SERIES DE FOURIER (1768-1830). 1

INTRODUCCIÓN.

EL TEMA DE SERIES DE FOURIER ES DE PRIMORDIAL IMPORTANCIA PARA EL ALUMNO, YA QUE LO APLICARÁ EN SUS CURSOS DE CIRCUITOS Y DE COMUNICACIONES, EN EL ESTUDIO DE SISTEMAS SOBRE LOS QUE ACTUAN PERTURBACIONES PERIÓDICAS DE TIPO GENERAL.

LA POSIBILIDAD DE EXPRESAR UNA SEÑAL PERIÓDICA EN TERMINOS DE UNA SERIE DE FOURIER, YA SEA EN SU FORMA TRIGONOMÉTRICA O COMPLEJA, LA CONVERGENCIA DE LAS MISMAS, LA OBTENCIÓN DE SUS ESPECTROS Y SU APLICACIÓN AL CONSIDERAR UN NÚMERO FINITO DE TÉRMINOS PARA APROXIMARNOS A ELLA.

• SEÑALES PERIÓDICAS. SEÑALES PARES E IMPARES. (2)

• EL SISTEMA TRIGONOMÉTRICO BÁSICO.

SEÑAL PERIÓDICAS

UNA SEÑAL PERIÓDICA f EXISTE SI UN NÚMERO $T \neq 0$, TAL QUE:

$$f(t+T) = f(t) \quad \forall t \in \mathbb{R}. \quad (1)$$

ESTE NÚMERO T SE LLAMA EL PERIODO DE LA SEÑAL. EN PARTICULAR, SI T ES EL MENOR NÚMERO POSITIVO QUE SATISFACE (1), DECIMOS QUE T ES EL PERIODO FUNDAMENTAL DE f.

LAS SEÑALES TRIGONOMÉTRICAS SENO Y COSENO SON PERIÓDICAS CON PERIODO FUNDAMENTAL $T = 2\pi$. ES DECIR,

$$\operatorname{sen}(t+2\pi) = \operatorname{sen} t.$$

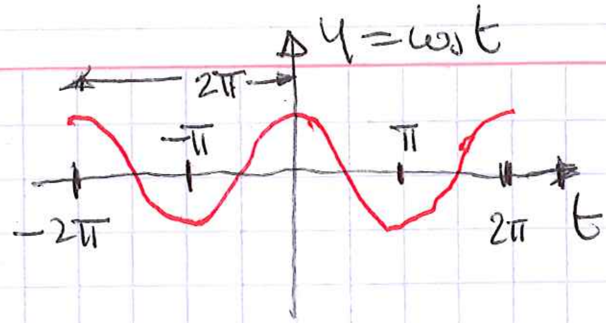
$$\cos(t+2\pi) = \cos t.$$

DEMOSTRAR $\operatorname{sen}(t+2\pi) = \operatorname{sen} t$

PRUEBA $f(t) = \operatorname{sen} t \Rightarrow f(t+2\pi) = \operatorname{sen}(t+2\pi)$.

$$\operatorname{sen}(t+2\pi) = \operatorname{sen} t \underbrace{\cos 2\pi}_{1} + \cos t \underbrace{\operatorname{sen} 2\pi}_{0} = \operatorname{sen} t$$

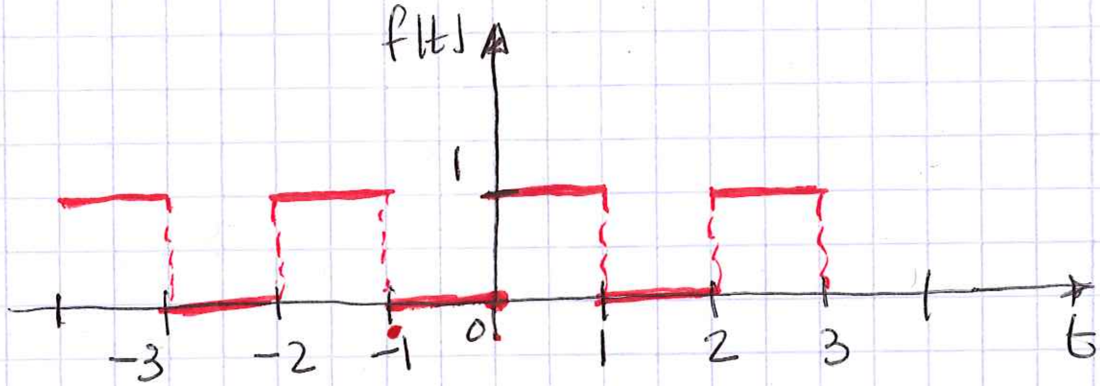
$$\therefore \operatorname{sen}(t+2\pi) = \operatorname{sen} t$$



• LA SEÑAL f DEFINIDA POR:

$$f(t)=\begin{cases}0, & si\ 2n-1<t<2n\\ 1, & si\ 2n<t<2n+1\end{cases} \qquad \text{con } n=0,\pm 1,\pm 2,\dots$$

ES PERIÓDICA CON PERIODO $T=2$.



- Si $n=0$

$$f(t)=\begin{cases}0, & si\ 2(0)-1<t<2(0)\\ 1, & si\ 2(0)<t<2(0)+1\end{cases} \Rightarrow =\begin{cases}0, & -1<t<0\\ 1, & 0<t<1\end{cases}$$

- Si $n=1$

$$f(t)=\begin{cases}0, & 2(1)-1<t<2(1)\\ 1, & 2(1)<t<2(1)+1\end{cases} =\begin{cases}0, & 1<t<2\\ 1, & 2<t<3.\end{cases}$$

- Si $n=-1$

$$f(t)=\begin{cases}0, & 2(-1)-1<t<2(-1)\\ 1, & 2(-1)<t<2(-1)+1\end{cases} =\begin{cases}0, & -3<t<-2\\ 1, & -2<t<-1.\end{cases}$$

SEÑAL CONSTANTE:

UNA SEÑAL CONSTANTE $f(t)=C$, ES PERIÓDICO CON PERIODO T, PARA CUALQUIER NÚMERO REAL T; ESTO ES.

$$f(t+T)=C=f(t),\ T\in\mathbb{R}\ y\ t\in\mathbb{R}.$$

DEF. SI UNA SEÑAL f ES PERIÓDICA CON PERIODO T, TENEMOS QUE:

$$f(t-T)=f\big((t-T)+T\big)=f(t);$$

POR TANTO, f ES DE PERIODO $-T$. ASIMISMO, APLICANDO REPETIDAMENTE (1), OBTENEMOS QUE:

$$f(t)=f(t+T)=f(t+2T)=f(t+3T)=\dots.$$

EN GENERAL:

$$f(t+nT)=f(t),\ t\in\mathbb{R},\ y\ n=0,\pm1,\pm2,\dots$$

POR EJEMPLO:

$$\operatorname{sen}(t+2n\pi)=\operatorname{sen}t.$$
$$\cos(t+2n\pi)=\cos t.$$
$$n=0,\pm1,\pm2,\dots$$

T. SEAN f y g SEÑALES PERIÓDICAS CON EL MISMO PERIODO T y α UN NÚMERO REAL. ENTONCES, fg, $f+g$ y αf SON PERIÓDICAS CON PERIODO T.

DEMOSTRACIÓN:

POR HIPÓTESIS:

$$f(t+T)=f(t) \text{ y } g(t+T)=g(t).$$

- $(fg)(t+T)=f(t+T)\,g(t+T)=f(t)g(t)=(fg)(t)$

$$\therefore (fg)(t+T)=(fg)(t).$$

$\blacksquare$

- $(f+g)(t+T)=f(t+T)+g(t+T)=f(t)+g(t)=(f+g)(t)$

$$\therefore (f+g)(t+T)=(f+g)(t).$$

$\blacksquare$

- TOMANDO $g=a$, SE CONCLUYE QUE TAMBIÉN af ES PERIÓDICA.

$\blacksquare$

EJEMPLOS:

- LA SEÑAL $f(t)=a\cos t+b\,\mathrm{sen}\,t$, $a,b\in\mathbb{R}$ ES PERIÓDICA CON PERIODO $T=2\pi$.
- LA SEÑAL $f(t)=5\,\mathrm{sen}\,3t$ TIENE PERIODO $T=\frac{2\pi}{3}$.

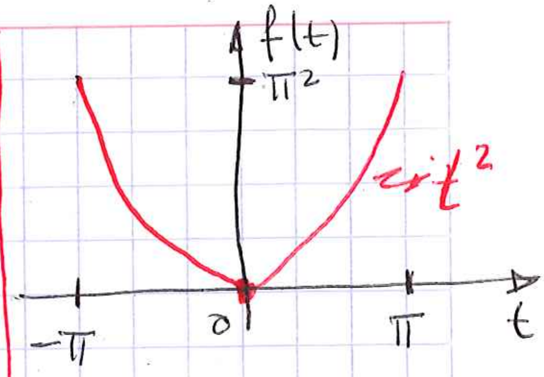
- GRAFIQUE LAS SEÑALES $f(t)$.

a) $f(t)=t^2$, $-\pi<t<\pi$.

b) $f(t)=\begin{cases}\pi, & \text{si } -\pi<t<0,\\ t, & \text{si } 0\le t<\pi.\end{cases}$

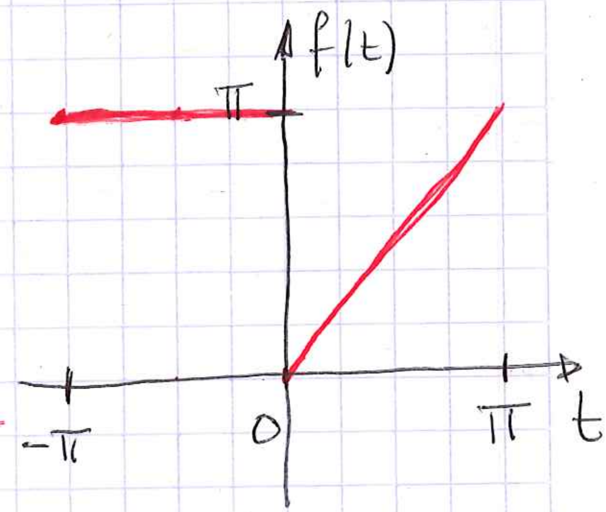
Sol: ⓐ $f(t)=t^2,\ -\pi<t<\pi.$

$-\pi<t<\pi$	$f(t)=t^2$
$t=-\pi$	$f(-\pi)=(-\pi)^2=\pi^2;\ (-\pi,\pi^2)$
$t=0$	$f(0)=0^2=0;\ (0,0)$
$t=\pi$	$f(\pi)=\pi^2;\ (\pi,\pi^2)$



ⓑ $f(t)=\begin{cases}\pi, & -\pi<t<0\\ t, & 0\le t<\pi\end{cases}$

$-\pi<t<0$	$f(t)=\pi$
$t=-\pi$	$f(-\pi)=\pi;\ (-\pi,\pi)$
$t=-\frac{1}{2}$	$f(-\frac{1}{2})=\pi;\ (-1/2,\pi)$
$t=0$	$f(0)=\pi;\ (0,\pi)$
$0\le t<\pi$	$f(t)=t$
$t=0$	$f(0)=0;\ (0,0)$
$t=\frac{\pi}{2}$	$f(\frac{\pi}{2})=\frac{\pi}{2};\ (\frac{\pi}{2},\frac{\pi}{2})$
$t=\pi$	$f(\pi)=\pi;\ (\pi,\pi)$



SERIE TRIGONOMÉTRICA

UNA SERIE TRIGONOMÉTRICA ES DE LA FORMA:

$$a_0 + \sum_{n=1}^{\infty} \left[a_n \cos nt + b_n \operatorname{sen} nt \right], \quad (*)$$

DONDE LOS COEFICIENTES a_n Y b_n SON NÚMEROS REALES.

T. SI UNA SERIE TRIGONOMÉTRICA CONVERGE Y SU SUMA ES LA SEÑAL f, ENTONCES f ES PERIÓDICA CON PERIODO $T = 2\pi$. EN PARTICULAR, SI LA SERIE

$$a_0 + \sum_{n=1}^{\infty} \left[a_n \cos \frac{n\pi t}{c} + b_n \operatorname{sen}\left(\frac{n\pi t}{c}\right) \right]$$

CONVERGE, SU SUMA ES UNA SEÑAL DE PERIODO $T = 2c$.

· SISTEMA TRIGONOMÉTRICO BÁSICO.

EL CONJUNTO DE SEÑALES EN LA SERIE $(*)$, ES DECIR, EL CONJUNTO

$$\{1, \cos nt, \operatorname{sen} nt ; n = 1, 2, 3, \dots\} =$$

$$\{1, \cos t, \operatorname{sen} t, \cos 2t, \operatorname{sen} 2t, \dots\}$$

SE DENOMINA SISTEMA TRIGONOMÉTRICO BÁSICO.

SEÑAL PAR Y SEÑAL IMPAR.

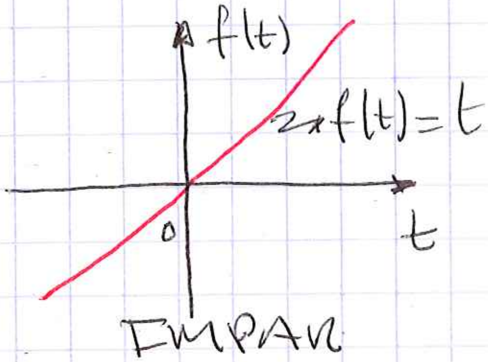
DADA UNA SEÑAL f, DECIMOS QUE f ES UNA SEÑAL PAR SI

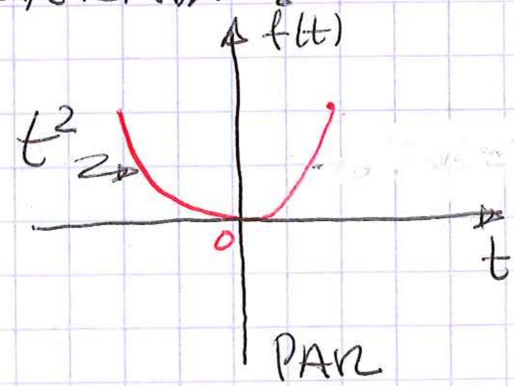
$$f(-t) = f(t), \ \forall \, t \in \mathbb{R}.$$

f ES UNA SEÑAL IMPAR SI:

$$f(-t) = -f(t).$$

POR DEFINICIÓN, LA GRÁFICA DE UNA SEÑAL PAR ES SIMÉTRICA CON RESPECTO AL EJE VERTICAL, MIENTRAS QUE LA DE UNA SEÑAL IMPAR LO ES CON RESPECTO AL ORIGEN DE COORDENADAS.





