Federal University of Technology Akure

Mechanical Engineering Department

AUTOMATIC CONTROL SYSTEM

(MEE 405)

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Learning Outcomes

At the end of this module, students are expected to:

(i) Understand the purpose and concept of a system model;

(ii) Understand the representation of a physical system by a mathematical model;

(iii) Understand the block diagram representation of control systems;

(iv) Use Laplace transforms to predict the response of a system to an input from the system model;

(v) Understand the concept of the root locus in predicting the response of more complex systems; and

(vi) Use the Routh-Hurwitz criterion to predict system stability.

Contents

CHAPTER 1

Development and Applications of Control Systems

1.1 Introduction

Systems modelling is used to predict the performance of a complex system from the known behaviour of its components (these may be levers, gearboxes, electric motors, hydraulic actuators, even jet engines). A practical application of systems modelling is in the design of an in-hub motor for an electric car as shown in Fig. 1.1.

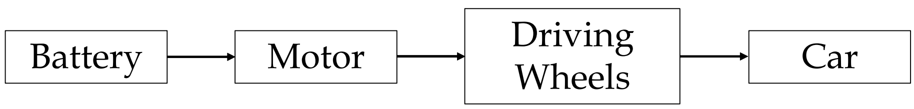


Figure 1.1: Processes of an in-hub motor for an electric car

Each of the processes entails a conversion - the battery supplies current to the motor, which converts the electrical energy (voltage $\times$ current) into motive power (torque T $\times$ angular velocity, ω) which is supplied to the wheels (diameter, coefficient of friction, axle weight) to provide traction force. The traction force is applied to the car, which has a mass, and if the car is moving, has to overcome the aerodynamic drag, which is dependent on the car's velocity. These conversions are characterised by transfer functions - these are mathematical expressions that relate each input to its corresponding output. The transfer function will change the input into an output every time.

Figure 1.2: Simple representation of a control system

A control system can therefore be defined as a system whose output is controlled to be at some desired value or to change in some desired way as regulated by the input to the system. Fig. 1.2 illustrates the relationship between the input, the control system and the output.

Some common examples of control systems are as follows:

(i) A thermostatically controlled domestic central heating system that keeps the house at a constant temperature.

(ii) The system which causes the table of a numerically controlled milling machine to move through a desired path in response to command signals generated by computer.

(iii) An engine or turbine governor system that limits changes in speed as the load changes.

(iv) The systems that cause the control surfaces on a large aircraft to move in response to movements of the joystick by the pilot (generation 1 was hydraulic control, generation 2 is fly-by-wire).

1.2 Classification of Control Systems

There are two classifications of control system:

(a) **Open-Loop Systems**

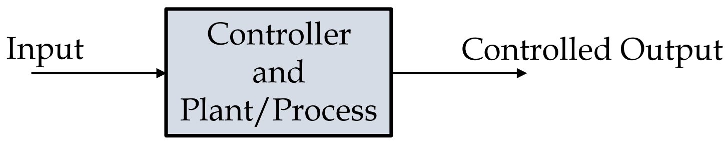


Figure 1.3: Open-loop control system

An Open-loop control system is a form of system which has no feedback information about the output to the controller, and hence, no comparison unit to compare output with the desired input. Therefore, it cannot make adjustment to correct the error or deviation from the desired output value. The controller in an open-loop system is essentially a clock operated switching device. This is the simplest form of control systems. On the basis of knowledge or experience about how the plant or process will behave a guess is made of what input is needed to give the desired output. Large variations can occur in the controlled output because changes in external circumstances are not taken into account.

Common examples of open-loop control systems are:

(i) Domestic central heating system without a room thermostart.

(ii) Domestic washing machine.

(iii) Traffic lights which changes at fixed interval of time.

(b) **Closed-Loop (Feedback) Systems**

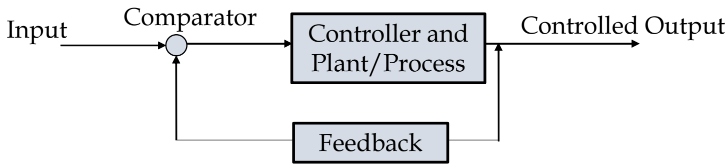


Figure 1.4: Closed-loop control system

In a closed-loop system the output is continuously monitored and fed back to be compared with the desired input signal. The difference between the measured output and the reference input, called the control difference or error signal, is fed into the controller, which then manipulates or modifies the error signal according to certain control laws, and sends a corrective/manipulative signal to the plant/Process. The corrective/manipulating signal tends to restore the plant output towards the required/desired value. Consequently the system continually attempts to reduce the error between the desired input and the output. The closed-loop system is therefore "error actuated". An essential feature of any closed-loop system is the use of feedback, which allows the output to be compared to the input.

The feedback can either be negative or positive. For a negative feedback system, error signal is measured as:

$$\text{Error signal} = \text{Reference input} - \text{Feedback signal} \tag{1.2.1}$$

while the error signal of a positive feedback system is computed as:

$$\text{Error signal} = \text{Reference input} + \text{Feedback signal} \tag{1.2.2}$$

Consequently, positive feedback systems find no practical application as error is never eliminated but rather built up in the system.

Closed-loop control systems can compensate for external disturbance or changes in the process being controlled. Whatever the error or deviation of the actual value from the desired value, the controller generates a new manipulating signal.

Examples of closed-loop control systems include:

(i) Traffic lights which are controlled by some devices which measure traffic flow and reacts accordingly.

(ii) Water closet of a domestic toilet in which a feedback is provided via the ball float which measures the water level after flushing is completed and closes the supply valve when the water reaches a pre-set level.

(iii) Domestic central heating system in which the thermostat turns ON or OFF so as to maintain the room temperature at a predefined/predetermined level.

(iv) Biological control system in which the body temperature is kept constant despite large variation in external conditions.

(v) Speed control system of a motor car (Fig. 1.5).

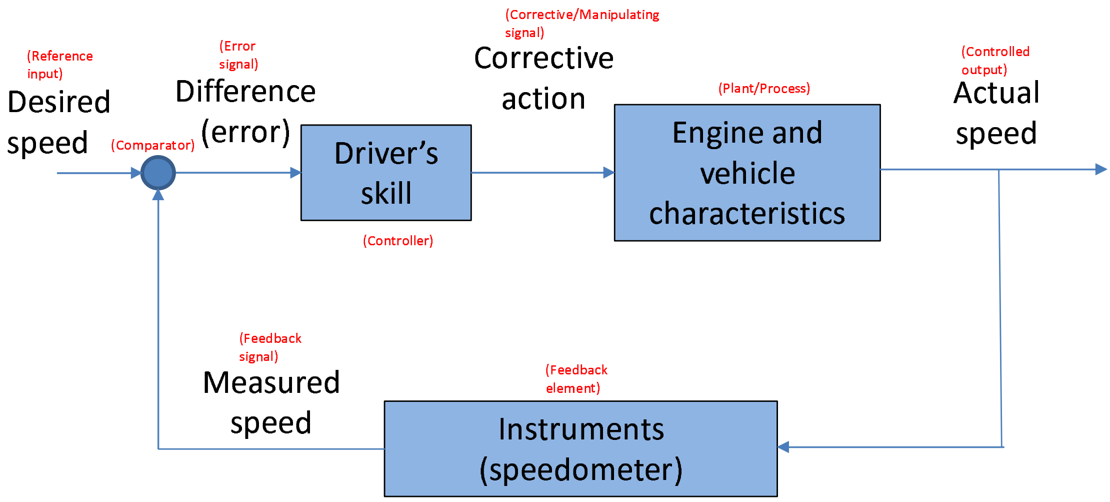


Figure 1.5: Speed control system of a motor car

1.3 Basic Components of Control Systems

A typical system will have a block diagram of the following form

Each box contains the transfer function of the element. A picture of the overall system can be built up from knowledge of its component parts and how they fit together. As represented in the block diagram, the following are the basic elements of the system:

(i) **Input Element:** This prepares the input signal for the comparator, especially when the input signal has to be converted to a desirable form before it is fed into the comparator.

(ii) **Comparator:** This is the error channel or comparison unit of the system. It measures and compares signals against what is expected and then sends out error

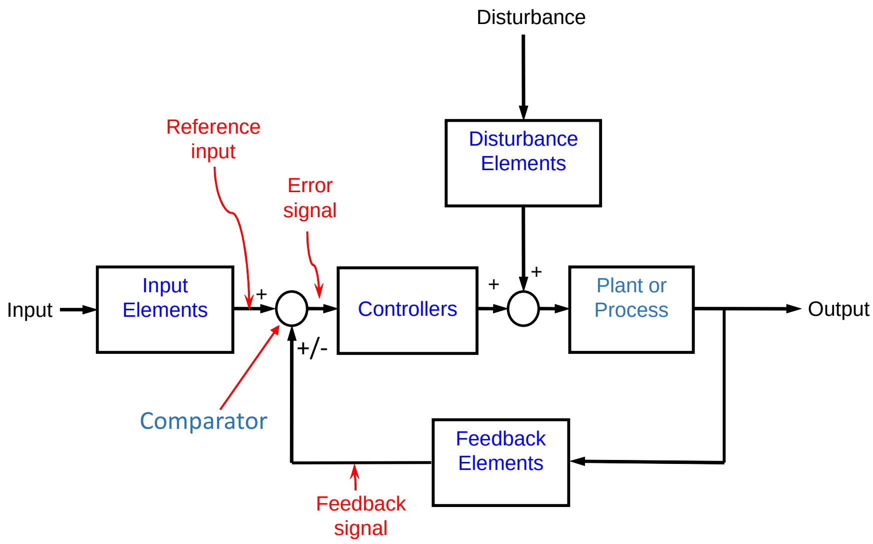


Figure 1.6: Block diagram of a typical control system

signal as applicable. For instance, it measures output signal and compares it with the input signal and sends out error signal if applicable.

(iii) **Controller:** The error signal from the comparator is received by the controller. It is then modified and sent out as a corrective signal to the plant or process.

(iv) **Plant or Process:** These are systems whose dynamic variables are being controlled.

(v) **Feedback Element:** These converts the output signal into a similar form as the reference/input signal to enable a comparison to be made (at the comparator) with the reference input.

1.4 Regulators and Servomechanisms

Regulators (also known as regulating systems) are closed-loop control systems designed to maintain certain plant output at/close to a desired constant value irrespective of disturbances. That is, their inputs are constant variables while outputs vary according to a desired value. Examples of regulators include air-conditioner (AC) system and automatic domestic heating system.

Servomechanisms (also known as SERVO) are closed-loop systems in which the controlled output is required to follow a time-varying reference input. That is, the output variables vary with changes in the input variables.

Chapter 2

Time Domain Control System Response

Time domain response of a control system is the response of the system, when subjected to certain input functions, with respect to time.

2.1 Basic Types of Input Functions

When studying control systems it is useful to consider the response of the system to certain 'standard' inputs. The response of different systems to the same standard input then forms a useful basis for comparing the performance of different systems. The basic input functions which determine the time response performance of control systems are discussed as follows.

(i) **Step input function**

A step input function (Fig. 2.1) can be expressed as

$$\begin{aligned} x(t) &= 0, \quad t < 0 \\ x(t) &= A, \quad t \geq 0 \end{aligned} \tag{2.1.1}$$

For a unit step function, $A = 1$.

(ii) **Ramp Input function**

A ramp input function (Fig. 2.2) can be expressed as

$$\begin{aligned} x(t) &= 0, \quad t < 0 \\ x(t) &= Kt, \quad t \geq 0 \end{aligned} \tag{2.1.2}$$

For a unit ramp function, $K = 1$.

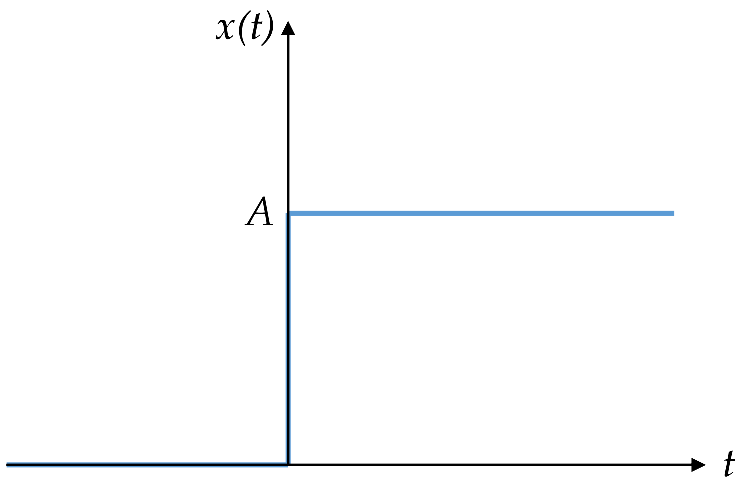


Figure 2.1: Step Input Function

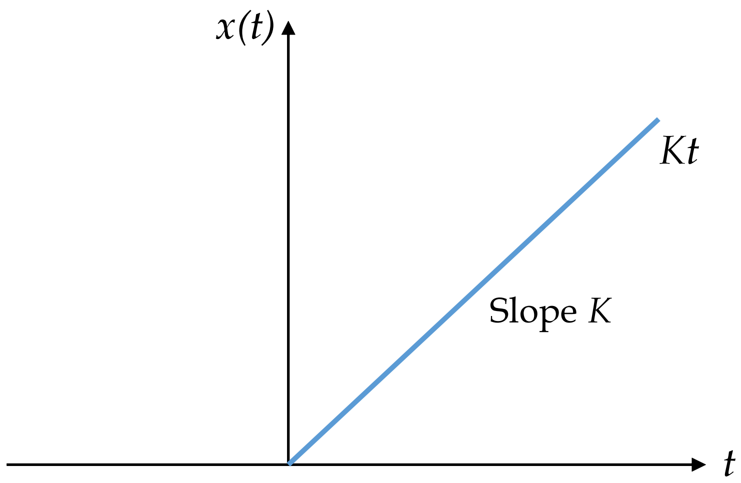


Figure 2.2: Ramp Input Function

(iii) **Impulse Input function**

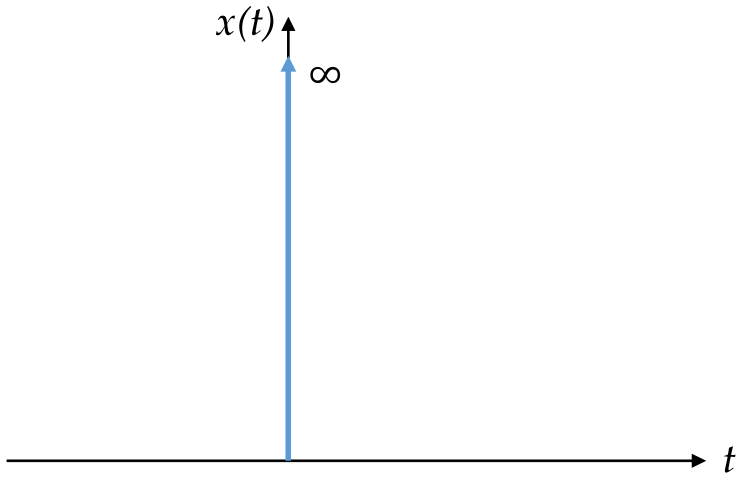


Figure 2.3: Impulse Input Function

An impulse input function (Fig. 2.3) can be expressed as

$$\begin{aligned} x(t) &= \infty, \quad t = 0 \\ x(t) &= 0, \quad t \neq 0 \end{aligned} \tag{2.1.3}$$

(iv) **Parabolic Input function**

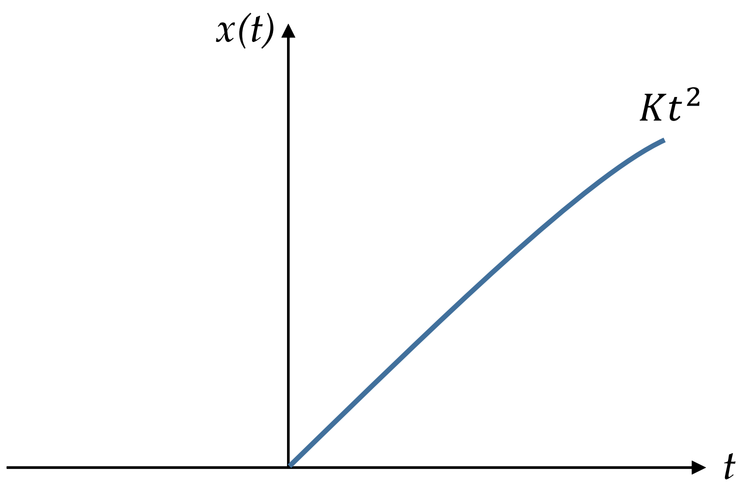


Figure 2.4: Parabolic Input Function

A parabolic input excitation (Fig. 2.4) can be expressed as

$$\begin{aligned} x(t) &= 0, \quad t < 0 \\ x(t) &= Kt^2, \quad t \geq 0 \end{aligned} \tag{2.1.4}$$

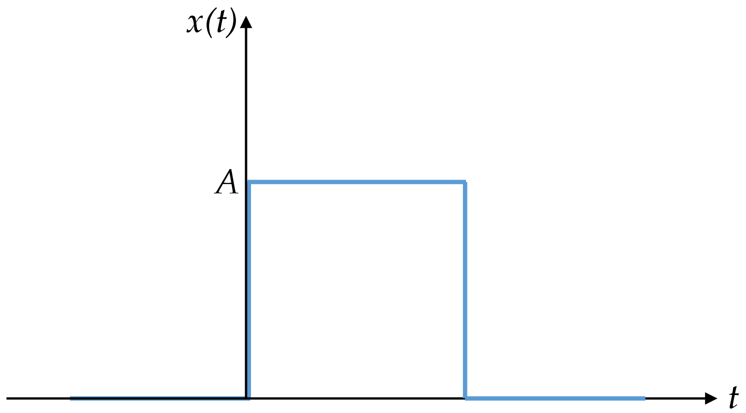


Figure 2.5: Pulse Input Function

(v) **Pulse Input function**

A pulse input function (Fig. 2.5) can be expressed as

$$\begin{aligned} x(t) &= 0, \quad t < 0 \\ x(t) &= A, \quad 0 \leq t < T \\ x(t) &= 0, \quad t \geq T \end{aligned} \tag{2.1.5}$$

(vi) **Sinusoidal Input function**

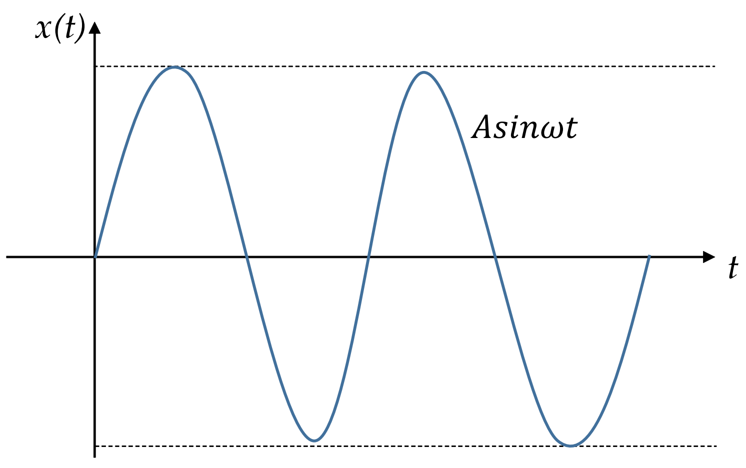


Figure 2.6: Sinusoidal Input Function

A sinusoidal input function (Fig. 2.6) can be expressed as

$$\begin{aligned} x(t) &= 0, \quad t < 0 \\ x(t) &= Asin\omega t, \quad t \geq 0 \end{aligned} \tag{2.1.6}$$

2.2 Laplace Transforms

The Laplace transform technique is a useful tool for the solution of differential equations and is widely used in control engineering, where it provides a convenient means of describing the transfer function of system components.

