V5A2 – RIGID ANALYTIC GEOMETRY
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WERN JUIN GABRIEL ONG

Preliminaries

These notes roughly correspond to the course **V5A2 - Rigid Analytic Geometry** taught by Prof. Jens Franke at the Universität Bonn in the Summer 2025 semester. These notes are LaTeX-ed after the fact with significant alteration and are subject to misinterpretation and mistranscription. Use with caution. Any errors are undoubtedly my own and any virtues that could be ascribed to these notes ought be attributed to the instructor and not the typist. These notes assume knowledge of the course on the same topic held in the Winter 2024-25 semester.

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Contents

1. Lecture 1 – 17th April 2025

We fix the following notation.

Notation 1.1. (i) K is a field complete with respect to a non-Archimedean norm.
(ii) We denote the Tate algebra

$$\mathbb{T}_n = \left\{ f \in \sum_{\alpha \in \mathbb{N}^n} f_\alpha X^\alpha : \forall \varepsilon > 0, |\{\alpha : |f_\alpha|_K \geq \varepsilon\}| < \infty \right\} \subseteq K[[X_1, \dots, X_n]]$$

the subring of convergent power series, with norm $\|f|\mathbb{T}_n\| = \max_{\alpha \in \mathbb{N}^n} |f_\alpha|$.

Remark 1.2. (i) The norm $|\cdot|_K$ extends uniquely to any algebraic extension of K.
(ii) The Tate algebra $\mathbb{T}_n$ is Noetherian, hence all ideals are closed.

Affinoid algebras are quotients of Tate algebras.

Definition 1.3 (Affiniod Algebra)**.** A K-algebra A is an affinoid K-algebra if it is of the form $\mathbb{T}_n/I$.

There is an induced norm on the Tate algebra known as the residual norm.

Definition 1.4 (Residual Norm)**.** Let A be an affinoid K-algebra. The residue norm of $a \in A$ is

$$\|a\| = \inf\{\|f|\mathbb{T}_n\| : \overline{f} = a\}.$$

Remark 1.5. Definition 1.4 is independent of the choice of representative.

As in algebraic geometry, affinoid algebras give rise to ringed spaces via the Tate spectrum. We discuss the construction by first defining the space, and the sheaf of rings on it.

Definition 1.6 (Tate Spectrum – Set)**.** Let A be an affinoid K-algebra. The set underlying the Tate spectrum $\mathrm{Sp}(A)$ is $\mathrm{mSpec}(A)$.

Remark 1.7. The Tate spectrum is endowed with the property that $[\kappa(x) : K] < \infty$, where $\kappa(x) = A/\mathfrak{m}_x$ is a field as the ideal $\mathfrak{m}_x$ corresponding to x is maximal.

The topology on the set is defined by rational sieves.

Definition 1.8 (Rational Open Set)**.** Let $\langle f_0, \dots, f_n \rangle_A = A$. The rational open associated to the generators $R_A(f_0|f_1, \dots, f_n)$ is given by

$$R_A(f_0|f_1, \dots, f_n) = \{x \in \mathrm{Sp}(A) : |f_0(x)| < |f_i(x)|, 1 \leq i \leq n\}.$$

Remark 1.9. Rational open subsets are preserved under finite intersection. For $\langle f_0, \dots, f_n \rangle_A, \langle g_0, \dots, g_m \rangle_A$ generators of A, the intersection

$$R_A(f_0|f_1, \dots, f_n) \cap R_A(g_0|g_1, \dots, g_m) = R_A(f_0 g_0|f_i g_j, 1 \leq i \leq n, 1 \leq j \leq m).$$

These rational open sets form the basis for the topology on the Tate spectrum $\mathrm{Sp}(A)$.

Definition 1.10 (Tate Spectrum – Topology)**.** Let A be an affinoid K-algebra. The set underlying the Tate spectrum $\mathrm{Sp}(A)$ has a topology with basis consisting of the rational open sets $R_A(f_0|f_1,\dots,f_n)$ and with Grothendieck topology obtained by enforcing quasicompactness of the rational open sets.

In some simple cases, the underlying space of the Tate spectrum admits a description.

Example 1.11. Let $K=\overline{K}$. $\mathrm{Sp}(\mathbb{T}_n)=(K^\circ)^n$, where K° is the subring of powerbounded elements of K. Each point $x\in\mathrm{Sp}(A)$ is taken to $(\xi_i)_{i=1}^n$ where ξ_i is the image of X_i in $K\cong\kappa(x)$ and an n-tuple of powerbounded elements of K is taken to the ideal of $\mathbb{T}_n$ consisting of functoins vanishing at that tuple. In this case, the basis for the ordinary topology on the Tate spectrum is identified with non-Archimedean balls $d(\xi,\nu)=\max_{1\leq i\leq n}|\xi_i-\nu_i|$.

We now want to define the structure sheaf on $\mathrm{Sp}(A)$ which will be valued in the category affinoid K-algebras Aff_K. This is a full subcategory of the category of K-algebras as all maps between affinoid K-algebras are automatically continuous.

