

**V4A2 – ALGEBRAIC GEOMETRY II**  
**SUMMER SEMESTER 2025**

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PRELIMINARIES

These notes roughly correspond to the course **V4A2 – Algebraic Geometry II** taught by Prof. Daniel Huybrechts at the Universität Bonn in the Summer 2025 semester. These notes are  $\text{\LaTeX}$ -ed after the fact with significant alteration and are subject to misinterpretation and mistranscription. Use with caution. Any errors are undoubtedly my own and any virtues that could be ascribed to these notes ought be attributed to the instructor and not the typist. Knowledge of commutative algebra, topology, and category theory will be assumed.

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## 1. LECTURE 1 – 7TH APRIL 2025

We begin by a consideration of the theory of smoothness, first in the local case. This is done by defining the sheaves of Kähler differentials on schemes – in the local picture, the module of differentials on a ring.

**Definition 1.1** (Derivation). Let  $B$  be an  $A$ -algebra and  $M$  a  $B$ -module. An morphism of  $A$ -modules  $D : B \rightarrow M$  is an  $A$ -derivation if it satisfies the Leibniz rule  $d(xy) = xd(y) + yd(x)$  for all  $x, y \in B$ . Denote the set of  $A$ -derivations in  $M$  by  $\text{Der}_A(B, M)$ .

**Remark 1.2.** It is necessary that  $M$  is a  $B$ -module, since the Leibniz rule involves elements of  $B$ .

**Remark 1.3.** Observe that the composition  $A \rightarrow B \rightarrow M$  is zero since  $a = a \cdot 1_B$  and computing we get  $d(a \cdot 1_B) = ad(1_B)$  by  $A$ -linearity, but on the other hand  $d(a \cdot 1_B) = ad(1_B) + 1_B d(a)$  by the Leibniz rule, so  $ad(1_B) = 0$  showing  $d(1_B) = 0$  and thus  $d(a) = 0$ .

The Kähler differentials of a ring map is the universal recipient of an  $A$ -algebra  $B$  in the following sense.

**Definition 1.4** (Module of Kähler Differentials). Let  $B$  be an  $A$ -algebra. The module of Kähler differentials of  $B$  over  $A$  is a  $B$ -module  $\Omega_{B/A}^1$  with an  $A$ -derivation  $d : B \rightarrow \Omega_{B/A}^1$  that is initial amongst  $B$ -modules receiving an  $A$ -derivation from  $B$ .

Unwinding the universal property, if  $M$  is a  $B$ -module receiving an  $A$ -derivation from  $B$  by  $f : B \rightarrow M$ , there is a unique factorization over  $\Omega_{B/A}^1$  as follows.

$$\begin{array}{ccc} \Omega_{B/A}^1 & \xrightarrow{\quad} & M \\ \uparrow \exists! & \nearrow & \\ B & & \end{array}$$

In particular, there is a bijection  $\text{Der}_A(B, M) \leftrightarrow \text{Hom}_{\text{Mod}_B}(\Omega_{B/A}^1, M)$  functorial in  $M$ .

**Proposition 1.5.** Let  $B$  be an  $A$ -algebra. The  $B$ -module  $\Omega_{B/A}^1$  and the  $A$ -derivation  $d : B \rightarrow \Omega_{B/A}^1$  exist and are unique up to unique isomorphism.

*Proof.* The module  $\Omega_{B/A}^1$  can be constructed as the free  $B$ -module on elements  $dx$  for  $x \in B$  modulo the relations generated by the Leibniz rule and  $da = 0$  for  $a \in A$ . Uniqueness up to unique isomorphism is clear from the universal property and Yoneda's lemma. ■

In special cases, the module of Kähler differentials can be described explicitly.

**Example 1.6.** Let  $A = k, B = k[x_1, \dots, x_n]$ .  $\Omega_{B/A}^1$  is a free module of rank  $n$  with basis  $dx_i$ . The map  $f \mapsto \sum_{i=1}^n \frac{\partial f}{\partial x_i} \cdot dx_i$  is an  $A$ -derivation and the map  $dx_i \mapsto x_i$  defines an isomorphism  $\Omega_{B/A}^1 \rightarrow B^{\oplus n}$ .

Kähler differentials are also fairly easy to understand in the case of ring localizations and ring quotients. These will be important in understanding the sheaves of Kähler differentials of open and closed immersions in the case of schemes, respectively.

**Proposition 1.7.** Let  $A$  be a ring.

- (i) If  $B = S^{-1}A$ , then  $\Omega_{B/A}^1 = 0$ .
- (ii) If  $B = A/I$  for  $I \subseteq A$  an ideal, then  $\Omega_{B/A}^1 = 0$ .

*Proof of (i).* We already have that  $da = 0$  for all  $a \in A$ . We then observe that writing  $a = s \cdot \frac{a}{s}$  we have

$$\begin{aligned} 0 = d(a) &= d\left(s \cdot \frac{a}{s}\right) = sd\left(\frac{a}{s}\right) + \frac{a}{s}d(s) \\ &= sd\left(\frac{a}{s}\right) \end{aligned} \quad s \in A \Rightarrow ds = 0$$

so  $sd(\frac{a}{s}) = 0$  and  $d(\frac{a}{s}) = 0$  whence the claim.  $\blacksquare$

*Proof of (ii).* The map  $A \rightarrow B$  is surjective, so this is precisely the situation Remark 1.3.  $\blacksquare$

We can additionally understand sheaves of Kähler differentials in towers. Let  $A \rightarrow B \rightarrow C$  be maps of rings. There is a natural  $C$ -linear map  $\Omega_{B/A}^1 \otimes_B C \rightarrow \Omega_{C/A}^1$  which is a  $C$ -module homomorphism induced by the diagram

$$\begin{array}{ccccc} B & \longrightarrow & C & \xrightarrow{d_{C/A}} & \Omega_{C/A}^1 \\ \downarrow d_{B/A} & & & \nearrow \exists! & \\ \Omega_{B/A}^1 & & & & \end{array}$$

where the top row is both  $A$  and  $B$ -linear inducing a unique  $B$ -module map  $\Omega_{B/A}^1 \rightarrow \Omega_{C/A}^1$ , considering the latter as a  $B$ -module. By the extension-restriction adjunction, however, we have

$$\mathrm{Hom}_{\mathrm{Mod}_B}(\Omega_{B/A}^1, \Omega_{C/A}^1|_B) \leftrightarrow \mathrm{Hom}_{\mathrm{Mod}_C}(\Omega_{B/A}^1 \otimes_B C, \Omega_{C/A}^1)$$

hence the data of the dotted map in the diagram above gives rise to a unique map  $\Omega_{B/A}^1 \otimes_B C \rightarrow \Omega_{C/A}^1$ . Arguing similarly, there is a  $C$ -linear map  $\Omega_{C/A}^1 \rightarrow \Omega_{C/B}^1$  induced by

$$\begin{array}{ccc} C & \xrightarrow{d_{C/B}} & \Omega_{C/B}^1 \\ \downarrow d_{C/A} & & \nearrow \exists! \\ \Omega_{C/A}^1 & & \end{array}$$

where the map is induced by the universal property as any  $B$ -derivation is also an  $A$ -derivation.

The maps in the preceding discussion assemble to give the following proposition.

**Proposition 1.8.** Let  $A \rightarrow B \rightarrow C$  be maps of rings. There is an exact sequence

$$\Omega_{B/A}^1 \otimes_B C \rightarrow \Omega_{C/A}^1 \rightarrow \Omega_{C/B}^1 \rightarrow 0.$$

*Proof.* The above discussion gives the existence of such maps, so it remains to show exactness at  $\Omega_{C/A}^1$  and surjectivity of the map  $\Omega_{C/A}^1 \rightarrow \Omega_{C/B}^1$ .

We begin with the latter, where by the quotient construction of Proposition 1.5 it suffices to observe that  $\Omega_{C/B}^1$  is a quotient of  $\Omega_{C/A}^1$ .

For the former, we note that for a fixed  $C$ -module  $M$  we have an exact sequence

$$0 \rightarrow \operatorname{Der}_B(C, M) \rightarrow \operatorname{Der}_A(C, M) \rightarrow \operatorname{Der}_A(B, M|_B)$$

since an  $A$ -derivation ( $d : C \rightarrow M$ ) is taken to the composite  $B \rightarrow C \rightarrow M$  which is zero when the map is also a  $B$ -derivation. Rewriting this using the universal property, this is

$$0 \rightarrow \operatorname{Hom}_{\operatorname{Mod}_C}(\Omega_{C/B}^1, M) \rightarrow \operatorname{Hom}_{\operatorname{Mod}_C}(\Omega_{C/A}^1, M) \rightarrow \operatorname{Hom}_{\operatorname{Mod}_C}(\Omega_{B/A}^1 \otimes_B C, M)$$

which by contravariant exactness of the Hom-functor (see [Stacks, Tag 0582] for the precise statement), is the claim.  $\blacksquare$

As a corollary, we can deduce the following fact about localizations.

**Corollary 1.9.** Let  $B$  be an  $A$ -algebra and  $S$  a multiplicative subset of  $B$ . Then  $S^{-1}\Omega_{B/A}^1 \cong \Omega_{S^{-1}B/A}^1$ .

*Proof.* Apply Proposition 1.8 to  $C = S^{-1}B$  and note that  $\Omega_{C/B}^1 = 0$  so the map  $S^{-1}\Omega_{B/A}^1 \rightarrow \Omega_{S^{-1}B/A}^1$  is surjective. To prove injectivity, we produce an inverse map which is an  $A$ -derivation of  $S^{-1}B$  to  $S^{-1}\Omega_{B/A}^1$  by  $d(\frac{b}{s}) \mapsto \frac{1}{s}d(b) - \frac{1}{s^2}bd(s)$  which by the universal property can be seen to be the inverse.  $\blacksquare$

Note that in general  $\Omega_{B/A}^1 \otimes_B C \rightarrow \Omega_{C/A}^1$  is rarely injective.

**Example 1.10.** Let  $A = k, B = k[x], C = k[x]/(x)$ . So  $\Omega_{B/A}^1 \cong Bdx$  but  $\Omega_{C/A} = \Omega_{k/k} = 0$ .

On the other hand, there are situations in which the exact sequence of Proposition 1.8 extends to a short exact sequence.

**Example 1.11.** Let  $B$  be an  $A$ -algebra and  $C = B[x_1, \dots, x_n]$ . We then have a split short exact sequence

$$0 \rightarrow \Omega_{B/A}^1 \otimes_B C \rightarrow \Omega_{C/A}^1 \rightarrow \Omega_{C/B}^1 \rightarrow 0$$

where denoting the map  $\Omega_{C/A}^1 \rightarrow \Omega_{C/B}^1$  by  $\varphi$ , we have the splitting  $\Omega_{C/A}^1 \rightarrow (\Omega_{B/A}^1 \otimes_B C) \oplus \Omega_{C/B}^1$  prescribed by the  $C$ -derivation  $f \mapsto d_{B/A}(f) + \varphi(f)$  under the bijection

$$\operatorname{Hom}_{\operatorname{Mod}_C}(\Omega_{C/A}^1, (\Omega_{B/A}^1 \otimes_B C) \oplus \Omega_{C/B}^1) \leftrightarrow \operatorname{Der}_A(C, (\Omega_{B/A}^1 \otimes_B C) \oplus \Omega_{C/B}^1).$$

The following proposition describes the behavior of the module of Kähler differentials with respect to tensor products.

**Proposition 1.12.** Let  $B, A'$  be  $A$ -algebras. Then there is an isomorphism of  $B$ -modules  $\Omega_{B/A}^1 \otimes_B (B \otimes_A A') \cong \Omega_{(B \otimes_A A')/A'}^1$ .

*Proof.* We contemplate the diagram

$$\begin{array}{ccc} B \otimes_A A' & \xrightarrow{\quad} & \Omega_{B/A}^1 \otimes_A A' \\ \downarrow & \nearrow \exists! & \\ \Omega_{(B \otimes_A A')/A'}^1 & & \end{array}$$

where the solid arrows are  $B \otimes_A A'$ -linear with  $\Omega_{B/A}^1 \otimes_A A' \cong (\Omega_{B/A}^1 \otimes_B (B \otimes_A A'))$  and the dotted arrow induced by the universal property of  $\Omega_{(B \otimes_A A')/A'}^1$ . By applying the tensor-hom adjunction and the universal property of derivations, prescribing an inverse map to the dotted arrow is equivalent to producing an  $A$ -derivation of  $B$  in  $\Omega_{(B \otimes_A A')/A'}^1$  and one observes that the map  $b \mapsto d_{(B \otimes_A A')/A'}(b \otimes 1)$  gives an inverse, whence the claim.  $\blacksquare$

We now treat the case of quotients.

**Proposition 1.13.** Let  $A \rightarrow B \rightarrow C$  be maps of rings where  $C \cong B/\mathfrak{b}$  for some ideal  $\mathfrak{b} \subseteq B$ . There is an exact sequence

$$\mathfrak{b}/\mathfrak{b}^2 \rightarrow \Omega_{B/A}^1 \otimes_B C \rightarrow \Omega_{C/A}^1 \rightarrow 0.$$

*Proof.* We first observe that  $\Omega_{C/B}^1 = 0$  by Proposition 1.7 and  $\Omega_{B/A}^1 \otimes_B C \cong \Omega_{B/A}^1/\mathfrak{b}\Omega_{B/A}^1$ .

We denote  $\mathfrak{b}/\mathfrak{b}^2 \rightarrow \Omega_{B/A}^1 \otimes_B C$  by  $\delta$ ,  $b \mapsto db \otimes 1$ . We first show  $\delta$  is well-defined. For this, we want to show that  $d(b_1 b_2) \otimes 1$  is zero for  $b_1, b_2 \in \mathfrak{b}$ . Indeed, using the Leibniz rule, we have

$$d(b_1 b_2) \otimes 1 = d(f_2) \otimes f_1 + d(f_1) \otimes f_2 \in \mathfrak{b}\Omega_{B/A}^1$$

hence zero in the quotient, showing the map is well-defined.

