• FUNCIONES DE VARIABLE COMPLEJA.

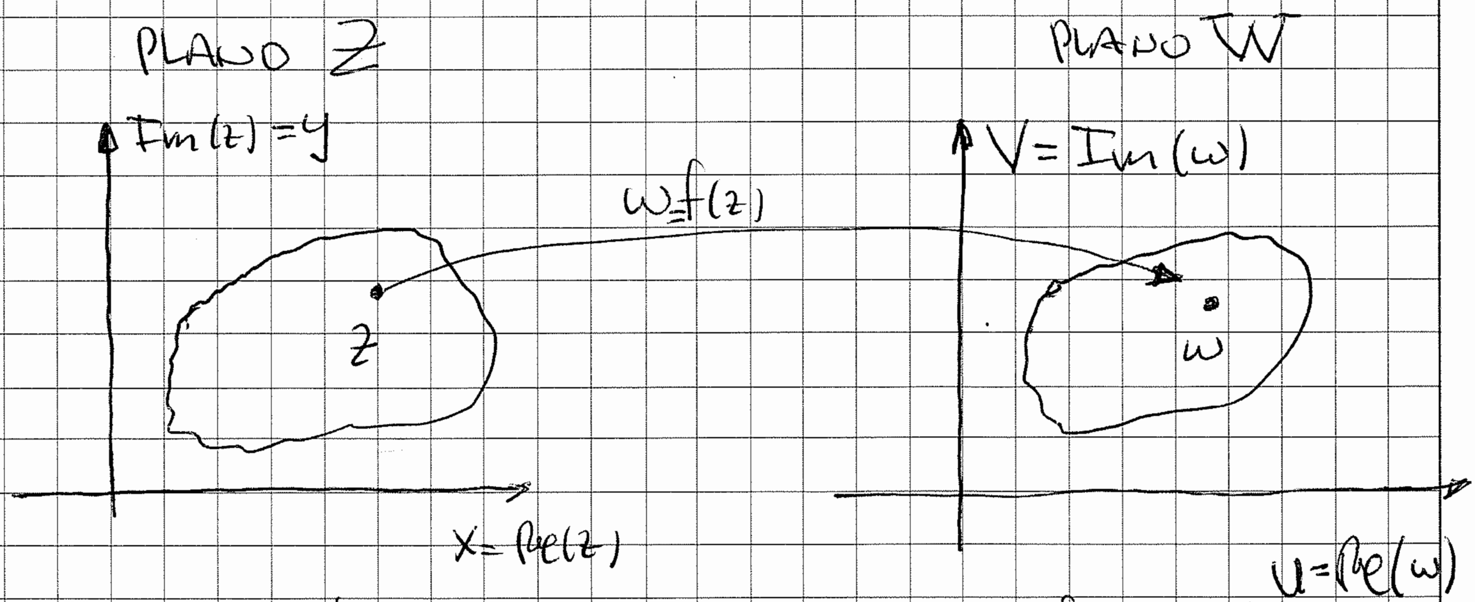
SE DICE f ES UNA FUNCIÓN DE UNA VARIABLE COMPLEJA z O FUNCIÓN COMPLEJA, LA IMAGEN w DE UN NÚMERO COMPLEJO z ES ALGÚN NÚMERO COMPLEJO $u+iv$, ES DECIR:

$$w = f(z) = u + iv = u(x,y) + i\,v(x,y),$$

DONDE u Y v SON LAS PARTES REAL E IMAGINARIA DE w, Y SON FUNCIONES DE VALORES REALES.

$u = u(x,y)$ PARTE REAL DE w.

$v = v(x,y)$ PARTE IMAGINARIA DE w.



$z = x + iy$.

$w = f(z) = u + iv$.

UNA FUNCIÓN COMPLEJA $w=f(z)$ PUEDE INTERPRETARSE COMO UN MAPEO O TRANSFORMACIÓN DEL PLANO Z AL PLANO W.

EJEMPLOS:

EXPRESAR LAS SIGUIENTES FUNCIONES COMPLEJAS EN LA FORMA $w=f(z)=u(x,y)+iv(x,y)$.

(1) $f(z)=z^2+4z \;\; ; \;\; z=x+iy$.

SOL:

$$f(z)=z^2+4z=(x+iy)^2+4(x+iy)=x^2+2ixy+i^2y^2+4x+i4y$$

$$=x^2+i2xy-y^2+4x+i4y=x^2-y^2+4x+i(2xy+4y)$$

$$=x^2-y^2+4x+i(2xy+4y)=u(x,y)+iv(x,y).$$

DONDE:

$$u=u(x,y)=x^2-y^2+4x$$

$$v=v(x,y)=2xy+4y.$$

(2) $f(z)=z+\operatorname{Re}(z)$

SOL:

$$f(z)=(x+iy)+x=2x+iy.=u(x,y)+iv(x,y).$$

DONDE:

$$u=u(x,y)=2x \quad y \quad v=v(x,y)=y.$$

(3) $f(z) = \dfrac{z-1}{z^2}$, $z = x+iy$

$$f(z) = \frac{z-1}{z} = \frac{x+iy-1}{x+iy} = \frac{(x-1)+iy}{x+iy} = \frac{(x-1)+iy}{x+iy}\frac{(x-iy)}{(x-iy)}$$

$$= \frac{(x-1)x - i(x-1)y + iyx - i^2y^2}{x^2 - i^2y^2} = \frac{(x-1)x - i(x-1)y + iyx + y^2}{x^2+y^2}$$

$$= \frac{(x-1)x + y^2 + i(-(x-1)y + yx)}{x^2+y^2} = \frac{(x-1)x+y^2}{x^2+y^2} + i\frac{(yx-(x-1)y)}{x^2+y^2}$$

DONDE:

$$u = u(x,y) = \frac{(x-1)x+y^2}{x^2+y^2}$$

$$v = v(x,y) = \frac{xy-(x-1)y}{x^2+y^2} = \frac{xy-xy+y}{x^2+y^2} = \frac{y}{x^2+y^2}$$

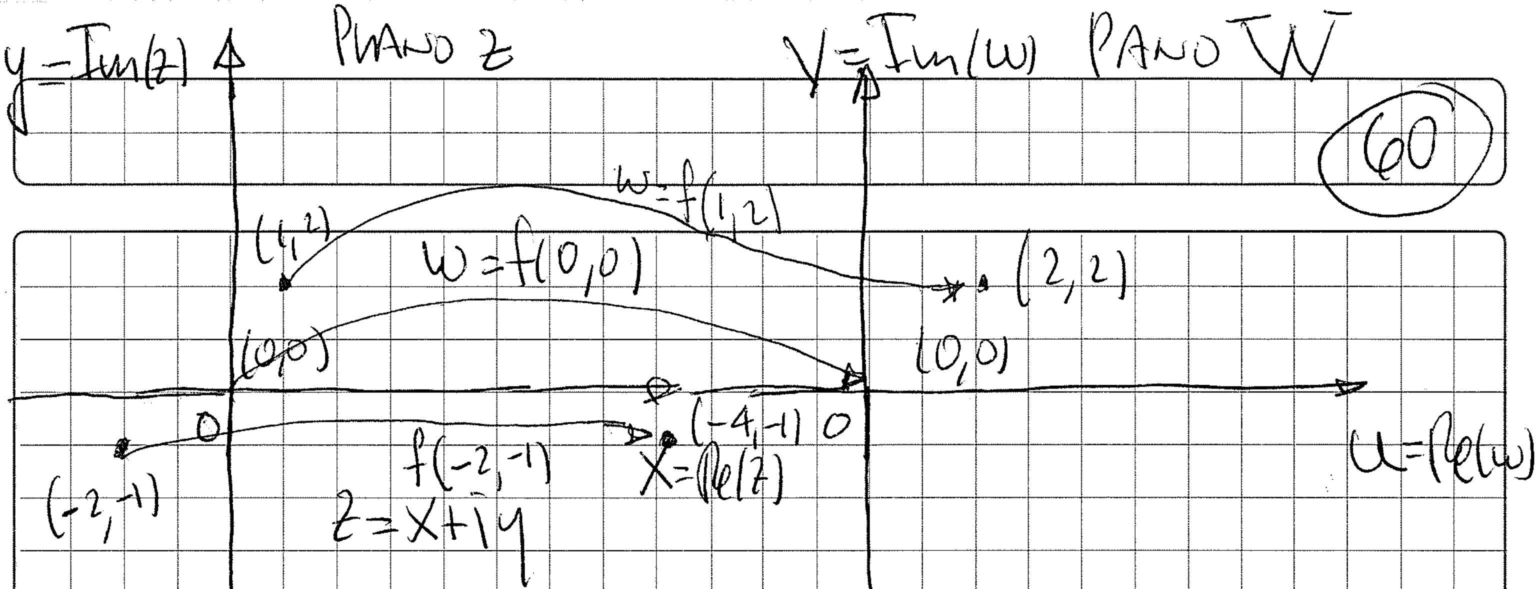
(4) DEL EJEMPLO (2) EFECTUE UNA GRÁFICA (BOSQUEJO) DE:

(2) $f(z) = 2x+iy \Rightarrow f(x+iy) = 2x+iy$,

si $x=0, y=0 \Rightarrow f(0,0) = 2(0)+i(0) = (0,0)$

si $x=1, y=2 \Rightarrow f(1,2) = 2(1)+i(2) = (2,2) = 2+i2 = (2,2)$

si $x=-2; y=-1 \Rightarrow f(-2,-1) = 2(-2)+i(-1) = -4-i = (-4,-1)$



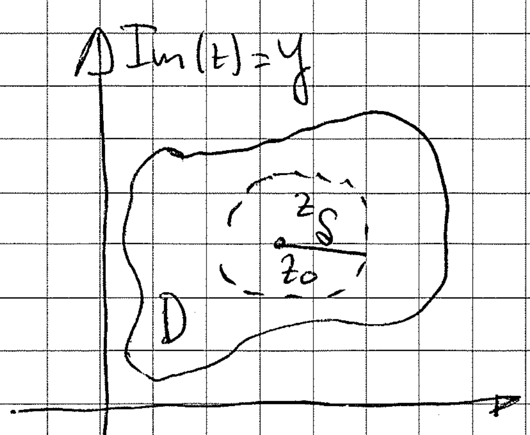
NO ES GRÁFICA ES UN BOSQUEJO.

• LÍMITE DE UNA FUNCIÓN.

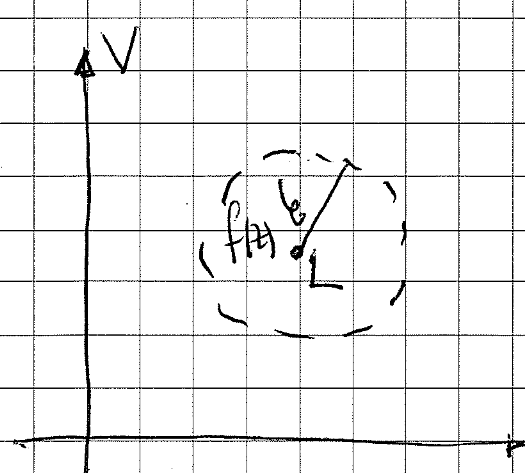
DEF. SEA LA FUNCIÓN f ESTÁ DEFINIDA EN UNA VECINDAD DE z_0, EXCEPTO POSIBLEMENTE EN EL MISMO z_0. ENTONCE SE DICE QUE f POSEE UN LÍMITE EN z_0, ESCRITO COMO:

$$\lim_{z \to z_0} f(z) = L$$

SI, PARA CADA $\varepsilon > 0$, EXISTE UNA $\delta > 0$ TAL QUE $|f(z) - L| < \varepsilon$ SIEMPRE QUE $0 < |z - z_0| \delta$.



VECINDAD δ



VECINDAD ε

• PROPIEDADES DEL LÍMITE.

TEOREMA. SEAN $\lim_{z \to z_0} f(z) = L_1$ y $\lim_{z \to z_0} g(z) = L_2$ (61)

ENTONCES:

SUMA Y RESTA

i) $$\lim_{z \to z_0} \left[f(z) \pm g(z) \right] = \lim_{z \to z_0} f(z) \pm \lim_{z \to z_0} g(z) = L_1 \pm L_2 .$$

PRODUCTO

ii) $$\lim_{z \to z_0} \left[f(z) g(z) \right] = \lim_{z \to z_0} f(z) \lim_{z \to z_0} g(z) = L_1 L_2 .$$

COCIENTE

iii) $$\lim_{z \to z_0} \left[\frac{f(z)}{g(z)} \right] = \frac{\lim_{z \to z_0} f(z)}{\lim_{z \to z_0} g(z)} = \frac{L_1}{L_2}, \quad L_2 \neq 0 .$$

• CONTINUIDAD EN UN PUNTO.

DEF UNA FUNCIÓN f ES CONTINUA EN UN PUNTO z_0 SI

$$\lim_{z \to z_0} f(z) = f(z_0) .$$

EJEMPLOS:

(1) LA FUNCIÓN POLINOMIAL

$$f(z) = a_n z^n + a_{n-1} z^{n-1} + \dots + a_2 z^2 + a_1 z + a_0 , \quad a_n \neq 0 ,$$

DONDE n ES UN ENTERO NO NEGATIVO Y LOS

a_i, $i = 0, 1, \ldots, n$ SON CONSTANTES COMPLEJAS, (62)

SE DENOMINA POLINOMIO DE GRADO n. LA FUNCIÓN POLINOMIAL ES CONTINUA EN CUALQUIER PUNTO.

• FUNCIÓN RACIONAL

$$f(z) = \frac{g(z)}{h(z)},$$

DONDE g y h SON FUNCIONES POLINOMIALES, ES CONTINUA, EXCEPTO EN AQUELLOS PUNTOS PARA LOS CUALES $h(z)$ ES CERO.