The structure sheaf is defined as follows.

Definition 1.12 (Tate Spectrum – Structure Sheaf)**.** Let A be an affinoid K-algebra. The functor

$$R_A(f_0|f_1,\dots,f_n)\mapsto A\left\langle\frac{\varepsilon}{f_0}\right\rangle\left\langle\frac{f_1}{f_0},\dots,\frac{f_n}{f_0}\right\rangle$$

where $\varepsilon\in K^\times$ such that $\max_{0\leq i\leq n}|f_i(x)|\geq|\varepsilon|$ for all $x\in\mathrm{Sp}(A)$ represents the functor $\mathsf{Rat}_A^{\mathsf{Opp}}\to\mathsf{Aff}_K$

$$F_\Omega(B)=\{\varphi\in\mathrm{Hom}_{\mathsf{Aff}_K}(A,B):\mathrm{Sp}(\varphi)(\mathrm{Sp}(B))\subseteq\Omega\}.$$

Summing up the preceding constructions, we have:

Definition 1.13 (Tate Spectrum – Ringed Space)**.** Let A be an affinoid K-algebra. The Tate spectrum is given by:

- Topological space $\mathrm{mSpec}(A)$ with basis for the topology given by rational open subsets $R_A(f_0|f_1,\dots,f_n)$ with $\langle f_0,\dots,f_n\rangle_A=A$.
- Sheaf of rings given by $R_A(f_0|f_1,\dots,f_n)\mapsto A\left\langle\frac{\varepsilon}{f_0}\right\rangle\left\langle\frac{f_1}{f_0},\dots,\frac{f_n}{f_0}\right\rangle$.

Here we used the fact that any sheaf on the base extends to a sheaf on the space.

Remark 1.14. (i) There are identifications $\mathcal{O}_{\mathrm{Sp}(\mathcal{O}_{\mathrm{Sp}(A)}(\Omega))}\cong\mathcal{O}_{\mathrm{Sp}(A)}|_\Omega$.

(ii) By Tate acyclicity, the higher cohomology of $\mathcal{O}_{\mathrm{Sp}(A)}$ vanishes.

We state some additional results surrounding Tate acyclicity.

Definition 1.15 (Laurent Order)**.** Let $\mathcal{S}$ be a sieve on $\mathrm{Sp}(A)$. We define the Laurent order $\mathfrak{o}_L(\mathcal{S})$ inductively as follows:

- $\mathfrak{o}_L(\mathcal{S})=0$ if and only if is the all sieve.
- $\mathfrak{o}_L(\mathcal{S})\leq k$ if there is $g\in\mathcal{O}_X(\Omega)$ such that the restriction sieves $\mathcal{S}|_{R_\Omega(g|1)}$ and $\mathcal{S}|_{R_\Omega(1|g)}$ have Laurent order at most k.

- $\mathcal{S}$ is of Laurent order k if k is the smallest number such that $\mathcal{S}$ is of Laurent order at most k

Finiteness of the Laurent order characterizes covering sieves.

Proposition 1.16. Let $\mathcal{S}$ be a sieve on $\mathrm{Sp}(A)$ for A an affinoid K-algebra. $\mathcal{S}$ is a covering sieve if and only if $\mathfrak{o}_L(\mathcal{S}) < \infty$.

This immediately gives a simple sufficient condition for Tate acyclicity.

Corollary 1.17. Let $\mathcal{F}$ be a sheaf of Abelian groups on $\mathrm{Sp}(A)$. If

$$0 \to \mathcal{F}(\Omega) \to \mathcal{F}(R_\Omega(g|1)) \oplus \mathcal{F}(R_\Omega(1|g)) \to \mathcal{F}(R_\Omega(g|1) \cap R_\Omega(1|g)) \to 0$$

is exact for all $\Omega \subseteq \mathrm{Sp}(A)$ rational and $g \in \mathcal{O}_{\mathrm{Sp}(A)}(\Omega)$ then $\mathcal{F}$ is acyclic.

We state two additional results concerning the unviersality of certain affinoid K-algebras. We first recall the following definitions.

Definition 1.18 (nat Ring)**.** Let A be a topological ring. A is a nat ring if it has a basis of neighborhoods of zero consisting of open subgroups.

Definition 1.19 (Tate Ring)**.** A nat ring A is Tate if if it has a powerbounded neighborhood of zero and has a topologically nilpotent unit known as a quasi-uniformizer.

In turn:

Proposition 1.20. Let A be a Tate ring and

$$A\langle f_1, \ldots, f_n \rangle = A\langle X_1, \ldots, X_n \rangle / \langle X_1 - f_1, \ldots, X_n - f_n \rangle.$$

$A\langle f_1, \ldots, f_n \rangle$ is initial among nat A-algebras B where $f_1, \ldots, f_n$ are powerbounded. Furthermore, $A\langle f_1, \ldots, f_n \rangle$ contains A as a dense subring.

