V5A4 – HABIRO COHOMOLOGY
SUMMER SEMESTER 2025

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Preliminaries

These notes roughly correspond to the course **V5A4 – Habiro Cohomology** taught by Prof. Peter Scholze at the Universität Bonn in the Summer 2025 semester. These notes are LaTeX-ed after the fact with significant alteration and are subject to misinterpretation and mistranscription. Use with caution. Any errors are undoubtedly my own and any virtues that could be ascribed to these notes ought be attributed to the instructor and not the typist. Recordings of the lecture are availible at the following link:

`archive.mpim-bonn.mpg.de/id/eprint/5155/`

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1. Lecture 1 – 11th April 2025

Recall that the construction of the Habiro ring of a number field [GS+24, Sch24] was motivated by an expectation of the instructor, circa 2017, that there exists some form of "Habiro cohomology." Within this larger aspriational framework, the Habiro ring of a number field serves as the zero-dimensional case where the variety is a discrete collection of points. More precisely, in the case of the Habiro ring of a number field, there are certian q-series related to pertubative Chern-Simons theory giving rise to an explicit approach to Habiro rings of number fields. In particular, these q-series from pertubative Chern-Simons theory as computed by Garoufalidis and Zagier arise as elements of the abstract Habiro ring of a number field.

The goal of this course, then, is to explicate this aspirational framework of Habiro cohomology that synthesizes the concrete approach of Garoufalidis-Zagier with the instructor's abstract approach. In particular, we will define a new explicit cohomology theory for algebraic varieties that has specializations to clasical cohomology theories: de Rham cohomology as well as p-adic étale cohomology, crystalline cohomology, and prismatic cohomology for all primes p. Moreover, this cohomology theory will extend to the rigid-analytic setting of Berkovich spaces.

Let recall a modern definiton of Weil-type cohomology theories for algebraic varieties: functors

$$\mathsf{Sch}_k^{\mathsf{sft}} \longrightarrow \mathsf{Pr}_A^{\mathsf{L}}$$

where $\mathsf{Sch}_k^{\mathsf{sft}}$ is the category of separated finite type schemes over k and $\mathsf{Pr}_A^{\mathsf{L}}$ the category of presentable A-linear categories with a six-functor formalism and satisfying the Künneth formula. In particular this exculedes some cohomology theories such as motivic cohomology.

The state of the art of Weil-type cohomology theories for algebraic varieties can be summarized in the following diagram.

The instructor remarks that this is his favorite diagram.

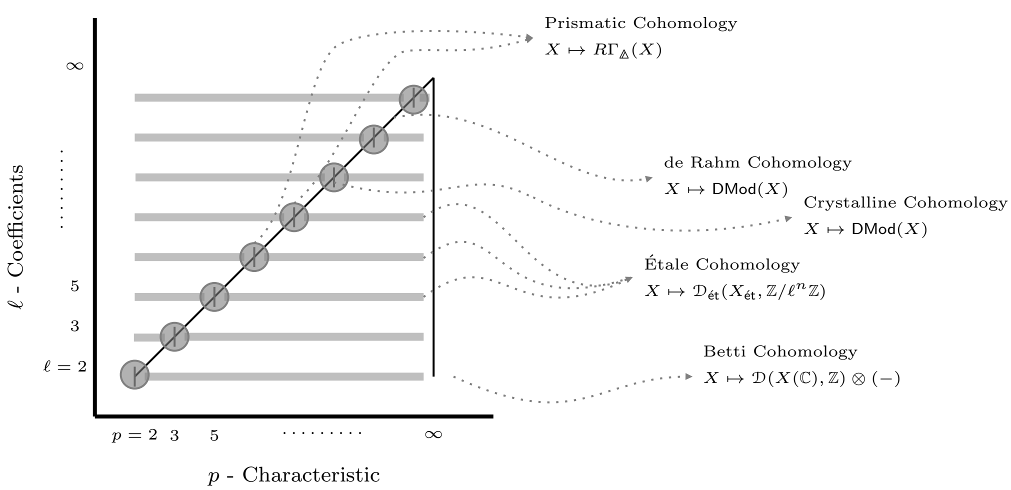

Figure 1. Cohomology theories for algebraic varieties. Or: the instructor's favorite diagram.

- Betti cohomology $X \mapsto \mathcal{D}(X(\mathbb{C}), \mathbb{Z}) \otimes (-)$ produces a cohomology theory for complex schemes. But coefficients can be taken in any field by base change.
- de Rham cohomology $X \mapsto \mathsf{DMod}(X)$ associating to a scheme its category of D-modules produces a cohomology theory for k-schemes (modulo technicalities). This produces a k-vector space for a k-scheme, hence has coefficients equal to the characteristic of the scheme.
- Étale cohomology as defined by Grothendieck $X \mapsto \mathcal{D}_{\text{ét}}(X_{\text{ét}}, \mathbb{Z}/\ell^n\mathbb{Z})$ produces for a k-scheme X, a cohomology theory with $\mathbb{Z}/\ell^n\mathbb{Z}$-coeffients with ℓ of characteristic distinct from that of k. That étale cohomology is able to produce cohomology in coefficients modulo powers of ℓ is represented by the thickening of the horizontal. Note that étale cohomology satisfies the Künneth formula, but not its categorical variant.
- Crystalline cohomology after Grothendieck, Berthelot, Caro, et. al. that associates to a k-scheme where k is of positive characteristic a cohomology theory $X \mapsto \mathsf{DMod}(X)$ that associates to X its category of arithmetic D-modules and which satisfies the categorical Künneth formula. This produces a module over the Witt vectors $W(k)$ of k for a k-scheme, and is represented by vertical thickenings at the characteristic.
- Prismatic cohomology was defined by Bhatt-Scholze [BS22] as a universal cohomology theory at the (p,p)-point by computing the structure sheaf cohomology of the prismatic site $X \mapsto R\Gamma_{\Delta}(X)$ where X is a scheme over $\mathcal{O}_K$ where K is a mixed characteristic local field which has coefficients valued in prisms.[1]

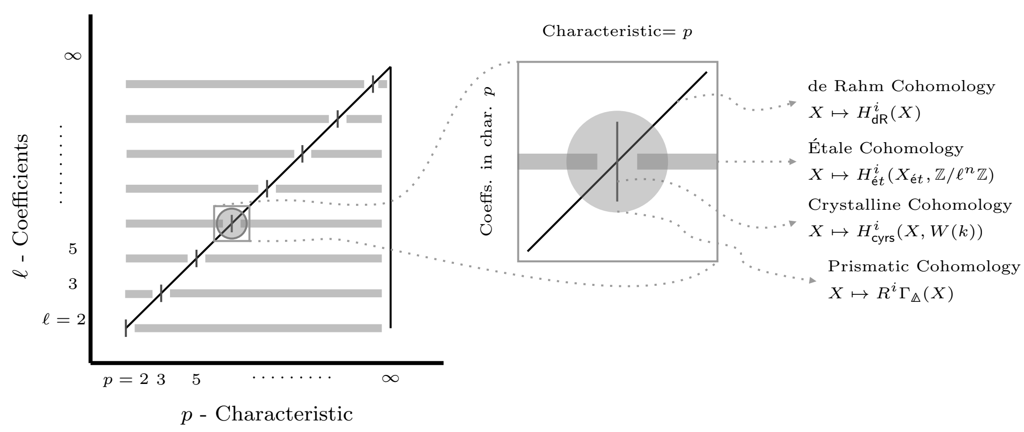

FIGURE 2. Prismatic cohomology at the (p,p)-point.

Moreover, the diagram reflects several important comparison phenomena between the abovementioned cohomology theories:

[1]It would be more precise to state this using "derived category of sheaves" associated to prismatic cohomology, namely the category of F-gauges a là Bhatt-Lurie [Bha22], but we do not comment on this further.

- The intersection of the lines corresponding to Betti and de Rham cohomology at the (∞, ∞)-point is substantiated by the comparison isomorphism between singular cohomology with $\mathbb{C}$-coefficients and de Rham cohomology via the Riemann-Hilbert correspondence.
- The intersection of the lines corresponding to étale and Betti cohomology at the (∞, p)-points are substantiated by the Artin's comparison isomorphism between étale and Betti cohomology.
- The intersection of the thickenings of crystalline cohomology meeting de Rham cohomology along the diagonal at the (p, p)-point is substantiated by the isomorphism between crystaline cohomology reduced modulo p and de Rham cohomology.
- Prismatic cohomology as depicted in Figure 1 admits specializations to de Rham, crystalline, and étale cohomology. Prismatic cohomology is additionally compatible with the structures of the various cohomology theories around the (p, p)-point, specializing to the action of the Frobenius in crystalline cohomology, the Hodge-Tate filtration in the case of de Rham cohomology, and the action of the absolute Galois group $\mathrm{Gal}(\overline{\mathbb{Q}_p}/\mathbb{Q}_p)$ in the case of étale cohomology.
- The "prismatization" at the (∞, ∞)-point is the content of classical complex Hodge theory, which considers Hodge filtrations on de Rham cohomology and associated objects.

Observe, then, that de Rham cohomology is the unifying cohomology theory on the diagonal, while prismatic cohomology only exists at a fixed prime. One then wonders if there is a way to unify the cohomology theories along the diagonal. This is provided by Habiro cohomology, at least in the positive characteristic case.

The instructor remarks that he is unsure how to unify Habiro cohomology with classical Hodge theory.

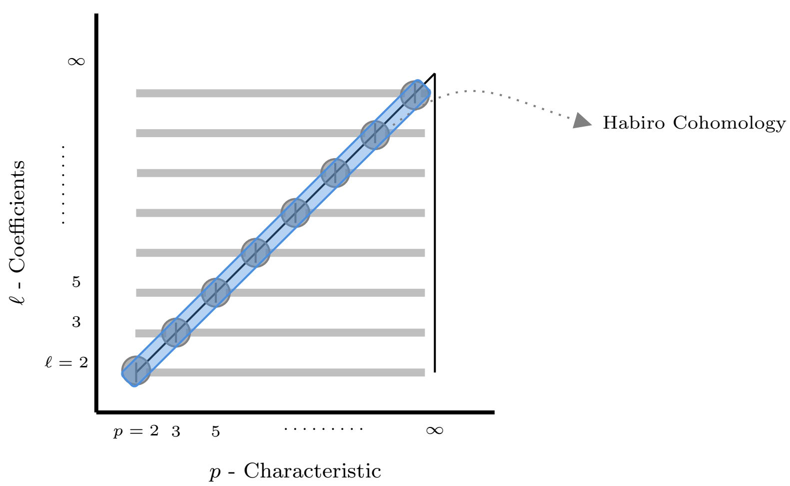

FIGURE 3. The role of Habiro cohomology highlighted in blue generalizing prismatic cohomology at all primes. Compare Figure 1.

