$. Note, however, that

$$\boldsymbol{H}_{G,G} = \int_{\mathcal{D}} \boldsymbol{r}_G^P \times \boldsymbol{v}_G^P \mathrm{d}m = \int_{\mathcal{D}} \boldsymbol{r}_G^P \times (\boldsymbol{v}_0^P - \boldsymbol{v}_0^G)\mathrm{d}m \tag{11.18}$$

$$= \int_{\mathcal{D}} \boldsymbol{r}_G^P \times \boldsymbol{v}_0^P \mathrm{d}m - \left(\int_{\mathcal{D}} \boldsymbol{r}_G^P \mathrm{d}m\right) \times \boldsymbol{v}_0^G = \boldsymbol{H}_{G,0} + \boldsymbol{0}, \tag{11.19}$$

so that $\boldsymbol{H}_{G,G} \equiv \boldsymbol{H}_{G,0}$. Finally we obtain:

$$\boldsymbol{H}_{A,0} = \boldsymbol{H}_{G,0} + m\boldsymbol{r}_A^G \times \boldsymbol{v}_0^G, \tag{11.20}$$

and, using Eq. (11.12),

$$\boldsymbol{H}_{A,0} = \overline{\overline{\mathcal{I}}}_G \cdot \boldsymbol{\omega}_{B0} + m\boldsymbol{r}_A^G \times \boldsymbol{v}_0^G, \tag{11.21}$$

This result can be interpreted as follows: the angular momentum of a rigid body about a point A is equal to the sum of two terms. The first one is the angular momentum of the rigid body due to the rotational motion about the center of mass G. The second term is due to the translational motion of the center of mass, i.e., the angular momentum about A of an equivalent point particle with the total mass of the body, m, located at G, and moving with the velocity of the center of mass, $\boldsymbol{v}_0^G$.

11.3 Kinetic energy

The kinetic energy of a differential element of mass $\mathrm{d}m$ with position $\boldsymbol{r}_0^P$ and velocity $\boldsymbol{v}_0^P$ is $\mathrm{d}T_0 = (1/2)(v_0^P)^2\mathrm{d}m$. Integrating over the mass distribution $\mathcal{D}(\boldsymbol{r}_0^P)$ of the material system B we obtain the total *kinetic energy*,

$$T_0 = \int_{\mathcal{D}} \mathrm{d}T_0 = \frac{1}{2}\int_{\mathcal{D}} (v_0^P)^2 \mathrm{d}m. \tag{11.22}$$

Again, in the case of a rigid body there are two simplifications of interest for the integral. Firstly, if we use the velocity field of the rigid body based as written in Eq. (11.5), we can obtain

$$\begin{aligned} T_0 &= \frac{1}{2}\int_{\mathcal{D}} (v_0^P)^2 \mathrm{d}m = \frac{1}{2}\int_{\mathcal{D}} (\boldsymbol{v}_0^{A(B)} + \boldsymbol{\omega}_{B0} \times \boldsymbol{r}_A^P) \cdot (\boldsymbol{v}_0^{A(B)} + \boldsymbol{\omega}_{B0} \times \boldsymbol{r}_A^P)\mathrm{d}m \\ &= \frac{1}{2}m(v_0^{A(B)})^2 + \left[\boldsymbol{\omega}_{B0} \times \left(\int_{\mathcal{D}} \boldsymbol{r}_A^P \mathrm{d}m\right)\right] \cdot \boldsymbol{v}_0^{A(B)} + \frac{1}{2}\int_{\mathcal{D}} (\boldsymbol{\omega}_{B0} \times \boldsymbol{r}_A^P) \cdot (\boldsymbol{\omega}_{B0} \times \boldsymbol{r}_A^P)\mathrm{d}m. \end{aligned} \tag{11.23}$$

The second term can be integrated into

$$\left[\boldsymbol{\omega}_{B0} \times \left(\int_{\mathcal{D}} \boldsymbol{r}_A^P \mathrm{d}m\right)\right] \cdot \boldsymbol{v}_0^{A(B)} = m\left(\boldsymbol{\omega}_{B0} \times \boldsymbol{r}_A^G\right) \cdot \boldsymbol{v}_0^{A(B)} \tag{11.24}$$

For the last term, recall that the dot product of two vector products can be expressed, Eq. (1.38), as:

$$(\boldsymbol{A} \times \boldsymbol{B}) \cdot (\boldsymbol{C} \times \boldsymbol{D}) = (\boldsymbol{A} \cdot \boldsymbol{C})(\boldsymbol{B} \cdot \boldsymbol{D}) - (\boldsymbol{A} \cdot \boldsymbol{D})(\boldsymbol{B} \cdot \boldsymbol{C}), \tag{11.25}$$

which in the present case leads to:

$$(\boldsymbol{\omega}_{B0} \times \boldsymbol{r}_A^P) \cdot (\boldsymbol{\omega}_{B0} \times \boldsymbol{r}_A^P) = (\boldsymbol{\omega}_{B0} \cdot \boldsymbol{\omega}_{B0})(\boldsymbol{r}_A^P \cdot \boldsymbol{r}_A^P) - (\boldsymbol{\omega}_{B0} \cdot \boldsymbol{r}_A^P)(\boldsymbol{r}_A^P \cdot \boldsymbol{\omega}_{B0}). \tag{11.26}$$

Proceeding in a similar way as for the angular momentum, the first term in the last expression can be written as

$$(\boldsymbol{\omega}_{B0}\cdot\boldsymbol{\omega}_{B0})(\boldsymbol{r}_A^P\cdot\boldsymbol{r}_A^P)=(\boldsymbol{\omega}_{B0}\cdot\boldsymbol{\omega}_{B0})(r_A^P)^2=\boldsymbol{\omega}_{B0}\cdot(r_A^P)^2\overline{\overline{\mathcal{U}}}\cdot\boldsymbol{\omega}_{B0}. \tag{11.27}$$

The second term, using again the tensor $\boldsymbol{r}_A^P\otimes\boldsymbol{r}_A^P$, is:

$$(\boldsymbol{\omega}_{B0}\cdot\boldsymbol{r}_A^P)(\boldsymbol{r}_A^P\cdot\boldsymbol{\omega}_{B0})=\boldsymbol{\omega}_{B0}\cdot(\boldsymbol{r}_A^P\otimes\boldsymbol{r}_A^P)\cdot\boldsymbol{\omega}_{B0}. \tag{11.28}$$

Introducing the last two expressions into the last term of Eq. (11.23),

$$\begin{aligned}\frac{1}{2}\int_{\mathcal{D}}(\boldsymbol{\omega}_{B0}\times\boldsymbol{r}_A^P)\cdot(\boldsymbol{\omega}_{B0}\times\boldsymbol{r}_A^P)\mathrm{d}m&=\frac{1}{2}\int_{\mathcal{D}}\left[\boldsymbol{\omega}_{B0}\cdot(r_A^P)^2\overline{\overline{\mathcal{U}}}\cdot\boldsymbol{\omega}_{B0}-\boldsymbol{\omega}_{B0}\cdot(\boldsymbol{r}_A^P\otimes\boldsymbol{r}_A^P)\cdot\boldsymbol{\omega}_{B0}\right]\mathrm{d}m\\&=\frac{1}{2}\boldsymbol{\omega}_{B0}\cdot\overline{\overline{\mathcal{I}}}_A\cdot\boldsymbol{\omega}_{B0}.\end{aligned} \tag{11.29}$$

In summary, we obtain

$$T_0=\frac{1}{2}\boldsymbol{\omega}_{B0}\cdot\overline{\overline{\mathcal{I}}}_A\cdot\boldsymbol{\omega}_{B0}+\frac{1}{2}m(v_0^{A(B)})^2+m\left(\boldsymbol{\omega}_{B0}\times\boldsymbol{r}_A^G\right)\cdot\boldsymbol{v}_0^{A(B)}. \tag{11.30}$$

Just like in the case of the angular momentum, this expression is notably useful when $A\equiv G$ so that $\boldsymbol{r}_A^G\equiv\boldsymbol{0}$ or when the point A is a fixed point of the rigid body so that $\boldsymbol{v}_0^{A(B)}=\boldsymbol{v}_0^A=\boldsymbol{0}$. When $A=G$, we can write:

$$T_0=\frac{1}{2}\boldsymbol{\omega}_{B0}\cdot\overline{\overline{\mathcal{I}}}_G\cdot\boldsymbol{\omega}_{B0}+\frac{1}{2}m(v_0^G)^2. \tag{11.31}$$

When A is a fixed point of the rigid body, we have:

$$T_0=\frac{1}{2}\boldsymbol{\omega}_{B0}\cdot\overline{\overline{\mathcal{I}}}_A\cdot\boldsymbol{\omega}_{B0}. \tag{11.32}$$

Secondly, returning to Eq. (11.22), there is another useful form for this integral. Using instead the velocity field of the rigid body referenced to the center of mass G, $\boldsymbol{v}_0^P=\boldsymbol{v}_0^G+\boldsymbol{\omega}_{B0}\times\boldsymbol{r}_G^P$, we can write:

$$(v_0^P)^2=\boldsymbol{v}_0^P\cdot\boldsymbol{v}_0^P=(v_0^G)^2+\left|\boldsymbol{\omega}_{B0}\times\boldsymbol{r}_G^P\right|^2+2\boldsymbol{v}_0^G\cdot(\boldsymbol{\omega}_{B0}\times\boldsymbol{r}_G^P). \tag{11.33}$$

Introducing this expression into Eq. (11.22) and applying similar simplifications as for the angular momentum in the previous Section leads to the following integrals, each of which can be easily reduced:

$$\frac{1}{2}\int_{\mathcal{D}}(v_0^G)^2\mathrm{d}m=\frac{1}{2}m(v_0^G)^2, \tag{11.34}$$

$$\int_{\mathcal{D}}\boldsymbol{v}_0^G\cdot(\boldsymbol{\omega}_{B0}\times\boldsymbol{r}_G^P)\mathrm{d}m=\boldsymbol{v}_0^G\cdot\left(\boldsymbol{\omega}_{B0}\times\int_{\mathcal{D}}\boldsymbol{r}_G^P\mathrm{d}m\right)=0, \tag{11.35}$$

$$\frac{1}{2}\int_{\mathcal{D}}\left|\boldsymbol{\omega}_{B0}\times\boldsymbol{r}_G^P\right|^2\mathrm{d}m=T_G. \tag{11.36}$$

where we have introduced T_G, the kinetic energy of the rigid body as seen from the parallel reference frame S_G. Finally, we obtain:

$$T_0=T_G+\frac{1}{2}m(v_0^G)^2. \tag{11.37}$$

This result states that the kinetic energy of the rigid body can be decomposed into the sum of two terms. The first one is the kinetic energy of the body in its rotational motion about the center of mass, T_G. The second one is due to the translational motion of the center of mass, i.e., the kinetic energy of an equivalent point particle with the total mass of the body, m, concentrated in it and moving with the velocity of the center of mass, $\boldsymbol{v}_0^G$.

12 Dynamics of the rigid body

In the previous chapters we have seen that the general motion of a rigid body can be represented as a superposition of a translation of any point in a body plus a rotation about that point, as expressed by the velocity field of the rigid body, and we have obtained useful expressions for the linear and angular momenta and the kinetic energy of a rigid body, relying heavily on the compact results for its inertia properties.

In this chapter we characterize the relation between the loads acting on a rigid body, the kinematics of that body, and its inertia properties. The resultant force is related to the translational motion, while the resultant moment of forces defines the rotational motion.

12.1 General equations for a material system

We begin by considering an arbitrary material system B with mass distribution $\mathcal{D}(\boldsymbol{r}_0^P)$. Afterwards we will consider the particular case of a rigid body by imposing that the velocity field satisfies Eq. (9.3). The fundamental axioms for the following derivation are Newton's Second Law, which governs the motion of each particle in the system, and the Third Law, which introduces the concept of action and reaction forces.

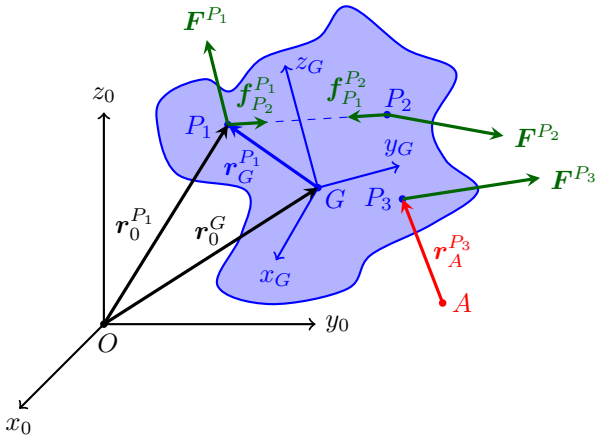

Figure 12.1: External $\boldsymbol{F}^{P_i}$ and internal $\boldsymbol{f}_{P_j}^{P_i}$ forcesacting on a material system.

Figure 12.1 shows a material system that we will regard as a collection of point particles labeled P_i with $i = 1, 2, \ldots, N$. For a continuous mass distribution, mass and force differentials would be used instead and the finite sums would become integrals, without affecting the final result. The figure also displays an inertial reference frame $S_0 : \{O; B_0\}$. We distinguish two types of forces acting on the system:

1. $\boldsymbol{f}_{P_j}^{P_i}$ are *internal forces* between two points of the material system under analysis. The superscript denotes the point of application of the force and subscript indicates which particle causes the force.

2. $\boldsymbol{F}^{P_i}$ are *external forces*, that is, the forces exerted on points of the material system by anything that is not part of the material system itself (i.e., the *environment*).

According to Newton's Third Law, any pair of internal forces such as the ones depicted on Fig. 12.1 are equal in magnitude and oppositely directed:

$$\boldsymbol{f}_{P_j}^{P_i} = -\boldsymbol{f}_{P_i}^{P_j}. \tag{12.1}$$

Furthermore, the internal forces are collinear, meaning that they have the same line of action, or in other words, that the internal force $\boldsymbol{f}_{P_j}^{P_i}$ is parallel to the vector that goes from P_i to P_j, i.e. $\boldsymbol{f}_{P_j}^{P_i} \parallel \boldsymbol{P_iP_j}$. The significance of this extra condition becomes apparent below when considering the

moment of force exerted by any pair internal forces about an arbitrary point A, that is, $\boldsymbol{r}_A^{P_i} \times \boldsymbol{f}_{P_j}^{P_i}$ and $\boldsymbol{r}_A^{P_j} \times \boldsymbol{f}_{P_i}^{P_j}$. Because the two forces have the same line of action, the moment of force of the pair cancels out:

$$\boldsymbol{r}_A^{P_i} \times \boldsymbol{f}_{P_j}^{P_i} = -\boldsymbol{r}_A^{P_j} \times \boldsymbol{f}_{P_i}^{P_j}. \tag{12.2}$$

The resultant force acting on each particle is the sum of the external and internal contributions, so Newton's Second Law for each particle states that in the inertial reference frame S_0:

$$\begin{aligned}
\boldsymbol{F}^{P_1} + \boldsymbol{f}_{P_2}^{P_1} + \boldsymbol{f}_{P_3}^{P_1} + \ldots + \boldsymbol{f}_{P_N}^{P_1} &= m^{P_1} \boldsymbol{a}_0^{P_1}, \\
\boldsymbol{F}^{P_2} + \boldsymbol{f}_{P_1}^{P_2} + \boldsymbol{f}_{P_3}^{P_2} + \ldots + \boldsymbol{f}_{P_N}^{P_2} &= m^{P_2} \boldsymbol{a}_0^{P_2}, \\
&\ldots \\
\boldsymbol{F}^{P_N} + \boldsymbol{f}_{P_1}^{P_N} + \boldsymbol{f}_{P_2}^{P_N} + \ldots + \boldsymbol{f}_{P_{N-1}}^{P_N} &= m^{P_N} \boldsymbol{a}_0^{P_N}.
\end{aligned} \tag{12.3}$$

Summing over all points in the system and using Eq. (12.1), all the internal forces cancel out, and we obtain

$$\boldsymbol{F} = \sum_{i=1}^{N} \boldsymbol{F}^{P_i} = \sum_{i=1}^{N} m^{P_i} \boldsymbol{a}_0^{P_i}, \tag{12.4}$$

where $\boldsymbol{F}$ (without sub- or superindices) is the *resultant of the external forces* acting on the material system.