EJEMPLOS

(1) $$f(z) = \frac{g(z)}{h(z)} = \frac{z}{z^2+1}.$$

RAÍCES Ó CEROS.

$$h(z) = z^2+1 = 0 \Rightarrow z^2 = -1; \quad z = \pm\sqrt{-1} = \pm i \text{ (RAÍCES)}$$

$f(z)$ ES CONTINUA PARA TODA z EXCEPTO PARA $z \neq i$ Y $z \neq -i$.

$$f(i) = \frac{(i)}{(i^2)+1} = \frac{i}{-1+1} = \frac{i}{0} = \infty \text{ NO ESTÁ DEFINIDO}$$

$$f(-i) = \frac{(-i)}{(-i)^2+1} = \frac{(-i)}{(-1)^2 i^2+1} = \frac{-i}{(1)(-1)+1} = \frac{-i}{-1+1} = \frac{-i}{0} = \infty$$

(2) $f(z) = \dfrac{z+1}{2z-1} = \dfrac{g(z)}{h(z)}$

$h(z) = 2z-1 = 0 \Rightarrow z = 1/2$. CERO Ó RAÍZ.

$$f(z) = \frac{z+1}{2z-1} = \frac{1/2+1}{2\left(\frac{1}{2}\right)-1} = \frac{3/2}{\frac{2}{2}-1} = \frac{\frac{3}{2}}{1-1} = \frac{3/2}{0} = \infty$$

• DERIVADA

DEF: SUPÓNGASE QUE LA FUNCIÓN COMPLEJA f SE DEFINE EN LA VECINDAD DE UN PUNTO z_0. LA DERIVADA DE f EN z_0 ES:

$$f'(z_0) = \lim_{\Delta z \to 0} \frac{f(z_0+\Delta z) - f(z_0)}{\Delta z},$$

SIEMPRE Y CUANDO EXISTA DICHO LÍMITE.

LA DERIVADA DE UNA FUNCIÓN $w = f(z)$ SE ESCRIBE $\dfrac{dw}{dz}$.

SI f ES DERIVABLE EN z_0, ENTONCES f ES CONTINUA EN z_0.

• REGLAS DE DERIVACIÓN

REGLAS DE LA CONSTANTE

$\dfrac{d}{dz}c = 0$; C = CONSTANTE COMPLEJA $C = C_1 + iC_2$.

$\dfrac{d}{dz}cf(z) = cf'(z)$

• REGLA DE LA SUMA Y RESTA

$$\frac{d}{dz}\left[f(z) \pm g(z)\right] = \frac{df(z)}{dz} \pm \frac{dg(z)}{dz} = f'(z) \pm g'(z).$$

• REGLA DEL PRODUCTO.

$$\frac{d}{dz}\left[f(z)g(z)\right] = f(z)\frac{dg(z)}{dz} + g(z)\frac{df(z)}{dz} = f(z)g'(z) + g(z)f'(z).$$

O BIEN

$$= f'(z)g(z) + f(z)g'(z).$$

• REGLA DEL COCIENTE

$$\frac{d}{dz}\left[\frac{f(z)}{g(z)}\right] = \frac{g(z)\frac{df(z)}{dz} - f(z)\frac{dg(z)}{dz}}{\left[g(z)\right]^2} = \frac{g(z)f'(z) - f(z)g'(z)}{\left[g(z)\right]^2}$$

O BIEN

$$= \frac{f'(z)g(z) - g'(z)f(z)}{\left[g(z)\right]^2}.$$

• REGLA DE LA CADENA.

$$\frac{d}{dz}f\left[g(z)\right] = f'\left[g(z)\right]g'(z).$$

• DERIVADA DE POTENCIAS.

$$\frac{d}{dz}z^n = n z^{n-1}\frac{dz}{dz} = n z^{n-1}(1) = n z^{n-1}$$, n ES UN ENTERO.

EJEMPLOS: CALCULAR LOS SIGUIENTES LÍMITES.

1) $$\lim_{z \to e^{i\pi/6}} \left[\left(z - e^{i\pi/6}\right) \frac{1}{z^6+1} \right] =$$

SOL:

$$= \lim_{z \to e^{i\pi/6}} \left[\left(z - e^{i\pi/6}\right) \frac{1}{z^6+1} \right] = \left(e^{i\pi/6} - e^{i\pi/6}\right) \frac{1}{\left(e^{i\pi/6}\right)^6+1}$$

$$= 0 \; \frac{1}{e^{i6\pi/6}+1} = \frac{0}{e^{i\pi}+1} = \frac{0}{\cos\pi + i\,\text{sen}\,\pi + 1} = \frac{0}{-1+i(0)+1}$$

$$= \frac{0}{-1+1} = \frac{0}{0} \text{ INDETERMINACIÓN.}$$

APLICANDO LA REGLA DE L'HOSPITAL.

SE DERIVA NUMERADOR Y DENOMINADOR INDEPENDIENTEMENTE RESPECTO A LA VARIABLE "z", ES DECIR:

$$\lim_{z \to e^{i\pi/6}} \left[\frac{\left(z - e^{i\pi/6}\right)}{z^6+1} \right] = \lim_{z \to e^{i\pi/6}} \left[\frac{(1-0)}{6z^5+0} \right] = \lim_{z \to e^{i\pi/6}} \left[\frac{1}{6z^5} \right]$$

APLICANDO LAS REGLAS DE LOS LÍMITES DEL COCIENTE

$$= \frac{\lim_{z \to e^{i\pi/6}} (1)}{\lim_{z \to e^{i\pi/6}} 6z^5} = \frac{1}{6\left(e^{i\pi/6}\right)^5} = \frac{1}{6\,e^{i5\pi/6}} = \frac{1}{6}\, e^{-5\pi/6}$$

$$= \frac{1}{6} e^{-i5\pi/6} \quad \therefore \lim_{z \to e^{i\pi/6}} \left[\frac{(z - e^{i\pi/6})}{z^6+1} \right] = \frac{1}{6} e^{-i5\pi/6}$$

16e

(2) $$\lim_{z \to 2i} \frac{z-2i}{z^4-16} = \frac{(2i)-2i}{(2i)^4-16} = \frac{2i-2i}{16i^4-16} = \frac{2i-2i}{16-16} = \frac{0}{0}$$

POR L'HOSPITAL.

$$\lim_{z \to 2i} \frac{z-2i}{z^4-16} = \lim_{z \to 2i} \left[\frac{1-0}{4z^3-0} \right] = \lim_{z \to 2i} \frac{1}{4z^3} = \frac{1}{4(2i)^3}$$

$$= \frac{1}{4 2^3 i^3} = \frac{1}{32 i^3} = \frac{1}{32 i i^2} = \frac{1}{-32 i} \frac{i}{i} = \frac{i}{-32 i^2} = \frac{i}{32}$$

$$\therefore \lim_{z \to 2i} \frac{z-2i}{z^4-16} = \frac{i}{32}$$

(3) $$\lim_{z \to 2i} \frac{(3z+1)(z-1)}{z^2+z-1}$$

Sol:

$$\lim_{z \to 2i} \frac{(3z+1)(z-1)}{z^2+z-1} = \frac{\lim_{z \to 2i} [(3z+1)(z-1)]}{\lim_{z \to 2i} [z^2+z-1]}$$

$$= \frac{[3(2i)+1][2i-1]}{(2i)^2+2i-1} = \frac{-12-6i+2i-1}{-4-1+2i} = \frac{-13-4i}{-5+2i}$$

4) $\lim\limits_{z\to 0} \dfrac{\operatorname{sen} z}{z}$

SOL: $\lim\limits_{z\to 0} \dfrac{\operatorname{sen} z}{z} = \dfrac{\operatorname{sen} 0}{0} = \dfrac{0}{0}$ INDETERMINACIÓN.

POR L'HOSPITAL.

$$\lim_{z\to 0} \frac{\operatorname{sen} z}{z} = \lim_{z\to 0} \frac{\cos z}{1} = \lim_{z\to 0} \cos z = \cos 0 = 1.$$

$$\therefore \lim_{z\to 0} \frac{\operatorname{sen} z}{z} = 1.$$

5) CALCULAR $\dfrac{df}{dz} = f'$ PARA LAS SIGUIENTES FUNCIONES COMPLEJAS.

1) $f(z) = 3z^3 + (z+1)^2 + z$

SOL: $$\frac{df}{dz} = \frac{d}{dz}\left(3z^3 + (z+1)^2 + z\right) = \frac{d}{dz}\left(3z^3 + z^2 + 2z + 1 + z\right)$$

$$= \frac{d}{dz}\left(3z^3 + z^2 + 3z + 1\right) = 9z^2 + 2z + 3 + 0 = 9z^2 + 2z + 3$$

2) $f(z) = \dfrac{z-1}{z+1}$

SOL:

$$\frac{df}{dz} = \frac{d}{dz}\left[\frac{z-1}{z+1}\right] = \frac{\frac{d}{dz}(z-1)\cdot(z+1) - (z-1)\frac{d}{dz}(z+1)}{(z+1)^2}$$

$$= \frac{z+1-z+1}{(z+1)^2} = \frac{2}{(z+1)^2}$$

③ $f(z) = \left((z+2)^3 + z^3\right)^4$

Sol.

68

$$\frac{df}{dz} = \frac{d}{dz}\left[(z+2)^3 + z^3\right]^4 = 4\left[(z+2)^3 + z^3\right]^3 \frac{d}{dz}\left[(z+2)^3 + z^3\right]$$

$$= 4\left[(z+2)^3 + z^3\right]^3 \left[3(z+2)^2 + 3z^2\right]$$

④ $f(z) = 2z\left[\frac{1}{(z+1)^2} + (z+1)^2\right]^2$

Sol:

$$\frac{df}{dz} = \frac{d}{dz}\left[2z\left[\frac{1}{(z+1)^2} + (z+1)^2\right]^2\right]$$

$$= 2\left[\frac{1}{(z+1)^2} + (z+1)^2\right]^2 + 2z\frac{d}{dz}\left[\frac{1}{(z+1)^2} + (z+1)^2\right]^2$$

$$= 2\left[\frac{1}{(z+1)^2} + (z+1)^2\right]^2 + 2z\,2\left[\frac{1}{(z+1)^2} + (z+1)^2\right]\left[\frac{-2}{(z+1)^3} + 2(z+1)\right]$$

FUNCIONES ANALÍTICAS

ANALITICIDAD EN UN PUNTO

DEF UNA FUNCIÓN COMPLEJA $w = f(z)$ ES ANALÍTICA EN UN PUNTO z_0 SI f ES DERIVABLE EN z_0 Y EN TODO PUNTO DE ALGUNA VECINDAD DE z_0.

UNA FUNCIÓN f ES ANALÍTICA EN UN DOMINIO D SI ES ANALÍTICA EN TODOS LOS PUNTOS DE D.

FUNCIÓN ENTERA.

UNA FUNCIÓN QUE ES ANALÍTICA EN CUALQUIER PUNTO z ES UNA FUNCIÓN ENTERA.

EJEMPLOS

1) LOS POLINOMIOS DE LA FORMA:

$f(z) = c_0 + c_1 z + c_2 z^2 + \ldots + c_n z^n$ TALQUE $c_0, \ldots, c_n$ = CTES COMPLEJAS

SON ANALÍTICAS EN EL PLANO COMPLEJO. A SU VEZ ES UNA FUNCIÓN ENTERA.

2) $f(z) = \dfrac{z}{z-3i}$, ES ANALÍTICA EXCEPTO EN LOS PUNTOS EN DONDE EL DENOMINADOR SE ANULA; EN $z = 3i$.

JUSTIFICACIÓN: EL DENOMINADOR

$z - 3i = 0 \Rightarrow z = 3i$

SUSTITUYENDO $z=3i$ EN $f(z)$, RESULTA:

$$f(3i)=\frac{3i}{3i-3i}=\frac{3i}{0}=\infty$$

TAMBIÉN NO ES UNA FUNCIÓN ENTERA.

(3) SEA $f(z)=\dfrac{z-4+3i}{z^2-6z+25}$, ES ANALÍTICA EXCEPTO

EN LOS PUNTOS EN DONDE EL DENOMINADOR SE ANULA;

ES DECIR: $z^2-6z+25=0 \Rightarrow z=3+4i$ y $z=3-4i$.

$$f(z)=\frac{z-4+3i}{z^2-6z+25} \Rightarrow f(3+4i)=\frac{3+4i-4+3i}{(3+4i)^2-6(3+4i)+25}$$

$$f(3+4i)=\frac{-1+7i}{9+24i-16-18-24i+25}=\frac{-1+7i}{0}=\infty$$

DENOMINADOR SE ANULA

$$z^2-6z+25=(3+4i)^2-6(3+4i)+25=9+24i-16-18-24i+25=0$$

TAMBIÉN NO ENTERA $f(z)$.

POR OTRA PARTE PARA:

$$f(3-4i)=\frac{3-4i-4+3i}{(3-4i)^2-6(3-4i)+25}=\frac{-1-i}{0}=\infty$$

TAMBIÉN $f(z)$ NO ES ENTERA, NI ANALÍTICA

ECUACIONES DE CAUCHY-RIEMANN (C-R).

TEOREMA 1. SEA $f(z) = u + iv = u(x,y) + iv(x,y)$ ES DERIVABLE EN UN PUNTO $z = x + iy$. ENTONCES EXISTEN DERIVADAS PARCIALES DE PRIMER ORDEN DE u y v EN EL PUNTO z QUE CUMPLEN CON LAS ECUACIONES DE CAUCHY-RIEMANN (C-R)

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \text{ó} \quad u_x = v_y$$

y

$$\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \quad \text{ó} \quad u_y = -v_x$$

(1)

TEOREMA 2. SI DOS FUNCIONES CONTINUAS DE VALORES REALES $u = u(x,y)$ y $v = v(x,y)$, DE DOS VARIABLES REALES x y y, TIENEN PRIMERAS DERIVADAS PARCIALES CONTINUAS QUE SATISFACEN LAS ECUACIONES DE CAUCHY-RIEMANN (1) EN ALGÚN DOMINIO D, ENTONCES LA FUNCIÓN COMPLEJA $f(z) = u + iv = u(x,y) + iv(x,y)$ ES ANALÍTICA EN D.

- LA DERIVADA DE $f'(z)$ DE $f(z)$ EN TÉRMINOS DE (C-R) SON:

$$f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = u_x + iv_x.$$

$$f'(z) = \frac{\partial v}{\partial y} - i\frac{\partial u}{\partial y} = v_y - iu_y.$$

• ECUACIONES DE CAUCHY-RIEMANN EN LA FORMA POLAR.

SEAN $z = re^{i\theta} = r(\cos\theta + i\,\mathrm{sen}\,\theta)$, $f(z) = u(r,\theta) + iV(r,\theta)$ 72

ENTONCES:

$$\frac{\partial u}{\partial r} = u_r = \frac{1}{r}\frac{\partial v}{\partial \theta} = \frac{1}{r}V_\theta \quad \text{ó} \quad u_r = \frac{1}{r}V_\theta$$

y

$$\frac{\partial v}{\partial r} = V_r = -\frac{1}{r}\frac{\partial u}{\partial \theta} = -\frac{1}{r}u_\theta \quad \text{ó} \quad V_r = -\frac{1}{r}u_\theta .$$

EJEMPLOS: 1 a) SI $f(z) = z^2 + 2z + 1$, HALLAR $f'(z)$.

