V4A2 – ALGEBRAIC GEOMETRY II
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WERN JUIN GABRIEL ONG

Preliminaries

These notes roughly correspond to the course **V4A2 – Algebraic Geometry II** taught by Prof. Daniel Huybrechts at the Universität Bonn in the Summer 2025 semester. These notes are LaTeX-ed after the fact with significant alteration and are subject to misinterpretation and mistranscription. Use with caution. Any errors are undoubtedly my own and any virtues that could be ascribed to these notes ought be attributed to the instructor and not the typist. Knowledge of commutative algebra, topology, and category theory will be assumed.

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1. Lecture 1 – 7th April 2025

We begin by a consideration of the theory of smoothness, first in the local case. This is done by defining the sheaves of Kähler differentials on schemes – in the local picture, the module of differentials on a ring.

Definition 1.1 (Derivation)**.** Let B be an A-algebra and M a B-module. An morphism of A-modules $D : B \to M$ is an A-derivation if it satisfies the Leibniz rule $\mathrm{d}(xy) = x\mathrm{d}(y) + y\mathrm{d}(x)$ for all $x, y \in B$. Denote the set of A-derivations in M by $\mathrm{Der}_A(B, M)$.

Remark 1.2. It is necessary that M is a B-module, since the Leibniz rule involves elements of B.

Remark 1.3. Observe that the composition $A \to B \to M$is zero since $a = a \cdot 1_B$ and computing we get $\mathrm{d}(a \cdot 1_B) = a\mathrm{d}(1_B)$ by A-linearity, but on the other hand $\mathrm{d}(a \cdot 1_B) = a\mathrm{d}(1_B) + 1_B\mathrm{d}(a)$ by the Leibniz rule, so $a\mathrm{d}(1_B) = 0$ showing $\mathrm{d}(1_B) = 0$ and thus $\mathrm{d}(a) = 0$.

The Kähler differentials of a ring map is the universal recipient of an A-algebra B in the following sense.

Definition 1.4 (Module of Kähler Differentials)**.** Let B be an A-algebra. The module of Kähler differentials of B over A is a B-module $\Omega^1_{B/A}$ with an A-derivation $\mathrm{d} : B \to \Omega^1_{B/A}$ that is initial amongst B-modules recieving an A-derivation from B.

Unwinding the universal property, if M is a B-module recieving an A-derivation from B by $f : B \to M$, there is a unique factorization over $\Omega^1_{B/A}$ as follows.

$$\begin{tikzcd}
\Omega^1_{B/A} \arrow[r] & M \\
B \arrow[u, dashed, "\exists!"] \arrow[ur] &
\end{tikzcd}$$

In particular, there is a bijection $\mathrm{Der}_A(B, M) \leftrightarrow \mathrm{Hom}_{\mathsf{Mod}_B}(\Omega_{B/A}, M)$ functorial in M.

Proposition 1.5. Let B be an A-algebra. The B-module $\Omega^1_{B/A}$ and the A-derivation $\mathrm{d} : B \to \Omega^1_{B/A}$ exist and are unique up to unique isomorphism.

Proof. The module $\Omega^1_{B/A}$ can be constructed as the free B-module on elements $\mathrm{d}x$ for $x \in B$ modulo the relations generated by the Leibniz rule and $\mathrm{d}a = 0$ for $a \in A$. Uniqueness up to unique isomorphism is clear from the universal property and Yoneda's lemma. ■

In special cases, the module of Kähler differentials can be described explicitly.

Example 1.6. Let $A = k, B = k[x_1, \ldots, x_n]$. $\Omega^1_{B/A}$ is a free module of rank n with basis $\mathrm{d}x_i$. The map $f \mapsto \sum_{i=1}^n \frac{\partial f}{\partial x_i} \cdot \mathrm{d}x_i$ is an A-derivation and the map $\mathrm{d}x_i \mapsto x_i$ defines an isomorphism $\Omega^1_{B/A} \to B^{\oplus n}$.

Kähler differentials are also fairly easy to understand in the case of ring localizations and ring quotients. These will be important in understanding the sheaves of Kähler differentials of open and closed immersions in the case of schemes, respectively.

Proposition 1.7. Let A be a ring.

(i) If $B = S^{-1}A$, then $\Omega^1_{B/A} = 0$.

(ii) If $B = A/I$ for $I \subseteq A$ an ideal, then $\Omega^1_{B/A} = 0$.

Proof of (i). We already have that $\mathrm{d}a = 0$ for all $a \in A$. We then observe that writing $a = s \cdot \frac{a}{s}$ we have

$$\begin{aligned} 0 = \mathrm{d}(a) = \mathrm{d}\left(s \cdot \frac{a}{s}\right) &= s\mathrm{d}\left(\frac{a}{s}\right) + \frac{a}{s}\mathrm{d}(s) \\ &= s\mathrm{d}\left(\frac{a}{s}\right) && s \in A \Rightarrow \mathrm{d}s = 0 \end{aligned}$$

so $s\mathrm{d}(\frac{a}{s}) = 0$ and $\mathrm{d}(\frac{a}{s}) = 0$ whence the claim. ■

Proof of (ii). The map $A \to B$ is surjective, so this is precisely the situation Remark 1.3. ■

We can additionally understand sheaves of Kähler differentials in towers. Let $A \to B \to C$ be maps of rings. There is a natural C-linear map $\Omega^1_{B/A} \otimes_B C \to \Omega^1_{C/A}$ which is a C-module homomorphism induced by the diagram

$$\begin{tikzcd} B \arrow[r] \arrow[d, "\mathrm{d}_{B/A}"'] & C \arrow[r, "\mathrm{d}_{C/A}"] & \Omega^1_{C/A} \\ \Omega^1_{B/A} \arrow[urr, dashed, "\exists!"'] & & \end{tikzcd}$$

where the top row is both A and B-linear inducing a unique B-module map $\Omega^1_{B/A} \to \Omega^1_{C/A}$, considering the latter as a B-module. By the extension-restriction adjunction, however, we have

$$\mathrm{Hom}_{\mathsf{Mod}_B}(\Omega_{B/A}, \Omega_{C/A}|_B) \leftrightarrow \mathrm{Hom}_{\mathsf{Mod}_C}(\Omega_{B/A} \otimes_B C, \Omega_{C/A})$$

hence the data of the dotted map in the diagram above gives rise to a unique map $\Omega_{B/A} \otimes_B C \to \Omega_{C/A}$. Arguing similarly, there is a C-linear map $\Omega^1_{C/A} \to \Omega^1_{C/B}$ induced by

$$\begin{tikzcd} C \arrow[r, "\mathrm{d}_{C/B}"] \arrow[d, "\mathrm{d}_{C/A}"'] & \Omega^1_{C/B} \\ \Omega^1_{C/A} \arrow[ur, dashed, "\exists!"'] & \end{tikzcd}$$

where the map is induced by the universal property as any B-derivation is also an A-derivation.

The maps in the preceding discussion assemble to give the following proposition.

Proposition 1.8. Let $A \to B \to C$ be maps of rings. There is an exact sequence

$$\Omega^1_{B/A} \otimes_B C \to \Omega^1_{C/A} \to \Omega^1_{C/B} \to 0.$$

Proof. The above discussion gives the existence of such maps, so it remains to show exactness at $\Omega^1_{C/A}$ and surjectivity of the map $\Omega^1_{C/A} \to \Omega^1_{C/B}$.

We begin with the latter, where by the quotient construction of Proposition 1.5 it suffices to observe that $\Omega^1_{C/B}$ is a quotient of $\Omega^1_{C/A}$.

For the former, we note that for a fixed C-module M we have an exact sequence

$$0 \to \mathrm{Der}_B(C, M) \to \mathrm{Der}_A(C, M) \to \mathrm{Der}_A(B, M|_B)$$

since an A-derivation $(\mathrm{d} : C \to M)$ is taken to the composite $B \to C \to M$ which is zero when the map is also a B-derivation. Rewriting this using the universal property, this is

$$0 \to \mathrm{Hom}_{\mathsf{Mod}_C}(\Omega^1_{C/B}, M) \to \mathrm{Hom}_{\mathsf{Mod}_C}(\Omega^1_{C/A}, M) \to \mathrm{Hom}_{\mathsf{Mod}_C}(\Omega^1_{B/A} \otimes_B C, M)$$

which by contravariant exatness of the Hom-functor (see [Stacks, Tag 0582] for the precise statement), is the claim. ∎

As a corollary, we can deduce the following fact about localizations.

Corollary 1.9. Let B be an A-algebra and S a multiplicative subset of B. Then $S^{-1}\Omega^1_{B/A} \cong \Omega^1_{S^{-1}B/A}$.

Proof. Apply Proposition 1.8 to $C = S^{-1}B$ and note that $\Omega^1_{C/B} = 0$ so the map $S^{-1}\Omega^1_{B/A} \to \Omega^1_{S^{-1}B/A}$ is surjective. To prove injectivity, we produce an inverse map which is an A-derivation of $S^{-1}B$ to $S^{-1}\Omega^1_{B/A}$ by $\mathrm{d}(\frac{b}{s}) \mapsto \frac{1}{s}\mathrm{d}(b) - \frac{1}{s^2}b\mathrm{d}(s)$ which by the universal property can be seen to be the inverse. ∎

Note that in general $\Omega^1_{B/A} \otimes_B C \to \Omega^1_{C/A}$ is rarely injective.

Example 1.10. Let $A = k, B = k[x], C = k[x]/(x)$. So $\Omega^1_{B/A} \cong B\mathrm{d}x$ but $\Omega_{C/A} = \Omega_{k/k} = 0$.

On the other hand, there are situations in which the exact sequence of Proposition 1.8 extends to a short exact sequence.

Example 1.11. Let B be an A-algebra and $C = B[x_1, \dots, x_n]$. We then have a split short exact sequence

$$0 \to \Omega^1_{B/A} \otimes_B C \to \Omega^1_{C/A} \to \Omega^1_{C/B} \to 0$$

where denoting the map $\Omega^1_{C/A} \to \Omega^1_{C/B}$ by φ, we have the splitting $\Omega^1_{C/A} \to (\Omega^1_{B/A} \otimes_B C) \oplus \Omega^1_{C/B}$ prescribed by the C-derivation $f \mapsto \mathrm{d}_{B/A}(f) + \varphi(f)$ under the bijection

$$\mathrm{Hom}_{\mathsf{Mod}_C}(\Omega_{C/A}, (\Omega_{B/A} \otimes_B C) \oplus \Omega_{C/B}) \leftrightarrow \mathrm{Der}_A(C, (\Omega_{B/A} \otimes_B C) \oplus \Omega_{C/B}).$$

The following proposition describes the behavior of the module of Kähler differentials with respect to tensor products.

Proposition 1.12. Let B, A' be A-algebras. Then there is an isomorphism of B-modules $\Omega^1_{B/A} \otimes_B (B \otimes_A A') \cong \Omega^1_{(B\otimes_A A')/A'}$.

Proof. We contemplate the diagram

$$\begin{tikzcd} B \otimes_A A' \arrow[r] \arrow[d] & \Omega^1_{B/A} \otimes_A A' \\ \Omega^1_{(B\otimes_A A')/A'} \arrow[ur, dashed, "\exists!"'] & \end{tikzcd}$$

where the solid arrows are $B \otimes_A A'$-linear with $\Omega^1_{B/A} \otimes_A A' \cong (\Omega^1_{B/A} \otimes_B (B \otimes_A A'))$ and the dotted arrow induced by the universal property of $\Omega^1_{(B\otimes_A A')/A'}$. By applying the tensor-hom adjunction and the universal property of derivations, prescribing an inverse map to the dotted arrow is equivalent to producing an A-derivation of B in $\Omega^1_{(B\otimes_A A')/A'}$ and one observes that the map $b \mapsto \mathrm{d}_{(B\otimes_A A')/A'}(b \otimes 1)$ gives an inverse, whence the claim. ■

We now treat the case of quotients.

Proposition 1.13. Let $A \to B \to C$ be maps of rings where $C \cong B/\mathfrak{b}$ for some ideal $\mathfrak{b} \subseteq B$. There is an exact sequence

$$\mathfrak{b}/\mathfrak{b}^2 \to \Omega^1_{B/A} \otimes_B C \to \Omega^1_{C/A} \to 0.$$

Proof. We first observe that $\Omega^1_{C/B} = 0$ by Proposition 1.7 and $\Omega^1_{B/A} \otimes_B C \cong \Omega^1_{B/A}/\mathfrak{b}\Omega^1_{B/A}$.

We denote $\mathfrak{b}/\mathfrak{b}^2 \to \Omega^1_{B/A} \otimes_B C$ by δ, $b \mapsto \mathrm{d}b \otimes 1$. We first show δ is well-defined. For this, we want to show that $\mathrm{d}(b_1 b_2) \otimes 1$ is zero for $b_1, b_2 \in \mathfrak{b}$. Indeed, using the Leibniz rule, we have

$$\mathrm{d}(b_1 b_2) \otimes 1 = \mathrm{d}(f_2) \otimes f_1 + \mathrm{d}(f_1) \otimes f_2 \in \mathfrak{b}\Omega_{B/A}$$

hence zero in the quotient, showing the map is well-defined.

The diagram is a complex as db maps to zero in $\Omega^1_{C/A}$. The kernel of $\Omega^1_{B/A} \to \Omega^1_{C/A}$ is generated by the B-submodule $\mathfrak{b}\Omega^1_{B/A}$ and the elements $\mathrm{d}b$ for $b \in \mathfrak{b}$, showing exactness of the complex in the middle. ■

This specializes to finite type algebras.

Corollary 1.14. Let C be a finite type A-algebra – that is, the quotient of $B = A[x_1, \ldots, x_n]$. Then $\Omega^1_{C/A}$ is a finitely generated C-module.

Proof. Set $B = A[x_1, \ldots, x_n]$ for which $C = B/\mathfrak{b}$. Exactness of the sequence in Proposition 1.13 gives a surjection $\Omega^1_{B/A} \otimes_B C \to \Omega^1_{C/A}$, and observing that $\Omega^1_{B/A} \otimes_B C \cong B^{\oplus n} \otimes_B C \cong C^{\oplus n}$ gives a surjection $C^{\oplus n} \to \Omega^1_{C/A}$, showing that it is finitely generated. ■

Let us consider the case of quotients of multivariate polynomial rings by a single polynomial.

Example 1.15. Let A be a ring, $B = A[x_1, \dots, x_n]$, and $C = B/(f)$ for $f \in B$. By Proposition 1.13 and Corollary 1.14, we have that $\Omega^1_{C/A}$ is the cokernel of the map $\delta : (f)/(f)^2 \to \Omega^1_{B/A} \otimes_B C \cong C^{\oplus n}$ of Proposition 1.13, so is the quotient $\left(\bigoplus_{i=1}^n C\mathrm{d}x_i\right)/\mathrm{d}f$.

We can also consider the case of k-algebras.

Corollary 1.16. Let A be a k-algebra and $\mathfrak{m}$ a maximal ideal in A such that $\kappa(\mathfrak{m}) = A/\mathfrak{m} \cong k$. Then $\Omega^1_{A/k} \otimes_k \kappa(\mathfrak{m}) \cong \mathfrak{m}/\mathfrak{m}^2$.

Proof. This is precisely Proposition 1.13 for $k \to k[x_1, \dots, x_n] \to A$, and the map $\mathfrak{m}/\mathfrak{m}^2 \to \Omega^1_{A/k} \otimes_k \kappa(\mathfrak{m})$ is a surjection between vector spaces of the same dimension, hence an isomorphism. ■

Note that this is the dual of the Zariski tangent space $\mathrm{Hom}_{\mathsf{Vec}_{\kappa(\mathfrak{m})}}(\mathfrak{m}/\mathfrak{m}^2, \kappa(\mathfrak{m}))$, motivating the connection to schemes.

2. Lecture 2 – 10th April 2025

We begin with an example.

Example 2.1. Let $A = k, B = k[x,y], C = B/\mathfrak{b}$ where $\mathfrak{b} = (xy)$. We have that $\operatorname{Spec}(B)$ is the affine plane $\mathbb{A}^2_k$ and $\operatorname{Spec}(C)$ is the union of the two coordinate axes. The exact sequence of Proposition 1.13 gives

$$\mathfrak{b}/\mathfrak{b}^2 \to \Omega^1_{k[x,y]/k} \otimes_{k[x,y]} C \to \Omega^1_{C/k} \to 0.$$

Explicitly identifying $\mathfrak{b}/\mathfrak{b}^2$ with the C-module $(xy)/(x^2y^2)$ module-isomorphic to C by $1 \mapsto xy$ and $\Omega^1_{k[x,y]/k}$ with the free $k[x,y]$-module $B\mathrm{d}x \oplus B\mathrm{d}y$, we observe that the map $\mathfrak{b}/\mathfrak{b}^2 \to \Omega^1_{k[x,y]/k} \otimes_{k[x,y]} C$ is given by $\overline{xy} \mapsto \mathrm{d}(xy) \otimes 1 = (x\mathrm{d}y + y\mathrm{d}x) \otimes 1$. This yields a map $C \to C \oplus C$ by $1 \mapsto (y,x)$ and the cokernel of this map is the kernel of the map $C \oplus C \to C$ by $(a,b) \mapsto ax - by$ so by exactness the image of $C \oplus C \to C$ is the ideal $(x,y) \subseteq C$ showing $\Omega^1_{C/k} \cong (x,y) \subseteq C$. Thus $\Omega_{C/k} \otimes_C \frac{k[x,y]}{(x,y)} \cong kx \oplus ky$ and in particular $\Omega^1_{C/k} \otimes_C k$ is of k-dimension 2. For all points $\mathfrak{p} \in \operatorname{Spec}(C) \setminus \{(x,y)\}$, we have $\Omega_{C/k} \otimes_C \kappa(\mathfrak{p}) \cong k$ extending the exact sequence above to a short exact sequence.

In what follows, we will use the following lemma for Kähler differentials of field extensions, the proof of which we omit.

Lemma 2.2. Let k be a field and K/k a separable extension. Then $\Omega^1_{K/k} = 0$.

This lemma, in conjunction with Proposition 1.13, shows that for maximal ideals of k-algebras A with separable residue field, the base change of the sheaf of Kähler differentials to k is isomorphic to the Zariski tangent space $\mathfrak{m}/\mathfrak{m}^2$.

Proposition 2.3. Let A be a finite type k-algebra and $\mathfrak{m} \subseteq A$ maximal with residue field $\kappa(\mathfrak{m})$ separable over k. There is an isomorphism $\frac{\mathfrak{m}}{\mathfrak{m}^2} \to \Omega^1_{A/k} \otimes_A k$.

Proof. Let $\varphi : k[x_1, \ldots, x_n] \to A$ with kernel $\ker(\varphi) = \mathfrak{a}$ and $\widetilde{\mathfrak{m}} = \varphi^{-1}(\mathfrak{m})$. This yields a surjective map $\frac{\widetilde{\mathfrak{m}}}{\widetilde{\mathfrak{m}}^2} \to \frac{\mathfrak{m}}{\mathfrak{m}^2}$ with kernel $\mathfrak{a}$. By Proposition 1.13 we have the following diagram with bottom row exact

$$\begin{tikzcd}
\mathfrak{a} \arrow[r] \arrow[d] & \frac{\widetilde{\mathfrak{m}}}{\widetilde{\mathfrak{m}}^2} \arrow[r] \arrow[d] & \frac{\mathfrak{m}}{\mathfrak{m}^2} \arrow[r] \arrow[d] & 0 \\
\frac{\mathfrak{a}}{\mathfrak{a}^2} \arrow[r] & \Omega^1_{k[x_1,\ldots,x_n]/k} \otimes_{k[x_1,\ldots,x_n]} k \arrow[r] & \Omega^1_{A/k} \otimes_A k \arrow[r] & 0
\end{tikzcd}$$

and noting the top row is the $- \otimes_{k[x_1,\ldots,x_n]} k$ of the bottom, a right-exact operation, we get the claim. ■

We deduce the following result which will be required for defining the sheaf of Kähler differentials on schemes more generally.

Corollary 2.4. Let B be an A-algebra and I the kernel of the map $B \otimes_A B \to B$ by $b_1 \otimes b_2 \mapsto b_1 b_2$. Then $\Omega^1_{B/A} \cong I/I^2$ as B-modules by $\mathrm{d}b \mapsto 1 \otimes b - b \otimes 1$, and where I/I^2 has the structure of a B-module by $b(b_1 \otimes b_2) = bb_1 \otimes b_2$.

Proof. We use the universal property of Kähler differentials. Defining δ by $b \mapsto 1 \otimes b - b \otimes 1$ we get the diagram

$$\begin{tikzcd} B \arrow[r, "\delta"] \arrow[d] & I/I^2 \\ \Omega^1_{B/A}. \arrow[ur, dashed, "\exists!"'] & \end{tikzcd}$$

Note for $b_1, b_2 \in B$, we compute

$$\delta(b_1 \cdot b_2) = 1 \otimes b_1 b_2 - b_1 b_2 \otimes 1 = b_1 \delta(b_2) + b_2 \delta(b_1) = b_1(1 \otimes b_2 - b_2 \otimes 1) + b_2(1 \otimes b_1 - b_1 \otimes 1).$$

The difference of the two expresssions is $(1 \otimes b_1 - b_1 \otimes 1)(1 \otimes b_2 - b_2 \otimes 1)$ the product of two elements of I, hence in I^2, hence vanishes in the quotient. Since this is a derivation, there exists an extension $\Omega^1_{B/A} \to I/I^2$.

To show surjectivity, we consider an element $\sum b_i \otimes b_i' \in I$ and compute

$$\begin{aligned} \sum b_i \otimes b_i' &= \sum b_i (1 \otimes b_i') \\ &= \sum b_i(1 \otimes b_i') - \underbrace{\left(\sum b_i b_i'\right)}_{=0} \otimes 1 \\ &= \sum b_i (1 \otimes b_i' - b_i' \otimes 1) \end{aligned}$$

showing it is surjective.

To show injectivity, we consider the diagram

$$\begin{tikzcd} B \arrow[r, "\delta"] \arrow[d, "\mathrm{d}"'] & M \\ \Omega^1_{B/A} \arrow[ur] \arrow[d] & \\ I/I^2 \arrow[uur, dashed] & \end{tikzcd}$$

The existence of a dotted arrow rendering the entire diagram commutative would imply the injectivity of δ for $M = \Omega^1_{B/A}$. Note that the B-module $B \oplus M$ can be given the structure of a free algebra by the map $b \mapsto b \oplus 0$ and multiplication $(b_1, m_1) \cdot (b_2, m_2) = (b_1 b_2, b_1 m_2 + b_2 m_1)$, which defines a B-algebra in which M is an ideal with square zero. We can define a map $\varphi : B \otimes_A B \to B \oplus M$ by $b_1 \otimes b_2 \mapsto (b_1 b_2, b_1 \delta(b_2))$ which is a homomorphism of A-algebras and where the image of the ideal I is zero since M^2 is zero. Thus an extension $\psi : I/I^2 \to M$ and the diagram commutes, yielding injectivity, and hence the claim. ∎

We now seek to define the sheaf of Kähler differentials on a scheme.

Definition 2.5 (Relative Kähler Differentials of a Scheme)**.** Let $f : X \to Y$ be a morphism of schemes. The sheaf of Kähler differentials of X is locally given by the $\mathcal{I}/\mathcal{I}^2$ of the locally closed embedding $X \to X \times_Y X$.

We will most often be interested in the situation where $Y = \operatorname{Spec}(k)$ and f is the structure map of f as a k-scheme.

Example 2.6. Let $X = \mathbb{A}^n_k$ over $\operatorname{Spec}(A)$. $\Omega^1_{X/A}$ is free of rank n as in Example 1.6.

Example 2.7. Let $X = \mathbb{P}^n_A$ over $\operatorname{Spec}(A)$. $\Omega^1_{X/A}$ is the locally free sheaf of rank n obtained by gluing the free sheaves of Example 2.6 on the distinguished affine opens $D_+(x_i)$ of $\mathbb{P}^n_A$.

The sheaf of Kähler differentials is another example of an interesting sheaf on schemes which is not the structure sheaf. In the case of projective space, the relationship between the sheaf of Kähler differentials is related to the structure sheaf by the Euler sequence.

Theorem 2.8 (Euler Sequence)**.** Let A be a ring. There is a short exact sequence of sheaves

$$0 \longrightarrow \Omega^1_{\mathbb{P}^n_A/A} \longrightarrow \mathcal{O}_{\mathbb{P}^n_A}(-1)^{\oplus n+1} \xrightarrow{(x_0,\dots,x_n)} \mathcal{O}_{\mathbb{P}^n_A} \longrightarrow 0$$

on $\mathbb{P}^n_A$.

Proof. We have $X = \operatorname{Proj}(B)$ with $B = A[x_0, \dots, x_n]$ and $\mathcal{O}_{\mathbb{P}^n_A}(-1) = \widetilde{B(1)}$. The map $\mathcal{O}_{\mathbb{P}^n_A}(-1)^{\oplus n+1} \to \mathcal{O}_{\mathbb{P}^n_A}$ is given by the module homomorphism given by the dot product map, and is surjective since $\bigcap_{i=0}^n V_+(x_i) = \emptyset$ and $- \otimes \mathcal{O}_{\mathbb{P}^n_A}(1)$ being right-exact. We show that the kernel of this map is $\Omega^1_{\mathbb{P}^n_A/A}$ affine-locally.

On $D_+(x_i)$, the map is given by localizations $B(-1)^{\oplus n+1}_{(x_i)} \to B_{(x_i)}$ and the kernel is free of rank n generated by $e_j - \frac{x_j}{x_i} e_i$ for $j \neq i$. In particular, the kernel is a free $\mathcal{O}_{D_+(x_i)}$-module generated by $\frac{1}{x_i} e_j - \frac{x_j}{x_i^2} e_i$ for $j \neq i$. Recall that $\Omega^1_{\mathbb{P}^n_A/A}$ is the free $\mathcal{O}_{D_+(x_i)}$-module spanned by $\mathrm{d}(\frac{x_0}{x_i}), \dots, \mathrm{d}(\frac{x_n}{x_i})$ and the isomorphism to the kernel of the map is given by $\mathrm{d}(\frac{x_j}{x_i}) \mapsto \frac{1}{x_i} e_j - \frac{x_j}{x_i^2} e_i$. The isomorphisms glue by inspection, giving the isomorphism of sheaves. ■

Example 2.9. Let $n = 1$. The Euler sequence gives $0 \to \Omega^1_{\mathbb{P}^1_A/A} \to \mathcal{O}_{\mathbb{P}^1_A}(-1)^{\oplus 2} \to \mathcal{O}_{\mathbb{P}^1_A} \to 0$ where passing to determinants gives $\det(\mathcal{O}_{\mathbb{P}^1_A}(-1)^{\oplus 2}) \cong \det(\mathcal{O}_{\mathbb{P}^1_A}) \otimes \det(\Omega^1_{\mathbb{P}^1_A})$ showing $\Omega^1_{\mathbb{P}^1_A/A} \cong \mathcal{O}_{\mathbb{P}^1_A}(-2)$.

Example 2.10. For $n > 1$m $\Omega^1_{\mathbb{P}^n_A/A}$ is never a direct sum of line bundles. We have $\det(\Omega^1_{\mathbb{P}^n_A/A}) \cong \mathcal{O}_{\mathbb{P}^n_A}(-n-1)$. Twisting by $\mathcal{O}_{\mathbb{P}^n_A}(1)$, we get a short exacts sequence

$$0 \to H^0(\mathbb{P}^n_A, \Omega^1_{\mathbb{P}^n_A/1}) \to H^0(\mathbb{P}^n_A, \mathcal{O}^{\oplus n+1}_{\mathbb{P}^n_A}) \to H^0(\mathbb{P}^n_A, \mathcal{O}_{\mathbb{P}^n_A}(1)) \to 0$$

since H^1 of all the sheaves in the Euler sequence vanishes. If $\Omega^1_{\mathbb{P}^n_A/A} \cong \bigoplus \mathcal{O}_{\mathbb{P}^n_A}(a_i)$ then $a_i \leq -2$ and $\sum a_i \leq -2n$ which has global sections, a contradiction for $n \geq 2$.

3. Lecture 3 – 14th April 2025

Recall that the cotangent sheaf on $\mathbb{A}^n_A, \mathbb{P}^n_A$ over $\mathrm{Spec}(A)$ are locally free sheaves by Examples 2.6 and 2.7. One is then led to consider for what other S-schemes X is $\Omega^1_{X/S}$ locally free. This is roughly captured by smoothness. Moreover, as suggested by Proposition 2.3, the notion of smoothness is connected to the Zariski tangent space, which in the case of algebraic geometry – unlike differential geometry – need not coincide with the geometric tangent space, especially in characteristic p situations.

We recall the definition of the Zariski tangent space.

Definition 3.1 (Zariski Tangent Space of a Ring)**.** Let A be a local ring with maximal ideal $\mathfrak{m}$ and residue field $\kappa = A/\mathfrak{m}$. The Zariski tangent space of A is the κ-vector space $(\frac{\mathfrak{m}}{\mathfrak{m}^2})^\vee = \mathrm{Hom}_{\mathsf{Vect}_\kappa}(\frac{\mathfrak{m}}{\mathfrak{m}^2}, \kappa)$.

Definition 3.2 (Zariski Tangent Space of a Scheme)**.** Let X be a scheme and $x \in X$ a point. The Zariski tangent space $T_{X,x}$ is the Zariski tangent space of the local ring $\mathcal{O}_{X,x}$.

Example 3.3. Let A be a ring and $\mathfrak{p} \subseteq \mathrm{Spec}(A)$. The Zariski tangent space $T_{\mathrm{Spec}(A),\mathfrak{p}}$ is given by $(\frac{\mathfrak{p}A_\mathfrak{p}}{(\mathfrak{p}A_\mathfrak{p})^2})^\vee$. This is a vector space over the field $\kappa(\mathfrak{p}) = A_\mathfrak{p}/\mathfrak{p}A_\mathfrak{p}$.