The Laplace transform of a function f(t) is written as F(s) and is defined as the integral of $f(t)\mathrm{e}^{-st}$ between the limits $t = 0$ and $t = \infty$. That is

$$\mathcal{L}[f(t)] = F(s) = \int_0^\infty f(t)e^{-st}dt \tag{2.2.1}$$

where $s = \sigma + j\omega$, a complex variable, is the Laplace operator and $f(t) = 0$ for $t < 0$.

Some useful theorems relating to Laplace Transforms are given as thus:

(i) Addition and Subtraction theorems

$$\mathcal{L}[f_1(t) \pm f_2(t)] = F_1(s) \pm F_2(s) \tag{2.2.2}$$

(ii) Multiplication by a constant

$$\mathcal{L}[kf(t)] = kF(s) \tag{2.2.3}$$

(iii) Shifting theorem

$$\text{if} \quad \mathcal{L}[f(t)] = F(s), \quad \text{then} \quad \mathcal{L}[f(t-\tau)] = \mathrm{e}^{-s\tau}F(s) \tag{2.2.4}$$

2.2.1 Examples of the use of Laplace Transforms

Example 1)
Determine the Laplace transform of a step input function.
Solution
For a step input,

$$f(t) = A \quad \text{for} \quad t \geq 0 \tag{2.2.5}$$

then from Eq. (2.2.1)

$$\begin{aligned} F(s) &= \int_0^\infty A * e^{-st}dt \\ &= -A\frac{\mathrm{e}^{-st}}{s}\bigg|_0^\infty \\ &= \frac{\mathbf{A}}{\mathbf{s}} \end{aligned} \tag{2.2.6}$$

for a unit step input function, where $A = 1$, Laplace transform is

$$F(s) = \frac{\mathbf{1}}{\mathbf{s}} \tag{2.2.7}$$

Example 2)

Determine the Laplace transform of a unit ramp input function.

Solution

For a unit ramp input,

$$f(t) = t \quad \text{for} \quad t \geq 0 \tag{2.2.8}$$

then from Eq. (2.2.1)

$$F(s) = \int_0^\infty t * e^{-st} dt \tag{2.2.9}$$

This can be solved using integration by part method:

$$\int u dv = uv - \int v du \tag{2.2.10}$$

From Eq. (2.2.9),

$$u = t \quad \text{and} \quad dv = e^{-st} dt \tag{2.2.11}$$

du and v can then be obtained as

$$du = dt \quad \text{and} \quad v = -\frac{\mathrm{e}^{-st}}{s} \tag{2.2.12}$$

Substituting Eqs. (2.2.11) and (2.2.12) in Eq. (2.2.10), then the Laplace transform (Eq. 2.2.9) can be obtained as

$$\begin{aligned} F(s) &= -\frac{t}{s}\mathrm{e}^{-st} + \int_0^\infty \frac{\mathrm{e}^{-st}}{s} dt \\ &= \left| -\frac{t}{s}\mathrm{e}^{-st} - \frac{\mathrm{e}^{-st}}{s^2} \right|_0^\infty \\ &= \frac{\mathbf{1}}{\mathbf{s^2}} \end{aligned} \tag{2.2.13}$$

Example 3)

Use the Laplace transform to determine the solution of:

$$\frac{d^2x(t)}{dt^2} + {\omega_n}^2 x(t) = \cos(pt) \tag{2.2.14}$$

with zero initial conditions.

Solution

Taking Laplace transform with zero initial conditions,

$$s^2X(s) + {\omega_n}^2 X(s) = \frac{s}{s^2 + p^2} \tag{2.2.15}$$

Rearranging gives:

$$X(s) = \frac{s}{(s^2 + p^2)(s^2 + \omega_n{}^2)} \tag{2.2.16}$$

Converting back to time domain gives:

$$x(t) = \frac{1}{\omega_n{}^2 - p^2}[\cos(pt) - \cos(\omega_n t)] \tag{2.2.17}$$

Useful Laplace transform pairs are given in Table 2.1.

Table 2.1: Table of Laplace Transforms

S/N	Time Function $f(t)$	Laplace Transform $F(s)$
1	$\frac{d^n f(t)}{dt^n}$ (when all initial conditions are zero)	$s^n F(s)$
2	$\int f(t)dt$ (when all initial conditions are zero)	$\frac{1}{s}F(s)$
3	$\delta(t)$ (unit impulse function at $t = 0$)	1
4	1 (unit step function)	$\frac{1}{s}$
5	t (unit ramp function)	$\frac{1}{s^2}$
6	t^n	$\frac{n!}{s^{n+1}}$
7	e^{-at}	$\frac{1}{s+a}$
8	$1-e^{-at}$	$\frac{a}{s(s+a)}$
9	$t^n e^{-at}$	$\frac{n!}{(s+a)^{n+1}}$
10	$sin(\omega t)$	$\frac{\omega}{s^2+\omega^2}$
11	$cos(\omega t)$	$\frac{s}{s^2+\omega^2}$
12	$e^{-at} sin(\omega t)$	$\frac{\omega}{(s+a)^2+\omega^2}$
13	$e^{-at} cos(\omega t)$	$\frac{s+a}{(s+a)^2+\omega^2}$

2.3 Transfer Function

The block diagram is the most common pictorial means of depicting control systems. For analysis purposes the function of the components of the system must be described mathematically and this is most often achieved by means of a transfer function.

The transfer function of a linear system is defined as the ratio of the Laplace transform of the output to the Laplace transform of the input when the initial conditions are zero.

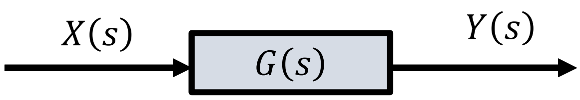


Figure 2.7: Block diagram for an element of a control system

The transfer function, $G(s)$, is thus given by

$$G(s) = \frac{Y(s)}{X(s)} \tag{2.3.1}$$

where $Y(s)$ and $X(s)$ are the Laplace transforms of the output and input respectively.

2.4 Differential Equation (DE) Models of Control Systems

2.4.1 First Order DE Model

The first order differential equation model of a system, whose input and output functions are respectively $x(t)$ and $y(t)$, can be generally described as

$$\tau \frac{dy(t)}{dt} + y(t) = G_{ss}x(t) \tag{2.4.1}$$

where the characterizing parameters τ and G_{ss} are respectively the **time constant** and **steady-state gain** of the system. Using the differential operator ($\frac{d}{dt} = D$), Eq. (2.4.1) can be expressed as

$$\tau Dy(t) + y(t) = G_{ss}x(t) \tag{2.4.2}$$

Rearranging Eq. (2.4.2), the time-domain transfer function of the first order DE model can be expressed as

$$\frac{y(t)}{x(t)} = \frac{G_{ss}}{\tau D + 1} \tag{2.4.3}$$

Using Table 2.1, the Laplace-domain transfer function, when all initial conditions are zero, can be represented as

$$\frac{Y(s)}{X(s)} = \frac{G_{ss}}{\tau s + 1} \tag{2.4.4}$$

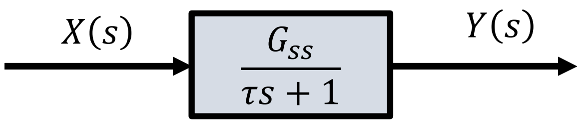


Figure 2.8: First Order Differential Equation Model

2.4.2 Second Order DE Model

The second order differential equation model of a system, whose input and output functions are respectively $x(t)$ and $y(t)$, can be generally described as

$$\frac{d^2y(t)}{dt^2} + 2\zeta\omega_n \frac{dy(t)}{dt} + \omega_n{}^2 y(t) = G_{ss}\omega_n{}^2 x(t) \tag{2.4.5}$$

where the characterizing parameters ζ, ω_n and G_{ss} are respectively the **damping ratio**, the **natural angular frequency** and the **steady-state gain** of the system. Using the differential operator (D), Eq. (2.4.5) can be expressed as

$$D^2y(t) + 2\zeta\omega_n Dy(t) + {\omega_n}^2 y(t) = G_{ss}{\omega_n}^2 x(t) \tag{2.4.6}$$

Rearranging Eq. (2.4.6), the time-domain transfer function of the second order DE model can be expressed as

$$\frac{y(t)}{x(t)} = \frac{G_{ss}{\omega_n}^2}{D^2 + 2\zeta\omega_n D + {\omega_n}^2} \tag{2.4.7}$$

Using Table 2.1, the Laplace-domain transfer function, when all initial conditions are zero, can be represented as

$$\frac{Y(s)}{X(s)} = \frac{G_{ss}{\omega_n}^2}{s^2 + 2\zeta\omega_n s + {\omega_n}^2} \tag{2.4.8}$$

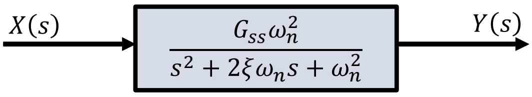


Figure 2.9: Second Order Differential Equation Model

2.4.3 Worked Examples

(1) Determine the characterizing parameters of the following:

(i)

$$2\frac{d^2v(t)}{dt^2} + 5\frac{dv(t)}{dt} + 8v(t) = 8u(t) \tag{2.4.9}$$

Solution

Rearrange the equation to conform to the standard equation of second order DE model (Eq. 2.4.6) by dividing through by 2

$$\frac{d^2v(t)}{dt^2} + \frac{5}{2}\frac{dv(t)}{dt} + 4v(t) = 4u(t) \tag{2.4.10}$$

and comparing Eq. (2.4.10) with

$$\frac{d^2y(t)}{dt^2} + 2\zeta\omega_n\frac{dy(t)}{dt} + {\omega_n}^2 y(t) = G_{ss}{\omega_n}^2 x(t) \tag{2.4.11}$$

then

$$2\zeta\omega_n = 2.5; \quad {\omega_n}^2 = 4; \quad G_{ss}{\omega_n}^2 = 4 \tag{2.4.12}$$

with the characterizing parameters (natural frequency, steady-state gain and damping ratio) obtained as

$$\omega_n = 2\text{rad/s}; \quad G_{ss} = 1; \quad \zeta = 0.625 \tag{2.4.13}$$

(ii)

$$3\frac{dw(t)}{dt} + 5w(t) = 8q(t) \tag{2.4.14}$$

Solution

Rearrange the equation to conform to the standard equation of first order DE model (Eq. 2.4.2) by dividing through by 5

$$\frac{3}{5}\frac{dw(t)}{dt} + w(t) = \frac{8}{5}q(t) \tag{2.4.15}$$

and comparing with

$$\tau\frac{dy(t)}{dt} + y(t) = G_{ss}x(t) \tag{2.4.16}$$

then the characterizing parameters (time constant and steady-state gain) can be obtained as

$$\tau = 0.6\text{s}; \quad G_{ss} = 1.6 \tag{2.4.17}$$

(2) Determine the characterizing parameters of the following, where D is a differential operator $\frac{d}{dt}$:

(i)

$$\frac{y(t)}{x(t)} = \frac{2}{D+5} \tag{2.4.18}$$

Solution

Rearrange the equation, to conform to the time-domain transfer function equation for 1st order system (Eq. 2.4.3), by dividing the numerator and the denominator of the RHS equation by 5

$$\frac{y(t)}{x(t)} = \frac{\frac{2}{5}}{\frac{1}{5}D+1} \tag{2.4.19}$$

Comparing with

$$\frac{y(t)}{x(t)} = \frac{G_{ss}}{\tau D+1} \tag{2.4.20}$$

then the characterizing parameters (time constant and steady-state gain) can be obtained as

$$\tau = 0.2\text{s}; \quad G_{ss} = 0.4 \tag{2.4.21}$$

(ii)

$$(D+3)y(t) = 2x(t) \tag{2.4.22}$$

Solution

Rearrange, the transfer function can be expressed as

$$\frac{y(t)}{x(t)} = \frac{2}{D+3} \tag{2.4.23}$$

Dividing the numerator and the denominator of the RHS equation by 3

$$\frac{y(t)}{x(t)} = \frac{\frac{2}{3}}{\frac{1}{3}D+1} \tag{2.4.24}$$

and comparing with

$$\frac{y(t)}{x(t)} = \frac{G_{ss}}{\tau D+1} \tag{2.4.25}$$

then the characterizing parameters (time constant and steady-state gain) can be obtained as

$$\tau = 0.33\text{s}; \quad G_{ss} = 0.67 \tag{2.4.26}$$

(iii)

$$\frac{y(t)}{x(t)} = \frac{40}{5D^2+7D+20} \tag{2.4.27}$$

Solution

Rearrange the equation, to conform to the time-domain transfer function equation for 2nd order system (Eq. 2.4.7), by dividing the numerator and the denominator of the RHS equation by 5

$$\frac{y(t)}{x(t)} = \frac{8}{D^2+\frac{7}{5}D+4} \tag{2.4.28}$$

Comparing with

$$\frac{y(t)}{x(t)} = \frac{G_{ss}{\omega_n}^2}{D^2+2\zeta\omega_n D+{\omega_n}^2} \tag{2.4.29}$$

then

$$2\zeta\omega_n = 7/5; \quad {\omega_n}^2 = 4; \quad G_{ss}{\omega_n}^2 = 8 \tag{2.4.30}$$

with the characterizing parameters (natural frequency, steady-state gain and damping ratio) obtained as

$$\omega_n = 2\text{rad/s}; \quad G_{ss} = 2; \quad \zeta = 0.35 \tag{2.4.31}$$

(3) Derive the transfer function of

(i)

$$\frac{d^2m(t)}{dt^2} + 4\frac{dm(t)}{dt} + 29m(t) = \frac{d^2q(t)}{dt^2} + q(t) \tag{2.4.32}$$

Solution

Using the differential operator D, the equation can be expressed as

$$D^2m(t) + 4Dm(t) + 29m(t) = D^2q(t) + q(t) \tag{2.4.33}$$

Making the input and the output functions as the subjects, the equation can be further rearranged as

$$(D^2 + 4D + 29)m(t) = (D^2 + 1)q(t) \tag{2.4.34}$$

Hence, the time-domain transfer function can be expressed as

$$\frac{m(t)}{q(t)} = \frac{D^2 + 1}{D^2 + 4D + 29} \tag{2.4.35}$$

and the Laplace-domain transfer function can be expressed as

$$\frac{m(t)}{q(t)} = \frac{s^2 + 1}{s^2 + 4s + 29} \tag{2.4.36}$$

2.5 Introduction to Transient and Steady-state Responses

The total time response of a system can be expressed as

$$\text{Total time response} = \text{Transient response} + \text{Steady-state response} \tag{2.5.1}$$

where

Transient response is the part of the system response which occurs immediately the forcing input is applied and it dies away as time approaches infinity.

and

Steady-state response is the part of the response which occurs when the transient response has died away and the response has settled to a constant value.

2.5.1 Transient Response - First Order Systems

The transfer function for a 1st order system is expressed as

$$\frac{Y(s)}{X(s)} = \frac{G_{ss}}{\tau s + 1} \tag{2.5.2}$$

As earlier stated, the responses of systems to standard inputs form a useful basis for comparing the performance of different systems. Responses of 1st order system to step, impulse and ramp inputs is considered in this section.