Proposition 1.21. Let A be a Tate ring and

$$A\left\langle \frac{1}{f_1}, \ldots, \frac{1}{f_n} \right\rangle = A\langle X_1, \ldots, X_n \rangle / \left\langle X_1 - \frac{1}{f_1}, \ldots, X_n - \frac{1}{f_n} \right\rangle.$$

$A\langle \frac{1}{f_1}, \ldots, \frac{1}{f_n} \rangle$ is initial among nat A-algebras B where $f_1, \ldots, f_n$ are units with $\frac{1}{f_1}, \ldots, \frac{1}{f_n}$ powerbounded. Furthermore, $A\langle \frac{1}{f_1}, \ldots, \frac{1}{f_n} \rangle$ contains $A[\frac{1}{f_1}, \ldots, \frac{1}{f_n}]$ as a dense subring.

We are now ready to define coherent sheaves.

We now begin marginal labeling, which follows the lecture.

We begin with the following preparatory result.

Proposition 1.22. Let A be an affinoid algebra and $\Omega \subseteq \mathrm{Sp}(A)$ a rational subset. Then:

Proposition 2.1

(i) For $B = \mathcal{O}_{\mathrm{Sp}(A)}(\Omega)$ and $\mathfrak{m} \in \mathrm{Sp}(B)$, there is an isomorphism of K-algebras $B_{\mathfrak{m}}^\wedge \cong A_{\widetilde{\mathfrak{m}}}^\wedge$ where $\widetilde{\mathfrak{m}}$ is the preimage of $\mathfrak{m}$ under the map $A \to B$ and $(-)_I^\wedge$ is the completion of a ring with respect to the ideal I.
(ii) B is flat as an A-algebra.

Proof of (i). We first show a claim:

(†) For all $n \in \mathbb{N}$, $A/\widetilde{\mathfrak{m}}^n \to B/\mathfrak{m}^n$ is an isomorphism.

Note that $B/\mathfrak{m}^n$ is initial amongst affinoid B-algebras C such that $\mathfrak{m}^n C = 0$, while $A/\mathfrak{m}^n$ is initial amongst affinoid A-algebras C' such that $\widetilde{\mathfrak{m}}^n C = 0$. For C as above, the image of $\mathrm{Sp}(C)$ in $\mathrm{Sp}(B)$ is $\mathfrak{m}$, while the image of $\mathrm{Sp}(C')$ in $\mathrm{Sp}(A)$ is $\widetilde{\mathfrak{m}} \in \Omega$. Applying the universal property twice, C' can be endowed uniquely with the structure of a B-algebra, and by $\kappa(\mathfrak{m}) \cong \kappa(\widetilde{\mathfrak{m}})$ it follows that C' is generated by $\widetilde{\mathfrak{m}}$ Thus both $A/\widetilde{\mathfrak{m}}^n, B/\mathfrak{m}^n$ satisfy the same universal property, hence isomorphic.

The desired claim follows from (†) by passage to the limit. ■

Proof of (ii). By a standard result in commutative algebra, it suffices to show $B_{\mathfrak{m}}$ is A-flat for all $\mathfrak{m} \in \mathrm{mSpec}(B)$. B being Noetherian, $B_{\mathfrak{m}}^\wedge$ is a faithfully flat $B_{\mathfrak{m}}$-algebra, whereby it is sufficient to show that $B_{\mathfrak{m}}^\wedge$ is flat over A. But $B_{\mathfrak{m}}^\wedge \cong A_{\widetilde{\mathfrak{m}}}^\wedge$ by (i), which is a flat A-module as A is Noetherian, giving the claim. ■

As in the case of algebraic geometry, coherent sheaves are defined as $\widetilde{(-)}$-ifications of finitely generated modules.

Definition 1.23 ($\widetilde{(-)}$)**.** Let A be an affinoid K-algebra and M a finitely generated A-module. The sheaf $\widetilde{M}$ is the sheafification of the presheaf $\Omega \mapsto M \otimes_A \mathcal{O}_{\mathrm{Sp}(A)}(\Omega)$ on rational open subsets. Definition 2.1

Exactness of the sequence

$$0 \to \mathcal{O}_{\mathrm{Sp}(A)}(\Omega) \to \mathcal{O}_{\mathrm{Sp}(A)}(R_\Omega(1|g)) \oplus \mathcal{O}_{\mathrm{Sp}(A)}(R_\Omega(g|1)) \to \mathcal{O}_{\mathrm{Sp}(A)}(R_\Omega(g|1, g^2)) \to 0$$

is preserved under $- \otimes_A M$ by Corollary 1.17. In particular, we have:

Proposition 1.24. Let A be an affinoid K-algebra and M a finitely generated A-module with associated sheaf $\widetilde{M}$. Then $\widetilde{M}(\Omega) = \mathcal{O}_{\mathrm{Sp}(A)}(\Omega) \otimes_A M$ and $H^p(\Omega, \widetilde{M}) = 0$ for all $p > 0$ and for all rational Ω. Proposition 2.2