Example 3.4. Let k be a field and $x \in \mathbb{A}^n_k(k)$ a closed k-rational point hence of the form $(x_1 - a_1, \ldots, x_n - a_n)$. Define a map $D_x : k[x_1, \ldots, x_n] \to \mathrm{Hom}_{\mathsf{Vect}_k}(k^n, k)$ by

$$f \mapsto \left[(\alpha_i)_{i=1}^n \mapsto \sum_{i=1}^n \alpha_i \frac{\partial f}{\partial x_i}(x) \right].$$

The map is k-linear and satisfies the Leibniz rule, hence defines a k-linear derivation which is a k-vector space. This defines an isomorphism between $(\frac{\mathfrak{m}}{\mathfrak{m}^2})^2$ and $\mathrm{Hom}_{\mathsf{Vect}_k}(k^n, k)$ by considering the Taylor expansion of a polynomial

$$f = f(x) + \sum_{i=1}^n \frac{\partial f}{\partial x_i}(x)(x_i - a_i) + \underbrace{O(x^2)}_{\in \mathfrak{m}^2}$$

hence the map is zero on $f \in \mathfrak{m}^2$ showing that $(\frac{\mathfrak{m}}{\mathfrak{m}^2})^\vee \to \mathrm{Hom}_{\mathsf{Vect}_k}(k^n, k)$ by $(x_i - a_i) \mapsto e_i^\vee$ is an injection between vector spaces of the same dimension and hence an isomorphism.

Example 3.5. In general, one can still define a map on non-rational points with target $\mathrm{Hom}_{\mathsf{Vect}_{\kappa(\mathfrak{p})}}(\kappa(\mathfrak{p})^n, \kappa(\mathfrak{p}))$ which may fail to be injective. Let k be a field of characteristic p and consider $(x^p - a) \subseteq k[x]$ which is maximal when $a^{1/p} \notin k$. We have $\frac{\mathfrak{m}}{\mathfrak{m}^2} = \frac{(x^p - a)}{(x^p - a)^2} \cong k$ which defines a map $\mathrm{Hom}_{\mathsf{Vect}_k}(k, k)$ by Example 3.4 which is the zero map as $px^{p-1} = 0$.

In what follows, we will use the following result for closed subschemes of affine spaces.

Proposition 3.6. Let $\mathfrak{a} \subseteq k[x_1, \ldots, x_n]$ be an ideal defining $X = V(\mathfrak{a}) \subseteq \mathbb{A}^n_k$. Let $x \in X(k) \subseteq \mathbb{A}^n_k(k)$. Then $T_{X,x}$ is the annihilator of the image of $\mathfrak{a}$ under D_x

Proof. We have a short exact sequence

$$0 \to \mathfrak{a} \to \widetilde{\mathfrak{m}} \to \mathfrak{m} \to 0$$

inducing

$$0 \to \frac{\mathfrak{a}}{\mathfrak{a} \cap \widetilde{\mathfrak{m}}^2} \to \frac{\widetilde{\mathfrak{m}}}{\widetilde{\mathfrak{m}}^2} \to \frac{\mathfrak{m}}{\mathfrak{m}^2} \to 0. \tag{3.1}$$

Applying the right-exact functor $\mathrm{Hom}_{\mathsf{Vect}_k}(-, k)$ we get

$$\left(\frac{\mathfrak{a}}{\mathfrak{a} \cap \widetilde{\mathfrak{m}}^2}\right)^\vee \to T^\vee_{\mathbb{A}^n_k,x} \to T^\vee_{X,x} \to 0$$

where the map $\left(\frac{\mathfrak{a}}{\mathfrak{a} \cap \widetilde{\mathfrak{m}}^2}\right)^\vee \to \mathrm{Hom}_{\mathsf{Vect}_k}(k^n, k)$ by taking $\mathfrak{a}$-derivations as in Example 3.4. ■

An analogous proof can be used to show that the Zariski tangent space is the cokernel of the Jacobian matrix.

Corollary 3.7. Let $\mathfrak{a} \subseteq k[x_1, \ldots, x_n]$ be an ideal defining $X = V(\mathfrak{a}) \subseteq \mathbb{A}^n_k$. Let $x \in X(k) \subseteq \mathbb{A}^n_k(k)$. Then $T^\vee_{X,x} \cong \mathrm{coker}(J_x)$ where J_x is the Jacobian at x.

Proof. We use the short exact sequence (3.1) and observe that the map $\frac{\mathfrak{a}}{\mathfrak{a} \cap \widetilde{\mathfrak{m}}^2} \to \frac{\widetilde{\mathfrak{m}}}{\widetilde{\mathfrak{m}}^2}$ is given by multiplication by the Jacobian, giving the claim. ■

Having related this to the Zariski tangent space, we want to relate the Jacobian matrix to the sheaf/module of Kähler differentials, an analogy suggested by Proposition 2.3.

Proposition 3.8. Let $\mathfrak{a} \subseteq k[x_1, \ldots, x_n]$ be an ideal defining $X = V(\mathfrak{a}) \subseteq \mathbb{A}^n_k$. Let $x \in X(k) \subseteq \mathbb{A}^n_k(k)$. Then the corank of the Jacobian J_x is equal to $\dim_{\kappa(x)} \Omega^1_{X/k} \otimes \kappa(x)$.

Proof. Applying $- \otimes \kappa(x)$ to the short exact sequence of Proposition 1.13, we have

$$\frac{\mathfrak{a}}{\mathfrak{a}^2} \otimes \kappa(x) \to \Omega^1_{\mathbb{A}^n_k/k} \otimes \kappa(x) \to \Omega^1_{X/k} \otimes \kappa(x) \to 0$$

which factors over the image of the Jacobian J_x. As such, we get that $\dim_{\kappa(x)} \Omega^1_{X/k} = n - \dim(\mathrm{im}(J_x)) = n - \mathrm{rank}(J_x)$ which is precisely the corank. ■

Moreover, for a general point x, the property of the Zariski tangent space being isomorphic to the scalar extension of the sheaf of Kähler differentials.

Proposition 3.9. Let $\mathfrak{a} \subseteq k[x_1, \ldots, x_n]$ be an ideal defining $X = V(\mathfrak{a}) \subseteq \mathbb{A}^n_k$. Let $x \in X$. $\kappa(x)$ is a separable extension of k if and only if $T^\vee_{X,x} \cong \Omega^1_{X/k} \otimes \kappa(x)$.

Proof. ($\Rightarrow$) If $\kappa(x)$ is separable over k, then $\Omega^1_{\kappa(x)/k} = 0$ by Lemma 2.2 so $\frac{\mathfrak{m}}{\mathfrak{m}^2} \to \Omega^1_{X,k} \otimes \kappa(x)$ is surjective, but this shows that we have a surjection of $\kappa(x)$-vector spaces of the same dimension, hence an isomorphism.

($\Leftarrow$) If the Zariski cotangent space is isomorphic to teh sheaf of differentials, then the cokernel $\Omega^1_{\kappa(x)/k}$ of the exact sequence Proposition 1.13 is zero, showing that $\kappa(x)/k$ is separable. ■

Note that the equality $\dim(T_{X,x}) = \dim_{\kappa(x)} \Omega^1_{X/k} \otimes \kappa(x)$ does not imply the natural map is an isomorphism when $\kappa(x)$ is not separable over k.

Example 3.10. Let k be a field of characteristic p and consider $(x^p - a) \subseteq k[x]$ for $a^{1/p} \notin k$. Denoting $X = V(x^p - a) \subseteq \mathbb{A}^1_k$, we have $\dim(T_{X,x}) = 1 = \dim_{\kappa(x)} \Omega^1_{X/k} \otimes \kappa(x)$ but $\Omega^1_{\kappa(x)/k}$ is nonzero as $\kappa(x)$ is not a separable extension of k.

We arrive at the notion of smoothness for schemes.

Definition 3.11 (Smooth Scheme)**.** Let X be a scheme of finite type over a field k. X is smooth of pure dimension d if

(i) each of the finitely many irreducible components of X are of dimension d, and
(ii) every point $x \in X$ is contained in an affine open neighborhood where the Jacobian matrix is of corank d.

Remark 3.12. Smoothness is a relative notion, determined by the structure map to $\mathrm{Spec}(k)$.

Remark 3.13. By Example 3.3, this construction is independent of the choice of chart.

Remark 3.14. It suffices to verify this condition on closed points, as if the Jacobian is rank-deficient at some non-closed point, then it is rank-deficient at any specialization.

Intuitively, we can view X locally as the fiber of a map $\mathbb{A}^n_k \to \mathbb{A}^r_k$ defined by the r polynomials $f_1, \dots, f_r \in k[x_1, \dots, x_n]$, and where x being in the fiber over zero implies that the tangent space $T_{X,x}$ is the kernel of the map $(f_1, \dots, f_r)$ hence equal to the dimension of the fiber. We introduce the notion of being geometrically smooth.

Definition 3.15 (Geometrically Smooth Scheme)**.** Let X be a scheme of finite type over a field k. X is geometrically smooth if the base change $X_{\overline{k}}$ to the algebraic closure is smooth over $\overline{k}$.

This is in fact equivalent to the condition of being smooth.

Lemma 3.16. Let X be a scheme of finite type over a field k. X is a smooth k-scheme if and only if it is geometrically smooth.

Proof. We use the Cartesian square

$$\begin{tikzcd} X_{\overline{k}} \arrow[r] \arrow[d] & X \arrow[d] \\ \mathrm{Spec}(\overline{k}) \arrow[r] & \mathrm{Spec}(k) \end{tikzcd}$$

where using Proposition 1.8, we have $\Omega^1_{X/k} \otimes \kappa(x) \cong \Omega^1_{X_{\overline{k}}/\overline{k}} \otimes \kappa(y)$ where y is the closed point corresponding to x in $X_{\overline{k}}$. This isomorphism of sheaves characterizes the Jacobian being full rank at x, y, hence the smoothness conditions are equivalent. ■

While smoothness depends on the structure of X as a k-scheme, it is closely related to the absolute notion of regularity.

We recall the relevant definitons from commutative algebra.

Definition 3.17 (Regular Local Ring)**.** Let $(A, \mathfrak{m})$ be a Noetherian local ring with residue field $\kappa = A/\mathfrak{m}$. A is a regular local ring if $\dim(A) = \dim_\kappa(\frac{\mathfrak{m}}{\mathfrak{m}^2})$.

Definition 3.18 (Regular Ring)**.** Let A be a Noetherian ring. A is a regular ring if for all primes $\mathfrak{p} \subseteq A$, the localization $A_\mathfrak{p}$ is a regular local ring.

Remark 3.19. Checking regularity of an arbitrary Noetherian ring can be done on maximal ideals by reasoning analogous to that of Remark 3.14.

This allows us to define regularity of schemes.

Definition 3.20 (Regular Scheme)**.** Let X be a locally Noetherian scheme. X is regular if for all closed points $x \in X$, the local ring $\mathcal{O}_{X,x}$ is regular.

Remark 3.21. By Definition 3.18, this is equivalent to each point admitting an affine neighborhood given by the Zariski spectrum of a regular ring.

Remark 3.22. In contrast to Remark 3.12, regularity is absolute and does not depend on any structure map of X.

The notions of regularity and smoothness are connected by the following proposition.

Proposition 3.23. Let X be a scheme of finite type over an algebraically closed field k. X is k-smooth if and only if X is regular.

Proof. Both conditions can be checked affine-locally, so without loss of generality, we can take $X = V(\mathfrak{a}) \subseteq \mathbb{A}^n_k$. By Proposition 3.6 the dimension of the Zariski tangent space of any $x \in X$ is dimension of the image of the map D_x defined in Example 3.4, which is equal to the rank of the Jacobian J_x. This is of rank equal to the Zariski tangent space (ie. X is regular) if and only if the corank of the Jacobian is $\dim(X)$ (ie. X is smooth). ■

Over general fields, smoothness implies regularity, but not the converse.

Corollary 3.24. Let X be a scheme of finite type over a field k. If X is k-smooth, then X is regular.

Proof. By locality, we reduce once more to $X = V(\mathfrak{a}) \subseteq \mathbb{A}^n_k$. By smoothness, $D_x(\mathfrak{a}) = \operatorname{rank}(J_x)$ and by the short exact sequence (3.1) we have

$$\dim_{\kappa(\mathfrak{m})}\left(\frac{\mathfrak{m}}{\mathfrak{m}^2}\right) = \dim_{\kappa(\mathfrak{m})}\left(\frac{\widetilde{\mathfrak{m}}}{\widetilde{\mathfrak{m}}^2}\right) - \dim_{\mathfrak{m}}\left(\frac{\mathfrak{a}}{\mathfrak{a} \cap \widetilde{\mathfrak{m}}^2}\right)$$

showing that the dimension of the Zariski tangent space at x is equal to the dimension of X, hence X is regular. ■

We now see an example of a regular non-smooth scheme.

Example 3.25. Let k be a field of characteristic p and consider $X = V(x^p - a) \subseteq \mathbb{A}^1_k$ and where $a^{1/p} \notin k$. $\mathrm{Spec}(\frac{k[x]}{(x^p-a)})$ is the Zariski spectrum of a field, hence regular, but X is not geometrically smooth and hence not smooth.

4. Lecture 4 – 17th April 2025

We prove another characterization of smoothness via freeness of the sheaf of Kähler differentials using the following lemma from commutative algebra, the proof of which we omit.

Lemma 4.1. Let A be a Noetherian integral local ring with fraction field K and residue field κ. If M is a finite A-module then M is free if and only if $\dim_K(M \otimes_A K) = \dim_\kappa(M \otimes_A \kappa)$.

We now state and prove the proposition.

Proposition 4.2. Let X be locally of finite type over a field k of pure dimension d. X is k-smooth if and only if $\Omega^1_{X/k}$ is locally free of dimension d.

Proof. By Lemma 3.16 and Proposition 3.23, X being k-smooth is equivalent to X being geometrically regular – ie. $X_{\overline{k}}$ being regular.

It thus suffices to show that that all closed points $y \in X_{\overline{k}}(\overline{k})$ $T_{X_{\overline{k}},y} = \dim(X)$. But X being regular and Lemma 4.1 show that this condition holds as

$$\dim(X) = \operatorname{trdeg}(K(X_{\overline{k}})) = \dim(\Omega^1_{X_{\overline{k}}/\overline{k}} \otimes \kappa(\eta)) = \dim(\Omega^1_{X_{\overline{k}}/\overline{k}} \otimes \kappa(y)) = \dim(T_{X_{\overline{k}},y})$$

as desired. ■

Corollary 4.3. Let X be a finite type integral k-scheme. X is smooth if and only if $\Omega^1_{X/k}$ is locally free and $\kappa(x)/k$ is separable for all closed points $x \in X$.

Proof. ($\Rightarrow$) By Proposition 4.2, $\Omega^1_{X/k}$ is locally free of rank the dimension of X so $\dim_{K(X)}(\Omega_{K(X)/k}) = \dim(X)$ showing that the residue field of each closed point is separable.

($\Leftarrow$) Using Proposition 4.2 the conclusion is immediate. ■

Corollary 4.4. Let X be a finite type k-scheme where k is a perfect field. X is smooth if and only if X is regular.

Proof. ($\Rightarrow$) Smoothness implies regularity in general by Corollary 3.24.

($\Leftarrow$) Let X be regular. Then X is locally integral. Smoothness is local on source so without loss of generality $X = \operatorname{Spec}(A)$ is integral and $x \in \operatorname{Spec}(A)$ maximal. $\kappa(x)/k$ is finite and separable as k was perfect. So $\Omega^1_{\kappa(x)/k} = 0$ and $\frac{\mathfrak{m}}{\mathfrak{m}^2} \cong \Omega^1_{X/k} \otimes \kappa(x)$. Applying Lemma 4.1 to $\Omega^1_{A/k}$, we have that the module of differentials is locally free, hence the claim. ■

We show that smoothess is in fact true on a nonempty open, and hence dense, subset of an integral scheme.

Proposition 4.5. Let X be an integral finite type k scheme and $K(X)/k$ separable. There exists a nonempty open set U such that U is k-smooth.

Proof. $\kappa(x)/k$ is separable if and only if $\dim \Omega_{\kappa(x)/k} = \dim(X)$. Recall that for X Noetherian and $\mathcal{F}$ coherent, the function $x \mapsto \dim_{\kappa(x)}(\mathcal{F} \otimes_{\mathcal{O}_{X,x}} \kappa(x))$ is uppersemicontinuous. Denote

$$Z_n = \{x \in X : \dim_{\kappa(x)}(\Omega^1_{X/k} \otimes \kappa(x))\}$$

which is closed.

Let $U = X \setminus Z_{\dim(X)+1}$ which is open and nonempty as it contains the generic point. We have $\Omega^1_{U/k} = \Omega^1_{X/k}|_U$ which is constant rank at each stalk, hence locally free, showing that U is smooth. ■

In particular this implies that the codimension of the non-smooth locus is at least 2 for X normal and $K(X)/k$ separable.

We consider the following example where the extension is not separable, and the smooth locus is empty.

Example 4.6. Let $(x^p - a) \subseteq k[x]$ where k is a field of characteristic p and $a^{1/p} \notin k$. The quotient $k[x]/(x^p - a)$ is a field but not separable over k and the smooth locus is empty.

We now discuss smoothness of subschemes.

Proposition 4.7. Let X be of finite type over a field k and $Y \subseteq X$ a closed irreducible subscheme. Y is k-smooth if and only if $\Omega^1_{Y/k}$ is locally free and the exact sequence $\mathcal{I}_Y/\mathcal{I}_Y^2 \to \Omega^1_{X/k}|_Y \to \Omega^1_{Y/k} \to 0$ extends on the left to a short exact sequence.

Proof. $(\Rightarrow)$ Consider

$$\begin{tikzcd}
\mathcal{I}_Y/\mathcal{I}_Y^2 \arrow[rr] \arrow[dr, two heads] & & \Omega^1_{X/k}|_Y \arrow[r] & \Omega^1_{Y/k} \arrow[r] & 0 \\
& K \arrow[ur, hook] & & & \\
0 \arrow[ur] & & & &
\end{tikzcd}$$

where $0 \to K \to \Omega^1_{X/k}|_Y \to \Omega^1_{Y/k} \to 0$ is a short exact sequence. Observe K is a locally free sheaf of rank equal to the codimension of Y in X. In particular, there are sections $s_1, \ldots, s_c$ whose images $\mathrm{d}s_1, \ldots, \mathrm{d}s_c$ generate $\Omega^1_{X/k}|_Y$ which locally freely generate K. Let $\mathcal{I}$ be the ideal generated by these sections $s_1, \ldots, s_c$ and $Y' = V(\mathcal{I})$ the associated vanishing locus which contains Y. Arguing similarly, we get

$$\begin{tikzcd}
\mathcal{I}'/\mathcal{I}'^2 \arrow[rr] \arrow[dr, two heads] & & \Omega^1_{X/k}|_{Y'} \arrow[r] & \Omega^1_{Y'/k} \arrow[r] & 0 \\
& K' \arrow[ur, hook] & & & \\
0 \arrow[ur] & & & &
\end{tikzcd}$$

with the short exact sequence $0 \to K' \to \Omega^1_{X/k}|_{Y'} \to \Omega^1_{Y'/k} \to 0$. Since $\mathrm{d}s_1, \ldots, \mathrm{d}s_c$ generate freely a subsheaf of $\Omega^1_{X/k}|_Y$, $\mathrm{d}s_1|_{Y'}, \ldots, \mathrm{d}s_c|_{Y'}$ generate freely a subsheaf of $\Omega^1_{X/k}|_{Y'}$ hence $\mathcal{I}'/\mathcal{I}'^2 \to K'$ is an isomorphism. By construction, $K'|_Y = K$ so

$\Omega^1_{Y'/k}|_Y \cong \Omega^1_{Y/k}$ the latter being locally free, so $\Omega^1_{Y'/k}$ is locally free in a neighborhood of Y. Y' is smooth of dimension equal to Y and smoothness implies local integrability so $\Omega^1_{Y/k}$ is locally free, yielding the claim.

($\Leftarrow$) It suffices to argue that the existence of a short exact sequence $0 \to \mathcal{I}_Y/\mathcal{I}_Y^2 \to \Omega^1_{X/k}|_Y \to \Omega^1_{Y/k} \to 0$ with $\Omega^1_{Y/k}$ locally free implies separability of $K(X)/k$. Note that under the hypotheses, the latter two terms of the short exact sequence are locally free, giving a splitting and showing in particular that $\mathcal{I}_Y/\mathcal{I}_Y^2$ is locally free of rank $r = \dim(X) - \operatorname{rank}(\Omega^1_{Y/k})$. In particular, we have $\dim(X) - r = \operatorname{rank}(\Omega^1_{Y/k}) \geq \dim(Y)$ and by $\mathcal{I}_Y$ being locally generated by r elements that $\dim(Y) \geq \dim(X) - r$ by Nakayama's lemma. This gives equality $\Omega^1_{Y/k} = \dim(Y)$ showing Y is smooth. ■

The preceding discussion of closed subschemes motivates the following definition of locally complete intersections.

Definition 4.8 (Locally Complete Intersection)**.** Let X be a smooth k scheme locally of finite type and $Y \subseteq X$ a closed subscheme. Y is a local complete intersection if $\mathcal{I}_Y$ is locally generated by c elements, where $c = \operatorname{codim}_X(Y) = \dim(X) - \dim(Y)$.

Smooth closed subschemes are in particular local complete intersections.

Example 4.9. Let X be a smooth k scheme locally of finite type and $Y \subseteq X$ a closed subscheme. If Y is smooth then Y is local complete intersection. Using Proposition 4.7, the short exact sequence

$$0 \to \mathcal{I}_Y/\mathcal{I}_Y^2 \to \Omega^1_{X/k}|_Y \to \Omega^1_{Y/k} \to 0$$

gives that $\mathcal{I}_Y/\mathcal{I}_Y^2$ is locally free of rank the codimension of Y in X.

Example 4.10. Being a local complete intersection does not imply reducedness or integrality. For $Y = V(x^2) \subseteq \mathbb{A}^1_k = \operatorname{Spec}(k[x])$ the ideal sheaf is generated by one element, but $k[x]/(x^2)$ is not reduced.

As suggested by the prevalence of quotients of ideal sheaves $\mathcal{I}_Y/\mathcal{I}_Y^2$ in the preceding discussion, this plays an important role in the study of smoothness of schemes.

Definition 4.11 (Conormal Sheaf)**.** Let X be a smooth k-scheme locally of finite type and $Y \subseteq X$ a local complete intersection closed subscheme. The conormal bundle $\mathcal{N}^\vee_{Y/X}$ is the sheaf $\mathcal{I}_Y/\mathcal{I}_Y^2$.

Dualizing yields the normal sheaf.

Definition 4.12 (Normal Sheaf)**.** Let X be a smooth k-scheme locally of finite type and $Y \subseteq X$ a local complete intersection closed subscheme. The normal bundle $\mathcal{N}_{Y/X}$ is the dual of the conormal sheaf $\underline{\operatorname{Hom}}_{\mathsf{Mod}_{\mathcal{O}_Y}}(\mathcal{I}_Y/\mathcal{I}_Y^2, \mathcal{O}_Y)$.

This gives the conormal bundle sequence.

Proposition 4.13 (Conormal Bundle Sequence)**.** Let X be a smooth k-scheme locally of finite type and $Y \subseteq X$ a local complete intersection closed subscheme. There is a short exact sequence

$$0 \to \mathcal{N}^\vee_{Y/X} \to \Omega^1_{X/k}|_Y \to \Omega^1_{Y/k} \to 0$$

of sheaves on Y.

Proof. This is immediate from Proposition 4.7. ■

Dual to the conormal and normal bundles, we can define tangent bundles.

Definition 4.14 (Tangent Bundle)**.** Let X be a smooth k-scheme locally of finite type. The tangent bundle $\mathcal{T}_X$ is $\underline{\mathrm{Hom}}_{\mathsf{Mod}_{\mathcal{O}_X}}(\Omega^1_{X/k}, \mathcal{O}_X)$.

In particular, we record the following observation.

Example 4.15. Let X be a smooth scheme. $(\mathcal{T}_X)_x = T_{X,x}$ the former being the tangent bundle and the latter the Zariski tangent space.

Recalling that dualizing preserves exactness, we have the normal bundle sequence.

Proposition 4.16 (Normal Bundle Sequence)**.** Let X be a smooth k-scheme locally of finite type and $Y \subseteq X$ a smooth closed subscheme. There is a short exact sequence

$$0 \to \mathcal{T}_Y \to \mathcal{T}_X|_Y \to \mathcal{N}_{Y/X} \to 0$$

of sheaves on Y.

Proof. This is obtained by dualizng Proposition 4.13. ■

The construction of the sheaf of Kähler differentials in fact is closely connected to Serre duality, as alluded to in the lectures of the preceding semester.

Definition 4.17 (Canonical Bundle)**.** Let X be a smooth k-scheme locally of finite type of pure dimension d. The canonical bundle of X, $\omega_{X/k}$ is $\Omega^d_{X/k} = \bigwedge^d \Omega^1_{X/k}$.

The normal bundle sequence immediately gives the relationship between the canonical bundle of a smooth scheme and a smooth subscheme.

Proposition 4.18 (Adjunction Formula)**.** Let X be a smooth k-scheme locally of finite type and $Y \subseteq X$ a smooth closed subscheme. Then

$$\omega_{Y/k} \cong \omega_{X/k}|_Y \otimes \det(\mathcal{N}_{Y/X}).$$

Proof. Note that for $0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to 0$ a short exact sequence, we have $\det(\mathcal{F}) \cong \det(\mathcal{F}') \otimes \det(\mathcal{F}'')$, in which case the statement follows from the conormal bundle sequence Proposition 4.13. ■

Example 4.19. Let X be a smooth k scheme, $\mathcal{L} \in \mathrm{Pic}(X)$, and $s \in H^0(X, \mathcal{L})$. $Y = (s)_0$ is a codimension 1 subscheme and there is an injective map $\mathcal{O}_X \xrightarrow{s} \mathcal{L}$. Defining the dual map $s^\vee : \mathcal{L}^\vee \to \mathcal{O}_X$ we can compose it with the surjection $\mathcal{O}_X \to \mathcal{O}_Y$ to observe that $\mathcal{I}_Y \cong \mathcal{L}^\vee$. Tensoring the short exact sequence $0 \to \mathcal{L}^\vee \to \mathcal{O}_X \to \mathcal{O}_Y \to 0$ by $\mathcal{L}^\vee$ yields a short exact sequence $0 \to \mathcal{L}^{\otimes -2} \to \mathcal{L}^\vee \to \mathcal{L}^\vee|_Y \to 0$ where $\mathcal{I}_Y/\mathcal{I}_Y^2 \cong \mathcal{L}^\vee|_Y$ implying $\mathcal{N}_{Y/X} \cong \mathcal{L}|_Y$. Now applying the normal bundle sequence, $\omega_{Y/k} \cong \omega_{X/k} \otimes \mathcal{L}|_Y \cong (\omega_{X/k} \otimes \mathcal{L})|_Y$.

Example 4.20. Let $X = \mathbb{P}^n_k$ and $Y = V_+(f)$ for $f \in \mathcal{O}_{\mathbb{P}^n_k}(d)$ for some $d \geq 0$. Recall that $\omega_{X/k} = \mathcal{O}_{\mathbb{P}^n_k}(-n-1)$. So $\omega_{Y/k} = \mathcal{O}_{\mathbb{P}^n_k}(d-n-1)|_Y$.

5. Lecture 5 – 24th April 2025

We begin a discussion of flatness, which intuitively corresponds to varying "nicely" in a family over the base. As always, we first define these in the local case.

Definition 5.1 (Flat Module)**.** Let A be a ring and M an A-module. M is a flat A-module if $- \otimes_A M$ is an exact functor.

Recall that for A-modules M, $- \otimes_A M$ is always a right exact functor so flatness is equivalent to being left exact, taking short exact sequences

$$0 \to N_1 \to N_2 \to N_3 \to 0$$

to short exact sequences

$$0 \to N_1 \otimes_A M \to N_2 \otimes_A M_2 \to N_3 \otimes_A N_3 \to 0.$$

In fact, it suffices to verify that for all ideals $\mathfrak{a} \subseteq A$ that $\mathfrak{a} \otimes_A M \to A \otimes_A M$ is injective.

We can apply this to the special case where an A-algebra B is considered as an A-module.

Definition 5.2 (Flat Algebra)**.** Let A be a ring and B an A-algebra. B is A-flat if B is flat as an A-module.

Let us recall some further properties of flatness.

Proposition 5.3. Let A be a ring.

(i) If A is a local ring and M is a finite A-module, then M is flat if and only if M is free.
(ii) If $S \subseteq A$ is a multiplicative subset, $A \to S^{-1}A$ is flat.
(iii) If $A \to B$ is a ring map and M is a flat A-module then $M \otimes_A B$ is a flat B-module.
(iv) If $A \to B$ is a flat ring map and N is a flat B-module then its restriction of scalars $N|_A$ is a flat A-module.
(v) Let M be an A-module. M is flat if and only if $M_\mathfrak{p}$ is a flat $A_\mathfrak{p}$-module for all prime ideals (maximal ideals).
(vi) Let $A \to B$ be a flat ring map between Noetherian local rings, and $b \in B$ such that $\overline{b} \in B/\mathfrak{m}_A B$ is a non-zerodivisor then $B/(b)$ is A-flat.
(vii) For a coCartesian diagram

$$\begin{tikzcd} A \arrow[r] \arrow[d] & B \arrow[d] \\ C \arrow[r] & B \otimes_C A \end{tikzcd}$$

and M a B-module that is A-flat, then $M \otimes_B (C \otimes_A B)$ is a flat C-module.

Proof. See [Stacks, Tag 00H9]. ■

We prove the following lemma about flatness on localizations at prime ideals.

Lemma 5.4. Let $\varphi : A \to B$ be a ring map. φ is flat if and only if for all primes $\mathfrak{q} \subseteq B$, the map $A_{\varphi^{-1}(\mathfrak{q})} \to B_{\mathfrak{q}} \cong A_{\varphi^{-1}(\mathfrak{q})} \otimes_A B$ is a flat ring map.