Another interesting result is obtained when we take moments on Newton's second law about an arbitrary point A. For each of the particles in the system we may write:

$$\begin{aligned}
\boldsymbol{r}_A^{P_1} \times \left(\boldsymbol{F}^{P_1} + \boldsymbol{f}_{P_2}^{P_1} + \boldsymbol{f}_{P_3}^{P_1} + \ldots + \boldsymbol{f}_{P_N}^{P_1} \right) &= m^{P_1} \boldsymbol{r}_A^{P_1} \times \boldsymbol{a}_0^{P_1}, \\
\boldsymbol{r}_A^{P_2} \times \left(\boldsymbol{F}^{P_2} + \boldsymbol{f}_{P_1}^{P_2} + \boldsymbol{f}_{P_3}^{P_2} + \ldots + \boldsymbol{f}_{P_N}^{P_2} \right) &= m^{P_2} \boldsymbol{r}_A^{P_2} \times \boldsymbol{a}_0^{P_2}, \\
&\ldots \\
\boldsymbol{r}_A^{P_N} \times \left(\boldsymbol{F}^{P_N} + \boldsymbol{f}_{P_1}^{P_N} + \boldsymbol{f}_{P_2}^{P_N} + \ldots + \boldsymbol{f}_{P_{N-1}}^{P_N} \right) &= m^{P_N} \boldsymbol{r}_A^{P_N} \times \boldsymbol{a}_0^{P_N}.
\end{aligned}$$

Summing for all particles in the material system, and using Eq. (12.2), the moments due to internal forces cancel out:

$$\boldsymbol{M}_A = \sum_{i=1}^{N} \boldsymbol{r}_A^{P_i} \times \boldsymbol{F}^{P_i} = \sum_{i=1}^{N} m^{P_i} \boldsymbol{r}_A^{P_i} \times \boldsymbol{a}_0^{P_i}. \tag{12.5}$$

where $\boldsymbol{M}_A$ is the *resultant moment of external forces* about point A acting on the material system.

In summary, only the external forces contribute to the resultant force and to the resultant moment about the arbitrary point A.

We now turn our attention to the inertial effects described by the right-hand side of equations Eq. (12.4) and Eq. (12.5). First, for the resultant force we replace the acceleration with the second derivative of position. Because the mass of each particle is constant, we may take the derivative sign outside of the sum, i.e.,

$$\boldsymbol{F} = \sum_{i=1}^{N} m^{P_i} \boldsymbol{a}_0^{P_i} = \frac{\mathrm{d}^2}{\mathrm{d}t^2} \sum_{i=1}^{N} m^{P_i} \boldsymbol{r}_0^{P_i}. \tag{12.6}$$

If the total mass of the material system is $m = \sum_{i=1}^{N} m^{P_i}$, using the definition of the center of mass, $m\boldsymbol{r}_0^G = \sum_{i=1}^{N} m^{P_i} \boldsymbol{r}_0^{P_i}$, leads to

$$\boldsymbol{F} = m \frac{\mathrm{d}^2 \boldsymbol{r}_0^G}{\mathrm{d}t^2} = m \frac{\mathrm{d}\boldsymbol{v}_0^G}{\mathrm{d}t} = m\boldsymbol{a}_0^G, \tag{12.7}$$

or, noting that the total linear momentum of the system is $\boldsymbol{p}_0 = m\boldsymbol{v}_0^G$,

$$\left. \frac{\mathrm{d}\boldsymbol{p}_0}{\mathrm{d}t} \right|_0 = \boldsymbol{F} = m\boldsymbol{a}_0^G, \tag{12.8}$$

From this result we conclude that, although Newton posed the Second Law for a single point particle, his equation also captures the behavior of the center of mass of a material system if we consider the

resultant of all external forces acting upon it. The relevance of this relation lies in its application to rigid bodies, in which case it addresses the translational part of the description of the motion the body.

We consider now the right-hand side of equation Eq. (12.5). For each particle P_i we may write:

$$\sum_{i=1}^{N} m^{P_i} \boldsymbol{r}_A^{P_i} \times \boldsymbol{a}_0^{P_i} = \sum_{i=1}^{N} (\boldsymbol{r}_0^{P_i} - \boldsymbol{r}_0^A) \times m^{P_i} \boldsymbol{a}_0^{P_i}. \tag{12.9}$$

Recall that the angular momentum of a point particle P_i with respect to an arbitrary point A is

$$\boldsymbol{H}_{A,0}^{P_i} = \boldsymbol{r}_A^{P_i} \times m^{P_i} \boldsymbol{v}_0^{P_i} = (\boldsymbol{r}_0^{P_i} - \boldsymbol{r}_0^A) \times m^{P_i} \boldsymbol{v}_0^{P_i}. \tag{12.10}$$

Differentiating:

$$\left. \frac{\mathrm{d}\boldsymbol{H}_{A,0}^{P_i}}{\mathrm{d}t} \right|_0 = (\boldsymbol{r}_0^{P_i} - \boldsymbol{r}_0^A) \times m^{P_i} \boldsymbol{a}_0^{P_i} + (\boldsymbol{v}_0^{P_i} - \boldsymbol{v}_0^A) \times m^{P_i} \boldsymbol{v}_0^{P_i} = \boldsymbol{r}_A^{P_i} \times m^{P_i} \boldsymbol{a}_0^{P_i} - \boldsymbol{v}_0^A \times m^{P_i} \boldsymbol{v}_0^{P_i} \tag{12.11}$$

Noting that the total angular momentum of the material system with respect to A, i.e., $\boldsymbol{H}_{A,0}$, is the sum of $\boldsymbol{H}_{A,0}^{P_i}$ for $i = 1, \ldots, N$, the right-hand side of equation (12.5) may expressed as

$$\sum_{i=1}^{N} m^{P_i} \boldsymbol{r}_A^{P_i} \times \boldsymbol{a}_0^{P_i} = \left. \frac{\mathrm{d}\boldsymbol{H}_{A,0}}{\mathrm{d}t} \right|_0 + \sum_{i=1}^{N} \boldsymbol{v}_0^A \times m^{P_i} \boldsymbol{v}_0^{P_i} = \left. \frac{\mathrm{d}\boldsymbol{H}_{A,0}}{\mathrm{d}t} \right|_0 + m \boldsymbol{v}_0^A \times \boldsymbol{v}_0^G, \tag{12.12}$$

where once again the definition of the center of mass has been used to simplify the last term. In conclusion, Eq. (12.5) can be rewritten as:

$$\left. \frac{\mathrm{d}\boldsymbol{H}_{A,0}}{\mathrm{d}t} \right|_0 = \boldsymbol{M}_A - m \boldsymbol{v}_0^A \times \boldsymbol{v}_0^G, \tag{12.13}$$

where $\boldsymbol{v}_0^A$ is the velocity of the geometric point A about which we have taken moments, which may or may not be a point of the rigid body. From this result we conclude that the angular momentum equation for a point particle, Eq. (4.52), is also applicable to a material system, if we use its total angular momentum and resultant moment of forces acting upon it. When the material system is a rigid body, this equation describes its rotational dynamics.

Expressions Eq. (12.8) and Eq. (12.13) can be regarded as the evolution equations for the total linear and angular momentum of the material system. They govern respectively the average translation and rotation motion of any system of particles. The details of the internal forces affect only how the individual particles in the system move with respect to these averages. For example, if our concern were with the effects of deformation of an elastic body, the internal forces would be stress resultants, and we would need to characterize how those quantities relate to the positions of the various particles. In the case of a free deformable body with N point particles, the number of degrees of freedom is $3N$, so Eqs. (12.8) and (12.13) alone are not enough to solve the dynamics of the material system, for which the individual equations Eqs. (12.3) need to be used. In the case of a free rigid body, however, the kinematic condition of rigidity reduces the number of degrees of freedom to only 6, and then only Eqs. (12.8) and (12.13) are needed to fully solve the dynamics of the body.

12.2 General equations for a rigid body

Direct application of Eq. (12.8) provides the acceleration of the center of mass of the body as a function of the resultant of the forces acting upon it. Integrating this vector equation would enable us to determine its velocity and position. The rotational part of the motion of the body is provided by the integration of Eq. (12.13). Clearly, the derivation has been general as to which point A we choose to take moments about. Since we have freedom to select A, in order to simplify the solution of the problem, it is often beneficial to select A such that the term $m \boldsymbol{v}_0^A \times \boldsymbol{v}_0^G$ vanishes in Eq. (12.13). There are two possibilities of interest:

1. Select point A to be the center of mass G, so that $\boldsymbol{v}_0^G = \boldsymbol{v}_0^A$ and the cross product vanishes. This is the point we shall use whenever the body of interest executes a general motion.

2. Select point A such that $\boldsymbol{v}_0^A = 0$. This, however, only leads to simple expressions if A is a body point, so that the expression of the tensor of inertia about A, which appears in the definition of $\boldsymbol{H}_{A,0}$, is simple. Hence, we will take this choice only when there is a fixed point A of the rigid body.

Formulation of the force equation of motion Eq. (12.8) already requires that we identify the center of mass. Thus, selecting the center of mass as the focal point for the moment equation would lead to a general procedure. However, there is an important reason why the second choice above is preferable for pure rotations when there exists a fixed point of the rigid body: preventing the pivot point A from moving requires the existence of reaction forces. By definition, reactions are not known in advance; however, these reaction forces do not cause a moment of forces about the pivot point, so writing the equation of evolution of $\boldsymbol{H}_{A,0}$ means that these unknown reactions will not appear in the rotational equations of motion.

Observe that in general the translational equations of motion, Eq. (12.8), and the rotational equations of motion, Eq. (12.13), are coupled together, meaning that they need to be solved simultaneously to solve for the 6 degrees of freedom of a free rigid body. Since each of the six scalar equations is a second-order differential equation, the combined order of the system of equations is 12. Consequently, a set of 12 initial conditions is required to determine the motion of the system once the equations have been integrated. Typically, these will be the initial values of the coordinates that we are using to represent the configuration of the body and their first derivatives; if our coordinates are the position coordinates of the center of mass G and three Euler angles, $x_0^G, y_0^G, z_0^G, \dot{x}_0^G, \dot{y}_0^G, \dot{z}_0^G$ and $\psi_0, \theta_0, \varphi_0, \dot{\psi}_0, \dot{\theta}_0, \dot{\varphi}_0$ would be required.

12.2.1 Generalization to include external torques

When there exists a pair of external forces acting on a rigid body, equal and opposite, each with a different line of action, their resultant is zero, but they cause a torque $\boldsymbol{Q}_k$, regardless of the point about which the moment of forces is computed. Sometimes, it is convenient to substitute such pair of forces in the formulation of the problem with the torque $\boldsymbol{Q}_k$ they produce. This approach allows to consider the external loads acting on the system of analysis as a general combination of external forces $\boldsymbol{F}^{P_i}$ and external torques $\boldsymbol{Q}_k$ with $k = 1, \ldots, M$. When this case is contemplated, one must redefine the resultant moment about a point A, Eq. (12.5), as

$$\boldsymbol{M}_A = \sum_{i=1}^{N} \boldsymbol{r}_A^{P_i} \times \boldsymbol{F}^{P_i} + \sum_{k=1}^{M} \boldsymbol{Q}_k. \tag{12.14}$$

After taking this change into account, Eq. (12.13) remains unchanged. Observe that if we considered the existence of internal torques $\boldsymbol{q}_{P_j}^{P_i}$, if any, would not affect in any way the obtained results for a rigid body.

12.2.2 Equivalence and reduction of systems of forces and moments

When working with general external forces and torques on a rigid body, it is useful to reduce them into a resultant force $\boldsymbol{F}$ and a resultant moment about a point $\boldsymbol{M}_A$ as we have done in the previous Section. We can say that the pair $\{\boldsymbol{F}; \boldsymbol{M}_A\}$ is *equivalent* to the whole set of actual loads $\{\boldsymbol{F}^{P_1}, \boldsymbol{F}^{P_1}, \ldots; \boldsymbol{Q}_1, \boldsymbol{Q}_2, \ldots\}$, as they provide the same information for the computation of the motion of the body. This can be done since the external forces acting on a rigid body are sliding vectors, and the external torques are free vectors. In general, we say that any two systems of forces and torques:

$$\begin{aligned} &\text{System 1:} \quad \{\boldsymbol{F}_1^{P_1}, \boldsymbol{F}_2^{P_2}, \ldots; \boldsymbol{Q}_1, \boldsymbol{Q}_2, \ldots\}, \\ &\text{System 2:} \quad \{\boldsymbol{F}_3^{P_3}, \boldsymbol{F}_4^{P_4}, \ldots; \boldsymbol{Q}_3, \boldsymbol{Q}_4, \ldots\}, \end{aligned}$$

are equivalent if they have the same force resultant ($\boldsymbol{F}$) and the same resultant moment ($\boldsymbol{M}_A$) at one point A, i.e.:

$$\begin{aligned} \boldsymbol{F}_1^{P_1} + \boldsymbol{F}_2^{P_2} + \ldots &= \boldsymbol{F}_3^{P_3} + \boldsymbol{F}_4^{P_4} + \ldots \\ \boldsymbol{r}_A^{P_1} \times \boldsymbol{F}_1^{P_1} + \boldsymbol{r}_A^{P_2} \times \boldsymbol{F}_2^{P_2} + \ldots + \boldsymbol{Q}_1 + \boldsymbol{Q}_2 + \ldots &= \boldsymbol{r}_A^{P_3} \times \boldsymbol{F}_3^{P_3} + \boldsymbol{r}_A^{P_4} \times \boldsymbol{F}_4^{P_4} + \ldots + \boldsymbol{Q}_3 + \boldsymbol{Q}_4 + \ldots \end{aligned}$$

Given the resultant force and the resultant moment about A of a system of forces, $\{\boldsymbol{F}; \boldsymbol{M}_A\}$, it is possible to "translate" it to an equivalent system $\{\boldsymbol{F}; \boldsymbol{M}_B\}$ where the moment is expressed about a different point B. Naturally, the resultant force $\boldsymbol{F}$ is the same for the two cases. The resultant moment of forces $\boldsymbol{M}_B$ about that point is related to $\boldsymbol{M}_A$ by the following expression:

$$\boldsymbol{M}_B = \boldsymbol{M}_A + \boldsymbol{BA} \times \boldsymbol{F} = \boldsymbol{M}_A + \boldsymbol{F} \times \boldsymbol{AB}. \tag{12.15}$$

An example of three equivalent load systems is shown in Fig. 12.2. Equation (12.15) is mathematically analogous to the equation of the velocity field of a rigid body, Eq. (9.3), if we identify $\boldsymbol{M}_A$ with $\boldsymbol{v}_0^A$ and $\boldsymbol{F}$ with $\boldsymbol{\omega}_{B0}$. For this reason, we can state that $M_{\parallel} = \boldsymbol{M}_A \cdot \boldsymbol{F}/F$ is a scalar constant of the system of forces that does not depend on the point A used to express the resultant moment of forces (by analogy with the parallel velocity $v_{0\parallel}$ in the velocity field of the rigid body), and that there is a line of reduction points (known as *central axis of the force system*) where the moment of forces is minimum and equal to $\boldsymbol{M}_A = M_{\parallel}\boldsymbol{F}/F$ (analogous to the instantaneous axis of rotation and slip of the velocity field of the rigid body). Note that it is possible to have $\boldsymbol{F} = 0$ and yet $\boldsymbol{M}_A \neq 0$, as it happens for example in the case of a *couple of forces*. Since the resultant force of a couple is zero, the point of reduction of the system is irrelevant in that case (analogous to a pure translation velocity field).