That is to say that Habiro cohomology, covering a neighborhood of the de Rham diagonal, specializes to prismatic cohomology at each prime, and spreads out further than prismatic cohomology along the horizontal étale branches in an appropriate sense.

The starting point of Habiro cohomology is the example of the q-de Rham prism, the definition of which we now recall.

Example 1.1. The q-de Rham prism is the prism $(\mathbb{Z}_p[[q-1]], [p]_q)$ where $[p]_q = \frac{1-p^n}{1-q}$ is the q-deformation of p with a Frobenius action by $q \mapsto q^p$. The quotient $\mathbb{Z}_p[[q-1]]/([p]_q)$ is precisely the quotient by the p-th cyclotomic polynomial and hence isomorphic to the cyclotomic extension $\mathbb{Z}_p[\zeta_p]$.

Computing the prismatic cohomology of $\mathbb{A}^1_{\mathbb{Z}_p[\zeta_p]}$ relative to the q-de Rham prism, one finds that this is computed by an obvious q-deformation of the de Rham complex. The cohomological comparisons of the preceding discussion suggest that there is a deformation of the de Rham complex given by

$$\nabla_q : \mathbb{Z}_p[\zeta_p][x][[q-1]] \longrightarrow \mathbb{Z}_p[\zeta_p][x][[q-1]]$$

by $x^n \mapsto [n]_q x^{n-1}$. It is not *a priori* clear why q-deformations appear in this setting. Moreover, the construction of prismatic cohomology over the q-de Rham prism is expected to be functorial in automorphisms of $\mathbb{A}^1_{\mathbb{Z}_p[\zeta_p]}$ but it is unclear if (and how) this construction is invariant under change of coordinates. Additionally, the q-deformation suggests that by removing p everywhere, one can find a construction independent that works for all primes p. In particular, the instructor conjectures in [Sch17] the following:

Conjecture 1.2 (Scholze; [Sch17, Conj. 1.1])**.** If R is a smooth $\mathbb{Z}$-algebra equipped with an étale map $\mathrm{Spec}(R) \to \mathbb{A}^d_{\mathbb{Z}}$, there is a cohomology theory for smooth proper varieties over R valued in finitely generated $R[[q-1]]$-modules with a q-connection.

The q-connection captures precisely the difficulties with coordinate transformations articulated above, and the specialization at $q = 1$ recovers the de Rham cohomology of X with a Gauss-Manin connection. This suggests that algebraic varieties have a canonical q-deformation with connection compatible with the Gauss-Manin connection on classical de Rham cohomology, and was proven after p-adic completion in [BS22] and in general by Ferdinand Wagner in [Wag24] using the machinery of adelic gluing.

Theorem 1.3 (Wagner; [Wag24, Thm. 1.7])**.** Let R be a smooth framed $\mathbb{Z}$-algebra. There is an isomorphism between the $(q-1)$-completed q-de Rham–Witt complex and the cohomology of the quotient of the q-Hodge complex by $(q^m - 1)$.

Let us consider an example of this phenomenon.

Example 1.4. Consider the Legendre family of elliptic curves X with affine model $y^2 = x(x-1)(x-\lambda)$ over $R = \mathbb{Z}[\frac{1}{2}, \lambda, \frac{1}{\lambda(1-\lambda)}]$. We have $H^1_{\mathsf{dR}}(X)$ free of rank 2, containing the Hodge filtration $\mathrm{Fil}^1_{\mathsf{Hdg}} = H^0(X, \Omega^1_{X/R})$ with canonical differential

$\omega = \frac{\mathrm{d}x}{y}$. Denoting ∇ the connection on $H^1_{\mathsf{dR}}(X)$, we have $\omega, \nabla(\omega)$ a basis of $H^1_{\mathsf{dR}}(X)$ and

$$\nabla^2(\omega) = \frac{1}{4\lambda(1-\lambda)} + \frac{2\lambda - 1}{\lambda(1-\lambda)}\nabla(\omega).$$

A horizontal section is $f(\lambda)\cdot\lambda(1-\lambda) - f'(\lambda)\lambda(1-\lambda)\nabla(\omega)$ for a certain hypergeometric function $f(\lambda) = \sum_{n\geq 0}\prod_{i=0}^{n-1}\left(\frac{i+\frac{1}{2}}{i+1}\right)^2\lambda^n$.

There is a q-analogue of hypergeometric functions.

Example 1.5. The q-hypergeometric function

$$\sum_{n\geq 0}\prod_{i=0}^{n-1}\left(\frac{[i+\frac{1}{2}]_q}{[i+1]_q}\right)^2\lambda^n$$

satisfies a second order q-difference equation that deforms the Picard-Fuchs equation whose solutions describe periods of elliptic curves [nLab-a].

The example suggests that there is a possible connection between q-hypergeometric functions – the q-analogue of hypergeometric functions – and q-deformations of de Rham cohomology.

In the case of de Rham cohomology as in Example 1.4, there is not only a connection ∇, but also a choice of canonical vector $\omega = \frac{\mathrm{d}x}{y}$ obtained by the filtration. Then considering the differential equation the class satisfies produces the desired differential equation – the module and connection alone are insufficient to produce the differential equation. The main barrier to considering the q-analogue, then, was the lack of choice of such a class.

Recent computations of Shirai [Shi20] and work of Garoufalidis-Wheeler remedy this by producing explicit classes in q-de Rham cohomology, allowing the procedure above to be repeated.

This course will consider what happens to these q-deformations when q approaches a root of unity ζ_m, knowing that it recovers the classical construction at $q = 1$. Working over the Habiro ring

$$\mathcal{H} = \lim_{m,n\geq 1}\mathbb{Z}[q]/(1-q^n)^m = \lim_n \mathbb{Z}[q]/(q;q)_n$$

allows us to consider specializations at different roots of unity.

One issue that arises in trying to naïvely generalize Habiro cohomology to schemes of higher dimension is that the specialization of prismatic cohomology over the q-de Rham prism at $q = 1$ recovers de Rham cohomology, but at other roots of unity recovers only Hodge cohomology – this does not put all roots of unity on equal footing. But if the q-de Rham cohomology could be modified to be Hodge cohomology in an appropriate manner. This was shown by Meyer-Wagner in [MW24].

Theorem 1.6 (Meyer-Wagner; [MW24, Thm. 1.7])**.** Let R be a p-torsion free p-complete ring which is a quasiregular quotient over $\mathbb{Z}_p$ and such that the Frobenius on R/p is semiperfect. If R admits a lift to a p-complete $\mathbb{E}_1$ ring spectrum $\mathbb{S}_R$ such that $R \simeq \mathbb{S}_R\widehat{\otimes}_{\mathbb{S}_p}\mathbb{Z}_p$ then the q-Hodge filtration on the p-complete derived

q-de Rham complex is a q-deformation of the Hodge filtration on the (ordinary) p-complete derived de Rham complex.

The proof of Meyer-Wagner once again leverages highly technical machinery, in particular the relationship bewteen prismatic cohomology and topological cyclic homology. However, there is a more computational way of achieving the same goal.

Here "ring stack" and "analytic" are to be taken in the sense of condensed mathematics [CS23].

Theorem 1.7 (Scholze)**.** There is an explicit ring stack over an analytic version of the Habiro ring yielding a full six-functor formalism.

Remark 1.8. This in particular yields a sheaf theory.

These are related to the constructions of the ring stacks for prismatic cohomology following Drinfeld [Dri20] and Bhatt-Lurie [BL22].

While multiplication is easy to define in this ring, addition is not: in particular, the instructor remarks that he spent a whole day computing what $1 + 1$ is in this ring.

2. Lecture 2 – 2nd May 2025

The goal of this course is to develop a theory of Habiro cohomology, a functor that associates to a smooth $\mathbb{Z}$-scheme X its Habiro cohomology – a module over the Habiro ring, or more generally its "category of constructible sheaves" which in this case we tentatively denote $\mathcal{D}_{\mathsf{Hab}}(X)$ of "variations of Habiro structure."

We begin with an exploration of what these structures are in terms of coordinates, and we will later show that the constructions we discuss are in fact independent of these coordinates. Let us make the notion of coordinates precise.

Definition 2.1 (Framed Algebra)**.** A framed algebra is a pair $(R, \square)$ where R is a smooth $\mathbb{Z}$-algebra and an étale map $\square : \mathrm{Spec}(R) \to \mathbb{A}^d_{\mathbb{Z}}$ or $\square : \mathrm{Spec}(R) \to \mathbb{G}^d_m$.

Remark 2.2. It is often simpler to consider the case where the coordinates are invertible, that is, the case of $\mathbb{G}^d_m$.

As a first pass, let us contemplate these constructions in the case where X is affine and equal to either $\mathbb{A}^d_{\mathbb{Z}}$ or $\mathbb{G}^d_m$ and only later consider the generalization to the case where X is étale over one of these spaces. Moreover, under these assumptions, we need not make any completions and one can work over $\mathbb{Z}[q^{\pm}]$.

Recall Habiro cohomology subsumes de Rham cohomology in an appropriate sense, and takes the q-derivative – the Gaussian q-analogue of the derivative – as an input. These q-derivatives were first investigated by Jackson [Jac10].