EJEMPLOS SEÑALES IMPARES.

SEÑAL SENO $\Rightarrow$. $\operatorname{sen}(-t) = -\operatorname{sen} t$

. $\operatorname{sen} \frac{n\pi t}{c}$. $\operatorname{sen} at$, $a \in \mathbb{R}$

SEÑALES PARES.

SEÑAL COSENO. $\Rightarrow$. $\cos(-t) = \cos t$

. $\cos \frac{n\pi t}{c}$. $\cos at$, $a \in \mathbb{R}$.

UNA SEÑAL CONSTANTE $c \neq 0$ ES PAR; LA ÚNICA SEÑAL QUE ES PAR E IMPAR SIMULTÁNEAMENTE ES LA SEÑAL CONSTANTE 0.

T) ÁLGEBRA DE SEÑALES PARES E IMPARES.

SEAN f Y g SEÑALES DADAS:

i) SI f Y g SON SEÑALES PARES $\Rightarrow$ $f+g$ Y fg TAMBIÉN LO SON.

ii) SI f Y g SON IMPARES $\Rightarrow$ $f+g$ ES IMPAR Y fg ES PAR.

iii) SI f ES PAR Y g ES IMPAR fg ES IMPAR.

DEMOSTRACIÓN.

i) POR HIPÓTESIS, $f(-t)=f(t)$ Y $g(-t)=g(t)$,

ENTONCES:

- $(f+g)(-t)=f(-t)+g(-t)=f(t)+g(t)=(f+g)(t)$.

$$\therefore (f+g)(-t)=(f+g)(t).$$

- $(fg)(-t)=f(-t)g(-t)=f(t)g(t)=(fg)(t)$.

$$\therefore (fg)(-t)=(fg)(t).$$

EJEMPLOS

- $\operatorname{sen} nt \cdot \cos mt$ son IMPARES con $[n = 1, 2, \ldots ; m = 0, 1, 2, \ldots$
- $\cos nt \cdot \cos mt$ son PARES $(n, m = 0, 1, 2, \ldots)$.
- $\operatorname{sen} nt \cdot \operatorname{sen} mt$ PARES $(n$ y $m = 1, 2, \ldots)$.

T Sea f una señal integrable y $c > 0$:

i) Si f es PAR, $\int_{-c}^{c} f(t)\,dt = 2\int_{0}^{c} f(t)\,dt$.

ii) Si f es IMPAR, $\int_{-c}^{c} f(t)\,dt = 0$.

COMENTARIO.

- $\operatorname{sen} x \cos y = \frac{1}{2}\left[\operatorname{sen}(x+y) + \operatorname{sen}(x-y)\right]$.
- $\cos x \operatorname{sen} y = \frac{1}{2}\left[\operatorname{sen}(x+y) - \operatorname{sen}(x-y)\right]$.
- $\cos x \cos y = \frac{1}{2}\left[\cos(x+y) + \cos(x-y)\right]$.
- $\operatorname{sen} x \operatorname{sen} y = \frac{1}{2}\left[\cos(x+y) - \cos(x-y)\right]$.

$$= -\frac{1}{2}\left[\cos(x-y) - \cos(x+y)\right]$$

ORTOGONALIDAD DEL SISTEMA TRIGONOMÉTRICO BÁSICO

T) LA INTEGRAL SOBRE EL INTERVALO $-\pi \leq t \leq \pi$ DEL PRODUCTO DE CUALQUIER PAR DE SEÑALES DISTINTAS DEL SISTEMA TRIGONOMÉTRICO BÁSICO

$$\{1, \cos t, \operatorname{sen} t, \cos 2t, \operatorname{sen} 2t, \ldots\}$$

ES CERO; ESTO ES,

i) $\displaystyle\int_{-\pi}^{\pi} \operatorname{sen} nt \cos mt \, dt = 0, \quad n = 1, 2, \ldots; \; m = 0, 1, 2, \ldots$

ii) $\displaystyle\int_{-\pi}^{\pi} \operatorname{sen} nt \operatorname{sen} mt \, dt = \begin{cases} \pi, & \text{si } m = n \\ 0, & \text{si } m \neq n. \end{cases} \quad m, n = 1, 2, \ldots$

iii) $\displaystyle\int_{-\pi}^{\pi} \cos nt \cos mt \, dt = \begin{cases} 0, & \text{si } m \neq n, \\ \pi, & \text{si } m = n = 1, 2, 3, \ldots \\ 2\pi, & \text{si } m = n = 0. \end{cases}$

DEMOSTRACIÓN i) $\displaystyle\int_{-\pi}^{\pi} \operatorname{sen} nt \cos mt \, dt = 0$

PRUEBA.

EL INTEGRANDO EN i) ES UNA SEÑAL IMPAR PUES SE TRATA DEL PRODUCTO DE UNA SEÑAL IMPAR ($\operatorname{sen} nt$)

POR UNA SEÑAL PAR ($\cos mt$).

LUEGO POR LO VISTO EN HOJA 10 DEL T i)

SI f ES IMPAR, $\int_{-c}^{c} f(t)\,dt = 0$

$$\therefore \int_{-\pi}^{\pi} \operatorname{sen} nt \cos mt\,dt = 0$$

DEMOSTRAR.

$$\text{ii)}\ \int_{-\pi}^{\pi} \operatorname{sen} nt \operatorname{sen} mt\,dt = \begin{cases} \pi, & \text{si } m = n \\ 0, & \text{si } m \neq n. \end{cases} \qquad m, n = 1, 2, \ldots$$

DEMOSTRACIÓN:

- Si $m = n$

$$I = \int_{-\pi}^{\pi} \operatorname{sen} nt \operatorname{sen} mt\,dt = \int_{-\pi}^{\pi} \operatorname{sen}^2 nt\,dt = \frac{1}{2}\int_{-\pi}^{\pi} (1 - \cos 2nt)\,dt$$

$$= \frac{1}{2}\left[t - \frac{1}{2n}\operatorname{sen} 2nt\right]_{-\pi}^{\pi}$$

HECHO $\operatorname{sen} k\pi = 0$

$$= \frac{1}{2}\left[\pi + \pi\right] = \frac{2\pi}{2} = \pi \qquad \therefore I = \pi$$

- Si $m \neq n$.

$$I = \int_{-\pi}^{\pi} \operatorname{sen} nt \operatorname{sen} mt\,dt = \frac{1}{2}\int_{-\pi}^{\pi} \left[\cos(n-m)t - \cos(n+m)t\right] dt$$

$$= \frac{1}{2}\int_{-\pi}^{\pi} \left[\cos(n-m)t - \cos(n+m)t\right] dt$$

$$I = -\frac{1}{2}\left[\frac{1}{(n-m)}\,\mathrm{sen}\,\cancel{(n-m)t}^{0} - \frac{1}{(n+m)}\,\mathrm{sen}\,\cancel{(n+m)t}\right]_{-\pi}^{\pi}$$

$$I = -\frac{1}{2}\left[0\right] = 0.$$

• SERIES DE FOURIER O TRIGONOMÉTRICO DE PERIODO $T = 2\pi$.

DEF: LA SERIE DE FOURIER DE UNA SEÑAL f DEFINIDA EN EL INTERVALO $[-\pi, \pi]$ Ó $-\pi \leq t \leq \pi$ ESTÁ DADA POR:

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left[a_n \cos nt + b_n \,\mathrm{sen}\, nt\right],$$

CON COEFICIENTES a_0, a_n y b_n LLAMADOS COEFICIENTES DE FOURIER Ó DE EULER-FOURIER DE LA SEÑAL f.

DONDE:

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\,dt.$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\cos nt\,dt \quad \text{si } n = 1, 2, \ldots$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\,\mathrm{sen}\, nt\,dt \quad \text{si } n = 1, 2, \ldots$$

DEMOSTRAR QUE: $a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\,dt$. 14

DEMOSTRACIÓN:

SABEMOS QUE:

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left[a_n \cos nt + b_n \operatorname{sen} nt\right]$$

MULTIPLICANDO AMBOS MIEMBROS POR $\cos mt$ OBTENEMOS.

$$f(t)\cos mt = \frac{a_0}{2}\cos mt + \sum_{n=1}^{\infty}\left[a_n \cos mt \cos nt + b_n \cos mt \operatorname{sen} mt\right],$$

INTEGRANDO

$$\int_{-\pi}^{\pi} f(t)\cos mt\,dt = \frac{a_0}{2}\int_{-\pi}^{\pi}\cos mt\,dt + \sum_{n=1}^{\infty}\left[a_n\int_{-\pi}^{\pi}\cos nt \cos mt\,dt + \cancel{b_n \int_{-\pi}^{\pi}\operatorname{sen} nt \cos mt\,dt}\right]$$

$$\boxed{\int_{-\pi}^{\pi}\operatorname{sen} nt \cos mt\,dt = 0,\quad n=1,2,\ldots;\ m=0,1,2\ldots}$$

LOS SEGUNDOS TÉRMINOS DE LA SERIE SE ANULAN, ESTO ES:

$$\int_{-\pi}^{\pi} f(t)\cos mt\,dt = \frac{a_0}{2}\int_{-\pi}^{\pi}\cos mt\,dt + \sum_{n=1}^{\infty} a_n\int_{-\pi}^{\pi}\cos nt \cos mt\,dt$$

SI $m=0$.

$$\int_{-\pi}^{\pi} f(t)\cancelto{1}{\cos 0}\,dt = \frac{a_0}{2}\int_{-\pi}^{\pi}\cancelto{1}{\cos 0}\,dt + \sum_{n=1}^{\infty} a_n\int_{-\pi}^{\pi}\cos nt\,\cancelto{1}{\cos 0}\,dt$$

$$\int_{-\pi}^{\pi} f(t)dt = \frac{a_0}{2}\int_{-\pi}^{\pi} dt + \sum_{n=1}^{\infty} a_n \int_{-\pi}^{\pi} \cos nt \, dt$$

$$\int_{-\pi}^{\pi} f(t)dt = \frac{a_0}{2} t \Big|_{-\pi}^{\pi} + \sum_{n=1}^{\infty} a_n \left\{ \cancel{\frac{\operatorname{sen} nt}{n}}^{\,0} \Big|_{-\pi}^{\pi} \right\}$$

$$\int_{-\pi}^{\pi} f(t)dt = \frac{a_0}{2}\left[\pi + \pi\right] + 0 = \frac{2\pi a_0}{2}$$

$$\int_{-\pi}^{\pi} f(t)dt = \frac{2\pi a_0}{2} = a_0 \pi$$

$$\therefore a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)dt$$

- CONDICIONES DE DIRICHLET

T) SEA f UNA SEÑAL PERIÓDICA CON PERIODO $T = 2\pi$ Y SUAVE POR PARTES EN EL INTERVALO $[-\pi, \pi]$. ENTONCES, LA SERIE DE FOURIER DE f

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left[a_n \cos nt + b_n \operatorname{sen} nt\right],$$

CON COEFICIENTES:

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\,dt$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\cos nt\,dt, \quad n=1,2,\ldots$$

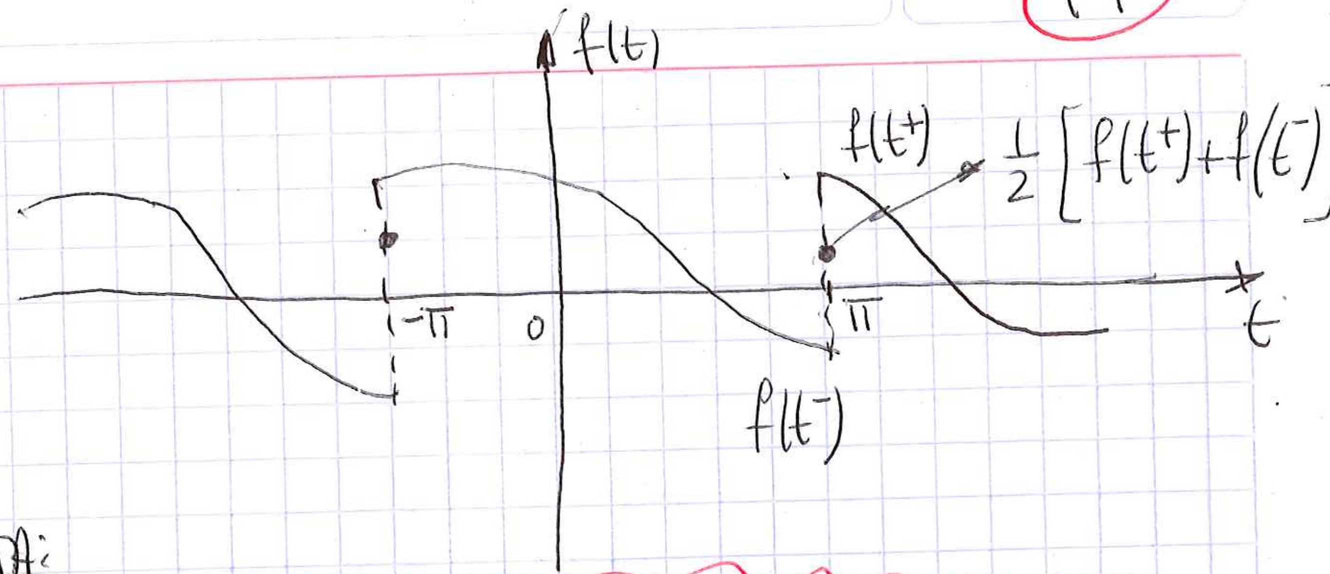
$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\,\text{sen}\,nt\,dt, \quad n=1,2,\ldots$$

CONVERGE A $\frac{1}{2}\left[f(t^+)+f(t^-)\right]$ EN CADA PUNTO DEL INTERVALO.

EL NÚMERO $\frac{1}{2}\left[f(t^+)+f(t^-)\right]$ ES LA MEDIA ARITMÉTICA DE LOS LÍMITES LATERALES DE f EN EL PUNTO t. POR TANTO, SI f ES CONTINUA EN t,

$$\frac{1}{2}\left[f(t^+)+f(t^-)\right] = \frac{1}{2}\left[f(t)+f(t)\right] = \frac{2f(t)}{2} = f(t).$$

EN OTRAS PALABRAS, BAJO LAS CONDICIONES DEL TEOREMA, LA SERIE DE FOURIER DE f CONVERGE A $f(t)$ SI f ES CONTINUA EN t, Y CONVERGE A $\frac{1}{2}\left[f(t^+)+f(t^-)\right]$ SI SI f ES DISCONTINUA EN t.