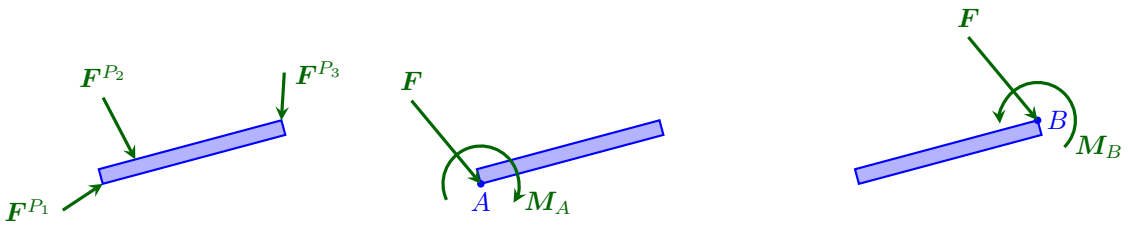

Figure 12.2: Three equivalent force systems on a rigid bar: (left) a number of forces applied at specific points; (middle) resultant force and torque about a point A; (right) resultant force and torque about a point B.

The ability to relate the moment of forces at two arbitrary points allows us, in particular, to write $\boldsymbol{M}_G$ at the center of mass G using the moment of forces about another point C. Combining Eq. (12.8) and Eq. (12.13):

$$\left.\frac{\mathrm{d}\boldsymbol{H}_{G,0}}{\mathrm{d}t}\right|_0 = \boldsymbol{M}_G = \boldsymbol{M}_C - \boldsymbol{CG} \times \boldsymbol{F}. \tag{12.16}$$

12.2.3 Rate of change of angular momentum

The main novelty that we find in the dynamics of a rigid body with respect to the dynamics of a point particle resides in the need to deal with the moment of force equation, Eq. (12.13). The treatment of the left hand side of this equation requires special attention.

Assuming that we have defined the point A about which we are taking moments as its center of mass G or as a fixed point of the rigid body, if any, then according to Eqs. (11.12) and (11.13) the total angular momentum of the body can be written as:

$$\boldsymbol{H}_{A,0} = \overline{\overline{\mathcal{I}}}_A \cdot \boldsymbol{\omega}_{B0}, \tag{12.17}$$

where $\overline{\overline{\mathcal{I}}}_A$ is the tensor of inertia of the body B about the body point A. Hence, the time derivative of the angular momentum of a rigid body in the left hand side of Eq. (12.13) always includes a term of the type:

$$\left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_0 \left(\overline{\overline{\mathcal{I}}}_A \cdot \boldsymbol{\omega}_{B0}\right), \tag{12.18}$$

i.e., the time derivative of a tensor times a vector in the inertial reference frame S_0. Importantly, if the body rotates relative to S_0, then the mass distribution $\mathcal{D}(\boldsymbol{r}_0^P)$ of the body changes in time with

respect to S_0, and therefore the inertia properties of the body are not constant in time as seen from S_0. In other words, if we take the time derivative in S_0 directly, then applying the chain rule,

$$\left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_0 \left(\overline{\overline{\mathcal{I}}}_A \cdot \boldsymbol{\omega}_{B0}\right) = \left.\frac{\mathrm{d}\overline{\overline{\mathcal{I}}}_A}{\mathrm{d}t}\right|_0 \cdot \boldsymbol{\omega}_{B0} + \overline{\overline{\mathcal{I}}}_A \cdot \boldsymbol{\alpha}_{B0} \tag{12.19}$$

Clearly, it is preferable to compute this time derivative in a reference frame where the spatial mass distribution remains constant, i.e., in a *body-fixed reference frame* $S_B = \{A; B_B\}$. This can be achieved by applying the Coriolis theorem, Eq. (3.8), to Eq. (12.18):

$$\left.\frac{\mathrm{d}\boldsymbol{H}_{A,0}}{\mathrm{d}t}\right|_0 = \left.\frac{\mathrm{d}\boldsymbol{H}_{A,0}}{\mathrm{d}t}\right|_B + \boldsymbol{\omega}_{B0} \times \boldsymbol{H}_{A,0}. \tag{12.20}$$

Now, since the mass distribution is fixed with respect to S_B, the inertia tensor is a constant inside the derivative, and can be taken out of it:

$$\left.\frac{\mathrm{d}\boldsymbol{H}_{A,0}}{\mathrm{d}t}\right|_0 = \overline{\overline{\mathcal{I}}}_A \cdot \boldsymbol{\alpha}_{B0} + \boldsymbol{\omega}_{B0} \times \left(\overline{\overline{\mathcal{I}}}_A \cdot \boldsymbol{\omega}_{B0}\right) \tag{12.21}$$

where we have taken into account that the time derivative of $\boldsymbol{\omega}_{B0}$ in S_B coincides with its time derivative in S_0, Eq. (3.9). Incidentally, by comparing Eqs. (12.19) and (12.21), it is apparent that

$$\left.\frac{\mathrm{d}\overline{\overline{\mathcal{I}}}_A}{\mathrm{d}t}\right|_0 \cdot \boldsymbol{\omega}_{B0} = \boldsymbol{\omega}_{B0} \times \left(\overline{\overline{\mathcal{I}}}_A \cdot \boldsymbol{\omega}_{B0}\right) \tag{12.22}$$

Additionally, if we decide to define the vector basis $B_B : \{\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3\}$ of S_B to coincide with the principal axes of inertia, the number of non-zero terms to compute is small (see §10.7.1), with the result that

$$\left.\frac{\mathrm{d}\boldsymbol{H}_{A,0}}{\mathrm{d}t}\right|_0 = \begin{bmatrix} I_1 & 0 & 0 \\ 0 & I_2 & 0 \\ 0 & 0 & I_3 \end{bmatrix}_{BB} \cdot \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{bmatrix}_B + \begin{bmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{bmatrix}_B \times \left(\begin{bmatrix} I_1 & 0 & 0 \\ 0 & I_2 & 0 \\ 0 & 0 & I_3 \end{bmatrix}_{BB} \cdot \begin{bmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{bmatrix}_B \right),$$

where all vector and tensor components are expressed in consistent bases. This leads to a form of the equations in this case known as *Euler's equations in the principal axes of inertia*:

$$\begin{aligned} \left.\frac{\mathrm{d}\boldsymbol{H}_{A,0}}{\mathrm{d}t}\right|_0 \cdot \boldsymbol{e}_1 &= I_1\alpha_1 - (I_2 - I_3)\omega_2\omega_3, \\ \left.\frac{\mathrm{d}\boldsymbol{H}_{A,0}}{\mathrm{d}t}\right|_0 \cdot \boldsymbol{e}_2 &= I_2\alpha_2 - (I_3 - I_1)\omega_3\omega_1, \\ \left.\frac{\mathrm{d}\boldsymbol{H}_{A,0}}{\mathrm{d}t}\right|_0 \cdot \boldsymbol{e}_3 &= I_3\alpha_3 - (I_1 - I_2)\omega_1\omega_2. \end{aligned} \tag{12.23}$$

The simplicity of the equations in body-fixed reference frames encourages us to compute $\overline{\overline{\mathcal{I}}}_A$, $\boldsymbol{\omega}_{B0}$ and $\boldsymbol{\alpha}_{B0}$ in a body-fixed vector basis instead of the inertial reference frame basis.. Moreover, the simpler appearance of Euler equations when using a body-fixed reference frame S_B that coincides with principal axes of inertia would suggest that we should always search for these special directions in the body.

However, it is important to realize that the identification of these directions in an asymmetrical body is an arduous task that involves diagonalizing a tensor, and that the choice of orientation of B_B also affects the ease with which $\boldsymbol{\omega}_{B0}$ and $\boldsymbol{\alpha}_{B0}$ are expressed. Hence, it may not be always worth to find and work in principal axes of inertia, as a general body-fixed reference frame may offer the same (or even less) mathematical complexity in the calculations.

This leads to the final observation that it is best to select the body-fixed reference frame based on its overall convenience to describe the inertia tensor, the angular velocity vector and the angular acceleration vector. If, and only if, our choice for S_B happens to correspond to principal axes, then we may use the system of equations given by Eqs. (12.23).

12.3 Equilibrium of a rigid body

The general equations of the rigid body discussed in previous sections can also be used to describe the existence of *equilibrium configurations*, i.e., to study the statics of the problem. In order to be in equilibrium, a body which is initially at rest (i.e. with zero linear and angular velocities, $\boldsymbol{v}_0^G = \boldsymbol{0}, \boldsymbol{\omega}_{B0} = \boldsymbol{0}$ at $t = 0$) must remain at rest, so that $\boldsymbol{v}_0^G = \boldsymbol{0}, \boldsymbol{\omega}_{B0} = \boldsymbol{0}$ for all times. Obviously, this means that there cannot be any linear or angular acceleration for the body, $\boldsymbol{a}_0^G = \boldsymbol{0}$ and $\boldsymbol{\alpha}_{B0} = \boldsymbol{0}$.

As a consequence of this, the left-hand side of the equations of motion is zero at all times. This leads to the condition that the force system acting on the body must fulfill in any equilibrium position or configuration:

$$\boldsymbol{F} = \boldsymbol{0}, \tag{12.24}$$

$$\boldsymbol{M}_A = \boldsymbol{0}. \tag{12.25}$$

From the previous discussion we can extract the following consequence: if a rigid body that is subject to a system of forces is in equilibrium, it will also be in equilibrium when it is subject to an equivalent system of forces.

12.4 Conservation of the mechanical energy of rigid bodies

We have studied the concepts of power, work and the principle of conservation of mechanical energy for the point particle (§4.5). These powerful ideas can be extended to rigid bodies.

Consider a rigid body composed of N particles as sketched in Fig. 12.1, subject to external forces $\boldsymbol{F}^{P_i}$. For the purpose of generality, consider also the existence of M external torques $\boldsymbol{Q}_k$. The rate of work done on the system per unit time (i.e., the *power*) is due only to the external forces and torques, as the work of internal ones cancels out by virtue of the Third Law of Newton. Observe that in this definition we have to consider the velocity vectors of the point of application of the force, and not the center of mass:

$$\dot{W}_0 = \sum_{i=1}^{N} \boldsymbol{F}^{P_i} \cdot \boldsymbol{v}_0^{P_i} + \sum_{k=1}^{M} \boldsymbol{Q}_k \cdot \boldsymbol{\omega}_{B0}. \tag{12.26}$$

Using the expression of the velocity field of a rigid body based on the body point A,

$$\boldsymbol{v}_0^{P_i} = \boldsymbol{v}_0^A + \boldsymbol{\omega}_{B0} \times \boldsymbol{r}_A^{P_i}, \tag{12.27}$$

it is possible to express the first term of the definition of power in Eq. (12.26) as:

$$\begin{aligned} \sum_{i=1}^{N} \boldsymbol{F}^{P_i} \cdot (\boldsymbol{v}_0^A + \boldsymbol{\omega}_{B0} \times \boldsymbol{r}_A^{P_i}) &= \sum_{i=1}^{N} \boldsymbol{F}^{P_i} \cdot \boldsymbol{v}_0^A + \sum_{i=1}^{N} \boldsymbol{F}^{P_i} \cdot (\boldsymbol{\omega}_{B0} \times \boldsymbol{r}_A^{P_i}) = \\ &= \left(\sum_{i=1}^{N} \boldsymbol{F}^{P_i} \right) \cdot \boldsymbol{v}_0^A + \left(\sum_{i=1}^{N} \boldsymbol{r}_A^{P_i} \times \boldsymbol{F}^{P_i} \right) \cdot \boldsymbol{\omega}_{B0}, \end{aligned} \tag{12.28}$$

and, therefore, we can finally rewrite Eq. (12.26) as

$$\dot{W}_0 = \boldsymbol{F} \cdot \boldsymbol{v}_0^A + \boldsymbol{M}_A \cdot \boldsymbol{\omega}_{B0}. \tag{12.29}$$

This result shows that the power developed by a system of forces and torques is equal to the power due to the resultant force on a point A and the power due to the resultant moment about A.

On the other hand, we can also write the power as the time rate of the total kinetic energy of the body:

$$\dot{W}_0 = \sum_{i=1}^{N} m^{P_i} \left. \frac{\mathrm{d}\boldsymbol{v}_0^{P_i}}{\mathrm{d}t} \right|_0 \cdot \boldsymbol{v}_0^{P_i} = \sum_{i=1}^{N} \frac{1}{2} m^{P_i} \frac{\mathrm{d}(\boldsymbol{v}_0^{P_i} \cdot \boldsymbol{v}_0^{P_i})}{\mathrm{d}t} = \frac{\mathrm{d}}{\mathrm{d}t} \left[\sum_{i=1}^{N} \frac{1}{2} m^{P_i} (v_0^{P_i})^2 \right] = \frac{\mathrm{d}T_0}{\mathrm{d}t}.$$

Therefore, the work done by all forces and torques acting on the rigid body as it moves from its initial position at time t_1 to its final position at time t_2 may be written as

$$W_0(t_1, t_2) = \int_{t_1}^{t_2} \dot{W}_0 \mathrm{d}t = \int_{t_1}^{t_2} \frac{\mathrm{d}T_0}{\mathrm{d}t} \mathrm{d}t = T_{0,2} - T_{0,1}, \tag{12.30}$$

where $T_{0,1}$ and $T_{0,2}$ are the kinetic energies of the rigid body at the mentioned instants, as seen from S_0. Recall that the kinetic energy of the rigid body can be computed from the expressions derived in §11.3. In particular, when A is a fixed point of the rigid body,

$$T_0 = \frac{1}{2}\boldsymbol{\omega}_{B0} \cdot \overline{\overline{\mathcal{I}}}_A \cdot \boldsymbol{\omega}_{B0}. \tag{12.31}$$

And, in general, we can always write:

$$T_0 = \frac{1}{2}m(v_0^G)^2 + \frac{1}{2}\boldsymbol{\omega}_{B0} \cdot \overline{\overline{\mathcal{I}}}_G \cdot \boldsymbol{\omega}_{B0}. \tag{12.32}$$

If all the external forces and torques that exert work are conservative, it is possible to write the work carried out by them between the two instants of time as the difference of the total potential energy, $W_0(t_1, t_2) = V_{0,1} - V_{0,2}$. In this case, we finally obtain the law of conservation of energy:

$$T_0 + V_0 = T_{0,2} + V_{0,2} = T_{0,1} + V_{0,1} = E_0. \tag{12.33}$$

As in the case of the point particle, the condition for this conservation equation to hold is that all forces that do work (i.e., those forces $\boldsymbol{F}^{P_i}$ that have a component parallel to $\boldsymbol{v}^{P_i}$) derive from a potential. The conservation of mechanical energy still applies if non-conservative forces exist but they are always perpendicular to the velocity of their application point (i.e., $\boldsymbol{F}^{P_i} \cdot \boldsymbol{v}_0^{P_i} = 0$). If torques $\boldsymbol{Q}_k$ are also being considered in the formulation of the problem, we can generalize this statement by saying that all torques that do work (i.e., those torques $\boldsymbol{Q}_k$ that have a component parallel to $\boldsymbol{\omega}_{B0}$) must derive from a potential, and that the conservation of mechanical energy still holds if those torques which are not conservative are always perpendicular to $\boldsymbol{\omega}_{B0}$ (i.e., $\boldsymbol{Q}_k \cdot \boldsymbol{\omega}_{B0} = 0$).