The diagram is a complex as  $db$  maps to zero in  $\Omega_{C/A}^1$ . The kernel of  $\Omega_{B/A}^1 \rightarrow \Omega_{C/A}^1$  is generated by the  $B$ -submodule  $\mathfrak{b}\Omega_{B/A}^1$  and the elements  $db$  for  $b \in \mathfrak{b}$ , showing exactness of the complex in the middle.  $\blacksquare$

This specializes to finite type algebras.

**Corollary 1.14.** Let  $C$  be a finite type  $A$ -algebra – that is, the quotient of  $B = A[x_1, \dots, x_n]$ . Then  $\Omega_{C/A}^1$  is a finitely generated  $C$ -module.

*Proof.* Set  $B = A[x_1, \dots, x_n]$  for which  $C = B/\mathfrak{b}$ . Exactness of the sequence in Proposition 1.13 gives a surjection  $\Omega_{B/A}^1 \otimes_B C \rightarrow \Omega_{C/A}^1$ , and observing that  $\Omega_{B/A}^1 \otimes_B C \cong B^{\oplus n} \otimes_B C \cong C^{\oplus n}$  gives a surjection  $C^{\oplus n} \rightarrow \Omega_{C/A}^1$ , showing that it is finitely generated.  $\blacksquare$

Let us consider the case of quotients of multivariate polynomial rings by a single polynomial.

**Example 1.15.** Let  $A$  be a ring,  $B = A[x_1, \dots, x_n]$ , and  $C = B/(f)$  for  $f \in B$ . By Proposition 1.13 and Corollary 1.14, we have that  $\Omega_{C/A}^1$  is the cokernel of the map  $\delta : (f)/(f)^2 \rightarrow \Omega_{B/A}^1 \otimes_B C \cong C^{\oplus n}$  of Proposition 1.13, so is the quotient  $(\bigoplus_{i=1}^n C dx_i) / df$ .

We can also consider the case of  $k$ -algebras.

**Corollary 1.16.** Let  $A$  be a  $k$ -algebra and  $\mathfrak{m}$  a maximal ideal in  $A$  such that  $\kappa(\mathfrak{m}) = A/\mathfrak{m} \cong k$ . Then  $\Omega_{A/k}^1 \otimes_k \kappa(\mathfrak{m}) \cong \mathfrak{m}/\mathfrak{m}^2$ .

*Proof.* This is precisely Proposition 1.13 for  $k \rightarrow k[x_1, \dots, x_n] \rightarrow A$ , and the map  $\mathfrak{m}/\mathfrak{m}^2 \rightarrow \Omega_{A/k}^1 \otimes_k \kappa(\mathfrak{m})$  is a surjection between vector spaces of the same dimension, hence an isomorphism.  $\blacksquare$

Note that this is the dual of the Zariski tangent space  $\mathrm{Hom}_{\mathrm{Vec}_{\kappa(\mathfrak{m})}}(\mathfrak{m}/\mathfrak{m}^2, \kappa(\mathfrak{m}))$ , motivating the connection to schemes.

## 2. LECTURE 2 – 10TH APRIL 2025

We begin with an example.

**Example 2.1.** Let  $A = k, B = k[x, y], C = B/\mathfrak{b}$  where  $\mathfrak{b} = (xy)$ . We have that  $\text{Spec}(B)$  is the affine plane  $\mathbb{A}_k^2$  and  $\text{Spec}(C)$  is the union of the two coordinate axes. The exact sequence of Proposition 1.13 gives

$$\mathfrak{b}/\mathfrak{b}^2 \rightarrow \Omega_{k[x,y]/k}^1 \otimes_{k[x,y]} C \rightarrow \Omega_{C/k}^1 \rightarrow 0.$$

Explicitly identifying  $\mathfrak{b}/\mathfrak{b}^2$  with the  $C$ -module  $(xy)/(x^2y^2)$  module-isomorphic to  $C$  by  $1 \mapsto xy$  and  $\Omega_{k[x,y]/k}^1$  with the free  $k[x, y]$ -module  $Bdx \oplus Bdy$ , we observe that the map  $\mathfrak{b}/\mathfrak{b}^2 \rightarrow \Omega_{k[x,y]/k}^1 \otimes_{k[x,y]} C$  is given by  $\overline{xy} \mapsto d(xy) \otimes 1 = (xdy + ydx) \otimes 1$ . This yields a map  $C \rightarrow C \oplus C$  by  $1 \mapsto (y, x)$  and the cokernel of this map is the kernel of the map  $C \oplus C \rightarrow C$  by  $(a, b) \mapsto ax - by$  so by exactness the image of  $C \oplus C \rightarrow C$  is the ideal  $(x, y) \subseteq C$  showing  $\Omega_{C/k}^1 \cong (x, y) \subseteq C$ . Thus  $\Omega_{C/k} \otimes_C \frac{k[x,y]}{(x,y)} \cong kx \oplus ky$  and in particular  $\Omega_{C/k}^1 \otimes_C k$  is of  $k$ -dimension 2. For all points  $\mathfrak{p} \in \text{Spec}(C) \setminus \{(x, y)\}$ , we have  $\Omega_{C/k} \otimes_C \kappa(\mathfrak{p}) \cong k$  extending the exact sequence above to a short exact sequence.

In what follows, we will use the following lemma for Kähler differentials of field extensions, the proof of which we omit.

**Lemma 2.2.** Let  $k$  be a field and  $K/k$  a separable extension. Then  $\Omega_{K/k}^1 = 0$ .

This lemma, in conjunction with Proposition 1.13, shows that for maximal ideals of  $k$ -algebras  $A$  with separable residue field, the base change of the sheaf of Kähler differentials to  $k$  is isomorphic to the Zariski tangent space  $\mathfrak{m}/\mathfrak{m}^2$ .

**Proposition 2.3.** Let  $A$  be a finite type  $k$ -algebra and  $\mathfrak{m} \subseteq A$  maximal with residue field  $\kappa(\mathfrak{m})$  separable over  $k$ . There is an isomorphism  $\frac{\mathfrak{m}}{\mathfrak{m}^2} \rightarrow \Omega_{A/k}^1 \otimes_A k$ .

*Proof.* Let  $\varphi : k[x_1, \dots, x_n] \rightarrow A$  with kernel  $\ker(\varphi) = \mathfrak{a}$  and  $\tilde{\mathfrak{m}} = \varphi^{-1}(\mathfrak{m})$ . This yields a surjective map  $\frac{\tilde{\mathfrak{m}}}{\tilde{\mathfrak{m}}^2} \rightarrow \frac{\mathfrak{m}}{\mathfrak{m}^2}$  with kernel  $\mathfrak{a}$ . By Proposition 1.13 we have the following diagram with bottom row exact

$$\begin{array}{ccccccc} \mathfrak{a} & \longrightarrow & \frac{\tilde{\mathfrak{m}}}{\tilde{\mathfrak{m}}^2} & \longrightarrow & \frac{\mathfrak{m}}{\mathfrak{m}^2} & \longrightarrow & 0 \\ \downarrow & & \downarrow & & \downarrow & & \\ \frac{\mathfrak{a}}{\mathfrak{a}^2} & \longrightarrow & \Omega_{k[x_1, \dots, x_n]/k}^1 \otimes_{k[x_1, \dots, x_n]} k & \longrightarrow & \Omega_{A/k}^1 \otimes_A k & \longrightarrow & 0 \end{array}$$

and noting the top row is the  $-\otimes_{k[x_1, \dots, x_n]} k$  of the bottom, a right-exact operation, we get the claim.  $\blacksquare$

We deduce the following result which will be required for defining the sheaf of Kähler differentials on schemes more generally.

**Corollary 2.4.** Let  $B$  be an  $A$ -algebra and  $I$  the kernel of the map  $B \otimes_A B \rightarrow B$  by  $b_1 \otimes b_2 \mapsto b_1 b_2$ . Then  $\Omega_{B/A}^1 \cong I/I^2$  as  $B$ -modules by  $db \mapsto 1 \otimes b - b \otimes 1$ , and where  $I/I^2$  has the structure of a  $B$ -module by  $b(b_1 \otimes b_2) = bb_1 \otimes b_2$ .



*Proof.* We use the universal property of Kähler differentials. Defining  $\delta$  by  $b \mapsto 1 \otimes b - b \otimes 1$  we get the diagram

$$\begin{array}{ccc} B & \xrightarrow{\delta} & I/I^2 \\ \downarrow & \nearrow \exists! & \\ \Omega_{B/A}^1 & & \end{array}$$

Note for  $b_1, b_2 \in B$ , we compute

$$\delta(b_1 \cdot b_2) = 1 \otimes b_1 b_2 - b_1 b_2 \otimes 1 = b_1 \delta(b_2) + b_2 \delta(b_1) = b_1(1 \otimes b_2 - b_2 \otimes 1) + b_2(1 \otimes b_1 - b_1 \otimes 1).$$

The difference of the two expressions is  $(1 \otimes b_1 - b_1 \otimes 1)(1 \otimes b_2 - b_2 \otimes 1)$  the product of two elements of  $I$ , hence in  $I^2$ , hence vanishes in the quotient. Since this is a derivation, there exists an extension  $\Omega_{B/A}^1 \rightarrow I/I^2$ .

To show surjectivity, we consider an element  $\sum b_i \otimes b'_i \in I$  and compute

$$\begin{aligned} \sum b_i \otimes b'_i &= \sum b_i(1 \otimes b'_i) \\ &= \sum b_i(1 \otimes b'_i) - \underbrace{\left(\sum b_i b'_i\right)}_{=0} \otimes 1 \\ &= \sum b_i(1 \otimes b'_i - b'_i \otimes 1) \end{aligned}$$

showing it is surjective.

To show injectivity, we consider the diagram

$$\begin{array}{ccc} B & \xrightarrow{\delta} & M \\ \downarrow d & \nearrow & \\ \Omega_{B/A}^1 & & \\ \downarrow & \nearrow \text{dotted} & \\ I/I^2 & & \end{array}$$

The existence of a dotted arrow rendering the entire diagram commutative would imply the injectivity of  $\delta$  for  $M = \Omega_{B/A}^1$ . Note that the  $B$ -module  $B \oplus M$  can be given the structure of a free algebra by the map  $b \mapsto b \oplus 0$  and multiplication  $(b_1, m_1) \cdot (b_2, m_2) = (b_1 b_2, b_1 m_2 + b_2 m_1)$ , which defines a  $B$ -algebra in which  $M$  is an ideal with square zero. We can define a map  $\varphi : B \otimes_A B \rightarrow B \oplus M$  by  $b_1 \otimes b_2 \mapsto (b_1 b_2, b_1 \delta(b_2))$  which is a homomorphism of  $A$ -algebras and where the image of the ideal  $I$  is zero since  $M^2$  is zero. Thus an extension  $\psi : I/I^2 \rightarrow M$  and the diagram commutes, yielding injectivity, and hence the claim. ■

We now seek to define the sheaf of Kähler differentials on a scheme.

**Definition 2.5** (Relative Kähler Differentials of a Scheme). Let  $f : X \rightarrow Y$  be a morphism of schemes. The sheaf of Kähler differentials of  $X$  is locally given by the  $\mathcal{I}/\mathcal{I}^2$  of the locally closed embedding  $X \rightarrow X \times_Y X$ .

We will most often be interested in the situation where  $Y = \text{Spec}(k)$  and  $f$  is the structure map of  $f$  as a  $k$ -scheme.

**Example 2.6.** Let  $X = \mathbb{A}_k^n$  over  $\text{Spec}(A)$ .  $\Omega_{X/A}^1$  is free of rank  $n$  as in Example 1.6.

**Example 2.7.** Let  $X = \mathbb{P}_A^n$  over  $\text{Spec}(A)$ .  $\Omega_{X/A}^1$  is the locally free sheaf of rank  $n$  obtained by gluing the free sheaves of Example 2.6 on the distinguished affine opens  $D_+(x_i)$  of  $\mathbb{P}_A^n$ .

The sheaf of Kähler differentials is another example of an interesting sheaf on schemes which is not the structure sheaf. In the case of projective space, the relationship between the sheaf of Kähler differentials is related to the structure sheaf by the Euler sequence.

**Theorem 2.8** (Euler Sequence). Let  $A$  be a ring. There is a short exact sequence of sheaves

$$0 \longrightarrow \Omega_{\mathbb{P}_A^n/A}^1 \longrightarrow \mathcal{O}_{\mathbb{P}_A^n}(-1)^{\oplus n+1} \xrightarrow{(x_0, \dots, x_n)} \mathcal{O}_{\mathbb{P}_A^n} \longrightarrow 0$$

on  $\mathbb{P}_A^n$ .

*Proof.* We have  $X = \text{Proj}(B)$  with  $B = A[x_0, \dots, x_n]$  and  $\mathcal{O}_{\mathbb{P}_A^n}(-1) = \widetilde{B(1)}$ . The map  $\mathcal{O}_{\mathbb{P}_A^n}(-1)^{\oplus n+1} \rightarrow \mathcal{O}_{\mathbb{P}_A^n}$  is given by the module homomorphism given by the dot product map, and is surjective since  $\bigcap_{i=0}^n V_+(x_i) = \emptyset$  and  $- \otimes \mathcal{O}_{\mathbb{P}_A^n}(1)$  being right-exact. We show that the kernel of this map is  $\Omega_{\mathbb{P}_A^n/A}^1$  affine-locally.

On  $D_+(x_i)$ , the map is given by localizations  $B(-1)_{(x_i)}^{\oplus n+1} \rightarrow B_{(x_i)}$  and the kernel is free of rank  $n$  generated by  $e_j - \frac{x_j}{x_i}e_i$  for  $j \neq i$ . In particular, the kernel is a free  $\mathcal{O}_{D_+(x_i)}$ -module generated by  $\frac{1}{x_i}e_j - \frac{x_j}{x_i^2}e_i$  for  $j \neq i$ . Recall that  $\Omega_{\mathbb{P}_A^n/A}^1$  is the free  $\mathcal{O}_{D_+(x_i)}$ -module spanned by  $d(\frac{x_0}{x_i}), \dots, d(\frac{x_n}{x_i})$  and the isomorphism to the kernel of the map is given by  $d(\frac{x_j}{x_i}) \mapsto \frac{1}{x_i}e_j - \frac{x_j}{x_i^2}e_i$ . The isomorphisms glue by inspection, giving the isomorphism of sheaves. ■

**Example 2.9.** Let  $n = 1$ . The Euler sequence gives  $0 \rightarrow \Omega_{\mathbb{P}_A^1/A}^1 \rightarrow \mathcal{O}_{\mathbb{P}_A^1}(-1)^{\oplus 2} \rightarrow \mathcal{O}_{\mathbb{P}_A^1} \rightarrow 0$  where passing to determinants gives  $\det(\mathcal{O}_{\mathbb{P}_A^1}(-1)^{\oplus 2}) \cong \det(\mathcal{O}_{\mathbb{P}_A^1}) \otimes \det(\Omega_{\mathbb{P}_A^1/A}^1)$  showing  $\Omega_{\mathbb{P}_A^1/A}^1 \cong \mathcal{O}_{\mathbb{P}_A^1}(-2)$ .