NOTA:

SUAVE O LISA POR PARTES.

DEF LA DERIVADA DE f SEA CONTINUA EN EL INTERVALO

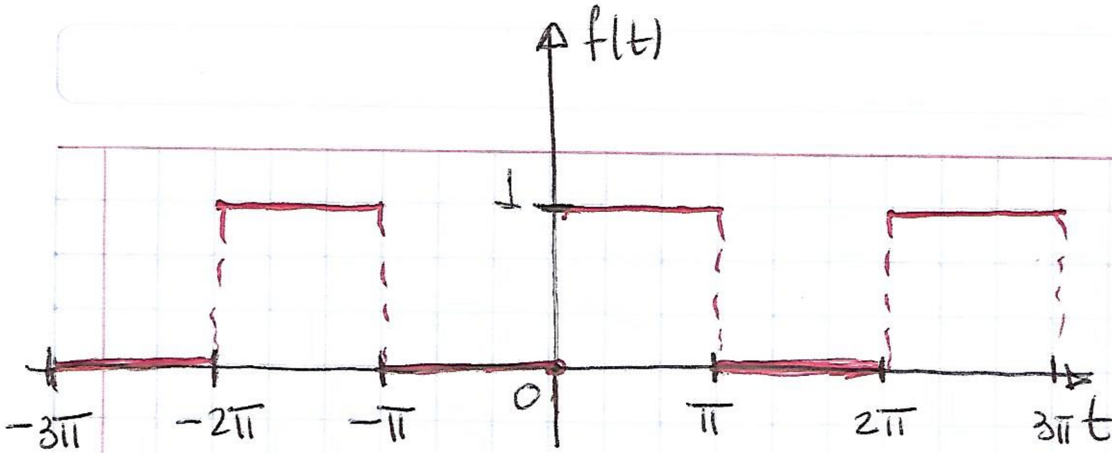
$-\pi \leq t \leq \pi$ ó $[-\pi, \pi]$

EJEMPLOS 1 GRAFICAR LA SEÑAL $f(t)$ Y DETERMINE LA SERIE DE FOURIER TRIGONOMÉTRICA PARA:

$$f(t) = \begin{cases} 0, & -\pi < t < 0 \\ 1, & 0 < t < \pi \end{cases} \quad \text{con } f(t+2\pi) = f(t).$$

SOL:

$$T = 2\pi \Rightarrow T = \pi - (-\pi) = \pi + \pi = 2\pi \;\therefore\; T = 2\pi$$



$-\pi < t < 0$	$f(t) = 0$
$t = -\pi$	$f(-\pi) = 0$
$t = -\pi/2$	$f(-\pi/2) = 0$
$t = 0$	$f(0) = 0$
$0 < t < \pi$	$f(t) = 1$
$t = 0$	$f(0) = 1$
$t = \pi/2$	$f(\pi/2) = 1$
$t = \pi$	$f(\pi) = 1$

CALCULAR LOS COEFICIENTES a_0, a_n y b_n:

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\,dt = \frac{1}{\pi}\int_{-\pi}^{0} \cancel{0\,dt}^{\,0} + \int_{0}^{\pi} \frac{1}{\pi}(1)\,dt$$

$$a_0 = \frac{1}{\pi}\int_{0}^{\pi} dt = \frac{1}{\pi}\, t\Big|_0^{\pi} = \frac{1}{\pi}\left[\pi - 0\right] = \frac{\pi}{\pi} = 1$$

$\therefore\ a_0 = 1$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\cos nt\,dt = \frac{1}{\pi}\left\{\int_{-\pi}^{0} \cancel{0\cos nt\,dt}^{\,0} + \int_{0}^{\pi} (1)\cos nt\,dt\right\}$$

$$= \frac{1}{\pi}\int_{0}^{\pi} \cos nt\,dt = \frac{1}{n\pi}\operatorname{sen} nt\Big|_0^{\pi} = \frac{1}{n\pi}\left[\operatorname{sen} n\pi - \operatorname{sen} 0\right] = \frac{0}{n\pi} = 0$$

$\therefore\ a_n = 0$ con $n = 1, 2, 3, \ldots$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\operatorname{sen} nt\,dt = \frac{1}{\pi}\left\{\int_{-\pi}^{0} \cancel{0\operatorname{sen} nt\,dt}^{\,0} + \int_{0}^{\pi} (1)\operatorname{sen} nt\,dt\right\}$$

$$b_n = \frac{1}{\pi}\int_0^{\pi} \operatorname{sen} nt\,dt = -\frac{1}{n\pi}\cos nt\Big|_0^{\pi}$$

$$= -\frac{1}{n\pi}\left[\cos n\pi - 1\right] = \frac{1}{n\pi}\left[-\cos n\pi + 1\right]$$

$$= \frac{1}{n\pi}\left[1 + \cos n\pi\right]$$

HECHO: $\operatorname{sen} n\pi = 0$ y $\cos n\pi = (-1)^n$

APLICANDO EL $\boxed{\text{HECHO.}}$

$$b_n = \frac{1}{n\pi}\left[1 - \cos n\pi\right] = \frac{1}{n\pi}\left[1-(-1)^n\right] \quad \text{con } n = 1, 2, 3, \ldots$$

SUSTITUYENDO $a_0 = 1$, $a_n = 0$ y $b_n = \frac{1}{n\pi}\left[1-(-1)^n\right]$ EN LA SERIE DE FOURIER TRIGONOMÉTRICO:

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left[a_n \cos nt + b_n \operatorname{sen} nt\right]$$

$$f(t) = \frac{1}{2} + \sum_{n=1}^{\infty}\left[0\cos nt + \frac{1}{n\pi}\left[1-(-1)^n\right]\operatorname{sen} nt\right]$$

$$f(t) = \frac{1}{2} + \frac{1}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}\left[1-(-1)^n\right]\operatorname{sen} nt, \quad f(t) = \begin{cases} 0, & -\pi < t \\ 1, & 0 < t \end{cases}$$

COMENTARIO

ESTE RESULTADO ES CORRECTO, PERO SI DESARROLLO LA SUMATORIA DEL COEFICIENTE $b_n = [1-(-1)^n]$ NOS RESULTA QUE:

$$b_n = 0 \text{ si } n \text{ es PAR } (2n) \qquad n=1,2,\ldots$$

$$b_n = 2 \text{ si } n \text{ es IMPAR } (2n-1)$$

PARA OMITIR EL CERO EN LA SUMATORIA, ESCRIBIMOS AHORA LA SERIE DE FOURIER COMO:

DONDE EXISTA UNA n SE TRANSFORMA EN (2n-1) EN LA SUMATORIA.

$$f(t) = \frac{1}{2} + \frac{1}{\pi} \sum_{n=1}^{\infty} \frac{1}{n} \left[1-(-1)^n\right] \operatorname{sen} nt$$

AHORA ES

$$f(t) = \frac{1}{2} + \frac{1}{\pi} \sum_{n=1}^{\infty} \frac{1}{(2n-1)} \, 2 \operatorname{sen}(2n-1)t$$

$$\therefore f(t) = \frac{1}{2} + \frac{2}{\pi} \sum_{n=1}^{\infty} \frac{1}{(2n-1)} \operatorname{sen}(2n-1)t.$$

• FENÓMENO DE GIBBS.

CUANDO UNA SEÑAL DADA SE APROXIMA MEDIANTE UNA SUMA PARCIAL DE LA SERIE DE FOURIER TRIGONOMÉTRICA, HABRÁ UN ERROR CONSIDERABLE EN LA VECINDAD DE UNA DISCONTINUIDAD, NO IMPORTA CUÁNTOS TÉRMINOS SE QUIEREN UTILIZAR. ESTE EFECTO SE CONOCE COMO FENÓMENO DE GIBBS.

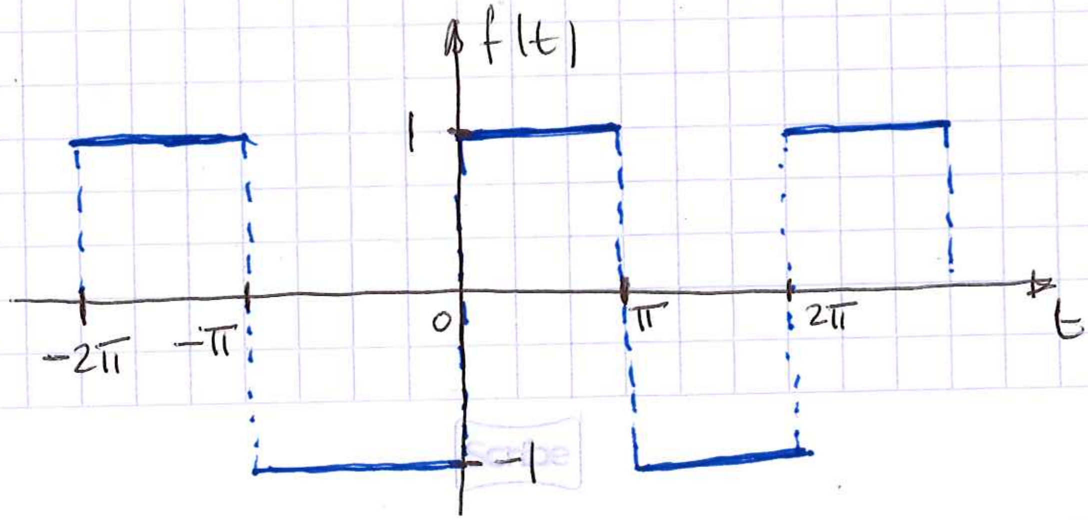
EJEMPLO.

CONSIDERE LA ONDA CUADRADA, ESTO ES:

$$f(t) = \begin{cases} -1, & -\pi < t < 0 \\ 1, & 0 < t < \pi. \end{cases} \quad \text{con } T = 2\pi.$$

- CALCULAR LA SERIE DE FOURIER TRIGONOMÉTRICA.
- ANALIZAR LA SUMA DE UN NÚMERO FINITO DE TÉRMINOS (ARMÓNICOS) DE LA SERIE DE FOURIER TRIGONOMÉTRICA.

SOL



- CALCULAREMOS PRIMERAMENTE LOS COEFICIENTES DE FOURIER TRIGONOMÉTRICOS.

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\,dt$$

$$a_0 = \frac{1}{\pi}\left\{\int_{-\pi}^{0}(-1)\,dt + \int_{0}^{\pi}(1)\,dt\right\} = \frac{1}{\pi}\left\{-t\Big|_{-\pi}^{0} + t\Big|_{0}^{\pi}\right\}$$

$$= \frac{1}{\pi}\left\{-1\left[0+\pi\right] + \left[\pi - 0\right]\right\} = \frac{1}{\pi}\left\{-\pi+\pi\right\} = \frac{0}{\pi} = 0$$

$$\therefore\ a_0 = 0.$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\cos nt\,dt \quad \text{si } n = 1, 2, \ldots$$

$$a_n = \frac{1}{\pi}\left\{\int_{-\pi}^{0}(-1)\cos nt\,dt + \int_{0}^{\pi}(1)\cos nt\,dt\right\}$$

$$\int \cos nt\,dt = \frac{\operatorname{sen} nt}{n} \;;\; \operatorname{sen} n\pi = 0 \;;\; \cos n\pi = 0$$

$$a_n = \frac{1}{\pi}\left\{-\frac{\operatorname{sen} nt}{n}\Big|_{-\pi}^{0} + \frac{\operatorname{sen} nt}{n}\Big|_{0}^{\pi}\right\}$$

$$a_n = \frac{1}{\pi}\left\{-\left[\cancel{\frac{\operatorname{sen} 0}{n}}^{0} - \cancel{\frac{\operatorname{sen}(-n\pi)}{n}}^{0}\right] + \left[\cancel{\frac{\operatorname{sen} n\pi}{n}}^{0} - \cancel{\frac{\operatorname{sen} 0}{n}}^{0}\right]\right\} = \frac{0}{\pi} = 0$$

$\therefore a_n = 0.$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\,\text{sen}\,nt\,dt \quad \text{si } n = 1, 2, 3, \ldots$$

$$b_n = \frac{1}{\pi}\left\{\int_{-\pi}^{0} (-1)\,\text{sen}\,nt\,dt + \int_{0}^{\pi} (1)\,\text{sen}\,nt\,dt\right\}$$

$$\boxed{\int \text{sen}\,nt\,dt = -\frac{\cos nt}{n} \quad ; \quad \cos n\pi = (-1)^n}$$

$$b_n = \frac{1}{\pi}\left\{\frac{\cos nt}{n}\Big|_{-\pi}^{0} - \frac{\cos nt}{n}\Big|_{0}^{\pi}\right\}$$

$$= \frac{1}{\pi}\left\{\left[\frac{\cos 0}{n} - \frac{\cos n\pi}{n}\right] - \left[\frac{\cos n\pi}{n} - \frac{\cos 0}{n}\right]\right\}$$

$$= \frac{1}{\pi}\left\{\frac{1}{n} - \frac{(-1)^n}{n} - \frac{(-1)^n}{n} + \frac{1}{n}\right\} = \frac{1}{\pi}\left\{\frac{2}{n} - \frac{2(-1)^n}{n}\right\}$$

$$b_n = \frac{2}{n\pi}\left[1 - (-1)^n\right]$$

$$\therefore b_n = \frac{2}{n\pi}\left[1 - (-1)^n\right].$$

$$\cos n\pi = (-1)^n = \begin{cases} -1 \text{ si } n \text{ IMPAR}(2n-1) \\ 1 \text{ si } n \text{ PAR}(2n) \end{cases} \Rightarrow \left[1-\cos n\pi\right] = \left[1-(-1)^n\right] = \begin{cases} 2 \text{ si } n \text{ IMPAR} \\ 0 \text{ si } n \text{ PAR.} \end{cases}$$

$$b_n = \frac{2}{n\pi}\left[1-(-1)^n\right]$$

$$b_1 = \frac{2}{\pi},\ b_2 = 0,\ b_3 = \frac{2}{3\pi},\ b_4 = 0,\ b_5 = \frac{2}{5\pi} \cdots b_n = \frac{2}{n\pi}$$

LOS COEFICIENTES DE FOURIER TRIGONOMÉTRICOS SON:

$$a_0 = 0$$

$$a_n = 0$$

$$b_n = \frac{2}{n\pi} \Rightarrow b_n = \frac{2}{\pi(2n-1)}2 = \frac{4}{\pi(2n-1)}$$

SUSTITUYENDO $a_0 = 0$, $a_n = 0$ y $b_n = \frac{4}{\pi(2n-1)}$ EN LA SERIE DE FOURIER TRIGONOMÉTRICA.