Definition 2.3 (q-Derivative)**.** Let R be $\mathbb{Z}[q^{\pm}][T_1, \ldots, T_d]$ or $\mathbb{Z}[q^{\pm}][T_1^{\pm}, \ldots, T_d^{\pm}]$. The q-derivative $\nabla_i^q : R \to R$ for $1 \le i \le d$ is defined by

$$\nabla_i^q(f(T_1, \ldots, T_d)) = \frac{f(T_1, \ldots, qT_i, \ldots, T_d) - f(T_1, \ldots, T_i, \ldots, T_d)}{qT_i - T_i}.$$

Remark 2.4. More explicitly, this operation is given on monomials by

$$\nabla_i^q(T_1^{n_1} \ldots T_d^{n_d}) = [n_i]_q \cdot T_1^{n_1} \ldots T_i^{n_i - 1} \ldots T_d^{n_d}$$

where $[n]_q = \frac{1-q^n}{1-q}$ is the Gaussian q-analogue of n.

Remark 2.5. ∇_i^q is closely related $\gamma_i : R \to R$ the automorphism by

$$T_j \mapsto \begin{cases} T_j & j \neq i \\ qT_i & j = i \end{cases}$$

allowing us to write $\nabla_i^q(f) = \frac{\gamma_i(f) - f}{(q-1)T_i}$.

The q-derivative does not satisfy the Leibniz rule on the nose, but does so up to a twist by the automorphism γ_i of Remark 2.5.

Lemma 2.6. Let R be $\mathbb{Z}[q^{\pm}][T_1, \ldots, T_d]$ or $\mathbb{Z}[q^{\pm}][T_1^{\pm}, \ldots, T_d^{\pm}]$. Then for $f, g \in R$ we have equalities

$$\nabla_i^q(fg) = \gamma_i(f) \cdot \nabla_i^q(g) + g \cdot \nabla_i^q(f) = f \cdot \nabla_i^q(g) + \gamma_i(g) \cdot \nabla_i^q(f).$$

Proof. We first show the second equality. We use Remark 2.5 to observe that the latter two terms are given by

$$\gamma_i(f)\cdot\frac{\gamma_i(g)-g}{(q-1)T_i}+g\cdot\frac{\gamma_i(f)-f}{(q-1)T_i}=\frac{\gamma_i(f)\gamma_i(g)-\gamma_i(f)g+\gamma_i(f)g-fg}{(q-1)T_i}=\frac{\gamma_i(f)\gamma_i(g)-fg}{(q-1)T_i}$$

and

$$f\cdot\frac{\gamma_i(g)-g}{(q-1)T_i}+\gamma_i(g)\frac{\gamma_i(f)-f}{(q-1)T_i}=\frac{\gamma_i(g)f-fg+\gamma_i(f)\gamma_i(g)-\gamma_i(g)f}{(q-1)T_i}=\frac{\gamma_i(f)\gamma_i(g)-fg}{(q-1)T_i}$$

respectively, which are evidently equal.

We now show the first equality. Note that γ_i is an automorphism $R \to R$, and in particular a homomorphism so $\gamma_i(fg) = \gamma_i(f)\gamma_i(g)$ in which case we have

$$\frac{\gamma_i(fg)-fg}{(q-1)T_i}=\frac{\gamma_i(f)\gamma_i(g)-fg}{(q-1)T_i}$$

whence the claim. ∎

We can now define the q-de Rham complex following Aomoto [Aom90].

Definition 2.7 (q-de Rham Complex of $\mathbb{A}^d_{\mathbb{Z}}$ and $\mathbb{G}^d_m$)**.** Let $R = \mathbb{Z}[q^{\pm}][\underline{T}]$ be $\mathbb{Z}[q^{\pm}][T_1, \dots, T_d]$ or $\mathbb{Z}[q^{\pm}][T_1^{\pm}, \dots, T_d^{\pm}]$. The q-de Rham complex $q\text{-}\Omega^{\bullet}_{R/\mathbb{Z}}$ of $\operatorname{Spec}(R)$ is the complex

$$0 \longrightarrow \mathbb{Z}[q^{\pm}][\underline{T}] \longrightarrow \mathbb{Z}[q^{\pm}][\underline{T}]^{\oplus d} \longrightarrow \bigoplus_{i<j} \mathbb{Z}[q^{\pm}][\underline{T}] \longrightarrow \cdots \tag{2.1}$$

$$\cdots \longrightarrow \mathbb{Z}[q^{\pm}][\underline{T}] \longrightarrow 0$$

with differentials given by the differentials for the Koszul complex of commuting operators $\nabla_1^q, \dots, \nabla_n^q$.

Remark 2.8. Recall that these are precisely the differentials for the classical de Rham complex. See [Stacks, Tag 0FKF] for an explicit description via equations.

Remark 2.9. Since the first differential $\mathbb{Z}[q^{\pm}][\underline{T}] \to \mathbb{Z}[q^{\pm}][\underline{T}]^{\oplus d}$ by $(\nabla_1^q, \dots, \nabla_d^q)$ does not satisfy the ordinary Leibniz rule, the complex (2.1) is not a differential graded algebra. Later, we will see that working in the derived (∞-)category, one can endow this with the structure of a commutative ring.

The complex (2.1) computes q-de Rham cohomology, or Aomoto-Jackson cohomology of $\operatorname{Spec}(R)$. But to compute Habiro cohomology, we use a closely related variant based on a modified q-derivative.

Definition 2.10 (Modified q-Derivative)**.** Let R be $\mathbb{Z}[q^{\pm}][T_1, \dots, T_d]$ or $\mathbb{Z}[q^{\pm}][T_1^{\pm}, \dots, T_d^{\pm}]$. The modified q-derivative is given by

$$\widetilde{\nabla}_i^q(f(T_1, \dots, T_d)) = \frac{f(T_1, \dots, qT_i, \dots, T_d) - f(T_1, \dots, T_i, \dots, T_d)}{T_i}.$$

Remark 2.11. In other words, $\widetilde{\nabla}_i^q(f) = (q-1)\nabla_i^q(f) = \frac{\gamma_i(f)-f}{T_i}$.

Recomputing everything using this modified derivative gives the q-Hodge complex.

Definition 2.12 (q-Hodge Complex of $\mathbb{A}^d_{\mathbb{Z}}$ and $\mathbb{G}^d_m$)**.** Let $R = \mathbb{Z}[q^{\pm}][\underline{T}]$ be $\mathbb{Z}[q^{\pm}][T_1, \dots, T_d]$ or $\mathbb{Z}[q^{\pm}][T_1^{\pm}, \dots, T_d^{\pm}]$. The q-Hodge complex $q\text{-}\mathsf{Hdg}_R$ of $\operatorname{Spec}(R)$ is the complex

$$0 \longrightarrow \mathbb{Z}[q^{\pm}][\underline{T}] \longrightarrow \mathbb{Z}[q^{\pm}][\underline{T}]^{\oplus d} \longrightarrow \bigoplus_{i<j} \mathbb{Z}[q^{\pm}][\underline{T}] \longrightarrow \dots \tag{2.2}$$
$$\dots \longrightarrow \mathbb{Z}[q^{\pm}][\underline{T}] \longrightarrow 0$$

with differentials given by the differentials for the Koszul complex of commuting operators $\widetilde{\nabla}^1_1, \dots, \widetilde{\nabla}^q_d$.

Remark 2.13. The nomenclature of Definitions 2.7 and 2.12 are justified by the fact that they recover the ordinary de Rham and Hodge complexes at $q = 1$.

Remark 2.14. An automorphism of $\mathbb{A}^d_{\mathbb{Z}}$ or $\mathbb{G}^d_m$ would give rise to an automorphism of the complexes (2.1) and (2.2), at least as an object in the derived category, but it is extremely difficult to understand these automorphisms from this explicit perspective.

The instructor remarks that he does not believe in non-flat connections. We will henceforth omit the adjective "flat."

In parallel to the correspondence between algebraic D-modules and modules with flat connection, one would expect the existence of a category of modules with an approrpiate connection to play the role of $\mathcal{D}_{\mathsf{Hab}}(X)$ alluded to earlier. To make this precise, we consider modules with q-connection. To simplify matters, we make these considerations on the Abelian and not ∞-categorical level.

Note that a q-connection is additional data on a module.

Definition 2.15 (q-Connections on Modules)**.** Let $R = \mathbb{Z}[q^{\pm}][\underline{T}]$ be $\mathbb{Z}[q^{\pm}][T_1, \dots, T_d]$ or $\mathbb{Z}[q^{\pm}][T_1^{\pm}, \dots, T_d^{\pm}]$. A module with (flat) q-connection is a $\mathbb{Z}[q^{\pm}][\underline{T}]$-module with commuting $\mathbb{Z}[q^{\pm}]$-linear operations $\nabla^q_{i,M} : M \to M$ which satisfy the q-Leibniz rule

$$\nabla^q_{i,M}(fm) = \gamma_i(f) \cdot \nabla^q_{i,M}(m) + \nabla^q_i(f) \cdot m$$

for $f \in \mathbb{Z}[q^{\pm}][\underline{T}]$ and $m \in M$.

Remark 2.16. To unwind any possible confusion between the similar-looking $\nabla^q_i : \mathbb{Z}[q^{\pm}][\underline{T}] \to \mathbb{Z}[q^{\pm}][\underline{T}], \nabla^q_{i,M} : M \to M$, we have

$$\underbrace{\underbrace{\gamma_i(f)}_{\in \mathbb{Z}[q^{\pm}][\underline{T}]} \cdot \underbrace{\nabla^q_{i,M}(m)}_{\in M}}_{\in M} + \underbrace{\underbrace{\nabla^q_i(f)}_{\in \mathbb{Z}[q^{\pm}][\underline{T}]} \cdot \underbrace{m}_{\in M}}_{\in M}$$

so everything type-checks.

Example 2.17. If $X = \mathbb{A}^1_{\mathbb{Z}}$ then recall that modules with connection are equivalent to modules over the Weyl algebra $\mathbb{Z}[q^{\pm}]\{T, \partial_q\}/(qT\partial_q - \partial_q T + 1)$ since we have the operators $T\partial_q, \partial_q T$ take T^n to $q[n]_q T^n$, $[n+1]_q T^n$, respectively, but $q[n]_q - [n+1]_q = q \cdot \frac{1-q^n}{1-q} - \frac{1-q^{n+1}}{1-q} = -1$. Passing to the associated-graded of the degree filtration, one gets commuting variables with the correct q-twists.

Similarly, we can construct modules with a modified q-connection.