**Example 2.10.** For  $n > 1$   $\Omega_{\mathbb{P}_A^n/A}^1$  is never a direct sum of line bundles. We have  $\det(\Omega_{\mathbb{P}_A^n/A}^1) \cong \mathcal{O}_{\mathbb{P}_A^n}(-n-1)$ . Twisting by  $\mathcal{O}_{\mathbb{P}_A^n}(1)$ , we get a short exact sequence

$$0 \rightarrow H^0(\mathbb{P}_A^n, \Omega_{\mathbb{P}_A^n/A}^1) \rightarrow H^0(\mathbb{P}_A^n, \mathcal{O}_{\mathbb{P}_A^n}^{\oplus n+1}) \rightarrow H^0(\mathbb{P}_A^n, \mathcal{O}_{\mathbb{P}_A^n}(1)) \rightarrow 0$$

since  $H^1$  of all the sheaves in the Euler sequence vanishes. If  $\Omega_{\mathbb{P}_A^n/A}^1 \cong \bigoplus \mathcal{O}_{\mathbb{P}_A^n}(a_i)$  then  $a_i \leq -2$  and  $\sum a_i \leq -2n$  which has global sections, a contradiction for  $n \geq 2$ .

## 3. LECTURE 3 – 14TH APRIL 2025

Recall that the cotangent sheaf on  $\mathbb{A}_A^n, \mathbb{P}_A^n$  over  $\text{Spec}(A)$  are locally free sheaves by Examples 2.6 and 2.7. One is then led to consider for what other  $S$ -schemes  $X$  is  $\Omega_{X/S}^1$  locally free. This is roughly captured by smoothness. Moreover, as suggested by Proposition 2.3, the notion of smoothness is connected to the Zariski tangent space, which in the case of algebraic geometry – unlike differential geometry – need not coincide with the geometric tangent space, especially in characteristic  $p$  situations.

We recall the definition of the Zariski tangent space.

**Definition 3.1** (Zariski Tangent Space of a Ring). Let  $A$  be a local ring with maximal ideal  $\mathfrak{m}$  and residue field  $\kappa = A/\mathfrak{m}$ . The Zariski tangent space of  $A$  is the  $\kappa$ -vector space  $(\frac{\mathfrak{m}}{\mathfrak{m}^2})^\vee = \text{Hom}_{\text{Vect}_\kappa}(\frac{\mathfrak{m}}{\mathfrak{m}^2}, \kappa)$ .

**Definition 3.2** (Zariski Tangent Space of a Scheme). Let  $X$  be a scheme and  $x \in X$  a point. The Zariski tangent space  $T_{X,x}$  is the Zariski tangent space of the local ring  $\mathcal{O}_{X,x}$ .

**Example 3.3.** Let  $A$  be a ring and  $\mathfrak{p} \subseteq \text{Spec}(A)$ . The Zariski tangent space  $T_{\text{Spec}(A), \mathfrak{p}}$  is given by  $(\frac{\mathfrak{p}A_{\mathfrak{p}}}{(\mathfrak{p}A_{\mathfrak{p}})^2})^\vee$ . This is a vector space over the field  $\kappa(\mathfrak{p}) = A_{\mathfrak{p}}/\mathfrak{p}A_{\mathfrak{p}}$ .

**Example 3.4.** Let  $k$  be a field and  $x \in \mathbb{A}_k^n(k)$  a closed  $k$ -rational point hence of the form  $(x_1 - a_1, \dots, x_n - a_n)$ . Define a map  $D_x : k[x_1, \dots, x_n] \rightarrow \text{Hom}_{\text{Vect}_k}(k^n, k)$  by

$$f \mapsto \left[ (\alpha_i)_{i=1}^n \mapsto \sum_{i=1}^n \alpha_i \frac{\partial f}{\partial x_i}(x) \right].$$

The map is  $k$ -linear and satisfies the Leibniz rule, hence defines a  $k$ -linear derivation which is a  $k$ -vector space. This defines an isomorphism between  $(\frac{\mathfrak{m}}{\mathfrak{m}^2})^\vee$  and  $\text{Hom}_{\text{Vect}_k}(k^n, k)$  by considering the Taylor expansion of a polynomial

$$f = f(x) + \sum_{i=1}^n \frac{\partial f}{\partial x_i}(x)(x_i - a_i) + \underbrace{O(x^2)}_{\in \mathfrak{m}^2}$$

hence the map is zero on  $f \in \mathfrak{m}^2$  showing that  $(\frac{\mathfrak{m}}{\mathfrak{m}^2})^\vee \rightarrow \text{Hom}_{\text{Vect}_k}(k^n, k)$  by  $(x_i - a_i) \mapsto e_i^\vee$  is an injection between vector spaces of the same dimension and hence an isomorphism.

**Example 3.5.** In general, one can still define a map on non-rational points with target  $\text{Hom}_{\text{Vect}_{\kappa(\mathfrak{p})}}(\kappa(\mathfrak{p})^n, \kappa(\mathfrak{p}))$  which may fail to be injective. Let  $k$  be a field of characteristic  $p$  and consider  $(x^p - a) \subseteq k[x]$  which is maximal when  $a^{1/p} \notin k$ . We have  $\frac{\mathfrak{m}}{\mathfrak{m}^2} = \frac{(x^p - a)}{(x^p - a)^2} \cong k$  which defines a map  $\text{Hom}_{\text{Vect}_k}(k, k)$  by Example 3.4 which is the zero map as  $px^{p-1} = 0$ .

In what follows, we will use the following result for closed subschemes of affine spaces.

**Proposition 3.6.** Let  $\mathfrak{a} \subseteq k[x_1, \dots, x_n]$  be an ideal defining  $X = V(\mathfrak{a}) \subseteq \mathbb{A}_k^n$ . Let  $x \in X(k) \subseteq \mathbb{A}_k^n(k)$ . Then  $T_{X,x}$  is the annihilator of the image of  $\mathfrak{a}$  under  $D_x$

*Proof.* We have a short exact sequence

$$0 \rightarrow \mathfrak{a} \rightarrow \tilde{\mathfrak{m}} \rightarrow \mathfrak{m} \rightarrow 0$$

inducing

$$(3.1) \quad 0 \rightarrow \frac{\mathfrak{a}}{\mathfrak{a} \cap \tilde{\mathfrak{m}}^2} \rightarrow \frac{\tilde{\mathfrak{m}}}{\tilde{\mathfrak{m}}^2} \rightarrow \frac{\mathfrak{m}}{\mathfrak{m}^2} \rightarrow 0.$$

Applying the right-exact functor  $\text{Hom}_{\text{Vect}_k}(-, k)$  we get

$$\left( \frac{\mathfrak{a}}{\mathfrak{a} \cap \tilde{\mathfrak{m}}^2} \right)^\vee \rightarrow T_{\mathbb{A}_k^n, x}^\vee \rightarrow T_{X, x}^\vee \rightarrow 0$$

where the map  $\left( \frac{\mathfrak{a}}{\mathfrak{a} \cap \tilde{\mathfrak{m}}^2} \right)^\vee \rightarrow \text{Hom}_{\text{Vect}_k}(k^n, k)$  by taking  $\mathfrak{a}$ -derivations as in Example 3.4.  $\blacksquare$

An analogous proof can be used to show that the Zariski tangent space is the cokernel of the Jacobian matrix.

**Corollary 3.7.** Let  $\mathfrak{a} \subseteq k[x_1, \dots, x_n]$  be an ideal defining  $X = V(\mathfrak{a}) \subseteq \mathbb{A}_k^n$ . Let  $x \in X(k) \subseteq \mathbb{A}_k^n(k)$ . Then  $T_{X, x}^\vee \cong \text{coker}(J_x)$  where  $J_x$  is the Jacobian at  $x$ .

*Proof.* We use the short exact sequence (3.1) and observe that the map  $\frac{\mathfrak{a}}{\mathfrak{a} \cap \tilde{\mathfrak{m}}^2} \rightarrow \frac{\tilde{\mathfrak{m}}}{\tilde{\mathfrak{m}}^2}$  is given by multiplication by the Jacobian, giving the claim.  $\blacksquare$

Having related this to the Zariski tangent space, we want to relate the Jacobian matrix to the sheaf/module of Kähler differentials, an analogy suggested by Proposition 2.3.

**Proposition 3.8.** Let  $\mathfrak{a} \subseteq k[x_1, \dots, x_n]$  be an ideal defining  $X = V(\mathfrak{a}) \subseteq \mathbb{A}_k^n$ . Let  $x \in X(k) \subseteq \mathbb{A}_k^n(k)$ . Then the corank of the Jacobian  $J_x$  is equal to  $\dim_{\kappa(x)} \Omega_{X/k}^1 \otimes \kappa(x)$ .

*Proof.* Applying  $-\otimes \kappa(x)$  to the short exact sequence of Proposition 1.13, we have

$$\frac{\mathfrak{a}}{\mathfrak{a}^2} \otimes \kappa(x) \rightarrow \Omega_{\mathbb{A}_k^n/k}^1 \otimes \kappa(x) \rightarrow \Omega_{X/k}^1 \otimes \kappa(x) \rightarrow 0$$

which factors over the image of the Jacobian  $J_x$ . As such, we get that  $\dim_{\kappa(x)} \Omega_{X/k}^1 = n - \dim(\text{im}(J_x)) = n - \text{rank}(J_x)$  which is precisely the corank.  $\blacksquare$

Moreover, for a general point  $x$ , the property of the Zariski tangent space being isomorphic to the scalar extension of the sheaf of Kähler differentials.

**Proposition 3.9.** Let  $\mathfrak{a} \subseteq k[x_1, \dots, x_n]$  be an ideal defining  $X = V(\mathfrak{a}) \subseteq \mathbb{A}_k^n$ . Let  $x \in X$ .  $\kappa(x)$  is a separable extension of  $k$  if and only if  $T_{X, x}^\vee \cong \Omega_{X/k}^1 \otimes \kappa(x)$ .

*Proof.* ( $\Rightarrow$ ) If  $\kappa(x)$  is separable over  $k$ , then  $\Omega_{\kappa(x)/k}^1 = 0$  by Lemma 2.2 so  $\frac{\mathfrak{m}}{\mathfrak{m}^2} \rightarrow \Omega_{X, k}^1 \otimes \kappa(x)$  is surjective, but this shows that we have a surjection of  $\kappa(x)$ -vector spaces of the same dimension, hence an isomorphism.

( $\Leftarrow$ ) If the Zariski cotangent space is isomorphic to the sheaf of differentials, then the cokernel  $\Omega_{\kappa(x)/k}^1$  of the exact sequence Proposition 1.13 is zero, showing that  $\kappa(x)/k$  is separable.  $\blacksquare$

Note that the equality  $\dim(T_{X,x}) = \dim_{\kappa(x)} \Omega_{X/k}^1 \otimes \kappa(x)$  does not imply the natural map is an isomorphism when  $\kappa(x)$  is not separable over  $k$ .

**Example 3.10.** Let  $k$  be a field of characteristic  $p$  and consider  $(x^p - a) \subseteq k[x]$  for  $a^{1/p} \notin k$ . Denoting  $X = V(x^p - a) \subseteq \mathbb{A}_k^1$ , we have  $\dim(T_{X,x}) = 1 = \dim_{\kappa(x)} \Omega_{X/k}^1 \otimes \kappa(x)$  but  $\Omega_{\kappa(x)/k}^1$  is nonzero as  $\kappa(x)$  is not a separable extension of  $k$ .

We arrive at the notion of smoothness for schemes.

**Definition 3.11** (Smooth Scheme). Let  $X$  be a scheme of finite type over a field  $k$ .  $X$  is smooth of pure dimension  $d$  if

- (i) each of the finitely many irreducible components of  $X$  are of dimension  $d$ , and
- (ii) every point  $x \in X$  is contained in an affine open neighborhood where the Jacobian matrix is of corank  $d$ .

**Remark 3.12.** Smoothness is a relative notion, determined by the structure map to  $\text{Spec}(k)$ .

**Remark 3.13.** By Example 3.3, this construction is independent of the choice of chart.

**Remark 3.14.** It suffices to verify this condition on closed points, as if the Jacobian is rank-deficient at some non-closed point, then it is rank-deficient at any specialization.

Intuitively, we can view  $X$  locally as the fiber of a map  $\mathbb{A}_k^n \rightarrow \mathbb{A}_k^r$  defined by the  $r$  polynomials  $f_1, \dots, f_r \in k[x_1, \dots, x_n]$ , and where  $x$  being in the fiber over zero implies that the tangent space  $T_{X,x}$  is the kernel of the map  $(f_1, \dots, f_r)$  hence equal to the dimension of the fiber. We introduce the notion of being geometrically smooth.

**Definition 3.15** (Geometrically Smooth Scheme). Let  $X$  be a scheme of finite type over a field  $k$ .  $X$  is geometrically smooth if the base change  $X_{\bar{k}}$  to the algebraic closure is smooth over  $\bar{k}$ .

This is in fact equivalent to the condition of being smooth.

**Lemma 3.16.** Let  $X$  be a scheme of finite type over a field  $k$ .  $X$  is a smooth  $k$ -scheme if and only if it is geometrically smooth.