Proof. ($\Rightarrow$) Suppose $A \to B$ is flat. Then any $A_{\varphi^{-1}(\mathfrak{q})} \to B_{\mathfrak{q}}$ is factored as $A_{\varphi^{-1}(\mathfrak{q})} \to B_{\varphi^{-1}(\mathfrak{q})} \cong A_{\varphi^{-1}(\mathfrak{q})} \otimes_A B \to (B_{\varphi^{-1}(\mathfrak{q})})_{\mathfrak{q}} \cong B_{\mathfrak{q}}$ which is a composition of a flat map with a localization, the latter flat, hence the composition is flat too.

($\Leftarrow$) Suppose we have an injection $M \to M'$ of A-modules. By exactness of localization, we have $M \otimes_A A_{\mathfrak{p}} \to M' \otimes_A A_{\mathfrak{p}}$. $B_{\mathfrak{q}}$ is $A_{\mathfrak{p}}$-flat so the map remains injective on base change to $B_{\mathfrak{q}}$ for all $\mathfrak{q} \subseteq B$ so taking $M = A, M' = B$ yields the claim in conjunction with Proposition 5.3 (v). ■

We now describe the geometric case.

Definition 5.5 (Flat Sheaves)**.** Let $f : X \to Y$ be a morphism of schemes and $\mathcal{F}$ a quasicoherent sheaf on X. $\mathcal{F}$ is flat over Y if $\mathcal{F}_x$ is flat as an $\mathcal{O}_{Y,f(x)}$-module.

Definition 5.6 (Flat Morphism)**.** Let $f : X \to Y$ be a morphism of schemes. f is a flat morphism if $\mathcal{O}_X$ is flat over Y.

We consider some examples.

Example 5.7. Let X be a scheme. $\mathrm{id}_X : X \to X$ is a flat morphism as affine-locally it is obtained by the identity ring map and rings are flat as modules over themselves.

Example 5.8. Let $\mathcal{F}$ be a coherent sheaf on X. $\mathcal{F}$ is a flat $\mathcal{O}_X$-module if and only if $\mathcal{F}$ is locally free. In the reduced case, this can be checked by the function $x \mapsto \dim_{\kappa(x)} \mathcal{F}_x \otimes_{\mathcal{O}_{X,x}} \kappa(x)$ being locally constant.

Example 5.9. Proposition 5.3 (ii) shows that open immersions are flat, though closed immersions are often not flat uness they are isomorphisms.

Proposition 5.10. Let $f : X \to Y$ be a morphism between locally Noetherian schemes. Then $\dim(\mathcal{O}_{X_y}, x) \geq \dim(\mathcal{O}_{X,x}) - \dim(\mathcal{O}_{Y,y})$ and equality occurs when f is flat.

Proof. We proceed by induction. Since the property is local on target, we can take Y to be affine, and since flatness is preserved by base change, we can take $Y = \mathrm{Spec}(\mathcal{O}_{Y,y})$ and $X = \mathrm{Spec}(\mathcal{O}_{X,x})$.

We proceed by induction on $\dim(Y)$. If $\dim(Y) = 0$, the nilradical is contained in the maximal ideal. Using the inclusion $\mathfrak{m}_x B \hookrightarrow \mathrm{Nil}(B)$ we have

$$\dim(\mathcal{O}_{X,x}) = \dim(B/\mathrm{Nil}(B)) = \dim(B/\mathfrak{m}_x B) = \dim(\mathcal{O}_{X_y,x})$$

by $(B/\mathrm{Nil}(B))/(\mathrm{Nil}(B)/\mathfrak{m}_x B) \cong B/\mathrm{Nil}(B)$ so we in fact have equality $\dim(\mathcal{O}_{X_y,x}) = \dim(\mathcal{O}_{X,x})$ as $\dim(\mathcal{O}_{Y,y}) = 0$. In particular, the desired inequality holds.

If $\dim(Y) \geq 1$, we can, without loss of generality, take Y to be reduced by base changing to the reduction – an operation that preserves all dimensions involved. Let $t \in \mathcal{O}_{Y,y}$ be neither a zerodivisor nor invertible with image $\bar{t}$ in $\mathcal{O}_{X,x}$. We then have

$$\dim(\mathcal{O}_{Y,y}/(t)) = \dim(\mathcal{O}_{Y,y}) - 1, \dim(\mathcal{O}_{X,x}/(\bar{t})) \geq \dim(\mathcal{O}_{X,x}) - 1$$

where if f is flat, then $\mathcal{O}_{X,x}$ is flat over $\mathcal{O}_{Y,y}$ and t is a non-zerodivisor in $\mathcal{O}_{X,x}$ giving $\dim(\mathcal{O}_{X,x}/(\bar{t})) = \dim(\mathcal{O}_{X,x}) - 1$.

Now set $Y' = \operatorname{Spec}(\mathcal{O}_{Y,y}/(t))$ and $X' = \operatorname{Spec}(\mathcal{O}_{X,x} \otimes_{\mathcal{O}_{Y,y}} (\mathcal{O}_{Y,y}/(t))) \cong \operatorname{Spec}(\mathcal{O}_{X,x}/(\bar{t}))$ where by the induction hypothesis we have

$$\dim(\mathcal{O}_{X'_y,x}) \geq \dim(\mathcal{O}_{X',x}) - \dim(\mathcal{O}_{Y',y}) \tag{5.1}$$

which is an equality if f is flat since flatness is preserved under base change. But on the fiber we have $X'_y = X_y$ since X_y is the fiber over $y \in \operatorname{Spec}(\mathcal{O}_{Y,y}/(t)) \subseteq \operatorname{Spec}(\mathcal{O}_{Y,y})$ so we have

$$\begin{aligned}
\dim(\mathcal{O}_{X'_y,x}) &\geq \dim(\mathcal{O}_{X',x}) - \dim(\mathcal{O}_{Y,y}) \\
&= (\dim(\mathcal{O}_{X,x}) - 1) - (\dim(\mathcal{O}_{Y,y}) - 1) \\
&= \dim(\mathcal{O}_{X,x}) - \dim(\mathcal{O}_{Y,y})
\end{aligned}$$

with equality in the flat case by the (5.1), as desired. ■

In the case of locally finite type schemes, this can be detected fiberwise.

Corollary 5.11. Let $f : X \to Y$ be a morphism of k-schemes where X, Y are locally of finite type, Y is irreducible, and X equidimensional. If for all $y \in Y$, X_y is equidimensional of $\dim(X) - \dim(Y)$ then f is flat and surjective.

Proof. Without loss of generality, let X be irreducible and take $x \in X$ closed. Then $\dim(X_y) = \dim(\mathcal{O}_{X_y,x})$. By hypothesis we have

$$\dim(\mathcal{O}_{X,x}) = \dim(X) - \dim(\overline{\{x\}})$$

which holds for finite-type k-algebras. On the other hand, we have

$$\begin{aligned}
\dim(\mathcal{O}_{Y,y}) &= \dim(Y) - \dim(\overline{\{y\}}) \\
&= \dim(Y) - \operatorname{trdeg}(\kappa(y)/k) \\
&= \dim(Y) - \dim\{\overline{x}\} \\
&= \dim(Y) - \dim(X) + \dim(\mathcal{O}_{X,y})
\end{aligned}$$

which gives the equality of Proposition 5.10 and from which the claim follows. ■

Moreover, flatness behaves especially well over smooth curves.

Proposition 5.12. Let $f : X \to Y$ be a morphism of schemes where Y is the spectrum of a discrete valuation. If f is flat, then $\overline{X_\eta} = X$.

In particular, there are no closed irreducible components $Z \subseteq Z$ over the (unique) closed point of Y.

Proof. ($\Rightarrow$) Suppose f is flat and there is $Z \subseteq Z$ irreducible in the fiber over the closed point $\mathfrak{m} \in Y$. Then $\dim(Z) \leq \dim(X_\mathfrak{m}) = \dim(X) - 1$. A contradiction to X equidimensional. Otherwise, we apply Corollary 5.11 to Z obtaining a contradiction once more.

($\Leftarrow$) This is [Har83, Prop III.9.7]. ■

6. Lecture 6 – 28th April 2025

As a corollary of the previous discussion about flatness, we show that flatness can be extended over a closed fiber.

Corollary 6.1. Let $f : X \to Y \setminus \{y\}$ be flat and projective with Y Dedekind and $y \in Y$ closed.

$$\begin{tikzcd} X \arrow[r] \arrow[d] & \mathbb{P}^n_{Y\setminus\{y\}} \arrow[d] \\ \overline{X} \arrow[r] & \mathbb{P}^n_Y \end{tikzcd}$$

Then the induced map $\overline{X} \to Y$ is flat.

Proof. Recall that a Dedekind scheme is a Noetherian regular integral scheme of dimension 1. We have the diagram of the statement of the corollary and by Proposition 5.12 we have $X_\eta = (\overline{X})_\eta \subseteq \overline{X}$ which is dense and X_η is dense in X so $X_\eta \subseteq \overline{X}$ is dense. This shows that $\overline{X}$ is flat over Y. ■

We show that images of flat maps are dense.

Proposition 6.2. Let $f : X \to Y$ be a flat map of schemes with Y irreducible. If $U \subseteq X$ is a nonempty open, then $f(U) \subseteq Y$ is dense.

Proof. Without loss of generality, $Y = \operatorname{Spec}(A)$ is affine and irreiducible so $A/\operatorname{Nil}(A)$ is integral and $\operatorname{Frac}(A/\operatorname{Nil}(A)) = K$. For $U = \operatorname{Spec}(B)$ affine, consider $B/\operatorname{Nil}(A)B = B \otimes_A (A/\operatorname{Nil}(A))$. Using the injection $A/\operatorname{Nil}(A) \hookrightarrow K$ and $- \otimes_A B$ being exact, we have an injective map $(A/\operatorname{Nil}(A)) \otimes_A B \hookrightarrow K \otimes_A B$ injective, where $K \otimes_A B = \mathcal{O}_X(U_\eta)$, here denoting $U_\eta = \operatorname{Spec}(B \otimes_A K)$.

It suffices to rpove that U_η is nonempty, or equivalently that $\mathcal{O}_X(U_\eta)$ is nonzero. If $K \otimes_A B$ was zero, then $\operatorname{Nil}(A)B = B$ and 1 is nilpotent, a contradiction. ■

We can show the following, weaker, version of a base change statement.

Proposition 6.3. Let $f : X \to \operatorname{Spec}(A)$ be a separated quasicompact morphism and $A \to A'$ a flat ring extension. The Cartesian diagram

$$\begin{tikzcd} X \times_A \operatorname{Spec}(A') \arrow[r, "g"] \arrow[d] & X \arrow[d, "f"] \\ \operatorname{Spec}(A') \arrow[r] & \operatorname{Spec}(A) \end{tikzcd}$$

induces for all quasicoherent sheaves $\mathcal{F}$ on X an isomorphism

$$H^0(X, \mathcal{F}) \otimes_A A' \longrightarrow H^0(X \times_A \operatorname{Spec}(A'), g^*\mathcal{F}).$$

Proof. Choose an affine open covering $\{U_i\}$ of X. The sheaf condition gives an exact sequence of A-modules

$$0 \to H^0(X, \mathcal{F}) \to \prod_i H^0(U_i, \mathcal{F}|_{U_i}) \to \prod_{i,j} H^0(U_{ij}, \mathcal{F}|_{U_{ij}}).$$

This remains exact after base changing to A', which precisely the exact sequence for $g^*\mathcal{F}$ on $X \times_A \operatorname{Spec}(A')$. ■

We omit the proof of the strong variant, which requires spectral sequences.

Proposition 6.4. Let $f : X \to Y$ be a separated quasicompact morphism and $g : Y' \to Y$ a flat morphism. The Cartesian diagram

$$\begin{tikzcd} X \times_Y Y' \arrow[r, "g'"] \arrow[d, "f'"'] & X \arrow[d, "f"] \\ Y' \arrow[r, "g"'] & Y \end{tikzcd}$$

induces for all quasicoherent sheaves $\mathcal{F}$ on X an isomorphism

$$g^* R^i f_* \mathcal{F} \cong R^i f'_* g'^* \mathcal{F}$$

for all $i \geq 0$. If further $Y = \operatorname{Spec}(A), Y' = \operatorname{Spec}(A')$ then $H^i(X, \mathcal{F}) \otimes_A A' \cong H^i(X', g'^*\mathcal{F})$.

Proof. See [Stacks, Tag 02KH]. ■

Having set up some basic constructions surrounding flatness, we discuss its relation to the Hilbert polynomial.

Definition 6.5 (Hilbert Polynomial)**.** Let X be projective k-scheme with a choice of embedding $i : X \to \mathbb{P}^n_k$ and $\mathcal{F}$ a coherent sheaf on X. The Hilbert polynomial of $\mathcal{F}$ is $P(X, \mathcal{F})(m) = \chi(X, \mathcal{F} \otimes \mathcal{O}_{\mathbb{P}^n_k}(m)|_X)$.

The Hilbert polynomial is in fact a numerical polynomial, that is, a map $\mathbb{Z} \to \mathbb{Z}$. Moreover, we can show that this polynomial characterizes flatness on the fibers. The proof of the statement is the globalization of the followimg lemma.

Lemma 6.6. Let A be an integral Noetherian local ring and $\mathcal{F}$ a coherent sheaf on $\mathbb{P}^n_A$. The following are equivalent:

Finish proof.

(a) $\mathcal{F}$ is flat over $\operatorname{Spec}(A)$.
(b) $H^0(X, \mathcal{F}(m))$ is a free A-module for m sufficiently large.
(c) The Hilbert polynomial $P(\mathbb{P}^n_{\kappa(\mathfrak{p})}, \mathcal{F}|_{\kappa(\mathfrak{p})})(m)$ is independent of $\mathfrak{p}$.

Proposition 6.7. Let $f : X \to Y$ be a projective morphism with Y integral Noetherian. Let $\mathcal{F}$ be a coherent sheaf on Y. f is flat if and only if the Hilbert polynomial of $\mathcal{F}$ on the fibers is constant.

Proof. The question is local on target, and flatness is preserved along base change, so it suffices to check the condition along the base change $\operatorname{Spec}(\mathcal{O}_{Y,y}) \to Y$, putting us in the situation of Lemma 6.6, in which case the desired statement is immediate. ■

Example 6.8. Blowups are not flat. Let f be the projection from the blowup of $\mathbb{A}^2_k$ at the origin to $\mathbb{A}^2_k$. The fiber over the origin is of a higher dimension.

This applies in particular to the structure sheaf.

Corollary 6.9. Let $f : X \to Y$ be a projective morphism with Y integral Noetherian. f is flat if and only if $P(X_y, \mathcal{O}_{X_y})(m)$ is constant.

Proof. This is an immediate application of Proposition 6.7 to $\mathcal{O}_X = \mathcal{F}$. ■

Moreover, we can show that flatness is open and universally open.

Proposition 6.10. Let $f : X \to Y$ be a finite type morphism of schemes with Y Noetherian. Then f is open and universally open.

Check.

Proof. We use the following input from the theory of spectral spaces:

- If $f : X \to Y$ is a finite type morphism with Y Noetherian then the image of f is constructible in Y[Stacks, Tag 054K].
- Let $V \subseteq Y$ be a subset. V is constructible and stable under specializations if and only if V is open in Y [Stacks, Tag 0542].

It suffices to show that $f(X)$ is open. We know already $f(X)$ is constructible so it suffices to show that $f(X)$ is stable under generalizations. Take $x \in X$ and $y = f(x) \in Y$. Assume $y \in \overline{\{y'\}}$. We want to show $y' \in f(X)$. Let $B = \mathcal{O}_{X,x}$ with $x \in U \subseteq X$ and $A = \mathcal{O}_{Y,y}$ which contains the prime ideal $\mathfrak{p}_{y'}$. The ring map $A \to B$ is a local homomorphism of local rings so $\mathfrak{m}_x \cap A = \mathfrak{m}_y$ and $\mathfrak{p}_{y'}B \subseteq \mathfrak{m}_y B \subseteq \mathfrak{m}_x \subsetneq B$ showing $B \otimes_A A_{\mathfrak{p}_{y'}} \neq 0$. If it were zero, then for all $b \in B$ there would be $t \in A_{\mathfrak{p}_{y'}}$ such that $tb = 0$ but the multiplication by t-map $A \to A$ is injective and remains injective after base-changing to B. This shows that $B \otimes_{A_{\mathfrak{p}_{y'}}} \kappa(\mathfrak{p}_{y'})$ is nonzero, and hence lies in the image. ■

7. Lecture 7 – 5th May 2025

We state without proof flatness in another setting.

Proposition 7.1 (Miracle Flatness)**.** Let X, Y be smooth integral k-schemes and $f : X \to Y$ a morphism such that $\dim(X_y)$ is constant. Then f is flat.

Proof. This is local, so we can reduce to the case of f induced by a ring map $B \to A$ which is [Stacks, Tag 00R4]. ■

Having discussed flatness, we can turn to a discussion of smoothness of morphisms, generalizing Definition 3.11.

Definition 7.2 (Smooth Morphism at a Point)**.** Let $f : X \to Y$ be a morphism locally of finite type. f is smooth at $x \in X$ if f is flat at x and the fiber $X_{f(x)}$ is smooth over $\kappa(f(x))$.

Remark 7.3. That is, $\mathcal{O}_{X,x}$ is a flat $\mathcal{O}_{Y,f(x)}$-module and $X_{f(x)} = X \times_Y \operatorname{Spec}(\kappa(f(x)))$ is a smooth $\kappa(f(x))$-scheme by the Cartesian diagram

$$\begin{tikzcd}
X_{f(x)} \arrow[r] \arrow[d] & X \arrow[d] \\
\operatorname{Spec}(\kappa(f(x))) \arrow[r] & Y.
\end{tikzcd}$$

This definition globalizes.

Definition 7.4 (Smooth Morphism)**.** Let $f : X \to Y$ be a morphism locally of finite type. f is smooth if f is smooth at all $x \in X$.

Example 7.5. Let W be a smooth k-scheme and Y any k-scheme. Let $X = W \times_k Y$ with $f : X \to Y$ the natural projection. f is smooth – by miracle flatness the dimensions of the fibers are constant and are W which is $\kappa(f(x))$-smooth since smoothness is preserved under base change.

Example 7.6. Let Y be any scheme. The projections $\mathbb{A}^n_Y \to Y, \mathbb{P}^n_Y \to Y$ are smooth. More generally for $\mathcal{E}$ locally free of finite rank on Y, the (total spaces of the) associated vector bundle $\mathbb{V}(\mathcal{E}) \to Y$ and projective bundle $\mathbb{P}(\mathcal{E}) \to Y$ are smooth.

Recall that for $f : X \to Y$ a morphism of schemes over k the Proposition 1.8 gives an exact sequence of sheaves

$$f^*\Omega^1_{Y/k} \to \Omega^1_{X/k} \to \Omega^1_{X/Y} \to 0.$$

As it turns out, smoothness of morphisms can be determined by $\Omega^1_{X/Y}$, in analogy to how smoothness of schemes is determined by $\Omega^1_{X/k}$. For this, we will require the following lemma.

Lemma 7.7. Let $f : X \to Y$ be a locally finite type morphism, $x \in X$ with $f(x) = y$ and $d = \dim(X_y)$. There exists a closed immersion $X \to W$ over Y such that W is smooth at x over y of dimension d.

Proof. Since f is locally of finite type, we can factor f as

$$\begin{tikzcd} X \arrow[rr, "i"] \arrow[dr, "f"'] & & \mathbb{A}^n_Y \arrow[dl, "p"] \\ & Y & \end{tikzcd}$$

where i is a closed immersion and p is the projection. As in Example 7.6, p is smooth so $\Omega^1_{\mathbb{A}^n_Y/Y}$ is locally free of rank n, but not of rank d. We find a closed subscheme of $\mathbb{A}^n_Y$ such that the sheaf of relative differentials is locally free of rank d.

Let $\mathcal{I}_X$ be the ideal sheaf of X in $\mathbb{A}^n_Y$ so by Proposition 1.13, we have an exact sequence

$$\mathcal{I}_X/\mathcal{I}_X^2 \to \Omega^1_{\mathbb{A}^n_Y/Y}|_{i(X)} \to \Omega_{X/Y} \to 0. \tag{7.1}$$

Now letting $\mathcal{J}$ be the ideal sheaf of $X_y \in \mathbb{A}^n_{\kappa(y)}$ we have an exact sequence

$$\mathcal{J} \to \Omega^1_{\mathbb{A}^n_{\kappa(y)}/\kappa(y)} \to \Omega^1_{X_y/\kappa(y)} \to 0 \tag{7.2}$$

where $\Omega^1_{\mathbb{A}^n_{\kappa(y)}/\kappa(y)}$ is locally free of rank n and $\Omega^1_{X_y/\kappa(y)}$ is locally free of rank d. Then

$$\ker\left(\Omega^1_{\mathbb{A}^n_{\kappa(y)}/\kappa(y)} \to \Omega^1_{X_y/\kappa(y)}\right)$$

is locally free of rank $n-d$. Base changing (7.1) and (7.2) by $\kappa(x)$, we get a diagram

$$\begin{tikzcd} \mathcal{I}_X/\mathcal{I}_X^2 \otimes \kappa(x) \arrow[r] \arrow[d, two heads] & \Omega^1_{\mathbb{A}^n_Y/Y}|_{i(X)} \otimes \kappa(x) \arrow[r] \arrow[d, "\wr"'] & \Omega^1_{X/Y} \otimes \kappa(x) \arrow[r] \arrow[d, "\wr"'] & 0 \\ \mathcal{J} \otimes \kappa(x) \arrow[r] & \Omega^1_{\mathbb{A}^n_{\kappa(y)}/\kappa(y)} \otimes \kappa(x) \arrow[r] & \Omega^1_{X_y/\kappa(y)} \otimes \kappa(x) \arrow[r] & 0 \end{tikzcd}$$

where the injectivity of the right two vertical arrows implies surjectivity of the leftmost vertical arrow.

If $n > d$, there is $g \in \mathcal{I}_X$ and $g_2, \dots, g_n \in \mathcal{O}_{\mathbb{A}^n_{\kappa(y)}}$ such that $\mathrm{d}g, \mathrm{d}g_2, \dots, \mathrm{d}g_n$ freely generate $\Omega^1_{\mathbb{A}^n_{\kappa(y)}/\kappa(y)} \otimes \kappa(x)$. By Nakayama's lemma, these lift to basis of $\Omega^1_{\mathbb{A}^n_{\kappa(y)}/\kappa(y)} \otimes \kappa(x')$ for x' in a neighborhood of x in X. Observe $X \subseteq V(g) = W \subseteq \mathbb{A}^n_Y$ then $\dim(W) = \dim(\mathbb{A}^n_Y) - 1$ and with $\Omega^1_{W'_y/\kappa(y)} = \Omega_{\mathbb{A}^n_{\kappa(y)}/\kappa(y)}/\mathrm{d}g$ which is locally free of rank $n-1$. Moreover, since g is a nonzerodivisor in $\mathcal{O}_{X,x}/\mathfrak{m}_y\mathcal{O}_{X,x}$, $W_y \to y$ is flat by Proposition 5.3 (vi).

Iterating this process, we arrive at the desired W. ∎

Example 7.8. Let $X = \{y\} \hookrightarrow Y$ a closed point. $\Omega^1_{X/Y} = 0$ and the fiber dimension is 0, but taking $W = Y$, $\mathrm{id}_Y : Y \to Y$ is smooth.

We now state and prove the desired result.

Proposition 7.9. Let $f : X \to Y$ be a morphism locally of finite type with all fibers of pure dimension d. Then f is smooth if and only if f is flat and $\Omega^1_{X/Y}$ is locally free of rank d.

Proof. Recall by Proposition 1.12 that for a Cartesian square

$$\begin{tikzcd} X' \arrow[r, "g"] \arrow[d] & X \arrow[d] \\ Y' \arrow[r] & Y \end{tikzcd}$$

we have $g^*\Omega^1_{X/Y} \cong \Omega_{X'/Y'}$ so in particular when $Y' = \mathrm{Spec}(\kappa(y))$ then $X' = X_y$ and we have $\Omega^1_{X_y/\kappa(y)} \cong \Omega^1_{X/Y}|_{X_y}$. We now begin the proof in earnest.

($\Rightarrow$) Now assume f is smooth. By definition, f is flat, and $\Omega^1_{X_y/\kappa(y)}$ is locally free of rank $\dim(X_y)$. In particular, the rank of $\Omega^1_{X_y/\kappa(y)} \otimes \kappa(x)$ is d for all $x \in X_y$. By base change, $\Omega^1_{X_y/\kappa(y)} \otimes \kappa(x)$ and $\Omega^1_{X/Y} \otimes \kappa(x)$ are of the same rank. If X is reduced, this already implies that $\Omega^1_{X/Y}$ is locally free of the correct rank.

If X is not reduced, we use the factorization produced by Lemma 7.7 to get a factorization

$$\begin{tikzcd} X \arrow[rr, "i"] \arrow[dr, "f"'] & & W \arrow[dl, "p"] \\ & Y & \end{tikzcd}$$

where i is a closed immersion and p is smooth with fiber dimension d. We have $X_y \subseteq W_y$ of the same dimension, but W_y is smooth hence reduced, so $X_y = W_y$ yielding the claim.

($\Leftarrow$) Suppose f is flat and $\Omega^1_{X/Y}$ is locally free of rank d. The restriction to the fiber $\Omega^1_{X_y/\kappa(y)}$ is locally free of rank $\dim(X_y)$ over $\kappa(y)$ hence smooth over $\kappa(y)$ showing the morphism is smooth. ■

As in the case of smoothness Proposition 4.5, the smooth locus of a morphism can be shown to be open on the source.

Proposition 7.10. Let $f : X \to Y$ be a dominant morphism of finite type integral schemes over k with $K(X)/K(Y)$ separable. There exists $U \subseteq X$ dense open such that $f|_U : U \to Y$ is smooth.

Proof. Since $K(X)/K(Y)$ is separable, $\Omega^1_{K(X)/K(Y)}$ is a $K(X)$-vector space of rank the transendence degree of $K(X)/K(Y)$, which is $d = \dim(X) - \dim(Y)$. By Lemma 4.1, we have that there is an open set U such that $\Omega^1_{X/Y}|_U \cong \Omega^1_{U/Y}$ is locally free of rank d.

By Proposition 6.10, we can by restricting further take U to be the flat locus of the morphism, which may be empty. On U, the fibers over $y \in f(U)$ are of dimension d by Corollary 5.11 (applied to U and not Y). So by Proposition 7.9, $f|_U$ is smooth. ■

Remark 7.11. The special case of Proposition 7.10 where $Y = \mathrm{Spec}(k)$ is precisely Proposition 4.5.

Let us consider some examples.

Example 7.12. A smooth morphism between smooth schemes extends the exact sequence of Proposition 1.8 to a short exact sequence. Let $f : X \to Y$ be a smooth morphism between smooth k-schemes. There is a short exact sequence

$$0 \to f^*\Omega^1_{Y/k} \to \Omega^1_{X/k} \to \Omega_{X/Y} \to 0. \tag{7.3}$$

To see this, we observe that since $\Omega^1_{X/Y}$ is locally free of rank $\dim(X) - \dim(Y)$ and $\Omega^1_{X/k}$ is locally free of rank $\dim(X)$ implying $f^*\Omega^1_{Y/k}$ locally free of rank the dimension of Y. This implies that $f^*\Omega^1_{Y/k} \to \ker\left(\Omega^1_{X/k} \to \Omega^1_{X/Y}\right)$ is surjective, but any surjection of locally free modules of the same rank is an isomorphism, whence the claim.

Example 7.13. Let $f : X \to Y$ be a smooth morphism between smooth k-schemes. Dualizing (7.3) of Example 7.12, we get

$$0 \to \mathcal{T}_{X/Y} \to \mathcal{T}_{X/k} \to f^*\mathcal{T}_{Y/k} \to 0$$

which on restriction to any closed fiber X_y we have

$$0 \to \mathcal{T}_{X/Y}|_{X_y} \cong \mathcal{T}_{X_y} \to \mathcal{T}_X|_{X_y} \to f^*\mathcal{T}_Y|_y \cong \mathcal{N}_{X_y/Y} \to 0$$

recovering the normal bundle sequence.

Observe that for $f : X \to Y$ a smooth morphism between smooth k-schemes with k-algebraically closed we have for all $x \in X$ an induced map on the Zariski tangent spaces $T_{X,x} \to T_{Y,f(x)} \otimes \kappa(x)$. In particular, if $x \in X(k)$ we have $\kappa(x) \cong k$ and thus $\kappa(y) \cong k$ as $\kappa(y)$ lies in the intermediate extension $\kappa(x)/\kappa(y)/k$. By the injectivity of the idnuced map $\mathfrak{m}_y/\mathfrak{m}_y^2 \to \mathfrak{m}_x/\mathfrak{m}_x^2$, we have surjectivity of the map on Zariski tangent spaces. This in fact determines smoothness of a morphism under appropriate hypotheses.

Proposition 7.14. Let $f : X \to Y$ be a morphism between smooth k-schemes with k algebraically closed. If the induced map $T_{X,x} \to T_{Y,f(x)}$ is surjective for all $x \in X(k)$ then f is smooth.

Proof. We show first that f is flat. Since flatness is an open condition, it suffices to show flatness on each closed point, since each point lies in an open neighborhood of some closed point.

By injectivity of $\mathfrak{m}_y/\mathfrak{m}_y^2 \hookrightarrow \mathfrak{m}_x/\mathfrak{m}_x^2$ we pick generators $(\overline{a}_1, \ldots, \overline{a}_n)$ where $n = \dim(Y)$ with images $\overline{b}_1, \ldots \overline{b}_n$ for $b_i \in \mathfrak{m}_x$. Injectivity implies that the $\overline{b}_i$ remain linearly independent in the quotient so $b_1, \ldots, b_n$ is a regular sequence in $\mathcal{O}_{X,x}$ – each b_i is a nonzerodivisor in $\mathcal{O}_{X,x}/(b_1, \ldots, b_{i-1})$. By induction, $\mathcal{O}_{X,x}/(b_1, \ldots, b_n)$ is flat over $\kappa(y) = \mathcal{O}_{Y,f(x)}/(a_1, \ldots, a_n)$. This gives flatness.