(i) Response to a unit step input, $x(t) = 1$

Consider a first order system subject to a unit step input. The Laplace transform of a unit step (Table 2.1) is

$$X(s) = \frac{1}{s} \tag{2.5.3}$$

Hence

$$Y(s) = \frac{G_{ss}}{s(\tau s + 1)} \tag{2.5.4}$$

Using partial fractions, this can be resolved into

$$Y(s) = \frac{G_{ss}}{s} - \frac{G_{ss}\tau}{\tau s + 1} \tag{2.5.5}$$

The time response (output) of the system, $\mathcal{L}^{-1}[Y(s)]$, can then be obtained as

$$y(t) = G_{ss}\mathcal{L}^{-1}[\frac{1}{s}] - G_{ss}\mathcal{L}^{-1}[\frac{1}{s + \frac{1}{\tau}}] \tag{2.5.6}$$

Using Table 2.1,

$$y(t) = G_{ss} - G_{ss}\mathrm{e}^{-\frac{1}{\tau}t} \tag{2.5.7}$$

$$y(t) = G_{ss}(1 - \mathrm{e}^{-\frac{t}{\tau}} \tag{2.5.8}$$

(ii) Response to a unit impulse input, $x(t) = \delta$

Consider a first order system subject to a unit impulse input. The Laplace transform of a unit impulse (Table 2.1) is

$$X(s) = 1 \tag{2.5.9}$$

Hence

$$Y(s) = \frac{G_{ss}}{\tau s + 1} \tag{2.5.10}$$

This can be resolved into

$$Y(s) = \frac{G_{ss}}{\tau s + 1} \tag{2.5.11}$$

The time response (output) of the system, $\mathcal{L}^{-1}[Y(s)]$, can then be obtained as

$$y(t) = \frac{G_{ss}}{\tau} \frac{1}{s + \frac{1}{\tau}} \tag{2.5.12}$$

Using Table 2.1,

$$y(t) = \frac{G_{ss}}{\tau} \mathrm{e}^{-\frac{t}{\tau}} \tag{2.5.13}$$

(iii) Response to a unit ramp input, $x(t) = t$

Consider a first order system subject to a unit ramp input. The Laplace transform of a unit ramp (Table 2.1) is

$$X(s) = \frac{1}{s^2} \tag{2.5.14}$$

Hence

$$Y(s) = \frac{G_{ss}}{s^2(\tau s + 1)} \tag{2.5.15}$$

Using partial fractions, this can be resolved into

$$Y(s) = G_{ss}[\frac{1}{s^2} - \frac{\tau}{s} + \frac{\tau^2}{\tau s + 1}] \tag{2.5.16}$$

The time response (output) of the system, $\mathcal{L}^{-1}[Y(s)]$, can then be obtained as

$$y(t) = G_{ss}[\mathcal{L}^{-1}[\frac{1}{s^2}] - \mathcal{L}^{-1}[\frac{\tau}{s}] + \mathcal{L}^{-1}[\frac{\tau^2}{\tau s + 1}]] \tag{2.5.17}$$

Using Table 2.1,

$$y(t) = G_{ss}(t - \tau + \tau \mathrm{e}^{-\frac{t}{\tau}}) \tag{2.5.18}$$

2.5.1.1 Worked Examples

(i) Determine the time response of a control system with the transfer function $\frac{2s+6}{(s+2)(s+4)}$, subject to a 2 unit ramp input.
Solution

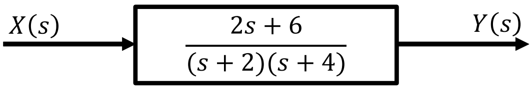


Figure 2.10

Transfer function

$$\frac{Y(s)}{X(s)} = \frac{2s+6}{(s+2)(s+4)} \tag{2.5.19}$$

For a 2 unit ramp input, the input function

$$x(t) = 2t \tag{2.5.20}$$

and

$$X(s) = \frac{2}{s^2} \tag{2.5.21}$$

Hence

$$\begin{aligned} Y(s) &= \frac{2}{s^2}\frac{2s+6}{(s+2)(s+4)} \\ &= \frac{4s+12}{s^2(s+2)(s+4)} \end{aligned} \tag{2.5.22}$$

By partial fractions, this can be resolved into

$$\frac{4s+12}{s^2(s+2)(s+4)} = \frac{A}{s} + \frac{B}{s^2} + \frac{C}{s+2} + \frac{D}{s+4} \tag{2.5.23}$$

Multiplying through by $s^2(s+2)(s+4)$

$$4s+12 = As(s+2)(s+4) + B(s+2)(s+4) + Cs^2(s+4) + Ds^2(s+2) \tag{2.5.24}$$

Expand the equation and collect like terms

$$4s+12 = s^3(A+C+D) + s^2(6A+B+4C+2D) + s(8A+6B) + 8B \tag{2.5.25}$$

Comparing the like terms, the unknown coefficients are obtained as

$$A = -\frac{5}{8}; \quad B = \frac{3}{2}; \quad C = \frac{1}{2}; \quad D = \frac{1}{8} \tag{2.5.26}$$

Hence

$$Y(s) = -\frac{5}{8}\frac{1}{s} + \frac{3}{2}\frac{1}{s^2} + \frac{1}{2}\frac{1}{s+2} + \frac{1}{8}\frac{1}{s+4} \tag{2.5.27}$$

Therefore the time response, $y(t) = \mathcal{L}^{-1}[Y(s)]$, can then be obtained as

$$y(t) = -\frac{5}{8} + \frac{3}{2}t + \frac{1}{2}e^{-2t} + \frac{1}{8}e^{-4t} \tag{2.5.28}$$

(ii) Determine the time response, $y(t)$ of a control system with the transfer function $\frac{3+7s-s^2}{2s^2+18}$, subject to an input function $x(t) = 2e^{2t}$.

Solution

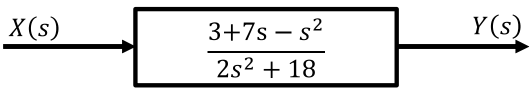


Figure 2.11

Transfer function

$$\frac{Y(s)}{X(s)} = \frac{3+7s-s^2}{2s^2+18} \tag{2.5.29}$$

For an input function

$$x(t) = 2e^{2t}, \tag{2.5.30}$$

$$X(s) = \frac{2}{s-2} \tag{2.5.31}$$

Hence

$$\begin{aligned} Y(s) &= \frac{2}{s-2}\frac{3+7s-s^2}{2s^2+18} \\ &= \frac{2}{s-2} \times \frac{3+7s-s^2}{2(s^2+9)} = \frac{3+7s-s^2}{(s-2)(s^2+9)} \end{aligned} \tag{2.5.32}$$

By partial fractions, this can be resolved into

$$\frac{3+7s-s^2}{(s-2)(s^2+9)} = \frac{A}{s-2} + \frac{Bs+C}{s^2+9} \tag{2.5.33}$$

Multiplying through by $(s-2)(s^2+9)$

$$3 + 7s - s^2 = A(s^2 + 9) + (Bs + C)(s - 2) \tag{2.5.34}$$

Expand the equation and collect like terms

$$3 + 7s - s^2 = s^2(A + B) + s(-2B + C) + (9A - 2C) \tag{2.5.35}$$

Comparing the like terms, the unknown coefficients are obtained as

$$A = 1; \quad B = -2; \quad C = 3 \tag{2.5.36}$$

Hence

$$Y(s) = \frac{1}{s-2} + \frac{-2s+3}{s^2+9} \tag{2.5.37}$$

Therefore the time response, $y(t) = \mathcal{L}^{-1}[Y(s)]$, can then be obtained as

$$\begin{aligned} y(t) &= \mathcal{L}^{-1}[\frac{1}{s-2}] + \mathcal{L}^{-1}[\frac{-2s+3}{s^2+9}] \\ &= \mathcal{L}^{-1}[\frac{1}{s-2}] + \mathcal{L}^{-1}[\frac{-2s}{s^2+3^2}] + \mathcal{L}^{-1}[\frac{3}{s^2+3^2}] \\ &= e^{2t} - 2\cos(3t) + \sin(3t) \end{aligned} \tag{2.5.38}$$

2.5.2 Transient Response - Second Order Systems

The transfer function for a 2nd order system, with a unit steady-state gain, is expressed as

$$\frac{Y(s)}{X(s)} = \frac{{\omega_n}^2}{s^2 + 2\zeta\omega_n s + {\omega_n}^2} \tag{2.5.39}$$

The transient behaviour of a second order control system is best evaluated in terms of its response to a unit step input. For a unit step input $x(t) = 1$,

$$X(s) = \frac{1}{s} \tag{2.5.40}$$

Hence

$$Y(s) = \frac{1}{s}\frac{{\omega_n}^2}{s^2 + 2\zeta\omega_n s + {\omega_n}^2} \tag{2.5.41}$$

This can be expressed as

$$Y(s) = \frac{{\omega_n}^2}{s(s-p_1)(s-p_2)} \tag{2.5.42}$$

where

$$p_1 = -\zeta\omega_n + \omega_n\sqrt{\zeta^2 - 1}; \quad p_1 = -\zeta\omega_n - \omega_n\sqrt{\zeta^2 - 1} \tag{2.5.43}$$

Using partial fractions

$$Y(s) = \frac{A_1}{s} + \frac{A_2}{s - p_1} + \frac{A_3}{s - p_2} \tag{2.5.44}$$

where

$$A_1 = 1; \quad A_2 = -\frac{1}{2} - \frac{\zeta}{2\sqrt{\zeta^2 - 1}}; \quad A_3 = -\frac{1}{2} + \frac{\zeta}{2\sqrt{\zeta^2 - 1}} \tag{2.5.45}$$

Therefore the time response, $y(t) = \mathcal{L}^{-1}[Y(s)]$, can then be obtained as

$$y(t) = A_1 + A_2 e^{p_1 t} + A_3 e^{p_2 t} \tag{2.5.46}$$

This solution, which is valid for damping ratio $\zeta \neq 1$, gives rise to two distinct types of transient response which depend on the value of ζ as:

(i) $\zeta > 1$: p_1 and p_2 are real and unequal. For this situation the response is **overdamped** (non-oscillatory with rapid amplitude decay).

(ii) $\zeta < 1$: p_1 and p_2 are complex conjugates as are A_2 and A_3. For this situation the response is **underdamped** (oscillatory with decaying amplitude). For this situation, p_1 and p_2 can be re-expressed as

$$p_1 = -\zeta\omega_n + i\omega_n\sqrt{1 - \zeta^2}; \quad p_1 = -\zeta\omega_n - i\omega_n\sqrt{1 - \zeta^2} \tag{2.5.47}$$

and the response can be re-written as

$$y(t) = 1 - \frac{e^{-\zeta\omega_n t}}{\sqrt{1 - \zeta^2}} \sin(\omega_n t\sqrt{1 - \zeta^2} + \phi) \tag{2.5.48}$$

where

$$\phi = \tan^{-1}\frac{\sqrt{1 - \zeta^2}}{\zeta} \tag{2.5.49}$$

(iii) $\zeta = 1$: p_1 and p_2 are real and equal ($= -\omega_n$). Here the response is said to be **critically damped** (extremely little oscillation with rapid amplitude decay) and its response can be given as

$$y(t) = 1 - (1 + \omega_n t)e^{-\omega_n t} \tag{2.5.50}$$

The transient response of the above three cases are summarised below

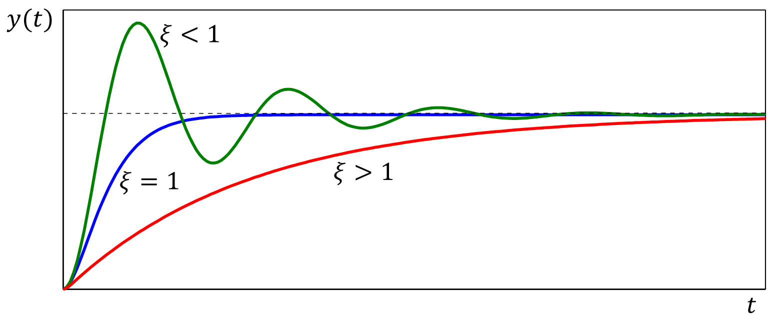


Figure 2.12: Responses of a second order system at different damping ratios

2.5.2.1 Practical Measures of the Transient Response

It is necessary to describe the step response of a system in terms of the parameters defined in the diagram below:

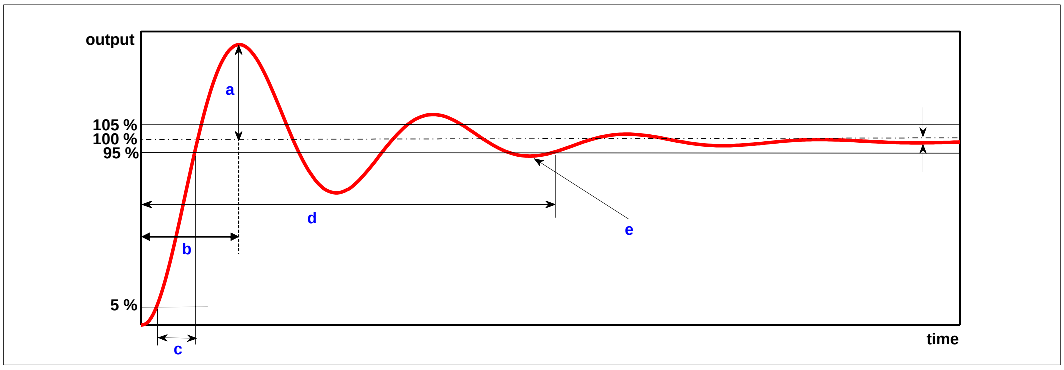


Figure 2.13: Practical measures of a system subject to a step excitation

(a) **Peak/Maximum Overshoot**: It is the maximum difference between the transient and the steady-state responses. It is expressed as a percentage of the steady state output (i.e. percentage peak/maximum overshoot).

$$\text{percentage peak overshoot} = 100e^{\frac{-\zeta\pi}{\sqrt{1-\zeta^2}}} \tag{2.5.51}$$

(b) **Peak Time** t_p: The time taken for the output (response) to rise from its initiation to the peak overshoot. It is expressed as

$$t_p = \frac{\pi}{\omega_n\sqrt{1-\zeta^2}} \tag{2.5.52}$$

(c) **Rise Time** t_r: The time taken for the output (response) to rise from 5% to 95% of step size. It is measured as

$$t_r = \frac{\pi}{2\omega_n\sqrt{1-\zeta^2}} \tag{2.5.53}$$

(d) **Settling Time** t_s: The time taken for the output to reach and remain within (say 5%) of steady-state value. E.g. within 5% of steady-state value,

$$0.05 = \mathrm{e}^{-\zeta\omega_n t_s} \tag{2.5.54}$$

Settling time is determined as

$$t_s = \frac{3}{\zeta\omega_n} \tag{2.5.55}$$

Within 3% of steady-state value,

$$0.03 = \mathrm{e}^{-\zeta\omega_n t_s} \tag{2.5.56}$$

Then the settling time is determined as

$$t_s = \frac{3.5}{\zeta\omega_n} \tag{2.5.57}$$

(e) **Number of Oscillations** N: This is the number of oscillations before the system settles to within a fixed percentage (5% say) of its steady state value. That is, the number of oscillations before reaching the settling time. It is expressed as

$$N = \frac{\text{settling time}}{\text{periodic time}} \tag{2.5.58}$$

E.g. For a 5% settling time, $t_s = \frac{3}{\zeta\omega_n}$ and periodic time $= \frac{2\pi}{\omega_d}$, oscillation numbers can be calculated as

$$N = \frac{3\omega_d}{2\pi\zeta\omega_n} \tag{2.5.59}$$

where $\omega_d = \omega_n\sqrt{1-\zeta^2}$ is the damped angular frequency. Therefore

$$N = \frac{3}{2\pi}\sqrt{\frac{1}{\zeta^2} - 1} \tag{2.5.60}$$

2.5.2.2 Worked Example

(1) A second order control system is described as

$$5D^2y(t) + 7Dy(t) + 20y(t) = 40x(t) \tag{2.5.61}$$

Determine the following:

(i) Undamped angular frequency (ii) Damping ratio

(iii) Damped angular frequency (iv) Time constant (vi) Rise time
(vii) the form of the transient response of the system, and using sketch, illustrate the transient behaviour.

Solution

Rearrange the equation to conform to the standard equation of second order DE model (Eq. 2.4.6) by dividing through by 5

$$\frac{d^2y(t)}{dt^2} + \frac{7}{5}\frac{dy(t)}{dt} + 4y(t) = 8x(t) \tag{2.5.62}$$

and comparing with the standard equation

$$\frac{d^2y(t)}{dt^2} + 2\zeta\omega_n\frac{dy(t)}{dt} + {\omega_n}^2 y(t) = G_{ss}{\omega_n}^2 x(t) \tag{2.5.63}$$

then

$$2\zeta\omega_n = 1.4; \quad {\omega_n}^2 = 4; \quad G_{ss}{\omega_n}^2 = 8 \tag{2.5.64}$$

(i) Undamped angular frequency ω_n is obtained as

$$\omega_n = 2\text{rad/s} \tag{2.5.65}$$

(ii) Damping ratio ζ is obtained as

$$\begin{aligned}\zeta &= \frac{1.4}{2\omega_n} \\ &= 0.35\end{aligned} \tag{2.5.66}$$

(iii) Damped angular frequency ω_d is obtained as

$$\begin{aligned}\omega_d &= \omega_n\sqrt{1-\zeta^2} \\ &= 2 \times \sqrt{1-0.35^2} \\ &= 1.87\text{rad/s}\end{aligned} \tag{2.5.67}$$

(iv) Time constant τ is obtained as

$$\begin{aligned}\tau &= \frac{1}{\zeta\omega_n} \\ &= 1.43\text{sec}\end{aligned} \tag{2.5.68}$$

(v) Rise time t_r is obtained as

$$\begin{aligned}t_r &= \frac{\pi}{2\omega_n\sqrt{1-\zeta^2}} \\ &= \frac{\pi}{2\omega_d} \\ &= 0.84\text{sec}\end{aligned} \tag{2.5.69}$$

(vi) Since the damping ratio $\zeta < 1$, the transient response is underdamped (Oscillatory with decaying amplitude). It is illustrated in Fig. (2.14).