*Proof.* We use the Cartesian square

$$\begin{array}{ccc} X_{\bar{k}} & \longrightarrow & X \\ \downarrow & & \downarrow \\ \text{Spec}(\bar{k}) & \longrightarrow & \text{Spec}(k) \end{array}$$

where using Proposition 1.8, we have  $\Omega_{X/k}^1 \otimes \kappa(x) \cong \Omega_{X_{\bar{k}}/\bar{k}}^1 \otimes \kappa(y)$  where  $y$  is the closed point corresponding to  $x$  in  $X_{\bar{k}}$ . This isomorphism of sheaves characterizes the Jacobian being full rank at  $x, y$ , hence the smoothness conditions are equivalent. ■

While smoothness depends on the structure of  $X$  as a  $k$ -scheme, it is closely related to the absolute notion of regularity.

We recall the relevant definitions from commutative algebra.

**Definition 3.17** (Regular Local Ring). Let  $(A, \mathfrak{m})$  be a Noetherian local ring with residue field  $\kappa = A/\mathfrak{m}$ .  $A$  is a regular local ring if  $\dim(A) = \dim_{\kappa}(\frac{\mathfrak{m}}{\mathfrak{m}^2})$ .

**Definition 3.18** (Regular Ring). Let  $A$  be a Noetherian ring.  $A$  is a regular ring if for all primes  $\mathfrak{p} \subseteq A$ , the localization  $A_{\mathfrak{p}}$  is a regular local ring.

**Remark 3.19.** Checking regularity of an arbitrary Noetherian ring can be done on maximal ideals by reasoning analogous to that of Remark 3.14.

This allows us to define regularity of schemes.

**Definition 3.20** (Regular Scheme). Let  $X$  be a locally Noetherian scheme.  $X$  is regular if for all closed points  $x \in X$ , the local ring  $\mathcal{O}_{X,x}$  is regular.

**Remark 3.21.** By Definition 3.18, this is equivalent to each point admitting an affine neighborhood given by the Zariski spectrum of a regular ring.

**Remark 3.22.** In contrast to Remark 3.12, regularity is absolute and does not depend on any structure map of  $X$ .

The notions of regularity and smoothness are connected by the following proposition.

**Proposition 3.23.** Let  $X$  be a scheme of finite type over an algebraically closed field  $k$ .  $X$  is  $k$ -smooth if and only if  $X$  is regular.

*Proof.* Both conditions can be checked affine-locally, so without loss of generality, we can take  $X = V(\mathfrak{a}) \subseteq \mathbb{A}_k^n$ . By Proposition 3.6 the dimension of the Zariski tangent space of any  $x \in X$  is dimension of the image of the map  $D_x$  defined in Example 3.4, which is equal to the rank of the Jacobian  $J_x$ . This is of rank equal to the Zariski tangent space (ie.  $X$  is regular) if and only if the corank of the Jacobian is  $\dim(X)$  (ie.  $X$  is smooth). ■

Over general fields, smoothness implies regularity, but not the converse.

**Corollary 3.24.** Let  $X$  be a scheme of finite type over a field  $k$ . If  $X$  is  $k$ -smooth, then  $X$  is regular.

*Proof.* By locality, we reduce once more to  $X = V(\mathfrak{a}) \subseteq \mathbb{A}_k^n$ . By smoothness,  $D_x(\mathfrak{a}) = \text{rank}(J_x)$  and by the short exact sequence (3.1) we have

$$\dim_{\kappa(\mathfrak{m})} \left( \frac{\mathfrak{m}}{\mathfrak{m}^2} \right) = \dim_{\kappa(\mathfrak{m})} \left( \frac{\tilde{\mathfrak{m}}}{\tilde{\mathfrak{m}}^2} \right) - \dim_{\mathfrak{m}} \left( \frac{\mathfrak{a}}{\mathfrak{a} \cap \tilde{\mathfrak{m}}^2} \right)$$

showing that the dimension of the Zariski tangent space at  $x$  is equal to the dimension of  $X$ , hence  $X$  is regular. ■

We now see an example of a regular non-smooth scheme.

**Example 3.25.** Let  $k$  be a field of characteristic  $p$  and consider  $X = V(x^p - a) \subseteq \mathbb{A}_k^1$  and where  $a^{1/p} \notin k$ .  $\text{Spec}(\frac{k[x]}{(x^p - a)})$  is the Zariski spectrum of a field, hence regular, but  $X$  is not geometrically smooth and hence not smooth.

## 4. LECTURE 4 – 17TH APRIL 2025

We prove another characterization of smoothness via freeness of the sheaf of Kähler differentials using the following lemma from commutative algebra, the proof of which we omit.

**Lemma 4.1.** Let  $A$  be a Noetherian integral local ring with fraction field  $K$  and residue field  $\kappa$ . If  $M$  is a finite  $A$ -module then  $M$  is free if and only if  $\dim_K(M \otimes_A K) = \dim_\kappa(M \otimes_A \kappa)$ .

We now state and prove the proposition.

**Proposition 4.2.** Let  $X$  be locally of finite type over a field  $k$  of pure dimension  $d$ .  $X$  is  $k$ -smooth if and only if  $\Omega_{X/k}^1$  is locally free of dimension  $d$ .

*Proof.* By Lemma 3.16 and Proposition 3.23,  $X$  being  $k$ -smooth is equivalent to  $X$  being geometrically regular – ie.  $X_{\bar{k}}$  being regular.

It thus suffices to show that that all closed points  $y \in X_{\bar{k}}(\bar{k})$   $T_{X_{\bar{k}},y} = \dim(X)$ . But  $X$  being regular and Lemma 4.1 show that this condition holds as

$$\dim(X) = \text{trdeg}(K(X_{\bar{k}})) = \dim(\Omega_{X_{\bar{k}}/\bar{k}}^1 \otimes \kappa(\eta)) = \dim(\Omega_{X_{\bar{k}}/\bar{k}}^1 \otimes \kappa(y)) = \dim(T_{X_{\bar{k}},y})$$

as desired. ■

**Corollary 4.3.** Let  $X$  be a finite type integral  $k$ -scheme.  $X$  is smooth if and only if  $\Omega_{X/k}^1$  is locally free and  $\kappa(x)/k$  is separable for all closed points  $x \in X$ .

*Proof.* ( $\Rightarrow$ ) By Proposition 4.2,  $\Omega_{X/k}^1$  is locally free of rank the dimension of  $X$  so  $\dim_{K(X)}(\Omega_{K(X)/k}) = \dim(X)$  showing that the residue field of each closed point is separable.

( $\Leftarrow$ ) Using Proposition 4.2 the conclusion is immediate. ■

**Corollary 4.4.** Let  $X$  be a finite type  $k$ -scheme where  $k$  is a perfect field.  $X$  is smooth if and only if  $X$  is regular.

*Proof.* ( $\Rightarrow$ ) Smoothness implies regularity in general by Corollary 3.24.

( $\Leftarrow$ ) Let  $X$  be regular. Then  $X$  is locally integral. Smoothness is local on source so without loss of generality  $X = \text{Spec}(A)$  is integral and  $x \in \text{Spec}(A)$  maximal.  $\kappa(x)/k$  is finite and separable as  $k$  was perfect. So  $\Omega_{\kappa(x)/k}^1 = 0$  and  $\frac{\mathfrak{m}}{\mathfrak{m}^2} \cong \Omega_{X/k}^1 \otimes \kappa(x)$ . Applying Lemma 4.1 to  $\Omega_{A/k}^1$ , we have that the module of differentials is locally free, hence the claim. ■

We show that smoothness is in fact true on a nonempty open, and hence dense, subset of an integral scheme.

**Proposition 4.5.** Let  $X$  be an integral finite type  $k$  scheme and  $K(X)/k$  separable. There exists a nonempty open set  $U$  such that  $U$  is  $k$ -smooth.

*Proof.*  $\kappa(x)/k$  is separable if and only if  $\dim \Omega_{\kappa(x)/k} = \dim(X)$ . Recall that for  $X$  Noetherian and  $\mathcal{F}$  coherent, the function  $x \mapsto \dim_{\kappa(x)}(\mathcal{F} \otimes_{\mathcal{O}_{X,x}} \kappa(x))$  is uppersemi-continuous. Denote

$$Z_n = \{x \in X : \dim_{\kappa(x)}(\Omega_{X/k}^1 \otimes \kappa(x))\}$$



which is closed.

Let  $U = X \setminus Z_{\dim(X)+1}$  which is open and nonempty as it contains the generic point. We have  $\Omega_{U/k}^1 = \Omega_{X/k}^1|_U$  which is constant rank at each stalk, hence locally free, showing that  $U$  is smooth. ■

In particular this implies that the codimension of the non-smooth locus is at least 2 for  $X$  normal and  $K(X)/k$  separable.

We consider the following example where the extension is not separable, and the smooth locus is empty.

**Example 4.6.** Let  $(x^p - a) \subseteq k[x]$  where  $k$  is a field of characteristic  $p$  and  $a^{1/p} \notin k$ . The quotient  $k[x]/(x^p - a)$  is a field but not separable over  $k$  and the smooth locus is empty.

We now discuss smoothness of subschemes.

**Proposition 4.7.** Let  $X$  be of finite type over a field  $k$  and  $Y \subseteq X$  a closed irreducible subscheme.  $Y$  is  $k$ -smooth if and only if  $\Omega_{Y/k}^1$  is locally free and the exact sequence  $\mathcal{I}_Y/\mathcal{I}_Y^2 \rightarrow \Omega_{X/k}^1|_Y \rightarrow \Omega_{Y/k}^1 \rightarrow 0$  extends on the left to a short exact sequence.

*Proof.* ( $\Rightarrow$ ) Consider

$$\begin{array}{ccccccc}
 \mathcal{I}_Y/\mathcal{I}_Y^2 & \longrightarrow & \Omega_{X/k}^1|_Y & \longrightarrow & \Omega_{Y/k}^1 & \longrightarrow & 0 \\
 & \searrow & \nearrow & & & & \\
 & & K & & & & \\
 & \nearrow & & & & & \\
 0 & & & & & & 
 \end{array}$$

where  $0 \rightarrow K \rightarrow \Omega_{X/k}^1|_Y \rightarrow \Omega_{Y/k}^1 \rightarrow 0$  is a short exact sequence. Observe  $K$  is a locally free sheaf of rank equal to the codimension of  $Y$  in  $X$ . In particular, there are sections  $s_1, \dots, s_c$  whose images  $ds_1, \dots, ds_c$  generate  $\Omega_{X/k}^1|_Y$  which locally freely generate  $K$ . Let  $\mathcal{I}$  be the ideal generated by these sections  $s_1, \dots, s_c$  and  $Y' = V(\mathcal{I})$  the associated vanishing locus which contains  $Y$ . Arguing similarly, we get

$$\begin{array}{ccccccc}
 \mathcal{I}'/\mathcal{I}'^2 & \longrightarrow & \Omega_{X/k}^1|_{Y'} & \longrightarrow & \Omega_{Y'/k}^1 & \longrightarrow & 0 \\
 & \searrow & \nearrow & & & & \\
 & & K' & & & & \\
 & \nearrow & & & & & \\
 0 & & & & & & 
 \end{array}$$

with the short exact sequence  $0 \rightarrow K' \rightarrow \Omega_{X/k}^1|_{Y'} \rightarrow \Omega_{Y'/k}^1 \rightarrow 0$ . Since  $ds_1, \dots, ds_c$  generate freely a subsheaf of  $\Omega_{X/k}^1|_Y$ ,  $ds_1|_{Y'}, \dots, ds_c|_{Y'}$  generate freely a subsheaf of  $\Omega_{X/k}^1|_{Y'}$  hence  $\mathcal{I}'/\mathcal{I}'^2 \rightarrow K'$  is an isomorphism. By construction,  $K'|_Y = K$  so

$\Omega_{Y'/k}^1|_Y \cong \Omega_{Y/k}^1$  the latter being locally free, so  $\Omega_{Y'/k}^1$  is locally free in a neighborhood of  $Y$ .  $Y'$  is smooth of dimension equal to  $Y$  and smoothness implies local integrability so  $\Omega_{Y/k}^1$  is locally free, yielding the claim.

( $\Leftarrow$ ) It suffices to argue that the existence of a short exact sequence  $0 \rightarrow \mathcal{I}_Y/\mathcal{I}_Y^2 \rightarrow \Omega_{X/k}^1|_Y \rightarrow \Omega_{Y/k}^1 \rightarrow 0$  with  $\Omega_{Y/k}^1$  locally free implies separability of  $K(X)/k$ . Note that under the hypotheses, the latter two terms of the short exact sequence are locally free, giving a splitting and showing in particular that  $\mathcal{I}_Y/\mathcal{I}_Y^2$  is locally free of rank  $r = \dim(X) - \text{rank}(\Omega_{Y/k}^1)$ . In particular, we have  $\dim(X) - r = \text{rank}(\Omega_{Y/k}^1) \geq \dim(Y)$  and by  $\mathcal{I}_Y$  being locally generated by  $r$  elements that  $\dim(Y) \geq \dim(X) - r$  by Nakayama's lemma. This gives equality  $\Omega_{Y/k}^1 = \dim(Y)$  showing  $Y$  is smooth. ■

The preceding discussion of closed subschemes motivates the following definition of locally complete intersections.

**Definition 4.8** (Locally Complete Intersection). Let  $X$  be a smooth  $k$  scheme locally of finite type and  $Y \subseteq X$  a closed subscheme.  $Y$  is a local complete intersection if  $\mathcal{I}_Y$  is locally generated by  $c$  elements, where  $c = \text{codim}_X(Y) = \dim(X) - \dim(Y)$ .

Smooth closed subschemes are in particular local complete intersections.

**Example 4.9.** Let  $X$  be a smooth  $k$  scheme locally of finite type and  $Y \subseteq X$  a closed subscheme. If  $Y$  is smooth then  $Y$  is local complete intersection. Using Proposition 4.7, the short exact sequence

$$0 \rightarrow \mathcal{I}_Y/\mathcal{I}_Y^2 \rightarrow \Omega_{X/k}^1|_Y \rightarrow \Omega_{Y/k}^1 \rightarrow 0$$

gives that  $\mathcal{I}_Y/\mathcal{I}_Y^2$  is locally free of rank the codimension of  $Y$  in  $X$ .