Use Proposition 1.8, we have

$$f^*\Omega^1_{Y/k} \to \Omega^1_{X/k} \to \Omega^1_{X/Y} \to 0$$

where the former two terms are locally free and surjectivity of the Zariski tangent space implies that the map $f^*\Omega^1_{Y/k} \to \Omega^1_{X/k}$ is fiberwise injective. Hence $\Omega^1_{X/Y}$ is locally free of $\dim(X) - \dim(Y)$ showing smoothness too. ∎

We hint towards an algebro-geometric version of Sard's theorem via the following lemma, which will be used in the proof of the result.

Lemma 7.15. Let $f : X \to Y$ be smooth finite type k-schemes with k of characterositic 0. Define

$$X_r = \overline{\left\{x \in X_{\mathrm{cl}} : \operatorname{rank}(T_{X,x} \to T_{Y,f(x)} \otimes \kappa(x)) \leq r\right\}}.$$

Then $\dim(\overline{f(X_r)}) \leq r$.

Proof. Pick an irreducible component $X' \subseteq \overline{X_r}$ such that $f' : X' \to Y'$ is dominant. Given X', Y' the induced reduced subscheme structure so X', Y' are integral. Using that $K(X')/K(Y')$ is separble since $K(Y')$ is of characteristic 0, there exists by Proposition 7.10 U' of X' such that $f'|_{U'} : U' \to Y'$ is smooth. Then take $x \in U' \cap X_r$ and contemplate the diagram

$$\begin{tikzcd} T_{U',x} \arrow[r, hook] \arrow[d, two heads] & T_{X,x} \arrow[d] \\ T_{Y',f(x)} \arrow[r, hook] & T_{Y,f(x)} \end{tikzcd}$$

where the injective horizontal maps and the right vertical map being of rank r imply surjectivity of the left vertical map and smoothness of $U' \to Y'$. Indeed, it suffices to take x to be k-rational, since the preceding discussion holds on base change.Thus $T_{Y',f(x)}$ is of rank at most r, showing $\dim(\overline{f(X_r)}) \leq r$, as desired. ∎

8. Lecture 8 – 8th May 2025

We can now state and prove the algebraic variant of Sard's theorem.

Theorem 8.1 (Algebraic Sard)**.** Let X, Y be smooth integral k-schemes with k of charactersitic 0 and $f : X \to Y$ a morphism of k-schemes. There exists a dense set $V \subseteq Y$ such that $f^{-1}(V) \to V$ is smooth.

Remark 8.2. Note that V is nonempty but $f^{-1}(V) = \emptyset$ can occur, for example where f is a closed embedding.

Proof. Denote the set

$$X_r = \overline{\left\{x \in X_{\mathrm{cl}} : \mathrm{rank}(T_{X,x} \to T_{Y,f(x)} \otimes \kappa(x)) \leq r\right\}}$$

where we have $\dim(\overline{f(X_r)}) \leq r$ by Lemma 7.15. Apply this to $r = \dim(Y) - 1$ so $\overline{f(X_{\dim(Y)-1})}$ is a proper closed subset. Let V be its open complement in Y which is nonempty since $\overline{f(X_{\dim(Y)-1})}$ is a proper closed subset. For all points $x \in X$, with image contained in V, we have that $T_{X,x} \to T_{Y,f(x)} \otimes \kappa(x)$ cannot have rank smaller than the dimension of Y, so the map is surjective as a map of $\kappa(x)$-vector spaces, hence smooth by Proposition 7.14. ■

There is an easy counterexample in positive characteristic.

Example 8.3. Let k be of characteristic p. The relative Frobenius $\mathbb{A}^2_k \to \mathbb{A}^2_k$ with all fibers non-reduced, so the morphism is nowhere smooth.

Algebraic Sard's theorem allows us to show that smoothness is generic on hyperplane sections of a scheme, in the sense that the set of smooth hyperplane sections of a quasiprojective scheme is dense in the set of hyperplanes. Let us be more precise: let $X \subseteq \mathbb{P}^n_k$ be a smooth quasiprojective variety. For a line bundle $\mathcal{L} \in \mathrm{Pic}(X)$ and $V \subseteq H^0(X, \mathcal{L})$ a basepoint free linear subspace, we can define a morphism $\varphi_V : X \to \mathbb{P}(V^\vee)$ with the property that $f^*\mathcal{O}_{\mathbb{P}(V^\vee)}(1) \cong \mathcal{L}$. There exists a nonempty open $U \subseteq V$ such that for all $s \in U$, $V_X(s) = \varphi^{-1}(V_+(s)) \subseteq X$ is smooth.

For this, we recall the following facts about the universal hypersurface.

Lemma 8.4. Let $\mathcal{U}_{n,1} \subseteq \mathbb{P}^n_k \times_k \mathbb{P}^N_k$ for $N = n$ be the universal hyperplane defined by $\sum_{i=0}^n a_i x_i$. Then $\mathrm{pr}_{\mathbb{P}^N_k}$ is a projective bundle of dimension $n - 1$.

Proof. It suffices to observe that the fiber over any point $(a_0, \ldots, a_n)$ is the hyperplane $V_+(\sum_{i=0}^n a_i x_i) \subseteq \mathbb{P}^n_k$. ■

With this in hand, we state and prove the desired result.

Theorem 8.5 (Bertini)**.** Let $X \subseteq \mathbb{P}^n_k$ be a smooth quasiprojective variety with k algebraically closed of characteristic 0. For a line bundle $\mathcal{L} \in \mathrm{Pic}(X)$ and $V \subseteq H^0(X, \mathcal{L})$ a basepoint free linear subspace. There is a nonempty open subset $U \subseteq V$ such that for all $s \in U$, $V_X(s) \subseteq X$ is smooth.

Proof. Construct the universal hyperplane

$$\begin{tikzcd} \mathbb{P}(V^\vee) \times_k \mathbb{P}^N_k & \mathcal{U}_{n,1} \arrow[l, hook'] \arrow[r, "\mathrm{pr}_{\mathbb{P}^N_k}"] \arrow[d, "\mathrm{pr}_{\mathbb{P}(V^\vee)}"'] & \mathbb{P}^N_k = |\mathcal{O}_{\mathbb{P}(V^\vee)}(1)| \\ & \mathbb{P}(V^\vee) & \end{tikzcd}$$

We have that $\mathrm{pr}_{\mathbb{P}^N_k} : \mathcal{U}_{n,1} \to \mathbb{P}^N_k$ is a projective bundle isomorphic to $\mathbb{P}(\mathcal{E})$ where

$$\mathcal{E} = \ker\left(H^0(\mathbb{P}(V^\vee), \mathcal{O}_{\mathbb{P}(V^\vee)}(1)) \otimes \mathcal{O}_{\mathbb{P}(V^\vee)}(1) \to \mathcal{O}_{\mathbb{P}(V^\vee)}(1)\right)$$

since we have a diagram

$$\begin{tikzcd} & & & \mathbb{P}(V^\vee) \\ X \times_k \mathbb{P}^n_k \arrow[r, "\varphi \times \mathrm{id}_{\mathbb{P}^N_k}"] & \mathbb{P}(V^\vee) \times_k \mathbb{P}^N_k & \mathcal{U}_{n,1} \arrow[l, hook'] \arrow[ur, "\mathrm{pr}_{\mathbb{P}(V^\vee)}"] \arrow[dr, "\mathrm{pr}_{\mathbb{P}^N_k}"'] & \\ \mathcal{X} = X \times_{\mathbb{P}(V^\vee)} \mathcal{U}_{n,1} \arrow[u, hook] & & & \mathbb{P}^N_k \end{tikzcd}$$

where in particular $\mathrm{pr}_{\mathbb{P}^N_k}$ is a $\mathbb{P}^{\dim(V)-2}_k$-bundle. Denote π the composite $\mathcal{X} \to \mathbb{P}^N_k$ obtained by the diagram above. Apply Theorem 8.1 to π so there exists $W \subseteq \mathbb{P}^N_k$ open such that $\pi^{-1}(W) \to W$ is smooth. For $g : V \setminus \{0\} \to (V \setminus \{0\})/k^\times$ be the quotient map and define $U = g^{-1}(W \cap (\mathbb{P}^n_k)_{\mathrm{cl}})$. Then for all $s \in U$, $V_+(s)$ is smooth. ∎

Remark 8.6. If k is not algebraically closed, we need that $W \cap (V\setminus\{0\})/k^\times = W(k)$ is nonempty.

Remark 8.7. The set U is open only if X is projective.

Remark 8.8. By passing to the Veronese embedding, the proof of Bertini's theorem holds for hypersurface sections.

We can now see some examples.

Example 8.9 (Katz)**.** Let $k = \mathbb{F}_q$. $V_+(\sum_{i=0}^n x_i y_i^q - x_i^q y_i) \subseteq \mathbb{P}^{2n+1}_k$ is smooth but no hyperplane section of X is smooth.

Example 8.10 (Poonen)**.** Fix $\mathbb{F}_q$. For $n \geq 2, d \geq 1$ there exists $X \subseteq \mathbb{P}^n_{\mathbb{F}_q}$ of degree d such that all hypersurface sections of degree at most d are singular.

Example 8.11 (Poonen)**.** Recall the definition of the zeta function of a scheme $\zeta_X(s) = \exp(\sum_{i=0}^\infty \frac{X(\mathbb{F}_{q^r})}{r} q^{-rs})$. Fix $X \subseteq \mathbb{P}^n_{\mathbb{F}_q}$ smooth.

$$\frac{\{f \in \mathbb{F}_q[x_0, \ldots, x_n]_d : V_+(f) \cap X \text{ smooth}\}}{\{f \in \mathbb{F}_q[x_0, \ldots, x_n]_d\}} \sim_{d \to \infty} \zeta_X(\dim(X)).$$

Example 8.12. Let $\mathcal{U}_{n,d}$ be the universal hypersurface of degree d in $\mathbb{P}^n_k$. There is a proper closed subset of $|\mathcal{O}_{\mathbb{P}^n_k}(d)|$ parametrizing the singular hypersurfaces of degree $(d-1)^{n+1}(n+3)$.

We begin a discussion of unramified morphisms.

Definition 8.13 (Unramified Morphism at a Point)**.** Let $f : X \to Y$ be a locally finite type morphism betewen locally Noetherian schemes. f is unramified at $x \in X$ with image $y = f(x)$ if $\mathfrak{m}_y\mathcal{O}_{X,x} = \mathfrak{m}_x$ and $\kappa(x)/\kappa(y)$ is separable.

Definition 8.14 (Unramified Morphism)**.** Let $f : X \to Y$ be a locally finite type morphism betewen locally Noetherian schemes. f is unramified if it is unramified at all $x \in X$.

Remark 8.15. Unramified morphisms are the algebro-geometric analogue of immersions in differential topology.

Let us now consider some examples.

Example 8.16. Let K/k be a separable algebraic extension. Then $\mathrm{Spec}(K) \to \mathrm{Spec}(k)$ is unramified.

Example 8.17. Let $L/K/\mathbb{Q}$ be number fields inducing on rings of integers $\mathbb{Z} \to \mathcal{O}_K \xrightarrow{\varphi} \mathcal{O}_L$. This induces a map $\mathrm{Spec}(\mathcal{O}_L) \to \mathrm{Spec}(\mathcal{O}_K)$. The map is unramified at the generic point and at all closed points $\mathfrak{q} \subseteq \mathcal{O}_L, \mathfrak{p} = \varphi^{-1}(\mathfrak{q})$ that $(\mathfrak{q} \cap \mathcal{O}_{K,\mathfrak{p}})\mathcal{O}_{L,\mathfrak{q}} = \mathfrak{q} \subseteq \mathcal{O}_{L,\mathfrak{q}}$.

Example 8.18. $\mathbb{A}^1_k \to \mathbb{A}^1_k$ by $x \mapsto x^2$ is ramified at (x) as $k[x]_{(x)}$ is not generated by x^2, but unramified at all nonzero points away from characteristic 2. Conversely, the morphism is ramified everywhere in characteristic 2.

Example 8.19. Closed and open immersions are always unramified.

Example 8.20. $\mathrm{Spec}(k[x]/(f)) \to \mathrm{Spec}(k)$ is unramifed over the factors f_i of f that $k[x]/(f_i)$ are separable.

As it turns out, unramifiedness can be tested on the relative Kähler differentials, or equivalently on the diagonal being an open embedding. To show this, we will first require the following lemma.

Lemma 8.21. Let $f : X \to Y$ be a locally finite type morphism betewen locally Noetherian schemes. f is unramified if and only if for all points $y \in Y$, the fiber X_y is reduced, locally finite, and for all $x \in X_y$ the field extension $\kappa(x)/\kappa(y)$ is separable.

Proof. ($\Rightarrow$) Assume that f is unramified. We have that $\mathcal{O}_{X_y,x} \cong \mathcal{O}_{X,x}/\mathfrak{m}_y\mathcal{O}_{X,x}$ where by unramifiedness, $\mathfrak{m}_y\mathcal{O}_{X,x} \cong \mathfrak{m}_x$ so $\mathcal{O}_{X_y,x} \cong \kappa(x)$ is reduced so the fiber is reduced and locally finite.

($\Leftarrow$) Suppose for all $y \in Y$, the fiber X_y is reduced, locally finite, and the extension of residue fields is separable. We have an injection $\kappa(y) \hookrightarrow \mathcal{O}_{X_y,x}$ so $\mathcal{O}_{X_y,x}$ is zero-dimensional, reduced, and locally finite so $\mathcal{O}_{X_y,x} = \kappa(x)$ showing $\mathfrak{m}_y\mathcal{O}_{X,x} = \mathfrak{m}_x$, as desired. ∎

9. Lecture 9 – 12th May 2025

We characterize unramifiedness in terms of triviality of the sheaf of relative Kähler differentials.

Proposition 9.1. Let $f : X \to Y$ be a locally finite type morphism betewen locally Noetherian schemes. f is unramified if and only if $\Omega^1_{X/Y} = 0$.

Proof. ($\Rightarrow$) For $y \in Y$, we have $\Omega^1_{X/Y}|_{X_y} = \Omega^1_{X_y/\kappa(y)}$ so $\Omega^1_{X/Y} \otimes \kappa(x) \cong \Omega^1_{X_y/\kappa(y)} \otimes \kappa(x)$. Lemma 8.21, we have locally $X_y = \operatorname{Spec}(\kappa(x))$ and $\kappa(x)/\kappa(y)$ separable, so $\Omega^1_{\kappa(x)/\kappa(y)} = 0$. So since for each $y \in Y$ we have $\Omega^1_{X_y/\kappa(y)} = 0$, $\Omega^1_{X/Y} = 0$ too.

($\Leftarrow$) Assume $\Omega^1_{X/Y} = 0$ and $X_y = \operatorname{Spec}(A)$ locally finite type over $\kappa(y)$. We have $\Omega^1_{X_y/\kappa(y)} = \Omega^1_{X/Y}|_{X_y} = 0$. Using the exact sequence

$$\mathfrak{m}_x/\mathfrak{m}_x^2 \to \Omega^1_{A/\kappa(y)} \to \Omega^1_{\kappa(x)/\kappa(y)} \to 0$$

we have $\Omega^1_{A/\kappa(y)} = 0$ by assumption and $\Omega^1_{\kappa(x)/\kappa(y)}$ by separability. Thus $\mathfrak{m}_x/\mathfrak{m}_x^2 = 0$ and using Nakayama's lemma, $\mathfrak{m}_x = 0$ as well. Thus $\mathfrak{m}_y\mathcal{O}_{X,x} = \mathfrak{m}_x$ showing $\mathfrak{m}_y = 0$. We can then conclude f is unramified by Lemma 8.21. ∎

Example 9.2. Let X be the nodal affine plane curve. The map from the normalization is unramified.

We show the property alluded to in Remark 8.15.

Lemma 9.3. Let $f : X \to Y$ be a locally finite type morphism betewen locally Noetherian schemes. If f is unramified, then $T_{X,x} \to T_{Y,y} \otimes \kappa(x)$ is injective.

Proof. Dualizing, we show that $\frac{\mathfrak{m}_y}{\mathfrak{m}_y^2} \otimes_{\mathcal{O}_{Y,y}} \kappa(x) \to \frac{\mathfrak{m}_x}{\mathfrak{m}_x^2}$ is surjective. We can write $\mathfrak{m}_x = \mathfrak{m}_y \otimes \mathcal{O}_{X,x}$ so there is an obvious surjection $\frac{(\mathfrak{m}_y \otimes \mathcal{O}_{X,x})}{(\mathfrak{m}_y \otimes \mathcal{O}_{X,x})^2} \to \frac{\mathfrak{m}_x}{\mathfrak{m}_x^2}$, whence the claim. ∎

Lemma 9.4. Let $f : X \to Y$ be a locally finite type morphism betewen locally Noetherian schemes. f is unramified if and only if $f^*\Omega^1_{Y/k} \to \Omega^1_{X/k}$ is surjective.

Proof. The cokernel of $f^*\Omega^1_{Y/k} \to \Omega^1_{X/k}$ has cokernel $\Omega^1_{X/Y}$ so the map is surjective if and only if $\Omega^1_{X/Y} = 0$, that is, if f is unramified. ∎

We can now define étaleness.

Definition 9.5 (Étale Morphism at a Point)**.** Let $f : X \to Y$ be a locally finite type morphism betewen locally Noetherian schemes and $x \in X$. f is étale at x if f is flat and unramified at x.

Definition 9.6 (Étale Morphism)**.** Let $f : X \to Y$ be a locally finite type morphism betewen locally Noetherian schemes. f is étale if f is flat and unramified.

The normalization of the nodal affine plane curve is not étale.

Example 9.7. Let X be the nodal affine plane curve. The map from the normalization is unramified, but not étale, as it is not flat.

We can characterize étaleness as follows.

Proposition 9.8. Let $f : X \to Y$ be a locally finite type morphism betewen locally Noetherian schemes. The map from the normalization is unramified. The following are equivalent:

(a) f is étale.
(b) f is flat and unramified.
(c) f is flat and $\Omega^1_{X/Y} = 0$.
(d) f is smooth of relative dimension 0.

Proof. (i)⇔(ii) This is the definition.
(ii)⇔(iii) This is Proposition 9.1.
(iii)⇒(iv) This is Proposition 7.9.
(iv)⇒(iii) Suppose f is smooth of relative dimension 0. Then f is flat by definition. Smoothness implies further that $\Omega^1_{X/Y}$ is locally free of rank $\dim(X) - \dim(Y) = 0$, hence trivial. ■

Étaleness gives surjectivity on tangent spaces, so we have that étale morphisms induce isomorphisms on Zariski tangent spaces since these are in particular unramified (cf. Lemma 9.3).

Lemma 9.9. Let $f : X \to Y$ be an étale morphism between locally Noetherian schemes. Then $T_{X,x} \to T_{Y,y} \otimes \kappa(x)$ is an isomorphism.

Proof. Étale morphisms are in particular unramifed so $T_{X,x} \to T_{Y,y} \otimes \kappa(x)$ is injective by Lemma 9.3. Flatness implies $\mathfrak{m}_y \otimes \mathcal{O}_{X,x} = \mathfrak{m}_y \mathcal{O}_{X,x} = \mathfrak{m}_x$ so the induced map on Zariski tangent spaces is surjective too. ■

Generalizing Lemma 9.4, we can show that for f étale, $f^*\Omega^1_{Y/k} \to \Omega^1_{X/k}$ is an isomorphism. For this we will require the following lemma.

Lemma 9.10. Let $f : X \to Y$ be an unramified morphism between locally finite type k-schemes. Then $\Delta_{X/Y}$ is an open immersion.

To do.

We now begin the proof in earnest.

Proposition 9.11. Let $f : X \to Y$ be an étale morphism between locally finite type k-schemes. Then $f^*\Omega^1_{Y/k} \to \Omega^1_{X/k}$ is an isomorphism.

Proof. We use the exact sequence

$$f^*\Omega^1_{Y/k} \to \Omega^1_{X/k} \to \Omega^1_{X/Y} \to 0$$

where $\Omega^1_{X/Y} = 0$ since étale morphisms are in particular unramified so $f^*\Omega^1_{X/k} \to \Omega^1_{Y/k}$ is surjective as in Lemma 9.4. It remains to show that the induced map on Kähler differentials is injective. If X, Y are k-smooth, then we are done, as $f^*\Omega^1_{X/k} \to \Omega^1_{Y/k}$ is a surjection of locally free sheaves of the same rank, hence an isomorphism. In the general case, we can reduce to where X, Y are affine and X is the spectrum of a finitely generated $\Gamma(Y, \mathcal{O}_Y)$-algebra. Using the diagrams

$$\begin{tikzcd} X \times_k X \arrow[r] \arrow[d] & X \arrow[d, "f"] \\ X \times_k Y \arrow[r] & Y \end{tikzcd} \qquad \begin{tikzcd} Y \times_k X \arrow[r] \arrow[d] & X \arrow[d, "f"] \\ Y \times_k Y \arrow[r] & Y \end{tikzcd}$$

we use the preservation of flatness under base change to observe that $X \times_k X \to X \times_k Y, Y \times_k X \to Y \times_k Y$ are flat, and under the isomorphism $X \times_k Y \cong Y \times_k X$ we have that the composite $X \times_k X \to Y \times_k Y$ is flat. Now for

$$\begin{tikzcd} X \arrow[r, hook, "\Delta_{X/Y}"] \arrow[dr, "f"'] & X \times_Y X \arrow[r, "i"] \arrow[d, "p"'] & X \times_k X \arrow[d] \\ & Y \arrow[r, "\Delta_Y"'] & Y \times_k Y \end{tikzcd}$$

with square Cartesian we have that p is flat and i is closed as Δ_Y is. Denote $\mathcal{I}, \mathcal{J}$ the ideal sheaves of i, Δ_Y, respectively. We have $p^*(\mathcal{J}/\mathcal{J}^2) \cong \mathcal{I}/\mathcal{I}^2$ by flatness of p and $p^*(\mathcal{J}/\mathcal{J}^2) \cong p^*\Omega^1_{Y/k}$ by definition. So we compute

$$\begin{aligned} f^*\Omega^1_{Y/k} &\cong \Delta^*_{X/Y}(p^*\Omega^1_{Y/k}) \\ &\cong \Delta^*_{X/Y}(\mathcal{I}/\mathcal{I}^2). \end{aligned}$$

By $i \circ \Delta_{X/Y} = \Delta_X : X \to X \times_k X$ and $\Delta_{X/Y}$ is an open immersion by Lemma 9.10, we have that $\Delta^*_{X/Y}(\mathcal{I}/\mathcal{I}^2) = \mathcal{K}/\mathcal{K}^2$ where $\mathcal{K}$ is the ideal sheaf of $X \to X \times_k X$, that is, $\Omega^1_{X/k}$, as desired. ∎

We now consider some geometric consequences of étaleness and unramfiedness.

Let X, Y be smooth integral k-schemes and $f : X \to Y$ a dominant morphism such that $K(X)/K(Y)$ is finite and $\alpha : f^*\Omega^1_{Y/k} \to \Omega^1_{X/k}$ the induced map on the Kähler differentials. Passing to the fiber at the generic point yields a morphism $\Omega^1_{K(Y)/k} \otimes K(X) \to \Omega^1_{K(X)/k}$ which is an isomorphism by separability, implying that $f^*\Omega^1_{Y/k} \otimes K(X) \to \Omega^1_{X/k} \otimes K(X)$ is an isomorphism as well. So the support of $\ker(\alpha)$ is a proper closed subset of X which is empty by $f^*\Omega^1_{Y/k}$ locally free. α induces a canonical map $f^*\omega_{Y/k} \to \omega_{X/k}$ which can be viewed as a global section $s \in H^0(X, f^*\omega^\vee_{Y/k} \otimes \omega_{X/k})$, the vanishing locus of which can be studied.

Definition 9.12 (Ramification Divisor)**.** Let $f : X \to Y$ be a dominant morphism for X, Y smooth integral k-schemes. The ramification divisor R_f is the vanishing $V_X(s)$ for $s \in H^0(X, f^*\omega^\vee_{Y/k} \otimes \omega_{X/k})$ uniquely determined by the map $f^*\omega_{Y/k} \to \omega_{X/k}$.

We record some elementary properties of the ramification divisor.

Lemma 9.13. *Let $f : X \to Y$ be a dominant morphism for X, Y smooth integral k-schemes with ramification divisor R_f.*

(i) $f|_{X \setminus R_f} : X \setminus R_f \to Y$ is unramified.
(ii) $\omega_{X/k} \cong f^*\omega_{Y/k} \otimes \mathcal{O}_X(R_f)$.

Proof of (i). On the complement of R_f the induced morphism on Kähler differentials is an isomorphism which can be checked stalkwise, so the sheaf of relative Kähler differentials vanishes, whence the claim. ■

Proof of (ii). We observe that $\mathcal{O}_X(R_f)$ is the determinant of the normal bundle of R_f in X, whence the claim follows by the adjunction formula Proposition 4.18. ■

Let us consider some examples.

Example 9.14. $\mathbb{A}^1_k \to \mathbb{A}^1_k$ by $x \mapsto x^2$ has ramification divisor (x). More generally for the map $x \mapsto x^n$ and with the characteristicof k not dividing n, the ramification divisor is (x) to order $n-1$.

Example 9.15. If f is étale, then $R_f = \emptyset$.

This is especially interesting in the case of curves, giving the Riemann-Hurwitz formula.

Proposition 9.16 (Riemann-Hurwitz Formula)**.** Let $f : X \to Y$ be a dominant morphism for X, Y smooth integral curves over k and with $K(X)/K(Y)$ separable. Then

$$2g_X - 2 = \deg(f)(2g_Y - 2) + \deg(R_f).$$

Proof. Note that $\omega_{X/k} = \Omega^1_{X/k}, \omega_{Y/k} = \Omega^1_{Y/k}$ are of degrees $2g_X - 2, 2g_Y - 2$, respectively. Then using Lemma 9.13 (ii), we compute

$$\begin{aligned} \deg(\omega_{X/k}) &= \deg\left(f^*\omega_{Y/k} \otimes \mathcal{O}_X(R_f)\right) \\ 2g_X - 2 &= \deg(f^*\omega_{Y/k}) + \deg(\mathcal{O}_X(R_f)) \\ &= \deg(f)(2g_Y - 2) + \deg(R_f) \end{aligned}$$

as desired. ■

Example 9.17. If f is étale then $2g_X - 2 = \deg(f)(2g_Y - 2)$ (cf. Example 9.15).

Example 9.18. A degree 2 map $\mathbb{P}^1_k \to \mathbb{P}^1_k$ by squaring has a ramification divisor of degree 2 given by $\{[0:1], [1:0]\}$.

Example 9.19. If $f : X \to \mathbb{P}^1_k$ is a degree 2 map of curves and $\deg(R_f) = 4$ then $g_X = 1$. If X has a rational point, then X is an elliptic curve. Without loss of generality, we can take the image of R_f to be $\{[0:1], [1:0], [1:1], [\lambda:1]\}$. λ determines X uniquely.

Another interesting consequence is that of Luröth's problem.

Corollary 9.20 (Luröth's Problem)**.** Let k be algebraically closed and X a smooth projective curve over k. For a tower $k \subsetneq K \subseteq L$ where L is purely transcendental of transcendence degree 1, then K is purely transcendental of transcendence degree 1 with $L = K$.

Proof. Recall that $k(t)$ is the function field of $\mathbb{P}^1_k$ and for L as above, there exists a unique smooth projective curve Y over k such that $K = K(Y)$. $K \subseteq k(t)$ can be viewed as induced by a morphism $\mathbb{P}^1_k \to X$ which is dominant. By the Riemann-Hurwitz formula, we have

$$-2 = \deg(f)(2g_X - 2) + \deg(R_f)$$

but the quantity on the right is ≥ 0 if $g_X > 0$ so $g_X = 0$, ie. $X \cong \mathbb{P}^1_k$ and the function fields are isomorphic. ■

10. Lecture 10 – 15th May 2025

Having stated and proved the Riemann-Hurwitz formula Proposition 9.16, we continue with a discussion of unramifiedness and étaleness in the case of curves and their consequences.

Let $f : X \to Y$ be a morphism of smooth projective integral curves over a field k. Let $x \in X$ and $y = f(x) \in Y$ be closed points. Recall that the stalks $\mathcal{O}_{X,x}, \mathcal{O}_{Y,y}$ are discrete valuation rings and that the map of local rings $\mathcal{O}_{Y,y} \to \mathcal{O}_{X,x}$ extends to a map to $\mathbb{Z}$ by taking the value of the image of the uniformizer π_y, for any local ring homomorphisms of discrete valuation rings takes the uniformizer of the source to a power of the uniformizer of the target. This preempts the notion of ramification for morphisms of curves.

Definition 10.1 (Ramification Points of Morphisms of Curves)**.** Let $f : X \to Y$ be a morphism of smooth projective integral curves over a field k. Let $x \in X$ and $y = f(x) \in Y$ be closed points. f is ramified at x if $\nu_x(\pi_y) = e_x > 1$ for ν_x the valuation on $\mathcal{O}_{X,x}$ and π_y the uniformizer of $\mathcal{O}_{Y,y}$.

Definition 10.2 (Tamely Ramified Morphism of Curves at a Point)**.** Let $f : X \to Y$ be a morphism of smooth projective integral curves over a field k ramifed at $x \in X$. f is tamely ramified at x if $\mathrm{char}(k) \nmid e_x = \nu_x(\pi_y)$ and $\kappa(x)/\kappa(y)$ is separable.

Definition 10.3 (Tamely Ramified Morphism of Curves)**.** Let $f : X \to Y$ be a morphism of smooth projective integral curves over a field k. f is a tamely ramified morphism of curves if it is tamely ramified at each ramification point $x \in X$.

This coincides with the language of unramifiedness Definitions 8.13 and 8.14.

Lemma 10.4. Let $f : X \to Y$ be a morphism of smooth projective integral curves over a field k. The following are equivalent:

(a) f is unramified in the sense of Definition 8.14: for all $x \in X$ with image $y \in Y$ $\mathfrak{m}_y\mathcal{O}_{X,x} = \mathfrak{m}_x$ and $\kappa(x)/\kappa(y)$ is separable.

(b) f is not ramified in the sense of Definition 10.1: for all $x \in X$ with image $y \in Y$, $\nu_x(\pi_y) = 1$ and $\kappa(x)/\kappa(y)$ is separable.