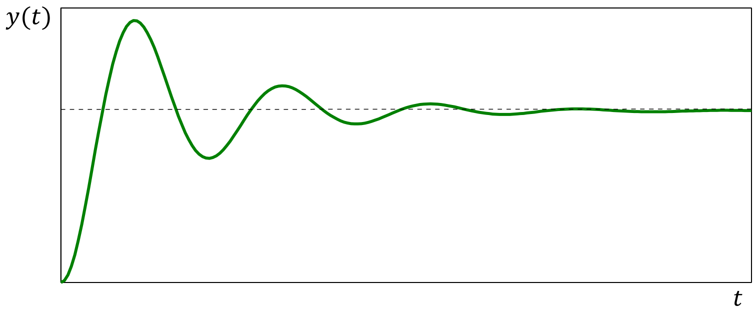


Figure 2.14

(2) A second order control system has a damping ratio of 0.4 and undamped angular frequency of 3 rad/s. Determine the following:
(i) Peak time (ii) Settling time within 5% of steady-state value
(iii) % Peak overshoot (iv) The differential equation of the system, when steady-state gain is 2.

Solution

Given

$$\zeta = 0.4; \quad \omega_n = 3rad/s \tag{2.5.70}$$

(i) Peak time, t_p is obtained as

$$\begin{aligned} t_p &= \frac{\pi}{\omega_n\sqrt{1-\zeta^2}} \\ &= 1.14\text{sec} \end{aligned} \tag{2.5.71}$$

(ii) Settling time, t_s
Within 5% of steady-state value

$$0.05 = e^{-\zeta\omega_n t_s} \tag{2.5.72}$$

Then Settling time is determined as

$$\begin{aligned} t_s &= \frac{3}{\zeta\omega_n} \\ &= 2.5\text{sec} \end{aligned} \tag{2.5.73}$$

(iii) % Peak overshoot is obtained as

$$\begin{aligned} \text{\% Peak overshoot} &= 100e^{\frac{-\zeta\pi}{\sqrt{1-\zeta^2}}} \\ &= 25.38\% \end{aligned} \tag{2.5.74}$$

(iv) The general differential equation for a second order system is given as

$$\frac{d^2y(t)}{dt^2} + 2\zeta\omega_n\frac{dy(t)}{dt} + {\omega_n}^2 y(t) = G_{ss}{\omega_n}^2 x(t) \tag{2.5.75}$$

Substituting $\zeta = 0.4$, $\omega_n = 3$ rad/s and $G_s s = 2$, then the DE becomes

$$\frac{d^2y(t)}{dt^2} + 2.4\frac{dy(t)}{dt} + 9y(t) = 18x(t) \tag{2.5.76}$$

CHAPTER 3

Block Diagram Models and Transfer Functions

Systems Engineers represent the components of a system as a cluster of blocks known as the block diagram. The lines connecting the blocks represent the signals, the summing junctions (where signals are algebraically summed) are represented with circles and the polarities of these signals are also considered. The forward path and the feedback path transfer functions are represented as $G(s)$ and $H(s)$ respectively.

3.1 Canonical Form of a Block Diagram

The aim of block diagram manipulation is to simplify the complex cluster of blocks in the simplest form known as the **canonical form**. Canonical form is the simplest form of a closed loop system in which a complex block diagram is simplified into a block diagram of a single forward path transfer function and a single feedback path transfer function. A canonical form block diagram is as shown below.

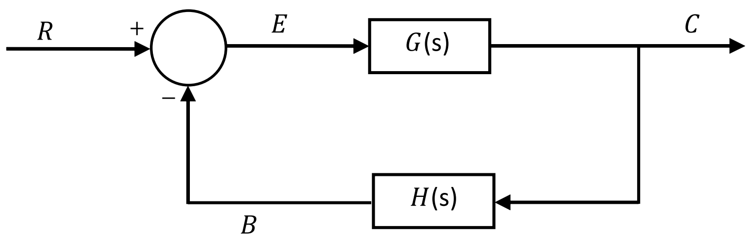
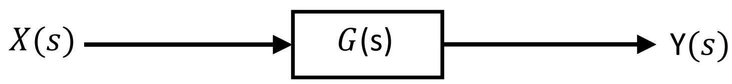


Figure 3.1: Canonical form of a block diagram

where R, E, C and B reference input, Error, Controlled output and Feedback signals respectively. Recall from

that the output signal can be represented in terms of the transfer function and the



input signal as

$$Y(s) = G(s)X(s) \tag{3.1.1}$$

Similarly, the following can be defined from the canonical block diagram

$$C = EG(s) \quad B = CH(s) \quad E = R - B \tag{3.1.2}$$

Combining these equations, the following ratios can be determined

(i) Control ratio/closed-loop transfer function

$$\begin{aligned} \text{Control ratio} &= \frac{\text{Controlled output signal}}{\text{Reference input signal}} \\ &= \frac{C}{R} \\ &= \frac{G(s)}{1 + G(s)H(s)} \end{aligned} \tag{3.1.3}$$

(ii) Error ratio

$$\begin{aligned} \text{Error ratio} &= \frac{\text{Error signal}}{\text{Reference input signal}} \\ &= \frac{E}{R} \\ &= \frac{1}{1 + G(s)H(s)} \end{aligned} \tag{3.1.4}$$

(iii) Feedback ratio

$$\begin{aligned} \text{Feedback ratio} &= \frac{\text{Feedback signal}}{\text{Reference input signal}} \\ &= \frac{B}{R} \\ &= \frac{G(s)H(s)}{1 + G(s)H(s)} \end{aligned} \tag{3.1.5}$$

(iv) Open-loop transfer function

$$\begin{aligned} \text{Open-loop transfer function} &= \frac{\text{Feedback signal}}{\text{Reference input signal}} \\ &= \frac{B}{E} \\ &= G(s)H(s) \end{aligned} \tag{3.1.6}$$

3.2 Block Diagram Algebra

The systematic procedure of block diagram algebra which can assist in the simplification of relatively complex block diagrams is considered in this section. The rules of block diagram algebra are illustrated as follows:

(i) **Elements in series**
As shown in Fig. 3.3.

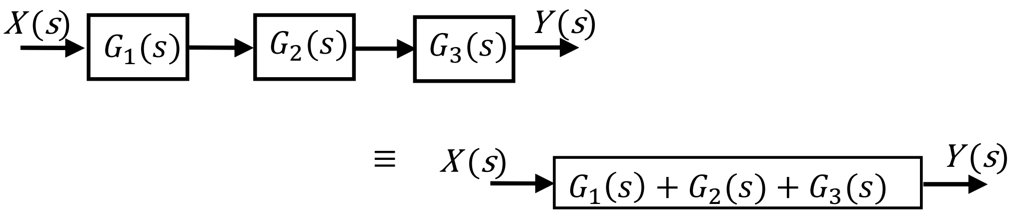


Figure 3.3

(ii) **Elements in parallel**
As shown in Fig. 3.4.

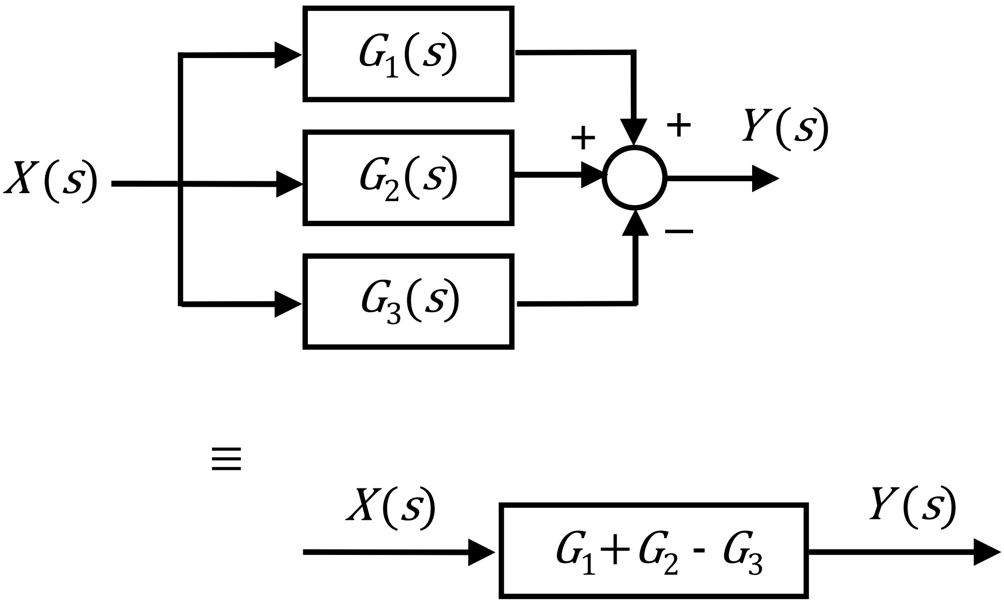


Figure 3.4

(iii) **Moving a branch point after a block**
As shown in Fig. 3.5.

(iv) **Moving a branch point before a block**
As shown in Fig. 3.6.

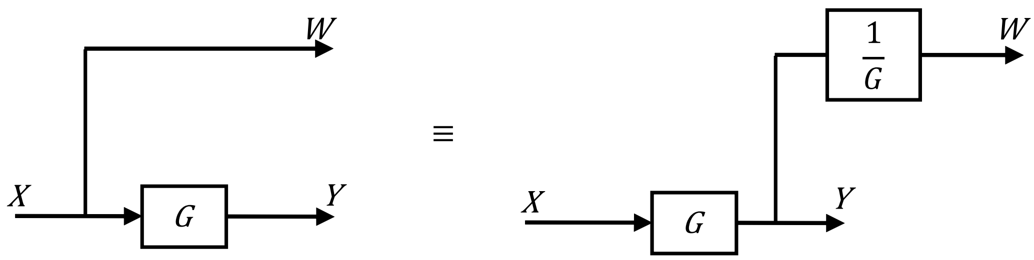


Figure 3.5

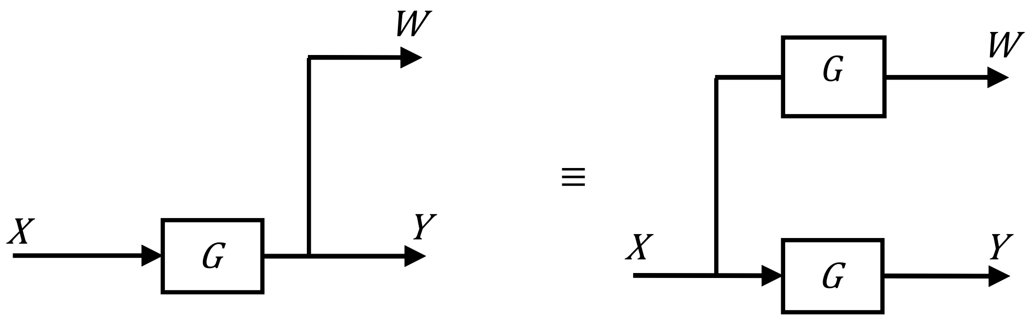


Figure 3.6

(v) **Moving a summing junction after a block**
As shown in Fig. 3.7.

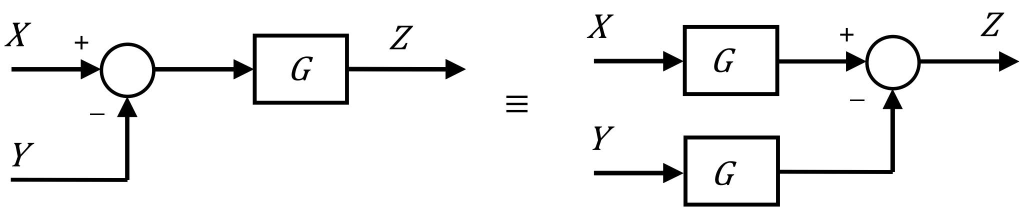


Figure 3.7

(vi) **Moving a summing junction before a block**
As shown in Fig. 3.8.

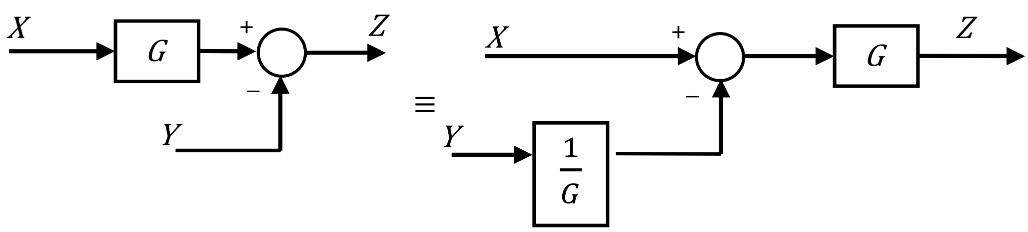


Figure 3.8

(vii) **Removing a block from a feedback loop**
As shown in Fig. 3.9.

(viii) **Combining adjacent summing junctions**

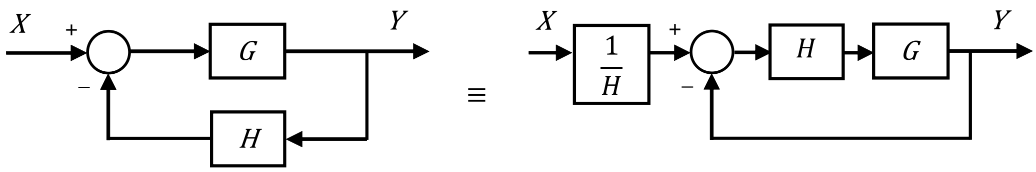


Figure 3.9

As shown in Fig. 3.10.

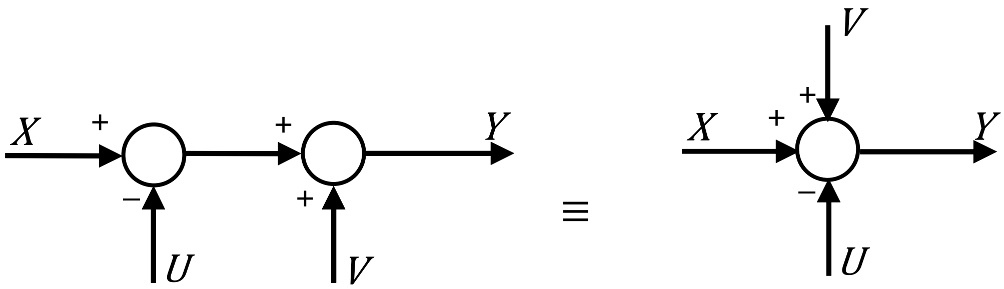


Figure 3.10

3.2.1 Worked Examples

(1) Reduce the block diagram (Fig. 3.11) and find the following:

(i) The feedforward transfer function (ii) The feedback transfer function (iii)

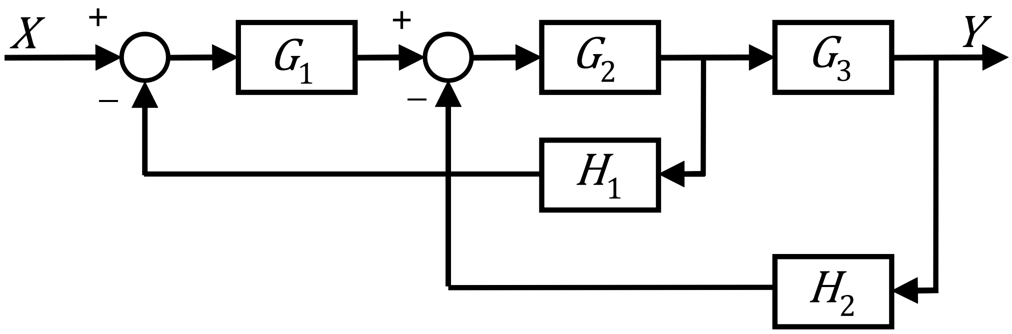


Figure 3.11

The open-loop transfer function (iv) The closed-loop transfer function.

Solution

Step 1: Rearrange (to avoid interlinking loops) by moving block H_1 beyond block G_3 as shown in Fig. 3.12.