Proof. By flatness of $\mathcal{O}_{\mathrm{Sp}(A)}(\Omega)$ as an A-algebra, exactness of the sequence for Ω rational, we have a short exact sequence

$$0 \to \mathcal{O}_{\mathrm{Sp}(A)}(\Omega) \to \mathcal{O}_{\mathrm{Sp}(A)}(\Omega_1) \oplus \mathcal{O}_{\mathrm{Sp}(A)}(\Omega_2) \to \mathcal{O}_{\mathrm{Sp}(A)}(\Omega_1 \cap \Omega_2) \to 0$$

and by flatness of $\mathcal{O}_{\mathrm{Sp}(A)}(\Omega)$ (vis. [Stacks, Tag 00M5]), we get that

$$0 \to \mathcal{O}_{\mathrm{Sp}(A)}(\Omega) \otimes_A M \to (\mathcal{O}_{\mathrm{Sp}(A)}(\Omega_1) \oplus \mathcal{O}_{\mathrm{Sp}(A)}(\Omega_2)) \otimes_A M \to \mathcal{O}_{\mathrm{Sp}(A)}(\Omega_1 \cap \Omega_2) \otimes_A M \to 0$$

is exact, so by noting that $M \otimes_A \mathcal{O}_{\mathrm{Sp}(A)}(\Omega) \cong \widetilde{M}(\Omega)$, we have that $\widetilde{M}$ is acyclic. ■

2. Lecture 2 – 24th April 2025

We define coherent sheaves.

Definition 2.2

Definition 2.1 (Coherent Sheaves)**.** Let A be an affinoid K-algebra and $\mathcal{F}$ an $\mathcal{O}_{\mathrm{Sp}(A)}$-module. $\mathcal{F}$ is coherent if it is of the form $\widetilde{M}$ for some finitely generated A-module M.

The coherence condition can be shown to be local in the sense that any sheaf of modules for which there exists a covering sieve consisting of a trivialization by a cover by rational opens on which the modules are finitely generated is coherent. We show this as a consequence of a sequence of results.

Proposition 2.3

Proposition 2.2. Let A be an affinoid K-algebra and Ω_1, Ω_2 a rational cover of $\mathrm{Sp}(A)$ with intersection Ω_{12}. If $f_{12} \in \mathcal{O}_{\mathrm{Sp}(A)}(\Omega_{12})$ then $f_{12} = f_1|_{\Omega_{12}} + f_2|_{\Omega_{12}}$ for $f_i \in \mathcal{O}_{\mathrm{Sp}(A)}(\Omega_i)$ with $\|f_i\| = O(\|f_{12}\|)$.

Proof. Omitted. ■

Lemma 2.1

Lemma 2.3. Let $\mathcal{M}$ be a sheaf of $\mathcal{O}_{\mathrm{Sp}(A)}$-modules on $\mathrm{Sp}(A)$ and Ω_1, Ω_2 a rational cover of $\mathrm{Sp}(A)$ on which $\mathcal{M}|_{\Omega_i} = M_i$ with M_i finitely generated. If $m_{12} \in \mathcal{M}(\Omega_{12})$ then there are $m_i \in \mathcal{M}(\Omega_i)$ with $\|m_i|\mathcal{M}(\Omega_i)\| = O(\|m_{12}\|)$ (the constant independent of m_{12}) and such that

$$\|m_{12} - m_1|_{\Omega_{12}} - m_2|_{\Omega_{12}}\| \leq \frac{1}{2}\|m_{12}\|.$$

Finish proof.

Lemma 2.2

Lemma 2.4. Let $\mathcal{M}$ be a sheaf of $\mathcal{O}_{\mathrm{Sp}(A)}$-modules on $\mathrm{Sp}(A)$ and Ω_1, Ω_2 a rational cover of $\mathrm{Sp}(A)$ on which $\mathcal{M}|_{\Omega_i} = M_i$ with M_i finitely generated. If $m_{12} \in \mathcal{M}(\Omega_{12})$ then

$$m_{12} = m_1|_{\Omega_{12}} + m_2|_{\Omega_{12}}$$

where $m_i \in \mathcal{M}(\Omega_i)$ and $\|m_i|\mathcal{M}(\Omega_i)\| = O(\|m_{12}|\mathcal{M}(\Omega_{12})\|)$ with the implied constant independent of m_{12}.

Proof. Let C be the implied constant of Lemma 2.3. We can define recursively

$$\begin{aligned} m_{12} = m_{12}^{(0)} &= m_1^{(0)}|_{\Omega_{12}} + m_2^{(0)}|_{\Omega_{12}} + m_{12}^{(1)} \\ m_{12}^{(1)} &= m_1^{(1)}|_{\Omega_{12}} + m_2^{(0)}|_{\Omega_{12}} + m_{12}^{(2)} \\ \vdots \quad & \qquad\qquad \vdots \end{aligned}$$

where $\|m_{12}^{(i+2)}|\mathcal{M}(\Omega_{12})\| \leq \frac{1}{2}\|m_{12}^{(i)}|\mathcal{M}(\Omega_{12})\|$ and $\|m_j^{(i)}|\mathcal{M}(\Omega_j)\| \leq C\|m_{12}^{(i)}|\mathcal{M}(\Omega_{12})\|$, hence $\|m_{12}^{(i)}|\mathcal{M}(\Omega_{12})\| \leq \frac{1}{2^i}\|m_{12}|\mathcal{M}(\Omega_{12})\|$ and $\|m_j^{(i)}|\mathcal{M}(\Omega_j)\| \leq \frac{C}{2^i}\|m_{12}|\mathcal{M}(\Omega_{12})\|$ and the assertion follows with $m_j = \sum_{i=0}^{\infty} m_j^{(i)} \in \mathcal{M}(\Omega_j)$ with the implied constant C. ■