SOL:

$$f(z) = z^2 + 2z + 1 \Rightarrow f'(z) = 2z + 2 = 2(x+iy) + 2$$
$$= 2x + i2y + 2 = 2x + 2 + i2y \quad (**)$$

b) VERIFICAR QUE $f'(z) = 2z + 2$, UTILIZANDO

SOL: $f'(z) = u_x + iV_x$ ó BIEN $f'(z) = V_y - iu_y$

$$f(z) = z^2 + 2z + 1 = (x+iy)^2 + 2(x+iy) + 1$$
$$= x^2 + i2xy - y^2 + 2x + i2y + 1 = (x^2 - y^2 + 2x + 1) + i(2xy + 2y)$$

DONDE: $u = u(x,y) = x^2 - y^2 + 2x + 1$

$V = V(x,y) = 2xy + 2y$

UTILIZANDO $f'(z) = u_x + iV_x$ — (*)

$$u = x^2 - y^2 + 2x + 1 \Rightarrow u_x = \frac{\partial}{\partial x}\left[x^2 - y^2 + 2x + 1\right] = \frac{\partial x^2}{\partial x} - \frac{\partial y^2}{\partial x} + 2\frac{\partial x}{\partial x} + \frac{\partial 1}{\partial x}$$

$$U_x = 2x - 0 + 2(1) + 0 = 2x + 2$$

$$V_x = \frac{\partial}{\partial x}(2xy - 2y) = 2y\frac{\partial x}{\partial x} - 2\frac{\partial y}{\partial x} = 2y(1) - 2(0)$$ 73

$$V_x = 2y - 0 = 2y.$$

SUSTITUYENDO $U_x = 2x+2$ y $V_x = 2y$ EN (*)

$$f'(z) = U_x + iV_x = 2x + 2 + i2y = 2(x + 1 + iy)$$

$$= 2(x + iy + 1) = 2(z+1) = 2z + 2$$ LO CUAL

RESULTA LO MISMO EN (**).

AHORA CON $f'(z) = V_y - iU_y$ — (***)

$$U = (x^2 - y^2 + 2x + 1) \Rightarrow U_y = \frac{\partial}{\partial y}\left[x^2 - y^2 + 2x + 1\right] = \frac{\partial x^2}{\partial y} - \frac{\partial y^2}{\partial y} + \frac{\partial 2x}{\partial y} + \frac{\partial 1}{\partial y}$$

$$U_y = 0 - 2y + 0 + 0 = -2y.$$

$$V = 2xy + 2y \Rightarrow V_y = \frac{\partial}{\partial y}\left[2xy + 2y\right] = 2x\frac{\partial y}{\partial y} + 2\frac{\partial y}{\partial y}$$

$$V_y = 2x(1) + 2(1) = 2x + 2.$$

SUSTITUYENDO $U_y = -2y$ y $V_y = 2x+2$ EN (***)

$$f'(z) = V_y - iU_y = 2x + 2 - i(-2y) = 2x + 2 + i2y$$

$$= 2(x + 1 + iy) = 2(x + iy + 1) = 2(z+1) = 2z + 2$$ ES

EL MISMO RESULTADO QUE (**)

(c) VERIFICAR QUE $f(z) = z^2 + 2z + 1 = (x^2 - y^2 + 2x + 1) + i(2xy + 2y)$ 74

CUMPLE LAS CONDICIONES DE CAUCHY - RIEMANN.

Sol:
$$f(z) = (x^2 - y^2 + 2x + 1) + i(2xy + 2y)$$

$$u = x^2 - y^2 + 2x + 1$$

$$v = 2xy + 2y$$

SABEMOS:
$$\left.\begin{array}{l} u_x = v_y \\ u_y = -v_x \end{array}\right\} (1)$$

$$u = x^2 - y^2 + 2x + 1$$

$$u_x = \frac{\partial}{\partial x}(x^2 - y^2 + 2x + 1) = 2x - 0 + 2 + 0 = 2x + 2$$

$$u_y = \frac{\partial}{\partial y}(x^2 - y^2 + 2x + 1) = 0 - 2y + 0 + 0 = -2y$$

AHORA $V = 2xy + 2y$

$$v_x = \frac{\partial}{\partial x}(2xy + 2y) = 2y + 0 = 2y$$

$$v_y = \frac{\partial}{\partial y}(2xy + 2y) = 2x + 2.$$

UTILIZANDO (1):

$$u_x = v_y \Rightarrow 2x + 2 = 2x + 2 \quad \therefore V_x \equiv V_y$$

$$u_y = -v_x \Rightarrow -2y = -(2y);\ -2y = -2y \quad \therefore u_y \equiv -v_x$$

$\therefore f(z)$ CUMPLE LAS CONDICIONES DE C-R y $f(z)$ ES ANALÍTICA.

② VERIFICAR LAS CONDICIONES DE C-R, SI:

$f(z) = (\bar{z} + 2i)^2 - 1$ CON $z = x + iy$

SOL:

$$f(z) = (\bar{z} + 2i)^2 - 1 = (x - iy + 2i)^2 - 1$$

$$= (x - iy)^2 + 4i(x - iy) + (2i)^2 - 1$$

$$= x^2 - i2xy - y^2 + i4x + 4y - 4 - 1 = x^2 - y^2 + 4y - 5 + i(4x - 2xy)$$

DONDE:

$$u = x^2 - y^2 + 4y - 5 \Rightarrow u_x = 2x \;;\; u_y = -2y + 4$$

$$v = 4x - 2xy \Rightarrow v_x = 4 - 2x \;;\; v_y = -2x$$

LAS CONDICIONES DE C-R $\left.\begin{array}{l} u_x = v_y \\ u_y = -v_x \end{array}\right\} (1)$

POR (1):

$$u_x = v_y \Rightarrow 2x \neq -2x \Rightarrow u_x \neq v_y$$

COMO SON DIFERENTES UNA DE LAS PROPIEDADES DE C-R, LUEGO $f(z)$ NO CUMPLE (1), ENTONCES $f(z)$ NO ES ANALÍTICA.

③ $f(z) = \dfrac{z+1}{z+4}$ SI $z = x + iy$

SOL:

$$f(z) = \frac{z+1}{z+4} = \frac{x+iy+1}{x+iy+4} = \frac{x+iy+1}{x+iy+4} = \frac{(x+1+iy)(x+4-iy)}{(x+4+iy)(x+4-iy)}$$

$$= \frac{x^2+y^2+5x+4}{x^2+8x+16+y^2} + i\,\frac{3y}{x^2+8x+16+y^2}$$

JUSTIFICACIÓN EL DENOMINADOR ES: (DIFERENCIA DE CUADRADOS)

$$(x+4+iy)(x+4-iy) = (x+4)^2 + y^2$$
$$= x^2 + 8x + 16 + y^2$$

EN NUMERADOR ES:

$$(x+1+iy)(x+4-iy) = x^2 + x + ixy + 4x - ixy + 4 + i4y - iy + y^2$$
$$= x^2 + 5x + 4 + y^2 + i3y$$

DONDE:

$$u = \frac{x^2+y^2+5x+4}{x^2+8x+16+y^2}$$

$$v = \frac{3y}{x^2+8x+16+y^2}$$

$$\left.\begin{array}{l} u_x = v_y \\ u_y = -v_x \end{array}\right\} (1)$$

$$u_x = \frac{(2x+5)(x^2+8x+16+y^2) - (2x+8)(x^2+y^2+5x+4)}{(x^2+8x+16+y^2)^2}$$

$$v_y = \frac{3(x^2+8x+16+y^2) - (3y)(2y)}{(x^2+8x+16+y^2)^2}$$

$$u_x = v_y \Rightarrow \quad \frac{3x^2+24x+48-3}{(x^2+8x+16+y^2)^2} = \frac{3x^2+24x+48-3}{(x^2+8x+16+y^2)}$$

$$U_y = \frac{2y(x^2+8x+16+y^2) - 2y(x^2+y^2+5x+4)}{(x^2+8x+16+y^2)^2}$$

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$$V_x = \frac{-3y(2x+8)}{(x^2+8x+16+y^2)^2}$$

$$U_y = -V_x \Rightarrow \frac{6yx+24y}{(x^2+8x+16+y^2)^2} = -\frac{(-6yx-24y)}{(x^2+8x+16+y^2)^2}$$

$$\frac{6yx+24y}{(x^2+8x+16+y^2)^2} = \frac{6yx+24y}{(x^2+8x+16+y^2)^2}$$

$\therefore$ $f(z)$ ES ANALÍTICA EXCEPTO EN $z=-4$.

$f(z) = \frac{z+1}{z+4} \Rightarrow$ DENOMINADOR $z+4=0 \Rightarrow z=-4$.

4 VERIFICAR LA CONDICIÓN DE C-R EN FORMA POLAR.

a $\quad f(z) = z^3$

SOL: $f(z) = z^3 = (re^{i\theta})^3 = r^3 e^{i3\theta} = r^3\cos 3\theta + i\, r^3 \operatorname{sen} 3\theta$

$U = U(r,\theta) = r^3 \cos 3\theta$

$V = V(r,\theta) = r^3 \operatorname{sen} 3\theta$.

$$V_r = -\frac{1}{r} U_\theta$$

$$U_r = \frac{1}{r} V_\theta = \frac{3}{r} r^3 \cos 3\theta = 3r^2 \cos 3\theta$$

$U_\theta = -3r^3 \operatorname{sen} 3\theta$; $V_r = 3r^2 \operatorname{sen} 3\theta$; $V_\theta = 3r^3 \cos 3\theta$.

CONDICIONES DE C-R POLAR.

$$U_r = \frac{1}{r} V_\theta \Rightarrow 3r^2 \cos 3\theta = \frac{1}{r} 3r^3 \cos 3\theta$$

$$3r^2 \cos 3\theta = 3r^2 \cos 3\theta$$

AHORA.

$$V_r = -\frac{1}{r} U_\theta \Rightarrow 3r^2 \operatorname{sen} 3\theta = -\frac{1}{r}(-3r^3 \operatorname{sen} 3\theta)$$

$$3r^2 \operatorname{sen} 3\theta = 3r^2 \operatorname{sen} 3\theta$$

$\therefore f(z) = z^3$ ES ANALÍTICA, CUMPLE LA CONDICIONES DE C-R.

(b) $f(z) = \dfrac{y - ix}{x^2+y^2}$.

SOL

SABEMOS QUE COORDENADAS POLARES:

$x = r\cos\theta$; $y = r \operatorname{sen}\theta$; $r^2 = x^2+y^2 \Rightarrow r = \sqrt{x^2+y^2}$.

$$f(z) = \frac{y - ix}{x^2+y^2} = \frac{y}{x^2+y^2} - i\frac{x}{x^2+y^2} = \frac{r \operatorname{sen}\theta}{r^2} - i\frac{r\cos\theta}{r^2}$$

$$f(z) = \frac{\operatorname{sen}\theta}{r} - i\frac{\cos\theta}{r}$$ (79)

DONDE:

$$U = \frac{\operatorname{sen}\theta}{r} \qquad V = -\frac{\cos\theta}{r}$$

$$U_r = -\frac{\operatorname{sen}\theta}{r^2} \; ; \; U_\theta = \frac{\cos\theta}{r}$$

$$V_r = \frac{\cos\theta}{r^2} \; ; \; V_\theta = \frac{\operatorname{sen}\theta}{r}$$

$$\left.\begin{aligned} U_r &= \frac{1}{r}V_\theta \\ V_r &= -\frac{1}{r}U_\theta \end{aligned}\right\} (1)$$

$$U_r = \frac{1}{r}V_\theta \Rightarrow -\frac{\operatorname{sen}\theta}{r^2} \neq \frac{1}{r}\frac{\operatorname{sen}\theta}{r} = \frac{1}{r}V_\theta$$

$\therefore$ $f(z)$ NO ES ANALÍTICA.

C) $$f(z) = \frac{x - iy}{x^2+y^2}$$

SOL:

$$f(z) = \frac{x}{x^2+y^2} - i\frac{y}{x^2+y^2} = \frac{r\cos\theta}{r^2} - i\frac{r\operatorname{sen}\theta}{r^2}$$

$$= \frac{\cos\theta}{r} - i\frac{\operatorname{sen}\theta}{r} \; ; \; r > 0.$$

$$U = \frac{\cos\theta}{r} \; ; \; V = -\frac{\operatorname{sen}\theta}{r}$$

$$U = \frac{\cos\theta}{r} \left| \begin{aligned} & U = r^{-1}\cos\theta \Rightarrow U_r = -r^{-2}\cos\theta = -\frac{\cos\theta}{r^2} \\ & U_\theta = -\frac{\operatorname{sen}\theta}{r} \end{aligned}\right.$$

$V = -r^{-1} \operatorname{sen}\theta \Rightarrow V_r = r^{-2} \operatorname{sen}\theta = \dfrac{\operatorname{sen}\theta}{r^2}$ (80)

$V = -\dfrac{\operatorname{sen}\theta}{r}$ | $V_\theta = -\dfrac{\cos\theta}{r}$,

CONDICIONES DE C-R.

$$U_r = \frac{1}{r} V_\theta \Rightarrow -\frac{\cos\theta}{r^2} = -\frac{1}{r}\,\frac{\cos\theta}{r} = -\frac{\cos\theta}{r^2}$$

$$V_r = -\frac{1}{r} U_\theta \Rightarrow \frac{\operatorname{sen}\theta}{r^2} = -\frac{1}{r}\left(-\frac{\operatorname{sen}\theta}{r}\right) = \frac{\operatorname{sen}\theta}{r^2}$$

FINALMENTE, $f(z)$ ES ~~UNA~~ UNA FUNCIÓN ANALÍTICA.