Definition 2.18 (Modified q-Connections on Modules)**.** Let $R = \mathbb{Z}[q^{\pm}][\underline{T}]$ be $\mathbb{Z}[q^{\pm}][T_1, \ldots, T_d]$ or $\mathbb{Z}[q^{\pm}][T_1^{\pm}, \ldots, T_d^{\pm}]$. A module with modified q-connection is a $\mathbb{Z}[q^{\pm}][\underline{T}]$-module with commuting $\mathbb{Z}[q^{\pm}]$-linear operations $\widetilde{\nabla}^q_{i,M} : M \to M$ which satisfy the q-Leibniz rule

$$\widetilde{\nabla}^q_{i,M}(fm) = \gamma_i(f) \cdot \widetilde{\nabla}^q_{i,M}(m) + \widetilde{\nabla}^q_i(f) \cdot m$$

for $f \in \mathbb{Z}[q^{\pm}][\underline{T}]$ and $m \in M$.

Remark 2.19. For a more in-depth discussion of modules with q-connection, see Morrow-Tsuji [MT21] and André [And01].

Remark 2.20. Let T_i be invertible. Unwinding the definition of the modified q-derivative, we have

$$\widetilde{\nabla}^q_{i,M}(fm) = \gamma_i(f) \cdot \widetilde{\nabla}^q_{i,M}(m) + (q-1)\nabla^q_i(f) \cdot m$$

where in particular we observe that the second summand has denominator T_i. Define a new operator

$$\widetilde{\widetilde{\nabla}}^q_{i,M} = T_i \cdot \widetilde{\nabla}^q_{i,M}$$

which satisfies

$$\begin{aligned} \widetilde{\widetilde{\nabla}}^q_{i,M}(fm) &= \gamma_i(f) \cdot \widetilde{\widetilde{\nabla}}^q_{i,M}(m) + (\gamma_i(f) - f)m \\ &= \gamma_i(f) \left(\widetilde{\widetilde{\nabla}}^q_{i,M}(m) + m \right) - fm. \end{aligned}$$

In particular,

$$\left(\widetilde{\widetilde{\nabla}}^q_{i,M} + \mathrm{id}_M \right)(fm) = \gamma_i(f) \left(\widetilde{\widetilde{\nabla}}^q_{i,M} + \mathrm{id}_M \right)(m)$$

so denoting $\gamma_{i,M} = \left(\widetilde{\widetilde{\nabla}}^q_{i,M} + \mathrm{id}_M \right)$, we have $\gamma_{i,M}(fm) = \gamma_i(f)\gamma_{i,M}(m)$ simplyfing the relation.

The preceding discussion of Remark 2.20 implies the following.

Corollary 2.21. Let $R = \mathbb{Z}[q^{\pm}][T_1^{\pm}, \ldots, T_d^{\pm}]$. There is an equivalence of categories between R-modules with modified q-connection and R-modules with commuting $\gamma_i : R \to R$-semilinear endomorphisms $\gamma_{i,M} : M \to M$.

Note that for $R = \mathbb{Z}[q^{\pm}][\underline{T}]$, $(-) \otimes_R (-)$ does not define a symmetric monoidal structure on the category of modules with q-connection: for $(M, \nabla^q_{i,M}), (N, \nabla^q_{i,N})$ two modules with q-connecition,

$$(M \otimes_R N, \nabla^q_{i,M} \otimes_R \mathrm{id}_N + \mathrm{id}_M \otimes_R \nabla^q_{i,N})$$

is not a module with q-connection. One needs instead to take the twist

$$(M \otimes_R N, \nabla^q_{i,M} \otimes_R \mathrm{id}_N + \gamma_{i,M} \otimes_R \nabla^q_{i,N}),$$

defining $\gamma_{i,M} : M \to M$ in an analogous way to Remark 2.20. While *a priori* appearing assymetric in M, N, there is in fact a canonical isomorphism between them.

Proposition 2.22. Let $R = \mathbb{Z}[q^{\pm}][T_1^{\pm}, \ldots, T_d^{\pm}]$. The category of R-modules with q-connection is symmetric monoidal.

Proof Outline. Using the equivalence of Corollary 2.21, the latter category is symmetric monoidal, hence the former can be promoted to a symmetric monoidal category. ■

Proposition 2.23. Let $R = \mathbb{Z}[q^{\pm}][T_1^{\pm}, \ldots, T_d^{\pm}]$. There is a fully faithful embedding from $(q-1)$-torsion free R-modules with q-connection and R-modules with modified q-connection by $(M, \nabla_{i,M}^q) \mapsto (M, \widetilde{\nabla}_{i,M}^q)$ with essential image those that are $(q-1)$-torsion free and such that $\widetilde{\nabla}_{i,M}^q \equiv 0 \pmod{(q-1)}$.

The discussion thus far has been done entirely in terms of coordinates. This prompts:

Question 2.24. To what extent are the cohomologies and categories discussed thus far independent of coordinates?

Let us consider the following example.

The instructor remarks that in the theory of analytic geometry the quotient would be the Tate elliptic curve for $d = 1$. See [CS23].

Example 2.25. Let $X = \mathbb{G}_m^d$. The modules with modified q-connection are quasicoherent sheaves on $(\mathbb{G}_m/q^{\mathbb{Z}})^d$ – the γ_i's act by multiplication by q on the coordinates so the data of the endomorphisms $\gamma_{i,M}$ on the modules prescribe descent data to the quotient stack (ie. as an fpqc quotient).

Let us relate the discussion of complexes Definitions 2.7 and 2.12, their cohomologies, and these categories of modules with (modified) q-connections.

Proposition 2.26. Let $R = \mathbb{Z}[q^{\pm}][T_1^{\pm}, \ldots, T_d^{\pm}]$.

(i) The q de Rham complex $q\text{-}\Omega^{\bullet}_{R/\mathbb{Z}}$ computes $R\mathrm{Hom}_{q\text{-}\mathsf{Mod}_R}(\mathbb{1}, \mathbb{1})$ in the derived category of modules with q-connection $q\text{-}\mathsf{Mod}_R$ on $\mathrm{Spec}(R)$.

(ii) The q-Hodge complex $q\text{-}\mathsf{Hdg}_R$ computes $R\mathrm{Hom}_{q\text{-}\widetilde{\mathsf{Mod}}_R}(\mathbb{1}, \mathbb{1})$ in the derived category of modules with modified q-connection $q\text{-}\widetilde{\mathsf{Mod}}_R$ on $\mathrm{Spec}(R)$.

Proof Outline of (i). Using the equvialence between modules with q-connection and modules over the Weyl algebra, we compute a resolution of the symmetric monoidal unit $\mathbb{Z}[q^{\pm}][\underline{T}]$ in the category of modules over the Weyl algebra – which precisely recovers the de Rham complex, whence the claim. ■

Example 2.27. Consider the case of $\mathbb{A}^1_{\mathbb{Z}}$ taking $R = \mathbb{Z}[q^{\pm}][T]$. We compute $R\mathrm{Hom}(\mathbb{Z}[q^{\pm}][T], \mathbb{Z}[q^{\pm}][T])$ as $R\mathrm{Hom}(-, \mathbb{Z}[q^{\pm}][T])$ of a free resolution of $\mathbb{Z}[q^{\pm}][T]$ in the category of modules over the Weyl algebra $\mathbb{Z}[q^{\pm}]\{T, \partial_q\}/(qT\partial_q - \partial_q T + 1)$ (vis. Example 2.17). This produces

$$0 \to \mathbb{Z}[q^{\pm}][T] \xrightarrow{\nabla_1^q} \mathbb{Z}[q^{\pm}][T] \to 0$$

which is the q-de Rham complex (after passing back to modules with q-connection along the equivalence).

Moreover, in the setting of higher algebra, these promote canonically to commutative algebra objects.

Corollary 2.28. The q-de Rham complex and q-Hodge complex have canonical structures as $\mathbb{E}_\infty$-rings.

3. Lecture 3 – 9th May 2025

In Lecture 2, we constructed the q-de Rham and q-Hodge complexes for $\mathbb{A}^d_{\mathbb{Z}}, (\mathbb{G}_m)^d$ using the q-derivatives ∇_i^q and modified q-derivatives $\widetilde{\nabla}_i^q$, respectively.

We now consider the construction of the q-de Rham and q-Hodge complexes more generally in the case where the T_i are invertible and using the logarithmic q-derivative $\nabla_i^{q,\log} = T_i \nabla_i^q$. We first define these in the case $R = \mathbb{Z}[q^{\pm}][\underline{T}^{\pm}] = \mathbb{Z}[q^{\pm}][T_1^{\pm}, \dots, T_d^{\pm}]$.

Definition 3.1 (Logarithmic q-Derivative)**.** Let $R = \mathbb{Z}[q^{\pm}][T_1^{\pm}, \dots, T_d^{\pm}]$. The logarithmic q-derivative $\nabla_i^{q,\log} : R \to R$ for $1 \leq i \leq d$ is defined by

$$\nabla_i^q(f) = \frac{\gamma_i(f) - f}{q-1}.$$

Definition 3.2 (Modified Logarithmic q-Derivative)**.** Let $R = \mathbb{Z}[q^{\pm}][T_1^{\pm}, \dots, T_d^{\pm}]$. The modified logarithmic q-derivative $\widetilde{\nabla}_i^{q,\log} : R \to R$ for $1 \leq i \leq d$ is defined by

$$\widetilde{\nabla}_i^q(f) = \gamma_i(f) - f.$$

Remark 3.3. The commutation relation for the ordinary logarithmic q-derivative are given by $\gamma_i T_i = qT_i\gamma_i$ since multiplying by T_i and applying the map $T_i \mapsto qT_i$ is the same as applying the map $T_i \mapsto qT_i$ and multiplying by qT_i.

Example 3.4. Using Remark 3.3, we deduce that category of logarithmic q-connections on $\mathbb{G}_m$ are modules over the ring $\mathbb{Z}[q^{\pm}]\{T^{\pm}, \gamma\}/(\gamma T - qT\gamma)$ (cf. Example 2.17).