Proof. (a)⇒(b) If $\mathfrak{m}_y\mathcal{O}_{X,x} = \mathfrak{m}_y$ for all $x \in X$ with image $y \in Y$ and $\kappa(x)/\kappa(y)$ is separable then the valuation $\nu_x(\pi_y) = 1$ as π_y generates $\mathfrak{m}_y \subseteq \mathcal{O}_{Y,y}$ and hence $\mathfrak{m}_x \subseteq \mathcal{O}_{X,x}$.

(b)⇒(a) Suppose $\nu_x(\pi_y) = 1$ then π_y generates $\mathfrak{m}_y \subseteq \mathcal{O}_{Y,y}$ and hence $\mathfrak{m}_x \subseteq \mathcal{O}_{X,x}$. ■

Ramification allows us to compute the size of finite automorphism groups of curves of genus at least 2.

Proposition 10.5. Let X be a smooth integral projective curve over an algebraically closed field k of characteristic 0 of genus at least 2. Then if $\mathrm{Aut}_k(X)$ is finite, $|\mathrm{Aut}_k(X)| \leq 84(g-1)$.

Proof. Recall the antiequivalence of categories between smooth integral projective curves over algebraically closed fields k and extensions of k of transcendence degree

1. Let $G \leq \mathrm{Aut}_k(X)$ be a subgroup. Define $K(Y) = K(X)^G$ which corresponds to a map $F : X \to Y$ of curves. By construction F, is G-equivariant so for all $y \in Y$ with fiber $\{x_1, \ldots, x_r\} \subseteq X$ G acts transitively on the fiber. In particular, the degree of F is the order of G as a group. Applying the Riemann-Hurwitz formula Proposition 9.16, we have

$$2g_X - 2 = |G|(2g_Y - 2 + \sum_{y \in Y} \frac{e_y - 1}{e_y}). \tag{10.1}$$

We consider several cases:

- If $g_Y \geq 2$ then $2g_Y - 2 \geq 2$ so (10.1) is at least 2, showing $|G| \leq g_X - 1$.
- If $g_Y = 1$ then $2g_Y - 2 = 0$ and $2g_X - 2 = \sum_{y \in Y} \frac{e_y - 1}{e_y}$ which is at least $\frac{1}{2}$ – if f is unramified then $2g_X - 2 = g_Y = 0$ a contradiction as the genus of X is at least 2 – so $\frac{1}{2}|G| \leq 2g_X - 2$ and thus $|G| \leq 4g_X - 1$.
- If $g_Y = 0$ then $2g_X = \sum_{y \in Y} \frac{e_y - 1}{e_y}$. Let $n = |\{y \in Y : e_y > 1\}|$. So

$$2g_X \geq \begin{cases} \geq \frac{1}{2} & n \geq 5 \\ \frac{1}{10} & n = 4 \\ \frac{1}{42} & n = 3. \end{cases}$$

This yields the claim. ∎

Remark 10.6. A deformation theory argument is used to show that the automorphism groups of curves are in fact finite, which we do not produce here.

We can see some (counter)examples of this phenomenon.

Example 10.7. Let $X = V_+(x^3y + y^3z + z^3x) \subseteq \mathbb{P}^2_k$ be the Klein quartic curve. X is of genus $\frac{(4-1)(4-2)}{2} = 3$ and $|\mathrm{Aut}_k(X)| = 168$. This is a Hurwitz curve of genus 3. There exist genera $g \in \mathbb{N}$ for which there is no Hurwitz curve of genus g – in particular, for $2 \leq g \leq 11$ only $g = 3, g = 7$ admit Hurwitz curves. In these cases $g_Y = 0$ in the proof above and the bound is attained.

Example 10.8. In positive characteristic, one can produce larger automorphism groups (which remain finite).

We consider some local properties of étale morphisms.

Proposition 10.9. Let $f : X \to Y$ be smooth of relative dimension d. Then for all $x \in X$ there is an affine neighborhood $U \subseteq X$ of x with image contained in an affine open $V \subseteq Y$ such that there exists a commutative diagram

$$\begin{tikzcd} X \arrow[d, "f"] & U \arrow[l, hook'] \arrow[r, "i"] \arrow[d] & \mathbb{A}^d_V \arrow[dl] \\ Y & V \arrow[l, hook'] & \end{tikzcd}$$

where i is étale.

Proof. See [Stacks, Tag 039P]. ∎

Example 10.10. Let X be a smooth k-scheme of dimension d. Then there locally exists a map to $\mathbb{A}^d_k$, but this map need not be an open immersion. Moreover, there are rarely open immersions $\mathbb{A}^d_k \to X$.

We introduce the notion of standard étaleness.

Definition 10.11 (Standard Étale)**.** Let $f : X \to Y$ be a locally finite type morphism between Noetherian affine schemes. f is standard étale if it is of the form $\mathrm{Spec}(A[t]_f/(g)) \to \mathrm{Spec}(A)$ where g is monic with dierivative invertible in the localization $A[t]_f/(g)$.

Any étale morphism can be factored over a standard étale one.

Proposition 10.12. Let $f : X \to Y$ be an étale morphism. Then for all $x \in X$ there exists an affine neighborhood $U \subseteq X$ of x with image contained in an affine open $V \subseteq Y$ such that $f|_U : U \to V$ is standard étale as a map of affine schemes.

Proof. See [Stacks, Tag 02GT]. ■

We consider formal variants of étale, smoothness, and unramifiedness. These are characterized very similarly to the valuative criterion.

Definition 10.13 (Formally Smooth)**.** Let $f : X \to Y$ be a locally finite type morphism between Noetherian affine schemes. f is formally smooth if for all solid diagrams

$$\begin{tikzcd} \mathrm{Spec}(A/I) \arrow[r] \arrow[d] & X \arrow[d] \\ \mathrm{Spec}(A) \arrow[r] \arrow[ur, dashed] & Y \end{tikzcd} \tag{10.2}$$

where I is a nilpotent ideal, there is at least one morphism $\mathrm{Spec}(A) \to X$ rendering the entire diagram commutative.

Definition 10.14 (Formally Unramified)**.** Let $f : X \to Y$ be a locally finite type morphism between Noetherian affine schemes. f is formally unramified if for all solid diagrams (10.2) where I is a nilpotent ideal, there is at most one morphism $\mathrm{Spec}(A) \to X$ rendering the entire diagram commutative.

Definition 10.15 (Formally Étale)**.** Let $f : X \to Y$ be a locally finite type morphism between Noetherian affine schemes. f is formally étale if for all solid diagrams (10.2) where I is a nilpotent ideal, there is a unique morphism $\mathrm{Spec}(A) \to X$ rendering the entire diagram commutative.

These agree with Definitions 7.4, 8.14 and 9.6 we have already seen, as we made these constructions in the case of f locally finite type betewen locally Noetherian schemes.

Proposition 10.16. Let $f : X \to Y$ be a locally finite type morphism between locally Noetherian schemes. Then:

(i) f is smooth in the sense of Definition 7.4 if and only if it is formally smooth in the sense of Definition 10.13.
(ii) f is unramified in the sense of Definition 8.14 if and only if it is formally unramified in the sense of Definition 10.14.
(iii) f is étale in the sense of Definition 9.6 if and only if it is formally étale in the sense of Definition 10.15.

Proof of (i). See [Stacks, Tag 02H6]. ■

Proof of (ii). See [Stacks, Tag 02HE]. ■

Proof of (iii). See [Stacks, Tag 02HM]. ■

11. Lecture 11 – 19th May 2025 – Interlude: The Étale Toplogy

We briefly discuss the étale topology on schemes. A comprehensive treatment would require an entire course.

Recall that the Zariski topology is intrinsically defined on a scheme, but it does come with some drawbacks. Two notable ones are that constant sheaves are acyclic on irreducible schemes, and that local triviality in the Zariski topology is at times too strong a condition in practice. We will see examples in what follows.

We begin by definign the small étale site.

Definition 11.1 (Small Étale Site)**.** Let X be a locally Noetherian scheme. Define $X_{\text{ét}}$ to be the full subcategory of $\mathsf{Sch}_{/X}$ spanned by objects $U \to X$ where $U \to X$ is étale.

Remark 11.2. A morphism in the small étale site $U \to U'$ is given by a diagram

$$\begin{tikzcd} U \arrow[rr] \arrow[dr] & & U' \arrow[dl] \\ & X & \end{tikzcd}$$

where $U \to X, U' \to X$ is étale, so $U \to U'$ is étale by cancellation for étale morphisms [Stacks, Tag 02GW]. Additionally, being étale is preserved under composition and base change, and isomorphisms are étale – these are the necessary conditions to define a Grothendieck pretopology.

We want the étale site to behave like a topological space. In particular, we want a notion of coverings.

Definition 11.3 (Étale Covering)**.** Let X be a locally Noetherian scheme and $X_{\text{ét}}$ its small étale site. If $U \in X_{\text{ét}}$ then a family of morphisms $\{U_i \to U\}_{i\in I}$ in $X_{\text{ét}}$ is a covering if $\sqcup_{i\in I} U_i \to U$ is surjective.

Let us consider a simple example.

Example 11.4. $\mathbb{A}^1_k \setminus \{0\} \to \mathbb{A}^1_k \setminus \{0\}$ by $z \mapsto z^2$ is étale. The two-element family

$$\left\{ \mathbb{A}^1_k \setminus \{0\} \xrightarrow{z\mapsto z^2} \mathbb{A}^1_k, \mathbb{A}^1_k \setminus \{1\} \xrightarrow{\mathrm{id}_{\mathbb{A}^1_k\setminus\{1\}}} \mathbb{A}^1_k \right\}$$

is an étale covering since the maps are jointly surjective.

We can compare this to the Zariski site associated to the Zariski topology.

Definition 11.5 (Zariski Site)**.** Let X be a locally Noetherian scheme. Define X_{Zar} to be the full subcategory of $\mathsf{Sch}_{/X}$ spanned by the objects $U \to X$ where $U \to X$ is an open immersion.

Remark 11.6. As before, open immersions satisfy cancellation, so any morphism $U \to U'$ in X_{Zar} is automatically an open immersion.

Note that open immersions are in particular étale so the Zariski site includes into the étale site – in other words, the étale site contains more morphisms than the Zariski site. The functor $F : X_{\textsf{ét}} \to X_{\textsf{Zar}}$ is continuous since the preimage $F^{-1}([U \to X])$ is a map in the étale topology for an open immersion $[U \to X]$ in $X_{\textsf{Zar}}$.

We can define sheaves and presheaves over the étale site.

Definition 11.7 (Étale Presheaves)**.** Let X be a locally Noetherian scheme and $X_{\textsf{ét}}$ its étale site. An étale presheaf on $X_{\textsf{ét}}$ is a functor $\mathcal{F} : X_{\textsf{ét}}^{\textsf{Opp}} \to \textsf{AbGrp}$.

Definition 11.8 (Étale Sheaves)**.** Let X be a locally Noetherian scheme and $X_{\textsf{ét}}$ its étale site. An étale sheaf on $X_{\textsf{ét}}$ is an étale presheaf $\mathcal{F}$ such that for all étale coverings $\{U_i \to U\}_{i \in I}$ in $X_{\textsf{ét}}$ the sequence

$$0 \longrightarrow \mathcal{F}(U) \longrightarrow \prod_{i \in I} \mathcal{F}(U_i) \rightrightarrows \prod_{i,j \in I} \mathcal{F}(U_i \times_U U_j)$$

is an equalizer.

We can define a functor $F_* : \textsf{Sh}(X_{\textsf{ét}}) \to \textsf{Sh}(X_{\textsf{Zar}})$ by $\mathcal{F} \mapsto [[U \to X] \mapsto \mathcal{F}(U)]$ by restriction. More generally for a morphism $f : X \to Y$ of locally Noetherian schemes there we can define $f_* : \textsf{Sh}(X_{\textsf{ét}}) \to \textsf{Sh}(Y_{\textsf{ét}})$, $f^{-1} : \textsf{Sh}(Y_{\textsf{ét}}) \to \textsf{Sh}(X_{\textsf{ét}})$.

Definition 11.9 (Global Sections of Étale Sheaves)**.** Let X be a locally Noetherian scheme and $\mathcal{F}$ a sheaf of Abelian groups on $X_{\textsf{ét}}$. The global sections of $\mathcal{F}$ is defined to be $\Gamma(X_{\textsf{ét}}, \mathcal{F}) = \mathrm{Hom}_{\textsf{PSh}(X_{\textsf{ét}})}(*, \mathcal{F})$ where $*$ is the final object in $\textsf{PSh}(X_{\textsf{ét}})$.

Moreover, the category of Abelian sheaves on a site has enough injectives [Stacks, Tag 01DL], so this allows us to define all higher cohomology groups.

Definition 11.10 (Cohomology of Étale Sheaves)**.** Let X be a locally Noetherian scheme and $\mathcal{F}$ a sheaf of Abelian groups on $X_{\textsf{ét}}$. The cohomology of $\mathcal{F}$ is defined to be $H^i(X_{\textsf{ét}}, \mathcal{F})$ is the cohomology of an injective resolution of $\mathcal{F}$.

Remark 11.11. The notation $H^i_{\textsf{ét}}(X, \mathcal{F})$ is also used in the literature.

We can make the analogous constructions for the structure sheaf, stalks, and derived pushforwards.

Remark 11.12. The diagram

$$\begin{tikzcd}
\textsf{Sh}(X_{\textsf{Zar}}) \arrow[r] \arrow[d, "H^i(X,-)"'] & \textsf{Sh}(X_{\textsf{ét}}) \arrow[d, "H^i(X_{\textsf{ét}},-)"] \\
\textsf{AbGrp} \arrow[r, dotted, "?"'] & \textsf{AbGrp}
\end{tikzcd}$$

need not commute, but does so for quasicoherent sheaves.

Let us consider more examples of étale sheaves.

Example 11.13. Consider the sheaves $\mathbb{G}_m, \mathbb{G}_a, \mu_n$ as sheaves on the étale site taking $[U \to X]$ to $\Gamma(U, \mathcal{O}_U^\times), \Gamma(U, \mathcal{O}_U), \{s \in \mathcal{O}_U(U) : s^n = 1_U\}$.

As a consequence of Remark 11.12, we have the following comparison between the ordinary and the étale Picard groups.

Proposition 11.14. Let X be a locally Noetherian scheme. There is an isomorphism of Abelian groups $H^1(X, \mathcal{O}_X^\times) \cong H^1(X_{\text{ét}}, \mathbb{G}_m)$.

Though already on the level of second cohomology, there are schemes X for which $H^2(X, \mathcal{O}_X^\times) \ncong H^2(X_{\text{ét}}, \mathbb{G}_m)$.

The étale sheaves $\mathbb{G}_m, \mu_n$ fit together in the Kummer sequence.

Proposition 11.15 (Kummer Sequence)**.** Let X be a locally Noetherian k scheme and n an integer not divisible by the characteristic of k. Then there is a short exact sequence of étale sheaves in $X_{\text{ét}}$

$$0 \to \mu_n \to \mathbb{G}_m \xrightarrow{(-)^n} \mathbb{G}_m \to 0.$$

Taking étale cohomology of the Kummer sequence of Proposition 11.15 yields a long exact sequence

$$\begin{tikzcd}
0 \arrow[r] & H^0(X_{\text{ét}}, \mu_n) \arrow[r] & H^0(X_{\text{ét}}, \mathbb{G}_m) \arrow[r] & H^0(X_{\text{ét}}, \mathbb{G}_m) \arrow[dll] \\
 & H^1(X_{\text{ét}}, \mu_n) \arrow[r] & H^1(X_{\text{ét}}, \mathbb{G}_m) \arrow[r] & H^1(X_{\text{ét}}, \mathbb{G}_m) \arrow[r] & \dots.
\end{tikzcd}$$

We provide a more explicit description the behavior of the first cohomology groups:

- $H^1(X_{\text{ét}}, \mu_n) = \{(\mathcal{L}, \varphi) : \varphi : \mathcal{L}^{\otimes n} \xrightarrow{\sim} \mathcal{O}_X\}$.
- $H^1(X_{\text{ét}}, \mathbb{G}_m) \to H^1(X_{\text{ét}}, \mathbb{G}_m)$ by $\mathcal{L} \mapsto \mathcal{L}^{\otimes n}$.

We can consider related phenomena.

Example 11.16. $H^1(X, \mathrm{PGL}_n) \ncong H^1(X_{\text{ét}}, \mathrm{PGL}_n)$. If $P \to X$ is is a morphism for which there exists an étale cover $\{U_i \to X\}_{i \in I}$ such that $P \times_X U_i \cong \mathbb{P}^{n-1}_{U_i}$, we cannot concldue that P is also Zariski-locally a projective space. It is possible that there exists no Zarksi open cover such that the fibers are locally projective spaces.

Example 11.17. Let $\mathbb{G}_m = P \to X = \mathbb{G}_m$. The squaring map is étale so we get a diagram

$$\begin{tikzcd}
\mathbb{G}_m \coprod \mathbb{G}_m \arrow[dr, "\sim"] \arrow[drr, bend left, "{(\mathrm{id},(-)^{-1})}"] \arrow[ddr, bend right, "{(\mathrm{id},\mathrm{id})}"'] & & \\
 & \mathbb{G}_m \times_{\mathbb{G}_m} \mathbb{G}_m \arrow[r] \arrow[d] & \mathbb{G}_m = P \arrow[d] \\
 & \mathbb{G}_m \arrow[r, "(-)^2"'] & \mathbb{G}_m = X
\end{tikzcd}$$

which is once again étale locally trivial, but not Zariski locally trivial as there does not exist a nonempty open subset $U \subseteq \mathbb{G}_m$ for which $P_U = U \coprod U$.

Example 11.18. Let X be an elliptic curve over $\mathbb{C}$ and $x_0 \in X[2]$ a 2-torsion point. The map $x \mapsto x + x_0$ defines a $\mathbb{Z}/2\mathbb{Z}$ action on X. Define Y the quotient of X by this automorphism so $X \to Y$ is étale of degree 2. But this map is not Zariski locally trivial. Every nonempty open subset of U has preimage which is not a disjoint union as X is irreducible.

The technology of the étale site also allows us to define an analogue of singular cohomology on schemes.

Theorem 11.19 (Artin – Comparison Isomorphism)**.** Let X be a finite type smooth $\mathbb{C}$-scheme. Let X^{an} the set $X(\mathbb{C})$ with the complex-analytic topology and Λ a finite Abelian group. There is an isomorphism $H^i(X_{\text{ét}}, \underline{\Lambda}) \cong H^i(X^{\mathsf{an}}, \Lambda)$ where on the left we are computing constant sheaf cohomology in the étale site and on the right singular cohomology of the complex manifold with Λ-coefficients.

The reason this holds true is that any étale morphism $U \to X$ induces open maps $B_i \to X^{\mathsf{an}}$ in the analytic topology where $\bigcup B_i$ form an analytic cover of U^{an} as a complex manifold.

Remark 11.20. The comparison map $H^1(X_{\text{ét}}, \mathbb{G}_m) \to H^1(X^{\mathsf{an}}, \mathcal{O}_X^\times)$ beteween étale line bundles and complex analytic ones is not in general an isomorphism, but is in the case where X is projective by Serre's GAGA principle.

We conclude with a discussion of the étale fundamental group.

Definition 11.21 (Finite Étale Category)**.** Let X be a locally Noetherian scheme. Denote $\mathsf{FET}(X)$ to be the full category of $\mathsf{Sch}_{/X}$ spanned by finite étale maps $Y \to X$.

This allows us to define the étale fundamental group as follows.

Definition 11.22 (Étale Fundamental Group)**.** Let X be a locally Noetherian scheme with $x \in X$ and $F : \mathsf{FET}(X) \to \mathsf{Sets}$ by $[Y \to X] \mapsto \mathrm{Mor}_{\mathsf{Sch}_{/X}}(\overline{x}, Y)$ where $\overline{x}$ is a finite separable extension of $\kappa(x)$. The étale fundamental group $\pi_1^{\text{ét}}(X, x)$ is defined to be the automorphism group of the functor F.

This does not agree in general with the topological fundamental group.

Example 11.23. Let X be a smooth projective surface over $\mathbb{C}$ and $\sigma \in \mathrm{Aut}(\mathbb{C})$. Consider the Cartesian square

$$\begin{tikzcd} X^\sigma \arrow[r] \arrow[d] & X \arrow[d] \\ \mathrm{Spec}(\mathbb{C}) \arrow[r, "\sigma"'] & \mathrm{Spec}(\mathbb{C}) \end{tikzcd}$$

The étale fundamental groups of X, X^σ agree, but there is no induced map on the analytic manifolds that makes the map on fundamental groups an isomorphism.

12. Lecture 12 – 22nd May 2025

We begin a discussion of blowups. Recall that Cartier divisors are sections of $\Gamma(X, \mathcal{K}_X^\times/\mathcal{O}_X^\times)$. Roughly speaking, this provides the data of collections of rational functions on an affine open cover with regular quotients. In this way, Cartier divisors play a fundamental role in the study of schemes. However, divisors may fail to be Cartier for two reasons: singularties and being of the wrong codimension.

Blowups give a "universal" construction to modify a scheme with its closed subscheme to an effective Cartier divisor.

Definition 12.1 (Blowup)**.** Let X be locally Noetherian and $Z \subseteq X$ a closed subscheme. The blowup of X along Z is a Cartesian diagram

$$\begin{tikzcd} E_Z X \arrow[r] \arrow[d] & \mathrm{Bl}_Z X \arrow[d] \\ Z \arrow[r] & X \end{tikzcd}$$

where the exceptional divisor $E_Z X$ is an efective Cartier divisor in $\mathrm{Bl}_Z X$ and final amongst cartier divisor-scheme pairs (D, W) fitting into Cartesian diagrams

$$\begin{tikzcd} D \arrow[r] \arrow[d] & W \arrow[d] \\ Z \arrow[r] & X. \end{tikzcd}$$

Remark 12.2. That is, for any Cartesian square

$$\begin{tikzcd} D \arrow[r] \arrow[d] & W \arrow[d] \\ Z \arrow[r] & X \end{tikzcd}$$

where D is an effective Cartier divisor in W, there exists a factorization of this diagram

$$\begin{tikzcd} D \arrow[r] \arrow[d] & W \arrow[d] \\ E_Z X \arrow[r] \arrow[d] & \mathrm{Bl}_Z X \arrow[d] \\ Z \arrow[r] & X \end{tikzcd}$$

where both squares are Cartesian.

Remark 12.3. Having defined blowups in Definition 12.1 by its universal property, it is unique up to unique isomorphism if it exists.

Remark 12.4. If $Z \subseteq X$ is Cartier, then the blowup is just X, since the pair $Z \to X$ trivially satisfies the desired universal property.

We can show that these exist, first affine-locally, then globally by gluing.

Lemma 12.5. Let A be a Noetherian ring and $I \subseteq Z$ an ideal. The blowup of $\operatorname{Spec}(A)$ along $V(I)$ exists.

Proof. Since A is Noetherian, I is finitely generated, say by $a_0, \ldots, a_n$. Consider $\operatorname{Proj}(\bigoplus_{d\geq 0} I^d)$. We show this satisfies the universal property.

Note $\beta^{-1}(I) = I \cdot \bigoplus_{d\geq 0} I^d = \mathcal{O}_{\operatorname{Proj}(\bigoplus_{d\geq 0} I^d)}(1)$ which is a Cartier divisor. We can define a map of graded rings $\varphi : A[x_0, \ldots, x_n] \to \bigoplus_{d\geq 0} I^d$ by $x_i \mapsto a_i$ which is by inspection a surjective morphism of graded rings. This induces contravariantly on Proj

$$\begin{tikzcd} \operatorname{Proj}(\bigoplus_{d\geq 0} I^d) \arrow[rr] \arrow[dr] & & \mathbb{P}^n_A \arrow[dl] \\ & \operatorname{Spec}(A) & \end{tikzcd}$$

where $\mathbb{P}^n_A \to \operatorname{Spec}(A)$ and $\operatorname{Proj}(\bigoplus_{d\geq 0} I^d) \to \mathbb{P}^n_A$ are both closed, so $\operatorname{Proj}(\bigoplus_{d\geq 0} I^d) \to \operatorname{Spec}(A)$ is closed and the kernel of φ is the ideal generated by homogeneous polynomials in $n+1$ variables which vanish at $(a_0, \ldots, a_n)$.

Let

$$\begin{tikzcd} D \arrow[r] \arrow[d] & W \arrow[d, "f"] \\ V(I) \arrow[r] & \operatorname{Spec}(A) \end{tikzcd}$$

be a Cartesian square where D is an effective Cartier divisor in W – that is, where $f^{-1}I \cdot \mathcal{O}_W = \mathcal{I}_D$. Since I is finitely generated by the a_i's, their images $s_0, \ldots, s_n$ in $\mathcal{O}_W$ generate $\mathcal{I}_D$. This induces a unique morphism $g : W \to \mathbb{P}^n_A$ over $\operatorname{Spec}(A)$ such that $g^*\mathcal{O}_{\mathbb{P}^n_A}(1) \cong \mathcal{I}_D$ with $s_i = g^{-1}(x_i)$. Moreover, this morphism factors over the closed subscheme $\operatorname{Proj}(\bigoplus_{d\geq 0} I^d) \subseteq \mathbb{P}^n_A$ – any element of the kernel $\ker(\varphi)$ of the morphism of graded rings is a homogeneous polynomial of degree m that vanishes on $(a_0, \ldots, a_n)$ and hence on $(s_0, \ldots, s_n)$ in $\Gamma(W, \mathcal{I}_D^m)$. This shows that $\operatorname{Proj}(\bigoplus_{d\geq 0} I^d)$ satisfies the desired universal property. ■

Remark 12.6. Let $f : X \to Y$ be any morphism and $Z \subseteq Y$ closed with sheaf of ideals $\mathcal{I}_Z$. $f^{-1}\mathcal{I}_Z$ does not necessarily agree with $f^*\mathcal{I}_Z = \mathcal{O}_X \otimes_{f^{-1}\mathcal{O}_Y} f^{-1}\mathcal{I}_Z$, and $f^*\mathcal{I}_Z$ need not even be a subsheaf of $\mathcal{O}_X$. There exists a morphism $f^*\mathcal{I}_Z \to \mathcal{O}_X$ whose image is $f^{-1}\mathcal{I}_Z$ the ideal sheaf which need not be an isomorphism.

We now treat the general case by gluing.

Theorem 12.7. Let X be locally Noetherian and $Z \subseteq X$ a closed subscheme. The blowup of X along Z exists.

Proof. Note that for $U \subseteq X$ open, $\operatorname{Bl}_{(Z\cap U)}U \cong \beta^{-1}(U)$ uniquely by the universal property. Covering X with affine opens, and the intersections of any two such affine opens with distinguished opens, existence and uniqueness of the blowup affine-locally Lemma 12.5 shows that the construction glues to the blowup of X. ■

Remark 12.8. If $U \subseteq X$ is open, then $\mathrm{Bl}_{U\cap Z}U = \beta^{-1}(U)$, and if $U = X \setminus Z$ then $\beta^{-1}(U) \to U$ is an isomorphism.

We want to consider how subschemes in X behave in the blowup.

Definition 12.9 (Strict Transform)**.** Let X be locally Noetherian, $Z \subseteq X$ a closed subscheme, and $\beta : \mathrm{Bl}_Z X \to X$ the blowup of X along Z. If $Y \subseteq X$ is a closed subscheme of X not contained in Z the total transform of Y is the scheme-theoretic preimage $Y \times_X \mathrm{Bl}_Z X$ of Y in $\mathrm{Bl}_Z X$.

Definition 12.10 (Total Transform)**.** Let X be locally Noetherian, $Z \subseteq X$ a closed subscheme, and $\beta : \mathrm{Bl}_Z X \to X$ the blowup of X along Z. If $Y \subseteq X$ is a closed subscheme of X not contained in Z the total transform of Y is $\widetilde{Y} = \overline{\beta^{-1}(Y \setminus (Y \cap Z))} \subseteq \mathrm{Bl}_Z X$.

Let us make some computations of the line bundles associated to the exceptional divisor of blowups.

Proposition 12.11. Let X be locally Noetherian, $Z \subseteq X$ a closed subscheme, and $\beta : \mathrm{Bl}_Z X \to X$ the blowup of X along Z. Denote the structure sheaf of $\mathrm{Bl}_Z X$ by $\mathcal{O}_\beta$. Then $\mathcal{O}_\beta(E_Z X) = \mathcal{O}_\beta(-1)$.

Proof. It suffices to observe that the ideal sheaf of $E_Z X$ is $\mathcal{O}_\beta(1)$, so twisting by this divisor gives the dual of the ideal sheaf $\mathcal{O}_\beta(E_Z X) \cong \mathcal{O}_\beta(-1)$ ■

Remark 12.12. The use of $\mathcal{O}_\beta$ for $\mathcal{O}_{\mathrm{Bl}_Z X}$ is justified as $\mathrm{Bl}_Z X$ is a projective bundle as the $\underline{\mathrm{Proj}}$ of the sheaf of graded algebras locally given by the Rees algebra as shown in the construction of the blowup Lemma 12.5.

Moreover, smoothess is preserved under blowups.

Proposition 12.13. Let X be locally Noetherian, $Z \subseteq X$ a closed subscheme, and $\beta : \mathrm{Bl}_Z X \to X$ the blowup of X along Z. If X and Z are smooth, then $\mathrm{Bl}_Z X$ is smooth, and $\mathcal{N}_{E_Z X/\mathrm{Bl}_Z X} = \mathcal{O}_{E_Z X}(-1)$. Prove this.

Example 12.14. Consider $\mathrm{Bl}_{\{(0,0)\}}\mathbb{A}^2_k$. Note that $\mathbb{A}^2_k = \mathrm{Spec}(k[x,y])$ and the defining ideal of $\{(0,0)\}$ is $I = (x,y)$. We know that $\mathrm{Bl}_{\{(0,0)\}}\mathbb{A}^2_k \subseteq \mathbb{P}^1_{\mathbb{A}^2_k} = \mathbb{P}^1_A$ where $A = k[x,y]$. We can define a ring map $\varphi : A[u,v] \mapsto \bigoplus_{d\geq 0} I^d$ by $u \mapsto x, v \mapsto y$. The kernel is generated by $uy - vx$ which defines the blowup as a closed subscheme of $\mathbb{P}^1_A$. We can easily see that the fiber over any point away from the origin is a single point, and the fiber over the origin is an entire $\mathbb{P}^1_k$. Indeed for any L a line through the origin defined by $\{y = tx\}$ the strict transform is the union of the line itself with the $\mathbb{P}^1_k$ over the origin, and the line intersects the $\mathbb{P}^1_k$ over the origin at its slope $[1 : t]$.