From this we deduce:

Corollary 2.1

Corollary 2.5. Let $\mathcal{M}$ be a sheaf of $\mathcal{O}_{\mathrm{Sp}(A)}$-modules on $\mathrm{Sp}(A)$ and Ω_1, Ω_2 a rational cover of $\mathrm{Sp}(A)$ on which $\mathcal{M}|_{\Omega_i} = M_i$ with M_i finitely generated. If $m_{12} \in \mathcal{M}(\Omega_{12})$ and $\varepsilon > 0$ then there are $m_i \in \mathcal{M}(\Omega_i)$ such that $m_{12} = m_1|_{\Omega_{12}} + m_2|_{\Omega_{12}}$ and $\|m_2|\mathcal{M}(\Omega_2)\| < \varepsilon$.

Proof. Choose $m_2' \in \mathcal{M}(\Omega_2)$ such that $\|m_{12} - m_2|_{\Omega_{12}}|\mathcal{M}(\Omega_{12})\| < \delta$ then $m_{12} - m_2' = m_1 + m_2''$ with $\|m_1|\mathcal{M}(\Omega_1)\| + \|m_2''|\mathcal{M}(\Omega_2)\| \leq C \cdot \delta$ with C as in Lemma 2.4. Then choose δ such that $C \cdot \delta < \varepsilon$. ■

Corollary 2.2

Corollary 2.6. Let $\mathcal{M}$ be a sheaf of $\mathcal{O}_{\mathrm{Sp}(A)}$-modules on $\mathrm{Sp}(A)$ and Ω_1, Ω_2 a rational cover of $\mathrm{Sp}(A)$ on which $\mathcal{M}|_{\Omega_i} = M_i$ with M_i finitely generated. There are $(\mu_j)_{j=1}^{N_1} \in \mathcal{M}(\Omega)$ such that the $\mu_j|_{\Omega_1}$ generate $\mathcal{M}(\Omega_1)$.

Proof. By Corollary 2.5, $m_j^{(1)}|_{\Omega_{12}} = \mu_j^1|_{\Omega_{12}} + \mu_j^{(2)}|_{\Omega_{12}}$ where $\mu_j^{(k)} \in \mathcal{M}(\Omega_k)$ and $\|\mu_j^{(1)}|\mathcal{M}(\Omega) \leq \varepsilon$ for any ε. Let

$$\begin{aligned} \mu_j|_{\Omega_1} &= m_j^{(1)} - \mu_j^{(1)} \\ \mu_j|_{\Omega_2} &= \mu_j^{(2)} \end{aligned}$$

then $\|\mu_j|_{\Omega_1} - m_j^{(1)}|\mathcal{M}(\Omega_j)\| \leq \varepsilon$ and when $\varepsilon = \frac{1}{2}$ the assertion follows. ■

It follows that the set

$$\{g^{-k} \cdot m_1|_{\Omega_{12}} | m_1 \in \mathcal{M}(\Omega_1)\}$$

is dense in $\mathcal{M}(\Omega_{12})$, recalling here that $\Omega_1 = R_A(1|g)$.

Corollary 2.3

Corollary 2.7. Let $\mathcal{M}$ be a sheaf of $\mathcal{O}_{\mathrm{Sp}(A)}$-modules on $\mathrm{Sp}(A)$ and $\Omega_1 = R_A(1|g), \Omega_2 = R_A(g|1)$ for $g \in A$ a rational cover of $\mathrm{Sp}(A)$ on which $\mathcal{M}|_{\Omega_i} = M_i$ with M_i finitely generated. If $m_{12} \in \mathcal{M}(\Omega_{12})$ and $\varepsilon > 0$ then

$$m_{12} = g^{-k} \cdot m_1|_{\Omega_{12}} + m_2|_{\Omega_{12}}$$

where $m_i \in \mathcal{M}(\Omega_i)$ and $\|m_2|\mathcal{M}(\Omega_2)\| \leq \varepsilon$.

Proof. One need only repeat the arguments of Corollary 2.5 ■

Corollary 2.4

Corollary 2.8. Let $\mathcal{M}$ be a sheaf of $\mathcal{O}_{\mathrm{Sp}(A)}$-modules on $\mathrm{Sp}(A)$ and $\Omega_1 = R_A(1|g), \Omega_2 = R_A(g|1)$ for $g \in A$ a rational cover of $\mathrm{Sp}(A)$ on which $\mathcal{M}|_{\Omega_i} = M_i$ with M_i finitely generated. There are $(\mu_j^{(2)})_{j=1}^{N_2} \in \mathcal{M}(\Omega)$ such that $\mu_j|_{\Omega_2}$ generate $\mathcal{M}(\Omega_2)$.