For example, it is easy to show that the weight force due to a constant gravity acceleration $\boldsymbol{g}$ is a conservative force. After integrating over the mass distribution $\mathcal{D}(\boldsymbol{r}_0^P)$ of the rigid body, the following potential energy is found

$$V_0 = mgz_0^G, \tag{12.34}$$

where g is the gravitational acceleration and z_0^G is the height of the center of mass above some arbitrarily selected reference.

Lastly, observe that reducing the actual system of forces and torques to a resultant force $\boldsymbol{F}$ and a moment of forces $\boldsymbol{M}_G$ about the center of mass, allows us to identify the two contributions separately. Since the translational dynamics of the rigid body is equivalent to that of a point particle concentrating the total mass m of the body and located at the center of mass, i.e., $\boldsymbol{F} = m\boldsymbol{a}_0^G$, it is also true that the work of the resultant force changes the translational kinetic energy of the body (first term Eq. (12.32)),

$$\int_{t_1}^{t_2} \boldsymbol{F} \cdot \boldsymbol{v}_0^G \mathrm{d}t = \frac{1}{2}m(v_0^G)^2\Big|_2 - \frac{1}{2}m(v_0^G)^2\Big|_1. \tag{12.35}$$

Then, we can also identify the work of the resultant moment of forces about G as the change in rotational kinetic energy of the body (second term Eq. (12.32)),

$$\int_{t_1}^{t_2} \boldsymbol{M}_G \cdot \boldsymbol{\omega}_{B0} \mathrm{d}t = \frac{1}{2}\boldsymbol{\omega}_{B0} \cdot \overline{\overline{\mathcal{I}}}_G \cdot \boldsymbol{\omega}_{B0}\Big|_2 - \frac{1}{2}\boldsymbol{\omega}_{B0} \cdot \overline{\overline{\mathcal{I}}}_G \cdot \boldsymbol{\omega}_{B0}\Big|_1. \tag{12.36}$$

13 Constraints, reactions, and systems of rigid bodies

Like in the case of a single point particle, the motion of a rigid body can be partially restricted by *kinematic constraints*, such as contact constraints with neighboring bodies. The physical mechanisms that ensure the fulfillment of the constraints are *reaction forces and torques*. Constraints reduce the effective number of degrees of freedom of motion of the free rigid body (6 degrees of freedom). In exchange, the components of the reaction forces and couples appear as new unknowns of the problem.

There can exist many different types of constraints between rigid bodies; our goal in this chapter is to understand how to "translate" these constraints into mathematical conditions on the linear and angular displacement and velocity vectors of each body, and how to define reaction forces and torques. In the following Sections, the treatment of constraints and reactions is described, and then it is illustrated for some particular cases. In particular, the constraints of rolling and rolling without slip are discussed in detail.

Lastly, in many problems we have to deal not just with a single rigid body, but with a *system of rigid bodies* (and/or particles). The various parts of the system may interact among each other through forces and torques that are internal to the system, but external to each of the parts. These forces and torques can be applied ones, such as gravity or electromagnetic forces, or can be unknown reactions. In this Chapter we also show that the dynamics analysis can be applied to the whole system collectively if we are only interested in the position of the global center of mass and the global angular momentum, or to each of its parts separately, if we need to obtain properties specific to each of its constituents or the value of the system-internal reaction forces and torques.

13.1 Constraint examples

A simple but common constraint condition arises when a body is permitted to execute only *planar motion*. By definition, planar motion means that all points in the body move in parallel planes, which can only happen if the angular velocity is always perpendicular to these planes. The restriction to planar motion reduces the possible motions of the rigid body to three: two translations in the directions of the plane, and a rotation along the normal direction to the plane. Therefore, the effective number of degrees of freedom is 3.

An inextensible *string* keeps the distance between two points constant, as long as it remains straight. It is common for strings to be combined with ideally frictionless pulleys, in which case the constraint must still consider that the total length of the string remains constant regardless of its shape. The constraint links the displacement of the attached points in the tangent direction to the string. This is effected by the reaction force named tension, which is always tangent to the string.

A *ball-and-socket joint* connecting bodies B_1 and B_2 in Fig. 13.1 allows each body to rotate freely about the central point C, so it does not impose a constraint on the orientation of either body. However, the center C of the ball is a body point common to both bodies, so the bodies must move in unison at this junction. This means that the velocity and acceleration of body point $C(B_1)$ and body point $C(B_2)$ of each body must match: $\boldsymbol{v}_0^{C(B_1)} = \boldsymbol{v}_0^{C(B_2)}$ and $\boldsymbol{a}_0^{C(B_1)} = \boldsymbol{a}_0^{C(B_2)}$. For the dynamics analysis it is necessary to consider three components of the resultant reaction force at C: R_x, R_y and R_z. Recall that, by virtue of Newton's third law, the reaction force on body B_1 will be the same but with opposite sign as on body B_2. Since the orientation of the body 2 is not constrained, no reaction torques appear at C.

In the case of a *pin connection* between bodies, also depicted in Fig. 13.1, the axis of rotation of the pin is a common axis Cz for both bodies B_1 and B_2, i.e., for any point P of this axis, has the same velocity and acceleration when seen as a point of body B_1 and as a point of body B_2: $\boldsymbol{v}_0^{P(B_1)} = \boldsymbol{v}_0^{P(B_2)}$ and $\boldsymbol{a}_0^{P(B_1)} = \boldsymbol{a}_0^{P(B_2)}$. With respect to the ball-and-socket joint, the pin connection introduces an additional constraint on the relative rotation of the two bodies: the only permitted rotation of body B_2 relative to body B_1 by the pin connection is about the pin axis Cz. Thus, the angular velocity and acceleration vectors along any directions $\boldsymbol{i}, \boldsymbol{j}$ perpendicular to Cz must match for the two bodies: $(\boldsymbol{\omega}_{B_10} - \boldsymbol{\omega}_{B_20}) \cdot \boldsymbol{i} = 0$, $(\boldsymbol{\omega}_{B_10} - \boldsymbol{\omega}_{B_20}) \cdot \boldsymbol{j} = 0$, $(\boldsymbol{\alpha}_{B_10} - \boldsymbol{\alpha}_{B_20}) \cdot \boldsymbol{i} = 0$, $(\boldsymbol{\alpha}_{B_10} - \boldsymbol{\alpha}_{B_20}) \cdot \boldsymbol{j} = 0$. When we consider the reaction force system due to the constraint, it is distributed along the pin axis so as to prevent the other two relative rotations. If we reduce the reaction force system to a resultant force and a moment of forces about a point C of the pin axis, $\boldsymbol{R}$ and $\boldsymbol{M}_C$, only the component of M_{Cz} about the pin axis Cz is zero; hence we need to compute the three components of the reaction force

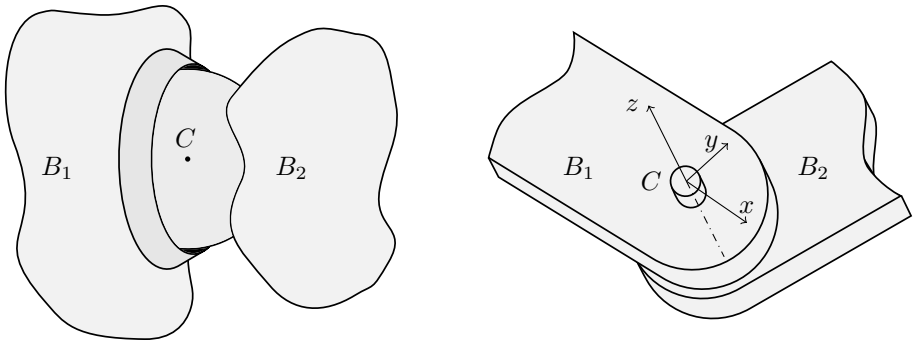

Figure 13.1: Examples of constraints. (left): a ball-and-socket joint. (right): a pin connection.

act at C, plus two moment components M_{Cx} and M_{Cy}. Once again, the reaction forces and couples on each body are equal and opposite to each other by the Third law of Newton.

Another connexion between bodies are *collars or sliders*. A collar that slides over a cylindrical and a prismatic bar is depicted in Fig. 13.2. We characterize the constraint condition in this case by defining a reference frame S_1 attached to the bar, with the z_1 axis aligned with the axis of that bar, and a reference frame S_2 attached to the body B_2 that has the collar, so that the axis Oz_2 coincides with Oz_1. In the first case, the collar can slide and rotate about the z_1 axis, but its displacement and rotations about the x_1 or y_1 with respect to the bar are impeded. Thus, the components of the linear and angular velocity of the two bodies in these directions must match. This is enforced by the corresponding reaction forces R_x and R_y and couples M_{Cx} and M_{Cy} along the x_1 and y_1 axes. In the second case, rotation about z_1 is also forbidden, so the reaction torque also has a component M_{Cz}.

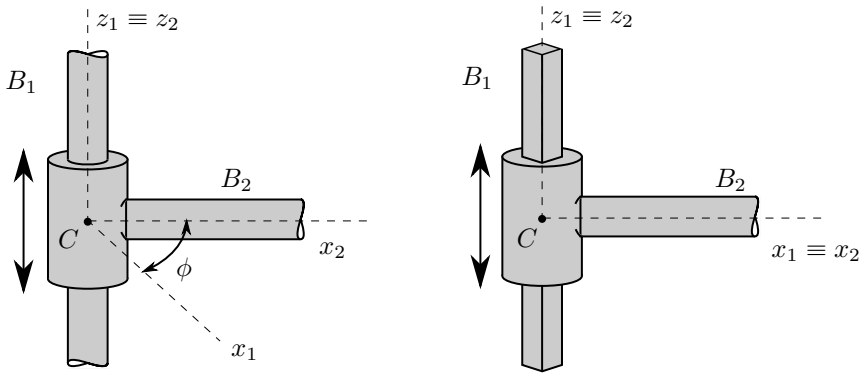

Figure 13.2: Collar or slider connector. (left): collar that allows sliding and rotating. (right): collar that allows only sliding.

Combinations of these mechanical connections are not uncommon. For instance a collar plus a pin connector. Adding the pin connector frees one of the rotational constraints. There are many other possibilities that will not be discussed here.

The general approach to study the kinematic constraints imposed by a mechanical linkage between two rigid bodies is first to infer the constraints in linear and angular displacement and velocity, and second to find the number of non-zero components of the reaction forces and couples, as illustrated above. The number of scalar equations imposed by the constraint must match the number of reaction unknowns (forces and couples); together with the remaining effective degrees of freedom, a maximum of 6 unknowns per constraint and body can exist.

13.2 Rolling motion

When two bodies B_1 and B_2 are in physical contact at a single body point C, they can *roll* relative to each other. Examples of these two cases include a sphere rolling on a plane or two cylinders rolling with respect to each other. There is always an identifiable contact normal direction given by unit vector $\boldsymbol{e}_n$, and a tangential plane given by unit vectors $\boldsymbol{e}_{t1}, \boldsymbol{e}_{t2}$. The contact point/line and the vector basis $B_C : \{\boldsymbol{e}_{t1}, \boldsymbol{e}_{t2}, \boldsymbol{e}_n\}$ can change as the bodies move relative to each other.

Figure 13.3 shows two planar rigid bodies rolling relative to each other, where the instantaneous contact point has been labeled C and the reference frame $S_C : \{C; B_C\}$ has been defined. A first constraint that two bodies in contact must satisfy is that they cannot penetrate each other: the normal velocities of the body points in contact,

$$(\boldsymbol{v}_0^{C(B_1)} - \boldsymbol{v}_0^{C(B_2)}) \cdot \boldsymbol{e}_n = 0. \tag{13.1}$$

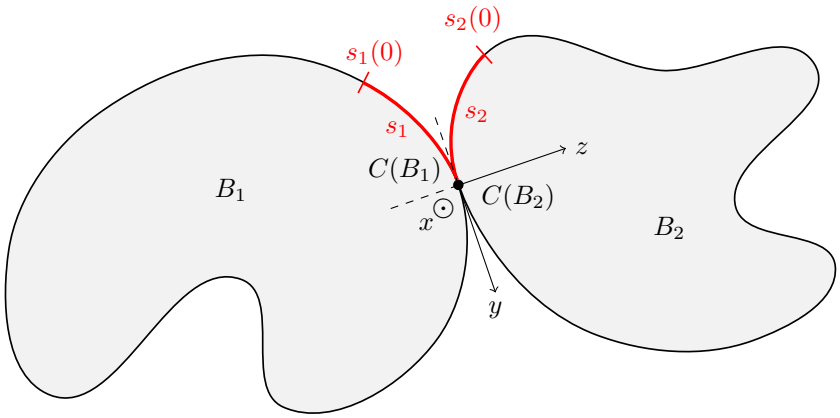

Figure 13.3: Rolling motion between two generic smooth bodies.

A case of rolling motion of special interest in mechanics is that of *rolling without slipping*. In this case, a second constraint that the two bodies must satisfy is that they may not slide with respect to each other. A good example of rolling without slipping is that of a rough rubber wheel on the road. When there is no slipping, the velocity components of the rigid body points in contact are also equal to each other. In other words, the two velocity vectors coincide:

$$\boldsymbol{v}_0^{C(B_1)} = \boldsymbol{v}_0^{C(B_2)}. \tag{13.2}$$

This is equivalent to saying that the contact point of B_1 has zero *relative* velocity with respect to B_2, and vice-versa. An equivalent way to characterize this condition is to consider that the arc lengths s_1 and s_2 traveled by the contact point along the perimeter of each body over a given period of time must be equal,

$$s_1(t) - s_1(0) = s_2(t) - s_2(0). \tag{13.3}$$

In the following, we illustrate the analysis of rolling motion by examining the simple case of a homogeneous disk B of radius R rolling on the floor F along one line, as shown in Fig. 13.4. Consider a reference frame $S_0 : \{O; B_0\}$ fixed to the floor, and a parallel reference frame $S_G : \{G; B_0\}$ with origin on the center of mass of the disk as defined in the figure. We call C the contact point of the disk with the floor.

If the disk *rolls and slips* (e.g. if it were a smooth disk on an ice surface), the only constraint on the motion of the disk would be the no-penetration constraint, meaning that, as long as the disk does not take off from the ground, the vertical position of its center of mass remains $y_0^G = 0$. Then, the problem has two independent degrees of freedom, associated to the horizontal displacement of the disk and its rotation about the Gz axis. We can represent the two degrees of freedom with two coordinates, e.g., the horizontal position of the center of mass, x_0^G and an angle θ that the disk has rotated since the beginning of the motion.