(5) VERIFICAR QUE $f(z) = \operatorname{sen}x \cosh y + i \cos x \operatorname{senh} y$ ¿ES UNA FUNCIÓN ANALÍTICA?

$$\frac{d}{dx}[\operatorname{senh} u] = [\cosh u]\,u'$$

$$\frac{d}{dx}[\cosh u] = [\operatorname{senh} u]\,u'$$

$$\frac{d}{dx}[\operatorname{tgh} u] = [\operatorname{sech}^2 u]\,u'$$

SOL

$$f(z) = \operatorname{sen}x \cosh y + i \cos x \operatorname{senh} y$$

$U = \operatorname{sen}x \cosh y$; $V = \cos x \operatorname{senh} y$

$U_x = \cos x \cosh y$; $U_y = \operatorname{sen}x \operatorname{senh} y$

$V_x = -\operatorname{sen}x \operatorname{senh} y$; $V_y = \cos x \operatorname{senh} y$

$$U_x = V_y \Rightarrow \cos x \cosh y \equiv \cos x \operatorname{senh} y$$

$$V_x = -U_y \Rightarrow -\operatorname{sen}x \operatorname{senh} y \equiv -\operatorname{sen}x \operatorname{senh} y \quad \therefore \operatorname{sen}x \operatorname{senh} y = \operatorname{sen}x \operatorname{senh} y$$

$\therefore$ CUMPLE $f(z)$ CON C-R, LUEGO $f(z)$ ES UNA FUNCIÓN ANALÍTICA

Funciones Armónicas.

Supóngase que $f(z) = u + iv = u(x,y) + iv(x,y)$ una función analítica en algún dominio D. Entonces, en cada punto de D.

Considere u y v tienen derivadas parciales de segundo orden continuas. Como f es analítica, las (C-R) se cumplen:

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \text{ó} \quad u_x = v_y$$

$$\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \quad \text{ó} \quad u_y = -v_x.$$

Derivando ambas partes de $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ respecto a x

y derivando ambos lados de $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$ respecto a y, obtenemos:

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 v}{\partial x \partial y}$$

$$\frac{\partial^2 u}{\partial y^2} = -\frac{\partial^2 v}{\partial y \partial x} = -\frac{\partial^2 v}{\partial x \partial y}$$

Pero las derivadas parciales son continuas, las derivadas mixtas son iguales. Sumando estas ecuaciones:

$$\boxed{\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=\frac{\partial^2 v}{\partial x\partial y}-\frac{\partial^2 v}{\partial y\partial x}=\frac{\partial^2 v}{\partial x\partial y}-\frac{\partial^2 v}{\partial x\partial y}=0} \quad (82)$$

$$\therefore \frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 V}{\partial y^2}=u_{xx}+u_{yy}=0 \quad -(*)$$

A TRAVÉS DE D.

LA ECUACIÓN (*) ES LLAMADA ECUACIÓN DIFERENCIAL PARCIAL DE LAPLACE Ó ECUACIÓN DE LAPLACE, TAMBIÉN SE LE CONOCE COMO ECUACIÓN DE POTENCIAL Y SE ACOSTUMBRA ESCRIBIRLA COMO:

$$\nabla^2 u=0 \quad \text{ó} \quad \Delta u=0 \quad -(**)$$

FUNCIÓN ARMÓNICA.

CUALQUIER FUNCIÓN QUE TIENE DERIVADAS PARCIALES CONTINUAS DE SEGUNDO Y QUE SATISFACE LA ECUACIÓN DE LAPLACE (**), SE LLAMA FUNCIÓN ARMÓNICA.

NOTACIÓN.

$$\nabla^2 u=\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=u_{xx}+u_{yy}=0$$

$$\nabla^2 V=\frac{\partial^2 V}{\partial x^2}+\frac{\partial^2 V}{\partial y^2}=V_{xx}+V_{yy}=0$$

• FUNCIÓN ARMÓNICA / FUNCIÓN ARMÓNICA CONJUGADA.

SI $f(z) = u + iv = u(x,y) + iv(x,y)$ ES ANALÍTICA EN UN DOMINIO D, ENTONCES u y v SON ARMÓNICAS EN D. AHORA SUPONGAMOS QUE $u = u(x,y)$ ES UNA FUNCIÓN DADA QUE ES ARMÓNICA EN D. ENTONCES ES POSIBLE HALLAR EN OCASIONES OTRA FUNCIÓN $v = v(x,y)$ QUE SEA ARMÓNICA EN D, DE FORMA QUE $u + iv = u(x,y) + iv(x,y)$ SEA UNA FUNCIÓN ANALÍTICA EN D. LA FUNCIÓN $v = v(x,y)$ SE DENOMINA UNA FUNCIÓN ARMÓNICA CONJUGADA DE $u = u(x,y)$.

EJEMPLOS

1 (a) VERIFIQUE QUE LA FUNCIÓN $u = u(x,y) = x^3 - 3xy^2 - 5y$ ES ARMÓNICA EN TODO EL PLANO COMPLEJO. (b) HALLE LA FUNCIÓN ARMÓNICA CONJUGADA DE $u = u(x,y)$.

SOL: $u = u(x,y) = x^3 - 3xy^2 - 5y$

$u = x^3 - 3xy^2 - 5y$

$u_x = 3x^2 - 3y^2$; $u_y = -6xy - 5$

$u_{xx} = 6x$; $u_{yy} = -6x$

SUSTITUYENDO $u_{xx} = 6x$ y $u_{yy} = -6x$ EN $(*)$:

$$u_{xx} + u_{yy} = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \nabla^2 u = 6x - 6x = 0.$$

ENTONCES $u = u(x,y)$ ES UNA FUNCIÓN ARMÓNICA.

COMENTARIO:

SI $\nabla^2 u \neq 0$, NO DA CERO, ENTONCES YA NO SE PUEDE CALCULAR LA FUNCIÓN ARMÓNICA CONJUGADA.

(b) ENCUENTRE LA FUNCIÓN ARMÓNICA CONJUGADA DE u; ES DECIR, HALLAR $V = V(x,y) = ?$

SOL: POR (C-R)

$$V_y = u_x$$

$$V_x = -u_y$$

$$V_y = u_x \Rightarrow V_y = 3x^2 - 3y^2$$

INTEGRANDO V_y RESPECTO A y.

$$\int V_y\,dy = 3x^2 \int dy - 3\int y^2\,dy$$

$$V = 3x^2 y - 3\frac{y^3}{3} + \phi(x) = 3x^2 y - y^3 + \phi(x)$$

$$V = 3x^2 y - y^3 + \phi(x) \Rightarrow V_x = 6xy + \phi'(x).$$

IGUALANDO $V_x = -U_y$.

$V_x = -U_y \Rightarrow 6xy + \phi'(x) = -(-6xy-5)$ 85

$6xy + \phi'(x) = 6xy + 5$; $\phi'(x) = 6xy + 5 - 6xy = 5$

$\phi'(x) = 5 \Rightarrow \phi(x) = 5x + C$

SUSTITUYENDO $\phi(x) = 5x + C$ EN $V = 3x^2y - y^3 + \phi(x)$

$$V = V(x,y) = 3x^2y - y^3 + 5x + C.$$

$\therefore f(z) = u + iv = u(x,y) + iv(x,y) = x^3 - 3xy^2 - 5y + i(3x^2y - y^3 + 5x + C)$

$$\therefore f(z) = x^3 - 3xy^2 - 5y + i(3x^2y - y^3 + 5x) + iC$$

COMPROBACIÓN.

$U = U(x,y) = x^3 - 3xy^2 - 5y$

$V = V(x,y) = 3x^2y - y^3 + 5x + C$

APLICANDO (C-R):

$U_x = 3x^2 - 3y^2$; $U_y = -6xy - 5$

$V_x = 6xy + 5$; $V_y = 3x^2 - 3y^2$.

UTILIZANDO: (C-R) NUEVAMENTE.

$V_y = U_x \Rightarrow 3x^2 - 3y^2 \equiv 3x^2 - 3y^2$

$V_x = -U_y \Rightarrow 6xy + 5 = -(-6xy - 5) = 6xy + 5 \equiv 6xy + 5$

$\therefore f(z)$ ES ANALÍTICA

• EXPRESAR $f(z)=x^3-3xy^2-5y+i(3x^2y-y^3+5x)$ EN TÉRMINOS DE LA VARIABLE z. 86

$$u=u(x,y)=x^3-3x^2y-5y \Rightarrow u(z,0)=z^3-3z(0)^2-5(0)=z^3$$

$$v=v(x,y)=3x^2y-y^3-5x \Rightarrow v(z,0)=3z^2(0)-0^3-5z=-5z.$$

FINALMENTE.

$$f(z)=u(z,0)+iv(z,0)=z^3+i5z.$$

$$\therefore f(z)=z^3+i5z$$

2 (a) SEA $u=u(x,y)=e^x\cos y$, VERIFICAR QUE u ES ARMÓNICA.

(b) ENCUENTRE LA FUNCIÓN ARMÓNICA CONJUGADA DE u; ES DECIR $v=v(x,y)$

SOL

(a) $u=u(x,y)=e^x\cos y$

$$u_x=e^x\cos y \quad ; \quad u_y=-e^x\operatorname{sen} y$$

$$u_{xx}=e^x\cos y \quad ; \quad u_{yy}=-e^x\cos y$$

$$\nabla^2 u=u_{xx}+u_{yy}=\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=e^x\cos y-e^x\cos y=0$$

$\therefore \nabla^2 u=0 \Rightarrow u=u(x,y)$ ES UNA FUNCIÓN ARMÓNICA.

(b) $u = e^{x}\cos y \Rightarrow u_x = e^{x}\cos y$

POR (C-R):

$$V_y = u_x \Rightarrow V_y = e^{x}\cos y$$

INTEGRANDO $V_y = e^{x}\cos y$ RESPECTO A y, OBTENEMOS QUE:

$$\int V_y = \int e^{x}\cos y = e^{x}\int \cos y = e^{x}\operatorname{sen} y + \phi(x)$$

$$\therefore V = e^{x}\operatorname{sen} y + \phi(x) \quad (*)$$

DONDE $\phi(x)$ ES UNA FUNCIÓN ARBITRARIA DE x.

DE $(*)$ $V = e^{x}\operatorname{sen} y + \phi(x) \Rightarrow V_x = e^{x}\operatorname{sen} y + \phi'(x)$.

NUEVAMENTE UTILIZANDO (C-R) E IGUALANDO.

$$V_x = -u_y \Rightarrow e^{x}\operatorname{sen} y + \phi'(x) = -(-e^{x}\operatorname{sen} y)$$

$$e^{x}\operatorname{sen} y + \phi'(x) = e^{x}\operatorname{sen} y$$

$$\phi'(x) = e^{x}\operatorname{sen} y - e^{x}\operatorname{sen} y = 0$$

$\therefore \phi'(x) = 0 \Rightarrow \phi(x) = C$ SUSTITUYENDO EN $(*)$:

$V = e^{x}\operatorname{sen} y + C$, Y LA FUNCIÓN $f(z) = u + iV$ CORRESPONDIENTE.

$$f(z) = u + iV = u(x,y) + iV(x,y) = e^{x}\cos y + i\,e^{x}\operatorname{sen} y + iC.$$

$$f(z) = e^{x}\left[\cos y + i\operatorname{sen} y\right] + iC = e^{z} + iC$$

RECUERDE:

$$e^{z} = e^{x+iy} = e^{x}e^{iy} = e^{x}\left[\cos y + i\,\text{sen}\,y\right]$$

OTRA FORMA:

$$u = e^{x}\cos y \Rightarrow u(z,0) = e^{z}\cos 0^{\circ} = e^{z}(1) = e^{z}.$$

$$v = e^{x}\,\text{sen}\,y \Rightarrow v(z,0) = e^{z}\,\text{sen}\,0^{\circ} = e^{z}(0) = 0$$

$$\therefore f(z) = u(z,0) + i\,v(z,0) = e^{z} + i(0) = e^{z}.$$

(c) COMPROBACIÓN:

$$f(z) = e^{z} + iC = e^{x}\cos y + i\,e^{x}\,\text{sen}\,y + iC$$

$$f(z) = e^{z} + iC \Rightarrow f'(z) = e^{z} + 0 = e^{z}$$ FUNCIÓN ENTERA.

POR OTRO LADO:

$$u = e^{x}\cos y \Rightarrow u_x = e^{x}\cos y \;;\; u_y = -e^{x}\,\text{sen}\,y$$

$$v = e^{x}\,\text{sen}\,y \Rightarrow v_x = e^{x}\,\text{sen}\,y \;;\; v_y = e^{x}\cos y$$

LAS ECUACIONES DE (C-R):

$$v_y = u_x \Rightarrow e^{x}\cos y \equiv e^{x}\cos y$$

$$v_x = -u_y \Rightarrow e^{x}\,\text{sen}\,y \equiv -(-e^{x}\,\text{sen}\,y) \equiv e^{x}\,\text{sen}\,y$$

f(z) CUMPLE LAS CONDICIONES DE (C-R), ES ANALÍTICA.

• FUNCIONES ELEMENTALES.

• FUNCIÓN EXPONENCIAL $e^z = \exp z$.

ANTERIORMENTE, UNA FUNCIÓN EXPONENCIAL CON UN EXPONENTE IMAGINARIO PURO, ES LA FÓRMULA DE EULER.

$$e^{iy} = \cos y + i \operatorname{sen} y \quad \forall\, y \in \mathbb{R} - (*)$$

DE (*) UTILIZANDO LAS SERIES DE POTENCIAS DE MACLAURIN PARA e^x Y SUSTITUYENDO x POR iy, SE TIENE QUE:

$$\boxed{e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots = \sum_{n=0}^{\infty} \frac{x^n}{n!}}$$

$$e^{iy} = \sum_{k=0}^{\infty} \frac{(iy)^k}{k!} = \frac{(iy)^0}{0!} + \frac{(iy)^1}{1!} + \frac{(iy)^2}{2!} + \frac{(iy)^3}{3!} + \frac{(iy)^4}{4!} + \cdots$$

$$= 1 + iy + \frac{(iy)^2}{2!} + \frac{(iy)^3}{3!} + \frac{(iy)^4}{4!} + \cdots$$

REACOMODANDO TÉRMINOS REALES E IMAGINARIOS, TAL QUE $i^2 = -1$

$$= \left(1 - \frac{y^2}{2!} + \frac{y^4}{4!} - \frac{y^6}{6!} + \cdots\right) + i\left(y - \frac{y^3}{3!} + \frac{y^5}{5!} - \frac{y^7}{7!} + \cdots\right)$$

SABEMOS QUE:

$$\cos y = \sum_{k=0}^{\infty} (-1)^k \frac{y^{2k}}{(2k)!} \quad y \quad \operatorname{sen} y = \sum_{k=0}^{\infty} (-1)^k \frac{y^{2k+1}}{(2k+1)!}.$$

$$e^{iy} = \cos y + i \operatorname{sen} y \Rightarrow e^{iy} = \cos y + i \operatorname{sen} y \quad \text{EULER.}$$

Si $z = x + iy$

$$e^{z} = e^{x+iy} = e^{x} e^{iy} = e^{x}(\cos y + i \operatorname{sen} y).$$

FUNCIÓN EXPONENCIAL e^{z}.