We undertake the task of constructing the q-de Rham and q-Hodge complexes for general smooth $\mathbb{Z}$-schemes X locally admitting an étale framing. For simplicity, we will restrict our attention to the case where $X = \operatorname{Spec}(R)$ with R a smooth $\mathbb{Z}$-algebra and $\square : X \to (\mathbb{G}_m)^d$ is étale (equivalently, $\mathbb{Z}[T_1^{\pm}, \dots, T_d^{\pm}] \to R$ étale).

If we were to mirror the constructions of Definitions 2.15 and 2.18, we would want to produce $R[q^{\pm}]$-modules with commuting semilinear endomorphisms $\gamma_{i,M} : M \to M$ (used to produce $\nabla^q_{i,M}, \widetilde{\nabla}^q_{i,M}$). This semilinearity ought be defined in terms of $\gamma_{i,R} : R[q^{\pm}] \to R[q^{\pm}]$ which extend γ_i on $\mathbb{Z}[q^{\pm}][\underline{T}^{\pm}]$, but there is no reason such maps should exist. Put in other – more geometric – terms, the automorphisms γ_i on $(\mathbb{G}_m)^d$ need not lift along the map $\square : X \to (\mathbb{G}_m)^d$.

Completion allows us to resolve this issue: after $(q-1)$-adic completion, there are unique such $\gamma_{i,R} : R[[q-1]] \to R[[q-1]]$ restricting to the identity modulo $(q-1)$. This is a consequence of the infintesmal lifting property for (formally) étale maps [Stacks, Tag 00UP]:

$$\begin{tikzcd}
\mathbb{Z}[[q-1]][\underline{T}^{\pm}] \arrow[r, "\square\circ\gamma_i"] \arrow[d, "\square"'] & R[[q-1]] \arrow[d, "(\mathrm{mod}\ (q-1))"] \\
R[[q-1]] \arrow[r, "(\mathrm{mod}\ (q-1))"'] \arrow[ur, dotted, "\exists!\gamma_{i,R}"] & R.
\end{tikzcd}$$

More formally, $\square : \mathbb{Z}[\underline{T}^{\pm}] \to R$ is étale, and étaleness is preserved under base change, so $\square : \mathbb{Z}[[q-1]][\underline{T}^{\pm}] \to R[[q-1]]$ is étale and $R[[q-1]] \to R$ is an infinitesmal thickening, so the desired lift exists rendering the entire diagram commutative. Geometrically, $(q-1)$-adic completion the automorphisms γ_i on $(\mathbb{G}_m)^d$ are infinitesmally close to the identity, hence lift uniquely along the framing map (that is, the framing map on schemes $\square : \mathrm{Spec}(R[[q-1]]) \to \mathbb{Z}[[q-1]][\underline{T}^{\pm}]$). This allows us to define (modified/logarithmic) q-derivatives and the notion of modules with (modified/logarithmic) q-connection. This notion is illustrated in the following equivalence of categories.

Lemma 3.5. There is an equivalence of categories

$$\left\{\begin{matrix}\text{étale } \mathbb{Z}[q^{\pm}][\underline{T}^{\pm}]/(q-1)^n \\ \text{algebras}\end{matrix}\right\} \simeq \left\{\begin{matrix}\text{étale } \mathbb{Z}[\underline{T}^{\pm}] \\ \text{algebras}\end{matrix}\right\}.$$

Proof. See [Stacks, Tag 039R]. ∎

Theorem 3.6 (Bhatt-Scholze, [BS22, §16]; Wagner, [Wag24, Thm. 1.5])**.** Let $(R, \square)$ be a smooth framed $\mathbb{Z}$-algebra. The q-de Rham complex $q\text{-}\Omega^{\bullet}_{(R,\square)/\mathbb{Z}[[q-1]]}$ given by

$$R[[q-1]] \xrightarrow{(\nabla_i^q)_{i=1}^d} \bigoplus_{i=1}^{d} R[[q-1]] \longrightarrow \dots$$

as an object of $\mathcal{D}(\mathbb{Z}[[q-1]])$ is canonically independent of the choice of coordinates.

Such coordinate independence is somewhat easy to deduce in the case where R is a $\mathbb{Q}$-algebra.

Example 3.7 ([Sch17, Lem. 4.1];[BMS18, Lem. 12.4])**.** Consider the case of a smooth framed $\mathbb{Q}$-algebra $(R, \square)$ where $\square : \mathrm{Spec}(R) \to \mathbb{G}_m$. We can use Taylor's theorem to write

$$f(qT) = f(T) + \log(q)(\nabla^{\log} f)(T) + \frac{1}{2}\log(q)^2((\nabla^{\log})^2 f)(T) + \dots$$

where $\log(q) = \sum_{n\geq 0}(-1)^{n-1}\frac{(q-1)^n}{n} \in \mathbb{Q}[[q-1]]$ so taking the difference of $f(qT)$ and $f(T)$, we find the operators $\nabla^{q,\log}, \widetilde{\nabla}^{q,\log}$ are given by

$$\nabla^{q,\log} = \frac{\log(q)}{(q-1)}(\nabla^{\log} f)(T) + \frac{1}{2}\frac{\log(q)^2}{(q-1)}((\nabla^{\log})^2 f)(T) + \dots$$
$$\widetilde{\nabla}^{q,\log} = \log(q)(\nabla^{\log} f)(T) + \frac{1}{2}\log(q)^2((\nabla^{\log})^2 f)(T) + \dots.$$

Using that $\widetilde{\nabla}^{\log} = \log(q)\nabla^{\log}$ we get

$$\widetilde{\nabla}^{q,\log} = \widetilde{\nabla}^{\log} + \frac{1}{2}(\widetilde{\nabla}^{\log})^2 + \dots$$

we get that $\widetilde{\nabla}^{q,\log} = \exp(\widetilde{\nabla}^{\log}) + 1$. In particular, for smooth framed $\mathbb{Q}$-algebras, the data of modified logarithmic q-connections are equivalent to modified logarithmic connections up to a transformation, and allow us to interpolate between the two structures.

Example 3.7 yields the following more general result.

Proposition 3.8. Let $(R, \square)$ be a smooth framed $\mathbb{Q}$-algebra. There is an symmetric monoidal equivalence of categories

$$\left\{\begin{matrix}(q-1)\text{-adically complete } R[[q-1]] \\ \text{-modules with } q\text{-connection}\end{matrix}\right\} \simeq \left\{\begin{matrix}(q-1)\text{-adically complete } R[[q-1]] \\ \text{-modules with connection}\end{matrix}\right\}.$$

Moreover, these categories are independent of choice of coordinates on $R[[q-1]]$.

Proof Outline. The computation of Example 3.7 in several variables (cf. [Sch17, Lem. 4.1]) shows an equivalence of data between modified logarithmic q-connections and modified logarithmic connections, and since we are working over $\mathbb{Q}$ and the torus, these are the same as ordinary (q-)connections. Thus for a fixed $(q-1)$-adically complete $R[[q-1]]$-module M with q-connection, there is a unique ordinary connection with which it can be endowed, and conversely.

The latter statement follows from the observation that the latter category of $(q-1)$-adically complete $R[[q-1]]$-modules with connection are visibly coordinate independent. ∎

As in the case of $(\mathbb{G}_m)^d$ in Proposition 2.26 (i), we have in this case the following result.

Corollary 3.9. Let $(R, \square)$ be a framed $\mathbb{Q}$-algebra and denote the category of $(q-1)$-adically complete $R[[q-1]]$-modules with q-connection by $q\text{-}\mathsf{Mod}_{R[[q-1]]}$. The q-de Rham complex $q\text{-}\Omega^\bullet_{(R,\square)/\mathbb{Q}}$ computes $R\mathrm{Hom}_{q\text{-}\mathsf{Mod}_{R[[q-1]]}}(\mathbb{1}, \mathbb{1})$ and is canonically independent of coordinates.

We seek to treat the case of modified q-connections expressing logarithmic connections in terms of ordinary ones, but the case of modified q-connections is more subtle as the convergence of the exponential becomes problematic.

To that end, we introduce the following notion.

Definition 3.10 (h-Connections)**.** Let R be a $\mathbb{Q}$-algebra. A h-connection over $R[h]$ is an $R[h]$-module M with a map $\widetilde{\nabla}_M : M \to M \otimes_R \Omega^1_{R/\mathbb{Q}}$ satisfying $(\widetilde{\nabla}_M)^2 : M \to M \otimes_R \Omega^2_{R/\mathbb{Q}}$ the zero map and

$$\widetilde{\nabla}_M(fm) = h \cdot \nabla(f) \cdot m + f \cdot \widetilde{\nabla}_M(m).$$

Remark 3.11. Such constructions are known as λ-connections in the literature and appear in Hodge and twistor theory. Specialization at $h = 1$ recovers the ordinary notion of a connection, while specialization at $h = 0$ gives a Higgs field (cf. [MT21, §2.3]).

We would like to see an analogue of Proposition 3.8.

Proposition 3.12. Let $(R, \square)$ be a smooth framed $\mathbb{Q}$-algebra. There is an symmetric monoidal equivalence of categories

$$\left\{\begin{matrix}(q-1)\text{-adically complete } R[[q-1]]\text{-modules} \\ \text{with modified } q\text{-connection s.t. } \widetilde{\nabla}^{q,\log}_{i,M}\text{'s are top. nil.}\end{matrix}\right\} \simeq \left\{\begin{matrix}h\text{-adically complete } R[[h]]\text{-modules} \\ \text{with } h\text{-connection s.t. } \widetilde{\nabla}_M \text{ is top. nil.}\end{matrix}\right\}$$

$$\left(M, (\widetilde{\nabla}^{q,\log}_{i,M})_{i=1}^d\right) \longleftarrow\!\!\mapstochar\!\!\longrightarrow \left(M, (\widetilde{\nabla}^{\log}_{i,M})_{i=1}^d\right)$$

where $h = (q-1)$ and $\widetilde{\nabla}^{q,\log}_{i,M} = \exp(\widetilde{\nabla}^{\log}_{i,M}) - 1$. Moreover, these categories are independent of choice of coordinates on $R[[q-1]] \cong R[[h]]$.

Proof Outline. Observing that the topological nilpotence of the operators imply convergence of the exponential, the proof outline of Proposition 3.8 goes through verbatim. ■

Remark 3.13. The topological nilpotence condition is typically satisfied in practice. Regardless, this is likely the best construction one can hope for – it is unlikely that one can produce an equivalence on larger categories.