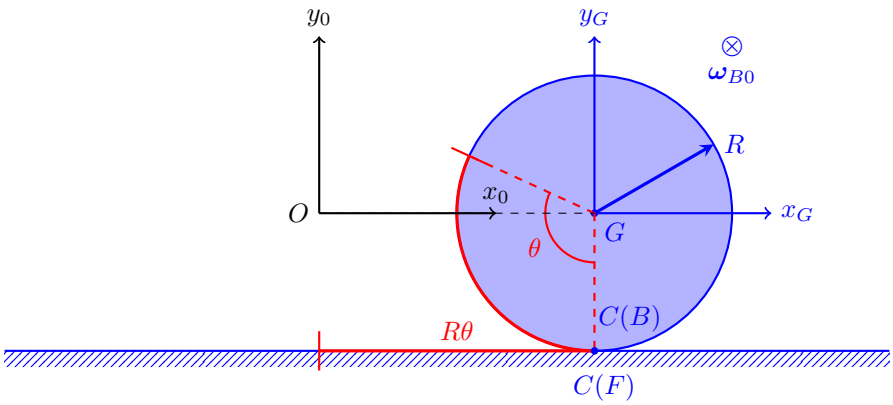

Figure 13.4: A disk B of radius R rolling without slip on a horizontal floor F.

If the disk *rolls without slipping*, however, there is a relation between x_0^G and θ due to the fact that the arc-length s traveled by the contact point C along the floor equals the arc-length of the disk perimeter given by angle θ. Since $s = x_0^G$, the no-slippage constraint reads

$$x_0^G = R\theta. \tag{13.4}$$

Clearly, the rolling without slipping constraint reduces the number of effective degrees of freedom in this case to one, and we only need one generalized coordinate to describe its state. It is often simpler, however, to work initially with the two coordinates, x_0^G and θ in this example, knowing that one of them is redundant, and then apply Eq. (13.4) to eliminate one of them as a function of the other.

One difficulty with a formulation based on arc-lengths is that it becomes increasingly difficult to use, or outright impossible, as the complexity of the motion increases. This is particularly true for 3D motion, as for example in the case of a disk that is rolling without slipping over the floor in a wobbly manner, as depicted in Fig. 13.5. The trace that point C leaves on the floor is now a complicated two-dimensional curve, meaning that now two coordinates are required to define the position of the contact point C in the plane.

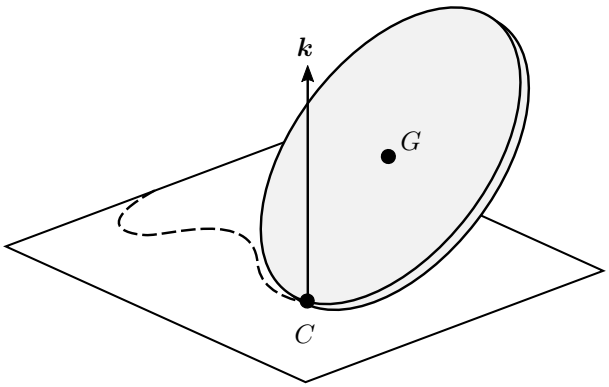

Figure 13.5: A more complex situation: a disk B of radius R rolling and wobbling three-dimensionally on the floor F.

In order to apply simultaneously the no-penetration and no-slippage constraint and obtain the kinematic relations of the disk in a general case like this one it is necessary to establish the mathematical connection between the two coordinates using the condition on the velocities, Eq. (13.2):

$$\boldsymbol{v}_0^{C(B)} = \boldsymbol{v}_0^{C(F)} = \boldsymbol{0}. \tag{13.5}$$

Observe that the geometrical point C which marks where the contact between the disk and the floor takes place is not a body point of either the disk nor the floor, thus the notation $C(B)$ and $C(F)$ to refer to the body point of the disk and the floor that instantaneously coincide with C, as introduced in §9.1.4. This equation states that the velocity of point $C(B)$ of the disk coincides with the velocity of point $C(F)$ of the floor, and, since the floor is at rest, its velocity is zero. In other words, $C(B)$ is a point of the *instantaneous axis of rotation* of the disk. Be aware that, if the ground were not at rest (e.g. if it were a moving conveyor belt), then $\boldsymbol{v}_0^{C(B)} \neq \boldsymbol{0}$.

It should be noted that, except in trivial cases such as the one of Fig. 13.4, it is not possible to integrate the velocity constraint Eq. (13.5) to obtain a relation between the coordinates like the one in Eq. (13.4). The rolling without slipping constraint is then a mathematical relation between the *derivatives* of the coordinates, not between the coordinates themselves. Whenever we cannot write a constraint in the closed form $f(q_1, q_2, q_3, \ldots) = 0$, we say that it is a *non-integrable* constraint. In these cases it is *necessary* to keep one or more redundant coordinates in the formulation of the problem, as it its not possible to eliminate them completely with the equation of the constraint, but only to eliminate their derivatives.

In mechanics, we are typically interested in obtaining the velocity and acceleration of the center of mass G, as well as the angular velocity and acceleration of the body, as a function of a set of independent coordinates. The reason for this is that the equations of dynamics are posed in terms of these kinematic quantities. Knowing the velocity of the body point $C(B)$, it is possible to use the expression of the velocity field of the rigid body to obtain a relation between $\boldsymbol{v}_0^G$ and $\boldsymbol{\omega}_{B0}$:

$$\boldsymbol{v}_0^G = \boldsymbol{v}_0^{C(B)} + \boldsymbol{\omega}_{B0} \times \boldsymbol{CG}. \tag{13.6}$$

After establishing the constraint Eq. (13.5), and with the help of the velocity field of the rigid body, it is always possible to eliminate the redundant coordinates introduced to pose the problem, if any, except for additive integration constants (which must be solved using the initial conditions).

When two bodies roll over each other, it is necessary to consider the existence of a reaction force at the contact point. In general, this reaction force can have a component along the normal direction and in the contact tangent plane, so three unknowns are required to describe it, one for each vector component along three different directions in space. Accordingly, it is usual to decompose the force into a normal reaction $\boldsymbol{N}$ and a friction force $\boldsymbol{R}$ contained within the contact tangent plane. The normal force ensures that the no-penetration constraint is satisfied, and in most cases must be regarded as a unilateral constraint, so that the two bodies can separate if $\boldsymbol{N}$ becomes zero. In general, there may also exist friction torques $\boldsymbol{Q}_R$ if the contact between the two bodies is not a single point.

Regarding friction $\boldsymbol{R}$, there are two distinct motion regimes, which need to be treated differently. Similarly to §6.6, in the most general situation, the rolling bodies may switch back and forth between the two scenarios, requiring us to detect those transitions to modify or solution strategy:

1. If there is no slipping between the bodies, then the friction force could have any value and any direction within the tangent plane, and the two components of vector $\boldsymbol{R}$ need to be retained in the problem as unknowns in exactly the same way as the magnitude of the normal reaction $\boldsymbol{N}$, and be computed as part of the solution. The modulus of the friction force must satisfy $|\boldsymbol{R}| < \mu|\boldsymbol{N}|$, where μ is the friction coefficient.

2. If there is slipping between the bodies, then the direction of the friction force is opposite to the direction of the slipping velocity of the contact point. The magnitude of the friction force is its maximum possible value, $|\boldsymbol{R}| = \mu|\boldsymbol{N}|$. In this case the friction force is known.

Consider again the case of a homogeneous disk that is rolling and slipping over a fixed flat surface as in Fig. 13.4, such that its center of mass G moves to the right with velocity $\boldsymbol{v}_0^G = v_0^G \boldsymbol{i}$. Let us consider that $0 < \mu < \infty$ so the disk can slip if forced to, and that the angular velocity $\boldsymbol{\omega}_{B0} = -\dot{\theta}\boldsymbol{k}$ is such that $v_0^G < |\dot{\theta}|R$. Then, the velocity of the contact point $C(B)$ is $\boldsymbol{v}_0^{C(B)} = v_0^G \boldsymbol{i} - \dot{\theta}R\boldsymbol{i} \neq 0$, directed towards the left. Therefore, the friction force, which opposes the direction of this velocity, is directed towards the right. Interestingly, in this case it acts in the same direction as the velocity of G, resulting in a net acceleration of the disk and a gain of its translational kinetic energy at the expense of its rotational kinetic energy. This is the case of a car at rest or low velocity when the driver suddenly accelerates and the wheels skids on the road briefly.

The transition from slipping to no-slip conditions takes place as soon as the relative velocity of the contact point is zero. Likewise, the transition from no-slip to slipping motion occurs when the magnitude of the required friction force to keep the no-slip condition would be larger than the maximum possible value, $\mu|\boldsymbol{N}|$.

13.3 Indeterminable reactions

There are some situations where it is impossible to fully determine the individual reaction forces that exist at each contact point between two rigid bodies. This is the case, for example, of a table resting on the floor, if the table has more $n > 3$ legs as shown in Fig 13.6. Calling P_i the contact point of leg i with the ground, we cannot determine the normal force at each point, $\boldsymbol{N}^{P_i}$. In fact, we can only determine some overall aspects of the reaction force system from the equilibrium conditions of the table: the sum of all reaction forces must counteract the table weight force $\boldsymbol{W}$, and the torque of all forces about a point, e.g. P_1, must be zero:

$$\sum_i^n \boldsymbol{N}^{P_i} + \boldsymbol{W} = \boldsymbol{0}; \tag{13.7}$$

$$\boldsymbol{P_1P_i} \times \sum_i^n \boldsymbol{N}^{P_i} + \boldsymbol{P_1G} \times \boldsymbol{W} = \boldsymbol{0}. \tag{13.8}$$

These two vector equations yield a total of only 3 independent scalar equations where the magnitudes N^{P_i} appear. Thus, when $n > 3$, the system is mathematically underdetermined. In this case we say that there are redundant constraints. Equilibrium problems like this one are referred to as hyperstatic problems.

In the general motion of rigid bodies, the underdetermination of the reaction forces arises whenever the number of reaction unknowns (reaction force and torque components) is greater than the number of available equations to solve them. In order to obtain the reactions in such cases it is necessary to abandon the model of the rigid body and study the elastic properties of materials. This issue is studied in courses on elasticity and structural analysis and is not discussed further here.

Regardless of the situation, it is always possible to reduce the system of reactions between two rigid bodies to a resultant reaction force and a resultant reaction torque about a chosen point A. These reduced loads can always be determined.

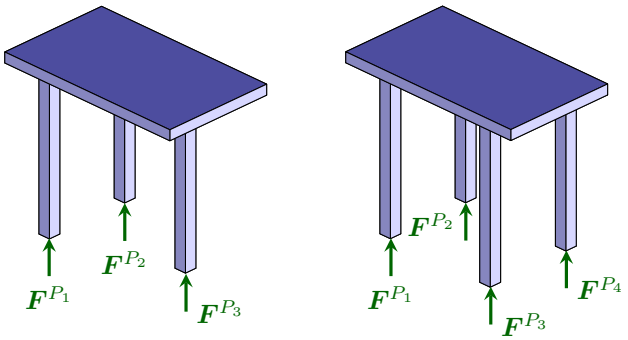

Figure 13.6: Example of determined (left) and undetermined (right) system of reaction forces on a rigid table with n legs resting on the floor.

13.4 General equations for a system of rigid bodies

As we have seen, the model of the rigid body, compared to studying individually each of its constituent particles, has important benefits: doing so reduces the number of kinematic variables and equations needed down to six in the case of a free rigid body, and enables us to ignore the internal interaction forces exerted between the particles. Both advantages result from acknowledging that the particles forming a rigid body are mutually constrained by the rigidity condition.

In a general problem, we may not have just a single rigid body, but a system of several rigid bodies and/or particles that may constrain the motion of each other (e.g. by means of ball-and-socket joints between two bodies, pin connections, etc), and interacting together with reaction and applied forces and torques. The total number of degrees of freedom of the system is the sum of independent degrees of freedom of its parts: for each free body, 6 degrees of freedom exist, and for each free particle, 3. For instance, a system with two bodies and one particle without any constraints require a total of 15 independent parameters to fully determine the configuration (position and orientation of the bodies, and position of the particle). Naturally, if constraints exist, the number of effective degrees of freedom will go down by one per scalar constraint equation, and the same number of new unknowns have to be introduced for the components of the reaction forces and torques.

Looking separately at each rigid body/particle in the system, we can pose the force and moment equations for each body (a total of six scalar relations per body), and the force equations for each particle (a total of 3 scalar ones per particle). The equations for the different parts are likely coupled together when constraints or interactions between them exist, so in general they need to be solved simultaneously.

In the same manner, it is also possible to consider *all bodies/particles* (or several of them) grouped together *as a single material system*, and write down the equations of motion of the combined system. This allows to obtain 6 scalar equations where, interestingly, the inter-body forces and the reaction forces and torques between the parts are now *internal forces and torques*, so they cancel out.

It is important to observe that the equations obtained for the whole system are just a linear combination of the equations of its parts, so the latter do no new information about the system. However, in many circumstances it may be desirable to replace one or more of the equations of a part by one or more of the equations of the whole system to facilitate the resolution of the problem. Note, nevertheless, that the only way to obtain the value of the reaction forces and torques between bodies is to study the equations of the different parts, as they do not appear in the equations of the whole system.

The bottom line is that we can apply the general equations of motion Eqs. (12.4) and (12.5) to any material system that we define, be it a single particle or body, a group of them, or the whole system. The terminology "internal" and "external," when referring to forces and torques depends on what material system we analyze.

In the example of Fig. 13.7, there are two rigid bodies, B_1 and B_2, connected by a ball-and-socket joint at the common body point C, and two external forces $\boldsymbol{F}^{P_1}$ and $\boldsymbol{F}^{P_2}$ applied at points P_1 and P_2. Additionally, the two bodies exert forces on each other, labeled $\boldsymbol{f}_{P_4}^{P_3}$, $\boldsymbol{f}_{P_3}^{P_4}$ acting respectively on P_3 and P_4: observe that by the action-reaction principle, these forces are equal but opposite to each other on each body, and that $\boldsymbol{f}_{P_4}^{P_3}$ and $\boldsymbol{f}_{P_3}^{P_4}$ share the same line of action.

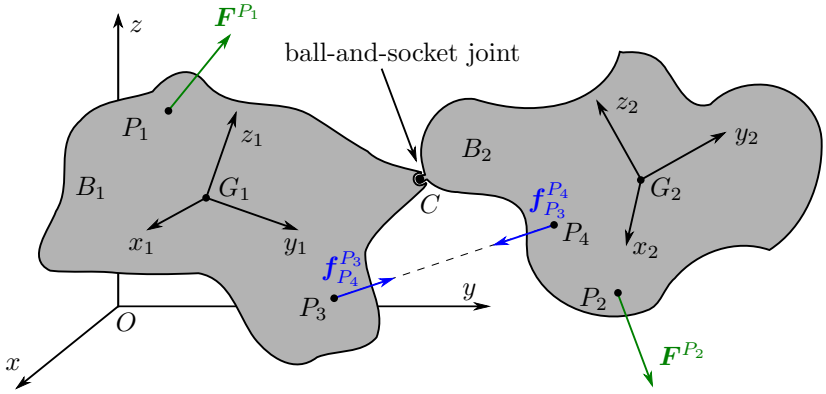

Figure 13.7: Example of a system of two rigid bodies, general view.