**Example 4.10.** Being a local complete intersection does not imply reducedness or integrality. For  $Y = V(x^2) \subseteq \mathbb{A}_k^1 = \text{Spec}(k[x])$  the ideal sheaf is generated by one element, but  $k[x]/(x^2)$  is not reduced.

As suggested by the prevalence of quotients of ideal sheaves  $\mathcal{I}_Y/\mathcal{I}_Y^2$  in the preceding discussion, this plays an important role in the study of smoothness of schemes.

**Definition 4.11** (Conormal Sheaf). Let  $X$  be a smooth  $k$ -scheme locally of finite type and  $Y \subseteq X$  a local complete intersection closed subscheme. The conormal bundle  $\mathcal{N}_{Y/X}^\vee$  is the sheaf  $\mathcal{I}_Y/\mathcal{I}_Y^2$ .

Dualizing yields the normal sheaf.

**Definition 4.12** (Normal Sheaf). Let  $X$  be a smooth  $k$ -scheme locally of finite type and  $Y \subseteq X$  a local complete intersection closed subscheme. The normal bundle  $\mathcal{N}_{Y/X}$  is the dual of the conormal sheaf  $\underline{\text{Hom}}_{\text{Mod}_{\mathcal{O}_Y}}(\mathcal{I}_Y/\mathcal{I}_Y^2, \mathcal{O}_Y)$ .

This gives the conormal bundle sequence.

**Proposition 4.13** (Conormal Bundle Sequence). Let  $X$  be a smooth  $k$ -scheme locally of finite type and  $Y \subseteq X$  a local complete intersection closed subscheme. There is a short exact sequence

$$0 \rightarrow \mathcal{N}_{Y/X}^\vee \rightarrow \Omega_{X/k}^1|_Y \rightarrow \Omega_{Y/k}^1 \rightarrow 0$$

of sheaves on  $Y$ .

*Proof.* This is immediate from Proposition 4.7. ■

Dual to the conormal and normal bundles, we can define tangent bundles.

**Definition 4.14** (Tangent Bundle). Let  $X$  be a smooth  $k$ -scheme locally of finite type. The tangent bundle  $\mathcal{T}_X$  is  $\underline{\text{Hom}}_{\text{Mod}_{\mathcal{O}_X}}(\Omega_{X/k}^1, \mathcal{O}_X)$ .

In particular, we record the following observation.

**Example 4.15.** Let  $X$  be a smooth scheme.  $(\mathcal{T}_X)_x = T_{X,x}$  the former being the tangent bundle and the latter the Zariski tangent space.

Recalling that dualizing preserves exactness, we have the normal bundle sequence.

**Proposition 4.16** (Normal Bundle Sequence). Let  $X$  be a smooth  $k$ -scheme locally of finite type and  $Y \subseteq X$  a smooth closed subscheme. There is a short exact sequence

$$0 \rightarrow \mathcal{T}_Y \rightarrow \mathcal{T}_X|_Y \rightarrow \mathcal{N}_{Y/X} \rightarrow 0$$

of sheaves on  $Y$ .

*Proof.* This is obtained by dualizing Proposition 4.13. ■

The construction of the sheaf of Kähler differentials in fact is closely connected to Serre duality, as alluded to in the lectures of the preceding semester.

**Definition 4.17** (Canonical Bundle). Let  $X$  be a smooth  $k$ -scheme locally of finite type of pure dimension  $d$ . The canonical bundle of  $X$ ,  $\omega_{X/k}$  is  $\Omega_{X/k}^d = \bigwedge^d \Omega_{X/k}^1$ .

The normal bundle sequence immediately gives the relationship between the canonical bundle of a smooth scheme and a smooth subscheme.

**Proposition 4.18** (Adjunction Formula). Let  $X$  be a smooth  $k$ -scheme locally of finite type and  $Y \subseteq X$  a smooth closed subscheme. Then

$$\omega_{Y/k} \cong \omega_{X/k}|_Y \otimes \det(\mathcal{N}_{Y/X}).$$

*Proof.* Note that for  $0 \rightarrow \mathcal{F}' \rightarrow \mathcal{F} \rightarrow \mathcal{F}'' \rightarrow 0$  a short exact sequence, we have  $\det(\mathcal{F}) \cong \det(\mathcal{F}') \otimes \det(\mathcal{F}'')$ , in which case the statement follows from the conormal bundle sequence Proposition 4.13. ■

**Example 4.19.** Let  $X$  be a smooth  $k$  scheme,  $\mathcal{L} \in \text{Pic}(X)$ , and  $s \in H^0(X, \mathcal{L})$ .  $Y = (s)_0$  is a codimension 1 subscheme and there is an injective map  $\mathcal{O}_X \xrightarrow{s} \mathcal{L}$ . Defining the dual map  $s^\vee : \mathcal{L}^\vee \rightarrow \mathcal{O}_X$  we can compose it with the surjection  $\mathcal{O}_X \rightarrow \mathcal{O}_Y$  to observe that  $\mathcal{I}_Y \cong \mathcal{L}^\vee$ . Tensoring the short exact sequence  $0 \rightarrow \mathcal{L}^\vee \rightarrow \mathcal{O}_X \rightarrow \mathcal{O}_Y \rightarrow 0$  by  $\mathcal{L}^\vee$  yields a short exact sequence  $0 \rightarrow \mathcal{L}^{\otimes -2} \rightarrow \mathcal{L}^\vee \rightarrow \mathcal{L}^\vee|_Y \rightarrow 0$  where  $\mathcal{I}_Y/\mathcal{I}_Y^2 \cong \mathcal{L}^\vee|_Y$  implying  $\mathcal{N}_{Y/X} \cong \mathcal{L}|_Y$ . Now applying the normal bundle sequence,  $\omega_{Y/k} \cong \omega_{X/k} \otimes \mathcal{L}|_Y \cong (\omega_{X/k} \otimes \mathcal{L})|_Y$ .

**Example 4.20.** Let  $X = \mathbb{P}_k^n$  and  $Y = V_+(f)$  for  $f \in \mathcal{O}_{\mathbb{P}_k^n}(d)$  for some  $d \geq 0$ . Recall that  $\omega_{X/k} = \mathcal{O}_{\mathbb{P}_k^n}(-n-1)$ . So  $\omega_{Y/k} = \mathcal{O}_{\mathbb{P}_k^n}(d-n-1)|_Y$ .

## 5. LECTURE 5 – 24TH APRIL 2025

We begin a discussion of flatness, which intuitively corresponds to varying “nicely” in a family over the base. As always, we first define these in the local case.

**Definition 5.1** (Flat Module). Let  $A$  be a ring and  $M$  an  $A$ -module.  $M$  is a flat  $A$ -module if  $- \otimes_A M$  is an exact functor.

Recall that for  $A$ -modules  $M$ ,  $- \otimes_A M$  is always a right exact functor so flatness is equivalent to being left exact, taking short exact sequences

$$0 \rightarrow N_1 \rightarrow N_2 \rightarrow N_3 \rightarrow 0$$

to short exact sequences

$$0 \rightarrow N_1 \otimes_A M \rightarrow N_2 \otimes_A M \rightarrow N_3 \otimes_A M \rightarrow 0.$$

In fact, it suffices to verify that for all ideals  $\mathfrak{a} \subseteq A$  that  $\mathfrak{a} \otimes_A M \rightarrow A \otimes_A M$  is injective.

We can apply this to the special case where an  $A$ -algebra  $B$  is considered as an  $A$ -module.

**Definition 5.2** (Flat Algebra). Let  $A$  be a ring and  $B$  an  $A$ -algebra.  $B$  is  $A$ -flat if  $B$  is flat as an  $A$ -module.

Let us recall some further properties of flatness.

**Proposition 5.3.** Let  $A$  be a ring.

- (i) If  $A$  is a local ring and  $M$  is a finite  $A$ -module, then  $M$  is flat if and only if  $M$  is free.
- (ii) If  $S \subseteq A$  is a multiplicative subset,  $A \rightarrow S^{-1}A$  is flat.
- (iii) If  $A \rightarrow B$  is a ring map and  $M$  is a flat  $A$ -module then  $M \otimes_A B$  is a flat  $B$ -module.
- (iv) If  $A \rightarrow B$  is a flat ring map and  $N$  is a flat  $B$ -module then its restriction of scalars  $N|_A$  is a flat  $A$ -module.
- (v) Let  $M$  be an  $A$ -module.  $M$  is flat if and only if  $M_{\mathfrak{p}}$  is a flat  $A_{\mathfrak{p}}$ -module for all prime ideals (maximal ideals).
- (vi) Let  $A \rightarrow B$  be a flat ring map between Noetherian local rings, and  $b \in B$  such that  $\bar{b} \in B/\mathfrak{m}_A B$  is a non-zero-divisor then  $B/(b)$  is  $A$ -flat.
- (vii) For a coCartesian diagram

$$\begin{array}{ccc} A & \longrightarrow & B \\ \downarrow & & \downarrow \\ C & \longrightarrow & B \otimes_C A \end{array}$$

and  $M$  a  $B$ -module that is  $A$ -flat, then  $M \otimes_B (C \otimes_A B)$  is a flat  $C$ -module.

*Proof.* See [Stacks, Tag 00H9]. ■

We prove the following lemma about flatness on localizations at prime ideals.

**Lemma 5.4.** Let  $\varphi : A \rightarrow B$  be a ring map.  $\varphi$  is flat if and only if for all primes  $\mathfrak{q} \subseteq B$ , the map  $A_{\varphi^{-1}(\mathfrak{q})} \rightarrow B_{\mathfrak{q}} \cong A_{\varphi^{-1}(\mathfrak{q})} \otimes_A B$  is a flat ring map.

*Proof.* ( $\Rightarrow$ ) Suppose  $A \rightarrow B$  is flat. Then any  $A_{\varphi^{-1}(\mathfrak{q})} \rightarrow B_{\mathfrak{q}}$  is factored as  $A_{\varphi^{-1}(\mathfrak{q})} \rightarrow B_{\varphi^{-1}(\mathfrak{q})} \cong A_{\varphi^{-1}(\mathfrak{q})} \otimes_A B \rightarrow (B_{\varphi^{-1}(\mathfrak{q})})_{\mathfrak{q}} \cong B_{\mathfrak{q}}$  which is a composition of a flat map with a localization, the latter flat, hence the composition is flat too.

( $\Leftarrow$ ) Suppose we have an injection  $M \rightarrow M'$  of  $A$ -modules. By exactness of localization, we have  $M \otimes_A A_{\mathfrak{p}} \rightarrow M' \otimes_A A_{\mathfrak{p}}$ .  $B_{\mathfrak{q}}$  is  $A_{\mathfrak{p}}$ -flat so the map remains injective on base change to  $B_{\mathfrak{q}}$  for all  $\mathfrak{q} \subseteq B$  so taking  $M = A, M' = B$  yields the claim in conjunction with Proposition 5.3 (v). ■

We now describe the geometric case.

**Definition 5.5** (Flat Sheaves). Let  $f : X \rightarrow Y$  be a morphism of schemes and  $\mathcal{F}$  a quasicoherent sheaf on  $X$ .  $\mathcal{F}$  is flat over  $Y$  if  $\mathcal{F}_x$  is flat as an  $\mathcal{O}_{Y,f(x)}$ -module.

**Definition 5.6** (Flat Morphism). Let  $f : X \rightarrow Y$  be a morphism of schemes.  $f$  is a flat morphism if  $\mathcal{O}_X$  is flat over  $Y$ .

We consider some examples.

**Example 5.7.** Let  $X$  be a scheme.  $\text{id}_X : X \rightarrow X$  is a flat morphism as affine-locally it is obtained by the identity ring map and rings are flat as modules over themselves.

**Example 5.8.** Let  $\mathcal{F}$  be a coherent sheaf on  $X$ .  $\mathcal{F}$  is a flat  $\mathcal{O}_X$ -module if and only if  $\mathcal{F}$  is locally free. In the reduced case, this can be checked by the function  $x \mapsto \dim_{\kappa(x)} \mathcal{F}_x \otimes_{\mathcal{O}_{X,x}} \kappa(x)$  being locally constant.

**Example 5.9.** Proposition 5.3 (ii) shows that open immersions are flat, though closed immersions are often not flat unless they are isomorphisms.

**Proposition 5.10.** Let  $f : X \rightarrow Y$  be a morphism between locally Noetherian schemes. Then  $\dim(\mathcal{O}_{X_y}, x) \geq \dim(\mathcal{O}_{X,x}) - \dim(\mathcal{O}_{Y,y})$  and equality occurs when  $f$  is flat.

*Proof.* We proceed by induction. Since the property is local on target, we can take  $Y$  to be affine, and since flatness is preserved by base change, we can take  $Y = \text{Spec}(\mathcal{O}_{Y,y})$  and  $X = \text{Spec}(\mathcal{O}_{X,x})$ .

We proceed by induction on  $\dim(Y)$ . If  $\dim(Y) = 0$ , the nilradical is contained in the maximal ideal. Using the inclusion  $\mathfrak{m}_x B \hookrightarrow \text{Nil}(B)$  we have

$$\dim(\mathcal{O}_{X,x}) = \dim(B/\text{Nil}(B)) = \dim(B/\mathfrak{m}_x B) = \dim(\mathcal{O}_{X_y,x})$$

by  $(B/\text{Nil}(B))/(\text{Nil}(B)/\mathfrak{m}_x B) \cong B/\text{Nil}(B)$  so we in fact have equality  $\dim(\mathcal{O}_{X_y,x}) = \dim(\mathcal{O}_{X,x})$  as  $\dim(\mathcal{O}_{Y,y}) = 0$ . In particular, the desired inequality holds.

If  $\dim(Y) \geq 1$ , we can, without loss of generality, take  $Y$  to be reduced by base changing to the reduction – an operation that preserves all dimensions involved. Let  $t \in \mathcal{O}_{Y,y}$  be neither a zerodivisor nor invertible with image  $\bar{t}$  in  $\mathcal{O}_{X,x}$ . We then have

$$\dim(\mathcal{O}_{Y,y}/(t)) = \dim(\mathcal{O}_{Y,y}) - 1, \dim(\mathcal{O}_{X,x}/(\bar{t})) \geq \dim(\mathcal{O}_{X,x}) - 1$$

where if  $f$  is flat, then  $\mathcal{O}_{X,x}$  is flat over  $\mathcal{O}_{Y,y}$  and  $t$  is a non-zerodivisor in  $\mathcal{O}_{X,x}$  giving  $\dim(\mathcal{O}_{X,x}/(\bar{t})) = \dim(\mathcal{O}_{X,x}) - 1$ .