DEF

$$e^{z} = e^{x+iy} = e^{x} e^{iy} = e^{x}(\cos y + i \operatorname{sen} y).$$

DE LA DEFINICIÓN ANTERIOR.

. Si HACEMOS $y = 0$ SE REDUCE A e^{x}; $\arg e^{z} = y$.

$$\left| e^{iy} \right| = \sqrt{\cos^2 y + \operatorname{sen}^2 y} = \sqrt{1} = 1.$$

. CALCULE: $e^{1.7+i4.2} = e^{1.7}\cos 4.2 + i\, e^{1.7} \operatorname{sen} 4.2$

$$e^{1.7+i4.2} = -2.6837 - i\,4.7710.$$

. $e^{i\pi} = e^{-i\pi} = \cos\pi + i \operatorname{sen}\pi = \cos(-\pi) + i \operatorname{sen}(-\pi)$

$= \cos(\pi) - i \operatorname{sen}\pi = (-1) + i(0) = -1 + 0 = -1$

. $e^{-\pi/2} = \cos\left(\frac{\pi}{2}\right) - i \operatorname{sen}\left(\frac{\pi}{2}\right) = 0 - i(1) = -i.$

. $e^{i\pi/2} = \cos\left(\frac{\pi}{2}\right) + i \operatorname{sen}\left(\frac{\pi}{2}\right) = 0 + i(1) = i.$

SABEMOS QUE: $e^z = e^x \cos y + i e^x \operatorname{sen} y$. (91)

DONDE:

$$u = e^x \cos y \quad y \quad v = e^x \operatorname{sen} y.$$

POR LO ANTERIOR, VIMOS e^z CUMPLE LAS CONDICIONES DE CAUCHY-RIEMANN EN TODOS LOS PUNTOS DE PLANO COMPLEJO.

$$u_x = e^x \cos y = v_y \quad y \quad u_y = -e^x \operatorname{sen} y = -v_x.$$

POR TANTO $f(z) = e^z$ ES ANALÍTICO PARA CUALQUIER z, EN OTRAS PALABRAS, $f(z)$ ES FUNCIÓN ENTERA.

TAMBIÉN QUE EN VARIABLE REAL; ES DECIR.

$$f'(z) = e^x \cos y + i(e^x \operatorname{sen} y) = e^x(\cos y + i \operatorname{sen} y) = f(z).$$

$$f'(z) = e^z = f(z).$$

$$e^{z+2\pi i} = e^z e^{2\pi i} = e^z$$

JUSTIFICACIÓN: $e^{z+2\pi i} = e^z e^{2\pi i} = e^z(\cos 2\pi + i \operatorname{sen} 2\pi)$

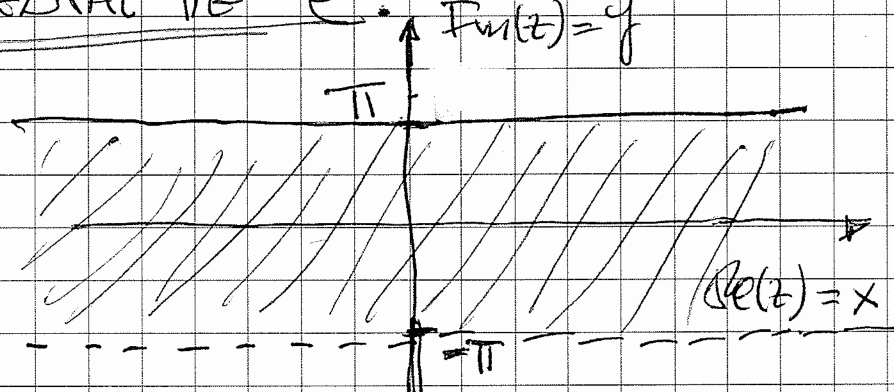
$$= e^z(1 + i(0)) = e^z(1) = e^z.$$

LO CUAL INDICA QUE e^z ES PERIÓDICO CON EL PERIODO

IMAGINARIO $i2\pi$. $\quad e^{z \pm 2n\pi i} = e^{z}$ si $n = 0, 1, \ldots$

DEBIDO A LA PERIODICIDAD, TODOS LOS VALORES QUE $w = e^{z}$ PUEDE TOMAR YA HAN SIDO TOMADOS EN LA FRANJA $-\pi < y \leq \pi$.

ESTA FRANJA INFINITA RECIBE EL NOMBRE DE REGIÓN FUNDAMENTAL DE e^{z}.



- $e^{z} e^{-z} = e^{0} = 1 \Rightarrow e^{z} \neq 0 \;\forall z$.
- $e^{z} = e^{x}(\cos y + i \operatorname{sen} y) \Rightarrow |e^{z}| = e^{x}$, $\arg e^{z} = y$.

$$e^{z} = e^{x}(\cos y + i \operatorname{sen} y) = e^{x}\cos y + i e^{x} \operatorname{sen} y$$

$$|e^{z}| = \sqrt{(e^{x}\cos y)^2 + (e^{x}\operatorname{sen} y)^2} = \sqrt{(e^{x})^2(\cos^2 y + \operatorname{sen}^2 y)}$$

$$= \sqrt{(e^{x})^2(1)} = e^{x} \quad \therefore \quad |e^{z}| = e^{x}$$ $\boxed{\,}$

HACIENDO $z_1 = x_1 + iy_1$ y $z_2 = x_2 + iy_2$, TENEMOS:

- $e^{z_1 z_2} = e^{z_1 + z_2}$

- $\dfrac{e^{z_1}}{e^{z_2}} = e^{z_1} \cdot e^{-z_2} = e^{z_1 - z_2}$.

· LOGARITMO DE UN NÚMERO COMPLEJO

DEF PARA CUALQUIER NÚMERO COMPLEJO $z \neq 0$, EXISTEN NÚMEROS COMPLEJOS $w = \ln z$ SI $z = e^{w}$. EN PARTICULAR, UN w ES UN NÚMERO COMPLEJO

$$w = \ln z = \operatorname{Ln}|z| + i \arg z = \operatorname{Ln} r + i\theta, \quad (*)$$

Y CUALQUIER w ESTÁ DADA POR:

$$\ln z = \operatorname{Ln} r + i\theta + i2n\pi \quad \text{si } n = 0, \pm 1, \pm 2, \quad (**)$$

DONDE:

$$z = r e^{i\theta} \; ; \; \theta = \arg z \quad \text{con } -\pi < \theta \leq \pi.$$

$$|z| = r = \sqrt{x^2 + y^2}$$

SEA:

$$w = \ln z = \operatorname{Ln} r + i(\theta + 2n\pi) \quad \text{si } n = 0, \pm 1, \pm 2, \ldots$$

EN PARTICULAR, EL NÚMERO w DADO POR $(*)$ SE LLAMA LOGARITMO DE z.

DE (*) NUEVAMENTE SI $n=0$ SE LLAMA VALOR PRINCIPAL DEL LOGARITMO DE Z.

DE (*)

$$w = \ln z = \operatorname{Ln} r + i(\theta + 2n\pi)$$

SI $n=0$

$$\therefore w = \ln z = \operatorname{Ln} r + i(\theta + 2(0)\pi)$$

$$w = \ln z = \operatorname{Ln} r + i\theta \qquad (***)$$

NOTE QUE $\ln z$ NO ES UN VALOR ÚNICO SINO UNA FUNCIÓN MULTIVALUADA DE Z.

EJEMPLOS:

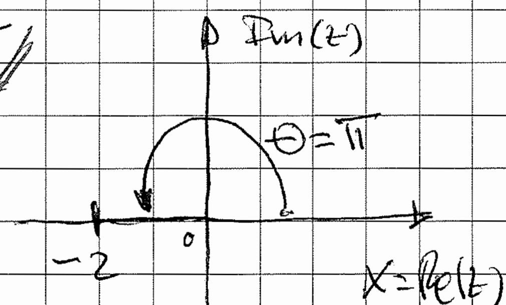
- DETERMINAR EL VALOR PRINCIPAL DEL LOGARITMO DE Z.

① $\ln(-2)$

SOL: DE (***)

$$w = \ln z = \operatorname{Ln} r + i\theta = \operatorname{Ln} 2 + i\pi = 0.693 + i\pi$$

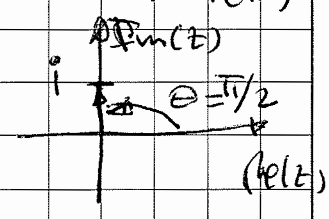
$$\therefore w = \ln z = 0.693 + i\pi$$



$r = \sqrt{(-2)^2} = \sqrt{2^2} = 2$

$\theta = \arg z = \pi \Rightarrow \theta = \pi$

② $w = \ln i.$



SOL:

$$z = i = (0,1) \Rightarrow r = |z| = \sqrt{0^2 + 1^2} = \sqrt{1} = 1$$

$\theta = \frac{\pi}{2}.$

POR (***):

$$w = \ln z = \operatorname{Ln} r + i\theta \Rightarrow w = \ln i = \operatorname{Ln} 1 + i\frac{\pi}{2} = 0 + i\frac{\pi}{2} = \frac{\pi}{2} i.$$

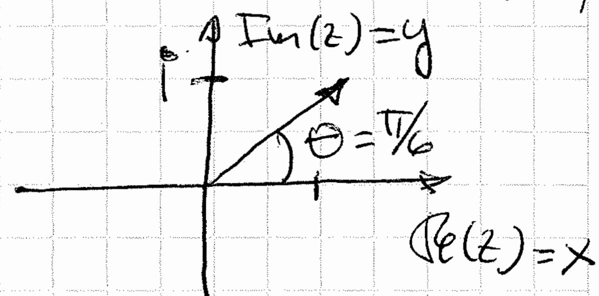
$$\therefore w = \ln i = i\frac{\pi}{2}.$$

2) HALLE TODOS LOS VALORES DE z TALES QUE: 95

Sol $e^{z} = \sqrt{3} + i$.

$$\ln e^{z} = \ln(\sqrt{3}+i) \;\Longrightarrow\; z = \ln(\sqrt{3}+i) = \ln(\sqrt{3},1)$$

$$z = \ln(\sqrt{3}+i) = \ln(\sqrt{3},1) = \operatorname{Ln} r + i(\theta + 2n\pi) \text{ si } n = 0, \pm 1, \pm 2, \ldots$$



$$r = \sqrt{(\sqrt{3})^2 + 1^2} = \sqrt{3+1} = \sqrt{4} = 2$$

$$\theta = \operatorname{tg}^{-1}\left[\frac{1}{\sqrt{3}}\right] = \frac{\pi}{6} = 30^{\circ}$$

$$z = \ln(\sqrt{3}+i) = \operatorname{Ln} 2 + i(\theta + 2n\pi) \quad \text{si } n = 0, \pm 1, \pm 2, \ldots$$

$$z = 0{,}6931 + i\left(\frac{\pi}{6} + 2n\pi\right) \quad \text{si } n = 0, \pm 1, \pm 2, \ldots$$

- PROPIEDADES.

a) $\ln(z_1 z_2) = \ln z_1 + \ln z_2$.

b) $\ln\left(\dfrac{z_1}{z_2}\right) = \ln z_1 - \ln z_2$.

c) $\dfrac{d}{dz} \operatorname{Ln} z = \dfrac{1}{z}$.

COMENTARIO: EN GENERAL LA PROPIEDAD a) NO SIEMPRE SE CUMPLE.

③ VERIFICAR a) CON $z_1 = (-1+i)$ y $z_2 = i$. (96)

SOL: $\ln(z_1 z_2) = \ln z_1 + \ln z_2$

$$\ln((-1+i)i) = \ln(-1+i) + \ln i$$

$$\ln(-i-1) = \ln(-1+i) + \ln i$$

$$\frac{1}{2}\ln 2 - i\frac{3}{4}\pi \neq \frac{1}{2}\ln 2 + i\frac{3}{4}\pi + i\frac{\pi}{2}.$$

$$\therefore \ln(z_1 z_2) \neq \ln z_1 + \ln z_2$$

- POTENCIAS GENERALES.

DEF: SI $z \neq 0$ y w ES UN NÚMERO COMPLEJO, DEFINIMOS z^w POR:

$$z^w = e^{w \ln z} \quad \text{TAL QUE } \ln z = \operatorname{Ln} r + i\theta + 2n\pi i$$

$$-\pi < \theta \leq \pi, \quad n = 0, \pm 1, \pm 2, \ldots$$

NOTE QUE SI w NO ES UN NÚMERO RACIONAL, ENTONCES:

$$e^{w \ln z} = e^{w[\operatorname{Ln} r + i\theta + 2n\pi i]} \quad \text{si } n = 0, \pm 1, \pm 2, \ldots$$

El VALOR PRINCIPAL DE z^w $(n=0)$ SE DEFINE COMO:

$$z^w = e^{w \ln z} \quad \text{si } \ln z = \operatorname{Ln} r + i\theta \Rightarrow z^w = e^{w \ln z} = e^{w[\operatorname{Ln} r + i\theta]} \quad (*)$$

DE AQUÍ EN ADELANTE $z^w = e^{w \ln z}$ EXCEPTO DONDE SE INDIQUE LO CONTRARIO. ENTONCES z^w ES UNA FUNCIÓN UNIVALUADA.

EJEMPLOS.