Once again, observing that the right hand side is coordinate independent, we get coordinate independence for modules with modified q-connections.

Corollary 3.14. Let $(R, \square)$ be a framed $\mathbb{Q}$-algebra and denote the category of $(q-1)$-adically complete $R[[q-1]]$-modules with modified q-connection where the operators $\widetilde{\nabla}^{q,\log}_{i,M}$ are topologically nilpotent by $q\text{-}\widetilde{\mathsf{Mod}}_{R[[q-1]]}$. The q-Hodge complex $q\text{-}\mathsf{Hdg}_{(R,\square)/\mathbb{Q}}$ given by

$$R[[q-1]] \xrightarrow{(\widetilde{\nabla}^q_i)_{i=1}^d} \bigoplus_{i=1}^d R[[q-1]] \longrightarrow \dots$$

computes $R\mathrm{Hom}_{q\text{-}\widetilde{\mathsf{Mod}}_{R[[q-1]]}}(\mathbb{1}, \mathbb{1})$ and is canonically independent of coordinates.

Remark 3.15. The operators are topologically nilpotent on the symmetric monoidal unit as they are defined by mutliplying the ordinary operator by $(q-1)$. Moreover, the Ext-terms classify extensions, which remain topologically nilpotent. Thus the $R\mathrm{Hom}(-,-)$ computation in this case remains unaffected by passage to the subcategory where the operations are topologically nilpotent.

Deferring the discussion of coordinate independence integrally – which can be done by similarly isolating subcategories of modules with convergence conditions on their q-connections – we seek to understand the preceding constructions of modules with (modified) q-connections not just in the $(q-1)$-adically complete but more generally in the Habiro case, namely at all roots of unity.

In the preceding discussion, $(q-1)$-adic completion allowed us to leverage étaleness of the map to produce a unique lift of the endomorphism on $\mathbb{Z}[q^{\pm}][\underline{T}^{\pm}]$ since γ_i was infinitesmally close to the identity after $(q-1)$-adic completion. But noticing that ζ_p is p-adically close to 1, we can attempt a similar approach using $(q-\zeta_p)$-adic completion.

Example 3.16. Let $(R, \square)$ be a framed $\mathbb{Z}$-algebra with $\square : \mathrm{Spec}(R) \to \mathbb{G}_m$. This gives a map

$$\mathbb{Z}[T^{\pm}]^{\wedge}_p[[q-1]] \longrightarrow R^{\wedge}_p[[q-1]]$$

which on specialization to $q = \zeta_p$ yields

$$\mathbb{Z}_p[\zeta_p]\langle T^{\pm}\rangle \longrightarrow R_p^{\wedge}[\zeta_p]$$

where using that ζ_p is close to 1 p-adically, $\gamma : \mathbb{Z}_p[\zeta_p]\langle T^{\pm}\rangle \to \mathbb{Z}_p[\zeta_p]\langle T^{\pm}\rangle$ by $T \mapsto \zeta_p T$ lifts uniquely to an endomorphism $\gamma_R : R_p^{\wedge}[\zeta_p] \to R_p^{\wedge}[\zeta_p]$. However, $R[\zeta_p] \hookrightarrow R_p^{\wedge}[\zeta_p]$ may not have image stable under γ_R, for example, $\mathbb{G}_m \setminus \{1\}$ is not stable under multiplication by ζ_p.

So as seen in the example above, we will require an alternative description. For this, we produce an endomorphism of $R_p^{\wedge}[\zeta_p]$ that does globalize to all of R using the Frobenius.

Example 3.17. The Frobenius map $\varphi : \mathbb{Z}[T^{\pm}] \to \mathbb{Z}[T^{\pm}]$ by $T \mapsto T^p$ lifts uniquely to $R_p^{\wedge}$ and reduces to the pth power map $\varphi_{R/(p)} : R/(p) \to R/(p)$ modulo (p). Via the Frobenius, we get an isomorphism

$$\begin{array}{ccc} R_p^{\wedge} \otimes_{\mathbb{Z}[T^{\pm}]} \mathbb{Z}[T^{\pm 1/p}] & \xrightarrow{\quad\varphi\quad} & R_p^{\wedge} \\ \\ T^{1/p} & \longmapsto & T \end{array}$$

since after p-completion $R_p^{\wedge}$ is finite free over itself – $R_p^{\wedge}$ contains itself as a subring where "only p-powers are allowed."

Example 3.18. Building on Example 3.17, we similarly have an isomorphism

$$\begin{array}{ccc} R_p^{\wedge} \otimes_{\mathbb{Z}[T^{\pm}]} \mathbb{Z}[\zeta_p][T^{\pm 1/p}] & \xrightarrow{\quad\varphi\quad} & R_p^{\wedge}[\zeta_p] \\ \\ T^{1/p} & \longmapsto & T \end{array}$$

where the automorphism $\gamma_{R_p^{\wedge}[\zeta_p]} : R_p^{\wedge}[\zeta_p] \to R_p^{\wedge}[\zeta_p]$ by $T \mapsto \zeta_p T$ lifts to the automorphism $\mathrm{id}_{R_p^{\wedge}} \otimes [T^{1/p} \mapsto \zeta_p T^{1/p}]$ on $R_p^{\wedge} \otimes_{\mathbb{Z}[T^{\pm}]} \mathbb{Z}[\zeta_p][T^{\pm 1/p}]$. Observe that since $T \in R_p^{\wedge} \otimes_{\mathbb{Z}[T^{\pm}]} \mathbb{Z}[\zeta_p][T^{\pm 1/p}]$ is sent to $T^p \in R_p^{\wedge}[\zeta_p]$, it is fixed by $\gamma_{R_p^{\wedge}[\zeta_p]}$ as $\gamma_{R_p^{\wedge}[\zeta_p]}(T^p) = (\zeta_p T)^p = T^p$. Evidently we have an inclusion into the p-completion

$$R \otimes_{\mathbb{Z}[T^{\pm}]} \mathbb{Z}[\zeta_p][T^{\pm 1/p}] \hookrightarrow R_p^{\wedge} \otimes_{\mathbb{Z}[T^{\pm}]} \mathbb{Z}[\zeta_p][T^{\pm 1/p}]$$

but since $\gamma_{R_p^{\wedge}[\zeta_p]}$ is given by the identity on the $R_p^{\wedge}$ factor, it is stable under the automorphism, resolving the issue posed at the end of Example 3.16.

This motivates the following definiton.

The notation here is that of the author, not of the instructor.

Definition 3.19 (Root of Unity Algebra)**.** Let $(R, \square)$ be a smooth framed $\mathbb{Z}$-algebra and $m \geq 1$. Define the mth root of unity algebra $R^{(m)}$ to be the étale $\mathbb{Z}[\zeta_m, \underline{T}^{\pm}]$-algebra

$$R^{(m)} = R \otimes_{\mathbb{Z}[\underline{T}^{\pm}]} \mathbb{Z}[\zeta_m][\underline{T}^{\pm 1/m}]$$

where $\mathbb{Z}[\zeta_m, \underline{T}^{\pm}]$-algebra structure on $R^{(m)}$ is by $T_i \mapsto T_i^{1/m}$ and equipped with the automorphism $\gamma_i^{(m)} = \mathrm{id}_R \otimes [T_i \mapsto \zeta_m T_i]$ lifting $T_i \mapsto \zeta_m T_i$ on $\mathbb{Z}[\zeta_m, \underline{T}^{\pm}]$.

This gives a well-defined construction.

Example 3.20. Let $X = \mathbb{G}_m \setminus \{1\} = \operatorname{Spec}(\mathbb{Z}[T^{\pm}, \frac{1}{1-T}])$. The associated root of unity algebra $R^{(m)}$ is given by $\mathbb{Z}[\zeta_m][T^{\pm}, \frac{1}{1-T^m}]$ and the Zariski spectrum is visibly stable under $T \mapsto \zeta_m T$.

By unique deformations of étale algebras, the construction of Definition 3.19 extends from the specialization at $q = \zeta_m$ to $\Phi_m(q)$-adic completion, where $\Phi_m(q)$ is the mth cyclotomic polynomial. More formally:

Once again, author's notation.

Definition 3.21 (Completed Root of Unity Algebra)**.** Let $(R, \square)$ be a smooth framed $\mathbb{Z}$-algebra and $m \geq 1$. Define the completed mth root of unity algebra R_m to be the formally étale $\mathbb{Z}[q^{\pm}][\underline{T}^{\pm}]^{\wedge}_{\Phi_m(q)}$-algebra $(R^{(m)}[q^{\pm}])^{\wedge}_{\Phi_m(q)}$ and equipped with the automorphism $\gamma_{i,R_m} : R_m \to R_m$ lifting $\gamma_i : \mathbb{Z}[q^{\pm}][\underline{T}^{\pm}]^{\wedge}_{\Phi_m(q)} \to \mathbb{Z}[q^{\pm}][\underline{T}^{\pm}]^{\wedge}_{\Phi_m(q)}$.

Remark 3.22. The lifting diagram is given by

$$\begin{tikzcd}
\mathbb{Z}[q^{\pm}][\underline{T}^{\pm}]^{\wedge}_{\Phi_m(q)} \arrow[r, "\square \circ \gamma_i"] \arrow[d, "\square"'] & R_m \arrow[d, "(\mathrm{mod}\ \Phi_m(q))"] \\
R_m \arrow[r, "(\mathrm{mod}\ \Phi_m(q))"'] \arrow[ur, dotted, "\exists! \gamma_{i,R_m}"] & R.
\end{tikzcd}$$

Having produced these étale algebras R_m and their endomorphisms γ_{i,R_m} for each root of unity ζ_m, we can similarly define derivatives and modules with (modified) connections over these rings. *A priori*, these give distinct constructions for each $m \geq 0$, but we can seek to combine them using a "Habiro ring"-like construction in a by gluing where roots of unity meet in positive characteristic. In particular, $\zeta_{pm} = \zeta_m$ in characteristic p, so the Frobenius gives an isomorphism

$$\begin{array}{ccc} R^{(pm)}/(p) \cong R/(p) \otimes_{\mathbb{F}_p[\underline{T}^{\pm}]} \mathbb{F}_p\left[\zeta_{pm}, \underline{T}^{\pm 1/pm}\right] & \longrightarrow & R/(p) \otimes_{\mathbb{F}_p[\underline{T}^{\pm}]} \mathbb{F}_p\left[\zeta_m, T^{\pm 1/m}\right] \cong R^{(m)}/(p) \\ T^{1/pm} & \longmapsto & T^{1/m}. \end{array} \tag{3.1}$$

Moreover, these constructions deform uniquely over the $\Phi_m(q)$-adic completions allowing us to define the Habiro ring of a framing and repeat the process to define categories of modules with (modified) q-connections over the Habiro ring of a framing $\mathcal{H}_{(R,\square)}$ which is an algebra over the Habiro ring of the torus $\mathcal{H}_{\mathbb{Z}[\underline{T}^{\pm}]}$, where we have defined the latter to be as follows.