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left[a_n \cos nt + b_n \operatorname{sen} nt\right]$$

$$f(t) = \cancel{\frac{0}{2}} + \sum_{n=1}^{\infty}\left[\cancel{0\cos nt} + \frac{4}{\pi(2n-1)}\operatorname{sen}(2n-1)t\right]$$

$$f(t) = \sum_{n=1}^{\infty} \frac{4}{\pi(2n-1)}\operatorname{sen}(2n-1)t$$

$$\therefore f(t) = \frac{4}{\pi}\sum_{n=1}^{\infty}\frac{\operatorname{sen}(2n-1)t}{(2n-1)}$$

$$f(t) = \frac{4}{\pi}\left[\operatorname{sen} t + \frac{1}{3}\operatorname{sen} 3t + \frac{1}{5}\operatorname{sen} 5t + \cdots\right]$$

LAS SUMAS PARCIALES DE LAS GRÁFICAS SON:

$$S_1 = S_1(t) = \frac{4}{\pi}\operatorname{sen} t$$

$$S_2 = S_2(t) = \frac{4}{\pi}\left(\operatorname{sen} t + \frac{1}{3}\operatorname{sen}\frac{t}{3}\right)$$

$$S_3 = S_3(t) = \frac{4}{\pi}\left(\operatorname{sen} t + \frac{1}{3}\operatorname{sen}\frac{t}{3} + \frac{1}{5}\operatorname{sen}\frac{t}{5}\right).$$

GRÁFICAMENTE SE MUESTRA COMO LAS SUMAS PARCIALE DE LOS COMPONENTES ARMÓNICOS VAN CONFORMANDO EL PERFIL DE LA ONDA (ONDA CUADRADA); ESTO ES:

$$f(t) = S_1 + S_2 + S_3 + \cdots + S_n = \sum_{n=1}^{\infty} S_n.$$

PODEMOS ESCRIBIR EN FORMA CONDENSADA LA SEÑAL DADA COMO:

$$f(t) = \frac{4}{\pi} \sum_{n=1}^{\infty} \frac{\operatorname{sen}(2n-1)t}{(2n-1)},$$

LA CUAL NOS REPRESENTA UNA SÍNTESIS DE LAS ONDAS; QUE ES LA SERIE DE FOURIER TRIGONOMÉTRICO ES UNA MEZCLA DE TÉRMINOS ARMÓNICOS. (VER FIGURA A.2).

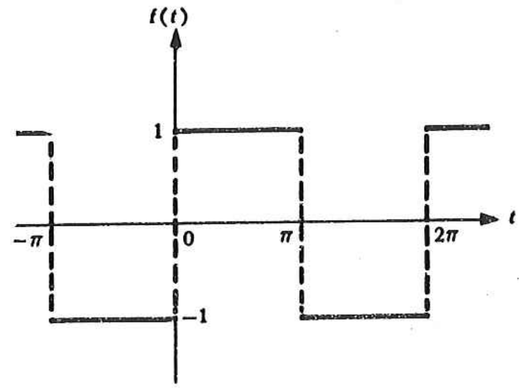


Figura A.1 La onda cuadrada del problema A.5.

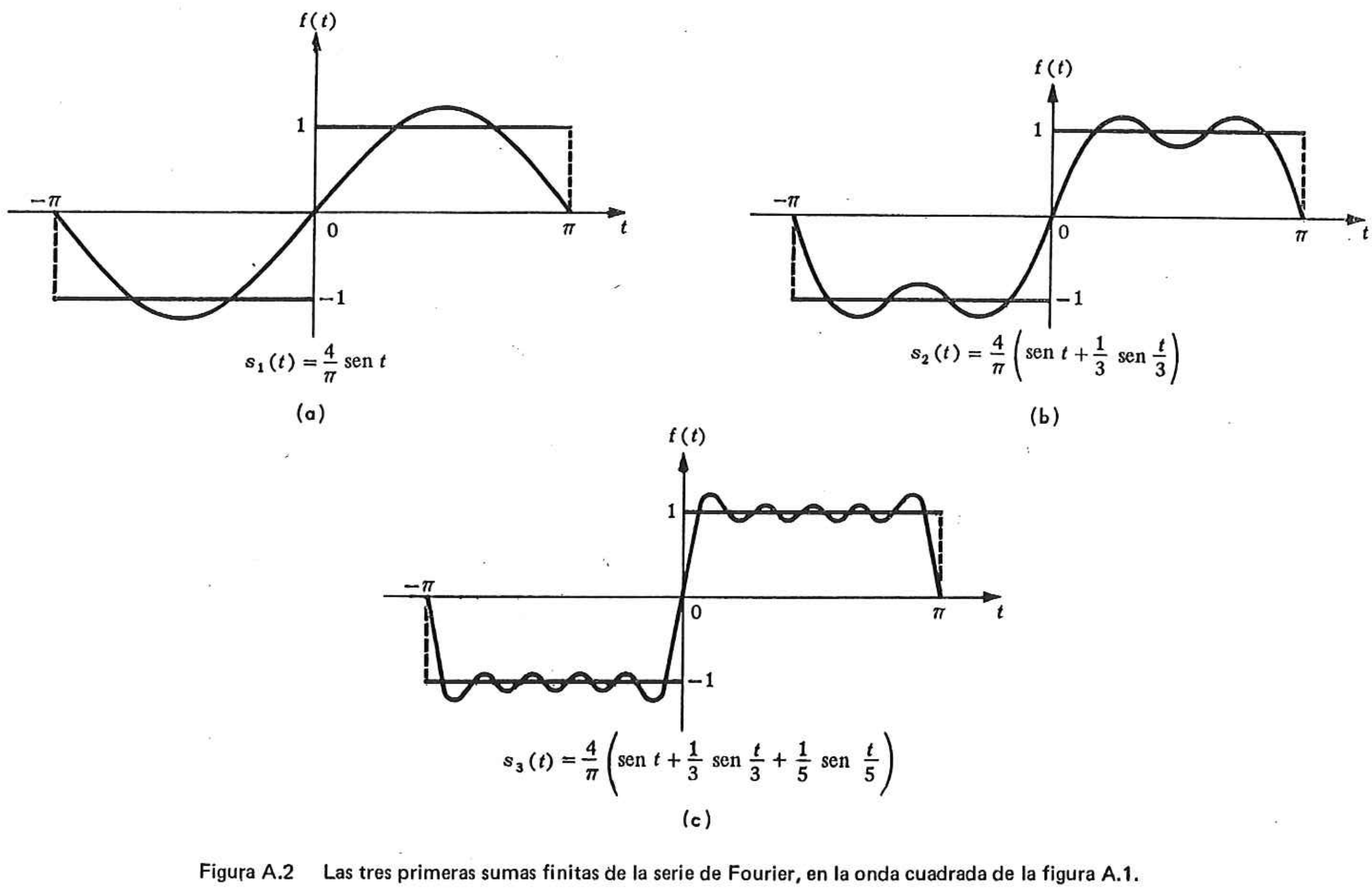


Figura A.2 Las tres primeras sumas finitas de la serie de Fourier, en la onda cuadrada de la figura A.1.

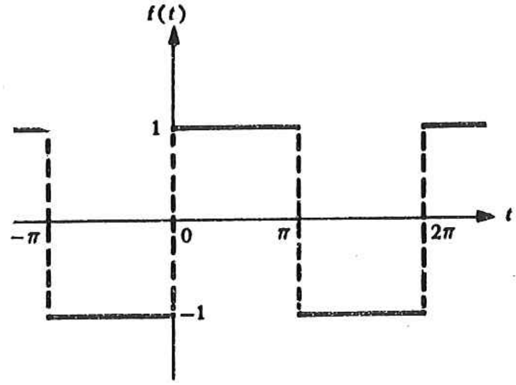


Figura A.1 La onda cuadrada del problema A.5.

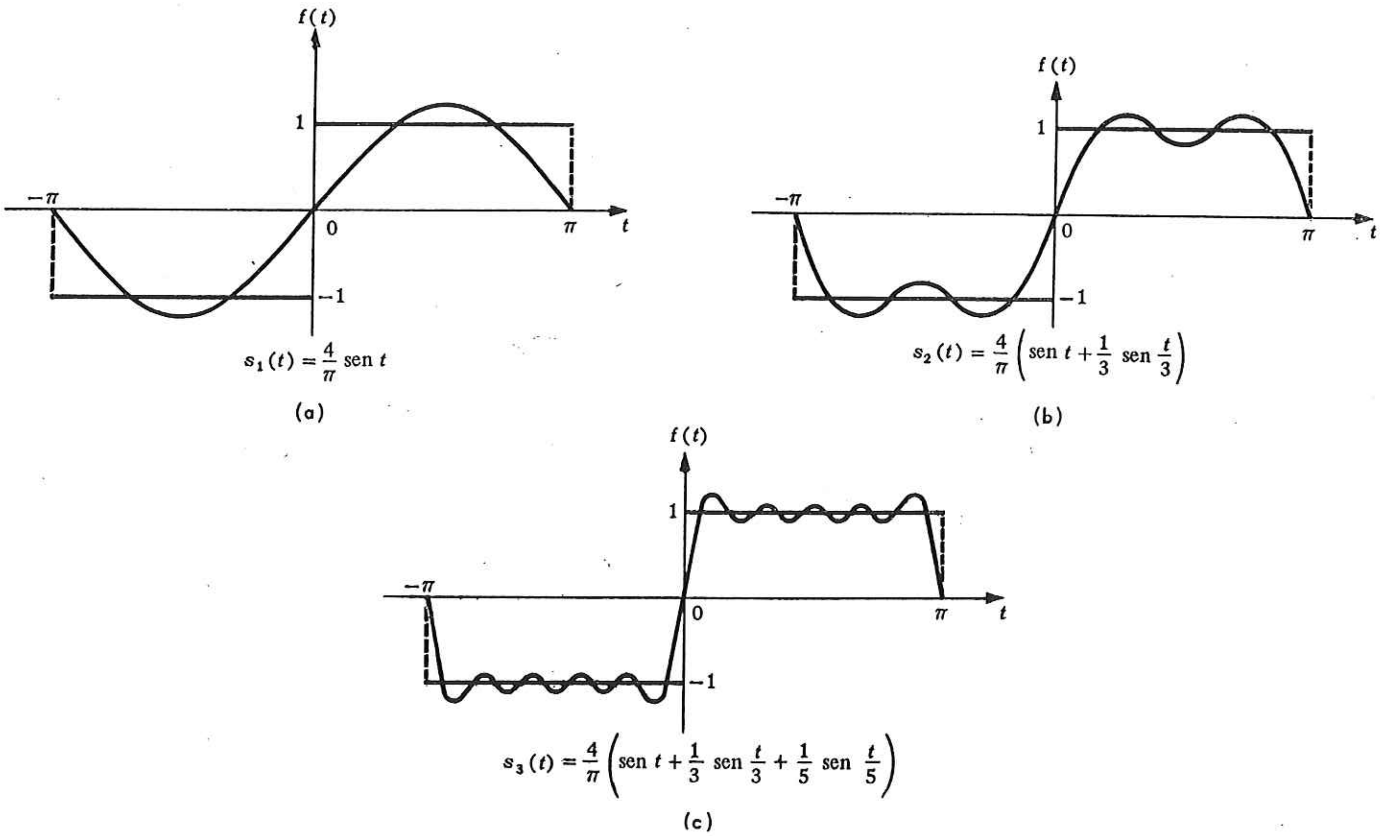


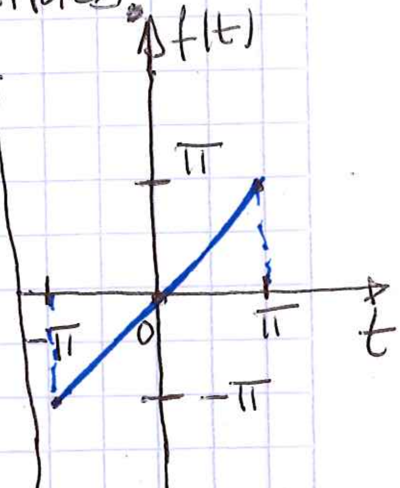
Figura A.2 Las tres primeras sumas finitas de la serie de Fourier, en la onda cuadrada de la figura A.1.

. GRAFICAR $f(t)=t$, si $-\pi \leq t \leq \pi$ si $T=2\pi$. 27

HALLAR LA SERIE DE FOURIER TRIGONOMÉTRICO.

SOL:

$-\pi \leq t \leq \pi$	$f(t)=t$
$t=-\pi$	$f(-\pi)=-\pi$, $(-\pi,\pi)$
$t=0$	$f(0)=0$, $(0,0)$
$t=\pi$	$f(\pi)=\pi$, (π,π)



LOS COEFICIENTES SON:

$$a_0=\frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\,dt$$

$$a_0=\frac{1}{\pi}\int_{-\pi}^{\pi} t\,dt=0$$ YA QUE $f(t)=t$ ES SEÑAL IMPAR.

$$a_0=0$$

$$a_n=\frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\cos nt\,dt=\frac{1}{\pi}\int_{-\pi}^{\pi} t\cos nt\,dt=0$$

$$a_n=0$$

$t\cos nt = (\text{IMPAR})(\text{PAR}) = \text{IMPAR}$ — $(-)$ $(+)$

IMPAR PAR

$$b_n=\frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\,\text{sen}\,nt\,dt=\frac{1}{\pi}\int_{-\pi}^{\pi} t\,\text{sen}\,nt\,dt$$

$t\,\text{sen}\,nt=(\text{IMPAR})(\text{IMPAR})=(\text{PAR})$ — $+$ $+$ $+$

$$b_n = \frac{2}{\pi}\int_0^{\pi} t \operatorname{sen} nt \, dt$$ 28

$$\boxed{\int t \operatorname{sen} at = \frac{\operatorname{sen} at}{a^2} - \frac{t \cos at}{a}}$$

$$b_n = \frac{2}{\pi}\left\{ \frac{\operatorname{sen} nt}{n^2}\Big|_0^{\pi} - t\frac{\cos nt}{n}\Big|_0^{\pi} \right\}$$

$$b_n = \frac{2}{\pi}\left\{ \cancel{\frac{\operatorname{sen} n\pi}{n^2}}^{0} - \cancel{\frac{\operatorname{sen} 0}{n^2}}^{0} - \left[\frac{\pi \cos n\pi}{n} - 0\right] \right\}$$

$$b_n = \frac{2}{\pi}\left\{ -\frac{\pi \cos n\pi}{n} \right\} = -\frac{2\cos n\pi}{n} = \frac{-2(-1)^n}{n} = \frac{2(-1)^{n+1}}{n}$$

$$\therefore\ b_n = \frac{-2(-1)^n}{n} = \frac{2(-1)^{n+1}}{n}$$

SUSTITUYENDO a_0, a_n y b_n EN LA SERIE DE FOURIER TRIGONOMÉTRICO

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left[a_n \cos nt + b_n \operatorname{sen} nt\right]$$

$$f(t) = \cancel{\frac{0}{2}}^{0} + \sum_{n=1}^{\infty}\left[\cancel{0 \cos nt}^{0} + \frac{2(-1)^{n+1}}{n} \operatorname{sen} nt\right]$$

$$f(t) = 2\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \operatorname{sen} nt.$$

A CONTINUACIÓN SE VERÁ OTRO TIPO DE SERIE DE FOURIER.