Step 2: As shown in Fig. 3.13

Step 3: As shown in Fig. 3.14

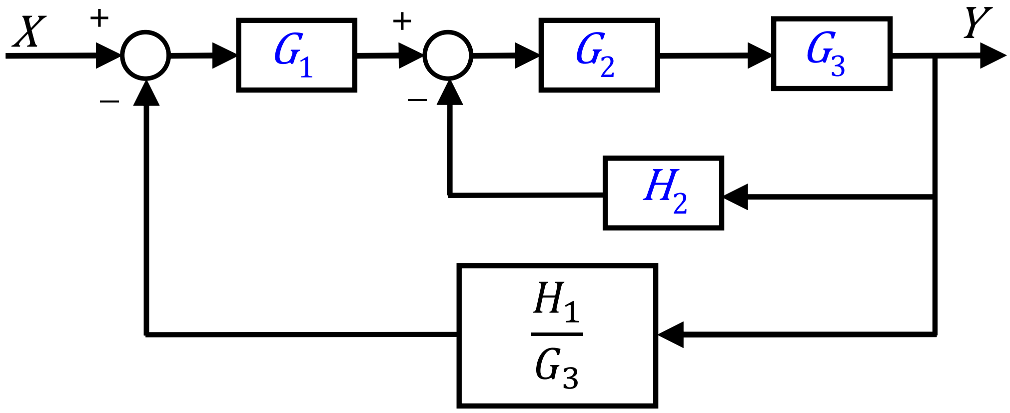


Figure 3.12

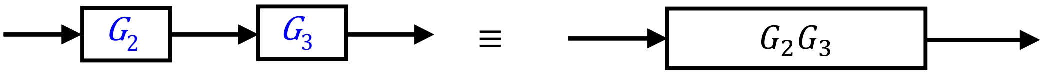


Figure 3.13

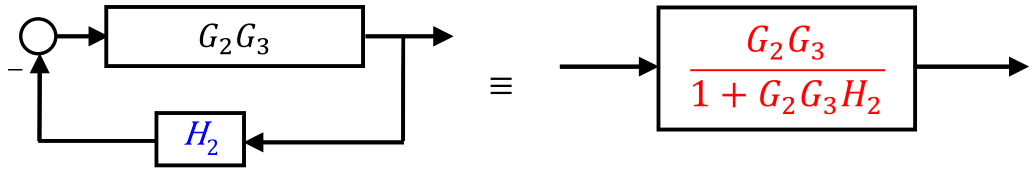


Figure 3.14

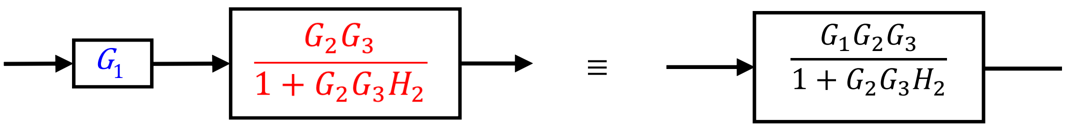


Figure 3.15

Step 4: As shown in Fig. 3.15

The canonical form of the block diagram is given as in Fig. 3.16

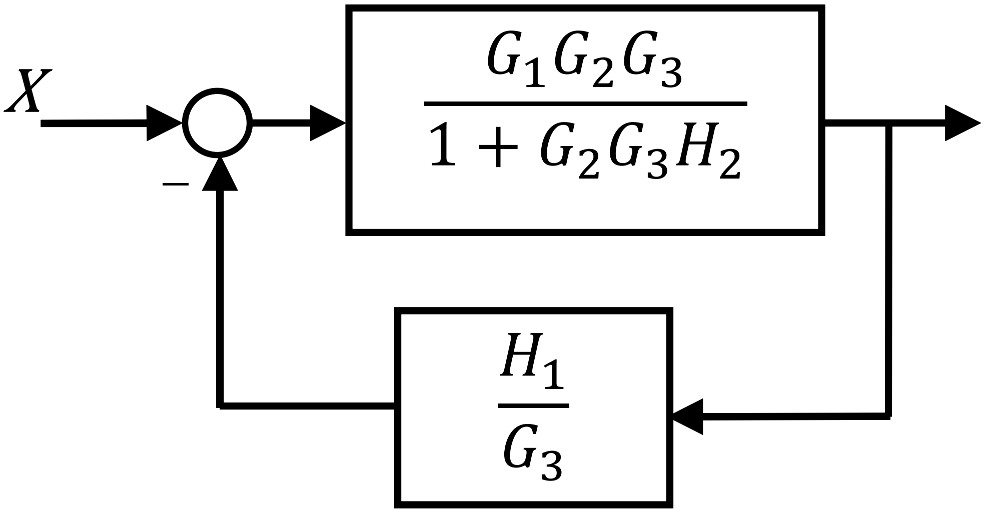


Figure 3.16

(i) The feedforward transfer function $G(s)$

$$G(s) = \frac{G_1G_2G_3}{1+G_2G_3H_2} \tag{3.2.1}$$

(ii) The feedback transfer function $H(s)$

$$H(s) = \frac{H_1}{G_3} \tag{3.2.2}$$

(iii) The open-loop transfer function G_{open}

$$\begin{aligned} G_{\text{open}} &= G(s)H(s) \\ &= \frac{G_1G_2H_1}{G_3+G_2G_3H_2} \end{aligned} \tag{3.2.3}$$

(iv) The closed-loop transfer function G_{closed}

$$\begin{aligned} G_{\text{closed}} &= \frac{G(s)}{1+G(s)H(s)} \\ &= \frac{G_1G_2G_3}{1+G_2G_3H_2+G_1G_2H_1} \end{aligned} \tag{3.2.4}$$

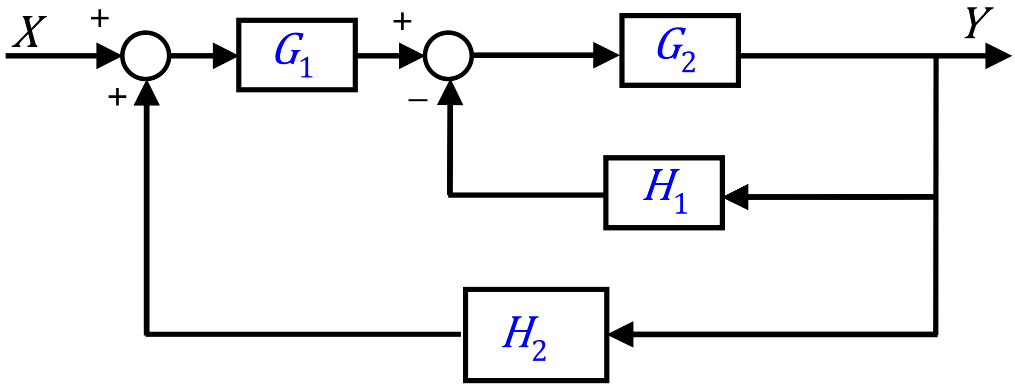


Figure 3.17

(2) Reduce the block diagram (Fig. 3.17) and find the overall transfer function

Solution

Step 1: As shown in Fig. 3.18

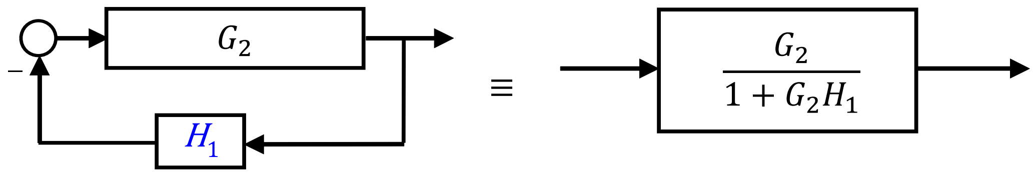


Figure 3.18

Step 2: As shown in Fig. 3.19

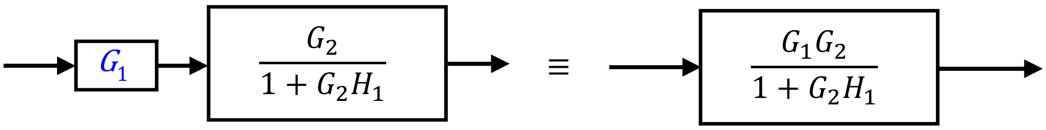


Figure 3.19

The canonical form of the block diagram is given as in Fig. 3.20

(3) The overall transfer function

$$\begin{aligned}\frac{Y(s)}{X(s)} &= \frac{G(s)}{1+G(s)H(s)} \\ &= \frac{G_1G_2}{1+G_2H_1+G_1G_2H_2}\end{aligned} \tag{3.2.5}$$

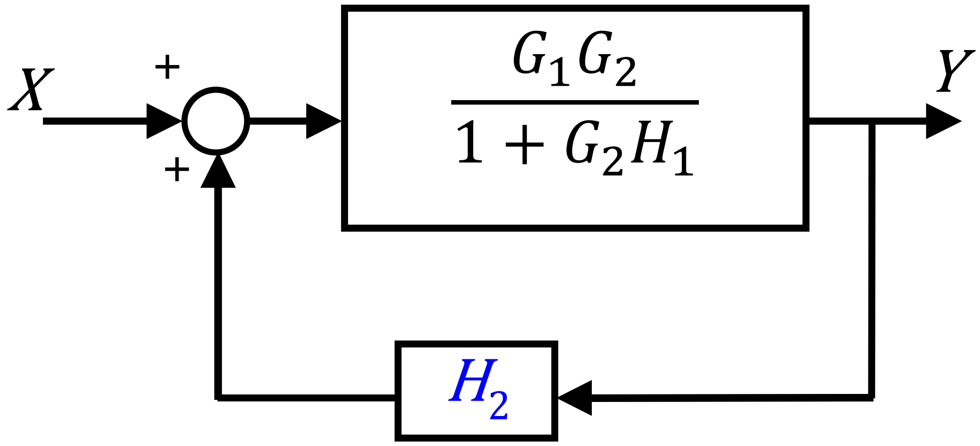


Figure 3.20

3.3 Disturbance signals and multiple inputs

A control system with more than one input signal acting simultaneously is referred to as a multiple input control system (Fig. 3.21). One or more of these multiple inputs can be in the form of disturbance input signals as shown in Fig. 3.21. The transfer function of a multiple input system can be computed using the principle of superposition theorem.

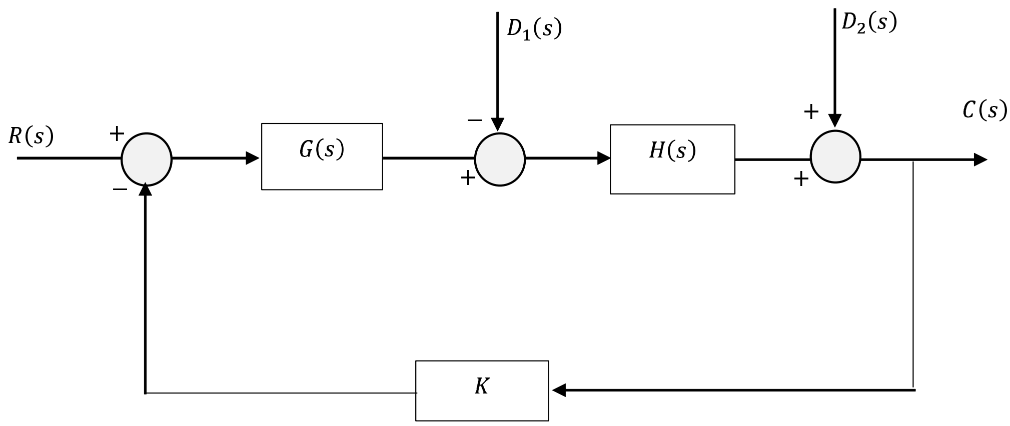


Figure 3.21

The superposition theorem states that the output ($Y(s)$) of a multiple input ($X_1(s), X_2(s), \ldots, X_n(s)$) system equals to the sum of the outputs due to each input signal acting alone while others are set to zero. That is the output of the system

$$Y(s) = \frac{Y(s)}{X_1(s)} X_1(s) + \frac{Y(s)}{X_2(s)} X_2(s) + \cdots + \frac{Y(s)}{X_n(s)} X_n(s) \tag{3.3.1}$$

is a function of the transfer functions due to each input signal. The procedure for the superposition theorem can be summarised as:

(i) Set all but one of the input signals to zero, turning the system into a single input system.

(ii) Determine the transfer function due to the one input signal.

(iii) Repeat steps (i) and (ii) for each of the inputs, in turn.

(iv) The overall transfer function of the system is then be determined as the algebraic sum of transfer functions due to each of the input signals.

3.3.1 Worked Example

Determine the output (response) of the system shown in Fig. 3.22.

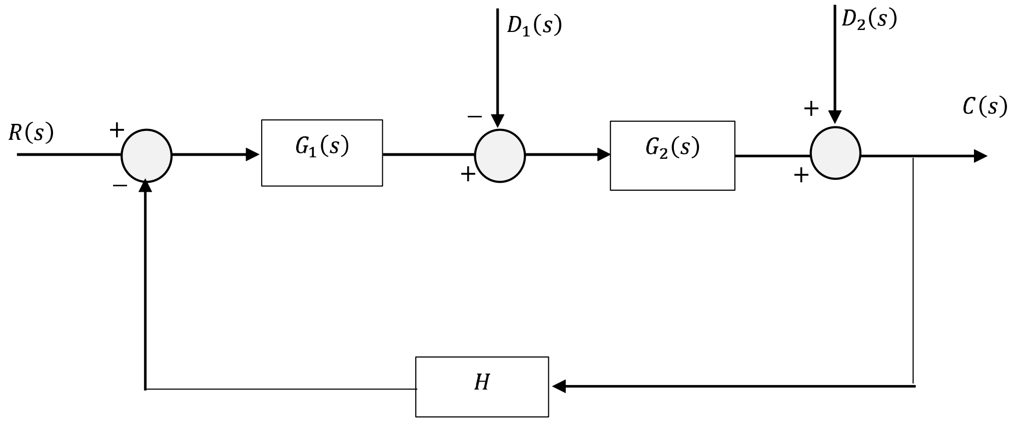


Figure 3.22

Solution

Step 1: Set $D_1(s)$ and $D_2(s)$ to zero and resolve the resulting block diagram into its canonical form as shown in Fig. 3.23.

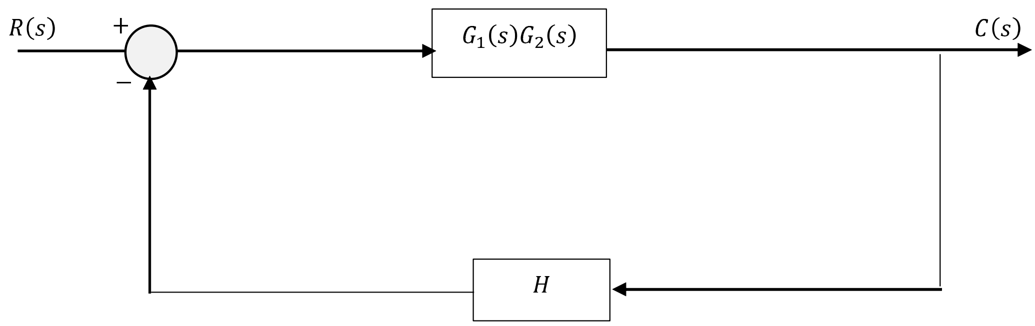


Figure 3.23

The transfer function due to the input function $R(s)$ is determined as

$$\frac{C(s)}{R(s)} = \frac{G_1(s)G_2(s)}{1 + G_1(s)G_2(s)H(s)} \tag{3.3.2}$$

Step 2: Set $R(s)$ and $D_2(s)$ to zero and resolve the resulting block diagram into its canonical form as shown in Fig. 3.24.

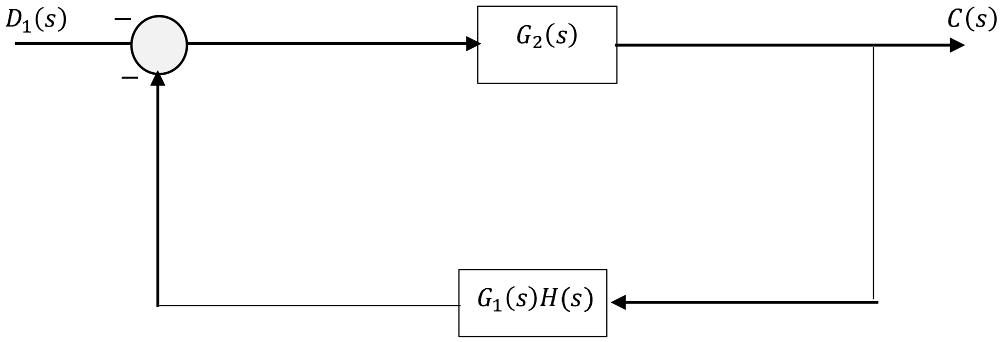


Figure 3.24

The transfer function due to the input function $D_1(s)$ is determined as

$$\frac{C(s)}{D_1(s)} = -\frac{G_2(s)}{1 + G_1(s)G_2(s)H(s)} \tag{3.3.3}$$

Step 3: Set $R(s)$ and $D_1(s)$ to zero and resolve the resulting block diagram into its canonical form as shown in Fig. 3.25.

