Reading Group: Higher Algebra

Jonas Heintze* and Yordan Toshev†

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The seminar aims to cover basics of Higher Algebra, mainly based upon [HW21, Chapter II]. The seminar ended last term by defining monoids and group objects in an ∞-category. Starting from there, we want to delve into the theory of stable ∞-categories, develop some of their properties and get some working knowledge. Among others, we want to examine the process of stabilisation, define monoidal structures (via ∞-operads) and localisations of stable ∞-categories and consider prime examples such as the category of spectra Sp and derived category of a ring $\mathcal{D}(R)$. These constructions appear in many modern approaches across different areas of mathematics, for example Stable Homotopy Theory (K-theory, chromatic homotopy theory), Arithmetic Geometry (p-adic Hodge theory, ...), Derived Algebraic Geometry etc.

Preliminaries

Knowledge of ∞-categories (the straightening-unstraightening equivalence, (co)limits, adjunctions, Yoneda's lemma, Kan extensions in the ∞-categorical setting) will be assumed. A good resource to get on track is [Wag24, Chapters 1-6]. Even if you already know the topics, please briefly revise them, to make sure we all start on the same page.

Schedule

The seminar meets at 18ct on Monday in Room N0.008. Each talk discusses roughly 10 pages of [HW21] and is divided between two speakers, each preparing a presentation of about 1h. The precise timing is left to the speakers. Speakers in brackets with (?) are not entirely fixed yet. You can still claim those talks if you want to.

№	Date	Topic	Speakers
0	07.04	Recollections on ∞-Engineering	Jonas,Yordan
1	14.04	Complete Segal Spaces	Henry
2	21.04	$\mathbb{E}_1$-Structures	
3	28.04	$\mathbb{E}_\infty$-Structures I	Sam, Daniel
4	05.05	$\mathbb{E}_\infty$-Structures II and Stable ∞-Categories	(Dave?)
5	12.05	Stable ∞-Categories II	Gabriel
6	19.05	∞-Operads I	Julius, Carl
7	26.05	∞-Operads II	Julius, Carl
8	02.06	∞-Operads III and Brave New Algebra	Mathieu
-	09.06	no talk	holidays
9	16.06	∞-Categories of Modules	
10	23.06	Relative Tensor Products	
11	30.06	Localisations of Ring Spectra	
12	07.07	(Open for recommendations)	(Melvin(?), Gabriel(?))
13	14.07	?	(Melvin(?))

*s6johei2@uni-bonn.de
†yordan.toshev@gmail.com

Syllabus

Talk 0: Recollections on ∞-Engineering

Talk 1: Complete Segal Spaces

State Theorem/Definition I.64 and give some intuition and Explanations I.65 to it. Work through I.66 to I.69 and sketch some proofs. Introduce the Quillen Q-construction and talk about I.71.

Talk 2: $\mathbb{E}_1$-Structures

Motivate (naive definition of) monoids in An as in II.0 and give the modern definition as in II.1. State Proposition 2.2 and outline the main steps of the proof. Motivate and state II.3. Summarise II.4 to II.5 with some proof ideas (if time permits). State II.9, II.10, but focus more on II.11. Mention briefly (5min) the example from p.71-76.

Talk 3: $\mathbb{E}_\infty$-Structures I

Motivate and work through II.15. Discuss II.16, II.17 (with very rough outlines of the proofs). Proof II.18 and make sure to mention II.18a. Discuss what is relevant from II.18b to be able to state II.21 (feel free to omit the proof). Discuss II.21a (with main focus on parts (b) and (c)).

Talk 4: $\mathbb{E}_\infty$-Structures II and Stable ∞-Categories I

Finish the discussion on $\mathbb{E}_\infty$-structures: II.19 and free $\mathbb{E}_\infty$-monoids. Introduce stable ∞-categories and prove II.23. Discuss homotopy groups, (co)homology of spectra and II.27. Use it to tease that Sp is a stable ∞-category.

Talk 5: Stable ∞-Categories II

State and prove II.28. Define II.29 and emphasize II.29a. Discuss II.30 and II.30a and sketch parts of the proofs, which you consider enlightening. Mention very briefly (max 10min) the ideas of II.31.(a)-(d). State as a (blackbox) result II.32. State II.33, II.34., II.34a (and prove at least one of them if time permits). Finish the discussion by II.35.

Talk 6: ∞-Operads I

Motivate the notion of ∞-operads via their 1-categorical precursors and discuss how to give an ∞-category a (symmetric) monoidal structure via ∞-operads. Discuss cartesian and cocartesian symmetric monoidal structures of ∞-categories. This entails p.105 - 117.

Talk 7: ∞-Operads II

Recall monoids over an ∞-operad, in particular II.43 and introduce algebras over an ∞-operad. Put emphasis on examples, e.g. II.45a-c. If time permits, sketch the proof of II.45a. Discuss Day convolution: II.46 and II.47. Conclude that Sp has a "nice" tensor product. This entails p.117-124.

Talk 8: ∞-Operads III and Brave New Algebra

Present II.48 and II.49. State II.50 and II.51 with emphasis on the examples, in particular II.51b. Introduce the tensor product on spectra, II.52, and use it to discuss II.53. Discuss II.54, II.55 (with proof) and finish with II.55a.

Talk 9: ∞-Categories of Modules

Discuss extensively II.45d-e and less extensively II.56, II.56a,c,e,f,g. State II.57 (and roughly sketch the proof, if time permits). State the Schwede-Shipley Theorem, II.58.

Talk 10: Tensor Products over Arbitrary Bases

Go over II.59 and state II.60. Discuss II.61 and II.62. Mention II.62a-c and sketch II.63. Discuss what you find interesting from the Miscellanea section.

Talk 11: Localisations of $\mathbb{E}_\infty$-Ring Spectra

III.1, III.2 (with sketch of proof), III.3, III.4 (and essential parts of proofs, if times permits), III.4a (discuss why it is not tautological, c.f. discussion in the proof). Finish with III.5 and preceding example.

Talk 12: Open for recommendations

(Verdier sequences p207-215?, triangulated structures somewhere in chapter IV(?))

References

[HW21] Fabian Hebestreit and Ferdinand Wagner. *Lecture Notes for Algebraic and Hermitian K-Theory*. visited on 01.03.2025. 2021. URL: https://florianadler.github.io/AlgebraBonn/KTheory.pdf.

[Wag24] Ferdinand Wagner. *∞-Categories in Topology*. visited on 01.03.2025, version from June 1, 2024. 2024. URL: https://florianadler.github.io/inftyCats/inftyCats.pdf.