SERIE Y COEFICIENTES DE FOURIER EN $[-C,C]$.

SUPONGAMOS QUE $f(t)$ ES UNA SEÑAL PERIÓDICA CON PERIODO $T=2C$ Y QUE $f(t)$ ES DIFERENCIABLE POR PARTES EN $[-C,C]$.

LOS COEFICIENTES DE FOURIER DE f EN EL INTERVALO $[-C,C]$ SON:

$$a_0=\frac{1}{C}\int_{-C}^{C} f(t)\,dt$$

$$a_n=\frac{1}{C}\int_{-C}^{C} f(t)\cos\frac{n\pi t}{C}\,dt \quad \text{si } n=1,2,\ldots$$

$$b_n=\frac{1}{C}\int_{-C}^{C} f(t)\,\text{sen}\frac{n\pi t}{C}\,dt \quad \text{si } n=1,2,\ldots$$

LA SERIE DE FOURIER DE f EN $[-C,C]$ ES:

$$f(t)=\frac{a_0}{2}+\sum_{n=1}^{\infty}\left[a_n\cos\frac{n\pi t}{C}+b_n\,\text{sen}\frac{n\pi t}{C}\right],$$

CONVERGE A $\frac{1}{2}\left[f(t^+)+f(t^-)\right]$ EN CADA PUNTO DE $[-C,C]$.

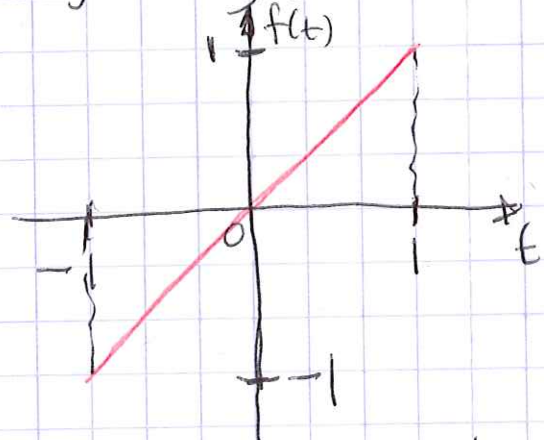
SI HACEMOS $C=\pi$, ESTA SERIE, SE REDUCE A LA DE FOURIER TRIGONOMÉTRICA DE UNA SEÑAL CON PERIODO $T=2\pi$.

EJEMPLO, GRAFICAR $f(t)$ Y DETERMINAR LA SERIE DE FOURIER PARA:

$$f(t) = t \text{ si } -1 \le t \le 1 \ ; \ f(t+2) = f(t).$$

Sol

$$T = 1-(-1) = 2 \Rightarrow 2C = 2 \quad \therefore C = 1$$



$$a_0 = \frac{1}{C}\int_{-C}^{C} f(t)\,dt = \frac{1}{1}\int_{-1}^{1} t\,dt = \int_{-1}^{1} t\,dt = 0 \ ; \ C = 1$$

POR SER $f(t) = t$ IMPAR $\Rightarrow a_0 = 0$

$$a_n = \frac{1}{C}\int_{-C}^{C} f(t)\cos\frac{n\pi t}{C}\,dt = \frac{1}{1}\int_{-1}^{1} t\cos n\pi t\,dt = 0$$

$t \cos n\pi t = (\overset{-}{\text{IMPAR}})(\overset{+}{\text{PAR}}) = (\overset{-}{\text{IMPAR}})$

$$\therefore a_n = 0$$

$$b_n = \frac{1}{C}\int_{-C}^{C} f(t)\,\text{sen}\frac{n\pi t}{C}\,dt = \frac{1}{1}\int_{-1}^{1} t\,\text{sen}\,n\pi t\,dt$$

$t\,\text{sen}\,n\pi t = (\overset{-}{\text{IMPAR}})(\overset{-}{\text{IMPAR}}) = (\overset{+}{\text{PAR}})$

$$b_n = 2\int_0^1 t \, sen\, n\pi t \, dt$$

$$\int t \, sen\, at \, dt = \frac{sen\, at}{a^2} - \frac{t \cos at}{a}, \quad a = n\pi.$$

$$b_n = 2\left[\frac{sen\, n\pi t}{n^2\pi^2}\Big|_0^1 - \frac{t\cos n\pi t}{n\pi}\Big|_0^1 \right]$$

$$b_n = 2\left[\cancel{\frac{sen\, n\pi}{n^2\pi^2}}^{\,0} - \cancel{\frac{sen\, 0}{n^2\pi^2}}^{\,0} - \left[\frac{(1)\cos n\pi}{n\pi} - 0 \right] \right]$$

$$b_n = 2\left[-\frac{\cos n\pi}{n\pi} \right] = -2\frac{(-1)^n}{n\pi} = \frac{2(-1)^{n+1}}{n\pi} \quad \therefore \; b_n = \frac{2(-1)^{n+1}}{n\pi}$$

SUSTITUYENDO a_0, a_n, b_n y $C=1$ EN LA SERIE DE FOURIER.

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left[a_n \cos\frac{n\pi t}{C} + b_n \frac{sen\, n\pi t}{C} \right]$$

$$f(t) = \cancel{\frac{0}{2}}^{\,0} + \sum_{n=1}^{\infty} \left[\cancel{0 \cos n\pi t}^{\,0} - \frac{2(-1)^{n+1}}{n\pi} sen\, n\pi t \right]$$

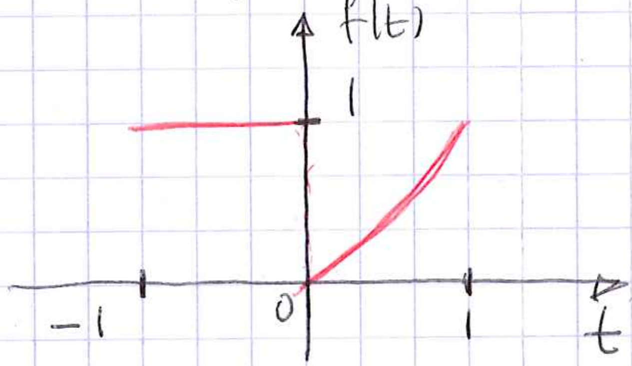
$$\therefore \; f(t) = -\frac{2}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} sen\, n\pi t.$$

• GRAFICAR $f(t)$ Y HALLAR LA SERIE DE FOURIER 32

PARA:

$$f(t) = \begin{cases} 1, & -1 < t < 0 \\ t, & 0 \leq t < 1 \end{cases}$$

Sol:



$T = 2C$; $T = 1-(-1) = 1+1 = 2$; $2C = 2 \Rightarrow C = 1$

$$a_0 = \frac{1}{C}\int_{-C}^{C} f(t)dt = \frac{1}{1}\int_{-1}^{1} f(t)dt = \int_{-1}^{0}(1)dt + \int_{0}^{1} t\,dt$$

$$= t\Big|_{-1}^{0} + \frac{t^2}{2}\Big|_{0}^{1} = (0+1) + \frac{1}{2} - 0 = 1 + \frac{1}{2} = \frac{2+1}{2} = \frac{3}{2}$$

$$\therefore a_0 = \frac{3}{2}$$

$$a_n = \frac{1}{C}\int_{-C}^{C} f(t)\cos\frac{n\pi t}{C}dt = \int_{-1}^{0}(1)\cos n\pi t\,dt + \int_{0}^{1} t\cos n\pi t\,dt$$

$$= \int_{-1}^{0}\cos n\pi t\,dt + \int_{0}^{1} t\cos n\pi t\,dt$$

$$\boxed{\int \cos at\,dt = \frac{\operatorname{sen} at}{a} \; ; \; \int t\cos at\,dt = \frac{\cos at}{a^2} + \frac{t\operatorname{sen} at}{a} \; ; \; a = n\pi}$$

$$a_n = \frac{\operatorname{sen} n\pi t}{n\pi}\Big|_{-1}^{0} + \frac{\cos n\pi t}{n^2\pi^2}\Big|_{0}^{1} + t\,\frac{\operatorname{sen} n\pi t}{n\pi}\Big|_{0}^{1}$$

$$= \frac{1}{n\pi}\left[\cancel{\operatorname{sen} 0}^{\,0} + \cancel{\operatorname{sen} n\pi}^{\,0}\right] + \frac{1}{n^2\pi^2}\left[\cos n\pi - \cos 0\right] + \frac{1}{n\pi}\left[\cancel{(1)\operatorname{sen} n\pi}^{\,0} - 0\right]$$

$$= \frac{1}{n^2\pi^2}\left[\cos n\pi - 1\right] = \frac{1}{n^2\pi^2}\left[(-1)^n - 1\right]$$

$$\left[(-1)^n - 1\right] = \begin{cases} -2, & \text{si } n \text{ IMPAR } (2n-1) \\ 0, & \text{si } n \text{ PAR } (2n). \end{cases}$$

$$a_n = \frac{1}{(2n-1)^2\pi^2}(-2) = -\frac{2}{\pi^2(2n-1)^2}$$

$$b_n = \frac{1}{c}\int_{-c}^{c} f(t)\operatorname{sen}\frac{n\pi t}{c}\,dt = \int_{-1}^{0}(1)\operatorname{sen} n\pi t\,dt + \int_{0}^{1} t\operatorname{sen} n\pi t\,dt$$

$$b_n = \int_{-1}^{0}\operatorname{sen} n\pi t\,dt + \int_{0}^{1} t\operatorname{sen} n\pi t\,dt$$

$$\int \operatorname{sen} at = -\frac{\cos at}{a}\ ;\ \int t\operatorname{sen} n\pi t\,dt = \frac{\operatorname{sen} at}{a^2} - \frac{t\cos at}{a}\ ;\ a = n\pi$$

$$b_n = -\frac{\cos n\pi t}{n\pi}\Big|_{-1}^{0} + \frac{\operatorname{sen} n\pi t}{n^2\pi^2}\Big|_{0}^{1} - t\,\frac{\cos n\pi t}{n\pi}\Big|_{0}^{1}$$

$$b_n = \frac{1}{n^2\pi^2}\left[\cancel{\operatorname{sen} n\pi}^{0} - \cancel{\operatorname{sen} 0}^{0}\right] - \frac{1}{n\pi}\left[(1)\cos n\pi - 0\right]$$

$$= -\frac{1}{n\pi}\left[\cos 0 - \cos(-n\pi)\right] - \frac{\cos n\pi}{n\pi}$$

$$\boxed{-\frac{\cos n\pi t}{n\pi}\Big|_{-1}^{0} = -\frac{1}{n\pi}\left[\cos 0^{\circ} - \cos(-n\pi)\right]}$$

$$= -\frac{1}{n\pi} + \frac{1}{n\pi}(-1)^n - \frac{(-1)^n}{n\pi} = -\frac{1}{n\pi}$$

$$b_n = -\frac{1}{n\pi} + \frac{(-1)^n}{n\pi} - \frac{(-1)^n}{n\pi} = -\frac{1}{n\pi} \quad \therefore b_n = -\frac{1}{n\pi}$$

SUSTITUYENDO a_0, a_n y b_n EN LA SERIE DE FOURIER.

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left[a_n \cos\frac{n\pi t}{c} + b_n \operatorname{sen}\frac{n\pi t}{c}\right] \text{ con } c=1$$

$$f(t) = \frac{3}{4} + \sum_{n=1}^{\infty}\left[\frac{1}{n^2\pi^2}\left[(-1)^n - 1\right]\cos n\pi t - \frac{1}{n\pi}\operatorname{sen} n\pi t\right]$$

$$f(t) = \frac{3}{4} + \sum_{n=1}^{\infty}\left[-\frac{2}{\pi^2(2n-1)^2}\cos(2n-1)\pi t - \frac{1}{n\pi}\operatorname{sen} n\pi t\right]$$

$$f(t) = \frac{3}{4} - \frac{1}{\pi}\sum_{n=1}^{\infty}\left[\frac{2}{\pi(2n-1)^2}\cos(2n-1)\pi t + \frac{1}{n}\operatorname{sen} n\pi t\right].$$

T) SEA f INTEGRABLE EN $[-c,c]$.

A) SI f ES PAR EN $[-c,c] \Rightarrow \int_{-c}^{c} f(t)dt = 2\int_{0}^{c} f(t)\,dt$.

B) SI f ES IMPAR EN $[-c,c] \Rightarrow \int_{-c}^{c} f(t)dt = 0$.

DONDE:

- $f(-t) = f(t)$ SEÑAL PAR.
- $f(-t) = -f(t)$ SEÑAL IMPAR.

T) SEA f UNA SEÑAL INTEGRABLE EN $[-c,c]$.

CASOS

A) SI f ES PAR, LA SERIE DE FOURIER DE f EN $[-c,c]$ ES:

$$f(t) = a_0 + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi t}{c}\right),$$

DONDE

$$a_0 = \frac{1}{c}\int_0^c f(t)\,dt,$$

$$a_n = \frac{2}{c}\int_0^c f(t)\cos\left(\frac{n\pi t}{c}\right)dt.$$

B) SI f ES IMPAR, SU SERIE DE FOURIER EN $[-c,c]$ ES:

$$f(t) = \sum_{n=1}^{\infty} b_n \operatorname{sen}\left(\frac{n\pi t}{c}\right).$$

DONDE: $$b_n = \frac{2}{c}\int_0^c f(t)\,\text{sen}\left(\frac{n\pi t}{c}\right)dt.$$

EJEMPLO.

- CALCÚLESE LA SERIE DE FOURIER PARA:

$$f(t) = \text{senh}\,t, \quad -\pi < t < \pi.$$

SOL:

$f(t) = \text{senh}\,t$ ES SEÑAL IMPAR.