The number of effective degrees of freedom taking into account the constraint is $6 + 6 - 3 = 9$, as there are three scalar constraints applied to the velocity components at point C in each body. There are three unknowns due related to the three components of the reaction force $\boldsymbol{R}$ at C exist and no couples, as it is a ball-and-socket joint.

A useful tool to analyze the reaction forces and torques between the bodies at the constraint point is an *exploded view* of the system. In this type of diagram, each body or particle is drawn separately from the rest, and the connections to the other parts of the system are replaced by reaction forces and torques as shown in Fig. 13.7. Care must be put to define unambiguously each component of the reaction force and torque, and the direction in which each component will be considered as "positive" in the calculations. By virtue of the Third law of Newton, the reaction force and torque are equal in magnitude and opposite on each of the rigid bodies.

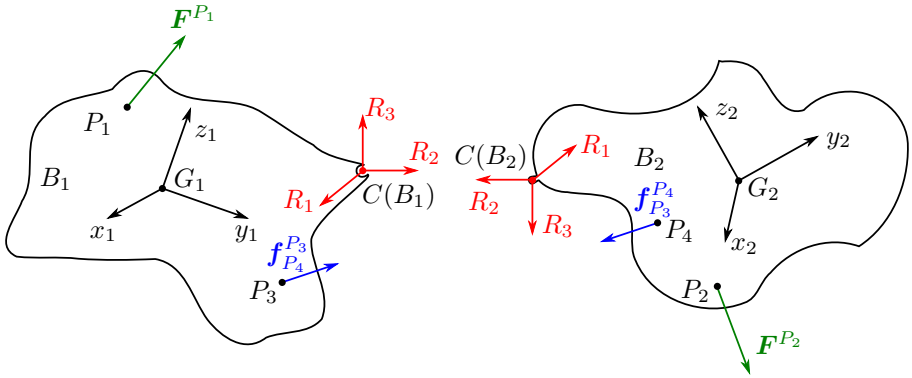

Figure 13.8: Exploded view of the two-body system of the previous example.

In the present example, we can label the reaction force at C on body B_1 as $\boldsymbol{R} = R_1\boldsymbol{i} + R_2\boldsymbol{j} + R_3\boldsymbol{k}$, where R_1, R_2, R_3 are three unknown values that must be computed as part of the solution. Then, for each of the two bodies in the example, the equations of motion are:

$$\text{Body } B_1\text{:} \quad \begin{cases} m^{B_1}\boldsymbol{a}_0^{G_1} = \boldsymbol{F}^{P_1} + \boldsymbol{R} + \boldsymbol{f}_{P_4}^{P_3}, \\ \left.\dfrac{\mathrm{d}\boldsymbol{H}_{G_1,0}^{B_1}}{\mathrm{d}t}\right|_0 = \boldsymbol{G_1P_1} \times \boldsymbol{F}^{P_1} + \boldsymbol{G_1C} \times \boldsymbol{R} + \boldsymbol{G_1P_3} \times \boldsymbol{f}_{P_4}^{P_3}, \end{cases} \tag{13.9}$$

$$\text{Body } B_2\text{:} \quad \begin{cases} m^{B_2}\boldsymbol{a}_0^{G_2} = \boldsymbol{F}^{P_2} - \boldsymbol{R} - \boldsymbol{f}_{P_4}^{P_3} \\ \left.\dfrac{\mathrm{d}\boldsymbol{H}_{G_2,0}^{B_2}}{\mathrm{d}t}\right|_0 = \boldsymbol{G_2P_2} \times \boldsymbol{F}^{P_2} - \boldsymbol{G_2C} \times \boldsymbol{R} - \boldsymbol{G_2P_4} \times \boldsymbol{f}_{P_4}^{P_3}, \end{cases} \tag{13.10}$$

where we have used $\boldsymbol{f}_{P_4}^{P_3} = -\boldsymbol{f}_{P_3}^{P_4}$.

Calling G the center of mass of the whole system $S = B_1 + B_2$, which satisfies

$$m^S\boldsymbol{r}_0^G = m^{B_1}\boldsymbol{r}_0^{G_1} + m^{B_2}\boldsymbol{r}_0^{G_2}, \tag{13.11}$$

with $m^S = m^{B_1} + m^{B_2}$, the equations of the resultant of forces and moment of forces for the whole system are

$$\text{Whole system:} \quad \begin{cases} m^S\boldsymbol{a}_0^G = \boldsymbol{F}^{P_1} + \boldsymbol{F}^{P_2}, \\ \left.\dfrac{\mathrm{d}\boldsymbol{H}_{G,0}^S}{\mathrm{d}t}\right|_0 = \boldsymbol{GP_1} \times \boldsymbol{F}^{P_1} + \boldsymbol{GP_2} \times \boldsymbol{F}^{P_2}, \end{cases} \tag{13.12}$$

where it should be noted that $\boldsymbol{H}_{G,0}^S = \boldsymbol{H}_{G,0}^{B_1} + \boldsymbol{H}_{G,0}^{B_2}$, i.e., the sum of the angular momenta of the two bodies about G, *not* about G_1 and G_2. It is easy to check that these two equations are linear combinations of Eqs. (13.9) and (13.10).

14 The Newtonian mechanics approach II

The velocity and acceleration fields of a rigid body are in fact the result of a strong kinematic constraint among the particles that integrate the body, namely, that the distance between any two of them must remain constant in time. After having introduced (*i*) this rigid body constraint and the identification of a rigid body with a body-fixed reference frame, (*ii*) the definition of total mass, linear momentum, angular momentum, and kinetic energy, together with the discussion of the inertia properties of the body, (*iii*) the equivalence of systems of forces and couples on a rigid body and the resultant force and moment of a system and (*iv*) the equations of motion of a rigid body, and in general, systems of particles and rigid bodies, we have all the ingredients and tools to pose and solve any problem in mechanics involving particles and rigid bodies.

In this chapter we revisit the scheme presented in Chapter 7 to complement it with the required changes to take into account rigid bodies. The resulting systematic approach to solve a system of particles and rigid bodies can be termed the Newtonian mechanics approach.

14.1 Preliminaries

The complexity of the problem being higher than before, it is now more necessary than ever to fully understand the problem. Do not attempt to start solving anything before first having read (and fully understood!) the totality of the problem and what is expected from you. Afterward, stop and ask yourself whether you know how to solve each of the questions before rushing ahead and begin writing.

One key element that can increase your chances of success in solving the problem is, once again, drawing sketches. As a general piece of advice, try at least to draw the whole system in the most generic configuration possible. But do not restrict yourself to just one sketch! Additional drawings with details of rotations, the initial configuration of the system, and exploded views of its parts are also very useful to visualize what is going on in the motion.

In the same spirit as in Chapter 7, you will need to ask yourself several key questions about the problem before you actually start solving it. They relate to the number of **degrees of freedom**, **constraints**, **coordinates** and **reference frames** to use:

1. How many rigid bodies and particles conform my system? What would the total number of degrees of freedom be if there were no constraints to their motion?

2. Constraints. Are there any? If so, how do they affect the motion of the components of the system? You need to translate those conditions into mathematical expressions that you can use. Try to visualize the type of motions allowed by the constraints.

3. What is the effective number of degrees of freedom of the system taking into account the constraints?

4. Select the set of independent coordinates that you are going to use to express the state of the system (i.e., its configuration or position). The number of independent coordinates must match the number of effective degrees of freedom, plus the number of non-integrable constraints, if any. The additional coordinates are redundant but cannot be fully eliminated, as the non-integrable constraints relate only their derivatives, but not the coordinates themselves.

   Even if all constraints are integrable, you may find it useful to define additional constraints with the condition that later on you explicitly state their dependency on the others (e.g. in problems with rolling without slipping constraints).

5. Define any reference frames and vector bases that you think will facilitate solving the problem. In particular, you should define a body-fixed reference frame for each rigid body in your system. Do not forget to explicitly define any new vectors you are going to use as a function of those that you already have.

After answering these questions you should also develop the general lines of how to tackle the problem. Are you going to need the equations of motion? In which reference frame do you want to write them down? Are you going to use the equation of conservation of mechanical energy (if applicable)? etc.

14.2 Kinematics and inertia properties

In the common case that we want to write the equations of motion, a necessary previous step is to write the relevant position vectors and linear and angular velocities and accelerations in terms of only the chosen coordinates, their derivatives, and time. It may be convenient to define the rotation of the body as the composition of three consecutive simple rotations (i.e., Euler angles, if we have chosen them as some of our coordinates).

For each rigid body in our system, we will likely need the position, velocity and acceleration of the center of mass in order to use the equation of forces in the next step. We will also need the total mass of the body. The expression of the velocity field and acceleration field of a rigid body may be useful at this step, specially if the velocity of some points of the body is already known (e.g. rolling without slip constraint).

In preparation for writing the equation of moments, we will need the angular velocity, the angular acceleration, and the tensor of inertia about a body point. This point must be chosen for each rigid body as the one about which angular momentum and moments of forces will be computed. This is typically the center of mass of the body or a fixed point, if any. It may be convenient to (re-)define the body-fixed reference frame of previous task with the origin located at this point. Often, the tensor of inertia can be computed from the tensor of inertia of the parts of the rigid body, and then applying Steiner's theorem.

14.3 Dynamics

In this step we must obtain the equations for the temporal evolution of our system configuration (also called "state"). Assuming we have already eliminated the derivatives of any redundant coordinates that we might have temporarily introduced in the previous steps, we will need as many equations as effective degrees of freedom (i.e., independent coordinates), plus additional equations for any extra unknown we may have introduced (e.g. components of reaction forces and torques).

Before we approach this, however, we need to address the following points regarding the forces and moments on the system:

1. Which **known** applied forces exist and what are their point of application? Express each load in terms of the chosen coordinates.

2. Do the constraints introduce any unknown **reaction** forces or torques? Identify which. Each component will add an unknown to the problem. Remember that the Third Law of Newton establishes that each force or torque on one body has an equal and opposite force or torque somewhere else in the universe, and that it is worth distinguishing between internal and external loads to the material system under analysis. A exploded view of the system may come in handy to complete this task.

3. If you plan on using a non-inertial reference frame to write down the equations of motion, then forces and moments of **inertia** exist. Note however that in this course we have not explored this way in the case of rigid bodies.

4. Compute the resultant force, and compute the resultant moment about the chosen point (in the case of a rigid body, this is typically the center of mass or a fixed point of the rigid body, if any).

As an aid in the determination of the reactions, recall that they are the physical mechanism that enforces the kinematic constraints. Thus, if e.g. a support prevents a point in the body from moving in a certain direction, then at that point there must be a reaction force exerted on the body along the forbidden direction. Similarly, a kinematic constraint on the rotation about a line is imposed by a reaction torque exerted about that line. As mentioned above, the reactions are not known in advance—they are extra unknowns that will appear in some or all of the equations of motion.

Once the resultant force and moment about the chosen point on each body are written down, we can proceed to write the equations of motion for each element in our system. Each rigid body yield six scalar equations (after projecting the force and moment equations in any three independent directions each); each particle produces three scalar equations. The total number of equations and

unknowns (coordinates plus reaction components) must match to declare the system of equations *closed*.

Sometimes, it may be preferable to write down the equations of force and moment for the whole system, or a subset of it, rather than each of its constituents. This removes from the equations any forces and couples that are internal to the system or subset. Remember, however, that this does not provide new independent equations with respect to the equations of each of the parts.

Usually it is easy to find a linear combination of the resulting equations that does not involve any unknown reactions. The result is a set of differential equations for the coordinates of our system, whose solution gives the motion of the system without regard to the reaction forces and torques. In order to solve these equations, one needs to impose **initial conditions** to each coordinate and its first derivative. These can be obtained from the information about the initial position and velocity of the rigid body. Analytic solutions of such equations are typically not available (only in a limited number of cases, such as when the equations are linear in the dependent variables). In most cases the solution is obtained using numerical techniques.

Finally, once the evolution of the coordinate variables is known, solving rest of independent equations would give the reaction components.

In some particular cases, the equations of motion for the coordinates of the system are directly integrable, resulting in conservation laws. A particular case of this is the conservation of some component of the linear or angular momentum, or the conservation of the mechanical energy. If the conditions for conservation of the mechanical energy apply, it is also possible to forgo the integration of one of the differential equations and substitute it for the conservation of the mechanical energy: as it is known, this equation is dependent upon the equations of motion, i.e. it does not add additional information.

14.4 Solving the problem

Most problems diverge at this point in the questions asked. Sometimes it is required to integrate the equations of motion or reduce them to first integrals; other times a qualitative analysis must be undertaken. For example, regions of existence of solution according to the available energy.

A common question pertains the existence of equilibrium configurations and their stability. As explained in §12.3, equilibrium states can be approached by setting all time derivatives equal to zero in the equations of motion. The study of stability about an equilibrium configuration requires analyzing small deviations from the equilibrium points. This can be done linearizing the equations of motion about that point. In the case of a single effective degree of freedom and a conservative problem, the second derivative of the potential function can also be used to discuss stability.

Linearizing the problem is sometimes asked as a question directly, in order to study small oscillations about a stable equilibrium point.

Part III
Atmospheric flight

15 Elements of atmospheric flight

In previous chapters we have presented the formulation and solution of classical mechanics problems. This is a fundamental skill for any aerospace engineer; one proof of this is that the subject of flight mechanics is built on top of classical mechanics.

Flight mechanics is a rather extensive branch of aerospace engineering, covering the mechanics of atmospheric flight vehicles (aircraft) and space vehicles (spacecraft). In turn, aircraft can be subdivided into heavier-than-air (aerodynes) and lighter-than-air (aerostats) crafts. Figure 15.1 shows that we can model these vehicles as point particles or rigid bodies, and that a number of aspects about them can be studied using these models.

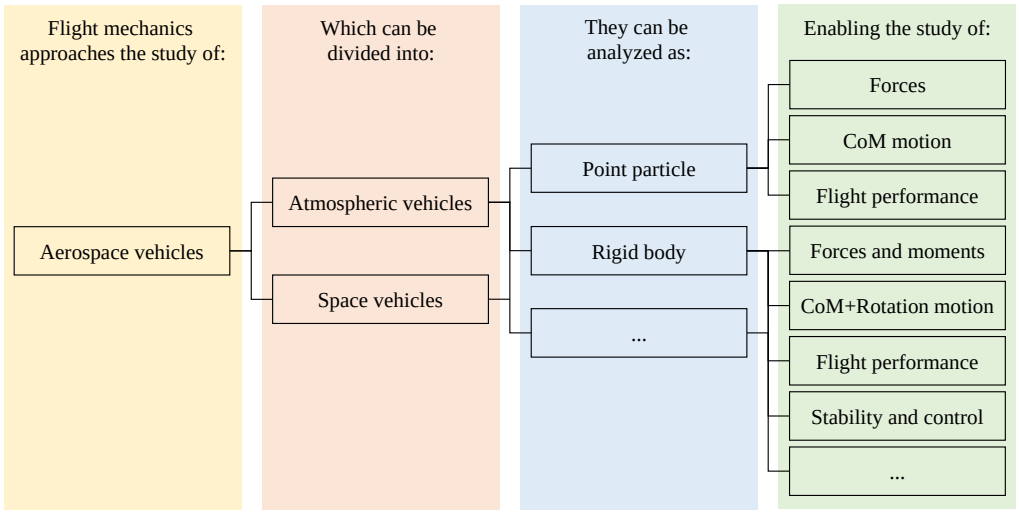

Figure 15.1: Diagram of flight mechanics and its different areas of study. Airplanes, which fall into the category of heavier-than-air atmospheric flight vehicles, can be studied as point particles or as rigid bodies in first approximation. These are the models introduced in this course.