Now set  $Y' = \operatorname{Spec}(\mathcal{O}_{Y,y}/(t))$  and  $X' = \operatorname{Spec}(\mathcal{O}_{X,x} \otimes_{\mathcal{O}_{Y,y}} (\mathcal{O}_{Y,y}/(t))) \cong \operatorname{Spec}(\mathcal{O}_{X,x}/(\bar{t}))$  where by the induction hypothesis we have

$$(5.1) \quad \dim(\mathcal{O}_{X'_y,x}) \geq \dim(\mathcal{O}_{X',x}) - \dim(\mathcal{O}_{Y',y})$$

which is an equality if  $f$  is flat since flatness is preserved under base change. But on the fiber we have  $X'_y = X_y$  since  $X_y$  is the fiber over  $y \in \operatorname{Spec}(\mathcal{O}_{Y,y}/(t)) \subseteq \operatorname{Spec}(\mathcal{O}_{Y,y})$  so we have

$$\begin{aligned} \dim(\mathcal{O}_{X'_y,x}) &\geq \dim(\mathcal{O}_{X',x}) - \dim(\mathcal{O}_{Y,y}) \\ &= (\dim(\mathcal{O}_{X,x}) - 1) - (\dim(\mathcal{O}_{Y,y}) - 1) \\ &= \dim(\mathcal{O}_{X,x}) - \dim(\mathcal{O}_{Y,y}) \end{aligned}$$

with equality in the flat case by the (5.1), as desired.  $\blacksquare$

In the case of locally finite type schemes, this can be detected fiberwise.

**Corollary 5.11.** Let  $f : X \rightarrow Y$  be a morphism of  $k$ -schemes where  $X, Y$  are locally of finite type,  $Y$  is irreducible, and  $X$  equidimensional. If for all  $y \in Y$ ,  $X_y$  is equidimensional of  $\dim(X) - \dim(Y)$  then  $f$  is flat and surjective.

*Proof.* Without loss of generality, let  $X$  be irreducible and take  $x \in X$  closed. Then  $\dim(X_y) = \dim(\mathcal{O}_{X_y,x})$ . By hypothesis we have

$$\dim(\mathcal{O}_{X,x}) = \dim(X) - \dim(\overline{\{x\}})$$

which holds for finite-type  $k$ -algebras. On the other hand, we have

$$\begin{aligned} \dim(\mathcal{O}_{Y,y}) &= \dim(Y) - \dim(\overline{\{y\}}) \\ &= \dim(Y) - \operatorname{trdeg}(\kappa(y)/k) \\ &= \dim(Y) - \dim\{\bar{x}\} \\ &= \dim(Y) - \dim(X) + \dim(\mathcal{O}_{X,y}) \end{aligned}$$

which gives the equality of Proposition 5.10 and from which the claim follows.  $\blacksquare$

Moreover, flatness behaves especially well over smooth curves.

**Proposition 5.12.** Let  $f : X \rightarrow Y$  be a morphism of schemes where  $Y$  is the spectrum of a discrete valuation. If  $f$  is flat, then  $\overline{X_\eta} = X$ .

In particular, there are no closed irreducible components  $Z \subseteq Z$  over the (unique) closed point of  $Y$ .

*Proof.* ( $\Rightarrow$ ) Suppose  $f$  is flat and there is  $Z \subseteq Z$  irreducible in the fiber over the closed point  $\mathfrak{m} \in Y$ . Then  $\dim(Z) \leq \dim(X_{\mathfrak{m}}) = \dim(X) - 1$ . A contradiction to  $X$  equidimensional. Otherwise, we apply Corollary 5.11 to  $Z$  obtaining a contradiction once more.

( $\Leftarrow$ ) This is [Har83, Prop III.9.7].  $\blacksquare$

## 6. LECTURE 6 – 28TH APRIL 2025

As a corollary of the previous discussion about flatness, we show that flatness can be extended over a closed fiber.

**Corollary 6.1.** Let  $f : X \rightarrow Y \setminus \{y\}$  be flat and projective with  $Y$  Dedekind and  $y \in Y$  closed.

$$\begin{array}{ccc} X & \longrightarrow & \mathbb{P}_{Y \setminus \{y\}}^n \\ \downarrow & & \downarrow \\ \overline{X} & \longrightarrow & \mathbb{P}_Y^n \end{array}$$

Then the induced map  $\overline{X} \rightarrow Y$  is flat.

*Proof.* Recall that a Dedekind scheme is a Noetherian regular integral scheme of dimension 1. We have the diagram of the statement of the corollary and by Proposition 5.12 we have  $X_\eta = (\overline{X})_\eta \subseteq \overline{X}$  which is dense and  $X_\eta$  is dense in  $X$  so  $X_\eta \subseteq \overline{X}$  is dense. This shows that  $\overline{X}$  is flat over  $Y$ . ■

We show that images of flat maps are dense.

**Proposition 6.2.** Let  $f : X \rightarrow Y$  be a flat map of schemes with  $Y$  irreducible. If  $U \subseteq X$  is a nonempty open, then  $f(U) \subseteq Y$  is dense.

*Proof.* Without loss of generality,  $Y = \text{Spec}(A)$  is affine and irreducible so  $A/\text{Nil}(A)$  is integral and  $\text{Frac}(A/\text{Nil}(A)) = K$ . For  $U = \text{Spec}(B)$  affine, consider  $B/\text{Nil}(A)B = B \otimes_A (A/\text{Nil}(A))$ . Using the injection  $A/\text{Nil}(A) \hookrightarrow K$  and  $- \otimes_A B$  being exact, we have an injective map  $(A/\text{Nil}(A)) \otimes_A B \hookrightarrow K \otimes_A B$  injective, where  $K \otimes_A B = \mathcal{O}_X(U_\eta)$ , here denoting  $U_\eta = \text{Spec}(B \otimes_A K)$ .

It suffices to prove that  $U_\eta$  is nonempty, or equivalently that  $\mathcal{O}_X(U_\eta)$  is nonzero. If  $K \otimes_A B$  was zero, then  $\text{Nil}(A)B = B$  and 1 is nilpotent, a contradiction. ■

We can show the following, weaker, version of a base change statement.

**Proposition 6.3.** Let  $f : X \rightarrow \text{Spec}(A)$  be a separated quasicompact morphism and  $A \rightarrow A'$  a flat ring extension. The Cartesian diagram

$$\begin{array}{ccc} X \times_A \text{Spec}(A') & \xrightarrow{g} & X \\ \downarrow & & \downarrow f \\ \text{Spec}(A') & \longrightarrow & \text{Spec}(A) \end{array}$$

induces for all quasicohherent sheaves  $\mathcal{F}$  on  $X$  an isomorphism

$$H^0(X, \mathcal{F}) \otimes_A A' \longrightarrow H^0(X \times_A \text{Spec}(A'), g^* \mathcal{F}).$$

*Proof.* Choose an affine open covering  $\{U_i\}$  of  $X$ . The sheaf condition gives an exact sequence of  $A$ -modules

$$0 \rightarrow H^0(X, \mathcal{F}) \rightarrow \prod_i H^0(U_i, \mathcal{F}|_{U_i}) \rightarrow \prod_{i,j} H^0(U_{ij}, \mathcal{F}|_{U_{ij}}).$$

This remains exact after base changing to  $A'$ , which precisely the exact sequence for  $g^*\mathcal{F}$  on  $X \times_A \text{Spec}(A')$ . ■

Having set up some basic constructions surrounding flatness, we discuss its relation to the Hilbert polynomial.

**Definition 6.4** (Hilbert Polynomial). Let  $X$  be projective  $k$ -scheme with a choice of embedding  $i : X \rightarrow \mathbb{P}_k^n$  and  $\mathcal{F}$  a coherent sheaf on  $X$ . The Hilbert polynomial of  $\mathcal{F}$  is  $P(X, \mathcal{F})(m) = \chi(X, \mathcal{F} \otimes \mathcal{O}_{\mathbb{P}_k^n}(m)|_X)$ .

The Hilbert polynomial is in fact a numerical polynomial, that is, a map  $\mathbb{Z} \rightarrow \mathbb{Z}$ . Moreover, we can show that this polynomial characterizes flatness on the fibers. The proof of the statement is the globalization of the following lemma.

**Lemma 6.5.** Let  $A$  be an integral Noetherian local ring and  $\mathcal{F}$  a coherent sheaf on  $\mathbb{P}_A^n$ . The following are equivalent:

- (a)  $\mathcal{F}$  is flat over  $\text{Spec}(A)$ .
- (b)  $H^0(X, \mathcal{F}(m))$  is a free  $A$ -module for  $m$  sufficiently large.
- (c) The Hilbert polynomial  $P(\mathbb{P}_{\kappa(\mathfrak{p})}^n, \mathcal{F}|_{\kappa(\mathfrak{p})})(m)$  is independent of  $\mathfrak{p}$ .

**Proposition 6.6.** Let  $f : X \rightarrow Y$  be a projective morphism with  $Y$  integral Noetherian. Let  $\mathcal{F}$  be a coherent sheaf on  $Y$ .  $f$  is flat if and only if the Hilbert polynomial of  $\mathcal{F}$  on the fibers is constant.

*Proof.* The question is local on target, and flatness is preserved along base change, so it suffices to check the condition along the base change  $\text{Spec}(\mathcal{O}_{Y,y}) \rightarrow Y$ , putting us in the situation of Lemma 6.5, in which case the desired statement is immediate. ■

**Example 6.7.** Blowups are not flat. Let  $f$  be the projection from the blowup of  $\mathbb{A}_k^2$  at the origin to  $\mathbb{A}_k^2$ . The fiber over the origin is of a higher dimension.

This applies in particular to the structure sheaf.

**Corollary 6.8.** Let  $f : X \rightarrow Y$  be a projective morphism with  $Y$  integral Noetherian.  $f$  is flat if and only if  $P(X_y, \mathcal{O}_{X_y})(m)$  is constant.

*Proof.* This is an immediate application of Proposition 6.6 to  $\mathcal{O}_X = \mathcal{F}$ . ■

Moreover, we can show that flatness is open and universally open.

**Proposition 6.9.** Let  $f : X \rightarrow Y$  be a finite type morphism of schemes with  $Y$  Noetherian. Then  $f$  is open and universally open.

*Proof.* We use the following input from the theory of spectral spaces:

- If  $f : X \rightarrow Y$  is a finite type morphism with  $Y$  Noetherian then the image of  $f$  is constructible in  $Y$  [Stacks, Tag 054K].
- Let  $V \subseteq Y$  be a subset.  $V$  is constructible and stable under specializations if and only if  $V$  is open in  $Y$  [Stacks, Tag 0542].

It suffices to show that  $f(X)$  is open. We know already  $f(X)$  is constructible so it suffices to show that  $f(X)$  is stable under generalizations. Take  $x \in X$  and  $y = f(x) \in Y$ . Assume  $y \in \overline{\{y'\}}$ . We want to show  $y' \in f(X)$ . Let  $B = \mathcal{O}_{X,x}$  with

Finish proof.

Check.



$x \in U \subseteq X$  and  $A = \mathcal{O}_{Y,y}$  which contains the prime ideal  $\mathfrak{p}_{y'}$ . The ring map  $A \rightarrow B$  is a local homomorphism of local rings so  $\mathfrak{m}_x \cap A = \mathfrak{m}_y$  and  $\mathfrak{p}_{y'}B \subseteq \mathfrak{m}_yB \subseteq \mathfrak{m}_x \subsetneq B$  showing  $B \otimes_A A_{\mathfrak{p}_{y'}} \neq 0$ . If it were zero, then for all  $b \in B$  there would be  $t \in A_{\mathfrak{p}_{y'}}$  such that  $tb = 0$  but the multiplication by  $t$ -map  $A \rightarrow A$  is injective and remains injective after base-changing to  $B$ . This shows that  $B \otimes_{A_{\mathfrak{p}_{y'}}} \kappa(\mathfrak{p}_{y'})$  is nonzero, and hence lies in the image. ■

## 7. LECTURE 7 – 5TH MAY 2025

We state without proof flatness in another setting.

**Proposition 7.1** (Miracle Flatness). Let  $X, Y$  be smooth integral  $k$ -schemes and  $f : X \rightarrow Y$  a morphism such that  $\dim(X_y)$  is constant. Then  $f$  is flat.

*Proof.* This is local, so we can reduce to the case of  $f$  induced by a ring map  $B \rightarrow A$  which is [Stacks, Tag 00R4].  $\blacksquare$

Having discussed flatness, we can turn to a discussion of smoothness of morphisms, generalizing Definition 3.11.

**Definition 7.2** (Smooth Morphism at a Point). Let  $f : X \rightarrow Y$  be a morphism locally of finite type.  $f$  is smooth at  $x \in X$  if  $f$  is flat at  $x$  and the fiber  $X_{f(x)}$  is smooth over  $\kappa(f(x))$ .

**Remark 7.3.** That is,  $\mathcal{O}_{X,x}$  is a flat  $\mathcal{O}_{Y,f(x)}$ -module and  $X_{f(x)} = X \times_Y \text{Spec}(\kappa(f(x)))$  is a smooth  $\kappa(f(x))$ -scheme by the Cartesian diagram

$$\begin{array}{ccc} X_{f(x)} & \longrightarrow & X \\ \downarrow & & \downarrow \\ \text{Spec}(\kappa(f(x))) & \longrightarrow & Y. \end{array}$$

This definition globalizes.

**Definition 7.4** (Smooth Morphism). Let  $f : X \rightarrow Y$  be a morphism locally of finite type.  $f$  is smooth if  $f$  is smooth at all  $x \in X$ .