Proof. Write

$$\begin{aligned} m_j^{(2)}|_{\Omega_{12}} &= g^{-k}\mu_j^{(1)}|_{\Omega_{12}} + \mu_j^{(2)}|_{\Omega_{12}} \\ \mu_j|_{\Omega_2} &= \mu_j^{(1)} \\ \mu_j|_{\Omega_2} &= g^k(m_j^{(2)} - \mu_j^{(2)}) \end{aligned}$$

and the assertion follows when $\varepsilon \leq \frac{1}{2}$ in which case $\|m_j^{(2)} - g^k \cdot \mu_j|\mathcal{M}(\Omega_2)\| \leq \frac{1}{2}$. ■

3. Lecture 3 – 8th May 2025

We prove the locality statement earlier alluded to.

Proposition 2.4

Proposition 3.1. Let $\mathcal{S}$ be a covering sieve of $\mathrm{Sp}(A)$ where $\mathcal{S}$ is generated by $\widetilde{M_\Omega}$ where M_Ω is a finitely generated $\mathcal{O}_{\mathrm{Sp}(A)}(\Omega)$-module. Then $\widetilde{M}$ is coherent.

Proof. By induction on Laurent order, it suffices to show that for $g \in A$ and $\Omega_1 = R_A(1|g), \Omega_2 = R_A(g|1)$ rational opens of $\mathrm{Sp}(A)$ on which $\mathcal{M}|_{\Omega_1} = M_1, \mathcal{M}|_{\Omega_2} = M_2$ are finitely generated $\mathcal{O}_{\mathrm{Sp}(A)}(\Omega_1), \mathcal{O}_{\mathrm{Sp}(A)}(\Omega_2)$-modules, that $\mathcal{M}$ is finitely generated too.

By Corollaries 2.6 and 2.8 there are sections $(m_i)_{i=1}^n$ generating $\mathcal{M}$ as an $\mathcal{O}_{\mathrm{Sp}(A)}$-module such that their restrictions to Ω_1, Ω_2 generate $\mathcal{M}|_{\Omega_1}, \mathcal{M}|_{\Omega_2}$. Consider

$$\mathcal{K} = \ker\left(\mathcal{O}_{\mathrm{Sp}(A)}^n \xrightarrow{(m_i)_{i=1}^N} \mathcal{M}\right).$$

We have $\mathcal{K}|_{\Omega_j} = \widetilde{K}_j$ where

$$K_j = \ker\left(\mathcal{O}_{\mathrm{Sp}(A)}(\Omega_j)^n \xrightarrow{(m_i|_{\Omega_j})_{i=1}^n} M_j\right).$$

Applying the same reasoning, we have that there are $(k_i)_{i=1}^m$ that generate $\mathcal{K}$ as an $\mathcal{O}_{\mathrm{Sp}(A)}$-module. It follows that $\mathcal{M}$ is the cokernel of

$$\mathcal{O}_{\mathrm{Sp}(A)}^m \xrightarrow{(k_i)_{i=1}^m} \mathcal{O}_{\mathrm{Sp}(A)}^n.$$

The universal property shows that $\mathcal{M}$ is isomorphic to the cokernel as it is an isomorphism of each Ω_j. Since $\widetilde{(-)}$ is exact, we obtain $\mathcal{M} = \widetilde{M}$ where M is the cokernel of $A^{\oplus m} \to A^{\oplus n}$ by the k_i's. ■

Remark 3.2. In general if $\mathcal{R}$ is a sheaf of rings on a site, we say an $\mathcal{R}$-module is finitely generated if there are fintely many global sections such that $\mathcal{R}^n \to \mathcal{M}$ is an epimorphism of sheaves. We say $\mathcal{M}$ is locally finitely generated if the objects on which $\mathcal{M}$ is finitely generated form =a covering sieve, and $\mathcal{M}$ is coherent if it is locally finitely generated and the kernels of the local maps $\mathcal{R}|_X^n \to \mathcal{M}|_X$ have finitely generated kernels.

Remark 3.3. On a one point space, this is the condition of the kernel being finitely generated. That is, that the ring is a coherent ring.

Example 3.4. Let us consider Remark 3.2 in the setting of $X = \mathrm{Sp}(A), \mathcal{R} = \mathcal{O}_X, \mathcal{M} = \widetilde{M}$ for M a finitely generated A-module. In this case, the kernel of the map $A^{\oplus n} \to M$ generate the kernel sheaf $\mathcal{O}_{\mathrm{Sp}(A)}^n \to \mathcal{M}$ so sheaves coherent in the sense of Definition 2.1 are coherent in the sense of Remark 3.2.

Dually, if $\mathcal{M}$ is coherent in the sense of Remark 3.2, we can use locality of coherence Proposition 3.1 to observe that the global sections generating $\mathcal{M}$ and the kernel sheaf $\mathcal{K}$ give rise to A-modules M, K such that M is the cokernel of $K \to A^n$.