We prove the property of Remark 12.8 for closed subschemes.

Lemma 12.15. Let X be locally Noetherian, $Z \subseteq X$ a closed subscheme, and $\beta : \mathrm{Bl}_Z X \to X$ the blowup of X along Z. Let $Y \subseteq X$ be a closed subscheme not contained in Z. Then $\mathrm{Bl}_{(Z\cap Y)}Y \cong \widetilde{Y}$ the strict transform of Y in the blowup of X.

Proof. We show this satisfies the universal property. Consider the following diagram.

$$\begin{tikzcd}
& W \arrow[rr] \arrow[ddrr, bend right, "g"] & & Y \arrow[rr] & & X \\
D \arrow[ur] \arrow[rr] \arrow[ddrr, bend right] & & Y \cap Z \arrow[ur] \arrow[rr] & & Z \arrow[ur] & \\
& & & \mathrm{Bl}_Z X \arrow[uurr, bend right] & & \\
& & E_Z X \arrow[ur] \arrow[uurr, bend right] & & &
\end{tikzcd}$$

We seek to show that the image of g is contained in $\widetilde{Y}$. But this is clear from the construction of $\widetilde{Y} = \overline{\beta^{-1}(Y \setminus (Y \cap Z))}$ and uniqueness from the universal property of the blowup $\mathrm{Bl}_Z X$. ■

We show some properties of the blowup morphism $\beta : \mathrm{Bl}_Z X \to X$.

Proposition 12.16. Let X be locally Noetherian, $Z \subseteq X$ a closed subscheme, and $\beta : \mathrm{Bl}_Z X \to X$ the blowup of X along Z. Then β is proper, birational, and surjective. Moreover, if X is projective, so too is $\mathrm{Bl}_Z X$.

Proof. Properness is local on target and β is locally given by the composition of a closed immersion and projection $\mathbb{P}^n_A \to \mathrm{Spec}(A)$, hence proper. Birationality is clear as $\mathrm{Bl}_Z X \setminus E_Z X \to X \setminus Z$ is an isomorphism between dense open sets of the sorce and target. If X is projective, then the structrure map of $\mathrm{Bl}_Z X$ is the composition of two projective morphisms. hence projective. ■

Remark 12.17. The blowup is rarely flat or unramified, hence almost never étale.

We consider some examples of blowups of curves, illustrating that they can be used to resolve singularties.

Example 12.18. Consider the nodal cubic $V(y^2 - x^3 - x^2) \subseteq \mathbb{A}^2_k$. Since this is singular at the origin, we can build on our computation of the blowup of $\mathbb{A}^2_k$ at the origin Example 12.14 to see that $\mathrm{Bl}_{\{(0,0)\}} C$ is given by $\{y^2 - x^3 - x^2, uy - vx\}$. On the $u \neq 0$ chart, the equations $y = xv, x^2v^2 = x^3 + x^2$ shows that $\mathrm{Bl}_{\{(0,0)\}} C$ meets the exceptional divisor at two points, since the latter equation is quadratic in the $\mathbb{P}^1_k$-variable v.

Example 12.19. The cuspidal curve $V(y^2 - x^3) \subseteq \mathbb{A}^2_k$ is similarly singular at the origin. Repeating the computation of Example 12.18, we get that $\mathrm{Bl}_{\{(0,0)\}} C$ is given by the equations $\{y^2 - x^3, uy - vx\}$ which on the $u \neq 0$ chart gives $\{y = xv, x^2v^2 = x^3\}$ showing that the blowup meets the exceptional divisor at one point with multiplicity 2.

13. Lecture 13 – 26th May 2025 – Interlude: On Algebraic Varieties

We consider the more classicial theory of algebraic varieties with the theory of schemes in hand. For this lecture, we fix an algebraically closed field $k = \overline{k}$ of characteristic zero.

Definition 13.1 (Affine Space)**.** Affine space $\mathbb{A}^n_k$ is the set underlying the k-vector space of dimension n endowed with the Zariski topology where sets of the form $V(\mathfrak{a}) = \{(x_1, \ldots, x_n) \in \mathbb{A}^n_k : f(x_1, \ldots, x_n) = 0, \forall f \in \mathfrak{a}\}$ for ideals $\mathfrak{a}$ are taken to be closed.

We can similarly define projective space.

Definition 13.2 (Projective Space)**.** Projective space $\mathbb{P}^n_k$ is the set underlying the projectivization of $k^{n+1} \setminus \{0\}$ endowed with the Zariski topology where sets of the form $V_+(\mathfrak{a}) = \{(x_0, \ldots, x_n) \in \mathbb{A}^n_k : f(x_0, \ldots, x_n) = 0, \forall f \in \mathfrak{a}\}$ for homogeneous ideal s$\mathfrak{a}$ are taken to be closed.

Remark 13.3. One easily verifies that Definitions 13.1 and 13.2 satisfies the axioms of the closed sets of a topological space.

This naturally recovers algebraic sets as those closed sets of affine and projective space with the Zariski topology.

Definition 13.4 (Algebraic Set)**.** A set $X \subseteq \mathbb{A}^n_k$ (resp. $X \subseteq \mathbb{P}^n_k$) is an affine (resp. projective) algebraic set if it is closed in the Zariski topology of $\mathbb{A}^n_k$ (resp. $\mathbb{P}^n_k$).

Varieties arise as a specific class of algebraic sets.

Definition 13.5 (Algebraic Variety)**.** An algebraic variety is an algebraic set whose underlying topological space is irreducible.

We can define quasiaffine and quasiprojective varieties as open subsets of affine and projective varieties, respectively.

Definition 13.6 (Quasiaffine Variety)**.** A quasiaffine variety is an open subset of an affine variety.

Definition 13.7 (Quasiprojective Variety)**.** A quasiprojective variety is an open subset of a projective variety.

Affine and projective varieties are themselves quasiaffine and quasiprojective, respectively, and quasiaffine varieties are quaisprojective by the open embedding of $\mathbb{A}^n_k \to \mathbb{P}^n_k$.

We can consider the category of all varieties.

Definition 13.8 (Category of Varieties)**.** The category Var_k has objects quasiprojective varieties and morphisms given by regular functions – those functions that are locally rational.

By the preceding discussion, Var_k contains affine, projective, and quasiaffine varieties.

Theorem 13.9. Let k be an algebraically closed field. There exists a fully fatihful embedding from k-varieties Var_k to k-schemes Sch_k

$$t : \mathsf{Var}_k \longrightarrow \mathsf{Sch}_k.$$

Outline of Proof. For a k-variety X, let $t(X)$ be its set of irreducible components endowed with the topology that sets of the form $t(Z)$ are closed for $Z \subseteq X$ closed. By reduction to the case of affine varieties, the map t takes X to $\mathrm{Spec}(A(X))$ with inverse (on the level of topological spaces) given by taking $\mathrm{mSpec}(\Gamma(t(X), \mathcal{O}_{t(X)}))$. The structure sheaf is induced by the obvious inclusion $\mathrm{mSpec}(A(X)) \to \mathrm{Spec}(A(X))$ for A the coordinate ring of X which can be seen to be fully faithful. ■

We can also define abstract varieties and consider its embedding into k-schemes.

Definition 13.10 (Abstract Varieties)**.** The category of abstract variteies AbsVar_k is the full category of Sch_k spanned by integral separated k-schemes of finite type.

Theorem 13.9 implies that there is a fully faithful embedding $\mathsf{Var}_k \to \mathsf{AbsVar}_k$ but this is not essentially surjective, with a counterexample given by Hironaka.

We now consider function fields of varieties.

Definition 13.11 (Function Field)**.** Let X be an algebraic variety. Its function field $K(X)$ is the field of equivalence classes of rational functions which agree on a nonempty open subset of their intersection.

The function field of a variety is equivalent to the function field of $t(X)$.

Definition 13.12 (Birational)**.** Two algebraic varieties X, Y are isomorphic if $K(X) \cong K(Y)$ as k-algebras.

This in turn allows us to define birationality of integral separated finite type k-schemes.

14. Lecture 14 – 2nd June 2025

We make preparations towards the proof of the theorem of formal functions, which allows the computation of stalks of derived pushforwards as limit of cohomology along thickened neighborhoods.

To wit, the tools of analytic geometry, in particular the idea of functions in a small open neighborhood of a point given by power series, can be captured in the setting of algebraic geometry using complete rings.

Definition 14.1 (Graded Construction)**.** Let A be a Noetherian ring and M an A-module with decreasing filtration

$$\cdots \subsetneq M_n \subsetneq \cdots \subsetneq M_1 \subsetneq M_0 = M. \tag{14.1}$$

The graded construction $\mathrm{gr}^\bullet(M)$ of M is the direct sum $\bigoplus_{0\le i\le n-1} M_i/M_{i+1}$.

Remark 14.2. Evidently we have by definition $\mathrm{gr}^i(M) \cong M_i/M_{i+1}$.

Definition 14.3 (Completion Along Filtration)**.** Let A be a Noetherian ring and M an A-module with decreasing filtration (14.1). The completion of M along this filtration is the limit $\widehat{M} = \lim_i M/M_i$.

The filtration induces a topology on M with basis given by the cosets $x + M_i$ for $x \in M, i \in \mathbb{N}$ which is Hausdorff if and only if $\bigcap_{i\ge 0} M_i = 0$.

Definition 14.4 (Equivalent Filtrations)**.** Let A be a Noetherian ring and M an A-module.

$$\cdots \subsetneq M_n \subsetneq \cdots \subsetneq M_1 \subseteq M_0 = M \tag{14.2}$$

$$\cdots \subsetneq M'_n \subsetneq \cdots \subsetneq M'_1 \subseteq M'_0 = M \tag{14.3}$$

be two filtrations of M. The filtrations (14.2) and (14.3) are equivalent if the systems are final in each other – for every M_n there exists M'_m such that $M'_m \subseteq M_n$ and for each M'_n there exists M_m such that $M_m \subseteq M'_n$.

Remark 14.5. Any two equivalent filtrations in the sense of Definition 14.4 induce the same topology on the module M.

In what follows, we will consider $I \subseteq A$ and $M_d = I^d M$.

Example 14.6. If $M = A$, the induced topology on A (as an A-module) is the I-adic topology.

Example 14.7. Let $M = A = k[x]$. There is a filtration of A by $I = (x)$. Two elements of A being close in the I-adic filtration imply that the coefficients of these polynomials agree in small degree, that is, that the polynomials are the same close to $0 \in k$.

Let A be a Noetherian ring, M a finite A-module, and $N \subseteq M$ an A-module. We seek to understand the filtration $\{I^d N\}_{d\ge 0}$ in terms of the filtration $\{I^d M \cap N\}_{d\ge 0}$. The Artin-Rees lemma gives us an equivalence beteween these two filtrations.

Theorem 14.8 (Artin-Rees Lemma)**.** Let A be a Noetherian ring, M a finitely generated A-module, $N \subseteq M$ a submodule, and $I \subseteq A$ an ideal. The filtrations $\{I^d M\}_{d \geq 0}$ and $\{I^d M \cap N\}_{d \geq 0}$ are equivalent as filtrations.

Proof. For any d, we have the containment $I^d N \subseteq I^d M \cap N$. By Definition 14.4, it suffices to find $m \geq 0$ such that $I^m M \cap N \subseteq I^d N$. For any $c \geq 0$ we have that $I^c M \cap N \subseteq N$ so it suffices to find $c \geq 0$ such that $I^{d+c} M \cap N \subseteq I^d(I^c M \cap N)$ in which case equality will hold.

Consider the graded ring $S = \bigoplus_{k \geq 0} I^k$ and the S-module $\widetilde{M} = \bigoplus_{k \geq 0} I^k M$. Since I is finitely generated, S is a Noetherian ring being the quotient of a finitely-generated A-algebra, and $\widetilde{M}$ is finite over S being generated by the generators of M over A. The submodule $\widetilde{N} = \bigoplus_{k \geq 0}(I^k M \cap N)$ of $\widetilde{M}$ is finitely generated over S as well. Let $x_1, \ldots, x_r$ be generators of $\widetilde{N}$ with x_j in some $I^{k_j} M \cap N$. By the finite generation hypothesis, we can take c such that $k_j < c$ for all $1 \leq j \leq r$. By construction, each $x \in I^{d+c} M \cap N$ can be written as $\sum_{j=1}^{r} \alpha_j x_j$ with $\alpha_j \in I^{d+c-k_j}$. This shows $x \in I^d(I^c M \cap N)$, whence the claim. ■

This "commutativity" of intersections with submodules allows us to show that rings map injectively into their completions.

Theorem 14.9 (Krull – Intersection)**.** Let A be a Noetherian local ring with maximal ideal $\mathfrak{m}$. Then $\bigcap_{d \geq 0} \mathfrak{m}^d = 0$.

Proof. Let $I = \bigcap_{d \geq 0} \mathfrak{m}^d$. We have that $I = \mathfrak{m}^d \cap I$ for every $d \geq 0$. By the Artin-Rees lemma Theorem 14.8, there exists $k \geq 0$ such that $I = \mathfrak{m}^k \cap I \subseteq \mathfrak{m} I$. Thus $\mathfrak{m} I = I$. By Nakayama's lemma, $I = 0$. ■

We immediately deduce the following.

Corollary 14.10. Let A be a Noetherian local ring with maximal ideal $\mathfrak{m}$. Then $A \to \widehat{A}$ is injective.

Proof. The kernel of the map $A \to \widehat{A}$ is given by $\bigcap_{d \geq 0} \mathfrak{m}^d$ – the intersection of the kernels of the component maps $A \mapsto A/\mathfrak{m}^d$ in the system

$$\begin{tikzcd}
A \arrow[dr] \arrow[drr] \arrow[drrr] & & & \\
\ldots \arrow[r] & A/\mathfrak{m}^3 \arrow[r] & A/\mathfrak{m}^2 \arrow[r] & A/\mathfrak{m}
\end{tikzcd}$$

By Theorem 14.9, the kernel is trivial, showing the map is injective. ■

We collect some important properties of complete rings.

Proposition 14.11. Let A be a Noetherian ring, M an A-module, and $I \subseteq A$ an ideal.

(i) The functor $\mathsf{Mod}_A \to \mathsf{Mod}_{\widehat{A}}$ by $M \mapsto \widehat{M}$ is exact on finitely generated modules.
(ii) If M is a finitely generated A-module, then $M \otimes_A \widehat{A} \to \widehat{M}$ is an isomorphism.

(iii) $\widehat{A}$ is flat over A.
(iv) $\widehat{A}$ is Noetherian.
(v) If A is a further a local ring, then $\widehat{A}$ with maximal ideal $\widehat{\mathfrak{m}}$ is a Noetherian local ring and $\mathrm{gr}^\bullet_{\mathfrak{m}}(A) \cong \mathrm{gr}^\bullet_{\widehat{\mathfrak{m}}}(\widehat{A})$.
(vi) Let A be a local Noetherian $k = A/\mathfrak{m}$-algebra of dimension d. The following are equivalent:
  (a) A is regular.
  (b) The local ring $\widehat{A}$ with maximal ideal $\widehat{\mathfrak{m}}$ of (v) is regular.
  (c) $\mathrm{gr}^\bullet_{\mathfrak{m}}(A) \cong \mathrm{gr}^\bullet_{\widehat{\mathfrak{m}}}(\widehat{A}) = k[x_1, \ldots, x_d]$.
(vii) Let A be a local Noetherian $k = A/\mathfrak{m}$-algebra of dimension d. Then $\widehat{A} \cong k[[x_1, \ldots, x_d]]$.

Proof of (i). If M is finitely generated, there is a surjection $A^{\oplus r} \to M$ yielding a surjection $\widehat{A}^{\oplus r} \to \widehat{M}$ showing that each $\widehat{M}$ is finitely generated over $\widehat{A}$. For $0 \to M_1 \to M_2 \to M_3 \to 0$ a short exact sequence of A-modules, we get a short exact sequence

$$0 \to I^d M_2 \cap M_1 \to I^d M_2 \to I^d M_3 \to 0$$

where applying the Artin-Rees lemma Theorem 14.8 and on passage to the limit we get the exact sequence $0 \to \widehat{M_1} \to \widehat{M_2} \to \widehat{M_3}$ since the limit need not preserve exactness on the right. It thus remains to prove exactness on the right. Let $x = (m_1, m_2, \ldots) \in \widehat{M_3}$ where $m_d \in M_3/I^d M_3$. By surjectivity of the map $\varphi : M_2 \to M_3$, we have $m_d = \varphi(m'_d)$ for some $m'_d \in M_2/I^d M_2$. Denote $\pi : M_1/(I^2 M_2 \cap M_1) \to M_1/(IM_2 \cap M_1)$ and $\rho_i : M_i/I^2 M_i \to M_i/IM_i$ for $i \in \{2, 3\}$. This gives the diagram

$$\begin{tikzcd}
0 \arrow[r] & M_1/(I^2M_2 \cap M_1) \arrow[r] \arrow[d, "\pi"'] & M_2/I^2M_2 \arrow[r] \arrow[d, "\rho_2"'] & M_3/I^2M_3 \arrow[r] \arrow[d, "\rho_3"'] & 0 \\
0 \arrow[r] & M_1/(IM_2 \cap M_1) \arrow[r] & M_2/IM_2 \arrow[r] & M_3/IM_3 \arrow[r] & 0
\end{tikzcd}$$

where π, ρ_2, ρ_3 are surjective. Then $m_1 = \rho_3(m_2)$ by definition so $\rho_2(m'_2), m'_1 \in M_2/IM_2$ have the same image in M_3/IM_3. That is,

$$\begin{tikzcd}
m'_2 \arrow[r, mapsto] \arrow[d, mapsto] & m_2 \arrow[d, mapsto] \\
m'_1 = \rho_2(m'_2) \arrow[r, mapsto] & m_1 = \rho_3(m_2)
\end{tikzcd}$$

for the rightmost square. Thus $\rho_2(m'_2) - m'_1 \in M_1/IM_1$. Since π is surjective, we can choose $m''_2 \in M_1/(I^2M_2 \cap M_1)$ such that $\pi(m''_2) = \rho_2(m'_2) - m'_1$. That is,

$$\begin{tikzcd}
m''_2 \arrow[r] \arrow[d] & m''_2 \arrow[d] \\
\rho_2(m'_2) - m'_1 \arrow[r] & \rho_2(m'_2) - m'_1 = 0
\end{tikzcd}$$

for the leftmost square. Substituting m_2' by $m_2' - m_2''$ we get surjectivity, and by induction we can lift each element of the sequence, giving surjectivity $\widehat{M_2} \to \widehat{M_3}$. ■

Proof of (ii). By the finite generation hypothesis, we have an exact sequence $A^{\oplus m} \to A^{\oplus n} \to M \to 0$. Tensoring with A yields a diagram

$$\begin{tikzcd}
\widehat{A}^{\oplus m} \arrow[r] \arrow[d, "\wr"'] & \widehat{A}^{\oplus n} \arrow[r] \arrow[d, "\wr"'] & M \otimes_A \widehat{A} \arrow[r] \arrow[d] & 0 \arrow[d, "\wr"'] \\
\widehat{A}^{\oplus m} \arrow[r] & \widehat{A}^{\oplus n} \arrow[r] & \widehat{M} \arrow[r] & 0
\end{tikzcd}$$

with exact rows, giving the isomorphism $M \otimes_A \widehat{A} \to \widehat{M}$ by the four-lemma. ■

Proof of (iii). This is immediate from (i) and (ii) which show that completion given by $- \otimes_A \widehat{A}$ is exact so $\widehat{A}$ is flat as an A-module. ■

Proof of (iv). Let $a_1, \dots, a_r$ be generators of I. Consider the map $A[[x_1, \dots, x_r]] \to \widehat{A}$ by $x_i \mapsto a_i$ induced by the map $A[x_1, \dots, x_r] \to A$ by $x_i \mapsto a_i$ – a surjective map from a Noetherian ring. This shows that $A[[x_1, \dots, x_r]] \to \widehat{A}$ is a surjection by (i). And since $A[[x_1, \dots, x_d]]$ is Noetherian, we get the claim. ■

Proof of (v). Let $\widehat{\mathfrak{m}} = \lim_{d \in \mathbb{N}} \mathfrak{m}/\mathfrak{m}^d$. By (i) we have $\widehat{A}/\widehat{\mathfrak{m}} \cong \widehat{A/\mathfrak{m}} \cong A/\mathfrak{m}$ since $\mathfrak{m}$ is trivial in $A/\mathfrak{m}$. Hence $\widehat{\mathfrak{m}} \subseteq \widehat{A}$ is maximal. To see that $\widehat{A}$ is local, it suffices to show that every element of $\widehat{A} \setminus \widehat{m}$ is a unit. Let $x = (a_1, a_2, \dots)$ be such an element. Necessarily $a_1 \in (A/\mathfrak{m})^\times$ is nonzero, and a_1 is the projection in $A/\mathfrak{m}$ of each a_i so each a_i is a unit, and $x^{-1}(a_1^{-1}, a_2^{-1}, \dots)$, as desired.

For the claim on the gradeds, it suffices to show that $A/\mathfrak{m}^d \cong \widehat{A} \cong \widehat{\mathfrak{m}}^d$ in which case the graded rings will agree in each degree. Note that the proof of the first part of the statement already gives $\mathrm{gr}^0_{\mathfrak{m}}(A) \cong \mathrm{gr}^0_{\widehat{\mathfrak{m}}}(\widehat{A})$. By (i) we have that $\widehat{A}/\widehat{\mathfrak{m}}^d \cong \widehat{A/\mathfrak{m}^d}$. Using that $\mathfrak{m}^n/\mathfrak{m}^d = 0$ for all $n \geq d$, we use the diagram

$$\begin{tikzcd}
0 \arrow[r] & \mathfrak{m}^d/\mathfrak{m}^{d+1} \arrow[r] \arrow[d] & A/\mathfrak{m}^{d+1} \arrow[r] \arrow[d, "\wr"] & A/\mathfrak{m}^d \arrow[r] \arrow[d, "\wr"] & 0 \\
0 \arrow[r] & \widehat{\mathfrak{m}}^d/\widehat{\mathfrak{m}}^{d+1} \arrow[r] & \widehat{A}/\widehat{\mathfrak{m}}^{d+1} \arrow[r] & \widehat{A}/\widehat{\mathfrak{m}}^d \arrow[r] & 0
\end{tikzcd}$$

with exact rows to deduce that $\mathfrak{m}^d/\mathfrak{m}^{d+1} \cong \widehat{\mathfrak{m}}^d/\widehat{\mathfrak{m}}^{d+1}$ so each $\mathrm{gr}^i_{\mathfrak{m}}(A) \cong \mathrm{gr}^i_{\widehat{\mathfrak{m}}}(\widehat{A})$ yielding the claim. ■

Proof of (vi). We show (a)⇔(c)⇔(b).

(a)⇒(c) Recall from Definition 3.17 that for A regular we have $\mathfrak{m} = (a_1, \dots, a_d)$. Denoting $k = A/\mathfrak{m}$ the residue field of A, we have a map $k[x_1, \dots, x_d] \to \mathrm{gr}^\bullet_{\mathfrak{m}}(A)$ by $x_i \mapsto a_i$. This is an isomorphism in degree 0 and 1, and therefore an isomorphism since $\mathrm{gr}^i_{\mathfrak{m}}(A) \cong \mathrm{Sym}^i(\mathfrak{m}/\mathfrak{m}^2) \cong \mathfrak{m}^i/\mathfrak{m}^{i+1}$ by regularity.

(c)⇒(a) If $\mathrm{gr}^\bullet_{\mathfrak{m}}(A) \cong k[x_1, \dots, x_d]$ then $\mathfrak{m}/\mathfrak{m}^2$ is generated by d elements, and A is regular.

(b)⇒(c) This is the argument of (a)⇒(c) verbatim. By regularity, we take $\widehat{\mathfrak{m}} = (\widehat{a}_1, \dots, \widehat{a}_d)$ and we have $k = A/\mathfrak{m} \cong \widehat{A}/\widehat{\mathfrak{m}}$ by (i). The map $k[x_1, \dots, x_d] \to \mathrm{gr}^\bullet_{\widehat{\mathfrak{m}}}(\widehat{A})$ is an isomorphism in degrees 0 and 1, and an isomorphism globally by $\mathrm{gr}^i_{\widehat{A}}(\widehat{A}) \cong \mathrm{Sym}^i(\widehat{\mathfrak{m}}/\widehat{\mathfrak{m}}^2) \cong \widehat{\mathfrak{m}}^i/\widehat{\mathfrak{m}}^{i+1} \cong \mathfrak{m}^i/\mathfrak{m}^{i+1}$ due to regularity.

(c)⇒(a) This is the argument of (c)⇒(a) verbatim. $\mathrm{gr}^\bullet_{\widehat{\mathfrak{m}}}(\widehat{A}) \cong k[x_1, \dots, x_d]$ is regular, so $\widehat{\mathfrak{m}}$ is generated by d elements, hence regular. ∎

Proof of (vii). If $\mathfrak{m} = (a_1, \dots, a_d)$, we have the map $k[x_1, \dots, x_d] \to \widehat{A}$ by $x_i \mapsto a_i$. We have an isomorphism $A/\mathfrak{m} \cong k$. So by (vi), the isomorphism of graded constructions $\mathrm{gr}^\bullet_{\mathfrak{m}}(A) \cong \mathrm{gr}^\bullet_{\widehat{\mathfrak{m}}}(\widehat{A}) = k[x_1, \dots, x_d]$ induce an isomorphism between the completions and $k[[x_1, \dots, x_d]]$. ∎

We conclude our discussion of completion by stating Cohen's structure theorem.

Theorem 14.12 (Cohen – Structure)**.** Let A be a complete Noetherian local ring with maximal ideal $\mathfrak{m}$ containing some field. Then there exixts a field k in A such that $k = A/\mathfrak{m}$ is the residue field of A, and $A \cong k[[x_1, \dots, x_d]]/I$ fo some $d \geq 0$ and ideal I. Furthermore, if A is regular, then $A \cong k[[x_1, \dots, x_d]]$.

Proof. See [Stacks, Tag 032A]. ∎

We consider some examples.

Example 14.13. Let $C = V(y^2 - x^3 - x^2) \subseteq \mathbb{A}^2_k, C' = V(y^2 - x^2) \subseteq \mathbb{A}^2_k$. Both of these curves are nodal at the origin, but their local rings $(k[x,y]/(y^2 - x^3 - x^2))_{(x,y)}, (k[x,y]/(y^2 - x^2))_{(x,y)}$ are not isomorphic. In a certain sense, they are not sufficiently "local" to capture the geometric behavior of these two being nodes. However, their completions at (x, y) are isomorphic.

15. Lecture 15 – 5th June 2025

We make preparations towards the proof of the theorem of formal functions.

Definition 15.1 (Faithfully Flat)**.** Let $\varphi : A \to B$ be a ring map. φ is faithfully flat if exactness of $0 \to M_1 \to M_2 \to M_3 \to 0$ in Mod_A is equivalent to exactness of $0 \to M_1 \otimes_A B \to M_2 \otimes_A B \to M_3 \otimes_A B \to 0$ in Mod_B.

We consider some properties thereof.

Proposition 15.2. Let $\varphi : A \to B$ be a ring map.

(i) If φ is faithfully flat, then φ is injective.
(ii) If φ is faithfully flat then for $I \subseteq A$ an ideal, $IB \cap A = I$.
(iii) If φ is a flat local homomorphism of local rings then φ is faithfully flat.
(iv) If A is a Noetherian local ring, then $A \hookrightarrow \widehat{A}$ is faithfully flat.

Proof of (i). Consider the morphism

$$\varphi \otimes \mathrm{id}_B = \widetilde{\varphi} : B = A \otimes_A B \to B \to B \otimes_A B.$$

Note that the composition $B \xrightarrow{\widetilde{\varphi}} B \otimes_A B \xrightarrow{b \otimes b' \mapsto bb'} B$ is the identity as $a \otimes b = \varphi(a)b \mapsto \varphi(a) \otimes b \mapsto \varphi(a)b$ for all $a \in A, b \in B$. Thus $\widetilde{\varphi}$ is injective. And since $\widetilde{\varphi}$ is injective (as a map of B-modules), φ is injective as well (as a map of A-modules). ∎

Proof of (ii). By (i), φ is a ring extension so we have $A \cap IB \subseteq I$ and thus a (not necessarily exact) sequence $0 \to IB \cap A \to A \to A/I \to 0$. But applying $\otimes_A B$ this yields $0 \to IB \to B \to B/IB \to 0$ which is exact, so $0 \to IB \cap A \to A \to A/I \to 0$ was exact to begin with. ∎

Proof of (iii). Let $\phi : M \to N$ be a homomorphism of A-modules such that its base extension $\widetilde{\phi} : M \otimes_A B \to N \otimes_A B$ is injective. We have that $\ker(\phi) \otimes_A B = 0$. Let $m \in \ker(\phi)$ and consider $I = \{a \in A : am = 0\} \subsetneq A$. The morphism $A \to \ker(\phi)$ by $a \mapsto am$ has kernel I so $A/I \hookrightarrow \ker(\phi)$. By flatness, $A/I \otimes_A B = B/IB \hookrightarrow \ker(\phi) \otimes_A B = 0$ so $B/IB = 0$. Since I is an ideal of the local ring A, either $I = 0$ or $I = \mathfrak{m}_A$. In the first case, $B = 0$ since the quotient by the trivial ideal is an isomorphism. In the second case $B/\mathfrak{m}B = 0$ implies $B = 0$ by Nakayama's lemma. This yields a contradiction in both cases as the zero ring is not a local ring. ∎

Proof of (iv). This is trivial from (iii) as Proposition 14.11 (iii) shows that $A \hookrightarrow \widehat{A}$ is flat. ∎

With the language of faithful flatness and completions in hand, we can prove the following characteriziation of étale morphisms in a special case.