In this chapter we introduce the basic definitions and concepts for the study of heavier-than-air aircraft motion in the next chapters (in particular, airplanes).

15.1 Airplane basic definitions

A modern airplane is composed of a *fuselage*, *wing*, *propulsive plant*, and *tail*. The aircraft has a longitudinal vertical plane of symmetry. The wing can be subdivided into left and right (half)-wings; the propulsive plant may consist of turbojets or turbofans contained in *nacelles* that are located under the wing or over the fuselage; several types of tails exist, but generally it is composed of a *horizontal* and a *vertical stabilizers*. Fig. 15.2 illustrates these concepts.

It is customary to define a right-handed body-fixed reference frame $S_B : \{G; B_B\}$, with origin on the aircraft center of mass G and $B_B : \{\boldsymbol{i}_B, \boldsymbol{j}_B, \boldsymbol{k}_B\}$, where $\boldsymbol{i}_B$ points along the forward airplane direction, $\boldsymbol{j}_B$ perpendicular to the plane of symmetry and toward the right side half-wing, and $\boldsymbol{k}_B$ downward. The orientation of the aircraft axes can be expressed by means of three Euler angles. These are usually chosen to be of the 3-2-1 type, and are called, in that order, *yaw*, *pitch* and *roll*.

Several control surfaces exist that allow the pilot to control the motion of the aircraft. Some common ones are:

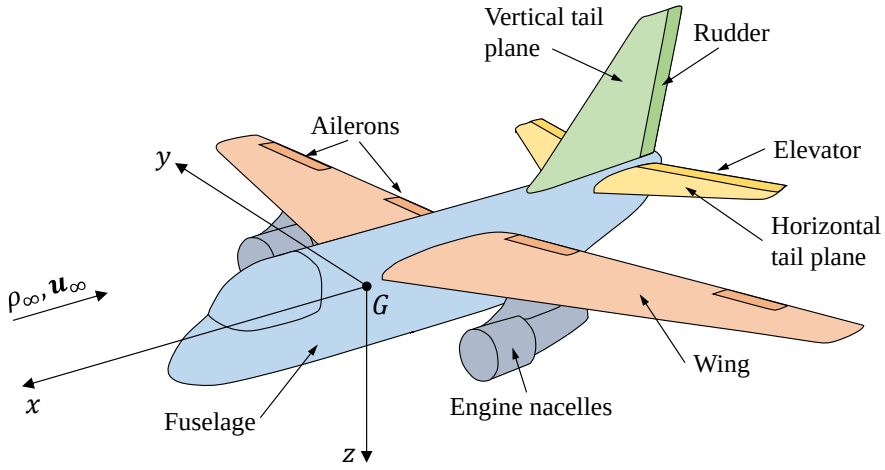

Figure 15.2: Sketch of the different parts of a conventional airplane and some of its control surfaces. The body-fixed reference frame used in flight mechanics analyses is also shown.

Ailerons: located on the trailing edge of the wing. When the left and right ailerons are deflected in opposite directions, they enable the control of aircraft *roll*. The ailerons on the wing tips are used at low speeds; the ailerons near the wing root are used at higher speeds.

Elevator: located on the trailing edge of the horizontal stabilizer. It is deflected symmetrically to control *pitch*.

Rudder: located on the trailing edge of the vertical stabilizer. It allows controlling the *yawing* motion of the aircraft.

Flaps: located on the trailing edge of the wing. These surfaces can be extended out of the wing to increase the lifting area of the aircraft and change their effective aerodynamic shape. This is typically necessary at low flight speed conditions, such as take-off and landing, to prevent stall.

Spoilers: located on the upper surface of the wing. They are opened to destroy lift upon landing and to increase drag.

Trim surfaces: they are located at the trailing edge of the rudder and the elevator. They are adjusted to control the aerodynamic moments and remove the required control force exerted by the pilot in certain flight conditions

Airplanes in a Canard configuration have a different layout of control surfaces. The *canards* are ahead of the wing and control pitch. The *elevons* are control surfaces on the trailing edge of the wing and provide pitch and roll control, combining the roles of elevators and ailerons. Finally, a rudder is used to control yaw.

The air conditions far from the aircraft are denoted with subindex ∞. For example, the air density of the unperturbed air stream is ρ_∞. From the body-fixed reference frame, air is seen to approach the aircraft with a velocity $\boldsymbol{u}_\infty$, which is equal and opposite to the flight velocity with respect to the atmosphere.

15.2 Forces on an aircraft

The system of forces acting on an aircraft flying in the atmosphere is a complex one. However, we can classify these forces into three different groups: *weight*, propulsive force or *thrust*, and *aerodynamic forces*.

15.2.1 Weight and thrust

Weight is a distributed body force $\mathrm{d}\boldsymbol{W} = \boldsymbol{g}\mathrm{d}m$, acting on each mass element $\mathrm{d}m$ of the aircraft. The total weight force, resulting from integrating over the body, is merely $\boldsymbol{W} = M\boldsymbol{g}$, where $\boldsymbol{g}$ is the gravitational acceleration vector pointing downward. The application line of the total weight force passes by the *center of gravity*, which in a uniform gravitational field coincides exactly with the center of mass.

If the aircraft is equipped with a propulsive plant (turbojets, turbofans, propellers, etc) then it can also experience a thrust force $\boldsymbol{T}$. The thrust force of airbreathing engines is a function, among others, of the air density and flight velocity: $\boldsymbol{T} = \boldsymbol{T}(\rho_\infty, u_\infty, \ldots)$.

The line of application of the thrust force of each engine is the axis of symmetry of the engine, but it is also possible in some cases to deflect thrust up or down in a small angle δ_T (this capability is known as *thrust vector control*). The total thrust force, resulting from adding together the contribution of each engine, is typically contained in the plane of symmetry of the aircraft. In the absence of more information about its direction, it is commonly assumed that $\boldsymbol{T}$ is applied parallel to $\boldsymbol{u}_\infty$ in the forward direction.

15.2.2 Aerodynamic forces

The aerodynamic forces result from the pressure differences and the air friction over the wet surfaces of the aircraft. The resultant of these forces is the *total aerodynamic force*, which is customarily decomposed into two contributions: *lift*, $\boldsymbol{L}$, and *drag*, $\boldsymbol{D}$. By definition, the lift force is the component of the aerodynamic force *perpendicular* to the incident air velocity: $\boldsymbol{L} \perp \boldsymbol{u}_\infty$, and the drag force is *parallel* to it: $\boldsymbol{D} \parallel \boldsymbol{u}_\infty$. Here, we shall assume that the lift force $\boldsymbol{L}$ is always contained in the plane of symmetry of the aircraft and therefore does not have any lateral component.

Experimental and theoretical evidence shows that, at least at subsonic velocities, the aerodynamic forces are proportional to the *dynamic pressure* of the air stream, $\rho_\infty u_\infty^2/2$. Furthermore, the forces scale with the characteristic area of the aircraft, typically chosen as the wing area S. In summary we may write:

$$L = \frac{1}{2}\rho_\infty u_\infty^2 S c_L \tag{15.1}$$

$$D = \frac{1}{2}\rho_\infty u_\infty^2 S c_D \tag{15.2}$$

where c_L and c_D are, respectively, the *lift* and *drag coefficients* of the aircraft computed for area S. These coefficients account for the contribution of all components in the aircraft to the generation of lift and drag: wing, fuselage, tail, landing gear, etc.

Lift is dominantly produced by the wing, and it is a function of the *angle of attack* α between the incident air stream and the plane of the wing. To a good degree of approximation, the lift coefficient is linear on α,

$$c_L = c_{L0} + c_{L\alpha}\alpha. \tag{15.3}$$

At very large angles of attack, however, the wing can stall and this equation fails to model the real behavior of the lift coefficient, which may drop suddenly beyond that point.

On the other hand, the drag coefficient can be modeled as a *parasitic drag* c_{D0} at zero lift plus an *induced drag* that grows quadratically with lift:

$$c_D = c_{D0} + K c_L^2, \tag{15.4}$$

where K is the *induced drag parameter*. Eq. (15.4) is known as the *drag polar* of the aircraft. There are more advanced forms of this equation that take into account asymmetries for positive and negative values of α and additional dependencies for the drag coefficient.

The intersection of the line of application of the resultant lift and drag forces and the reference plane of the wing is called the *center of pressure*. The aerodynamic force system does not create any moment of forces about this point. The location of the center of pressure depends on the angle of attack α, making it unappealing for the study of aerodynamic forces in general flight configurations. Instead, a point termed *aerodynamic center* is used, about which the aerodynamic moment is nonzero in general but it is *constant* with the angle of attack. If the wing has a chord c, this point is typically located at $c/4$ from the leading edge of the wing.

15.2.3 Aerodynamic efficiency

The *aerodynamic efficiency* E is one of the key performance parameters of an aircraft, which affects for instance the required fuel consumption to maintain a given flight condition. It is defined as the lift-to-drag ratio of the aircraft:

$$E = \frac{L}{D} = \frac{c_L}{c_D} = \frac{c_L}{c_{D0} + K c_L^2}. \tag{15.5}$$

The aerodynamic efficiency has a well defined maximum value $E_{\max}$, which can be found by differentiation:

$$\frac{\mathrm{d}E}{\mathrm{d}c_L} = \frac{c_{D0} - K c_L^2}{\left(c_{D0} + K c_L^2\right)^2} = 0 \tag{15.6}$$

$$c_{L,\mathrm{opt}} = \sqrt{\frac{c_{D0}}{K}}; \quad E_{\max} = \frac{1}{2\sqrt{c_{D0} K}} \tag{15.7}$$

Clearly, the lower c_{D0} and K are, the higher the (maximum) aerodynamic efficiency. Aircraft designers go to great efforts to reduce the parasitic and induced drag coefficients in order to make the aircraft more aerodynamically efficient.

16 Motion of an aircraft as a point particle

In this chapter we apply our knowledge of classical mechanics to study several basic motions of an airplane as a point particle. This enables the characterization of the motion of the center of mass G of the airplane without regard to its orientation and rotation. For many purposes, this simple model suffices to obtain useful results. In particular, this approach enables the study of the *flight performances* of the aircraft as a point particle.

The airplane is assumed to be a rigid body with a plane of symmetry, flying in an atmosphere at rest. We shall perform the analysis of motion in the body-fixed reference frame $Gxyz$ introduced in the previous chapter (see Fig. 15.2). Observe that in this reference frame the air far from the aircraft is moving with a velocity $\boldsymbol{u}_\infty = -u_\infty \boldsymbol{i}$ with respect to the aircraft, which is equal and opposite to the flight velocity of the aircraft with respect to the atmosphere.

In general, this body-fixed reference frame will be non-inertial, so that inertial forces $\boldsymbol{F}_I$ have to be included in the analysis. The equation of forces therefore reads:

$$\boldsymbol{T} + \boldsymbol{L} + \boldsymbol{D} + \boldsymbol{W} + \boldsymbol{F}_I = 0. \tag{16.1}$$

Last but not least, we shall only consider airplane motions for which the following conditions apply:

1. Only short periods of the motion of the aircraft are considered so that we can neglect the small variation of mass due to fuel consumption.
2. The lift force is contained in the plane of symmetry of the aircraft.
3. The thrust force $\boldsymbol{T}$ is along the Gx_B axis.
4. The control surfaces of the aircraft have been positioned and trimmed so that the resultant moment of forces at G is equal to zero.

It is this last point, in particular, which enables us to ignore here the rotational motion of the aircraft and focus on the motion of its center of mass G as a point particle by solving Eq. (16.1). The motion of the aircraft as a full rigid body must study as well its rotational motion, which is left for future courses.

16.1 Uniform level flight

The simplest flight condition is that of a cruising aircraft flying at a constant altitude and velocity. The sketch of forces on the center of mass for this situation is shown in Fig. 16.1.

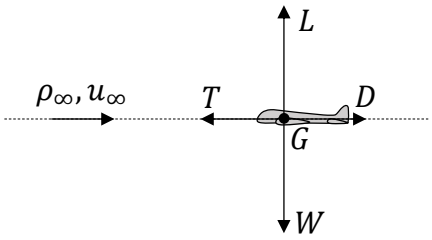

Figure 16.1: Force balance in the uniform level flight conditions.

In this case the flight path angle γ is zero and there are no forces of inertia. The equations of motion projected along $\boldsymbol{i}$ and $\boldsymbol{j}$ then read

$$W - L = 0, \tag{16.2}$$
$$T - D = 0, \tag{16.3}$$

which can be written as

$$W = \frac{1}{2}\rho_\infty u_\infty^2 S c_L = \frac{1}{2}\rho_\infty u_\infty^2 S(c_{L0} + c_{L\alpha}\alpha), \tag{16.4}$$

$$T = \frac{1}{2}\rho_\infty u_\infty^2 S(c_{D0} + K c_L^2). \tag{16.5}$$

For a given aircraft weight W, a change in the flight speed u_∞ means that a change in the lift coefficient c_L, and thus on the angle of attack α, is required to maintain this flight condition: a higher u_∞ requires a lower c_L and therefore a lower α. Observe that changing u_∞ and c_L also affects the magnitude of the required thrust, T.

Combining these two equations we can obtain the expression of the required thrust magnitude as a function of the characteristics of the aircraft and the flight speed:

$$T = \frac{1}{2}\rho_\infty u_\infty^2 S c_{D0} + K\frac{2W^2}{\rho_\infty u_\infty^2 S}. \tag{16.6}$$

This expression reveals that thrust has two terms: one to compensate the parasitic drag (first term on the right hand side) and another to compensate the induced drag (second term). Each of these terms has a very distinct behavior with flight speed: parasitic drag increases with u_∞^2 whereas induced drag decreases as u_∞^{-2} (after factoring in the required variation of c_L to maintain the flight condition as explained above).

The required angle of attack and thrust force are plotted on Fig. 16.2 as functions of the flight speed u_∞. Several conclusions can be drawn from these results. Firstly, if an airplane is flying at a given speed, e.g. operating point A in the diagram, and the pilot wants to increase the speed to an operating point B, it is necessary to simultaneously change the thrust and angle of attack accordingly. While the required angle of attack always decreases with increasing flight speed, the required thrust is non-monotonic, as shown in Fig. 16.2.

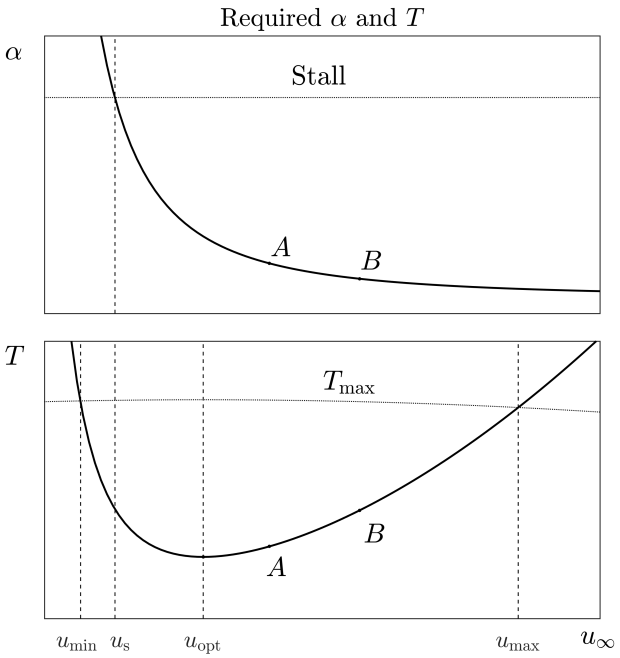

Figure 16.2: Required angle of attack α and thrust force T as a function of flight speed u_∞ in uniform level flight.