Definition 3.23 (Habiro Ring of $(\mathbb{G}_m)^d$)**.** The Habiro ring of $(\mathbb{G}_m)^d$ is given by the completion $\mathcal{H}_{\mathbb{Z}[\underline{T}^{\pm}]} = \lim_{n \in \mathbb{N}} \mathbb{Z}[q^{\pm}][\underline{T}^{\pm}]^{\wedge}_{\Phi_n(q)}$ where $\Phi_n(q)$ is the nth cyclotomic polynomial.

4. Lecture 4 – 23rd May 2025

Using the gluing procedure of (3.1) allows us to correct for the overspecification of prescribing a local algebra $R^{(m)}$ for each positive integer m in characteristic p – that is, gluing $R^{(m)}, R^{(m')}$ where m_0 is coprime to p and $m = m_0 p^a, m' = m_0 p^b$ using the Frobenius.

More generally, we can define the Habiro ring of a smooth framed $\mathbb{Z}$-algebra $(R, \square)$ by passing to the limit of the rings R_n where there are surjective transition maps $R_{pm} \to R_m$ given by the (necessarily unique) lift of the isomorphism (3.1) along the (necessarily unique) deformation of étale algebras $R^{(m)}$ to R_m.

Definition 4.1 (Habiro Ring of Framed Algebra)**.** Let $(R, \square)$ be a smooth framed $\mathbb{Z}$-algebra. The Habiro ring $\mathcal{H}_{(R,\square)}$ is given by the limit

$$\mathcal{H}_{(R,\square)} = \lim_{n\in\mathbb{N}} R_n$$

where R_n is the completed root of unity algebra of Definition 3.21.

Proposition 4.2. Let $(R, \square)$ be a smooth framed $\mathbb{Z}$-algebra. The Habiro ring $\mathcal{H}_{(R,\square)}$ of $(R, \square)$ is given by

$$\mathcal{H}_{(R,\square)} = \left\{ (f_m)_{m\geq 1} \in \prod_{m\geq 1} R^{(m)}[[q-\zeta_m]] : \substack{\forall m\in\mathbb{N},\ \forall p \text{ prime} \\ \varphi_p(f_{pm}) = f_m \in (R^{(m)})^\wedge_p[[q-\zeta_m]] \cong (R^{(pm)})^\wedge_p[[q-\zeta_{pm}]]} \right\} \tag{4.1}$$

where φ_p lifts the Frobenius on $R^{(m)}/(p)$ by raising each variable to the p-th power and fixes q and ζ_m.

Remark 4.3. There is an obvious map from the Habiro ring of the torus Definition 3.23 $\mathcal{H}_{\mathbb{Z}[\underline{T}^{\pm}]} \to \mathcal{H}_{(R,\square)}$ endowing the Habiro ring of $(R, \square)$ with the structure of a $\mathcal{H}_{\mathbb{Z}[\underline{T}^{\pm}]}$-algebra.

Let us consider some explicit elements of the Habiro ring.

Example 4.4. Let $R = \mathbb{Z}[T_1, \ldots, T_d, \frac{1}{1-T_1-\cdots-T_d}]$ with framing $\square : \mathbb{Z}[T_1, \ldots, T_d] \to R$. The element

$$\sum_{k_1,\ldots,k_d\geq 0} \begin{bmatrix} k_1+\cdots+k_d \\ k_1 \ \ldots \ k_d \end{bmatrix}_q T_1^{k_1} \ldots T_d^{k_d} \in \mathbb{Z}[q][[\underline{T}]]$$

is an element of the Habiro ring $\mathcal{H}_{(R,\square)}$ where

$$\begin{bmatrix} k_1+\cdots+k_d \\ k_1 \ \ldots \ k_d \end{bmatrix}_q = \frac{(q;q)_{k_1+\cdots+k_d}}{(q;q)_{k_1}\ldots(q;q)_{k_d}}$$

is the q-deformation of the multinomial $\binom{k_1+\cdots+k_d}{k_1\ldots k_d}$. More generally, explicit elements of the Habiro ring can be constructed by considering q-deformations of rational functions (vis. Example 1.4 and surrounding discussion).

The lecture contained a fairly substantive sketch of the proof Example 4.4, which the author has defered to Appendix A for continuity of exposition.

Returning to a discussion of Habiro cohomology of a smooth $\mathbb{Z}$-algebra with framing $\square : \mathrm{Spec}(R) \to (\mathbb{G}_m)^d$, we recall that there are lifts of the automorphism γ_i to $\mathcal{H}_{(R,\square)}$: more explicitly, for a section $(f_m)_{m\geq 0}$, the action γ_i acts by $(f_m)_{m\geq 1} \mapsto (\gamma_i^{(m)}(f_m))_{m\geq 1}$ where $\gamma_i^{(m)}$ is the automorphism given in Definition 3.19.

This produces a $\mathbb{Z}^d$-action on $\mathcal{H}_{(R,\square)}$, and we can define Habiro-Hodge cohomology to be the group cohomology of the action of $\mathbb{Z}^d$ on $\mathcal{H}_{(R,\square)}$.

Definition 4.5 (q-Habiro-Hodge Cohomology)**.** Let $(R,\square)$ be a smooth framed $\mathbb{Z}$-algebra. The q-Habiro-Hodge cohomology is the cohomology of the complex $q\text{-}\mathcal{H}\mathsf{dg}_{(R,\square)}$ given by

$$\mathcal{H}_{(R,\square)} \xrightarrow{(\gamma_i-1)_{i=1}^d} \bigoplus_{i=1}^d \mathcal{H}_{(R,\square)} \longrightarrow \cdots.$$

For this to be functorial, we would expect this to be coordinate independent, at least at the level of derived categories. As a first step, we study the cohomology of the complex modulo $(1-q^m)$ – that is, at specalizations to roots of unity.

If $m=1$, then $\mathcal{H}_{(R,\square)}/(1-q) \cong R$ and all differentials are zero, so

$$H^i\left(q\text{-}\mathcal{H}\mathsf{dg}_{(R,\square)}/(1-q)\right) \cong R^{\oplus\binom{d}{i}} \cong \Omega^i_{R/\mathbb{Z}}$$

and is therefore independent of coordinates since the middle term is so.

Remark 4.6. While *a priori* we only have a isomorphism to a free module of a certain rank, there is additonal structure that allows us to identify this with the module of Kähler differentials: the Bockstein map associated to the triangle

$$q\text{-}\mathcal{H}\mathsf{dg}_{(R,\square)}/(1-q) \xrightarrow{\times(1-q)} q\text{-}\mathcal{H}\mathsf{dg}_{(R,\square)}/(1-q)^2 \longrightarrow q\text{-}\mathcal{H}\mathsf{dg}_{(R,\square)}/(1-q) \longrightarrow \left(q\text{-}\mathcal{H}\mathsf{dg}_{(R,\square)}/(1-q)\right)[1]$$

inducing

$$H^i\left(q\text{-}\mathcal{H}\mathsf{dg}_{(R,\square)}/(1-q)\right) \longrightarrow H^{i+1}\left(q\text{-}\mathcal{H}\mathsf{dg}_{(R,\square)}/(1-q)\right)$$

which gives a derivation

$$H^0\left(q\text{-}\mathcal{H}\mathsf{dg}_{(R,\square)}/(1-q)\right) \longrightarrow H^1\left(q\text{-}\mathcal{H}\mathsf{dg}_{(R,\square)}/(1-q)\right)$$

and hence an isomorphism $H^1\left(q\text{-}\mathcal{H}\mathsf{dg}_{(R,\square)}/(1-q)\right) \to \Omega^1_{R/\mathbb{Z}}$. In addition, the ring structure on cohomology induces the structure of a commutative differential graded algebra on $H^\bullet\left(q\text{-}\mathcal{H}\mathsf{dg}_{(R,\square)}/(1-q)\right)$ and this structure is in fact independent of coordinates on the nose and not just up to quasi-isomorphism.

For general m, $H^\bullet\left(q\text{-}\mathcal{H}\mathsf{dg}_{(R,\square)}/(1-q^m)\right)$ has the strucuture of a commutative differential graded algebra that is coordinate independent.

Theorem 4.7 (Wagner; [Wag22, Prop. 5.7])**.** Let R be a smooth framed $\mathbb{Z}$-algebra. There is a canonical surjection

$$W_m(R)[q]/(1-q^m) \longrightarrow H^0\left(q\text{-}\mathcal{H}\mathsf{dg}_{(R,\square)}/(1-q^m)\right)$$

inducing

$$\Omega_{W_m(R)[q]/(1-q^m)} \longrightarrow H^\bullet\left(q\text{-}\mathcal{H}\mathsf{dg}_{(R,\square)}/(1-q)\right)$$

which is coordinate independent, degreewise surjective, and with kernel independent of coordinates.

Proof Outline. For every commutative differential graded algebra B receiving a map from a commutative ring A in 0th cohomology, there is an induced map from the initial commutative differential graded algebra generated by A to B – the latter being the de Rham complex. ■

This produces a description of $H^i\left(q\text{-}\mathcal{H}\mathsf{dg}_{(R,\square)}/(1-q)\right)$ that is visibly independent of coordinates, being the quotient of coordinate-independent objects.