Proposition 15.3. Let $\varphi : A \to B$ be a local homomorphism of Noetherian local rings where A, B are furthermore $k = A/\mathfrak{m}_A \cong B/\mathfrak{m}_B$-algebras. Then φ is étale if and only if $\widehat{\varphi} : \widehat{A} \to \widehat{B}$ is an isomorphism.

Proof. ($\Rightarrow$) Since φ is étale, it is in particular flat and unramified Definition 9.5. Thus φ is injective by Proposition 15.2 (i). Moreover, since φ is uramified, we have $\mathfrak{m}_A B = \mathfrak{m}_B$. Thus since $A/\mathfrak{m}_A \cong B/\mathfrak{m}_B$ we have

$$B \cong A + \mathfrak{m}_B \cong A + \mathfrak{m}_A B \cong A + \mathfrak{m}_A(A + \mathfrak{m}_A B) = A + \mathfrak{m}_A^2 B \cong \dots$$

so $B \cong A + \mathfrak{m}_B^d$ for each $d \geq 0$. The homomorphism $\varphi_d : A/\mathfrak{m}_A^d \to B/\mathfrak{m}_B^d$ fits into a commutative diagram

$$\begin{tikzcd} A \arrow[rr, two heads] \arrow[dr, two heads] & & A/\mathfrak{m}_A^d \arrow[dl, "\varphi_d"] \\ & B/\mathfrak{m}_B^d & \end{tikzcd}$$

where the composition $A \to B \to B/\mathfrak{m}_B^d$ is surjective being the composition of an injective and a surjective map. Thus by cancellation, φ_d is surjective too. The kernel of φ_d is $(A \cap \mathfrak{m}_B^d) \cap \mathfrak{m}_A^d$, but $A \cap \mathfrak{m}_B^d = A \cap \mathfrak{m}_A^d B = \mathfrak{m}_A^d$ by (i) above, so φ_d is an isomorphism for each $d \geq 0$. This induces an isomorphism on each of the terms of the filtration, and thus an isomorphism on completion.

($\Leftarrow$) Suppose that $\widehat{\varphi} : \widehat{A} \to \widehat{B}$ is an isomorphism. Then we have by Proposition 14.11 (v) that

$$\mathrm{gr}^\bullet_{\mathfrak{m}_A}(A) \cong \mathrm{gr}^\bullet_{\widehat{\mathfrak{m}_A}}(\widehat{A}) \cong \mathrm{gr}^\bullet_{\widehat{\mathfrak{m}_B}}(\widehat{B}) \cong \mathrm{gr}^\bullet_{\mathfrak{m}_B}(B).$$

and in particular $\mathfrak{m}_A/\mathfrak{m}_A^2 \cong \widehat{\mathfrak{m}_A}/\widehat{\mathfrak{m}_A}^2 \cong \widehat{\mathfrak{m}_B}/\widehat{\mathfrak{m}_B}^2 \cong \mathfrak{m}_B/\mathfrak{m}_B^2$. In particular, $\mathfrak{m}_B \cong \mathfrak{m}_A B + \mathfrak{m}_B^2$. We have finite B-mdoules $\mathfrak{m}_A B \subseteq \mathfrak{m}_B$ both finite B-modules with $\mathfrak{m}_B = \mathfrak{m}_A B + \mathfrak{m}_B^2$ so by Nakayama's lemma we have $\mathfrak{m}_A B = \mathfrak{m}_B$. This shows that φ is unramified. It remains to show φ is flat. We have a commutative diagram

$$\begin{tikzcd} A \arrow[r, "\varphi"] \arrow[d] & B \arrow[d] \\ \widehat{A} \arrow[r, "\widehat{\varphi}"'] & \widehat{B} \end{tikzcd}$$

with vertical maps completions at $\mathfrak{m}_A, \mathfrak{m}_B$, respectively. $A \to \widehat{A}$ is faithfully flat by Proposition 14.11 (iii) and Proposition 15.2 (iii), and $\widehat{\varphi}$ is an isomorphism hence faithfully flat, and $B \to \widehat{B}$ is faithfully flat once again by Proposition 14.11 (iii) and Proposition 15.2 (iii). It suffices to show that for $M \hookrightarrow N$ an injection of A-modules that $M \otimes_A B \to N \otimes_A B$ is injective as a map of B-modules. Suppose to the contrary that $M \otimes_A B \to N \otimes_A B$ is not injective, yielding an exact sequence of B-modules

$$0 \to \ker(M \otimes_A B \to N \otimes_A B) \to M \otimes_A B \to N \otimes_A B$$

and by faithful flatness of $B \to \widehat{B}$ an exact sequence

$$0 \to \ker(M \otimes_A B \to N \otimes_A B) \otimes_B \widehat{B} \to M \otimes_A \widehat{B} \to N \otimes_A \widehat{B}$$

of $\widehat{B}$-modules. We have that $\ker(M \otimes_A B \to N \otimes_A B) \otimes_B \widehat{B} = 0$ as $A \to \widehat{A} \to \widehat{B}$ is faithfully flat. So by faithful flatness of $B \to \widehat{B}$, $\ker(M \otimes_A B \to N \otimes_A B) = 0$ as well, showing that $- \otimes_A B$ preserves injectivity, and thus flatness of $A \to B$. ■

Remark 15.4. The statement of Proposition 15.3 should be thought of to be the situation $f : X \to Y$ a morphism of k-schemes, $x \in X(k)$, $y = f(x)$, with the induced local homomorphism of local rings $\mathcal{O}_{Y,y} \mapsto \mathcal{O}_{X,x}$, showing that étaleness can be checked on the induced morphism $\widehat{\mathcal{O}_{Y,y}} \to \widehat{\mathcal{O}_{X,x}}$.

We now set up the statement of the theorem of formal functions: a result that allows us to compute cohomology of stalks in terms of a limit of cohomologies of thickenings. Let $f : X \to Y$ be a proper morphism of locally Noetherian schemes. For $\mathcal{F} \in \mathsf{Coh}(X)$ a coherent sheaf, is higher direct image $R^i f_* \mathcal{F} \in \mathsf{Coh}(Y)$. For $y \in Y$, we have a Cartesian square

$$\begin{tikzcd}
X_y \arrow[r, "\iota'"] \arrow[d, "f'"'] & X \arrow[d, "f"] \\
\mathrm{Spec}(\kappa(y)) \arrow[r, "\iota"'] & Y
\end{tikzcd}$$

which by left exactness of global sections induces a natural transformation of functors $(\iota^* \circ f_*)(-) \to (f'_* \circ \iota'^*)(-)$. Observe that for $\mathcal{F} \in \mathsf{Coh}(X)$ applying the source to $\mathcal{F}$ yields $(\iota^* \circ f_*)(\mathcal{F})$ which can be identified with $(f_*\mathcal{F})_y$, and applying the target functor to $\mathcal{F}$ can be similarly identified with $\Gamma(X_y, \mathcal{F}|_{X_y})$ recalling that ι'^* is restriction and f'_* the direct image to a point computes cohomology. Using the universal property of δ-functors, we can pass to δ-functors to get morphisms $(R^i f_* \mathcal{F})_y \to H^i(X_y, \mathcal{F}|_{X_y})$ for all $i \geq 0$. For any $n \geq 0$ we can more generally consider the Cartesian diagram

$$\begin{tikzcd}
X_y^{(n)} \arrow[r] \arrow[d, "f_n"'] & X \arrow[d, "f"] \\
\mathrm{Spec}(\mathcal{O}_{Y,y}/\mathfrak{m}_y^n) \arrow[r] & Y
\end{tikzcd}$$

where $X_y^{(n)}$ is the n-th order thickening of the fiber X_y. Repeating the argument for δ-functors above, we get for $\mathcal{F} \in \mathsf{Coh}(X)$ morphisms $(f_*\mathcal{F}) \otimes_{\mathcal{O}_{Y,y}} (\mathcal{O}_{Y,y}/\mathfrak{m}_y^n) \to \Gamma(X_y^{(n)}, \mathcal{F}|_{X_y^{(n)}})$ inducing

$$(R^i f_* \mathcal{F}) \otimes_{\mathcal{O}_{Y,y}} (\mathcal{O}_{Y,y}/\mathfrak{m}_y^n) \to H^i(X_y^{(n)}, \mathcal{F}_{X_y^{(n)}}) \tag{15.1}$$

for each fixed n. Moreover, these maps are compatible with the restriction maps on thickenings $X_y^{(n-1)} \hookrightarrow X_y^{(n)}$ in the sense that there are commutative diagrams

$$\begin{tikzcd}
\left(R^i f_* \mathcal{F}\right) \otimes_{\mathcal{O}_{Y,y}} \left(\mathcal{O}_{Y,y}/\mathfrak{m}_y^n\right) \arrow[r] \arrow[d] & H^i\left(X_y^{(n)}, \mathcal{F}|_{X_y^{(n)}}\right) \arrow[d] \\
\left(R^i f_* \mathcal{F}\right) \otimes_{\mathcal{O}_{Y,y}} \left(\mathcal{O}_{Y,y}/\mathfrak{m}_y^{n-1}\right) \arrow[r] & H^i\left(X_y^{(n-1)}, \mathcal{F}|_{X_y^{(n-1)}}\right)
\end{tikzcd}$$

induced by the diagram

$$\begin{tikzcd}
X_y^{(n-1)} \arrow[r] \arrow[d] & X_y^{(n)} \arrow[r] \arrow[d] & X \arrow[d] \\
\operatorname{Spec}(\mathcal{O}_{Y,y}/\mathfrak{m}_y^{n-1}) \arrow[r] & \operatorname{Spec}(\mathcal{O}_{Y,y}/\mathfrak{m}_y^{n}) \arrow[r] & Y
\end{tikzcd}$$

with rightmost square and outer rectangle Cartesian, implying that the leftmost square is Cartesian. The maps of (15.1) assemble to a diagram

$$\begin{tikzcd}
\vdots \arrow[d] & \vdots \arrow[d] \\
\left(R^i f_* \mathcal{F}\right) \otimes_{\mathcal{O}_{Y,y}} \left(\mathcal{O}_{Y,y}/\mathfrak{m}_y^n\right) \arrow[r] \arrow[d] & H^i\left(X_y^{(n)}, \mathcal{F}|_{X_y^{(n)}}\right) \arrow[d] \\
\left(R^i f_* \mathcal{F}\right) \otimes_{\mathcal{O}_{Y,y}} \left(\mathcal{O}_{Y,y}/\mathfrak{m}_y^{n-1}\right) \arrow[r] \arrow[d] & H^i\left(X_y^{(n-1)}, \mathcal{F}|_{X_y^{(n-1)}}\right) \arrow[d] \\
\vdots \arrow[d] & \vdots \arrow[d] \\
(R^i f_* \mathcal{F})_y \arrow[r] & H^i\left(X_y, \mathcal{F}|_{X_y}\right)
\end{tikzcd}$$

and hence induces a map on the limits

$$(\widehat{R_i f_* \mathcal{F}}) \longrightarrow \lim_{n \in \mathbb{N}} H^i\left(X_y^{(n)}, \mathcal{F}|_{X_y^{(n)}}\right) \tag{15.2}$$

where $(\widehat{R^i f_* \mathcal{F}})$ is the completion of $(R^i f_* \mathcal{F})_y$ as an $\mathcal{O}_{Y,y}$-module with respect to the ideal $\mathfrak{m}_y$. The theorem of formal functions states that (15.2) is an isomorphism. Before we formally state and prove the theorem, we make a few reductions necessary for the proof.

Lemma 15.5. Let $f : X \to Y$ be a proper morphism of locally Noetherian schemes, $\mathcal{F} \in \mathsf{Coh}(X)$, and $y \in Y$. Consider the Cartesian square

$$\begin{tikzcd} W \arrow[r] \arrow[d, "g"'] & X \arrow[d, "f"] \\ \operatorname{Spec}(\mathcal{O}_{Y,y}) \arrow[r] & Y. \end{tikzcd} \tag{15.3}$$

Then there are isomorphisms $(R^i f_* \mathcal{F}) \otimes_{\mathcal{O}_{Y,y}} (\mathcal{O}_{Y,y}/\mathfrak{m}_y^n) \cong (R^i g_* \mathcal{F}|_W) \otimes_{\mathcal{O}_{Y,y}} (\mathcal{O}_{Y,y}/\mathfrak{m}_y^n)$ and $H^i(X_y^{(n)}, \mathcal{F}|_{X_y^{(n)}}) \cong H^i(W_y^{(n)}, \mathcal{F}|_{W_y^{(n)}})$. In particular, $(\widehat{R^i f_* \mathcal{F}}) \cong (\widehat{R^i g_* \mathcal{F}|_W})$.

Proof. For any $V \subseteq Y$ affine containing y, the morphism $\operatorname{Spec}(\mathcal{O}_{Y,y}) \to Y$ factors as $\operatorname{Spec}(\mathcal{O}_{Y,y}) \to V \to Y$. We have that $\operatorname{Spec}(\mathcal{O}_{Y,y}) \to V$ is flat as it is a localization, and $V \to Y$ flat as it is an open immersion. In particular the bottom horizontal map of (15.3) is flat. By flat base change Proposition 6.4 we have isomorphisms $R^i f_* \mathcal{F} \cong R^i g_* \mathcal{F}|_W$.

For the second isomorphism, we use the diagram

$$\begin{tikzcd} X_y^{(n)} \arrow[r] \arrow[d] & W \arrow[r] \arrow[d] & X \arrow[d, "f"] \\ \operatorname{Spec}(\mathcal{O}_{Y,y}/\mathfrak{m}_y^n) \arrow[r] & \operatorname{Spec}(\mathcal{O}_{Y,y}) \arrow[r] & Y \end{tikzcd}$$

where the rightmost square and outer rectangle are Cartesian. This implies that the leftmost square is Cartesian giving an isomorphism $W_y^{(n)} \cong X_y^{(n)}$ inducing the isomorphism on cohomology $H^i(X_y^{(n)}, \mathcal{F}|_{X_y^{(n)}}) \cong H^i(W_y^{(n)}, \mathcal{F}|_{W_y^{(n)}})$.

The final statement is obtained from the first on passage to the limit. ■

Lemma 15.6. Let $f : X \to Y$ be a proper morphism of locally Noetherian schemes with $Y = \operatorname{Spec}(A)$ for a Noetherian local ring A with maximal ideal $\mathfrak{m}$, $\mathcal{F} \in \mathsf{Coh}(X)$, and $y \in Y$. There is an isomorphism $(\widehat{R^i f_* \mathcal{F}}) \cong H^i(X, \mathcal{F}) \otimes_A \widehat{A}$ and $H^i(X_{\mathfrak{m}}^{(n)}, \mathcal{F}|_{X_{\mathfrak{m}}^{(n)}}) \cong H^i(X, \mathcal{F}/\mathfrak{m}^n \mathcal{F})$.

Proof. Since $Y = \operatorname{Spec}(A)$ is affine, $R^i f_* \mathcal{F}$ is a coherent sheaf on $\operatorname{Spec}(A)$ corresponding to a unique finitely generated A-module $H^i(X, \mathcal{F})$. The isomorphism $(\widehat{R^i f_* \mathcal{F}}) \cong H^i(X, \mathcal{F}) \otimes_A \widehat{A}$ is immediate from Proposition 14.11 (ii). For the second isomorphism, we use the Cartesian square

$$\begin{tikzcd} X_{\mathfrak{m}}^{(n)} \arrow[r, "j_n"] \arrow[d] & X \arrow[d, "f"] \\ \operatorname{Spec}(A/\mathfrak{m}^n) \arrow[r] & Y \end{tikzcd}$$

where we use that j_n is affine as it is obtained from the affine morphism $\operatorname{Spec}(A/\mathfrak{m}^n) \to \operatorname{Spec}(A) = Y$ by base change along f. Along affine morphisms there is an isomorphism $H^i(X_{\mathfrak{m}}^{(n)}, \mathcal{F}|_{X_{\mathfrak{m}}^{(n)}}) \cong H^i(X, \mathcal{F}/\mathfrak{m}^n \mathcal{F})$ since j_{n*} is the quotient by $\mathfrak{m}^n$. ■

We are now prepared to state and prove the theorem of formal functions.

Theorem 15.7 (Formal Functions)**.** Let $f : X \to Y$ be a proper morphism of locally Noetherian schemes, $\mathcal{F} \in \mathsf{Coh}(X)$, and $y \in Y$. Denote the $\mathfrak{m}_y$-adic completion of the $\mathcal{O}_{Y,y}$-module $(R^i f_*\mathcal{F})_y$ by $(\widehat{R^i f_*\mathcal{F}})$ and $X_y^{(n)}$ the n-th order thickening of the fiber $X \times_Y \mathrm{Spec}(\mathcal{O}_{Y,y}/\mathfrak{m}_y^n)$. The morphism (15.2) is an isomorphism

$$(\widehat{R^i f_*\mathcal{F}}) \cong \lim_{n\in\mathbb{N}} H^i(X_y^{(n)}, \mathcal{F}|_{X_y^{(n)}}).$$

Proof. Without loss of generality, we can take Y to be affine by Lemma 15.5. We can then apply Lemma 15.6 in which case we can take $Y = \mathrm{Spec}(A)$ with $A = \mathcal{O}_{Y,y}$ and observe it suffices to prove that there is an isomorphism

$$H^i(X, \mathcal{F}) \otimes_A \widehat{A} \longrightarrow \lim_{n\in\mathbb{N}} H^i\left(X, \mathcal{F}/\mathfrak{m}^n\mathcal{F}\right).$$

Note that we have a short exact sequence $0 \to \mathfrak{m}^n\mathcal{F} \to \mathcal{F} \to \mathcal{F}/\mathfrak{m}^n\mathcal{F} \to 0$ inducing by the long exact sequence in cohomology

$$H^i(X, \mathfrak{m}_n\mathcal{F}) \xrightarrow{a_n} H^i(X, \mathcal{F}) \xrightarrow{b_n} H^i(X, \mathcal{F}/\mathfrak{m}^n\mathcal{F}) \xrightarrow{c_n} H^{i+1}(X, \mathfrak{m}^n\mathcal{F}) \xrightarrow{d_n} H^{i+1}(X, \mathcal{F}).$$

We seek to show that the induced maps $\beta_n H^i(X, \mathcal{F})/\mathrm{im}(a_n) \to H^i(X, \mathcal{F}/\mathfrak{m}^n\mathcal{F})$ which on passage to the limit gives

$$\beta : \lim_{n\in\mathbb{N}} H^i(X, \mathcal{F})/\mathrm{im}(a_n) \to \lim_{n\in\mathbb{N}} H^i(X, \mathcal{F}/\mathfrak{m}^n\mathcal{F})$$

is isomorphic to $H^i(X, \mathcal{F}) \otimes_A \widehat{A} \to \lim_{n\in\mathbb{N}} H^i\left(X, \mathcal{F}/\mathfrak{m}^n\mathcal{F}\right)$.

We first show that $\lim_{n\in\mathbb{N}} H^i(X, \mathcal{F})/\mathrm{im}(a_n) \cong H^i(X, \mathcal{F}) \otimes_A \widehat{A}$. By the Artin-Rees lemma Theorem 14.8, it suffices to show that the filtrations $\{\mathrm{im}(a_n)\}$ and $\{\mathfrak{m}^n H^i(X, \mathcal{F})\}$ of $H^i(X, \mathcal{F})$ are equivalent in the sense of Definition 14.4. Observe that for every $n \geq 0$ that $\mathfrak{m}^n H^i(X, \mathcal{F}) \subseteq \mathrm{im}(a_n)$ as for each $x \in \mathfrak{m}^n$ we have a commutative diagram

$$\begin{tikzcd}
H^i(X, \mathcal{F}) \arrow[rr, "\cdot x"] \arrow[dr, "\cdot x"'] & & H^i(X, \mathcal{F}). \\
& H^i(X, \mathfrak{m}^n\mathcal{F}) \arrow[ur, "a_n"'] &
\end{tikzcd}$$

Conversely for each $n \geq 0$ there is $m \geq 0$ such that $\mathrm{im}(a_m) \subseteq \mathfrak{m}^n H^i(X, \mathcal{F})$. Consider the A-algebra $\bigoplus_{j\geq 0} \mathfrak{m}^j$ and the $\mathcal{O}_X$-algebra $\mathcal{A} = \bigoplus_{j\geq 0} \mathfrak{m}^j\mathcal{O}_X$. We have a Cartesian diagram

$$\begin{tikzcd}
X' \cong \underline{\mathrm{Spec}}(\mathcal{A}) \arrow[r] \arrow[d] & X \arrow[d, "f"] \\
\mathrm{Spec}(S) \arrow[r] & Y.
\end{tikzcd}$$

The morphism $\mathrm{Spec}(S) \to Y$ is affine, so $X' \to X$ is affine and since $X' \to \mathrm{Spec}(S)$ is proper the quasicoherent sheaf $\mathcal{E} \bigoplus_{j\geq 0} \mathfrak{m}^j\mathcal{F}$ on X is finitely generated as an $\mathcal{A}$-module and thus $\mathcal{E}$ is the direct image of $\mathcal{E}'$ on X'. But since $X' \to X$ is affine, we

have
$$H^i(X',\mathcal{E}') = H^i(X,\mathcal{E}) = H^i(X,\mathcal{F}) \oplus H^i(X,\mathfrak{m}\mathcal{F}) \oplus \dots.$$
By coherence of $\mathcal{E}'$, the cohomology $H^i(X,\mathcal{E}')$ is finitely generated as an S-module. By the Artin-Rees lemma Theorem 14.8, the filtration $\{a_j\left(H^i(X,\mathfrak{m}^j\mathcal{F})\right)\}$ of $H^i(X,\mathcal{F})$ is $\mathfrak{m}$-stable in the sense that $\mathfrak{m}\cdot a_j\left(H^i(X,\mathfrak{m}^j\mathcal{F})\right) = a_{j+1}\left(H^i(X,\mathfrak{m}^{j+1}\mathcal{F})\right)$ for all $j \geq c$ for some fixed $c \geq 0$. So
$$\operatorname{im}(a_{n+c}) = a_{n+c}\left(H^i(X,\mathfrak{m}^{n+c}\mathcal{F})\right) = \mathfrak{m}^n \cdot a_c\left(H^i(X,\mathfrak{m}^c\mathcal{F})\right) \subseteq \mathfrak{m}^n H^i(X,\mathcal{F})$$
showing that the filtrations agree – that is, $\lim_{n\in\mathbb{N}} H^i(X,\mathcal{F})/\operatorname{im}(a_n) \cong H^i(X,\mathcal{F})\otimes_A \widehat{A}$.

To show that β is an isomorphism, we have a short exact sequence
$$0 \to H^i(X,\mathcal{F})/\operatorname{im}(a_n) \xrightarrow{\beta_n} H^i(X,\mathcal{F}/\mathfrak{m}^n\mathcal{F}) \to \ker(d_n) \to 0.$$
Observe that the map $H^i(X,\mathcal{F})/\operatorname{im}(a_{n+1}) \to H^i(X,\mathcal{F})/\operatorname{im}(a_n)$ is surjective for each $n \geq 0$. By Proposition 14.11 (i), we get an exact sequence on passage to limits
$$0 \to \lim_{n\in\mathbb{N}} H^i(X,\mathcal{F})/\operatorname{im}(a_n) \xrightarrow{\beta} \lim_{n\in\mathbb{N}} H^i(X,\mathcal{F}/\mathfrak{m}^n\mathcal{F}) \to \lim_{n\in\mathbb{N}} \ker(d_n) \to 0.$$
It remains to show that $\lim_{n\in\mathbb{N}}\ker(d_n) = 0$. The multiplication maps $\mathfrak{m}\times\ker(d_n) \to \ker(d_{n+1})$, so $\mathcal{Q} = \bigoplus_{j\geq 0}\ker(d_n)$ is an S-module for which we choose a set of homogeneous generators with degree at most N. Since $\ker(d_n) = \operatorname{im}(c_n)$, and the image of c_n is zero after multiplication by $\mathfrak{m}^n$ as $\mathcal{F}/\mathfrak{m}^n\mathcal{F}$ is. Thus $\ker(d_n)$ is zero after multiplication by $\mathfrak{m}^n$ as well. That is, $\mathcal{Q}$ is zero after multiplication by $\mathfrak{m}^N S$ of S. Consider the composition $\mathfrak{m}^r \otimes \ker(d_n) \xrightarrow{\times} \ker(d_{n+r}) \xrightarrow{\mathfrak{m}^{n+r}\mathcal{F}\hookrightarrow\mathfrak{m}^n\mathcal{F}} \ker(d_n)$. The multiplication map is surjective for $r \geq 0$ and $n \in \mathbb{N}$. So the composition is zero if $r \geq n$. The restriction $\ker(d_{n+r}) \to \ker(d_n)$ is zero if $r, n \geq N$. In particular the limit vanishes, giving the desired isomorphsim. ∎

We conclude with a quick corollary of the theorem of formal functions.

Corollary 15.8. Let $f : X \to Y$ be a proper morphism of locally Noetherian schemes and r the maximal dimension of the fibers of f. Then $R^i f_*\mathcal{F} = 0$ for all $i > r$ and $\mathcal{F} \in \mathsf{Coh}(X)$.

Proof. Fix $y \in Y$ and $i > r$. For every $n \geq 1$ the topological space underlying $X_y^{(n)}$ is homeomorphic to X_y of dimension at most r. Thus $H^i(X_y^{(n)}, \mathcal{F}|_{X_y^{(n)}}) = 0$ for all n. By the theorem on formal functions $(\widehat{R^i f_*\mathcal{F}}) = 0$ and the morphism to the completion is an injection, so $(R^i f_*\mathcal{F})_y = 0$. But Y was arbitrary, so $R^i f_*\mathcal{F} = 0$. ∎

16. Lecture 16 – 16th June 2025

We discuss some consequences of the theorem of formal functions Theorem 15.7. We begin with the following definition.

Definition 16.1 ($\mathcal{O}$-Connected)**.** Let $f : X \to Y$ be a proper morphism of locally Noetherian schemes. f is $\mathcal{O}$-connected if $f_*\mathcal{O}_X \cong \mathcal{O}_Y$.

The first consequence of the theorem of formal functions is that $\mathcal{O}$-connectedness implies connectedness of fibers in the ordinary sense.

Proposition 16.2. Let $f : X \to Y$ be an $\mathcal{O}$-connected morphism of locally Noetherian schemes. Then the fibers X_y for all $y \in Y$ are connected.

Proof. Suppose to the contrary that X_y is not connected for some y. For such y, we can reduce to by induction to the case of two connected components and write $X_y = X_1 \sqcup X_2$ with X_1, X_2 both open and closed in the fiber. Define $e_i \in H^0(X_y, \mathcal{O}_{X_y})$ to be the function taking value 1 on X_i and 0 otherwise. We have $\widehat{\mathcal{O}_Y} \cong (\widehat{f_*\mathcal{O}_X}) \cong \lim_{n\in\mathbb{N}} H^0(X_y^{(n)}, \mathcal{F}|_{X_y^{(n)}})$ with the first isomorphism by $\mathcal{O}$-connectedness and the second by the theorem of formal functions Theorem 15.7 since f is proper as it is $\mathcal{O}$-connected. Note that there are elements satisfiying the conditions of e_1, e_2 in each term of the limit $H^0(X_y^{(n)}, \mathcal{F}|_{X_y^{(n)}})$. So $\widehat{\mathcal{O}_Y}$ contains elements e_1, e_2 such that $e_1e_2 = 0$ so $e_1, e_2 \in \mathfrak{m}_y$ which implies $e_1 + e_2 = 1 \in \mathfrak{m}_y$, a contradiction. ∎

Remark 16.3. The converse of Proposition 16.2 is false. $\mathrm{Spec}(A/I) \subseteq \mathrm{Spec}(A)$ has connected fibers since the empty set is connected, but is clearly not $\mathcal{O}$-connected as the direct image of the structure sheaf is the algebra sheaf $\widetilde{A/I}$.

The theorem of formal functions gives ways to factorize morphisms: the Stein factorization, Zariski's main theorem, and Grothendieck's variant of Zariski's main theorem.

Theorem 16.4 (Stein – Factorization)**.** Let $f : X \to Y$ be a proper morphism of locally Noetherian schemes. f admits a factorization as $X \xrightarrow{g} Y' \xrightarrow{\pi} Y$ with g $\mathcal{O}$-connected and π finite.

Proof. Define $Y' = \underline{\mathrm{Spec}}(f_*\mathcal{O}_X)$ and let $\pi : Y' \to Y$ be the structure morphism. By properness of f, $f_*\mathcal{O}_X$ is finitely generated as an $\mathcal{O}_Y$ module showing π is finite. We want to show that $g : X \to Y'$ is $\mathcal{O}$-connected. By the adjunction $\mathrm{Mor}_{\mathsf{Sch}_Y}(X, \underline{\mathrm{Spec}}(\mathcal{A})) \cong \mathrm{Mor}_{\mathsf{Alg}_{\mathcal{O}_Y}}(\mathcal{A}, f_*\mathcal{O}_X)$ natural in $\mathcal{O}_Y$-algebras $\mathcal{A}$, we get a unique map $g : X \to Y'$ induced by the identity along that equivalence. To see that g is $\mathcal{O}$-connected, it suffices to check after composition with π. $(\pi \circ g)_*\mathcal{O}_X \cong \pi_*\mathcal{O}_{Y'} \cong f_*\mathcal{O}_X$ ∎

For Zariski's main theorem, we will need to introduce some additional language.

Definition 16.5 (Birational Morphism)**.** Let $f : X \to Y$ be a morphism. f is a birational morphism if f admits an inverse as a rational map.

In particular, $f|_U : U \to V$ is an isomorphism for nonempty dense open sets $U \subseteq X, V \subseteq Y$.

Moreover, we will require the following preparatory lemma.

Lemma 16.6. Let $f : X \to Y$ be a finite birational morphism between integral varieties such that Y is normal. Then f is an isomorphism.