**Example 7.5.** Let  $W$  be a smooth  $k$ -scheme and  $Y$  any  $k$ -scheme. Let  $X = W \times_k Y$  with  $f : X \rightarrow Y$  the natural projection.  $f$  is smooth – by miracle flatness the dimensions of the fibers are constant and are  $\dim W$  which is  $\kappa(f(x))$ -smooth since smoothness is preserved under base change.

**Example 7.6.** Let  $Y$  be any scheme. The projections  $\mathbb{A}_Y^n \rightarrow Y, \mathbb{P}_Y^n \rightarrow Y$  are smooth. More generally for  $\mathcal{E}$  locally free of finite rank on  $Y$ , the (total spaces of the) associated vector bundle  $\mathbb{V}(\mathcal{E}) \rightarrow Y$  and projective bundle  $\mathbb{P}(\mathcal{E}) \rightarrow Y$  are smooth.

Recall that for  $f : X \rightarrow Y$  a morphism of schemes over  $k$  the Proposition 1.8 gives an exact sequence of sheaves

$$f^* \Omega_{Y/k}^1 \rightarrow \Omega_{X/k}^1 \rightarrow \Omega_{X/Y}^1 \rightarrow 0.$$

As it turns out, smoothness of morphisms can be determined by  $\Omega_{X/Y}^1$ , in analogy to how smoothness of schemes is determined by  $\Omega_{X/k}^1$ . For this, we will require the following lemma.

**Lemma 7.7.** Let  $f : X \rightarrow Y$  be a locally finite type morphism,  $x \in X$  with  $f(x) = y$  and  $d = \dim(X_y)$ . There exists a closed immersion  $X \rightarrow W$  over  $Y$  such that  $W$  is smooth at  $x$  over  $y$  of dimension  $d$ .

*Proof.* Since  $f$  is locally of finite type, we can factor  $f$  as

$$\begin{array}{ccc} X & \xrightarrow{i} & \mathbb{A}_Y^n \\ & \searrow f & \swarrow p \\ & Y & \end{array}$$

where  $i$  is a closed immersion and  $p$  is the projection. As in Example 7.6,  $p$  is smooth so  $\Omega_{\mathbb{A}_Y^n/Y}^1$  is locally free of rank  $n$ , but not of rank  $d$ . We find a closed subscheme of  $\mathbb{A}_Y^n$  such that the sheaf of relative differentials is locally free of rank  $d$ .

Let  $\mathcal{I}_X$  be the ideal sheaf of  $X$  in  $\mathbb{A}_Y^n$  so by Proposition 1.13, we have an exact sequence

$$(7.1) \quad \mathcal{I}_X/\mathcal{I}_X^2 \rightarrow \Omega_{\mathbb{A}_Y^n/Y}^1|_{i(X)} \rightarrow \Omega_{X/Y} \rightarrow 0.$$

Now letting  $\mathcal{J}$  be the ideal sheaf of  $X_y \in \mathbb{A}_{\kappa(y)}^n$  we have an exact sequence

$$(7.2) \quad \mathcal{J} \rightarrow \Omega_{\mathbb{A}_{\kappa(y)}^n/\kappa(y)}^1 \rightarrow \Omega_{X_y/\kappa(y)}^1 \rightarrow 0$$

where  $\Omega_{\mathbb{A}_{\kappa(y)}^n/\kappa(y)}^1$  is locally free of rank  $n$  and  $\Omega_{X_y/\kappa(y)}^1$  is locally free of rank  $d$ . Then

$$\ker \left( \Omega_{\mathbb{A}_{\kappa(y)}^n/\kappa(y)}^1 \rightarrow \Omega_{X_y/\kappa(y)}^1 \right)$$

is locally free of rank  $n - d$ . Base changing (7.1) and (7.2) by  $\kappa(x)$ , we get a diagram

$$\begin{array}{ccccccc} \mathcal{I}_X/\mathcal{I}_X^2 \otimes \kappa(x) & \longrightarrow & \Omega_{\mathbb{A}_Y^n/Y}^1|_{i(X)} \otimes \kappa(x) & \longrightarrow & \Omega_{X/Y}^1 \otimes \kappa(x) & \longrightarrow & 0 \\ \downarrow & & \downarrow & & \downarrow & & \\ \mathcal{J} \otimes \kappa(x) & \longrightarrow & \Omega_{\mathbb{A}_{\kappa(y)}^n/\kappa(y)}^1 \otimes \kappa(x) & \longrightarrow & \Omega_{X_y/\kappa(y)}^1 \otimes \kappa(x) & \longrightarrow & 0 \end{array}$$

where the injectivity of the right two vertical arrows implies surjectivity of the leftmost vertical arrow.

If  $n > d$ , there is  $g \in \mathcal{I}_X$  and  $g_2, \dots, g_n \in \mathcal{O}_{\mathbb{A}_{\kappa(y)}^n}$  such that  $dg, dg_2, \dots, dg_n$  freely generate  $\Omega_{\mathbb{A}_{\kappa(y)}^n/\kappa(y)}^1 \otimes \kappa(x)$ . By Nakayama's lemma, these lift to basis of  $\Omega_{\mathbb{A}_{\kappa(y)}^n/\kappa(y)}^1 \otimes \kappa(x')$  for  $x'$  in a neighborhood of  $x$  in  $X$ . Observe  $X \subseteq V(g) = W \subseteq \mathbb{A}_Y^n$  then  $\dim(W) = \dim(\mathbb{A}_Y^n) - 1$  and with  $\Omega_{W_y/\kappa(y)}^1 = \Omega_{\mathbb{A}_{\kappa(y)}^n/\kappa(y)}/dg$  which is locally free of rank  $n - 1$ . Moreover, since  $g$  is a nonzerodivisor in  $\mathcal{O}_{X,x}/\mathfrak{m}_y \mathcal{O}_{X,x}$ ,  $W_y \rightarrow y$  is flat by Proposition 5.3 (vi).

Iterating this process, we arrive at the desired  $W$ . ■

**Example 7.8.** Let  $X = \{y\} \hookrightarrow Y$  a closed point.  $\Omega_{X/Y}^1 = 0$  and the fiber dimension is 0, but taking  $W = Y$ ,  $\text{id}_Y : Y \rightarrow Y$  is smooth.

We now state and prove the desired result.

**Proposition 7.9.** Let  $f : X \rightarrow Y$  be a morphism locally of finite type with all fibers of pure dimension  $d$ . Then  $f$  is smooth if and only if  $f$  is flat and  $\Omega_{X/Y}^1$  is locally free of rank  $d$ .

*Proof.* Recall by Proposition 1.12 that for a Cartesian square

$$\begin{array}{ccc} X' & \xrightarrow{g} & X \\ \downarrow & & \downarrow \\ Y' & \longrightarrow & Y \end{array}$$

we have  $g^*\Omega_{X/Y}^1 \cong \Omega_{X'/Y'}^1$  so in particular when  $Y' = \text{Spec}(\kappa(y))$  then  $X' = X_y$  and we have  $\Omega_{X_y/\kappa(y)}^1 \cong \Omega_{X/Y}^1|_{X_y}$ . We now begin the proof in earnest.

( $\Rightarrow$ ) Now assume  $f$  is smooth. By definition,  $f$  is flat, and  $\Omega_{X_y/\kappa(y)}^1$  is locally free of rank  $\dim(X_y)$ . In particular, the rank of  $\Omega_{X_y/\kappa(y)}^1 \otimes \kappa(x)$  is  $d$  for all  $x \in X_y$ . By base change,  $\Omega_{X_y/\kappa(y)}^1 \otimes \kappa(x)$  and  $\Omega_{X/Y}^1 \otimes \kappa(x)$  are of the same rank. If  $X$  is reduced, this already implies that  $\Omega_{X/Y}^1$  is locally free of the correct rank.

If  $X$  is not reduced, we use the factorization produced by Lemma 7.7 to get a factorization

$$\begin{array}{ccc} X & \xrightarrow{i} & W \\ & \searrow f & \swarrow p \\ & Y & \end{array}$$

where  $i$  is a closed immersion and  $p$  is smooth with fiber dimension  $d$ . We have  $X_y \subseteq W_y$  of the same dimension, but  $W_y$  is smooth hence reduced, so  $X_y = W_y$  yielding the claim.

( $\Leftarrow$ ) Suppose  $f$  is flat and  $\Omega_{X/Y}^1$  is locally free of rank  $d$ . The restriction to the fiber  $\Omega_{X_y/\kappa(y)}^1$  is locally free of rank  $\dim(X_y)$  over  $\kappa(y)$  hence smooth over  $\kappa(y)$  showing the morphism is smooth.  $\blacksquare$

As in the case of smoothness Proposition 4.5, the smooth locus of a morphism can be shown to be open on the source.

**Proposition 7.10.** Let  $f : X \rightarrow Y$  be a dominant morphism of finite type integral schemes over  $k$  with  $K(X)/K(Y)$  separable. There exists  $U \subseteq X$  dense open such that  $f|_U : U \rightarrow Y$  is smooth.

*Proof.* Since  $K(X)/K(Y)$  is separable,  $\Omega_{K(X)/K(Y)}^1$  is a  $K(X)$ -vector space of rank the transcendence degree of  $K(X)/K(Y)$ , which is  $d = \dim(X) - \dim(Y)$ . By Lemma 4.1, we have that there is an open set  $U$  such that  $\Omega_{X/Y}^1|_U \cong \Omega_{U/Y}^1$  is locally free of rank  $d$ .

By Proposition 6.9, we can by restricting further take  $U$  to be the flat locus of the morphism, which may be empty. On  $U$ , the fibers over  $y \in f(U)$  are of dimension  $d$  by Corollary 5.11 (applied to  $U$  and not  $Y$ ). So by Proposition 7.9,  $f|_U$  is smooth.  $\blacksquare$

**Remark 7.11.** The special case of Proposition 7.10 where  $Y = \text{Spec}(k)$  is precisely Proposition 4.5.

Let us consider some examples.

**Example 7.12.** A smooth morphism between smooth schemes extends the exact sequence of Proposition 1.8 to a short exact sequence. Let  $f : X \rightarrow Y$  be a smooth morphism between smooth  $k$ -schemes. There is a short exact sequence

$$(7.3) \quad 0 \rightarrow f^*\Omega_{Y/k}^1 \rightarrow \Omega_{X/k}^1 \rightarrow \Omega_{X/Y} \rightarrow 0.$$

To see this, we observe that since  $\Omega_{X/Y}^1$  is locally free of rank  $\dim(X) - \dim(Y)$  and  $\Omega_{X/k}^1$  is locally free of rank  $\dim(X)$  implying  $f^*\Omega_{Y/k}^1$  locally free of rank the dimension of  $Y$ . This implies that  $f^*\Omega_{Y/k}^1 \rightarrow \ker(\Omega_{X/k}^1 \rightarrow \Omega_{X/Y}^1)$  is surjective, but any surjection of locally free modules of the same rank is an isomorphism, whence the claim.

**Example 7.13.** Let  $f : X \rightarrow Y$  be a smooth morphism between smooth  $k$ -schemes. Dualizing (7.3) of Example 7.12, we get

$$0 \rightarrow \mathcal{T}_{X/Y} \rightarrow \mathcal{T}_{X/k} \rightarrow f^*\mathcal{T}_{Y/k} \rightarrow 0$$

which on restriction to any closed fiber  $X_y$  we have

$$0 \rightarrow \mathcal{T}_{X/Y}|_{X_y} \cong \mathcal{T}_{X_y} \rightarrow \mathcal{T}_X|_{X_y} \rightarrow f^*\mathcal{T}_Y|_y \cong \mathcal{N}_{X_y/Y} \rightarrow 0$$

recovering the normal bundle sequence.

Observe that for  $f : X \rightarrow Y$  a smooth morphism between smooth  $k$ -schemes with  $k$ -algebraically closed we have for all  $x \in X$  an induced map on the Zariski tangent spaces  $T_{X,x} \rightarrow T_{Y,f(x)} \otimes \kappa(x)$ . In particular, if  $x \in X(k)$  we have  $\kappa(x) \cong k$  and thus  $\kappa(y) \cong k$  as  $\kappa(y)$  lies in the intermediate extension  $\kappa(x)/\kappa(y)/k$ . By the injectivity of the idnuced map  $\mathfrak{m}_y/\mathfrak{m}_y^2 \rightarrow \mathfrak{m}_x/\mathfrak{m}_x^2$ , we have surjectivity of the map on Zariski tangent spaces. This in fact determines smoothness of a morphism under appropriate hypotheses.

**Proposition 7.14.** Let  $f : X \rightarrow Y$  be a morphism between smooth  $k$ -schemes with  $k$  algebraically closed. If the induced map  $T_{X,x} \rightarrow T_{Y,f(x)}$  is surjective for all  $x \in X(k)$  then  $f$  is smooth.

*Proof.* We show first that  $f$  is flat. Since flatness is an open condition, it suffices to show flatness on each closed point, since each point lies in an open neighborhood of some closed point.

By injectivity of  $\mathfrak{m}_y/\mathfrak{m}_y^2 \hookrightarrow \mathfrak{m}_x/\mathfrak{m}_x^2$  we pick generators  $(\bar{a}_1, \dots, \bar{a}_n)$  where  $n = \dim(Y)$  with images  $\bar{b}_1, \dots, \bar{b}_n$  for  $b_i \in \mathfrak{m}_x$ . Injectivity implies that the  $\bar{b}_i$  remain linearly independent in the quotient so  $b_1, \dots, b_n$  is a regular sequence in  $\mathcal{O}_{X,x}$  – each  $b_i$  is a nonzerodivisor in  $\mathcal{O}_{X,x}/(b_1, \dots, b_{i-1})$ . By induction,  $\mathcal{O}_{X,x}/(b_1, \dots, b_n)$  is flat over  $\kappa(y) = \mathcal{O}_{Y,f(x)}/(a_1, \dots, a_n)$ . This gives flatness.

Use Proposition 1.8, we have

$$f^*\Omega_{Y/k}^1 \rightarrow \Omega_{X/k}^1 \rightarrow \Omega_{X/Y}^1 \rightarrow 0$$

where the former two terms are locally free and surjectivity of the Zariski tangent space implies that the map  $f^*\Omega_{Y/k}^1 \rightarrow \Omega_{X/k}^1$  is fiberwise injective. Hence  $\Omega_{X/Y}^1$  is locally free of  $\dim(X) - \dim(Y)$  showing smoothness too. ■

We hint towards an algebro-geometric version of Sard's theorem via the following lemma, which will be used in the proof of the result.