$T = 2\pi$; $T = \pi - (-\pi) = \pi + \pi = 2\pi$; $2c = 2\pi \Rightarrow c = \pi$

$$b_n = \frac{2}{c}\int_0^c f(t)\,\text{sen}\left(\frac{n\pi t}{c}\right)dt \text{ TAL QUE } f(t) = \text{senh}\,t = \frac{e^t - e^{-t}}{2}.$$

$$b_n = \frac{2}{\pi}\int_0^\pi \left[\frac{e^t - e^{-t}}{2}\right]\text{sen}\left(\frac{n\pi t}{\pi}\right)dt$$

$$= \frac{2}{2\pi}\int_0^\pi \left[e^t - e^{-t}\right]\text{sen}\,nt\,dt = \frac{1}{\pi}\int_0^\pi \left[e^t - e^{-t}\right]\text{sen}\,nt\,dt$$

$$\int e^{at}\,\text{sen}\,bt\,dt = \frac{e^{at}\left[a\,\text{sen}\,bt - b\cos bt\right]}{a^2+b^2}; \quad a = 1,\ b = n$$

$$b_n = \frac{1}{\pi}\left\{\int_0^\pi e^t\,\text{sen}\,nt\,dt - \int_0^\pi e^{-t}\,\text{sen}\,nt\,dt\right\}$$

$$A = \int_0^{\pi} e^{t}\,\text{sen}\,nt\,dt.$$

$$= \frac{e^{t}[\text{sen}\,nt - n\cos nt]}{1+n^2}\Bigg|_0^{\pi} = \frac{1}{n^2+1}\left\{ e^{t}\,\text{sen}\,nt - e^{t} n \cos nt \right\}\Bigg|_0^{\pi}$$

$$= \frac{1}{n^2+1}\left\{ e^{t}\,\text{sen}\,nt\Big|_0^{\pi} - e^{t} n\cos nt\Big|_0^{\pi} \right\}$$

$$= \frac{1}{n^2+1}\left\{ \left(e^{\pi}\cancelto{0}{\text{sen}\,n\pi} - e^{0}\cancelto{0}{\text{sen}\,n\pi}\right) - \left(e^{\pi} n\cos n\pi - e^{0} n \cos 0\right) \right\}$$

$$= \frac{1}{n^2+1}\left\{ -e^{\pi} n(-1)^n + n \right\} = \frac{n}{n^2+1}\left\{ -e^{\pi}(-1)^n + 1 \right\}$$

AHORA: $B = \int_0^{\pi} e^{-t}\,\text{sen}\,nt\,dt$; $a = -1$ y $b_n = n$.

$$= \frac{e^{-t}[-\text{sen}\,nt - n\cos nt]}{n^2+1}\Bigg|_0^{\pi}$$

$$= \frac{1}{n^2+1}\left[-e^{-t}\,\text{sen}\,nt\Big|_0^{\pi} - e^{-t} n\cos nt\Big|_0^{\pi} \right]$$

$$= \frac{1}{n^2+1}\left[-\left(e^{-\pi}\cancelto{0}{\text{sen}\,n\pi} - e^{0}\cancelto{0}{\text{sen}\,0}\right) - \left(e^{-\pi} n\cos n\pi - e^{0} n\cos 0\right) \right]$$

$$= \frac{1}{n^2+1}\left[-e^{-\pi} n(-1)^n + n\right] = \frac{n}{n^2+1}\left[-e^{-\pi}(-1)^n + 1\right]$$

$$b_n = \frac{1}{\pi}\left\{ A - B \right\} = \frac{1}{\pi}\left\{ \frac{n}{n^2+1}\left\{-e^{\pi}(-1)^n+1\right\} - \frac{n}{n^2+1}\left\{-e^{-\pi}(-1)^n+1\right\}\right\}$$

$$= \frac{n}{\pi(n^2+1)}\left\{-e^{\pi}(-1)^n + 1 + e^{-\pi}(-1)^n - 1\right\} = \frac{n}{\pi(n^2+1)}\left\{-e^{\pi}(-1)^n + e^{-\pi}(-1)^n\right\}$$

$$= \frac{n(-1)^n}{\pi(n^2+1)}\left[-e^{\pi} + e^{-\pi}\right] = \frac{-n(-1)^n}{\pi(n^2+1)}\left[e^{\pi} - e^{-\pi}\right]\frac{2}{2}$$

$$= \frac{-2n(-1)^n}{\pi(n^2+1)}\left[\frac{e^{\pi} - e^{-\pi}}{2}\right] = \frac{2n(-1)^{n+1}}{\pi(n^2+1)} \operatorname{senh}\pi$$

$$\therefore b_n = \frac{2n(-1)^{n+1}}{\pi(n^2+1)} \cdot \operatorname{senh}\pi.$$

LA SERIE DE FOURIER.

$$f(t) = \sum_{n=1}^{\infty} b_n \operatorname{sen} nt = \frac{2}{\pi}\operatorname{senh}\pi \sum_{n=1}^{\infty} \frac{n(-1)^{n+1}}{n^2+1} \operatorname{sen} nt.$$

Serie de Fourier de medio rango de cosenos y de senos.

Def. Serie de Fourier de cosenos en $[0,c]$.

Si f es integrable en $[0,c]$, la serie de Fourier de cosenos de f en $[0,c]$ es:

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi t}{c}\right),$$

Donde:

$$a_0 = \frac{2}{c}\int_0^c f(t)\,dt.$$

$$a_n = \frac{2}{c}\int_0^c f(t)\cos\left(\frac{n\pi t}{c}\right)dt.$$

Def. Serie de Fourier de senos en $[0,c]$.

Si f es integrable en $[0,c]$, la serie de Fourier en senos de f en $[0,c]$ es:

$$f(t) = \sum_{n=1}^{\infty} b_n \operatorname{sen}\left(\frac{n\pi t}{c}\right),$$

Donde:

$$b_n = \frac{2}{c}\int_0^c f(t)\operatorname{sen}\left(\frac{n\pi t}{c}\right)dt.$$

EJEMPLOS: CALCÚLESE LA SERIE FOURIER DE SENOS Y COSENOS PARA:

$$f(t) = 2 \text{ si } 0 < t < \pi.$$

SOL.

SERIE EN SENOS CON $C = \pi$.

$$b_n = \frac{2}{\pi}\int_0^{\pi} f(t)\,\text{sen}\left(\frac{n\pi t}{C}\right)dt = \frac{2}{\pi}\int_0^{\pi} 2\,\text{sen}\left(\frac{n\pi t}{\pi}\right)dt$$

$$b_n = \frac{4}{\pi}\int_0^{\pi} \text{sen}(nt)\,dt = -\frac{4}{n\pi}\cos nt\Big|_0^{\pi} = -\frac{4}{n\pi}\left[\cos n\pi - \cos 0\right]$$

$$b_n = \frac{4}{n\pi}\left[-(-1)^n + 1\right] = \frac{4}{n\pi}\left[1-(-1)^n\right]$$

$$\therefore b_n = \frac{4}{n\pi}\left[1-(-1)^n\right]$$

LA SERIE DE FOURIER DE SENOS.

$$f(t) = \sum_{n=1}^{\infty} b_n\,\text{sen}\left(\frac{n\pi t}{C}\right)$$

$$2 \cong \sum_{n=1}^{\infty} \frac{4}{n\pi}\left[1-(-1)^n\right]\text{sen}\,nt$$

FINALMENTE, $$2 \cong \frac{8}{\pi}\sum_{n=1}^{\infty}\frac{\text{sen}(2n-1)t}{(2n-1)}.$$

Serie de Fourier en cosenos.

Sol: $C = \pi$,

$$a_0 = \frac{2}{C}\int_0^C f(t)\,dt = \frac{2}{\pi}\int_0^{\pi} 2\,dt = \frac{4}{\pi} t\Big|_0^{\pi} = \frac{4}{\pi}\left[\pi - 0\right] = \frac{4\pi}{\pi} = 4$$

$$\therefore a_0 = 4$$

$$a_n = \frac{2}{C}\int_0^C f(t)\cos\left(\frac{n\pi t}{C}\right)dt \quad \text{con } C = \pi.$$

$$a_n = \frac{2}{\pi}\int_0^{\pi} 2\cos nt\,dt = \frac{4}{n\pi}\left[\operatorname{sen} nt\right]_0^{\pi} = \frac{4}{n\pi}\left[\operatorname{sen} n\pi - \operatorname{sen} 0\right] = 0$$

$$\therefore a_n = 0$$

La serie de Fourier.

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi t}{C}\right) \quad \text{si } C = \pi.$$

$$f(t) = \frac{4}{2} + \sum_{n=1}^{\infty} 0 \cos nt = \frac{4}{2} + 0 = 2$$

$$\therefore f(t) = 2$$

- ENCONTRAR LA SERIE DE FOURIER DE SENOS

PARA:

$$f(t) = \cos t, \quad 0 < t < \pi/2.$$

SOL:

$$f(t) = \sum_{n=1}^{\infty} b_n \operatorname{sen}\left(\frac{n\pi t}{c}\right); \quad c = \frac{\pi}{2}.$$

$$b_n = \frac{2}{c}\int_0^c f(t) \operatorname{sen}\left(\frac{n\pi t}{c}\right)$$

$$b_n = \frac{2}{\frac{\pi}{2}}\int_0^{\pi/2} \cos t \operatorname{sen}(2nt)\,dt = \frac{4}{\pi}\int_0^{\pi/2} \cos t \operatorname{sen}(2nt)\,dt$$

$$= \frac{4}{\pi}\int_0^{\pi/2} \operatorname{sen}(2nt)\cos t\,dt$$

$$\int \operatorname{sen} pt \cos qt = -\frac{\cos(p-q)t}{2(p-q)} - \frac{\cos(p+q)t}{2(p+q)}; \quad p = 2n,\ q = 1.$$

$$b_n = \frac{2}{\pi}\left\{-\frac{\cos(2n-1)t}{(2n-1)}\Bigg|_0^{\pi/2} - \frac{\cos(2n+1)t}{(2n+1)}\Bigg|_0^{\pi/2}\right\}$$

$$b_n = \frac{2}{\pi}\left\{-\frac{1}{(2n-1)}\left[\cos(2n-1)\frac{\pi}{2} - \cos 0\right] - \frac{1}{(2n+1)}\left[\cos(2n+1)\frac{\pi}{2} - \cos 0\right]\right\}$$

$$= \frac{2}{\pi}\left\{-\frac{1}{(2n-1)}\left[\cos(2n-1)\frac{\pi}{2} - 1\right] - \frac{1}{(2n+1)}\left[\cos(2n+1)\frac{\pi}{2} - 1\right]\right\}$$

$$b_n = \frac{2}{\pi}\left\{ -\frac{1}{2n-1}\left[\cancel{\cos\left(n\pi - \frac{\pi}{2}\right)}^{0} - 1\right] - \frac{1}{2n+1}\left[\cancel{\cos\left(n\pi - \frac{\pi}{2}\right)}^{0} - 1\right] \right\}$$

$$\cos(A \pm B) = \cos A \cos B \mp \operatorname{sen} A \operatorname{sen} B \; ; \; A = n\pi \text{ y } B = \frac{\pi}{2}.$$

$$\cos\left(n\pi - \frac{\pi}{2}\right) = \cos n\pi \cancel{\cos\frac{\pi}{2}}^{0} - \cancel{\operatorname{sen} n\pi}^{0} \operatorname{sen}\frac{\pi}{2} = 0$$

$$\cos\left(n\pi + \frac{\pi}{2}\right) = \cos n\pi \cancel{\cos\frac{\pi}{2}}^{0} - \cancel{\operatorname{sen} n\pi}^{0} \operatorname{sen}\frac{\pi}{2} = 0$$

$$b_n = \frac{2}{\pi}\left\{ -\frac{1}{2n-1}[-1] - \frac{1}{2n+1}[-1] = \frac{1}{2n-1} + \frac{1}{2n+1} \right.$$

$$= \frac{2}{\pi}\left\{ \frac{2n+1+2n-1}{4n^2-1} \right\} = \frac{2}{\pi(4n^2-1)}[4n] = \frac{8n}{\pi(4n^2-1)}$$

$$\therefore b_n = \frac{8n}{\pi(4n^2-1)}.$$

SUSTITUYENDO b_n EN LA SERIE DE FOURIER EN SENOS.

$$f(t) = \sum_{n=1}^{\infty} b_n \operatorname{sen} 2nt = \sum_{n=1}^{\infty} \frac{8n}{\pi(4n^2-1)} \operatorname{sen} 2nt$$

$$\therefore f(t) = \frac{8}{\pi} \sum_{n=1}^{\infty} \frac{n}{(4n^2-1)} \operatorname{sen} 2nt$$

FORMA COMPLEJA DE LA SERIE DE FOURIER Ó EXPONENCIAL.

CASO A

PARA SEÑALES DE PERIODO 2C, LA FORMA COMPLEJA DE LA SERIE DE FOURIER DEFINIDA EN EL INTERVALO $(-c,c)$ Ó $T=2C$, ES:

$$f(t)=\sum_{n=-\infty}^{\infty}{}' C_n e^{\frac{in\pi t}{c}}$$

$$f(t)=C_0+\sum_{n=1}^{\infty}\left[C_n e^{\frac{in\pi t}{c}}+C_{-n} e^{\frac{in\pi t}{c}}\right]$$

$$f(t)=C_0+\sum_{n=-\infty}^{\infty} C_n e^{\frac{in\pi t}{c}},$$

DONDE

$$C_n=\frac{1}{2C}\int_{-c}^{c} f(t)\, e^{-\frac{in\pi t}{c}}\,dt;\ n=0,\pm1,\pm2,\ldots$$

$$C_0=\frac{1}{2C}\int_{-c}^{c} f(t)\,dt.$$

EL SÍMBOLO $\sum_{n=-\infty}^{\infty}{}'$ SIGNIFICA QUE n RECORRE TODOS LOS NÚMEROS ENTEROS, EXCEPTO EL CERO.

HECHOS.

POR EULER.

$$\cos nt = \frac{1}{2}\left[e^{int} + e^{-int}\right]$$

$$\operatorname{sen} nt = \frac{1}{2i}\left[e^{int} - e^{-int}\right].$$

$$e^{-int} = \cos nt - i\operatorname{sen} nt$$

$$e^{int} = \cos nt - i\operatorname{sen} nt.$$

EJEMPLOS.