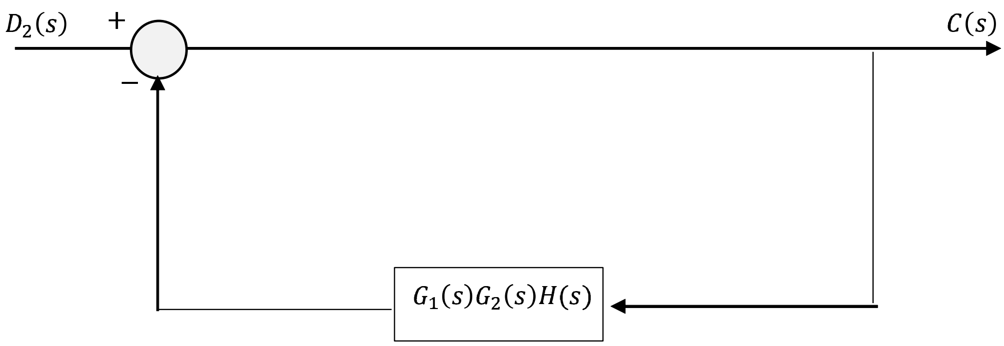


Figure 3.25

The transfer function due to the input function $D_2(s)$ is determined as

$$\frac{C(s)}{D_2(s)} = \frac{1}{1 + G_1(s)G_2(s)H(s)} \tag{3.3.4}$$

Step 4: Using the principle of superposition, the output would be given by

$$C(s) = \frac{C(s)}{R(s)}R(s) + \frac{C(s)}{D_1(s)}D_1(s) + \frac{C(s)}{D_2(s)}D_2(s)$$

$$= \frac{G_1(s)G_2(s)}{1 + G_1(s)G_2(s)H(s)}R(s) - \frac{G_2(s)}{1 + G_1(s)G_2(s)H(s)}D_1(s) + \frac{1}{1 + G_1(s)G_2(s)H(s)}D_2(s) \tag{3.3.5}$$

3.4 Practice Questions

(1) Using block reduction algebra, find the closed-loop transfer function of the block diagrams shown in Figs. 3.26 and 3.27

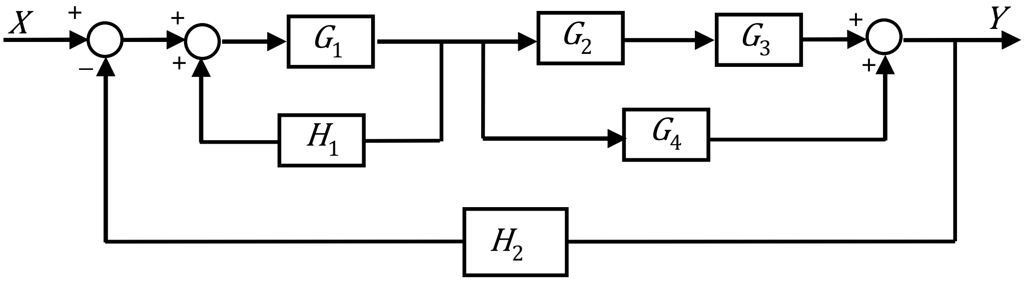


Figure 3.26

(a)

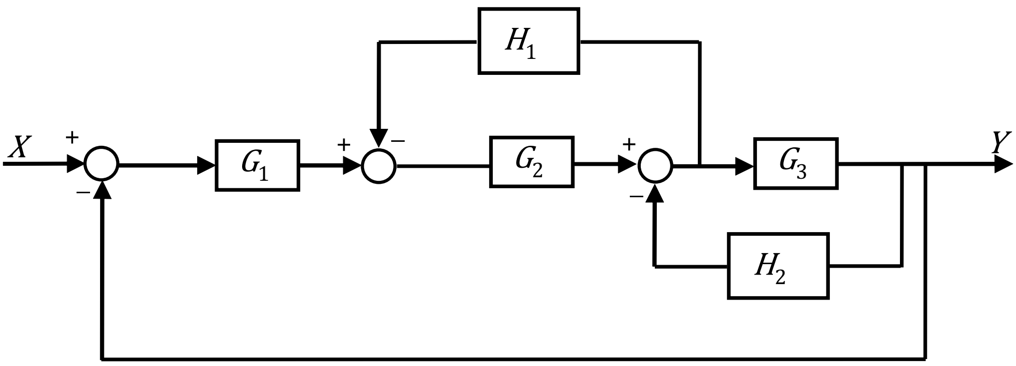


Figure 3.27

(b)

(2) Determine the output of the system shown in Fig. 3.28 in terms of reference input X and the disturbance inputs D_1 and D_2.

(3) For the control system shown in Fig. 3.29, determine the

(a) feedforward transfer function

(b) error ratio

(c) open-loop transfer function

(d) closed-loop transfer function

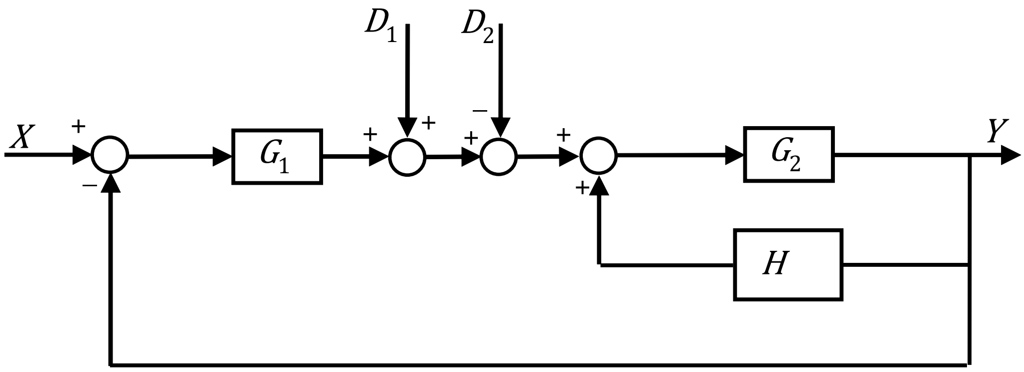


Figure 3.28

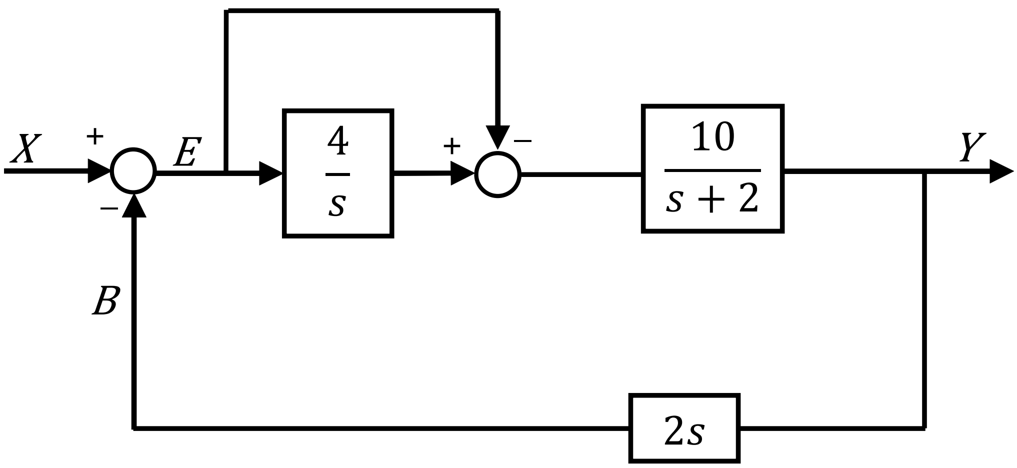


Figure 3.29

CHAPTER 4

Stability Analysis of Control Systems

4.1 Introduction

A system is said to be stable, if its output is under control.. Otherwise, it is said to be unstable. A stable system produces a bounded output for a given bounded input. The response of a stable system is as shown in Fig. 4.1.

The response is that of a first order control system subject to a unit step input. This response has the values between 0 and 1. So, it is a bounded output. The unit step signal has the value of one for all positive values of t including zero. So, it is a bounded input. Therefore, the first order control system is stable since both the input and the output are bounded.

4.2 Types of Stable Systems

Systems based on stability can be classified as thus:

1 **Absolutely Stable System**
If a system is stable for all the range of system components values, then it is said to be absolutely stable. The open-loop control system is absolutely stable if all the poles of the open-loop transfer function are present in the left half of s-plane. Similarly, the closed-loop control system is absolutely stable if all the poles of the closed-loop transfer function are present in the left half of the s-plane.

2 **Conditionally Stable System**
If a system is stable for a certain range of the system components values, then it is said to be conditionally stable.

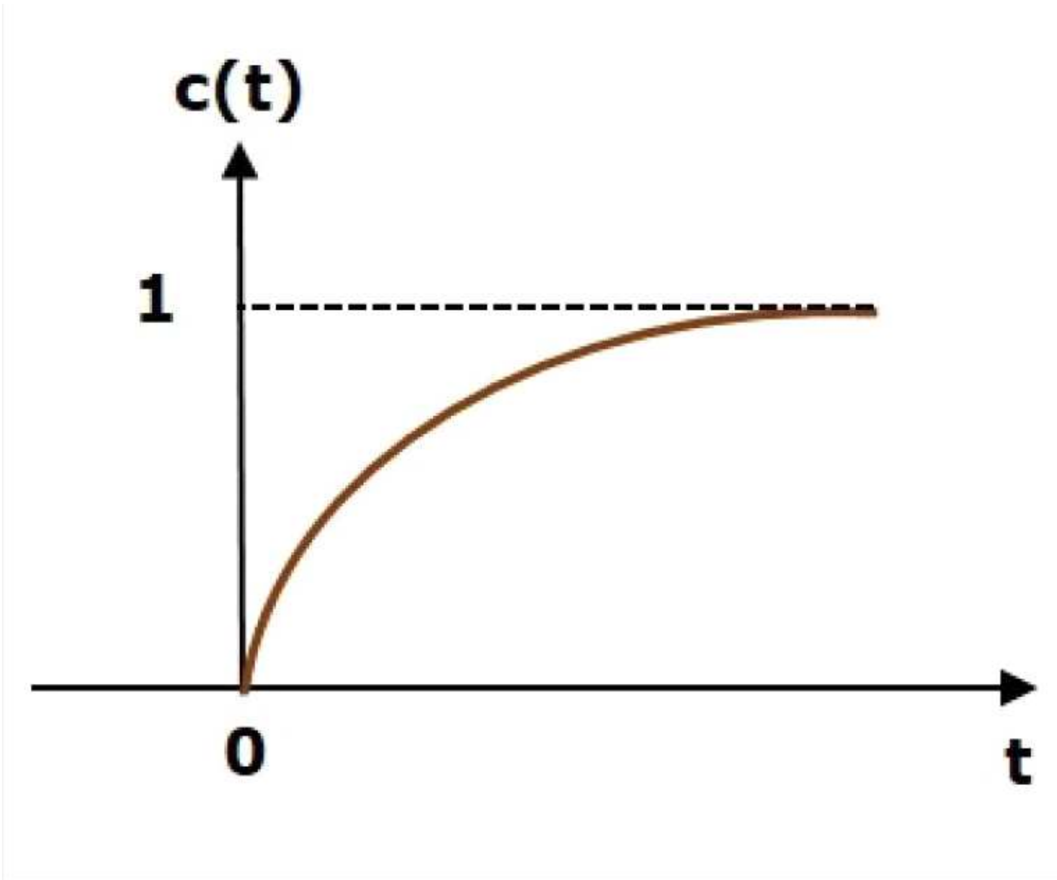


Figure 4.1

3 **Marginally Stable System**
If a system is stable by producing signals with constant amplitude and frequency of oscillation for bounded inputs, then it is said to be marginally stable. A open-loop control system is marginally stable if any two poles of the system's transfer function are present on the imaginary axis. Similarly, a closed-loop control system is marginally stable if any two poles of the system's transfer function are present on the imaginary axis.

4.3 Characteristic Equation, Poles and Zeros

As earlier defined, the transfer function $G(s)$ of a closed-loop control system can be expressed as the ratio of the output $Y(s)$ to the input $X(s)$, where X and Y are polynomials of Laplace operator s.

$$G(s) = k\frac{s^m + a_{m-1}s^{m-1} + a_{m-2}s^{m-2} + \cdots + a_1 s^1 + a_0}{s^n + b_{n-1}s^{n-1} + b_{n-2}s^{n-2} + \cdots + b_1 s^1 + b_0} \tag{4.3.1}$$

where $k = \frac{a_m}{b_n}$ is the system gain constant. The Laplace operator is a complex number, with a real component σ and an imaginary component ω, expressed as

$$s = \sigma + \mathrm{j}\omega \tag{4.3.2}$$

The polynomial equations can be expressed as products of linear equations, hence, the transfer function equation can be expressed in the form

$$G(s) = \frac{(s - z_1)(s - z_2)\dots(s - z_m)}{(s - p_1)(s - p_2)\dots(s - p_n)} \tag{4.3.3}$$

Definitions:

Characteristic equation: This is the equation that is obtained when the denomination of a transfer function equation is set to zero. For instance, the characteristic equation for the transfer function given in Eq. (4.3.1) is

$$s^n + b_{n-1}s^{n-1} + b_{n-2}s^{n-2} + \cdots + b_1 s^1 + b_0 = 0 \tag{4.3.4}$$

Poles: These are the values of s which make the transfer function infinite or undefined. That is, the roots of the characteristic equation. They are denoted as p_1, p_2, ..., p_n.

Zeros: These are the values of s which make the transfer function equal to zero. That is, the roots of the transfer function's numerator. They are denoted as z_1, z_2, ..., z_m.

s-Plane: This is a rectangular plane, used to represent the Laplace operator ($s = \sigma + j\omega$), with the horizontal axis as the real axis (σ) and the vertical axis as the imaginary axis (ω).)

Pole-Zero Plot: This is the plot of poles and zeros of a transfer function on the s-plane, in which the positions of the poles are marked with crosses (x) and that of the zeros with small circles (o).

4.3.1 Worked Example

Determine the poles and zeros of a control system modelled by the differential equation

$$\frac{3d^2y(t)}{dt^2} + 3\frac{dy(t)}{dt} - 18y(t) = 2\frac{dx(t)}{dt} - x(t) \tag{4.3.5}$$

where $x(t)$ and $y(t)$ are respectively the input and the output functions.

Solution

Using the D-operator, the differential equation can be expressed as

$$(3D^2 + 3D - 18)y(t) = (2D - 1)x(t) \tag{4.3.6}$$

Converting the D-operator to the Laplace operator, the transfer function of the system can be expressed as

$$G(s) = \frac{Y(s)}{X(s)} = \frac{2s - 1}{3s^2 + 3s - 18} \tag{4.3.7}$$

Re-write the transfer function in the form expressed by Eq. (4.3.3)

$$G(s) = \frac{2}{3}\frac{(s - \frac{1}{2})}{(s-2)(s+3)} \tag{4.3.8}$$

(i) The Poles

The poles are determined through the characteristics equations

$$(s-2)(s+3) = 0 \tag{4.3.9}$$

Therefore the poles are

$$p_1 = 2 \quad p_2 = -3 \tag{4.3.10}$$

(ii) The Zeros

The zeros are determined by setting the numerator of the transfer functions to zero. That is

$$(s - \frac{1}{2}) = 0 \tag{4.3.11}$$

Therefore there is only one zero

$$z_1 = \frac{1}{2} \tag{4.3.12}$$

4.4 Routh-Hurwitz Stability Criteria

The stability of a system is dependent upon whether or not the roots of a characteristic equation (poles of the system transfer function) lie in the right hand half of the s-plane. The presence of a root with a positive real part means that the output of a system will grow indefinitely with time after a disturbance is applied and the system is unstable. The Routh-Hurwitz criteria provide the simplest method to determine system stability, based on a straightforward algebraic manipulation of the characteristic function. In general the characteristic equation $P(s)$ can be written as:

$$P(s) = a_0 s^n + a_1 s^{n-1} + a_2 s^{n-2} + \cdots + a_{n-1} s^1 + a_n = 0 \tag{4.4.1}$$

where $P(s)$ is an nth order polynomial. For first and second order systems (n=1 or 2) the roots of this equation can be calculated easily. However, for higher order systems the task of determining the roots can be more time-consuming. For such systems it is convenient to consider the Routh-Hurwitz criteria which provide a means of determining whether any of the roots lie to the right of the imaginary axis in the s-plane, without having to determine the values of these roots. The Routh-Hurwitz criteria are:

(i) A necessary but not sufficient condition that no root of Eq. (4.4.1) lies in the right half of the s-plane is that all of the coefficients a_0, a_1, ..., a_n, are non-zero and have the same sign. Thus, provided that a_0 is positive, if one of the other coefficients is negative, or one of the powers of s is absent, then at least one root of the characteristic function lies to the right of the imaginary axis and therefore the system is unstable.

(ii) Provided that condition (i) is satisfied, then the necessary and sufficient condition that no root of Eq. (4.4.1) lies on the right hand side of the s-plane is that the Hurwitz determinants of the polynomial must be positive, where the Hurwitz determinants are given by:

$$D_1 = a_1, \quad D_2 = \begin{vmatrix} a_1 & a_3 \\ a_0 & a_2 \end{vmatrix}$$

$$D_3 = \begin{vmatrix} a_1 & a_3 & a_5 \\ a_0 & a_2 & a_4 \\ 0 & a_1 & a_3 \end{vmatrix}, \quad D_4 = \begin{vmatrix} a_1 & a_3 & a_5 & a_7 \\ a_0 & a_2 & a_4 & a_6 \\ 0 & a_1 & a_3 & a_5 \\ 0 & 0 & a_2 & a_4 \end{vmatrix} \tag{4.4.2}$$

For an equation of order n, there will be n determinant equations. Some of the arithmetic involved in calculating these determinants can be avoided by using the Routh array which is formed as follows:

$$\begin{array}{c|ccccc}
s^n & a_0 & a_2 & a_4 & a_6 & \dots \\
s^{n-1} & a_1 & a_3 & a_5 & a_7 & \dots \\
s^{n-2} & b_1 & b_2 & b_3 & \dots & \dots \\
s^{n-3} & c_1 & c_2 & c_3 & \dots & \dots \\
s^{n-4} & d_1 & d_2 & d_3 & \dots & \dots \\
\dots & \dots & \dots & \dots & \dots & \dots \\
\dots & \dots & \dots & \dots & \dots & \dots \\
s^0 & \dots & \dots & \dots & \dots & \dots
\end{array}$$

The first two rows are formed directly from the coefficients of Eq. (4.4.1), while the values in the third and subsequent rows are calculated as follows:

$$\begin{aligned}
b_1 = \frac{a_1a_2 - a_0a_3}{a_1} \quad b_2 = \frac{a_1a_4 - a_0a_5}{a_1} \quad b_3 = \frac{a_1a_6 - a_0a_7}{a_1} \quad \dots \\
c_1 = \frac{b_1a_3 - a_1b_2}{b_1} \quad c_2 = \frac{b_1a_5 - a_1b_3}{a_1} \quad \dots \quad \dots \\
d_1 = \frac{c_1b_2 - b_1c_2}{c_1} \quad \dots \quad \dots \quad \dots
\end{aligned} \tag{4.4.3}$$

On completion, the array has $n+1$ rows and the last row is always indicated by s^0. Every change of sign in the first column of the array indicates the presence of a root which lies to the right of the imaginary axis in the s-plane. Hence, for the system to be stable, all values of the first column must be positive. The procedure breaks down if either:

(i) A zero appears in the first column;
Or

(ii) A complete row of zeros appears so that the array cannot be completed.