Proof. Finite morphisms are in particular affine, so without loss of generality we can take $Y = \mathrm{Spec}(A), X = \mathrm{Spec}(B)$ and f induced by the ring map $A \to B$. Since f is dominant, $A \hookrightarrow B$ is injective and by birationality, we have $A \hookrightarrow B \hookrightarrow \mathrm{Frac}(B) = \mathrm{Frac}(A)$ with the last equality by birationality. A is normal, hence integrally closed, and B is an A-algebra that is finite as an A-module so $A \hookrightarrow B$ is an integral extension, but A was integrally closed, so $A \cong B$ yielding the claim. ■

We can now state and prove Zariski's main theorem in earnest.

Theorem 16.7 (Zariski – Main)**.** Let $f : X \to Y$ be a proper birational morphism between integral varieties such that Y is normal. Then f is $\mathcal{O}$-connected.

Proof. Apply the Stein factorization Theorem 16.4 to get f as a composite $X \xrightarrow{g} Y' \xrightarrow{\pi} Y$ where g is $\mathcal{O}$-connected and π is finite. g is $\mathcal{O}$-connected and we claim that π is an isomorphism. π is birational as $\pi \circ (g \circ f^{-1} \circ \pi) = \pi$ shows that $g \circ f^{-1}$ is an inverse to π as a rational map. In particular, π is birational and proper as it is finite, so π is surjective, and $g \circ f^{-1} \circ \pi = \pi^{-1}$. In particular, π is a finite birational morphism between integral varities with normal target so Lemma 16.6 completes the proof. ■

Example 16.8. Let X be smooth and projective over k and $\beta : \widetilde{X} \to X$ be the blowing up of X smooth at a k-rational point x. We have $(R^i\beta_*\mathcal{O}_{\widetilde{X}})_{x'} = 0$ for all $i > 0$ and $x' \neq x$ since β is an isomorphism there. It remains tos how that the stalk at x is also zero. For X of dimension d, we have $E_xX \cong \mathbb{P}^{d-1}_k$ and by the theorem of formal functions $(\widehat{R^i\beta_*\mathcal{O}_{\widetilde{X}}}) = 0$ as the structure sheaf cohomology of projective space is acyclic and thus is zero on every thickeing of the fiber.

Note that in general the blowups have cohomology that differs from the base.

We now show Grothendieck's generalization of Zariski's main theorem. For this we introduce some further language.

Definition 16.9 (Isolated in Fiber)**.** Let $f : X \to Y$ be a proper morphism and $x \in X$ with image $f(x) = y \in Y$. x is isolated in its fiber if x is an irreducible connected component of the fiber X_y.

Let us consider some elementary properties of this notion.

Lemma 16.10. Let $f : X \to Y$ be an $\mathcal{O}$-connected morphism. Then:

(i) f is surjective.
(ii) $x \in X$ is isolated in its fiber if and only if f is unramified at x.

Proof of (i). Properness implies that the image of X in Y is closed and $\mathcal{O}$-connectedness implies f is dominant, that is, has dense image. But any closed dense subset of Y is Y itself, hence f is surjective. ■

Proof of (ii). ($\Rightarrow$) Suppose that f is unramified at x. We obtain by base change $f|_{X_y} : X_y \to \mathrm{Spec}(\kappa(y))$ which is quasifinite as f is unramified at x. Since further f is locally of finite type, x is isolated in its fiber.

($\Leftarrow$) If x is isolated in its fiber, we seek to show $f^\sharp : \mathcal{O}_Y \to f_*\mathcal{O}_X$ is an isomorphism. But this is precisely $\mathcal{O}$-connectedness and implies unramifiedness. It suffices to show that for every $U \subseteq X$ containing x there is $V \subseteq Y$ open such that $f^{-1}(V) \subseteq U$. But by properness, $X \setminus U$ is closed so $f(X \setminus U)$ is closed. So $Y \setminus f(X \setminus U)$ satisfies the required conditions. ■

We are now prepared to show Grothendieck's variant of Zariski's main theorem.

Theorem 16.11 (Grothendieck – Zariski's Main Theorem)**.** Let $f : X \to Y$ be a proper morphism. Then there exists a diagram

$$\begin{tikzcd}
 & X \arrow[dr, "g"] \arrow[dd, bend left, "f" near end] & \\
X_0 \arrow[ur, hook] \arrow[rr, hook, bend right=10, "i"] \arrow[dr, "f|_{X_0}"'] & & Y' \arrow[dl, "\pi"] \\
 & Y &
\end{tikzcd}$$

where:

- $X_0 \subseteq X$ is the open subset of points isolated in their fiber,
- $f|_{X_0}$ factors as $\pi \circ i$ where i is an open embedding and i is finite,
- and $f = \pi \circ g$.

Proof. Take the Stein factorization $X \xrightarrow{g} Y' \xrightarrow{\pi} X$. Let X_0 be the set of points isolated in their fiber with respect to f. But X_0 is also the set of points isolated in their fiber with respect to g as π is finite. Every point of Y' is isolated in its fiber and X_0 is the set of $x \in X$ such that g is unramified at x by Lemma 16.10 (ii) and thus X_0 is open by openness of the unramified locus.

It remains to show $g|_{X_0} : X_0 \to Y' \setminus g(X \setminus X_0)$ is an isomorphism. By the proof of Lemma 16.10 (i) $g|_{X_0}$ is surjective and injective an injective by Proposition 16.2 and using that each point is isolated in its fiber and injective. So g is a homeomorphism that is $\mathcal{O}$-connected, and hence an isomorphism. ■

We conclude with a proof of the equivalence of some conditions of morphisms.

Corollary 16.12. Let $f : X \to Y$ be a morphism of locally Noetherian schemes. The following are equivalent:

(i) f is finite.
(ii) f is affine and proper.
(iii) f is proper and quasifinite.

Proof. (a)$\Rightarrow$(b) and (a)$\Rightarrow$(c) are immediate.

(b)$\Rightarrow$(a) If f is affine and proper, and for $V = \mathrm{Spec}(A) \subseteq Y$ with preimage $U = f^{-1}(V) = \mathrm{Spec}(B) \subseteq X$ since f is affine, we have by properness of $f|_U$that

$\pi_*\mathcal{O}_U$ is a coherent $\mathcal{O}_V$-module so B is finitely generated over A showing $f|_U$ and thus f is finite.

(c)⇒(a) Suppose f is proper and quasifinite. So $X = X_0$ where X_0 is the set of points isolated in their fiber since if $x \in X$ with image $y = f(x)$ we have $f|_{X_y} : X_y \to \operatorname{Spec}(\kappa(y))$ is proper and quasifinite and hence finite with X_y the disjoint union of the Zariski spectra of finite extensions of $\kappa(y)$. By Theorem 16.11, f admits a factorization as $\pi \circ g$ where $g : X \to Y$ is an open immersion and π is finite. But g is proper and hence a closed immersion, thus finite as the composition of two finite morphisms. ■

17. Lecture 17 – 23rd June 2025

We begin a discussion of base-change theorems. The setting is as follows: for $f : X \to Y$ projective with Y Noetherian and $\mathcal{F} \in \mathsf{Coh}(X)$we have that $R^i f_*\mathcal{F}$ are coherent and for $\operatorname{Spec}(A) \subseteq Y$ an affine open subset there are isomorphisms $R^i f_*\mathcal{F}|_{\operatorname{Spec}(A)} \cong H^i(X \times_Y \operatorname{Spec}(A), \mathcal{F}|_{X\times_Y \operatorname{Spec}(A)})$. The theorem of formal functions Theorem 15.7 states that the completion $H^i(X \times_Y \operatorname{Spec}(A), \mathcal{F}|_{X\times_Y \operatorname{Spec}(A)})$ at y can be computed as the limit of cohomologies of the fiber $\lim_{n\in\mathbb{N}} H^i(X_y^{(n)}, \mathcal{F}|_{X_y^{(n)}})$ though it is not in general true that $R^i f_*\mathcal{F} \otimes \kappa(y) \cong H^i(X_y, \mathcal{F}|_{X_y})$. We seek to understand the connection between the stalk of $R^i f_*\mathcal{F}$ and $H^i(X_y, \mathcal{F}_y)$, or, more generally, for a Cartesian square

$$\begin{CD} X' @>{g'}>> X \\ @V{f'}VV @VV{f}V \\ Y' @>>{g}> Y \end{CD} \tag{17.1}$$

the relationship between $g^* R^i f_*\mathcal{F}$ and $R^i f'_* g'^*\mathcal{F}$ for $\mathcal{F} \in \mathsf{Coh}(X)$. Evidently, the relationship between $R^i f_*\mathcal{F}\otimes\kappa(y)$ and $H^i(X_y, \mathcal{F}|_{X_y})$ is recovered from the preceding discussion by taking $g : \operatorname{Spec}(\kappa(y)) = Y' \to Y = \operatorname{Spec}(\mathcal{O}_{Y,y})$.

In the case where $\mathcal{F} \in \mathsf{Coh}(X)$ is Y-flat, this is the content of the proper base change theorem. We follow the exposition of [Mum08, p. 46], keeping the notation of (17.1).

Theorem 17.1 (Proper Base Change)**.** Let $f : X \to Y$ be a projective morphism with $Y = \operatorname{Spec}(A)$ a Noetherian affine scheme and $\mathcal{F} \in \mathsf{Coh}(X)$ that is Y-flat. There exists a finite complex $K^\bullet = 0 \to K^0 \to K^1 \to \cdots \to K^n \to 0$ of finite projective A-modules K^i such that for all $A \to B$ there is an isomorphism

$$H^i(X \times_Y \operatorname{Spec}(B), g'^*\mathcal{F}) \cong H^i(K^\bullet \otimes_A B).$$

We defer the proof to the subsequent lecture, and turn now to a discussion of applications.

For our first application, we will require the following lemma.

Lemma 17.2. Let A be a Noetherian ring and $\varphi : M_1 \to M_2$ a morphism between finite A-modules. Then

$$\{\mathfrak{p} \in \operatorname{Spec}(A) : \varphi \otimes \mathrm{id}_{\kappa(\mathfrak{p})} : M_1 \otimes_A \kappa(\mathfrak{p}) \to M_2 \otimes_A \kappa(\mathfrak{p}) \text{ is the zero map}\} \subseteq \operatorname{Spec}(A)$$

is closed.

We now state and prove the proposition of interest.

Proposition 17.3. Let $f : X \to Y$ be projective with Y Noetherian and $\mathcal{F} \in \mathsf{Coh}(X)$ Y-flat. Then:

- (i) The function $Y \to \mathbb{Z}$ by $y \mapsto h^i(X_y, \mathcal{F}|_{X_y})$ is upper semicontinuous: for all c the set of points where $h^i(X_y, \mathcal{F}|_{X_y}) \geq c$ is closed in Y.
- (ii) The function $Y \to \mathbb{Z}$ by $y \mapsto \chi(X_y, \mathcal{F}|_{X_y})$ is locally constant.

Proof of (i). By locality on target, we can, without loss of generality, take $Y = \operatorname{Spec}(\mathcal{O}_{Y,y}), Y' = \operatorname{Spec}(\kappa(y))$ and apply Theorem 17.1 to observe that $H^i(X_y, \mathcal{F}|_{X_y}) \cong H^i(K^\bullet \otimes_A \kappa(y))$. In particular

$$\begin{aligned} h^i(X_y, \mathcal{F}|_{X_y}) &= \dim(\ker(d^i \otimes \mathrm{id}_{\kappa(y)})) - \dim(\operatorname{im}(d^i \otimes \mathrm{id}_{\kappa(y)})) \\ &= \dim\ker(d^i \otimes \mathrm{id}_{\kappa(y)}) - \dim(\operatorname{im}(d^i \otimes \mathrm{id}_{\kappa(y)})) - \dim(\operatorname{im}(d^{i-1} \otimes \mathrm{id}_{\kappa(y)})) \end{aligned}$$

so to show that h^i is upper semicontinuous it suffices to show that $\dim(\operatorname{im}(d^i \otimes \mathrm{id}_{\kappa(y)})) + \dim(\operatorname{im}(d^{i-1} \otimes \mathrm{id}_{\kappa(y)}))$ is lower semicontinuous. Applying Lemma 17.2 to $M_1 = \bigwedge^c K^i, M_2 = \bigwedge^c K^{i+1}$, we know that $\dim(\operatorname{im}(d^i \otimes \mathrm{id}_{\kappa(y)})) < c$ if and only if $\bigwedge^c(d^i \otimes \mathrm{id}_{\kappa(y)}) = 0$ if and only if $\varphi \otimes \mathrm{id}_{\kappa(y)} = 0$ where $\varphi : \bigwedge^c K^i \to \bigwedge^c K^{i+1}$. In our case, we can take $\bigwedge^c K^i = A^{\oplus n}, \bigwedge^c K^{i+1} = A^{\oplus m}$ so $\varphi \otimes \kappa(\mathfrak{p})$ is given by an $n \times m$ matrix which vanishes if and only if each entry of the matrix lies in $\mathfrak{p}$. That is, the function $\varphi \otimes \mathrm{id}_{\kappa(y)}$ decreases in rank on the closed subset defined by the vanishing locus of the matrix entries, ie. is lower semicontinuous, showing the claim. ■

Proof of (ii). Using the second line of the displayed equation in the proof of (i), we have

$$\begin{aligned} \chi(X_y, \mathcal{F}|_{X_y}) &= \sum_{i \geq 0} (-1)^i h^i(X_y, \mathcal{F}|_{X_y}) \\ &= \sum_{i \geq 0} (-1)^i \dim(K^i \otimes \mathrm{id}_{\kappa(y)}) \\ &= \sum_{i \geq 0} (-1)^i \operatorname{rank}(K^i) \end{aligned}$$

which is constant by Theorem 17.1. ■

Remark 17.4. We note that we may not drop the Y-flatness assumption on $\mathcal{F} \in \mathsf{Coh}(X)$. Let $f : \mathrm{Bl}_0\mathbb{A}^2_k \to \mathbb{A}^2_k$ and $E_0\mathbb{A}^2_k \cong \mathbb{P}^1_k$. Let $\mathcal{F} \cong \mathcal{O}_{\mathrm{Bl}_0\mathbb{A}^2_k}(E_0\mathbb{A}^2_k)$. $\mathcal{F}$ is not $\mathbb{A}^2_k$-flat as f is not so – the fiber dimension is not constant. For a closed point $y \in \mathbb{A}^2_k$, $(\mathrm{Bl}_0\mathbb{A}^2_k)_y$ is either a point or $\mathbb{P}^1_k$ so

$$\mathcal{F}_y = \begin{cases} \operatorname{Spec}(\kappa(y)) & y \neq 0 \\ \mathcal{O}_{\mathrm{Bl}_0\mathbb{A}^2_k}(E_0\mathbb{A}^2_k)|_{E_0\mathbb{A}^2_k} & y = 0. \end{cases}$$

Proposition 17.3 (ii) also requires the Y-flatness assumption. $\mathrm{id}_X : X \to X$ for X connected is a flat morphism. For $x \in X$ a closed point, the skyscraper sheaf $\mathcal{F} = \iota_x\kappa(x)$ is a coherent sheaf but $\mathcal{F}$ is not flat as

$$h^0(X_y, \mathcal{F}_y) = \begin{cases} 0 & y \neq x \\ 1 & y = x \end{cases}$$

and $h^i(X_y, \mathcal{F}_y) = 0$ for all $i \geq 1$ so the Euler characteristic is

$$\chi(X_y, \mathcal{F}_y) = \begin{cases} 1 & y \neq x \\ 0 & y = x \end{cases}$$

which is not constant.

We now relate constancy of the cohomological dimension functor with local freeness of the higher direct images.

Proposition 17.5. Let $f : X \to Y$ be a projective morphism with Y Noetherian and $\mathcal{F} \in \mathsf{Coh}(X)$ Y-flat. Then TFAE:

(a) $y \mapsto h^i(X_y, \mathcal{F}|_{X_y})$ is constant.

(b) $R^i f_* \mathcal{F}$ is locally free and there is an isomorphism $(R^i f_* \mathcal{F})_y \cong H^i(X_y, \mathcal{F}|_{X_y})$.

Proof. (a)⇒(b) This is the statement of Example 5.8.

(b)⇒(a) If $R^i f_* \mathcal{F}$ is locally free and $R^i f_* \mathcal{F} \cong H^i(X_y, \mathcal{F}|_{X_y})$ then the local cohomological dimension is constant. ∎

Corollary 17.6. Let $f : X \to Y$ be a projective morphism with Y Noetherian and $\mathcal{F} \in \mathsf{Coh}(X)$ Y-flat. If $y \mapsto h^i(X_y, \mathcal{F}|_{X_y})$ is constant then $R^{i-1} f_* \mathcal{F} \otimes \kappa(y) \cong H^{i-1}(X_y, \mathcal{F}|_{X_y})$.

Remark 17.7. Note that in Corollary 17.6 $R^{i-1} f_* \mathcal{F} \otimes \kappa(y)$ need not be locally free – that is, $h^{i-1}(X_y, \mathcal{F}|_{X_y})$ need not be (locally) constant.

Corollary 17.8. Let $f : X \to Y$ be projective with Y Noetherian and $\mathcal{F} \in \mathsf{Coh}(X)$ Y-flat. If $H^i(X_y, \mathcal{F}|_{X_y}) = 0$ for all $y \in Y$ then $R^{i-1} f_* \mathcal{F} \otimes \kappa(y) \cong H^i(X_y, \mathcal{F}|_{X_y})$.

We can also show a vanishing result.

Proposition 17.9. Let $f : X \to Y$ be a projective morphism with Y Noetherian and $\mathcal{F} \in \mathsf{Coh}(X)$ Y-flat. Fix $i_0 \in \mathbb{N}$. If $R^i f_* \mathcal{F} = 0$ for all $i \geq i_0$ then $H^i(X_y, \mathcal{F}_y) = 0$ for all $i \geq i_0$.

Proof. We know that $H^i(X_y, \mathcal{F}|_{X_y}) = 0$ for $i > \dim(X)$. So we can assume $H^i(X_y, \mathcal{F}|_{X_y}) = 0$ for all $i > i_0$. By Corollary 17.8, $R^{i-1} f_* \mathcal{F} \otimes \kappa(y) \cong H^{i-1}(X_y, \mathcal{F}|_{X_y}) \cong H^i(X_y, \mathcal{F}|_{X_y}) = 0$ giving the claim. ∎

An especially nice situation is when the map $g : Y' \to Y$ of (17.1) is flat.

Theorem 17.10 (Flat Base Change)**.** Let $f : X \to Y$ be a projective morphism with Y Noetherian, $\mathcal{F} \in \mathsf{Coh}(X)$ Y-flat, and $g : Y' \to Y$ flat inducing a Cartesian diagram

$$\begin{tikzcd} X' \arrow[r, "g'"] \arrow[d, "f'"'] & X \arrow[d, "f"] \\ Y' \arrow[r, "g"'] & Y. \end{tikzcd}$$

Then the natural morphism $g^* R^i f_* \mathcal{F} \to R^i f'_* g'^* \mathcal{F}$ is an isomorphism.

We have encountered this statement before as Proposition 6.4.

Proof. Isomorphisms of quasicoherent sheaves can be checked affine-locally. It thus suffices to consider the case $Y' = \operatorname{Spec}(A'), Y = \operatorname{Spec}(A)$ affine. Then the assertion follows by observing

$$H^i(K^\bullet) \otimes_A A' \longrightarrow H^i(K^\bullet \otimes_A A')$$

is an isomorphism by flatness of A' over A. ∎

18. Lecture 18 – 26th July 2025

We prove Theorem 17.10. We begin with some preparatory homological algebra matters.

Definition 18.1 (Quasi-Isomorphism)**.** Let A be an Abelian category and $C^\bullet, C'^\bullet$ two A-chain complexes. $C^\bullet$ is quasiisomorphic to $C'^\bullet$ if $H^i(C^\bullet) \cong H^i(C'^\bullet)$.

Definition 18.2 (Mapping Cone)**.** Let A be an Abelian category and $\phi^\bullet : C^\bullet \to C'^\bullet$ a morphism of A-chain complexes. Then the mapping cone $\mathrm{Cone}(\phi)^\bullet$ is the complex beginning in degree -1 with terms $C^i \oplus C'^{i+1}$ with differentials $(\partial^i, -d^{i+1} + \psi^{i+1})$.

With this language in hand, we can prove the relevant lemmata.

Lemma 18.3. Let A be a Noetherian ring and $C^\bullet$ be a finite complex of A-modules such that all $H^i(C^\bullet)$ are also finitely generated A-modules. Then there exists a finite complex $K^\bullet$ and a morphism $K^\bullet \to C^\bullet$ in $\mathrm{Ch}(\mathsf{Mod}_A)$ such that K^i are free for $i \geq 1$ and $K^\bullet$ is quasi-isomorphic to $C^\bullet$. Moreover, if all C^i's are flat, then so too are the K^i's.

Proof. Assume that we have already constructed

$$\begin{tikzcd}
K^{m+1} \arrow[r, "d^{m+1}"] \arrow[d] & K^{m+2} \arrow[r, "d^{m+2}"] \arrow[d] & \dots \arrow[r] & K^n \arrow[r] \arrow[d] & 0 \\
C^{m+1} \arrow[r, "\partial^{m+1}"'] & C^{m+2} \arrow[r, "\partial^{m+2}"'] & \dots \arrow[r] & C^n \arrow[r] & 0
\end{tikzcd}$$

with K^i free and $H^i(K^\bullet) \cong H^i(C^\bullet)$ for all $i \geq m+2$ and $\ker(d^{m+1}) \twoheadrightarrow H^m(C^\bullet)$. Since K^{m+1} is finite and A is Noetherian we can set

$$M = \ker(\ker(d^{m+1}) \to H^{m+1}(C^\bullet))$$

which is also finite. Pick free modules K', K'' with morphisms $\alpha : K' \twoheadrightarrow M, K'' \to H^{m+1}(C^\bullet)$ inducing the map β below

$$\begin{tikzcd}
K'' \arrow[r] \arrow[dr, dashed, "\beta"'] & H^m(C^\bullet) \\
& \ker(\partial^m) \arrow[u, two heads]
\end{tikzcd}$$

which exists by K'' being free and in particular both injective and projective. Set $K^m = K' \oplus K''$ which is free being the direct sum of free modules. Consider

$$\begin{tikzcd}
K^m \arrow[rr, two heads, "{(\alpha,0)}"] \arrow[dr, "\beta"] \arrow[dd, "\widetilde{\alpha}"'] \arrow[ddrrr, bend left] & & M \arrow[r, hook] & K^{m+1} \arrow[dd] \\
& \ker(\partial^m) \arrow[dl, hook] & & \\
C^m \arrow[rrr, "\partial^m"'] & & & C^{m+1}
\end{tikzcd}$$

and where the curved arrow factors over $\mathrm{im}(\partial^m)$. We have that $M \subseteq \ker(d^{m+1})$ surjecting onto $H^{m+1}(C^\bullet)$. So $H^{m+1}(K^\bullet) \cong \ker(d^{m+1})/M$ injects into $H^{m+1}(C^\bullet)$. Since this was surjective by assumption, injectivity here yields an isomorphism. That β is surjective it suffices to observe that $\ker(d^m) \twoheadrightarrow H^m(C^\bullet)$. This extends the construction by one step and by iterating the process we get

$$\begin{tikzcd}
K^0 \arrow[r, "d^0"] \arrow[d, "\psi^0"'] & K^1 \arrow[r, "d^1"] \arrow[d] & \dots \arrow[r] & K^n \arrow[r] \arrow[d] & 0 \\
C^0 \arrow[r, "\partial^0"'] & C^1 \arrow[r, "\partial^1"'] & \dots \arrow[r] & C^n \arrow[r] & 0
\end{tikzcd}$$

where all K^i are free and $K^\bullet$ quasi-isomorphic to $C^\bullet$ in positive degree. Replacing K^0 by $K^0/\ker(d_0) \cap \ker(\psi^0)$, we get a quasiisomorphism in degree 0 as well but K^0 is no longer free.

It remains to show the second statement that K^0 in particular is flat when the C^i's are so. For this, we note that there are morphisms of complexes $K^\bullet \to C^\bullet \to \mathrm{Cone}(\psi)^\bullet \to K^\bullet[1]$ and $C^i \hookrightarrow \mathrm{Cone}(\psi)^i \twoheadrightarrow K^{i+1}$ inducing a long exact sequence in cohomology

$$\cdots \to H^i(K^\bullet) \xrightarrow{\sim} H^i(C^\bullet) \to H^i(\mathrm{Cone}(\psi)^\bullet) \to H^{i+1}(K^\bullet) \xrightarrow{\sim} H^{i+1}(C^\bullet) \to \dots$$

where the indicated isomorphisms are by the quasiisomorphism of the complex. Examining terms in low degrees, we have that each $\mathrm{Cone}(\psi)^i$ for $i \geq 0$ is $C^i \oplus K^{i+1}$ the sum of flat modules, hence flat. So K^0 is flat too. ■

Lemma 18.4. Let A be a Noetherian ring and $K^\bullet, C^\bullet$ complexes fo flat A-modules such that $\psi^\bullet : K^\bullet \to C^\bullet$ is a quasi-isomorphism. Then for all $A \to B$, the induced map $\psi \otimes \mathrm{id}_B : K^\bullet \otimes_A B \to C^\bullet \otimes_A B$ is a quasi-isomorphism.

Proof. We form the mapping cone fitting into

$$K^\bullet \to C^\bullet \to \mathrm{Cone}(\psi)^\bullet \to K^\bullet[1]$$

and since ψ is a quasi-isomorphism, $\mathrm{Cone}(\psi)^\bullet$ is an acyclic complex, so tensoring with B, we get $\mathrm{Cone}(\psi)^\bullet \otimes \mathrm{id}_B \cong \mathrm{Cone}(\psi \otimes \mathrm{id}_B)$ acyclic so $K^\bullet \otimes_A B$ is quasi-isomorphic to $C^\bullet \otimes_A B$. ■

We are now prepared to show Theorem 17.1.

Proof of Theorem 17.1. Let $X = \bigcup_{i=1}^n U_i$ where $U_i = \mathrm{Spec}(A_i)$ be a finite cover of X by affine open schemes. We can form the Čech complex which computes sheaf cohomology since X is projective (and in particular separated) over $\mathrm{Spec}(A)$. The Čech complex is termwise finitely generated by projectivity and termwise flat by flatness of $\mathcal{F}$ so for $A \to A'$, $X' = X \times_{\mathrm{Spec}(A)} \mathrm{Spec}(A')$, X' admits an open cover by $\bigcup_{i=1}^n U_i'$ where $U_i' = \mathrm{Spec}(A_i \otimes_A A')$. Here, the result follows immediately from Lemmas 18.3 and 18.4. ■

We turn to a discussion of some applications.

Proposition 18.5. Let X be projective and geometrically integral over k and T of finte type over k. For $\mathcal{L} \in \operatorname{Pic}(X \times_k T)$ the set

$$Z = \{t \in T : \mathcal{L}|_{X \times_k \operatorname{Spec}(\kappa(t)) \cong \mathcal{O}_{X \times_k \operatorname{Spec}(\kappa(t))}}\}$$

is closed.

Proof. Recall that if $\mathcal{M}$ is an invertible sheaf on a proper integral scheme X over a field, $\mathcal{M}$ is trivial if and only if $H^0(X, \mathcal{M}) \neq 0$ and $H^0(X, \mathcal{M}^\vee) \neq 0$. Since $H^0(X, \mathcal{O}_X)$ is a field, we can write

$$Z = \{t \in T : H^0(X \times_k \operatorname{Spec}(\kappa(t)), \mathcal{L}_t) \neq 0\} \cap \{t \in T : H^0(X \times_k \operatorname{Spec}(\kappa(t)), \mathcal{L}_t^\vee) \neq 0\}$$

where we denote $\mathcal{L}_t = \mathcal{L}|_{X \times_k \operatorname{Spec}(\kappa(t))}$. By Proposition 17.3, both of these sets are closed, so Z being their intersection is closed too. ■

Proposition 18.6. Let $f : X \to Y$ be a surjective flat projective morphism with integral fibers and Y a reduced Noetherian scheme. If $\mathcal{L} \in \operatorname{Pic}(X)$ such that $\mathcal{L}|_{X_y} \cong \mathcal{O}_{X_y}$ for all $y \in Y$ then $\mathcal{L} \cong f^*\mathcal{M}$ for some $\mathcal{M} \in \operatorname{Pic}(Y)$.

Proof. Set $\mathcal{M} = f_*\mathcal{L}$ and by Proposition 17.5 and reducedness of Y, $h^0(X_y, \mathcal{L}_y) = 1$ implies $f_*\mathcal{L}$ is invertible and the fibers $\mathcal{M} \otimes \kappa(y) = H^0(X_y, \mathcal{L}_y)$. Using adjunction, we can define a natural morphism $f^*\mathcal{M} = f^*f_*\mathcal{L} \to \mathcal{L}$ which is an isomorphism on each fiber and hence globally. ■

Corollary 18.7. Let $f : X \to Y$ be a surjective flat projective morphism with integral fibers and Y a reduced Noetherian scheme. If $\mathcal{L}_1, \mathcal{L}_2$ are invertible sheaves such that $\mathcal{L}_1|_{X_y} \cong \mathcal{L}_2|_{X_y}$ for each $y \in Y$ then there exists $\mathcal{M} \in \operatorname{Pic}(Y)$ such that $\mathcal{L}_1 \cong f^*\mathcal{M} \otimes \mathcal{L}_2$.

Proof. Apply Proposition 18.6 to $\mathcal{L}_1 \otimes \mathcal{L}_2^\vee$. ■

References

[Har83] Robin Hartshorne. *Algebraic geometry. Corr. 3rd printing.* English. Vol. 52. Grad. Texts Math. Springer, Cham, 1983.

[Mum08] David Mumford. *Abelian varieties. With appendices by C. P. Ramanujam and Yuri Manin.* English. Corrected reprint of the 2nd ed. 1974. New Delhi: Hindustan Book Agency/distrib. by American Mathematical Society (AMS); Bombay: Tata Institute of Fundamental Research, 2008. ISBN: 978-81-85931-86-9.

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Universität Bonn, Bonn, D-53113
Email address: `wgabrielong@uni-bonn.de`
URL: `https://wgabrielong.github.io/`