This concludes our discussion of coherent sheaves.

Recall that $\mathrm{Sp}(\mathbb{T}_1)$ for K algebraically closed has van der Put points ξ given by the balls of radius $\leq R$ for $R \in |K| \subseteq \mathbb{R}_{\geq 0}$. Denote $\mathfrak{K}_{\leq R}$ of all balls $K_{\leq R}(X)$ for $x \in \mathrm{Sp}(A)$. For a van der Put point ξ of $\mathrm{Sp}(A)$, we can define M_ξ to be the set of all $r \in [0, 1) \cap |K^\times|$ for which there exists an $x \in \mathfrak{K}_{\leq r} \cap \xi$ – that is, ξ contains a ball of radius r. Denote $K_{\leq R}(\xi)$ be set of rational open sets of radius at most R in the van der Put point ξ.

Example 3.5. If R is arbitrarily small, then $K_{\leq R}(\xi) = \{x\}$. In this case, $x \in \Omega$ if and only if $R(f_0|f_1, \ldots, f_n) = \Omega \in \xi$ if and only if $\nu(f_0) \geq \nu(f_i)$ where $\nu(f) = |f(x)|$ for all $1 \leq i \leq n$.

4. Lecture 4 – 15th May 2025

We consider the van der Put points of a Tate spectrum in terms of the adic spectrum.

The construction of the adic spectrum begins with Huber pairs.

Definition 4.1 (Huber Pair)**.** A Huber pair is a pair (A, A^+) where:

- A is a nat ring with an open bounded subring hose topology is I-adic for some finitely generated ideal I.
- $A^+ \subseteq A^\circ$ is an open integrally closed subring.

Definition 4.2 (Morphism of Huber Pairs)**.** A morphism of Huber pairs $\varphi : (A, A^+) \to (B, B^+)$ is a continuous homomorphism of rings $\varphi : A \to B$ such that $\varphi(A^+) \subseteq B^+$.

Remark 4.3. The subring of powerbounded elements A° is always an integrally closed subring of A. In particular, if A is Huber, (A, A°) is a Huber pair. If further A is Tate, then for any morphism of Huber pairs $\varphi : (A, A^\circ) \to (B, B^\circ)$ the condition that $\varphi(A^\circ) \subseteq B^\circ$ is automatic and B is also Tate.

The adic spectrum is defined as a certain subspace of the space of valuations.

Definition 4.4 (Valuation)**.** A valuation on a ring A is a map $\nu : A \to \Gamma \cup \{0\}$, where Γ is written multiplicatively, such that:

(i) $0 < \gamma$ for all $\gamma \in \Gamma$.
(ii) $\nu(ab) = \nu(a) \cdot \nu(b)$.
(iii) $\nu(a + b) \leq \max\{\nu(a), \nu(b)\}$.
(iv) $\nu(1) = 1$.

Definition 4.5 (Support of Valuation)**.** Let ν be a valuation on a ring A. The support $\mathrm{supp}(\nu)$ of ν is the prime ideal $\{a \in A : \nu(a) = 0\} \subseteq A$.

Remark 4.6. We assume that Γ is generated by the image of $R \setminus \mathrm{supp}(\nu)$. In this case, $\nu \simeq \widetilde{\nu}$ if and only if there is a unique isomorphism of groups $\tau : \Gamma \to \widetilde{\Gamma}$ of groups $\widetilde{\nu} = \tau \circ \nu$.

We consider some additional constructions related to valuations.

Definition 4.7 (Convex Subset)**.** A subset X of Γ is convex if and only if for all $\gamma, \gamma' \in X$, $[\gamma, \gamma']_\Gamma \subseteq X$.

Definition 4.8 (Rank of Valuation)**.** Let ν be a valuation on a ring A. The rank of ν is the number of convex subgroups.

Remark 4.9. The set of convex subgroups is linearly ordered by inclusion.

We can define continuous valuations by topologizing $\Gamma \cup \{0\}$ with the topology that all elements of Γ are open points and the collection of half-open intervals $\{[0, \gamma)_\gamma : \gamma \in \Gamma\}$ form a neighborhood basis of 0.

Definition 4.10 (Continuous Valuation)**.** Let ν be a valuation on a ring A. ν is a continuous valuation if it is continuous as a map where $\Gamma \cup \{0\}$ is equipped with the topology where all elements of Γ are open points and the collection of half-open intervals $\{[0, \gamma)_\Gamma : \gamma \in \Gamma\}$ form a neighborhood basis of 0.

Remark 4.11. The following conditions are equivalent to a valuation being continuous:

- If $\nu(a) \neq 0$ then $\{a' \in A : \nu(a') < \nu(a)\}$ is open.
- The set $\{a \in A : \nu(a) < \gamma\}$ is open for $\gamma \in \Gamma$.

Definition 4.12 (Space of Continuous Valuations)**.** Let A be a nat ring and let $\mathrm{Cont}(A)$ be the set of continuous valuations equipped with the topology that the rational open subsets

$$R_{\mathrm{Cont}(A)}(a_0|a_1, \dots, a_n) = \{\nu/\sim, \nu \text{ cts.} : \nu(a_0) \neq 0, \nu(a_i) \leq \nu(a_0) \ \forall 1 \leq i \leq n\}$$

such that $(a_0, \dots, a_n)$ is an open ideal in A.