For this course, examples will avoid these exceptions; however, there are standard techniques where these problems can be overcome.

4.4.1 Worked Example

The characteristic equation of a control system is given as

$$P(s) = 2s^3 + 4s^2 + 4s + 12 = 0 \tag{4.4.4}$$

Is the system stable or unstable? If it is unstable, how many roots lie in the right half of the s-plane?

Solution

Given that the coefficients of the characteristic equation are non-zero and have the same sign, the stability of the system can be investigated using criterion (2): Provided that condition (1) is satisfied, then the necessary and sufficient condition that no root of equation (1) lies on the right hand side of the s-plane is that the Hurwitz determinants of the polynomial must be positive. Routh array is constructed for the system:

$$\begin{array}{c|ccccc}
s^n & a_0 & a_2 & a_4 & a_6 & \dots \\
s^{n-1} & a_1 & a_3 & a_5 & a_7 & \dots \\
s^{n-2} & b_1 & b_2 & b_3 & \dots & \dots \\
s^{n-3} & c_1 & c_2 & c_3 & \dots & \dots \\
s^{n-4} & d_1 & d_2 & d_3 & \dots & \dots \\
\dots & \dots & \dots & \dots & \dots & \dots \\
\dots & \dots & \dots & \dots & \dots & \dots \\
s^0 & \dots & \dots & \dots & \dots & \dots
\end{array}$$

From the characteristic equation, we need a 3×4 Routh Table:

$$\begin{aligned}
b_1 = \frac{a_1 a_2 - a_0 a_3}{a_1} = \frac{16 - 24}{4} = -2 \quad b_2 = \frac{a_1 a_4 - a_0 a_5}{a_1} = 0 \quad b_3 = 0 \\
c_1 = \frac{b_1 a_3 - a_1 b_2}{b_1} = \frac{-24}{-2} = 12 \quad c_2 = \frac{b_1 a_5 - a_1 b_3}{a_1} = 0
\end{aligned} \tag{4.4.5}$$

$$\begin{array}{c|ccc}
s^3 & 2 & 4 & 0 \\
s^2 & 4 & 12 & 0 \\
s^1 & \text{-4} & 0 & 0 \\
s^0 & 12 & 0 & 0
\end{array}$$

There are two sign changes in the first column, therefore the system is unstable and two roots of the characteristic equation will lie on the right half of the s-plane.

Chapter 5

Frequency Response Analysis: Nyquist Stability Criterion

5.1 Frequency Response Analysis

Frequency Response Function

This is the expression obtained when the system transfer function, G(s), is converted to frequency domain by replacing the Laplace operator 's' by 'jω'. It is expressed in terms of real and imaginary components as

$$G(j\omega) = \Re[G(j\omega)] + \Im[G(j\omega)] \tag{5.1.1}$$

Magnification/Amplitude Ratio

This is the ratio of the amplitude of steady state sinusoidal output to the amplitude of sinusoidal input. It is the magnitude or gain of the frequency response function. It is expressed as

$$M(\omega) = \left|G(j\omega)\right| \tag{5.1.2}$$

Phase Shift

This is the relative angle by which the output shifts from the input and is defined as

$$\tan\phi = \frac{\Im[G(j\omega)]}{\Re[G(j\omega)]} \tag{5.1.3}$$

5.1.1 Worked Examples

(1) For a system with the transfer function $G(s) = \frac{1}{s+2}$, determine the following:

(i) Frequency response function

(ii) Amplitude magnification; and

(iii) Phase shift

Solution

$$G(s) = \frac{1}{s+2} \tag{5.1.4}$$

(i) Frequency response function

$$G(j\omega) = \frac{1}{2+j\omega} \tag{5.1.5}$$

rationalise and simplify

$$G(j\omega) = \frac{1}{2+j\omega} \times \frac{2-j\omega}{2-j\omega} \tag{5.1.6}$$
$$= \frac{2-j\omega}{4+\omega^2} \tag{5.1.7}$$

(ii) Amplitude magnification/Gain
Split the FRF into components

$$G(j\omega) = \frac{2-j\omega}{4+\omega^2} \tag{5.1.8}$$
$$= \frac{2}{4+\omega^2} + j\frac{-\omega}{4+\omega^2} \tag{5.1.9}$$

Hence

$$M(\omega) = \sqrt{\left(\frac{2}{4+\omega^2}\right)^2 + \left(\frac{-\omega}{4+\omega^2}\right)^2} \tag{5.1.10}$$
$$= \frac{1}{\sqrt{4+\omega^2}} \tag{5.1.11}$$

(iii) Phase shift

$$G(j\omega) = \frac{2}{4+\omega^2} + j\frac{-\omega}{4+\omega^2} \tag{5.1.12}$$
$$\tan\phi = \frac{\frac{-\omega}{4+\omega^2}}{\frac{2}{4+\omega^2}} \tag{5.1.13}$$
$$= \frac{-\omega}{2} \tag{5.1.14}$$

(2) For a system with the transfer function $G(s) = \frac{640}{s(s+1)(4s+3)}$, determine the following:

(i) Frequency response function

(ii) Amplitude magnification; and

(iii) Phase shift

when $\omega = 3$.

Solution

$$G(s) = \frac{640}{s(s+1)(4s+3)} \tag{5.1.15}$$

(i) Frequency response function

$$G(j\omega) = \frac{640}{j\omega(1+j\omega)(3+j4\omega)} \tag{5.1.16}$$

(ii) Amplitude magnification/Gain

$$M(\omega) = \left|G(j\omega)\right| = \frac{640}{\omega\sqrt{(1+\omega^2)}\sqrt{(9+16\omega^2)}} \tag{5.1.17}$$

$$\left|G(j\omega)\right|_{\omega=3} = \frac{640}{3\times\sqrt{10}\times\sqrt{(9+144)}} \tag{5.1.18}$$

$$= 5.45 \tag{5.1.19}$$

(iii) Phase shift

$$\phi(\omega) = 0^\circ - [90^\circ + \tan^{-1}(\omega) + \tan^{-1}(\frac{4\omega}{3})] \tag{5.1.20}$$

$$= -(90^\circ + 71.6^\circ + 76^\circ) \tag{5.1.21}$$

$$= -237.6^\circ \tag{5.1.22}$$

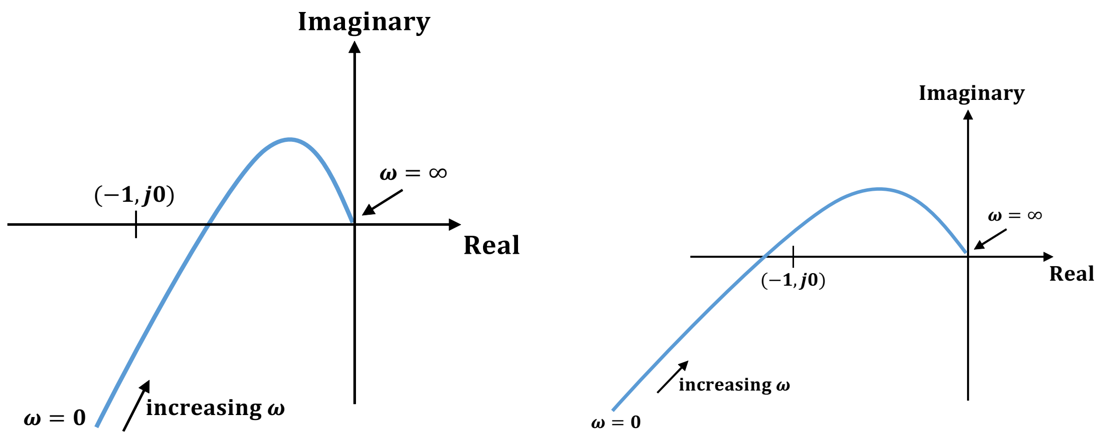
5.2 Nyquist Stability Criterion

Nyquist Diagram

This is a graphical method of determining the stability of a closed-loop system by investigating the properties of the plot of the open-loop transfer function. It is a polar plot of the open-loop frequency response of a system for frequencies ranging from $-\infty$ to $+\infty$.

Nyquist Stability Criterion

A control system is stable in the closed-loop, if the Nyquist curve plotted from the open-loop frequency response function, in the direction of increasing frequency, ω, does not enclose the critical point $(-1, j0)$. That's the gain is greater than -1 at $\phi = -180^\circ$. Nyquist curves depicting stability and instability are shown in Figure 5.1.



(a) Stable **(b)** Unstable

Figure 5.1: Illustration of Nyquist Stability Criterion

Gain Margin GM

This is the reciprocal of the gain (magnitude of the open-loop transfer function) when the phase shift $\phi = -180^\circ$.

$$GM = \frac{1}{\left|G(j\omega)\right|_{\phi=-180^\circ}} \tag{5.2.1}$$

This is the factor by which the system gain can be multiplied before instability sets in.

Phase Margin ϕ_{PM}

This is expressed as -180° minus the phase angle of the intersection which the critical point curve makes with the Nyquist plot.

$$\phi_{PM} = -180^\circ - \phi \tag{5.2.2}$$

It is the angle through which the Nyquist plot must be rotated in so that the unit magnitude point on the plot passes through the critical point $(-1, j0)$. It is also the angle by which ϕ lags or leads the -180° line. Illustrations of Phase Margin are shown in Figure 5.2.

From Figure 5.2;

Case 1:

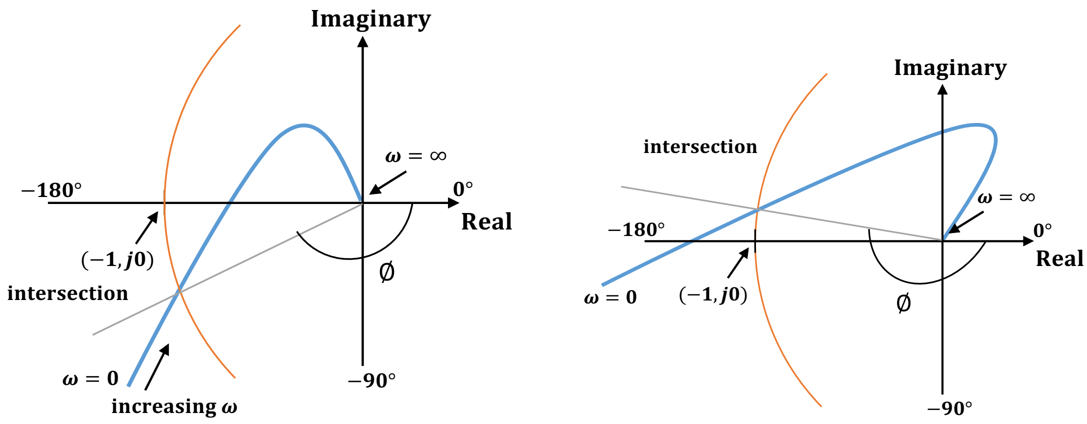
$$\phi_{PM} = -180^\circ - \phi = \text{-ve} \quad \text{(i.e. Lagging)} \tag{5.2.3}$$

Case 2:

$$\phi_{PM} = -180^\circ - \phi = \text{+ve} \quad \text{(i.e. Leading)} \tag{5.2.4}$$

A system is relatively stable if these conditions are satisfied:

(1) Gain Margin $GM > 1$; and



(a) Case1 **(b)** Case2

Figure 5.2: Illustration of Phase Margin

(2) Phase Margin ϕ_{PM} lags the -180° line (i.e. $-180^\circ - \phi$ is -ve)

Chapter 6

Controllers and Compensation Design

6.1 Controller

Controller

This is a control element which receives the error signal, modifies it according to certain control laws, and sends out a corrective signal to the plant.

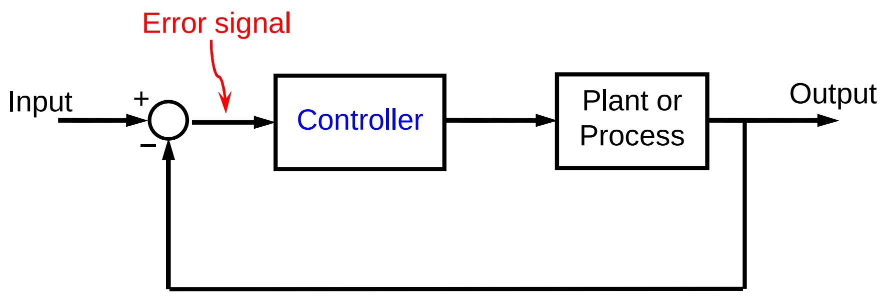


Figure 6.1: Illustration of a Controller

Control law

This is the relationship between the input and the output of the controller. There are three types; proportional, integral and derivative. They are shown in Figure 6.2.

Control objectives

These include

(i) To minimize the steady-state error

(ii) To minimize the settling time

(iii) To minimize the peak overshoot

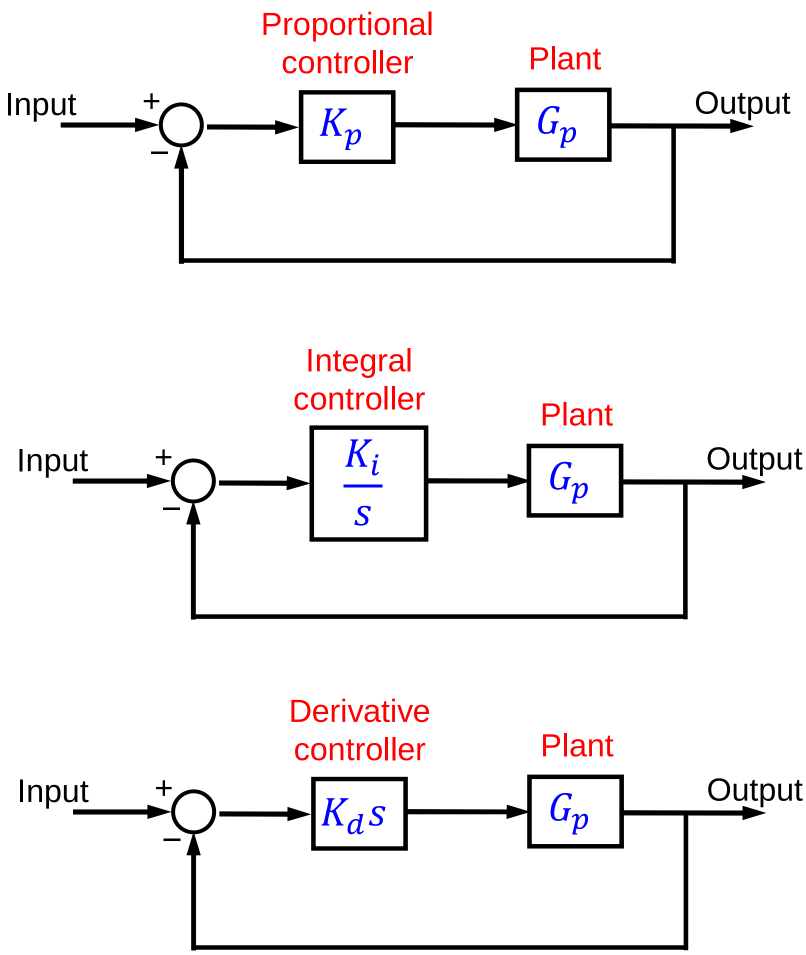


Figure 6.2: Illustration of Control Laws

6.1.1 Types of Controller

A controller may utilize:

(1) **PI Controller**

This utilizes proportional and integral control laws, as shown in Figure 6.3.

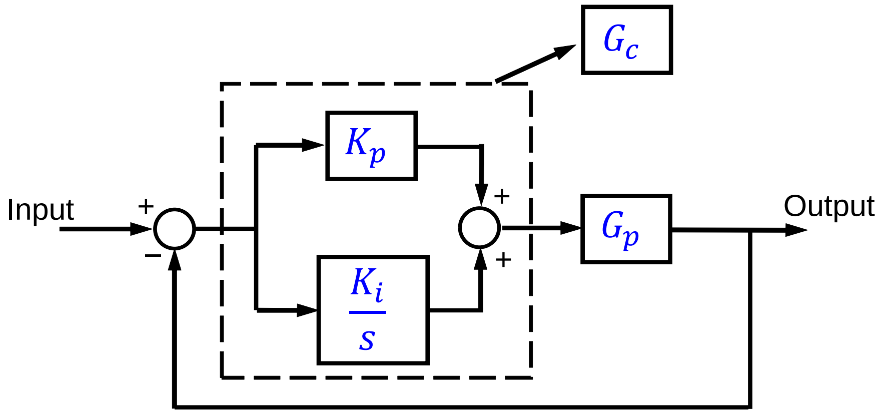


Figure 6.3: Illustration of PI Controller

where

$$G_c(s) = K_p + \frac{K_i}{s} \tag{6.1.1}$$

With the integral time constant, $\tau_i = \frac{K_p}{K_i}$, then

$$G_c(s) = K_p + \frac{K_p}{\tau_i s} \tag{6.1.2}$$

(2) **PD Controller**

This utilizes proportional and derivative control laws, as shown in Figure 6.4.