In fact we can do better. For any R, there is a notion of q-Witt vectors $q\text{-}W_m(R)$ and q-de Rham-Witt complexes $q\text{-}W_m\Omega_R$ which is a commutative differential graded algebra with first term $q\text{-}W_m(R)$ isomorphic to $H^\bullet\left(q\text{-}\mathcal{H}\mathsf{dg}_{(R,\square)}/(1-q^m)\right)$.

Theorem 4.8 (Wagner; [Wag22, Thm. 5.7])**.** Let R be a smooth framed $\mathbb{Z}$-algebra. There is an isomorphism

$$q\text{-}W_m\Omega_R^\bullet \longrightarrow H^\bullet\left(q\text{-}\mathcal{H}\mathsf{dg}_{(R,\square)}/(1-q)\right).$$

Remark 4.9. This is related to the classical construction of the de Rham-Witt complex, though the sense in which the preceding constructions are q-deformations are quite subtle.

Remark 4.10. One can often reduce to the case of computing on the torus, since many of the constructions "commute with étale maps" in the sense that they are preserved under étale base change.

Based on this, one might hope that these complexes are independent of coordinates.

Example 4.11. Let $R = \mathbb{Z}[T^{\pm}]$. The q-Habiro-Hodge complex is given by

$$\mathbb{Z}[q][T^{\pm}]/(1-q^m) \xrightarrow{\gamma-1} \mathbb{Z}[q][T^{\pm}]/(1-q^m)$$

by $T^k \mapsto (q^k-1)T^k$. We can compute the kernel of this map – the 0th cohomology – by noting that the map preserves the degree of T, we can compute the kernel in each degree to see that it is given by

$$\bigoplus_{k\in\mathbb{Z}}\left(\frac{\frac{q^m-1}{q^{\gcd(k,m)}-1}\mathbb{Z}[q]}{(q^m-1)\mathbb{Z}[q]}\right)T^k \cong \bigoplus_{k\in\mathbb{Z}}\left(\mathbb{Z}[q]/(1-q^{\gcd(k,m)})\mathbb{Z}[q]\right)T^k.$$

We similarly compute first cohomology to see it is also given by

$$\bigoplus_{k\in\mathbb{Z}}\left(\mathbb{Z}[q]/(1-q^{\gcd(k,m)})\mathbb{Z}[q]\right)T^k.$$

Indeed, when $m=p$ is prime, the 0th cohomology is a subring of $\mathbb{Z}[q][T^{\pm}]/(1-q^p)$ (hence a subring of $\mathbb{Z}[T^{\pm}]\times\mathbb{Z}[\zeta_p][T^{\pm p}] \subseteq \mathbb{Z}[T^{\pm}]\times\mathbb{Z}[\zeta_p][T^{\pm}]$) and is generated by T^p and $[p]_qT^i$ for $1\le i\le p-1$.

The computations of Example 4.11 is suggestive of a connection to Witt vectors since the cohomology lies in the product of rings $\mathbb{Z}[T^{\pm}]\times\mathbb{Z}[\zeta_p][T^{\pm p}]$. Recall that for a p-torsion free ring R, the p-th Witt vectors $W_p(R)$ consists of elements $(x_0, x_1, \dots)$

has ghost maps $\mathrm{gh}_1, \mathrm{gh}_p : W_p(R) \to R$ by $(x_0, x_1, \dots) \mapsto x_0$ and $(x_0, x_1, \dots) \mapsto x_0^p + px_1$, respectively. The image of $(\mathrm{gh}_1, \mathrm{gh}_p) : W_p(R) \to R \times R$ consists precisely of those pairs $(x, y) \in R \times R$ where $y \equiv x^p \pmod{p}$.

Proposition 4.12 (Wagner)**.** Let $R = \mathbb{Z}[T^{\pm}]$ with the identity framing and $q\text{-}\mathcal{H}\mathsf{dg}_{(R,\square)}$ its q-Habiro-Hodge complex. There is a canonical embedding

$$W_p(R) \hookrightarrow H^0\left(q\text{-}\mathcal{H}\mathsf{dg}_{(R,\square)}/(1-q^p)\right)$$

rendering the diagram

$$\begin{tikzcd}
\varphi_p(x_0) + [p]_q x_1 & & & \\
& H^0\left(q\text{-}\mathcal{H}\mathsf{dg}_{(R,\square)}/(1-q^p)\right) \arrow[r, hook] & \mathbb{Z}[T^{\pm}] \times \mathbb{Z}[\zeta_p][T^{\pm p}] & \\
& W_p(R) \arrow[u, hook] \arrow[r, "(\mathrm{gh}_1, \mathrm{gh}_p)"'] & R \times R \arrow[u] & \\
(x_0, x_1) \arrow[uuu, mapsto] \arrow[rrr, mapsto] & & & (x_0, x_0^p + px_1)
\end{tikzcd}$$

commutative.

Remark 4.13. On the q-Habiro-Hodge cohomologies, we can relate the different specializations by Frobenii and Verschiebungen

$$H^i\left(q\text{-}\mathcal{H}\mathsf{dg}_{(R,\square)}/(1-q^{mk})\right) \underset{V_k = \times\frac{1-q^{mk}}{1-q^m}}{\overset{F_k}{\rightleftarrows}} H^i\left(q\text{-}\mathcal{H}\mathsf{dg}_{(R,\square)}/(1-q^{m})\right).$$

More generally, we have the following.

Proposition 4.14. Let R be a flat $\mathbb{Z}$-algebra. There is a commutative diagram

$$\begin{tikzcd}
& W_m(R) \arrow[r, hook, "(\mathrm{gh}_d)_{d \mid m}"] \arrow[d, hook] \arrow[dl] & \prod_{d \mid m} R \arrow[d, hook] \\
W_m(R)[q]/(1-q^m) \arrow[r, two heads] & q\text{-}W_m(R) \arrow[r, hook, "(q\text{-}\mathrm{gh}_d)_{d \mid m}"'] & \prod_{d \mid m} R[\zeta_d]
\end{tikzcd}$$

where the Frobenii and Verschiebungen are defined on $q\text{-}W_m(R)$.

Remark 4.15. There are no restriction maps on the q-Witt vectors $q\text{-}W_m(R)$.

This shows that on the level of cohomology, the q-Habiro-Hodge complex is coordinate independent after specialization. However, due to a theorem of Wagner, this is the best we can do: there is no way to make the q-Habiro-Hodge complex itself coordinate independent in the derived category in such a way that remains coordinate independent on specialization.

Appendix A. Explicit Elements of the Habiro Ring (d'après Garoufalidis-Wheeler)

This appendix contains the proof sketch of Example 4.4. The interested reader is encouraged to consult [GW25] for further details.

We seek to show that for $R = \mathbb{Z}[T_1, \dots, T_d, \frac{1}{1-T_1-\dots-T_d}]$ that the element

$$\sum_{k_1,\dots,k_d \geq 0} \begin{bmatrix} k_1+\dots+k_d \\ k_1 \dots k_d \end{bmatrix}_q T_1^{k_1} \dots T_d^{k_d} \in \mathbb{Z}[q^{\pm}][[\underline{T}]] \tag{A.1}$$

lies in $\mathcal{H}_{(R,\square)}$ where $\square : \mathbb{Z}[q^{\pm}][\underline{T}] \to R$ is the obvious map. For this, it sufficse to show that $R^{(m)}[[q-\zeta_m]] \subseteq \mathbb{Z}[\zeta_m][[\underline{T}, q-\zeta_m]]$.

We have that

$$R^{(m)} = \mathbb{Z}\left[T_1, \dots, T_d, \frac{1}{1-T_1^m-\dots-T_d^m}\right]$$

and that the Frobenius gluing is already completely determined by the injectivity $R^{(m)}[[q-\zeta_m]] \hookrightarrow \mathbb{Z}[\zeta_m][[\underline{T}, q-\zeta_m]]$ as it can be checked after $\underline{T}$-adic completion. Note, furthermore, that

$$\mathbb{Z}\left[T_1, \dots, T_d, \frac{1}{1-T_1^m-\dots-T_d^m}\right] = \mathbb{Q}\left[T_1, \dots, T_d, \frac{1}{1-T_1^m-\dots-T_d^m}\right] \bigcap \mathbb{Z}[[T_1, \dots, T_d]]$$

as subrings of $\mathbb{Q}[[T_1, \dots, T_d]]$, so it suffices to verify the statement rationally. Using $q = \zeta_m \exp(h)$, we get an isomorphism $\mathbb{Q}(\zeta_m)[[q-\zeta_m]] \cong \mathbb{Q}(\zeta_m)[[h]]$ and seek to develop (A.1) as a power series in h. Using that

$$\begin{bmatrix} k_1+\dots+k_d \\ k_1 \dots k_d \end{bmatrix}_q = \binom{k_1+\dots+k_d}{k_1 \dots k_d} \cdot O(h)$$

where $O(h)$ is a power series in h with coefficients in $\mathbb{Q}[k_1, \dots, k_d]$, we have that each term in the power series expansion in h at $m = 1$ is of the form

$$\sum_{k_1,\dots,k_d \geq 0} \binom{k_1+\dots+k_d}{k_1 \dots k_d} P(k_1, \dots, k_d) T_1^{k_1} \dots T_d^{k_d}. \tag{A.2}$$

We then use the following lemma.

Lemma A.1. Let $P(k_1, \dots, k_d) \in \mathbb{Q}[k_1, \dots, k_d]$ as in (A.2) lies in

$$R = \mathbb{Q}\left[T_1, \dots, T_d, \frac{1}{1-T_1-\dots-T_d}\right].$$

Proof. Without loss of generality, we can take P to be a monomial $k_1^{a_1} \dots k_d^{a_d}$. We get, up to a constant, that $(\nabla_1^{\log})^{a_1} \dots (\nabla_d^{\log})^{a_d}$ of $\frac{1}{1-T_1-\dots-T_d}$ lies in R. ■

More generally the power series expansion at m is given by

$$\sum_{k_1,\dots,k_d \geq 0} \binom{mk_1+\dots+mk_d}{mk_1 \dots mk_d} P(k_1, \dots, k_d) T_1^{mk_1} \dots T_d^{mk_d} \tag{A.3}$$

which by similar arguments can be shown to lie in R as well.

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