Secondly, and as follows from the above observations, there is a flight speed u_{opt} that gives the minimum required thrust. From Eqs. (16.2) and (16.3) we may write:

$$T = \frac{D}{L}W = \frac{W}{E}. \tag{16.7}$$

Then, the minimum required thrust level is seen to coincide with the point of maximum aerodynamic efficiency, $E = E_{\max}$.

Thirdly, a given propulsive plant can only produce thrust up to a maximum value, $T_{\max}$, which is itself a function of the flight speed u_∞ and the air density at infinity, ρ_∞. From the diagram in Fig. 16.2, it is apparent that there is a range of flight velocities on which the aircraft can fly for this reason. Outside of the range defined by the minimum and maximum flight velocities, $u_{\min}$ and $u_{\max}$ for a given ρ_∞, the propulsive system is incapable of generating sufficient thrust to maintain the level flight condition. If, for any reason, the aircraft is momentarily outside of this valid flight range, it will start to descend.

Fourthly, since the maximum available thrust $T_{\max}$ from airbreathing engines decreases with decreasing air density ρ_∞, it decreases with increasing flight altitude h. As we increase h, the range of possible flight velocities $[u_{\min}, u_{\max}]$ shrinks, and eventually becomes a single point. That limit altitude at which the aircraft can cruise is known as the *ceiling* of the aircraft.

Lastly, there is another limitation to flight speed, which is the one associated to the maximum α before stalling the aircraft. This gives another lower bound to the flight velocity of the aircraft, the stall speed u_s. Depending on the aircraft, propulsive plant, and flight altitude, the effective lower bound to the flight velocity may be u_s or $u_{\min}$.

16.2 Uniform ascend or descend

We consider now an aircraft climbing or descending with constant velocity in a rectilinear trajectory with a small flight path angle $\gamma \ll 1$ as shown in Fig. 16.3. Once again, there are no inertial forces in this case, and the equations of motion read:

$$W\cos\gamma - L = 0, \tag{16.8}$$

$$T - D - W\sin\gamma = 0. \tag{16.9}$$

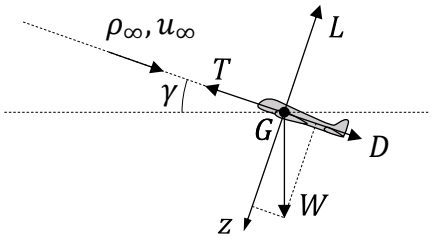

Figure 16.3: Force balance in the ascend flight condition. The sketch of forces in the descend is analogous.

Linearizing the equations under the condition $\gamma \ll 1$ these equations become

$$W - L = 0, \tag{16.10}$$

$$T - D - W\gamma = 0. \tag{16.11}$$

Equation (16.10) indicates that for a given aircraft weight W, the lift coefficient is essentially the same as the one in uniform level flight (and in consequence, the drag coefficient too). Referring to Eq. (16.3) we may therefore write the required thrust in this new flight condition as $T = T_{\text{ulf}} + W\gamma$, where $T_{\text{ulf}} = D$ is the thrust magnitude required to maintain uniform level flight. In other words, the *excess* (or defect) of thrust force compared to the straight level flight condition gives us the flight path angle:

$$\gamma = \frac{T - T_{\text{ulf}}}{W}. \tag{16.12}$$

This expression shows that it is possible to control the flight path angle of the aircraft trajectory by changing the thrust force only (i.e., without changing the angle of attack, or the lift coefficient). Climbing demands a larger thrust force than uniform level flight, whereas descending lowers the required thrust.

16.3 Gliding

Uniform, rectilinear gliding at constant velocity is a flight condition of an aircraft without thrust, and it can be considered a particular case of the uniform descend described above. Figure 16.4 shows this flight condition. Defining the gliding angle as the opposite of the flight path angle, $\gamma_g = -\gamma$, the equations of motion are

$$L = W\cos\gamma_g, \tag{16.13}$$

$$D = W\sin\gamma_g. \tag{16.14}$$

The ratio of these two equations gives the gliding angle,

$$\tan\gamma_g = \frac{D}{L} = \frac{1}{E} \quad \Rightarrow \quad \gamma_g \simeq \frac{1}{E}. \tag{16.15}$$

Therefore, the descent or gliding angle γ_g is seen to depend solely on the aerodynamic efficiency. The minimum gliding angle occurs at the flight speed for which the efficiency is optimal, $\gamma_{g,\min} = 1/E_{\max}$.

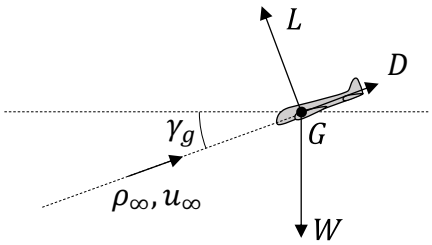

Figure 16.4: Force balance sketch during gliding.

16.4 Looping

When the aircraft motion describes a vertical circle of radius R at constant speed as in Fig. 16.5 we refer to it by the term *looping*. If the circle is small, we may neglect the variation of air density along the trajectory. In this condition the center of mass of the aircraft experiences a centripetal acceleration as seen from inertial space. In the body-fixed reference frame, a corresponding inertia force exists, and the equations of motion now read:

$$L = W\cos\gamma + M\frac{u_\infty^2}{R}, \tag{16.16}$$

$$D = T - W\sin\gamma, \tag{16.17}$$

where γ now goes from 0 to 2π as the aircraft completes the loop.

We define the wing load factor n as the ratio between lift and weight, $n = L/W$. It is a measurement of the mechanical loads on the structure of the aircraft, and a maximum $n_{\max}$ exists for which a given aircraft model is certified, which should not be overpassed during any maneuver. Using Eq. (16.16), it is possible to obtain the load factor as a function of γ:

$$n(\gamma) = \cos\gamma + \frac{u_\infty^2}{gR}. \tag{16.18}$$

Clearly, it has a maximum at $\gamma = 0$ (the lowest point in the loop), where $n = u_\infty^2/(gR) + 1$, and a minimum at $\gamma = \pi$ (highest point in the loop), where $n = u_\infty^2/(gR) - 1$ (a value that can be negative). For the looping maneuver to be carried out safely, it is required that $u_\infty^2/(gR) + 1 \leqslant n_{\max}$. It is apparent that increasing the looping velocity u_∞ or decreasing the loop radius R makes the load factor larger. Acrobatic and combat aircraft typically have a large $n_{\max}$, enabling them to perform more demanding maneuvers.

A similar procedure as the one used in §16.1 can be used here to identify the required thrust and lift coefficient at any point of the loop, and to detect whether the maneuver is possible for a $T_{\max}$ and $c_{L,\max}$.

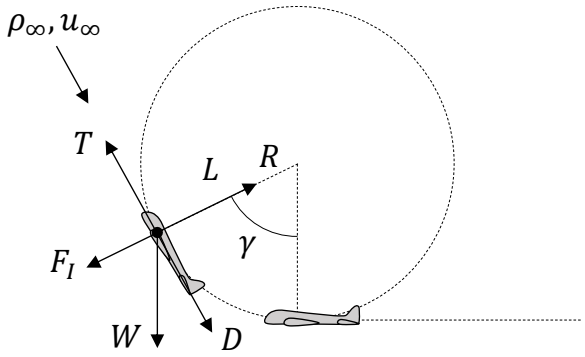

Figure 16.5: Force balance sketch during a looping maneuver or vertical turn.

16.5 Horizontal turns

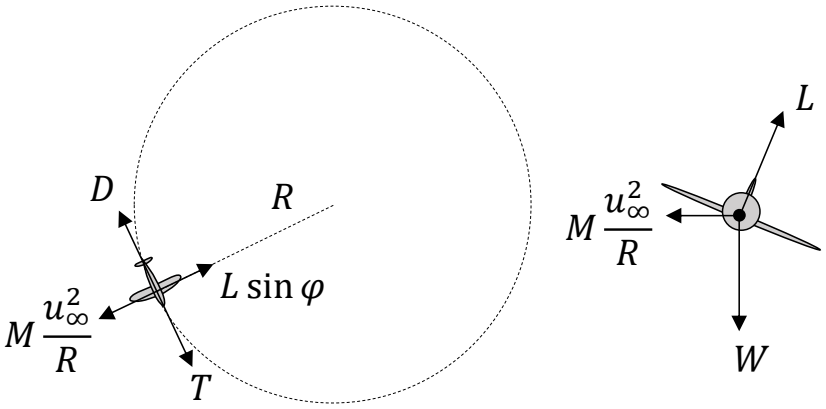

Figure 16.6: Force balance sketch during a coordinated horizontal turn.

Another common maneuver is the *coordinated horizontal turn*, where the aircraft motion describes a horizontal circle of radius R at constant speed. To do so, the pilot induces a roll angle φ so that the lift force has a component along the radial direction to produce the required centripetal acceleration. The diagram of forces of the maneuver is sketched in Fig. 16.6. Since the origin of the body-fixed reference frame is accelerated, the corresponding force of inertia has to be taken into account in the equation of motion of G, which projected along $B_B : \{\boldsymbol{i}_B, \boldsymbol{j}_B, \boldsymbol{k}_B\}$ reads:

$$T - D = 0, \tag{16.19}$$

$$L\sin\varphi - M\frac{u_\infty^2}{R} = 0, \tag{16.20}$$

$$W - L\cos\varphi = 0. \tag{16.21}$$

From Eq. (16.21) we obtain the load factor in this maneuver, $n = 1/\cos\varphi$. Comparing the required lift coefficient with that of uniform level flight, $c_{L,\text{ulf}}$ we note that $c_L = nc_{L,\text{ulf}}$, i.e., it is n times larger since a higher L force is now needed to compensate weight and create the centripetal acceleration for the turn. From Eq. (16.19), the required thrust force likewise increases, as the induced drag is therefore larger than in the uniform level flight conditions:

$$T = \frac{1}{2}\rho_\infty u_\infty^2 S c_{D0} + n^2 K \frac{2W^2}{\rho_\infty u_\infty^2 S}. \tag{16.22}$$

Finally, Eq. (16.20) can be used to find the radius of curvature as a function of the roll angle φ and the flight velocity,

$$R = \frac{u_\infty^2}{g \tan \varphi}. \tag{16.23}$$

The same considerations to the structural limit on n, the propulsive limit on T, and the aerodynamic limit on c_L as before apply here.

Another type of maneuver that sets the aircraft on a horizontal circle at constant speed is achieved by yawing the aircraft with a rudder command so that the incident stream comes at a slip angle β with respect to the plane of symmetry of the aircraft. In this case, the lift force stays on the vertical plane, and the centripetal acceleration is afforded completely by an increase of the thrust force. The equations of motion for this case are:

$$T \cos \beta - D = 0, \tag{16.24}$$

$$T \sin \beta - M \frac{u_\infty^2}{R} = 0, \tag{16.25}$$

$$W - L = 0. \tag{16.26}$$

While this turning maneuver is possible, the coordinated horizontal turn explained before is always preferred, since a large slip angle and a much higher value of the thrust force is required to perform the turn in this other way. In a typical aircraft, the lift force is much higher than the thrust force, and a much larger centripetal acceleration is achieved with lift by rolling the aircraft than with thrust by inducing a comparable slip angle. Therefore, this last type of turn is only used for slight course changes with a very large curvature radius and at low flight velocity.

17 Integral figures of merit

The expressions for uniform level flight that we have derived in the previous chapter together with the information about the fuel consumption of the propulsive plant can be used to determine the time that an aircraft can remain in the air without landing and the distance it can cover without refueling. This time is known as *endurance*, and the distance is termed the *range* of the aircraft. These integral performances are given by *Breguet's formulas*.

Clearly, to perform the analysis of this Chapter we have to consider the variation in mass of the aircraft with time as the fuel is consumed. The propulsive plant uses an amount of fuel per unit time $\dot{m}_f$, also known as *fuel mass flow rate*. After combustion, the flue gases are ejected out, which means that the mass of the aircraft changes in time as

$$\frac{\mathrm{d}M}{\mathrm{d}t} = -\dot{m}_f. \tag{17.1}$$

The specific fuel consumption SFC is defined as the ratio between $\dot{m}_f$ and T:

$$\mathrm{SFC} = \frac{\dot{m}_f}{T}, \tag{17.2}$$

and it can be considered a characteristic parameter of the propulsive plant which varies only slightly with the operating point of the engines.

17.1 Cruise endurance

Equation (17.1) provides the variation of aircraft mass with time. The endurance of the aircraft is the time that passes from the initial instant where the fuel tanks are full ($M = M_0 = M_d + M_f$, i.e. *dry mass* plus *fuel mass*) until these tanks are empty ($M = M_d$). Integrating:

$$t_e = \int_0^{t_e} \mathrm{d}t = -\int_{M_0}^{M_d} \frac{1}{T \cdot \mathrm{SFC}} \mathrm{d}M. \tag{17.3}$$

In cruise conditions, $T = W/E$ as explained in §16.1. Then,

$$t_e = -\int_{M_0}^{M_d} \frac{E}{g \cdot \mathrm{SFC}} \frac{\mathrm{d}M}{M}. \tag{17.4}$$

To be able to integrate this expression we need to know the variation with time of E and SFC. If we assume that the angle of attack remains constant and that SFC is not changed noticeably, then the endurance of the aircraft is given by

$$t_e = \left(\frac{1}{g}\right)\left(\frac{c_L}{c_D}\right)\left(\frac{1}{\mathrm{SFC}}\right)\left[\ln\left(\frac{M_0}{M_d}\right)\right]. \tag{17.5}$$

Clearly, to maximize the endurance, we want to fly at maximum aerodynamic efficiency $E_{\max} = (c_L/c_D)|_{\max}$ and keep the SFC as low as possible.

17.2 Cruise range

For an atmosphere at rest, the distance covered by an aircraft in a differential time interval is

$$\mathrm{d}s = u_\infty \mathrm{d}t. \tag{17.6}$$

In uniform level flight $u_\infty^2 = 2W/(\rho_\infty S c_L)$ and $T = W/E$. Using Eq. (17.1), assuming c_L and SFC remain constant, and integrating:

$$\begin{aligned} s &= -\int_{M_0}^{M_d} \frac{u_\infty}{T \cdot \mathrm{SFC}} \mathrm{d}M = -\int_{M_0}^{M_d} \sqrt{\frac{2}{g\rho_\infty S}} \frac{\sqrt{c_L}/c_D}{\mathrm{SFC}} \frac{\mathrm{d}M}{\sqrt{M}} \\ &= \left(\sqrt{\frac{8}{g\rho_\infty S}}\right)\left(\frac{\sqrt{c_L}}{c_D}\right)\left(\frac{1}{\mathrm{SFC}}\right)\left(\sqrt{M_0} - \sqrt{M_d}\right). \end{aligned} \tag{17.7}$$

As it can be observed, the maximum range occurs at the maximum of $\sqrt{c_L}/c_D$, not at the maximum aerodynamic efficiency. Clearly, a more efficient propulsive plant (i.e., a lower SFC) results in a longer range.