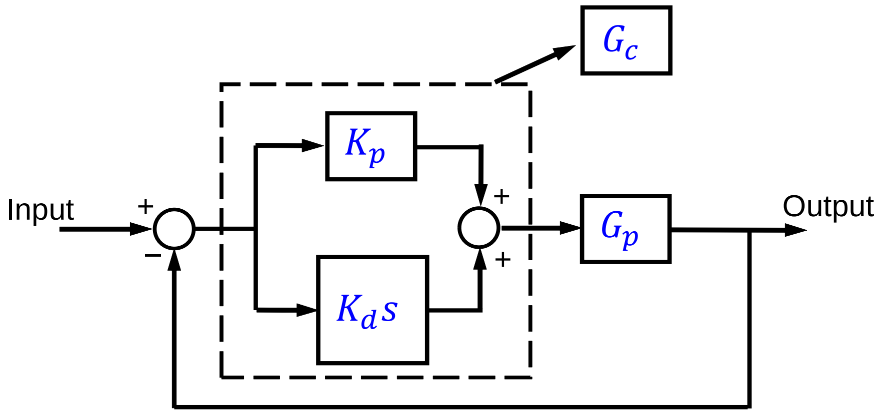


Figure 6.4: Illustration of PD Controller

where

$$G_c(s) = K_p + K_d s \tag{6.1.3}$$

With the derivative time constant, $\tau_d = \frac{K_d}{K_p}$, then

$$G_c(s) = K_p + K_p \tau_d s \tag{6.1.4}$$

(3) **PID Controller**

This utilizes proportional, integral and derivative control laws, as shown in Figure 6.5.

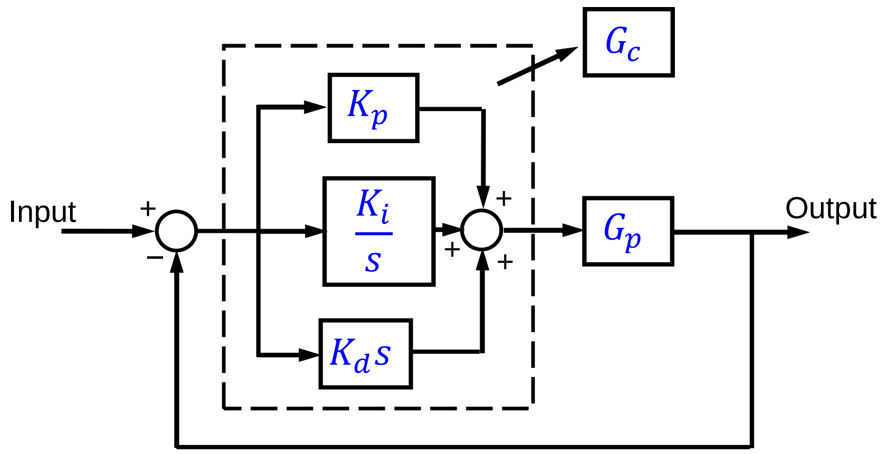


Figure 6.5: Illustration of PID Controller

where

$$G_c(s) = K_p + \frac{K_i}{s} + K_d s \tag{6.1.5}$$

With the integral and derivative time constants, $\tau_i = \frac{K_p}{K_i}$ and $\tau_d = \frac{K_d}{K_p}$, then

$$G_c(s) = K_p + \frac{K_p}{\tau_i s} + K_p \tau_d s \tag{6.1.6}$$

6.2 Compensation Design

The dynamics of controllers (K_p, K_i and K_d) and plant are sometimes impossible to alter, in order to obtain a satisfactory system performance specifications.

Compensation techniques may be used by inserting certain control devices to the control system loop in order to enhance the performance of the system. For instance, a suitable compensation design can stabilize an unstable system or increase its stability margin.

6.2.1 Types of Compensator

(1) **Cascade/series Compensator**

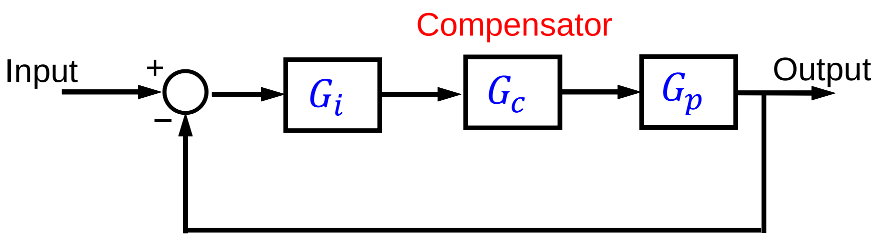


Figure 6.6: Illustration of Cascade/series Compensator

(a) **Cascade Lag Compensator:** also known as phase lag compensator. Its transfer function is given as

$$G_c(s) = \frac{k(s+z)}{s+p} \tag{6.2.1}$$

where $z > p$. That is, it introduces a real pole nearer to the origin than a real zero (as shown in Figure 6.7.

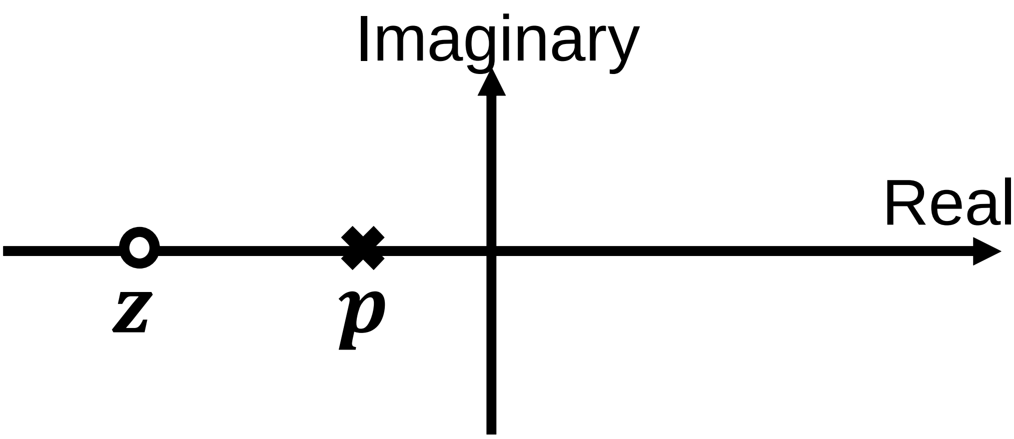


Figure 6.7: Illustration of the Pole-Zero Plot for Cascade/series Lag Compensator

Advantage: It permits increase in gain to reduce steady-state error.

(b) **Cascade Lead Compensator:** also known as phase lead compensator. Its transfer function is given as

$$G_c(s) = \frac{k(s+z)}{s+p} \tag{6.2.2}$$

where $p > z$. That is, it introduces a real zero nearer to the origin than a real pole (as shown in Figure 6.8.

Advantage: It permits increase in response of the system without loss of stability.

(c) **Cascade Lag-Lead Compensator:** in which the cascade lag compensator and cascade lead compensator are connected in series in order to utilize the advantages of both types of compensator.

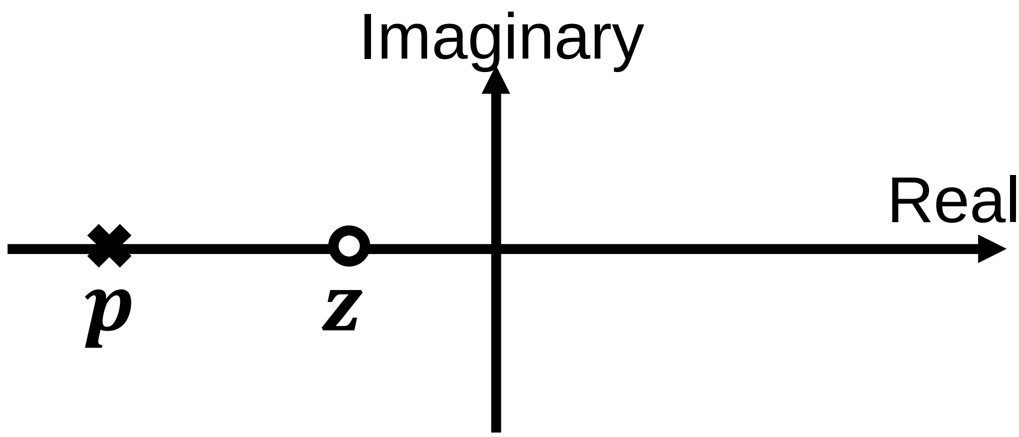


Figure 6.8: Illustration of the Pole-Zero Plot for Cascade/series Lead Compensator

(2) **Feed Forward Compensator**

This is useful in reducing the effect of disturbances in a system response. It is otherwise known as feed-forward disturbance compensator. It is illustrated in Figure 6.9

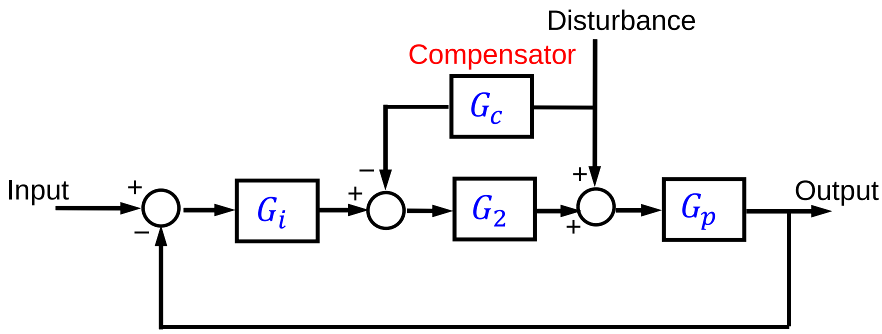


Figure 6.9: Illustration of Feed Forward Compensator

(3) **Feedback Compensator**

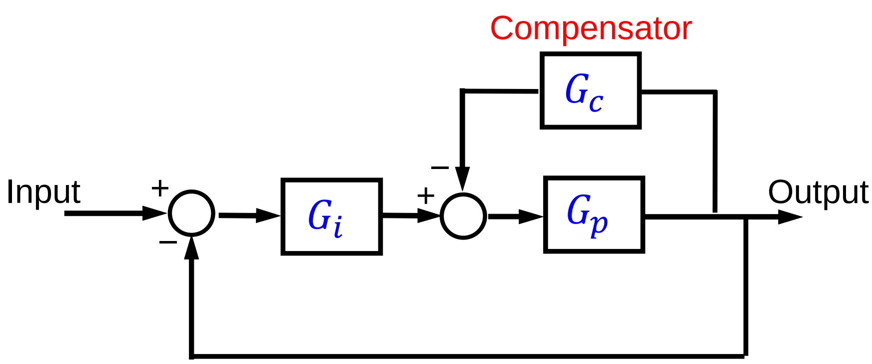


Figure 6.10: Illustration of Feedback Compensator

CHAPTER 7

Practice Questions

(1) The transfer function of a control system is given by the expression $\frac{1}{1+2s}$. For an applied step input function $q(t) = 3$ units, determine for the system (i) the response $m(t)$ and (ii) the steady state error.

(2) A speed control system with input variable $Q(s)$ and system output $C(s)$ is configured in the unity feedback and consists of a proportional controller of gain $K = 1$ and a motor modelled by the transfer function $G_p = \frac{1}{2s+2}$. (i) Determine the closed-loop transfer function and hence deduce that the steady state gain $= 0.33$ and time constant $= 0.67$ sec. (ii) Determine the system response $C(t)$ when input function $q(t) = 4t$ is applied to the control system.

(3) Differentiate between transient response and steady state response. The total response of a control system is given as $Y(s) = \frac{1}{s^2} - \frac{2}{s} + \frac{3}{0.5s+0.25}$. Determine the total response in the time domain $y(t)$ and hence separate this system response into its transient and steady state response components.

(4) (a) Sketch the unit step responses for a second order system under different damping conditions. (b) Determine the values of steady state gain; time constant; damping ratio; undamped natural angular frequency of oscillation; 5% settling time as applicable to the control systems modelled by the following differential equations:

(i) $5\frac{d^2v(t)}{dt^2} + 7\frac{dv(t)}{dt} + 20v(t) = 40u(t)$

(ii) $12\frac{dm(t)}{dt} + 40m(t) = 36q(t)$

(5) An accelerometer modelled as a second order system has a damping factor of 0.6, and is observed to oscillate at a frequency of $\omega_d = 80$ rad/s. (i) Deduce that undamped natural frequency $\omega_n = 80$ rad/s and peak overshoot $= 9.5\%$

(approximately). (ii) Given that the steady state gain $G_{ss} = 2$, output $= B(s)$ and input $= C(s)$, determine the differential equation model for the accelerometer.

(6) A feedback system consists of a plant of transfer function G_p, which is under the control action of a controller of transfer function G_c. The feedback loop contains an element of transfer function $H(s)$. Disturbance input $D_1(s)$ (with positive signal) is introduced between the plant and controller, and another disturbance input $D_2(s)$ (with negative signal) between the plant and feedback element. The reference input is $U(s)$ while the controlled output is $V(s)$. Using the Superposition Theorem, establish that:
$V(s) = \frac{1}{1+G_cG_pH}[G_cG_pU(s) + G_pD_1(s) + G_cG_pHD_2(s)$

(7) Consider the control system described by the block diagram shown in Figure 7.1. Determine the canonical feedback configuration, and hence deduce that: (i) error ratio $= \frac{s+1}{s+3}$ (ii) control ratio $= \frac{2}{s(s+3)}$.

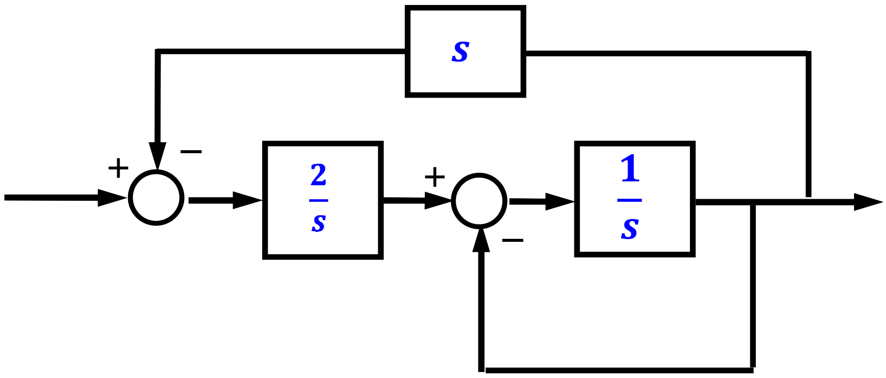


Figure 7.1

(8) A feedback system consists of a plant modelled by the transfer function $G(s)$ and a controller modelled by $G_c(s)$. Two feedback elements of transfer functions $H_1(s)$ and $H_2(s)$ are connected in series. (i) Sketch the system (ii) If a disturbance input $D_1(s)$ is introduced between the controller and the plant, and another disturbance $D_2(s)$ of negative signal occurs between the two feedback elements, establish the relation between the output variable $C(s)$ and the reference input variable $R(s)$ and the disturbance inputs.

(9) Determine the poles and zeros of a feedback system whose transfer function is given as $\frac{12-3s^2}{2s^4+16s^3+50s^2}$.

(10) A unity feedback system has a forward path transfer function $G_t = \frac{s+1}{s(8s^2+5)}$. Investigate the stability of this system by applying each of the following analytical methods: (i) the Pole-Zero diagram (ii) the Routh Hurwitz stability criterion.

(11) A servomechanism of controlled variable $Q_c(s)$ and reference input $Q_r(s)$ is described by the expression $Q_r(s) = \frac{(s^4+s^3-3s^2-s+2)}{(s^2+4s+4)} Q_c(s)$. (i) Determine the transfer function and characteristic equation for the servomechanism. (ii) By considering the Routh array, determine the stability of the system.

(12) A control system has an open-loop transfer function $\frac{K}{4s+1}$. Show that the magnitude and phase shift of the open-loop frequency response function are given by $\left|G(j\omega\right| = \frac{K}{\sqrt{1+16\omega^2}}$ and $\phi(\omega) = -\tan^{-1} 4\omega$ radians respectively.
Using the given magnitude and phase expressions above, and taking $K = 10$, evaluate: (i) the gain when $\omega = 20$ rad/s. (ii) the gain margin.

(13) Determine the (i) frequency response function (ii) its magnitude and (iii) phase shift for a feedback system with overall transfer function $\frac{2}{s+3}$.

(14) A speed control feedback loop in unity feedback form consists of a proportional controller of gain K and a motor of pure gain = 6. (a) Calculate the closed loop gain and steady state error as percentage of the desired speed, if the controller gain is (i) 1, (ii) 5 and (iii) 10. (b) If the gain of the motor increases by 20%, compute the new closed loop gains for controller gain $K =$ (i) 1, (ii) 5 and (iii) 10.

(15) A system has the following open-loop frequency response

ω (rad/s)	∞	17.3	8.48	3.56	2.06	1.41	0.97	0.56	0.24	0.12	0
Gain	0	0.01	0.04	0.20	0.46	0.71	0.97	1.26	1.45	1.49	1.50
ϕ	0°	-10°	-20°	-45°	-70°	-90°	-110°	-135°	-160°	-170°	-180°

(i) Using any suitable scale, plot the Nyquist diagram (ii) determine the phase margin (iii) determine the gain margin of the system. Would the system be stable in the closed-loop condition?

(16) A control system has a open-loop transfer function $\frac{K}{s(2s+1)(4s+1)}$. Show that the gain margin is $\frac{3}{4K}$ and hence determine the values of K which make the system unstable.

(17) A PID control system is used to control a plant having a transfer function of $\frac{1}{s(s+3+j)(s+3-j)}$. If the PID controller has an integral time constant of 4 sec and a derivative time constant of 1 sec, what will be the open-loop poles and zeros of the control system? (Hint: Integral time constant $\tau_i = \frac{K_p}{K_i}$ and Derivative time constant $\tau_d = \frac{K_d}{K_p}$).