(1) ENCUENTRE EL VALOR DE :

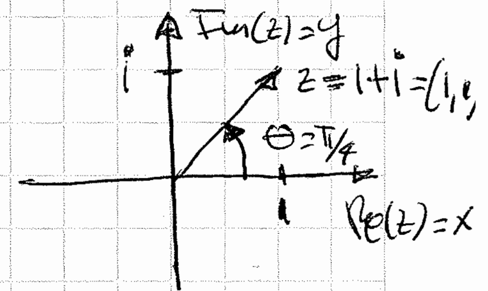
SOL:

$$\left[(1+i)^{(1-i)}\right]^{(1+i)}.$$

ANALIZANDO LOS EXPONENTES: $(1-i)(1+i)=1-i^2=1-(-1)=1+1=2.$

ENTONCES:

$$\left[(1+i)^{(1-i)}\right]^{(1+i)}=(1+i)^2$$



LUEGO: $z=1+i$ y $w=2$

$$r=\sqrt{1^2+1^2}=\sqrt{2} \qquad \theta=tg^{-1}\left[\frac{1}{1}\right]=\frac{\pi}{4}=45°.$$

DE (*)

$$z^w=e^{w\ln z}=e^{w[\ln r+i\theta]}$$

$$(1+i)^2=e^{2\left[\ln\sqrt{2}+i\frac{\pi}{4}\right]}=e^{2\ln\sqrt{2}}\,e^{i\frac{2\pi}{4}}=e^{\frac{2}{2}\ln 2}\,e^{i\frac{\pi}{2}}$$

$$=e^{\ln 2}\,e^{i\pi/2}=2\left[e^{i\pi/2}\right]=2\left[\cos\frac{\pi}{2}+i\,sen\frac{\pi}{2}\right]$$

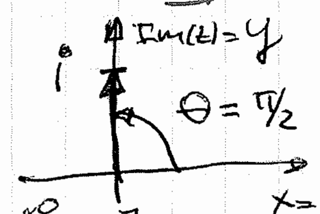
$$=2\left[\cos\frac{\pi}{2}+i\,sen\frac{\pi}{2}\right]=2\left[0+i(1)\right]=2[i]=2i$$

OTRO MÉTODO (ALGEBRAICO).

$$(1+i)^2=1^2+2i+i^2=1+2i-1=0+2i=2i$$

(2) HALLE EL VALOR DE i^{2i}.

SOL $z=i=(0,1)$; $w=2i$



$$r=\sqrt{0^2+1^2}=1 \ ; \ \theta=\frac{\pi}{2}$$

DE (*)

$$i^{2i}=e^{w[\ln r+i\theta]}=e^{2i\left[\cancel{\ln 1}^{\,0}+i\frac{\pi}{2}\right]}=e^{2i[i\pi/2]}=e^{-\pi}=0{,}043$$

4.2. THE TRIGONOMETRIC FUNCTIONS.

Since

$$e^{iy} = \cos y + i \sin y \quad \text{and} \quad e^{-iy} = \cos y - i \sin y,$$

subtracting and adding these equations we obtain

$$\sin y = \frac{e^{iy} - e^{-iy}}{2i}, \qquad \cos y = \frac{e^{iy} + e^{-iy}}{2}. \tag{4.2.1}$$

These real trigonometric functions will be extended to the domain of a complex variable by the following

Definition 4.2.1. Given any complex number z, we define

$$\sin z = \frac{e^{iz} - e^{-iz}}{2i}, \qquad \cos z = \frac{e^{iz} + e^{-iz}}{2}. \tag{4.2.2}$$

Taking z to be real, we note that these equations are consistent with equations (4.2.1). Also, note that $\sin z$ and $\cos z$ are both periodic with period 2π.

Using Theorems 4.1.4 and 3.5.1 we have the following

—*THEOREM* 4.2.1. The functions $\sin z$ and $\cos z$ are analytic for all values of z. Moreover

$$\frac{d}{dz}(\sin z) = \cos z, \qquad \frac{d}{dz}(\cos z) = -\sin z. \tag{4.2.3}$$

Definition 4.2.2. A point z_0 for which $f(z_0) = 0$ is called a *zero of the function* $f(z)$.

—*THEOREM* 4.2.2. The zeros of the functions $\sin z$ and $\cos z$ are given respectively by

$$z = n\pi \quad \text{and} \quad z = \frac{\pi}{2} + n\pi, \tag{4.2.4}$$

where $n = 0, \pm 1, \pm 2, \cdots$.

Proof. If $\sin z = 0$, then from (4.2.2) we obtain $\exp(2iz) = 1$. We then have in view of part (c) of Theorem 4.1.2

$$2iz = 2n\pi i \quad \text{or} \quad z = n\pi \qquad (n = 0, \pm 1, \pm 2, \cdots).$$

If $\cos z = 0$, then from (4.2.2) we obtain $\exp(2iz) = -1$. Consequently, $z = \pi/2 + n\pi$ $(n = 0, \pm 1, \pm 2, \cdots)$ in virtue of Exercise 4.1.1. The theorem is thus established.

Note that the only zeros of the complex sine and cosine functions are the real numbers that appear already as the zeros of the real sine and cosine functions.

We shall say that a domain D is *symmetric* with respect to the origin, if for every point z in D the point $-z$ is also in D.

Definition 4.2.3. Let $w = f(z)$ be a function defined in a domain D which is symmetric with respect to the origin. If $f(-z) = f(z)$ for all values of z in D, then $f(z)$ is called an *even function*; if $f(-z) = -f(z)$ for all values of z in D, then $f(z)$ is called an *odd function*.

From (4.2.2) we see that $\sin z$ and $\cos z$ are respectively odd and even functions:

$$\sin(-z) = -\sin z \quad \text{and} \quad \cos(-z) = \cos z. \tag{4.2.5}$$

The other trigonometric functions are given by the following

Definition 4.2.4. Given the complex number z, we define

$$\begin{aligned} \tan z &= \frac{\sin z}{\cos z} \text{ for } z \neq \frac{\pi}{2} + n\pi, \\ \cot z &= \frac{\cos z}{\sin z} \text{ for } z \neq n\pi, \\ \sec z &= \frac{1}{\cos z} \text{ for } z \neq \frac{\pi}{2} + n\pi, \\ \csc z &= \frac{1}{\sin z} \text{ for } z \neq n\pi, \end{aligned} \tag{4.2.6}$$

where in all cases $n = 0, \pm 1, \pm 2, \cdots$.

$\tan z$, $\cot z$ have period π, while $\sec z$, $\csc z$ have period 2π.

Utilizing Theorems 4.2.1 and 3.5.1, we may readily establish the following

—*THEOREM* 4.2.3. The functions $\tan z$, $\cot z$, $\sec z$, and $\csc z$ are analytic functions of z except for those values of z excluded by Definition 4.2.4. Moreover

$$\begin{aligned} \frac{d}{dz}(\tan z) &= \sec^2 z \text{ for } z \neq \frac{\pi}{2} + n\pi, \\ \frac{d}{dz}(\cot z) &= -\csc^2 z \text{ for } z \neq n\pi, \\ \frac{d}{dz}(\sec z) &= \sec z \tan z \text{ for } z \neq \frac{\pi}{2} + n\pi, \\ \frac{d}{dz}(\csc z) &= -\csc z \cot z \text{ for } z \neq n\pi, \end{aligned} \tag{4.2.7}$$

THEOREM 4.2.4. If $z = x + iy$, then

$$\sin z = \sin x \cosh y + i \cos x \sinh y, \tag{4.2.8}$$

$$\cos z = \cos x \cosh y - i \sin x \sinh y. \tag{4.2.9}$$

Proof. Verification of (4.2.8). Using (4.2.2) and (4.1.1), and recalling the definitions that $\sinh y = (e^y - e^{-y})/2$ and $\cosh y = (e^y + e^{-y})/2$, y real, we obtain

$$\begin{aligned} 2i \sin z &= e^{iz} - e^{-iz} \\ &= e^{-y}(\cos x + i \sin x) - e^{y}(\cos x - i \sin x) \\ &= i \sin x(e^y + e^{-y}) - \cos x(e^y - e^{-y}) \\ &= 2i \sin x \cosh y - 2 \cos x \sinh y, \end{aligned}$$

from which (4.2.8) now follows.

The proof for $\cos z$ is similar to that given for $\sin z$.

THEOREM 4.2.5. If $z = x + iy$, then

$$\sin iy = i \sinh y, \qquad \cos iy = \cosh y, \tag{4.2.10}$$

$$\sin \bar{z} = \overline{\sin z}, \qquad \cos \bar{z} = \overline{\cos z}, \tag{4.2.11}$$

$$|\sin z|^2 = \sin^2 x + \sinh^2 y, \tag{4.2.12}$$

$$|\cos z|^2 = \cos^2 x + \sinh^2 y. \tag{4.2.13}$$

Proof. Verification of (4.2.10). If we substitute $z = 0 + iy$ into (4.2.8) and (4.2.9), we obtain (4.2.10).

Verification of (4.2.11). Replace z by $\bar{z}$ in (4.2.8) and (4.2.9) and recall that $\cosh(-y) = \cosh y$ and $\sinh(-y) = -\sinh y$; we obtain

$$\sin \bar{z} = \sin x \cosh y - i \cos x \sinh y = \overline{\sin z},$$

$$\cos \bar{z} = \cos x \cosh y + i \sin x \sinh y = \overline{\cos z}.$$

Verification of (4.2.12). Utilizing (4.2.8) and Definition 1.3.2, and recalling the identity $\cosh^2 y - \sinh^2 y = 1$, we obtain

$$\begin{aligned} |\sin z|^2 &= \sin^2 x \cosh^2 y + \cos^2 x \sinh^2 y \\ &= \sin^2 x(1 + \sinh^2 y) + \cos^2 x \sinh^2 y \\ &= \sin^2 x + \sinh^2 y. \end{aligned}$$

Similarly, one verifies (4.2.13), and the theorem is established.

Remark 4.2.1. From (4.2.12) and (4.2.13) we see that the absolute values of $\sin z$ and $\cos z$ can be made as large as we please; however, when z is real, the absolute values of $\sin z$ and $\cos z$ are never greater than unity.

Remark 4.2.2. Using properties of the exponential function, one may show directly that the standard identities for the trigonometric functions of a real variable x extend to the case of a complex variable z. Thus we have, for example,

$$\sin^2 z + \cos^2 z = 1, \tag{4.2.14}$$

$$\sin(z_1 \pm z_2) = \sin z_1 \cos z_2 \pm \cos z_1 \sin z_2, \tag{4.2.15}$$

$$\cos(z_1 \pm z_2) = \cos z_1 \cos z_2 \mp \sin z_1 \sin z_2, \tag{4.2.16}$$

$$\sin\left(\frac{\pi}{2} - z\right) = \cos z, \tag{4.2.17}$$

$$\sin 2z = 2 \sin z \cos z, \tag{4.2.18}$$

$$\cos 2z = \cos^2 z - \sin^2 z. \tag{4.2.19}$$

EXAMPLE 4.2.1. Let us show that

$$\tan(z_1 + z_2) = \frac{\tan z_1 + \tan z_2}{1 - \tan z_1 \tan z_2}. \tag{4.2.20}$$

Solution. From (4.2.15) and (4.2.16) we have, as in the real case,

$$\begin{aligned} \tan(z_1 + z_2) &= \frac{\sin(z_1 + z_2)}{\cos(z_1 + z_2)} \\ &= \frac{\sin z_1 \cos z_2 + \cos z_1 \sin z_2}{\cos z_1 \cos z_2 - \sin z_1 \sin z_2} \\ &= \frac{\tan z_1 + \tan z_2}{1 - \tan z_1 \tan z_2}. \end{aligned}$$

EXERCISES 4.2

1. Prove Theorem 4.2.3.
2. Establish identities (4.2.14) to (4.2.19).
3. Prove (4.2.9).
4. Prove that $\exp(iz) = \cos z + i \sin z$ and $\exp(-iz) = \cos z - i \sin z$.
5. Prove that
$$\cos z_2 - \cos z_1 = -2 \sin\left(\frac{z_2 + z_1}{2}\right) \sin\left(\frac{z_2 - z_1}{2}\right).$$
6. Prove that
$$\sin z_2 - \sin z_1 = 2 \cos\left(\frac{z_2 + z_1}{2}\right) \sin\left(\frac{z_2 - z_1}{2}\right).$$
7. Prove that $|\cos z|^2 = \cos^2 x + \sinh^2 y$.

8. Show that the functions $\sin \bar{z}$ and $\cos \bar{z}$ are not analytic functions of z in any domain D.
9. Show that $|\sin z| \leq \cosh y$ and $|\sin z| \geq |\sinh y|$.
10. Show that $|\cos z| \leq \cosh y$ and $|\cos z| \geq |\sinh y|$.
11. Show that if $|z| \leq 1$, then $|\cos z| < 2$ and $|\sin z| < \frac{6}{5}|z|$, $z \neq 0$.
12. Show that if w is an analytic function of z, then $\sin w$ and $\cos w$ are also analytic functions of z, and

$$\frac{d}{dz}(\sin w) = \cos w \frac{dw}{dz}, \qquad \frac{d}{dz}(\cos w) = -\sin w \frac{dw}{dz}.$$

13. Prove results for the functions in Theorem 4.2.3 similar to those given in Exercise 12 above.
14. Find the roots of the equation $\cos z = 2$.
15. Find the roots of the equation $\sin z = \cosh k$, where k is a real constant.
16. Prove that if $\sin z_1 = \sin z_2$, then either

$$z_1 = z_2 + 2n\pi \quad \text{or} \quad z_1 = (2n+1)\pi - z_2, \quad \text{where } n \text{ is an integer.}$$

17. Prove that if $\cos z_1 = \sin z_2$, then either

$$\frac{\pi}{2} - z_1 = z_2 + 2n\pi \text{ or } \frac{\pi}{2} - z_1 = (2n+1)\pi - z_2, \quad \text{where } n \text{ is an integer.}$$

18. Prove that if $\cos z_1 = \sin z_2$, then

$$z_1 = (-1)^{k+1} z_2 + \frac{\pi}{2} + k\pi, \qquad \text{where } k \text{ is an integer.}$$

19. Prove that $\tan z_1 = \tan z_2$ if and only if $z_1 = z_2 + n\pi$, where n is an integer.
20. Prove that

$$\tan z = \frac{\sin 2x + i \sinh 2y}{\cos 2x + \cosh 2y}.$$

21. Show that the equation $\tan z = cz$, where c is real, has no complex roots of the form $z = x + iy$, $x \neq 0$, $y \neq 0$.
22. Show that if $\sin(x + iy) = \csc(u + iv)$, where x, y, u, v are real, then

    (a) $\sin x \cosh y = \dfrac{\sin u \cosh v}{\cosh^2 v - \cos^2 u}$, $\cos x \sinh y = -\dfrac{\cos u \sinh v}{\cosh^2 v - \cos^2 u}$;

    (b) $\tan x \coth y + \tan u \coth v = 0$;

    (c) $e^{iz} = i \tan \dfrac{w}{2}$ $(z = x + iy,\ w = u + iv)$;

    (d) $\tan x = -\sin u \operatorname{csch} v$;

    (e) $\tanh y \cosh v = \cos u$;

    (*f*) $\tanh v \cosh y = \cos x$.