(1) ENCUÉNTRESE LA FORMA COMPLEJA O EXPONENCIAL DE LA SERIE DE FOURIER DE LA SEÑAL PERIÓDICA CUYA DEFINICIÓN EN UN PERIODO ES:

$$f(t) = e^{-t}, \ -1 < t < 1 \quad \text{con } C = 1.$$

SOL:

$$c_0 = \frac{1}{2C}\int_{-C}^{C} f(t)\,dt = \frac{1}{2(1)}\int_{-1}^{1} e^{-t}\,dt = \frac{1}{2}\int_{-1}^{1} e^{-t}\,dt$$

$$\boxed{\int e^{at}\,dt = \frac{e^{at}}{a} \quad \text{si } a = -1}$$

$$C_0 = \frac{1}{2}\left[-e^{-t}\right]_{-1}^{1} = \frac{1}{2}\left[-\left(e^{-1} - e^{1}\right)\right]$$

$$= \frac{1}{2}\left[-e^{-1} + e^{1}\right] = \frac{1}{2}\left[-0.37 + 2.7\right] = 0.33.$$

$$\therefore C_0 = 0.33$$

$$C_n = \frac{1}{2c}\int_{-c}^{c} f(t)\, e^{-\frac{in\pi t}{c}}\, dt \quad \text{si } c = 1.$$

$$C_n = \frac{1}{2}\int_{-1}^{1} e^{-t}\, e^{-in\pi t}\, dt = \frac{1}{2}\int_{-1}^{1} e^{-(1+ni\pi)t}\, dt$$

$$C_n = \frac{1}{2}\left\{\frac{1}{-(1+in\pi)}\, e^{-(1+in\pi)t}\Big|_{-1}^{1}\right\}$$

$$C_n = \frac{1}{2}\left\{-\frac{1}{(1+in\pi)}\left[e^{-(1+in\pi)} - e^{(1+in\pi)}\right]\right\}$$

$$C_n = \frac{1}{2}\left\{-\frac{1}{(1+in\pi)}\left[e^{-1}e^{-in\pi} - e^{1}e^{in\pi}\right]\right\}$$

$$C_n = \frac{1}{2}\left\{\frac{1}{(1+in\pi)}\left[-e^{-1}e^{-in\pi} + e\, e^{in\pi}\right]\right\}$$

$$C_n = \frac{e\,e^{in\pi} - e^{-1}e^{-in\pi}}{2(1+in\pi)}$$

$$e^{i\pi} = \cos\pi + i\,\mathrm{sen}\,\pi = -1 \Rightarrow e^{in\pi} = e^{-in\pi} = (-1)^n$$

$$C_n = \frac{e(-1)^n - e^{-1}(-1)^n}{2(1+in\pi)} = \frac{(-1)^n}{(1+in\pi)}\,\frac{e-e^{-1}}{2}$$

$$C_n = \frac{(-1)^n}{(1+in\pi)}\,\mathrm{senh}\,1 = (-1)^n\,\mathrm{senh}\,1\left[\frac{1}{(1+in\pi)}\right]$$

$$C_n = (-1)^n\,\mathrm{senh}\,1\left[\frac{1}{(1+in\pi)}\,\frac{(1-in\pi)}{(1-in\pi)}\right]$$

$$C_n = \frac{(-1)^n(1-in\pi)\,\mathrm{senh}\,1}{1-(-1)n^2\pi^2} = \frac{(-1)^n(1-in\pi)\,\mathrm{senh}\,1}{1+n^2\pi^2}$$

$$\therefore\ C_n = \frac{(-1)^n(1-in\pi)\,\mathrm{senh}\,1}{1+n^2\pi^2}$$

SUSTITUYENDO C_0 Y C_n EN LA SERIE DE FOURIER COMPLEJA.

$$f(t) = C_0 + \sum_{n=-\infty}^{\infty} C_n\, e^{\frac{in\pi t}{c}} \quad \text{con } c = 1.$$

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$$f(t) = 0.33 + \sum_{n=-\infty}^{\infty} \frac{(-1)^n(1-in\pi)\,\mathrm{senh}1}{1+n^2\pi^2} e^{in\pi t}$$

COMPROBAR EL SIGUIENTE HECHO COMO EJERCICIO.

HALLAR C_{-n} CON $C=1$.

$$C_{-n} = \frac{1}{2}\int_{-C}^{C} e^{-t} e^{in\pi t}\, dt \Rightarrow \dots$$

$$\Rightarrow \boxed{C_{-n} = \frac{(-1)^n(1+in\pi)\,\mathrm{senh}1}{1+n^2\pi^2}}$$

2 DEL EJEMPLO 1 CON $C_0 = 0.33$, $C_n = \frac{(-1)^n(1-in\pi)\,\mathrm{senh}1}{1+n^2\pi^2}$

Y $C_{-n} = \frac{(-1)^n(1+in\pi)\,\mathrm{senh}1}{1+n^2\pi^2}$ ESCRIBIR EN TÉRMINOS

DE LA SERIE DE FOURIER TRIGONOMÉTRICA REAL.

DONDE:

$$C_0 = \frac{a_0}{2} \Rightarrow a_0 = 2C_0$$

$$a_n = C_n + C_{-n}$$

$$b_n = i(C_n - C_{-n})$$

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$$c_0 = \frac{a_0}{2} \Rightarrow a_0 = 2c_0$$

$$a_0 = 2(0.33) = 0.66$$

$$a_n = c_n + c_{-n} = \frac{(-1)^n(1-in\pi)\,\text{senh}\,1}{1+n^2\pi^2} + \frac{(-1)^n(1+in\pi)\,\text{senh}\,1}{1+n^2\pi^2}$$

$$= \frac{(-1)^n\,\text{senh}\,1}{1+n^2\pi^2}\left\{1-in\pi+1+in\pi\right\} = \frac{(-1)^n 2\,\text{senh}\,1}{1+n^2\pi^2}$$

$$\therefore a_n = \frac{(-1)^n 2\,\text{senh}\,1}{1+n^2\pi^2}$$

$$b_n = i(c_n - c_{-n}) = i\left[\frac{(-1)^n(1-in\pi)\,\text{senh}\,1}{1+n^2\pi^2} - \frac{(-1)^n(1+in\pi)\,\text{senh}\,1}{1+n^2\pi^2}\right]$$

$$= \frac{(-1)^n\,\text{senh}\,1}{1+n^2\pi^2}\, i\left[(1-in\pi)-1-in\pi\right] =$$

$$= \frac{(-1)^n\,\text{senh}\,1}{1+n^2\pi^2}\, i\left[-2in\pi\right] = \frac{(-1)^n\,\text{senh}\,1}{1+n^2\pi^2}\, 2n\pi$$

$$\therefore b_n = \frac{(-1)^n 2n\pi\,\text{senh}\,1}{1+n^2\pi^2}$$

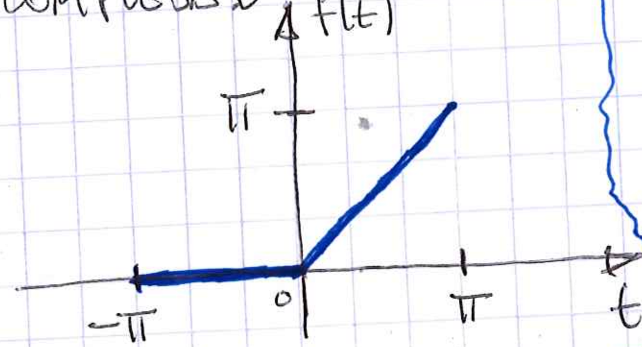
SUSTITUYENDO a_0, a_n y b_n EN LA SERIE DE FOURIER TRIGONOMÉTRICA.

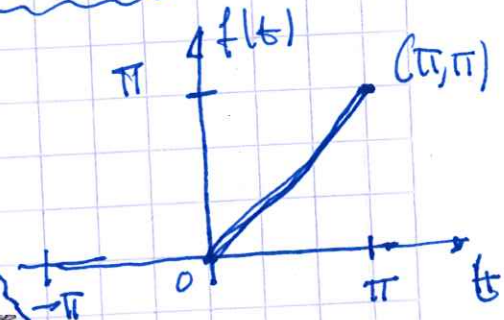
$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left[a_n \cos\frac{n\pi t}{c} + b_n \,\text{sen}\frac{n\pi t}{c}\right] \text{ con } c=1$$

$$f(t) = 0.33 + \sum_{n=1}^{\infty}\left[\frac{(-1)^n 2\,\text{senh}\,1}{1+n^2\pi^2}\cos n\pi t + \frac{(-1)^n 2n\pi\,\text{senh}\,1}{1+n^2\pi^2}\,\text{sen}\, n\pi t\right] \quad 50$$

$$\therefore f(t) = 0.33 + \sum_{n=1}^{\infty}\left[\frac{(-1)^n 2\,\text{senh}\,1}{1+n^2\pi^2}\cos n\pi t + \frac{(-1)^n 2n\pi\,\text{senh}\,1}{1+n^2\pi^2}\,\text{sen}\, n\pi t\right]$$

3) DE LA FIGURA SIGUIENTE, DETERMINE LA SERIE DE FOURIER COMPLEJA O EXPONENCIAL.





Sol

$(x_1, y_1) = (0,0)$

$(x_2, y_2) = (\pi, \pi)$

$$f(t) - y_1 = \frac{y_2 - y_1}{x_2 - x_1}(t - x_1)$$

$$f(t) - 0 = \frac{\pi - 0}{\pi - 0}(t - 0)$$

$$f(t) = \frac{\pi}{\pi} t$$

$\therefore f(t) = t,\ 0 < t < \pi.$

$$f(t) = \begin{cases} 0, & -\pi < t < 0 \\ t, & 0 < t < \pi \end{cases}$$

CON: $C = \pi$,

$$c_n = \frac{1}{2C}\int_{-C}^{C} f(t)\, e^{-\frac{in\pi t}{C}}\, dt$$

$$c_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)\, e^{-int}\, dt$$

$$c_n = \frac{1}{2\pi}\left[\cancelto{0}{\int_{-\pi}^{0} 0\, e^{-int}\, dt} + \int_{0}^{\pi} t\, e^{-int}\, dt\right] = \frac{1}{2\pi}\int_{0}^{\pi} t\, e^{-int}\, dt$$

$$\int t e^{at} dt = \frac{e^{at}}{a}\left[t - \frac{1}{a}\right] \text{ si } a = -in$$

S1

$$c_n = \frac{1}{2\pi}\left[-\frac{e^{int}}{in}\left[t + \frac{1}{in}\right]\right]_0^{\pi} = -\frac{1}{2\pi}\left[\frac{e^{-int}t}{in} + \frac{e^{-int}}{i^2n^2}\right]_0^{\pi}$$

$$= -\frac{1}{2\pi}\left[\frac{e^{-int}t}{in}\Big|_0^{\pi} - \frac{e^{-int}}{n^2}\Big|_0^{\pi}\right]$$

$$= -\frac{1}{2\pi}\left[\frac{1}{in}\left(\pi e^{-in\pi} - 0e^{0}\right) - \frac{1}{n^2}\left(e^{-in\pi} - e^{0}\right)\right]$$

$$\boxed{e^{i\pi} = \cos\pi + i\,\text{sen}\,\pi = -1 \Rightarrow e^{in\pi} = e^{-in\pi} = (-1)^n}$$

$$= -\frac{1}{2\pi}\left[\frac{1}{in}\pi(-1)^n - \frac{1}{n^2}\left((-1)^n - 1\right)\right]$$

$$= \frac{1}{2\pi}\left[-\frac{\pi(-1)^n}{in} + \frac{1}{n^2}\left((-1)^n - 1\right)\right] = \frac{1}{2\pi}\left[\frac{-\pi(-1)^n}{in}\frac{i}{i} + \frac{1}{n^2}\left((-1)^n - 1\right)\right]$$

$$= \frac{1}{2\pi}\left[\frac{-\pi(-1)^n i}{i^2 n} + \frac{1}{n^2}\left((-1)^n - 1\right)\right] = \frac{1}{2\pi}\left[\frac{-\pi(-1)^n i}{-n} + \frac{1}{n^2}\left((-1)^n - 1\right)\right]$$

$$= \frac{1}{2\pi}\left[\frac{i\pi(-1)^n}{n} + \frac{1}{n^2}\left((-1)^n - 1\right)\right] \quad \therefore c_n = \frac{1}{2\pi}\left[\frac{i\pi(-1)^n}{n} + \frac{1}{n^2}\left((-1)^n - 1\right)\right]$$

$$C_0 = \frac{1}{2c}\int_{-c}^{c} f(t)\,dt = \frac{1}{2\pi}\int_0^{\pi} t\,dt$$

$$C_0 = \frac{1}{2\pi}\frac{t^2}{2}\Big|_0^{\pi} = \frac{1}{2\pi}\,\frac{1}{2}\left[\pi^2 - 0\right] = \frac{1}{4}\frac{\pi^2}{\pi} = \frac{\pi}{4}$$

$$\therefore\ C_0 = \frac{\pi}{4}$$

SUSTITUYENDO C_0 Y C_n EN LA SERIE DE FOURIER COMPLEJA

$$f(t) = C_0 + \sum_{n=-\infty}^{\infty} C_n\, e^{\frac{in\pi t}{c}} \quad \text{si } c = \pi.$$

$$f(t) = \frac{\pi}{4} + \sum_{n=-\infty}^{\infty} \frac{1}{2\pi}\left[\frac{i\pi(-1)^n}{n} + \frac{1}{n^2}\left((-1)^n - 1\right)\right] e^{int}$$

CASO B

- SERIE DE FOURIER COMPLEJA O EXPONENCIAL.

LA SERIE DE FOURIER COMPLEJA O EXPONENCIAL DE f CON PERIODO T, SE DEFINE COMO:

$$\sum_{n=-\infty}^{\infty} c_n e^{in\omega_0 t} = c_0 + \sum_{n=1}^{\infty} c_n e^{in\omega_0 t} + \sum_{n=1}^{\infty} c_{-n} e^{in\omega_0 t}$$

$$= c_0 + \sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} c_n e^{in\omega_0 t}.$$

DONDE:

$$\omega_0 = \frac{2\pi}{T} \quad (\text{FRECUENCIA})$$

$$c_n = \frac{1}{T}\int_0^T f(t)\, e^{-in\omega_0 t}\, dt \text{ si } n=0,\pm 1,\pm 2,\ldots$$

$$c_0 = \frac{1}{T}\int_0^T f(t)\,dt.$$

LOS NÚMEROS c_n SON LOS COEFICIENTES DE FOURIER COMPLEJOS O EXPONENCIALES.

NOTA:

$$\cos n\omega_0 t = \frac{e^{in\omega_0 t} + e^{-in\omega_0 t}}{2}$$

$$\operatorname{sen} n\omega_0 t = \frac{e^{in\omega_0 t} - e^{-in\omega_0 t}}{2i}.$$

- ESPECTRO DE FRECUENCIA Ó AMPLITUD.