**Lemma 7.15.** Let  $f : X \rightarrow Y$  be smooth finite type  $k$ -schemes with  $k$  of characteristic 0. Define

$$X_r = \overline{\{x \in X_{\text{cl}} : \text{rank}(T_{X,x} \rightarrow T_{Y,f(x)} \otimes \kappa(x)) \leq r\}}.$$

Then  $\dim(\overline{f(X_r)}) \leq r$ .

*Proof.* Pick an irreducible component  $X' \subseteq \overline{X_r}$  such that  $f' : X' \rightarrow Y'$  is dominant. Given  $X', Y'$  the induced reduced subscheme structure so  $X', Y'$  are integral. Using that  $K(X')/K(Y')$  is separable since  $K(Y')$  is of characteristic 0, there exists by Proposition 7.10  $U'$  of  $X'$  such that  $f'|_{U'} : U' \rightarrow Y'$  is smooth. Then take  $x \in U' \cap X_r$  and contemplate the diagram

$$\begin{array}{ccc} T_{U',x} & \hookrightarrow & T_{X,x} \\ \downarrow & & \downarrow \\ T_{Y',f(x)} & \hookrightarrow & T_{Y,f(x)} \end{array}$$

where the injective horizontal maps and the right vertical map being of rank  $r$  imply surjectivity of the left vertical map and smoothness of  $U' \rightarrow Y'$ . Indeed, it suffices to take  $x$  to be  $k$ -rational, since the preceding discussion holds on base change. Thus  $T_{Y',f(x)}$  is of rank at most  $r$ , showing  $\dim(\overline{f(X_r)}) \leq r$ , as desired. ■

## 8. LECTURE 8 – 8TH MAY 2025

We can now state and prove the algebraic variant of Sard's theorem.

**Theorem 8.1** (Algebraic Sard). Let  $X, Y$  be smooth integral  $k$ -schemes with  $k$  of characteristic 0 and  $f : X \rightarrow Y$  a morphism of  $k$ -schemes. There exists a dense set  $V \subseteq Y$  such that  $f^{-1}(V) \rightarrow V$  is smooth.

**Remark 8.2.** Note that  $V$  is nonempty but  $f^{-1}(V) = \emptyset$  can occur, for example where  $f$  is a closed embedding.

*Proof.* Denote the set

$$X_r = \overline{\{x \in X_{\text{cl}} : \text{rank}(T_{X,x} \rightarrow T_{Y,f(x)} \otimes \kappa(x)) \leq r\}}$$

where we have  $\dim(\overline{f(X_r)}) \leq r$  by Lemma 7.15. Apply this to  $r = \dim(Y) - 1$  so  $\overline{f(X_{\dim(Y)-1})}$  is a proper closed subset. Let  $V$  be its open complement in  $Y$  which is nonempty since  $\overline{f(X_{\dim(Y)-1})}$  is a proper closed subset. For all points  $x \in X$ , with image contained in  $V$ , we have that  $T_{X,x} \rightarrow T_{Y,f(x)} \otimes \kappa(x)$  cannot have rank smaller than the dimension of  $Y$ , so the map is surjective as a map of  $\kappa(x)$ -vector spaces, hence smooth by Proposition 7.14. ■

There is an easy counterexample in positive characteristic.

**Example 8.3.** Let  $k$  be of characteristic  $p$ . The relative Frobenius  $\mathbb{A}_k^2 \rightarrow \mathbb{A}_k^2$  with all fibers non-reduced, so the morphism is nowhere smooth.

Algebraic Sard's theorem allows us to show that smoothness is generic on hyperplane sections of a scheme, in the sense that the set of smooth hyperplane sections of a quasiprojective scheme is dense in the set of hyperplanes. Let us be more precise: let  $X \subseteq \mathbb{P}_k^n$  be a smooth quasiprojective variety. For a line bundle  $\mathcal{L} \in \text{Pic}(X)$  and  $V \subseteq H^0(X, \mathcal{L})$  a basepoint free linear subspace, we can define a morphism  $\varphi_V : X \rightarrow \mathbb{P}(V^\vee)$  with the property that  $f^* \mathcal{O}_{\mathbb{P}(V^\vee)}(1) \cong \mathcal{L}$ . There exists a nonempty open  $U \subseteq V$  such that for all  $s \in U$ ,  $V_X(s) = \varphi^{-1}(V_+(s)) \subseteq X$  is smooth.

For this, we recall the following facts about the universal hypersurface.

**Lemma 8.4.** Let  $\mathcal{U}_{n,1} \subseteq \mathbb{P}_k^n \times_k \mathbb{P}_k^N$  for  $N = n$  be the universal hyperplane defined by  $\sum_{i=0}^n a_i x_i$ . Then  $\text{pr}_{\mathbb{P}_k^N}$  is a projective bundle of dimension  $n - 1$ .

*Proof.* It suffices to observe that the fiber over any point  $(a_0, \dots, a_n)$  is the hyperplane  $V_+(\sum_{i=0}^n a_i x_i) \subseteq \mathbb{P}_k^n$ . ■

With this in hand, we state and prove the desired result.

**Theorem 8.5** (Bertini). Let  $X \subseteq \mathbb{P}_k^n$  be a smooth quasiprojective variety with  $k$  algebraically closed of characteristic 0. For a line bundle  $\mathcal{L} \in \text{Pic}(X)$  and  $V \subseteq H^0(X, \mathcal{L})$  a basepoint free linear subspace. There is a nonempty open subset  $U \subseteq V$  such that for all  $s \in U$ ,  $V_X(s) \subseteq X$  is smooth.

*Proof.* Construct the universal hyperplane

$$\begin{array}{ccc} \mathbb{P}(V^\vee) \times_k \mathbb{P}_k^N & \longleftrightarrow & \mathcal{U}_{n,1} \xrightarrow{\text{pr}_{\mathbb{P}_k^N}} \mathbb{P}_k^N = |\mathcal{O}_{\mathbb{P}(V^\vee)}(1)| \\ & & \downarrow \text{pr}_{\mathbb{P}(V^\vee)} \\ & & \mathbb{P}(V^\vee) \end{array}$$

We have that  $\text{pr}_{\mathbb{P}_k^N} : \mathcal{U}_{n,1} \rightarrow \mathbb{P}_k^N$  is a projective bundle isomorphic to  $\mathbb{P}(\mathcal{E})$  where

$$\mathcal{E} = \ker (H^0(\mathbb{P}(V^\vee), \mathcal{O}_{\mathbb{P}(V^\vee)}(1)) \otimes \mathcal{O}_{\mathbb{P}(V^\vee)}(1) \rightarrow \mathcal{O}_{\mathbb{P}(V^\vee)}(1))$$

since we have a diagram

$$\begin{array}{ccccc} & & & & \mathbb{P}(V^\vee) \\ & & & & \uparrow \text{pr}_{\mathbb{P}(V^\vee)} \\ & & & & \mathcal{U}_{n,1} \\ & & & & \downarrow \text{pr}_{\mathbb{P}_k^N} \\ & & & & \mathbb{P}_k^N \\ X \times_k \mathbb{P}_k^n & \xrightarrow{\varphi \times \text{id}_{\mathbb{P}_k^n}} & \mathbb{P}(V^\vee) \times_k \mathbb{P}_k^N & \longleftrightarrow & \mathcal{U}_{n,1} \\ \uparrow & & & & \\ \mathcal{X} = X \times_{\mathbb{P}(V^\vee)} \mathcal{U}_{n,1} & & & & \end{array}$$

where in particular  $\text{pr}_{\mathbb{P}_k^N}$  is a  $\mathbb{P}_k^{\dim(V)-2}$ -bundle. Denote  $\pi$  the composite  $\mathcal{X} \rightarrow \mathbb{P}_k^N$  obtained by the diagram above. Apply Theorem 8.1 to  $\pi$  so there exists  $W \subseteq \mathbb{P}_k^N$  open such that  $\pi^{-1}(W) \rightarrow W$  is smooth. For  $g : V \setminus \{0\} \rightarrow (V \setminus \{0\})/k^\times$  be the quotient map and define  $U = g^{-1}(W \cap (\mathbb{P}_k^n)_{\text{cl}})$ . Then for all  $s \in U$ ,  $V_+(s)$  is smooth.  $\blacksquare$

**Remark 8.6.** If  $k$  is not algebraically closed, we need that  $W \cap (V \setminus \{0\})/k^\times = W(k)$  is nonempty.

**Remark 8.7.** The set  $U$  is open only if  $X$  is projective.

**Remark 8.8.** By passing to the Veronese embedding, the proof of Bertini's theorem holds for hypersurface sections.

We can now see some examples.

**Example 8.9** (Katz). Let  $k = \mathbb{F}_q$ .  $V_+(\sum_{i=0}^n x_i y_i^q - x_i^q y_i) \subseteq \mathbb{P}_k^{2n+1}$  is smooth but no hyperplane section of  $X$  is smooth.

**Example 8.10** (Poonen). Fix  $\mathbb{F}_q$ . For  $n \geq 2, d \geq 1$  there exists  $X \subseteq \mathbb{P}_{\mathbb{F}_q}^n$  of degree  $d$  such that all hypersurface sections of degree at most  $d$  are singular.

**Example 8.11** (Poonen). Recall the definition of the zeta function of a scheme  $\zeta_X(s) = \exp(\sum_{i=0}^\infty \frac{X(\mathbb{F}_{q^r})}{r} q^{-rs})$ . Fix  $X \subseteq \mathbb{P}_{\mathbb{F}_q}^n$  smooth.

$$\frac{\{f \in \mathbb{F}_q[x_0, \dots, x_n]_d : V_+(f) \cap X \text{ smooth}\}}{\{f \in \mathbb{F}_q[x_0, \dots, x_n]_d\}} \sim_{d \rightarrow \infty} \zeta_X(\dim(X)).$$



**Example 8.12.** Let  $\mathcal{U}_{n,d}$  be the universal hypersurface of degree  $d$  in  $\mathbb{P}_k^n$ . There is a proper closed subset of  $|\mathcal{O}_{\mathbb{P}_k^n}(d)|$  parametrizing the singular hypersurfaces of degree  $(d-1)^{n+1}(n+3)$ .

We begin a discussion of unramified morphisms.

**Definition 8.13** (Unramified Morphism at a Point). Let  $f : X \rightarrow Y$  be a locally finite type morphism between locally Noetherian schemes.  $f$  is unramified at  $x \in X$  with image  $y = f(x)$  if  $\mathfrak{m}_y \mathcal{O}_{X,x} = \mathfrak{m}_x$  and  $\kappa(x)/\kappa(y)$  is separable.

**Definition 8.14** (Unramified Morphism). Let  $f : X \rightarrow Y$  be a locally finite type morphism between locally Noetherian schemes.  $f$  is unramified if it is unramified at all  $x \in X$ .

**Remark 8.15.** Unramified morphisms are the algebro-geometric analogue of immersions in differential topology.

Let us now consider some examples.

**Example 8.16.** Let  $K/k$  be a separable algebraic extension. Then  $\text{Spec}(K) \rightarrow \text{Spec}(k)$  is unramified.

**Example 8.17.** Let  $L/K/\mathbb{Q}$  be number fields inducing on rings of integers  $\mathbb{Z} \rightarrow \mathcal{O}_K \xrightarrow{\varphi} \mathcal{O}_L$ . This induces a map  $\text{Spec}(\mathcal{O}_L) \rightarrow \text{Spec}(\mathcal{O}_K)$ . The map is unramified at the generic point and at all closed points  $\mathfrak{q} \subseteq \mathcal{O}_L, \mathfrak{p} = \varphi^{-1}(\mathfrak{q})$  that  $(\mathfrak{q} \cap \mathcal{O}_{K,\mathfrak{p}}) \mathcal{O}_{L,\mathfrak{q}} = \mathfrak{q} \subseteq \mathcal{O}_{L,\mathfrak{q}}$ .

**Example 8.18.**  $\mathbb{A}_k^1 \rightarrow \mathbb{A}_k^1$  by  $x \mapsto x^2$  is ramified at  $(x)$  as  $k[x]_{(x)}$  is not generated by  $x^2$ , but unramified at all nonzero points away from characteristic 2. Conversely, the morphism is ramified everywhere in characteristic 2.

**Example 8.19.** Closed and open immersions are always unramified.

**Example 8.20.**  $\text{Spec}(k[x]/(f)) \rightarrow \text{Spec}(k)$  is unramified over the factors  $f_i$  of  $f$  that  $k[x]/(f_i)$  are separable.

As it turns out, unramifiedness can be tested on the relative Kähler differentials, or equivalently on the diagonal being an open embedding. To show this, we will first require the following lemma.

**Lemma 8.21.** Let  $f : X \rightarrow Y$  be a locally finite type morphism between locally Noetherian schemes.  $f$  is unramified if and only if for all points  $y \in Y$ , the fiber  $X_y$  is reduced, locally finite, and for all  $x \in X_y$  the field extension  $\kappa(x)/\kappa(y)$  is separable.

*Proof.* ( $\Rightarrow$ ) Assume that  $f$  is unramified. We have that  $\mathcal{O}_{X_y,x} \cong \mathcal{O}_{X,x}/\mathfrak{m}_y \mathcal{O}_{X,x}$  where by unramifiedness,  $\mathfrak{m}_y \mathcal{O}_{X,x} \cong \mathfrak{m}_x$  so  $\mathcal{O}_{X_y,x} \cong \kappa(x)$  is reduced so the fiber is reduced and locally finite.

( $\Leftarrow$ ) Suppose for all  $y \in Y$ , the fiber  $X_y$  is reduced, locally finite, and the extension of residue fields is separable. We have an injection  $\kappa(y) \hookrightarrow \mathcal{O}_{X_y,x}$  so  $\mathcal{O}_{X_y,x}$  is zero-dimensional, reduced, and locally finite so  $\mathcal{O}_{X_y,x} = \kappa(x)$  showing  $\mathfrak{m}_y \mathcal{O}_{X,x} = \mathfrak{m}_x$ , as desired. ■

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