23. Use formula (b) of Exercise 3.8.16 to check Exercises 3.8.6 and 3.8.9.

4.3. THE HYPERBOLIC FUNCTIONS

Definition 4.3.1. Given any complex number z, we define

$$\sinh z = \frac{e^z - e^{-z}}{2}, \qquad \cosh z = \frac{e^z + e^{-z}}{2}. \tag{4.3.1}$$

Taking z to be real, we note that these equations are consistent with those for the hyperbolic sine and cosine with real arguments:

$$\sinh x = \frac{e^x - e^{-x}}{2}, \qquad \cosh x = \frac{e^x + e^{-x}}{2}. \tag{4.3.2}$$

Observe that $\sinh z$ and $\cosh z$ are periodic with period $2\pi i$.

Using Theorems 4.1.3 and 3.5.1, we have the following

THEOREM 4.3.1. The functions $\sinh z$ and $\cosh z$ are analytic for all values of z. Moreover

$$\frac{d}{dz}(\sinh z) = \cosh z, \qquad \frac{d}{dz}(\cosh z) = \sinh z. \tag{4.3.3}$$

Utilizing (4.3.1), (4.1.6), and Exercise 4.1.1, we have the following

THEOREM 4.3.2. The zeros of the functions $\sinh z$ and $\cosh z$ are given respectively by

$$z = n\pi i \quad \text{and} \quad z = \left(n + \frac{1}{2}\right)\pi i \tag{4.3.4}$$

where $n = 0, \pm 1, \pm 2, \cdots$.

Note that the zeros of $\sinh z$ and $\cosh z$ are pure imaginary numbers. From (4.3.1) and Definition 4.2.3, we see that $\sinh z$ and $\cosh z$ are respectively odd and even functions:

$$\sinh(-z) = -\sinh z \quad \text{and} \quad \cosh(-z) = \cosh z. \tag{4.3.5}$$

The other hyperbolic functions are given by the following

Definition 4.3.2. Given the complex number z, we define

$$\begin{aligned}
\tanh z &= \frac{\sinh z}{\cosh z} \text{ for } z \neq \left(n + \frac{1}{2}\right)\pi i,\\
\coth z &= \frac{\cosh z}{\sinh z} \text{ for } z \neq n\pi i,\\
\operatorname{sech} z &= \frac{1}{\cosh z} \text{ for } z \neq \left(n + \frac{1}{2}\right)\pi i,\\
\operatorname{csch} z &= \frac{1}{\sinh z} \text{ for } z \neq n\pi i,
\end{aligned} \tag{4.3.6}$$

where in all cases $n = 0, \pm 1, \pm 2, \cdots$.

Tanh z, coth z have period πi, while sech z, csch z have period $2\pi i$.

Utilizing Theorems 4.3.1 and 3.5.1, we may establish the following

——*THEOREM* 4.3.3. The functions $\tanh z$, $\coth z$, sech z and csch z are analytic functions of z except for those values of z excluded by Definition 4.3.2. Moreover

$$\begin{aligned} &\frac{d}{dz}(\tanh z) = \operatorname{sech}^2 z \text{ for } z \neq \left(n+\frac{1}{2}\right)\pi i, \\ &\frac{d}{dz}(\coth z) = -\operatorname{csch}^2 z \text{ for } z \neq n\pi i, \\ &\frac{d}{dz}(\operatorname{sech} z) = -\operatorname{sech} z \tanh z \text{ for } z \neq \left(n+\frac{1}{2}\right)\pi i, \\ &\frac{d}{dz}(\operatorname{csch} z) = -\operatorname{csch} z \coth z \text{ for } z \neq n\pi i, \end{aligned} \tag{4.3.7}$$

where in all cases $n = 0, \pm 1, \pm 2, \cdots$.

Using techniques similar to those employed to establish Theorem 4.2.4, one may establish the following

——*THEOREM* 4.3.4. If $z = x + iy$, then

$$\sinh z = \cos y \sinh x + i \sin y \cosh x, \tag{4.3.8}$$
$$\cosh z = \cos y \cosh x + i \sin y \sinh x. \tag{4.3.9}$$

By comparing (4.2.2) and (4.3.1), and also utilizing the above theorem, one may readily establish the following

——*THEOREM* 4.3.5. If $z = x + iy$, then

$$\sinh(iz) = i \sin z, \qquad \sin(iz) = i \sinh z, \tag{4.3.10}$$
$$\cosh(iz) = \cos z, \qquad \cos(iz) = \cosh z, \tag{4.3.11}$$
$$\sinh \bar{z} = \overline{\sinh z}, \qquad \cosh \bar{z} = \overline{\cosh z}, \tag{4.3.12}$$
$$\begin{aligned} &|\sinh z|^2 = \sin^2 y + \sinh^2 x, \\ &|\cosh z|^2 = \cos^2 y + \sinh^2 x. \end{aligned} \tag{4.3.13}$$

Remark 4.3.1. By means of the relations (4.3.10) and (4.3.11), all the properties of hyperbolic functions enumerated in this section may be derived from the corresponding properties of the trigonometric functions; and vice versa.

Also, we may easily verify the following identities:

$$\cosh^2 z - \sinh^2 z = 1, \tag{4.3.14}$$
$$\sinh(z_1 \pm z_2) = \sinh z_1 \cosh z_2 \pm \cosh z_1 \sinh z_2, \tag{4.3.15}$$
$$\cosh(z_1 \pm z_2) = \cosh z_1 \cosh z_2 \pm \sinh z_1 \sinh z_2, \tag{4.3.16}$$
$$\sinh\left(\frac{\pi}{2} i - z\right) = i \cosh z, \tag{4.3.17}$$
$$\sinh 2z = 2 \sinh z \cosh z, \tag{4.3.18}$$
$$\cosh 2z = \cosh^2 z + \sinh^2 z. \tag{4.3.19}$$

EXERCISES 4.3

1. Prove Theorem 4.3.1.
2. Prove Theorem 4.3.2.
3. Prove Theorem 4.3.3.
4. Prove Theorem 4.3.4.
5. Prove Theorem 4.3.5.
6. Verify (4.3.14)-(4.3.19).
7. Prove that if $z = x + iy$, then
$$\tanh z = \frac{\sinh x \cosh x + i \sin y \cos y}{\cos^2 y \cosh^2 x + \sin^2 y \sinh^2 x}.$$
8. Show that if $\tanh(x + iy) = u + iv$, where x, y, u, v are real, then
$$u = \frac{\sinh 2x}{\cosh 2x + \cos 2y}, \qquad v = \frac{\sin 2y}{\cosh 2x + \cos 2y}.$$
9. Find the values of z for which $\sinh z = -i$; $\sinh z = -1$.
10. Show that if w is an analytic function of z, then $\sinh w$ and $\cosh w$ are also analytic functions of z, and
$$\frac{d}{dz}(\sinh w) = \cosh w \frac{dw}{dz}, \qquad \frac{d}{dz}(\cosh w) = \sinh w \frac{dw}{dz}.$$
11. Prove Theorem 4.3.4 by using Theorems 4.3.5 and 4.2.4.
12. Prove Theorem 4.3.2 by using Theorems 4.3.5 and 4.2.2.

4.4. THE LOGARITHMIC FUNCTION. We observed in Theorem 4.1.2 that e^w (where $w = a + bi$, a, b real numbers) is never zero. We now ask whether there exist other values that e^w cannot assume. The following theorem shows that e^w attains all values except the value zero. We shall use the symbol $\operatorname{Log}|z|$ to mean the real natural logarithm of the positive number $|z|$, $z = x + iy \neq 0$.

COMENTARIO: (98)

LAS HOJAS QUE ESTÁN EN INGLÉS ES DEL TEMA DE FUNCIONES ELEMENTALES DE VARIABLE COMPLEJA. EFECTUAREMOS ALGUNAS DEMOSTRACIONES A CONTINUACIÓN.

VER HOJA (99).

EJEMPLOS: DEMOSTRAR QUE:

1) $\operatorname{sen} z = \operatorname{sen} x \cosh y + i \cos x \operatorname{senh} y$.

DEMOSTRACIÓN: HECHO:

$$\boxed{\operatorname{sen} z = \frac{1}{2i}\left[e^{iz} - e^{-iz}\right] \quad \text{si} \quad z = x+iy}$$

$$\operatorname{sen} z = \frac{1}{2i}\left[e^{iz} - e^{-iz}\right] = \frac{1}{2i}\left[e^{i(x+iy)} - e^{-i(x+iy)}\right]$$

$$= \frac{e^{ix} e^{i^2 y} - e^{-ix} e^{-i^2 y}}{2i} = \frac{e^{ix} e^{-y} - e^{-ix} e^{y}}{2i}$$

$$= \frac{e^{-y}\left[\cos x + i \operatorname{sen} x\right] - e^{y}\left[\cos x - i \operatorname{sen} x\right]}{2i}$$

$$= \frac{e^{-y}\cos x + i e^{-y} \operatorname{sen} x - e^{y} \cos x + i e^{y} \operatorname{sen} x}{2i}$$

$$= \frac{e^{-y}\cos x - e^{y}\cos x + i\left[e^{-y}\operatorname{sen} x + e^{y}\operatorname{sen} x\right]}{2i}$$

$$= \frac{\cos x\left[e^{-y} - e^{y}\right] + i \operatorname{sen} x\left[e^{-y} + e^{y}\right]}{2i}$$

$$= \frac{\cos x\left[e^{-y} - e^{y}\right]}{2i} + i \operatorname{sen} x \frac{\left[e^{-y} + e^{y}\right]}{2i}$$

$$= \cos x \frac{\left[e^{-y} - e^{y}\right]}{2i} + \operatorname{sen} x \frac{\left[e^{-y} + e^{y}\right]}{2}$$

HECHO:

$$\boxed{\begin{aligned} \cos iy &= \frac{[e^{-y}+e^{y}]}{2} = \cosh y. \\ \operatorname{sen} iy &= \frac{[e^{-y}-e^{y}]}{2i} = i\operatorname{senh} y \end{aligned}}$$

UTILIZANDO EL HECHO.

$$\operatorname{sen} z = \cos x \, i\operatorname{senh} y + \operatorname{sen} x \cosh y$$

$$\therefore \operatorname{sen} z = \operatorname{sen} x \cosh y + i \cos x \operatorname{senh} y$$

2 DEMOSTRAR:

a $\cos iy = \cosh y$ b $\operatorname{sen} iy = i\operatorname{senh} y.$

PRUEBA.

a HECHO.

$$\boxed{\begin{aligned} \cos z &= \cos x \cosh y - i \operatorname{sen} x \operatorname{senh} y \\ \operatorname{sen} z &= \operatorname{sen} x \cosh y + i \cos x \operatorname{senh} y. \end{aligned}}$$

Si $X=0$

$$\cos z = \cos(x+iy) = \cos(0+iy) = \cos iy = \cancel{\cos 0}^{1} \cosh y - i \cancel{\operatorname{sen} 0}^{0} \operatorname{senh} y$$

$$= (1)\cosh y - i(0)\operatorname{senh} y = \cosh y + 0 = \cosh y$$

$$\therefore \cos iy = \cosh y.$$

Si $X=0$:

(b) $\operatorname{sen} z = \operatorname{sen}(x+iy) = \operatorname{sen}(0+iy) = \operatorname{sen} iy$

$\operatorname{sen} iy = \cancel{\operatorname{sen} 0}^{0} \cosh y + i \cancel{\cos 0}^{1} \operatorname{senh} y = i \operatorname{senh} y$

$\therefore \operatorname{sen} iy = i \operatorname{senh} y$ ∎

(3) DEMOSTRAR:

$\operatorname{sen}(-z) = -\operatorname{sen} z$ FUNCIÓN <u>IMPAR</u>.

HECHO

$$\operatorname{sen}(z_1 \pm z_2) = \operatorname{sen} z_1 \cos z_2 \pm \cos z_1 \operatorname{sen} z_2$$
$$\cos(z_1 \pm z_2) = \cos z_1 \cos z_2 \mp \operatorname{sen} z_1 \operatorname{sen} z_2$$

PRUEBA:

HACIENDO: $z_1 = 0$ y $z_2 = z$, POR EL <u>HECHO</u>.

$\operatorname{sen}(z_1 \pm z_2) = \operatorname{sen}(0 \pm z) = \operatorname{sen}(-z) = \cancel{\operatorname{sen} 0}^{0} \cos z - \cancel{\cos 0}^{1} \operatorname{sen} z$

$\operatorname{sen}(-z) = (0)\cos z - (1)\operatorname{sen} z = -\operatorname{sen} z$.

$\therefore \operatorname{sen}(-z) = -\operatorname{sen} z$ ∎

(4) DEMOSTRAR.

$\cos 2z = \cos^2 z - \operatorname{sen}^2 z$.

DEMOSTRACIÓN.

POR EL <u>HEHO</u> ANTERIOR.

HACIENDO: $z_1 = z_2 = z$.

$$\cos(z_1 \pm z_2) = \cos z_1 \cos z_2 \mp \operatorname{sen} z_1 \operatorname{sen} z_2$$

$$\cos(z+z) = \cos 2z = \cos z \cos z - \operatorname{sen} z \operatorname{sen} z = \cos^2 z - \operatorname{sen}^2 z$$

$$\therefore \cos 2z = \cos^2 z - \operatorname{sen}^2 z.$$

5 PROBAR: $|\operatorname{sen} z|^2 = \operatorname{sen}^2 x + \operatorname{senh}^2 y$.

PRUEBA: HECHO:

$$\operatorname{sen} z = \operatorname{sen} x \cosh y + i \cos x \operatorname{senh} y$$
$$\cosh^2 y - \operatorname{senh}^2 y = 1$$

$$|z|^2 = z\bar{z}$$

(*) $\overline{z_1 + z_2} = \bar{z}_1 + \bar{z}_2$.