Remark 4.13. In particular if A is Tate then the only open ideal is the ring itself, and $(a_0, \dots, a_n) = A$.

Remark 4.14. For general topological rings, these rational open subsets may fail to be closed under finite intersection, but this always holds for Huber rings.

We recall the following result.

Proposition 4.15. Let A be a nat ring and let $\mathrm{Cont}(A)$ be the space of continuous valuations of A. $\mathrm{Cont}(A)$ is a spectral space with quasicompact basis given by the rational open subsets.

This allows us to construct the adic spectrum.

Corollary 4.16. Let A^+ be an integrally closed subring of a nat ring A. The subspace

$$\bigcap_{a \in A^+} R_{\mathrm{Cont}(A)}(1|a) = \{\nu/\sim, \nu \text{ cts.} : \nu(a) \leq 1 \ \forall a \in A^+\}$$

is spectral.

Proof. This is a proconstructible subset of a spectral space, hence spectral. ∎

We can finally define the adic spectrum.

Definition 4.17 (Affinoid Adic Space)**.** Let (A, A^+) be a Huber pair. The adic spectrum $\mathrm{Spa}(A, A^+)$ of (A, A^+) is the subspace

$$\bigcap_{a \in A^+} R_{\mathrm{Cont}(A)}(1|a) = \{\nu/\sim, \nu \text{ cts.} : \nu(a) \leq 1 \ \forall a \in A^+\}.$$

Remark 4.18. When A is a Tate ring,

$$R_{\mathrm{Cont}(A)}(f_0|f_1, \dots, f_n) = \{\nu/\sim, \nu \text{ cts.} : \nu(f_i) \leq \nu(f_0) \ \forall 1 \leq i \leq n\}.$$

Henceforth we take A to be an affinoid K-algebra.

Example 4.19. If $\kappa \in K$ and $|\kappa| = 1$ then $K^\circ \subseteq A^\circ$ and $1/\kappa \in K^\circ \subseteq A^\circ$ so $\nu(\kappa) \leq 1, \nu(1/\kappa) \leq 1$ showing $\nu(\kappa) = 1$ and $K^\times$ is a subgroup of Γ.

Up to replacing ν by an equivalent valuation as in Remark 4.6, we can assume that $|K^\times| \subseteq \Gamma$ which is typically not convex. If $t \in \mathbb{R}$ such that $t^n \in |K^\times|$ for some $n \in \mathbb{N}$, say $\gamma \leq t$ (resp. $\gamma < t$) if and only if $\gamma^n \leq t^n$ (resp. $\gamma^n < t^n$). If $t \in \mathbb{R}_{>0}$ and there is no positive integer n with $t^n \in |K^\times|$ exists, say $\gamma \leq t$ if and only if $\gamma < t$ if and only if there exists $n \in \mathbb{N}$ and $\kappa \in K^\times$ such that $\gamma^r \leq |\kappa|^r < t$. For general real numbers t, γ, the inequality $t < \Gamma$ is dealt with in the same way.

In what follows, we will use the following fact.

Lemma 4.20. Let A be an affinoid K-algebra. If $\| - |A\|$ is a residual norm, then a valuation ν on A such that $\nu((K^\circ)^\times) \subseteq \{1\}$. ν is continuous if and only if there exists $c \in \mathbb{R}$ such that $\nu(a) \leq c\|a|A\|$ and $|K^\times|$ is cofinal in Γ – for all $\gamma \in \Gamma$, there is $\varepsilon \in K^\times$ such that $|\varepsilon| \leq \gamma$.

We will soon show that the van der Put points of the Tate $\mathrm{Sp}(A)$ is homeomorphic as a G_+-space to $\mathrm{Spa}(A, A^\circ)$ by constructing a specific bijection, and show that $\mathrm{Spa}(A, A^\circ)$ has Krull dimension is that of A.

Example 4.21. Let $A = \mathbb{T}_1$ and $A^+ = K^\circ + A^{\circ\circ}$. (A, A^+) is a Huber pair and $\mathrm{Spa}(A, A^\circ) \setminus \mathrm{Spa}(A, A^+)$ has exactly one element ν where $\nu(f) = (\|f|\mathbb{T}_1\| - \kappa_f)$ where for $f = \sum_{j \geq 0} f_j T^j \in \mathbb{T}_1 \setminus \{0\}$, $\kappa_f = \max\{j : |f_j| = \|f|\mathbb{T}_1\|\}$.

Remark 4.22. The pair (A, A^+) is not necessarily topologically of finite type over (K, K°).

The proofs of the abovementioned result will require a careful treatment of power-bounded elements and the proof of the result on Krull dimension is also fairly subtle.

References

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Universität Bonn, Bonn, D-53113
Email address: `wgabrielong@uni-bonn.de`
URL: `https://wgabrielong.github.io/`