APLICANDO EL HECHO (*) SE TIENE QUE:

$$|\operatorname{sen} z|^2 = \operatorname{sen} z \, \overline{\operatorname{sen} z}$$

$$|\operatorname{sen} z|^2 = \operatorname{sen} z \, \overline{\operatorname{sen} z} = (\operatorname{sen} x \cosh y + i \cos x \operatorname{senh} y)\left(\overline{\operatorname{sen} x \cosh y + i \cos x \operatorname{senh} y}\right)$$

$$= (\operatorname{sen} x \cosh y + i \cos x \operatorname{senh} y)\left(\overline{\operatorname{sen} x \cosh y} + \overline{i \cos x \operatorname{senh} y}\right)$$

$$= (\operatorname{sen} x \cosh y + i \cos x \operatorname{senh} y)(\operatorname{sen} x \cosh y - i \cos x \operatorname{senh} y)$$

EFECTUANDO EL PRODUCTO EJ: $(A+B)(A-B) = A^2 - B^2;\ i^2 = -$

$$\operatorname{sen}^2 x \cosh^2 y + \cos^2 x \operatorname{senh}^2 y$$

PERO: $\cosh^2 y - \operatorname{senh}^2 y = 1 \Rightarrow \cosh^2 y = 1 + \operatorname{senh}^2 y$.

$$= \operatorname{sen}^2 x\left[1 + \operatorname{senh}^2 y\right] + \cos^2 x \operatorname{senh}^2 y$$

$$= \operatorname{sen}^2 x + \operatorname{sen}^2 x \operatorname{senh}^2 y + \cos^2 x \operatorname{senh}^2 y$$

$$= \operatorname{sen}^2 x + \left(\operatorname{sen}^2 x + \cos^2 x\right) \operatorname{senh}^2 y$$

$$= \operatorname{sen}^2 x + (1) \operatorname{senh}^2 y = \operatorname{sen}^2 x + \operatorname{senh}^2 y$$

$$\therefore \; |\operatorname{sen} z|^2 = \operatorname{sen}^2 x + \operatorname{senh}^2 y$$

⑥ DEMOSTRAR QUE: $\frac{d}{dz}(\cos z) = -\operatorname{sen} z$.

DEMOSTRACIÓN:

HECHO:

$$\boxed{\operatorname{sen} z = \frac{e^{iz} - e^{-iz}}{2i}} \qquad \cos z = \frac{e^{iz} + e^{-iz}}{2}$$

$$\frac{d}{dz}(\cos z) = \frac{d}{dz}\left[\frac{e^{iz} + e^{-iz}}{2}\right] = \frac{1}{2}\left\{\frac{d e^{iz}}{dz} + \frac{d e^{-iz}}{dz}\right\}$$

$$= \frac{1}{2}\left\{e^{iz}(i) - e^{-iz}(i)\right\} = \frac{i}{2}\left[e^{iz} - e^{-iz}\right]\frac{i}{i}$$

$$= \frac{i^2}{2}\left[\frac{e^{iz} - e^{-iz}}{i}\right] = -\left[\frac{e^{iz} - e^{-iz}}{2i}\right] = -\operatorname{sen} z$$

$$\therefore \frac{d}{dz}(\cos z) = -\operatorname{sen} z$$

(7) DEMOSTRAR:

$$w = \cos^{-1} z = -i \ln\left[z + \sqrt{z^2 - 1}\right]$$

DEMOSTRACIÓN:

$$z = \cos w = \frac{e^{iw} + e^{-iw}}{2}$$

HALLAR e^{iw} EN TÉRMINOS DE UNA ECUACIÓN CUADRÁTICA.

$$z = \frac{e^{iw} + e^{-iw}}{2} \Rightarrow 2z = e^{iw} + e^{-iw}$$

$$2z e^{iw} = (e^{iw} + e^{-iw}) e^{iw}$$

$$2z e^{iw} = e^{2iw} + e^{0} ; \quad 2z e^{iw} = e^{2iw} + 1$$

$$e^{2iw} - 2z e^{iw} + 1 = 0$$

$$(e^{iw})^2 - 2z e^{iw} + 1 = 0 \quad \text{(ECUACIÓN CUADRÁTICA).}$$

$$\text{Si } A = e^{iw} \Rightarrow A^2 - 2zA + 1 = 0$$

$$a = 1 ; \quad b = -2z ; \quad c = 1$$

$$A = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{2z \pm \sqrt{(-2z)^2 - 4(1)(1)}}{2(1)}$$

$$A = \frac{2z \pm \sqrt{4z^2 - 4}}{2} = \frac{2z \pm \sqrt{4(z^2 - 1)}}{2} = \frac{2z \pm 2\sqrt{z^2 - 1}}{2}$$

$$= \frac{2z + 2\sqrt{z^2-1}}{2} = z + \sqrt{z^2-1}$$

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$A = z + \sqrt{z^2-1}$ PERO $A = e^{iw}$

$$e^{iw} = z + \sqrt{z^2-1}$$

TOMANDO LOGARITMO EN AMBOS MIEMBROS Y DEPEJANDO A iw.

$$\ln e^{iw} = \ln\left[z + \sqrt{z^2-1}\right]$$

$$iw = \ln\left[z + \sqrt{z^2-1}\right] \;;\; w = \frac{1}{i}\ln\left[z + \sqrt{z^2-1}\right]$$

$$w = \frac{1}{i}\cdot\frac{i}{i}\ln\left[z + \sqrt{z^2-1}\right] = \frac{i}{i^2}\ln\left[z + \sqrt{z^2-1}\right]$$

$$w = -i\ln\left[z + \sqrt{z^2-1}\right]$$

$$\therefore\; w = -i\ln\left[z + \sqrt{z^2-1}\right]$$

COMENTARIO

- $\operatorname{senh}^{-1} z = \ln\left[z + \sqrt{z^2+1}\right]$.
- $\cosh^{-1} z = \ln\left[z + \sqrt{z^2-1}\right]$.
- $\operatorname{tgh}^{-1} z = \frac{1}{2}\ln\left[\frac{1+z}{1-z}\right]$.

8) DEMOSTRAR LAS RAÍCES ENÉSIMAS DE UN NÚMERO COMPLEJO.

$$w = z^{1/n} = \sqrt[n]{r}\left[\cos\left(\frac{\theta}{n} + k\left(\frac{2\pi}{n}\right)\right) + i\,\text{sen}\left(\frac{\theta}{n} + k\left(\frac{2\pi}{n}\right)\right)\right] \text{ si } k = 0, 1, \ldots, n-1.$$

$$w = \sqrt[n]{r}\, e^{i\left[\frac{\theta}{n} + k\left(\frac{2\pi}{n}\right)\right]} \quad \text{con } k = 0, 1, \ldots, n-1.$$

DEMOSTRACIÓN:

SEA EL NÚMERO w LLAMADO RAÍZ ENÉSIMA DE UN NÚMERO COMPLEJO z SI $w^n = z \Rightarrow w = \sqrt[n]{z} = z^{1/n}$.

CONSIDERE $w = R(\cos\phi + i\,\text{sen}\,\phi)$ Y A FIN DE HALLAR LOS n VALORES DE $\sqrt[n]{z}$, CONSIDERE LA FORMA MÁS GENERAL DADO POR:

$$z = r\left[\cos(\theta + 2k\pi) + i\,\text{sen}(\theta + 2k\pi)\right] \text{ si } k = 0, 1, \ldots, n-1.$$

UTILIZANDO LA FÓRMULA DE MOIVRE.

$$w^n = R^n(\cos n\phi + i\,\text{sen}\, n\phi)$$

$$w^n = z \Rightarrow R^n(\cos n\phi + i\,\text{sen}\, n\phi) = r\left[\cos(\theta + 2k\pi) + i\,\text{sen}(\theta + 2k\pi)\right] \quad (*)$$

IGUALANDO LOS VALORES ABSOLUTOS Y LOS ARGUMENTOS EN AMBOS LADOS DE $(*)$ SE TIENE QUE:

$$R^n = r \Rightarrow R = \sqrt[n]{r}$$

$$n\phi = \theta + 2k\pi \Rightarrow \phi = \frac{\theta + 2k\pi}{n} \cdot (**)$$

CONSECUENTEMENTE, PARA $z \neq 0$, $z^{1/n} = \sqrt[n]{z}$ TIENE LOS n DIFERENTES VALORES.

SUSTITUYENDO $w = z^{1/n}$, $R = \sqrt[n]{r} = r^{1/n}$ y $\phi = \dfrac{\theta + 2k\pi}{n}$ EN:

$$w = R(\cos\phi + i\,\text{sen}\,\phi)$$

$$w = z^{1/n} = \sqrt[n]{r}\left[\cos\left(\frac{\theta+2k\pi}{n}\right) + i\,\text{sen}\left(\frac{\theta+2k\pi}{n}\right)\right]$$

$$w = z^{1/n} = r^{1/n}\left[\cos\left(\frac{\theta}{n} + \frac{2k\pi}{n}\right) + i\,\text{sen}\left(\frac{\theta}{n} + \frac{2k\pi}{n}\right)\right]$$

$$\therefore w = z^{1/n} = r^{1/n} e^{i\left[\frac{\theta}{n} + k\left(\frac{2\pi}{n}\right)\right]} \quad \text{si } k = 0, 1, 2, \ldots n-1.$$

■

(9) EVALÚE: $tg(\pi - 2i)$

SOL:

HECHOS:

$$\text{sen}\,z = \frac{e^{iz} - e^{-iz}}{2i} \; ; \; \cos z = \frac{e^{iz} + e^{-iz}}{2}.$$

$$tg(\pi-2i)=\frac{sen(\pi-2i)}{cos(\pi-2i)}=\frac{e^{i(\pi-2i)}-e^{-i(\pi-2i)}}{2i}$$ 108

$$=\frac{\dfrac{e^{i(\pi-2i)}-e^{-i(\pi-2i)}}{2i}}{\dfrac{e^{i(\pi-2i)}+e^{-i(\pi-2i)}}{2}}=\frac{e^{i(\pi-2i)}-e^{-i(\pi-2i)}}{\left(e^{i(\pi-2i)}+e^{-i(\pi-2i)}\right)i}$$

$$=\frac{e^{i\pi}e^{2}-e^{-i\pi}e^{-2}}{\left(e^{i\pi}e^{2}+e^{-i\pi}e^{-2}\right)i}$$

$$e^{i\pi}=\cos\pi+sen\pi=-1+i(0)=-1\ ;\ e^{-i\pi}=\cos\pi-i\,sen\pi=-1$$

$$\therefore e^{i\pi}=e^{-i\pi}=-1$$

$$tg(\pi-2i)=\frac{-e^{2}+e^{-2}}{\left(-e^{2}-e^{-2}\right)i}=\frac{-\left(e^{2}-e^{-2}\right)}{-\left(e^{2}+e^{-2}\right)i}=\frac{e^{2}-e^{-2}}{\left(e^{2}+e^{-2}\right)i}$$

$$=\frac{-7.3890+0.1353}{(7.3890+0.1353)i}=\frac{-7.2537}{7.524i}\cdot\frac{i}{i}=0.9640i$$

$$\therefore tg(\pi-2i)=0.9640i$$

TAREA 2 AVANZADAS

T-2-1

1. ESCRIBIR LAS SIGUIENTES FUNCIONES COMPLEJAS EN LA FORMA: $w = f(z) = u + iv = u(x,y) + i\,v(x,y)$

(a) $f(z) = z^2 + z + 3$ (R) $u = x^2 - y^2 + x$; $v = 2xy + y$.

(b) $f(z) = z + \bar{z}$ (R) $u = 2x$; $v = 0$.

(c) $f(z) = z + \frac{1}{z}$ (R) $u = x + \frac{x}{x^2+y^2}$; $v = y - \frac{y}{x^2+y^2}$.

(d) $f(z) = \frac{1-z}{1+z}$ (R) $u = \frac{1-x^2-y^2}{(1+x)^2+y^2}$; $v = \frac{-2y}{(1+x)^2+y^2}$.

(e) $f(z) = z^{1/2}$. (R)

$$u = (x^2+y^2)^{1/4} \cos\left[\frac{\cos^{-1}\left(\frac{x}{\sqrt{x^2+y^2}}\right)}{2}\right]$$

$$v = (x^2+y^2)^{1/4} \operatorname{sen}\left[\frac{\cos^{-1}\frac{x}{\sqrt{x^2+y^2}}}{2}\right].$$

RECUERDE: $f(z) = z^{1/2} = (r e^{i\theta})^{1/2} = r^{1/2} e^{i\theta/2}$.

(f) $f(z) = z^3$ (R) $u = x^3 - 3xy^2$; $v = 3x^2y - y^3$.

TAREA 2 — AVANZADAS

T-2-2

(2) HAGA UN BOSQUEJO PARA $f(z)=z^2$ CON $z_1=2+3i$ y $z_2=-3-2i$ RESPECTIVAMENTE.

(R) $f(z_1)=-5+i12=(-5,12)$; $f(z_2)=5+i12=(5,12)$

(3) DETERMINE LA IMAGEN DEL CONJUNTO DADO BAJO LA TRANSFORMACIÓN O MAPEO QUE SE INDICA.

(a) CONSIDÉRESE LA REGIÓN $0\le x\le 2$ y $0\le y\le 4$, EN EL PLANO Z. BAJO LA TRANSFORMACIÓN O MAPEO DE:

$$w=(1+i)z+(2+i)$$

SOL:

LA REGIÓN $0\le x\le 2$ y $0\le y\le 4$ REPRESENTA UN RECTÁNGULO EN EL PLANO Z.

- 2
  - 4 (2,4)
  - 0 (2,0)
- 0
  - 4 (0,4)
  - 0 (0,0)

POR OTRO LADO:

$$w=(1+i)z+(2+i)=(1+i)(x+iy)=x+iy+ix+i^2y+2+i$$

$$w=x+iy+ix-y+2i=(x-y+2)+i(x+y+1).$$

DONDE:

$$\left.\begin{array}{l} u(x,y)=u=x-y+2 \\ v(x,y)=v=x+y+1 \end{array}\right\rangle (1)$$

SUSTITUYENDO LOS PUNTOS DEL RECTÁNGULO $(0,0), (2,0), (2,4)$ y $(0,4)$ EN (1)

T-2-3

- si $(x,y)=(0,0) \Rightarrow u = x-y+2 = 0-0+2 = 2$
  $V = x+y+1 = 0+0+1 = 1 \Rightarrow (2,1)$

- si $(x,y)=(2,0)$
  $u = x-y+2 = 2-0+2 = 4$
  $V = x+y+1 = 2+0+1 = 3 \Rightarrow (4,3) = 4+3i$

- si $(x,y)=(2,4)$
  $u = 2-4+2 = 0$
  $V = 2+4+1 = 7 \Rightarrow (0,7) = i7$

- si $(x,y)=(0,4)$
  $u = x-y+2 = 0-4+2 = -2$
  $V = x+y+1 = 0+4+1 = 5 \Rightarrow (-2,5) = -2+i5$