EN EL ESTUDIO DE LAS SEÑALES PERIÓDICAS DE TIEMPO, SE CONSIDERA DE MUCHA UTILIDAD EN ANÁLISIS ESPECTRAL DE DIVERSAS FORMAS DE ONDA. SI f ES PERIÓDICA Y TIENE UN PERIODO FUNDAMENTAL T, LA GRÁFICA DE LOS PUNTOS $(n\omega, |C_n|)$, PARA $n = 0, \pm 1, \pm 2, \ldots$, DONDE ω ES LA FRECUENCIA ANGULAR FUNDAMENTAL Y LOS C_n SON LOS COEFICIENTES DE FOURIER COMPLEJOS O EXPONENCIALES, SE LLAMA ESPECTRO DE FRECUENCIA O AMPLITUD DE f.

PARA CONVERTIR UNA SERIE DE FOURIER COMPLEJA A LA FORMA TRIGONOMÉTRICA DE LA SERIE DE FOURIER ES:

$$C_0 = \frac{a_0}{2} \Rightarrow a_0 = 2C_0$$

$$a_n = C_n + C_n^*$$

$$b_n = i(C_n - C_n^*) \; ; \; * \text{ CONJUGADO DE UN NÚMERO COMPLEJO.}$$

EJEMPLO.

4 a HALLESE LA FORMA COMPLEJA O EXPONENCIAL DE LA SERIE DE FOURIER, PARA:

$$f(t) = \begin{cases} 1, & 0 < t < 1 \\ 0, & 1 < t < 2. \end{cases}$$

SOL:

$$T = 2 \; ; \; \omega_0 = \frac{2\pi}{2} = \frac{2\pi}{2} = \pi \Rightarrow \omega_0 = \pi.$$

$$C_0 = \frac{1}{T}\int_0^T f(t)\,dt = \frac{1}{2}\int_0^2 f(t)\,dt = \frac{1}{2}\left[\int_0^1 (1)\,dt + \cancel{\int_1^2 0\,dt}^{\,0}\right]$$

$$C_0 = \frac{1}{2}\int_0^1 dt = \frac{1}{2}t\Big|_0^1 = \frac{1}{2}\left[1-0\right] = \frac{1}{2} \quad \therefore C_0 = \frac{1}{2}.$$

$$C_n = \frac{1}{T}\int_0^T f(t)\,e^{-in\omega_0 t}\,dt = \frac{1}{2}\int_0^2 f(t)\,e^{-in\omega_0 t}\,dt$$

$$C_n = \frac{1}{2}\int_0^1 (1)\,e^{-in\pi t}\,dt = \frac{1}{2}\int_0^1 e^{-in\pi t}\,dt = \frac{1}{2}\left[-\frac{1}{in\pi}e^{-in\pi t}\right]$$

$$= \frac{1}{2}\left[-\frac{1}{in\pi}\left\{e^{-in\pi} - e^{0}\right\}\right] = \frac{1}{2}\left[-\frac{1}{in\pi}\left\{(-1)^n - 1\right\}\right]$$

$$C_n = \frac{1}{2}\left[\frac{1}{in\pi}\left(-(-1)^n + 1\right)\right] = \frac{1}{2}\left[\frac{1-(-1)^n}{in\pi}\right]$$

$$C_n = \frac{1}{2}\left[\frac{2}{i(2n-1)\pi}\right] = \frac{1}{i\pi(2n-1)} \cdot \frac{i}{i}$$

$$= \frac{i}{i^2\pi(2n-1)} = -\frac{i}{\pi(2n-1)} \quad \therefore C_n = -\frac{i}{\pi(2n-1)}.$$

SUSTITUYENDO C_0 y C_n EN LA SERIE DE FOURIER COMPLEJA.

$$f(t) = C_0 + \sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} C_n e^{in\omega_0 t}$$

$$C_0 = \frac{1}{2}, \quad C_n = -\frac{i}{\pi(2n-1)} \quad y \quad \omega_0 = \pi.$$

$$f(t) = \frac{1}{2} - \sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} \frac{i}{\pi(2n-1)} e^{i(2n-1)\pi t}$$

$$f(t) = \frac{1}{2} - \frac{i}{\pi} \sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} \frac{e^{i(2n-1)\pi t}}{(2n-1)}.$$

b) HALLAR LA SERIE DE FOURIER TRIGONOMÉTRICA.

$$C_0 = \frac{1}{2} \quad ; \quad C_n = -\frac{i}{\pi(2n-1)} \quad y \quad \omega_0 = \pi.$$

$$c_0 = \frac{a_0}{2} \Rightarrow a_0 = 2c_0 = 2\left(\frac{1}{2}\right) = 1$$

$$\therefore\ a_0 = 1.$$

$$a_n = c_n + c_n^* = -\frac{i}{\pi(2n-1)} + \frac{i}{\pi(2n-1)} = 0$$

$$\therefore\ a_n = 0.$$

$$b_n = i\left[c_n - c_n^*\right] = i\left[-\frac{i}{\pi(2n-1)} - \frac{i}{\pi(2n-1)}\right]$$

$$b_n = -\frac{i^2}{\pi(2n-1)} - \frac{i^2}{\pi(2n-1)} = \frac{1}{\pi(2n-1)} + \frac{1}{\pi(2n-1)} = \frac{2}{\pi(2n-1)}$$

$$\therefore\ b_n = \frac{2}{\pi(2n-1)}.$$

LA SERIE TRIGONOMÉTRICA DE FOURIER ES.

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left[a_n \cos n w_0 t + b_n \operatorname{sen} n w_0 t\right]$$

$a_0 = 1$, $a_n = 0$, $b_n = \frac{2}{\pi(2n-1)}$ y $w_0 = \pi$.

$$f(t) = \frac{1}{2} + \sum_{n=1}^{\infty}\left[\cancel{0 \cos n\pi t}^{\,0} + \frac{2}{\pi(2n-1)} \operatorname{sen}(2n-1)\pi t\right]$$

$$f(t) = \frac{1}{2} + \sum_{n=1}^{\infty} \frac{2}{\pi(2n-1)} \operatorname{sen}(2n-1)\pi t$$

$$\therefore f(t) = \frac{1}{2} + \frac{2}{\pi} \sum_{n=1}^{\infty} \frac{\operatorname{sen}(2n-1)\pi t}{(2n-1)}$$

RECUERDE

- SEA $z \in \mathbb{C} \Rightarrow z = (x,y) = x + iy$; $\bar{z} = x - iy$
- EL MÓDULO DE z ES : $|z| = \sqrt{x^2+y^2}$

(C) BOSQUEJAR LA GRÁFICA DEL ESPECTRO DE FRECUENCIA Ó AMPLITUD.

SOL

C_0 y $C_n \in \mathbb{C}$, ESTO ES:

$$C_0 = \frac{1}{2} = \frac{1}{2} + i(0) = \frac{1}{2} = \left(\frac{1}{2}, 0\right)$$

$$C_n = -\frac{i}{\pi(2n-1)} = 0 - i\frac{1}{\pi(2n-1)} = \left(0, -\frac{1}{\pi(2n-1)}\right)$$

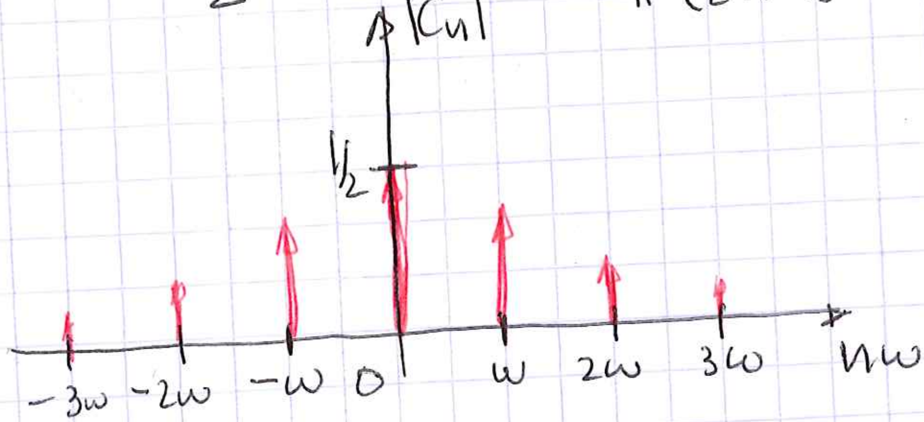
MÓDULO DE C_n.

$$|C_n| = \sqrt{0^2 + \left(\frac{1}{\pi(2n-1)}\right)^2} = \sqrt{\left(\frac{1}{\pi(2n-1)}\right)^2} = \frac{1}{\pi(2n-1)}$$

$$\therefore |C_n| = \frac{1}{\pi(2n-1)}$$

YA QUE: $C_0 = \frac{1}{2}$; $|C_n| = \frac{1}{\pi(2n-1)}$

$|C_n| = \frac{1}{\pi(2n-1)}$



$|C_1| = \frac{1}{\pi(1)} = 0.31$

$|C_2| = \frac{1}{3\pi}$

$|C_3| = \frac{1}{5\pi}$; $|C_4| = \frac{1}{7\pi}$

5) a) HALLAR LA SERIE DE FOURIER EN FORMA COMPLEJA O EXPONENCIAL PARA LA SIGUIENTE SEÑAL:

$$f(t) = A \operatorname{sen} \pi t \,, \quad 0 < t < 1 \,, \quad f(t+1) = f(t)$$

SOL: $T = 1$; $\omega_0 = \frac{2\pi}{T} = \frac{2\pi}{1} = 2\pi \Rightarrow \omega_0 = 2\pi$.

$$C_0 = \frac{1}{T}\int_0^T f(t)\,dt = \frac{1}{1}\int_0^1 A \operatorname{sen}\pi t\,dt = A\int_0^1 \operatorname{sen}\pi t\,dt$$

$$C_0 = -A\frac{\cos\pi t}{\pi}\Big|_0^1 = -\frac{A}{\pi}\left[\cos\pi - \cos 0\right] = -\frac{A}{\pi}\left[-1-1\right] = \frac{2A}{\pi}$$

$$\therefore\ C_0 = \frac{2A}{\pi}$$

$$C_n = \frac{1}{T}\int_0^T f(t)\, e^{-in\omega_0 t}\,dt \quad \text{si } T=1 \text{ y } \omega_0 = 2\pi$$

$$C_n = \int_0^1 A \operatorname{sen}\pi t\, e^{-in2\pi t}\,dt = A\int_0^1 \operatorname{sen}\pi t\, e^{-in2\pi t}\,dt$$

$$\boxed{\operatorname{sen}\pi t = \frac{e^{i\pi t} - e^{-i\pi t}}{2i}}$$

$$C_n = A\int_0^1 \left[\frac{e^{i\pi t} - e^{-i\pi t}}{2i}\right] e^{-in2\pi t}\,dt$$

$$C_n = \frac{A}{2i}\left[\int_0^1 \left\{ e^{i\pi t} e^{-in2\pi t} - e^{-i\pi t} e^{-in2\pi t} \right\} dt\right]$$

$$C_n = \frac{A}{2i}\left[\int_0^1 e^{i\pi t} e^{-in2\pi t}\,dt - \int_0^1 e^{-i\pi t} e^{-in2\pi t}\,dt\right]$$

$$C_n = \frac{A}{2i}\left[\int_0^1 e^{-i\pi(2n-1)t}\,dt - \int_0^1 e^{-i\pi(2n+1)t}\,dt\right]$$

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$$C_n = \frac{A}{2i}\left[\left.\frac{e^{-i\pi(2n-1)t}}{-i\pi(2n-1)}\right|_0^1 - \left.\frac{e^{-i\pi(2n+1)t}}{-i\pi(2n+1)}\right|_0^1\right]$$

$$C_n = \frac{A}{2i}\left[-\frac{1}{i\pi(2n-1)}\left\{e^{-i\pi(2n-1)} - e^{0}\right\} + \frac{1}{i\pi(2n+1)}\left\{e^{-i\pi(2n+1)} - e^{0}\right\}\right]$$

$$C_n = \frac{A}{2i}\left[-\frac{1}{i\pi(2n-1)}\left\{e^{-i\pi(2n-1)} - 1\right\} + \frac{1}{i\pi(2n+1)}\left\{e^{-i\pi(2n+1)} - 1\right\}\right]$$

$$C_n = \frac{A}{2i}\left[-\frac{1}{i\pi(2n-1)}\left\{e^{-i\pi 2n}\, e^{i\pi} - 1\right\} + \frac{1}{i\pi(2n+1)}\left\{e^{-i\pi 2n}\, e^{-i\pi} - 1\right\}\right]$$

$$e^{\pm i 2\pi n} = 1 \;;\; e^{i\pi} = e^{-i\pi} = \cos\pi + i\,\mathrm{sen}\,\pi = -1 + i(0) = -1$$

$$C_n = \frac{A}{2i}\left[-\frac{1}{i\pi(2n-1)}\left\{(1)(-1) - 1\right\} + \frac{1}{i\pi(2n+1)}\left\{(1)(-1) - 1\right\}\right]$$

$$C_n = \frac{A}{2i}\left[-\frac{1}{i\pi(2n-1)}\left\{-1-1\right\} + \frac{1}{i\pi(2n+1)}\left\{-1-1\right\}\right]$$

$$C_n = \frac{A}{2i}\left[-\frac{1}{i\pi(2n-1)}\{-2\} + \frac{1}{i\pi(2n+1)}\{-2\}\right]$$

$$C_n = \frac{-2A}{2i\,i\pi}\left[-\frac{1}{(2n-1)} + \frac{1}{(2n+1)}\right] = \frac{-2A}{2i^2\pi}\left[-\frac{1}{(2n-1)} + \frac{1}{(2n+1)}\right]$$

$$C_n = \frac{A}{\pi}\left[\frac{-2n-1+2n-1}{4n^2-1}\right] = \frac{A}{\pi}\left[\frac{-2}{(4n^2-1)}\right] = -\frac{2A}{\pi(4n^2-1)}$$

$$\therefore\ C_n = -\frac{2A}{\pi(4n^2-1)}$$

SI $n=0 \Rightarrow$ $C_0 = \frac{-2A}{4\pi n^2-\pi} = \frac{-2A}{4\pi(0)^2-\pi} = -\frac{2A}{-\pi} = \frac{2A}{\pi}$

$C_0 = \frac{2A}{\pi}$ ES EL MISMO RESULTADO CALCULADO ANTERIORMENTE.

LA SERIE DE FOURIER COMPLEJA CON $C_0 = \frac{2A}{\pi}$ y $C_n = \frac{-2A}{\pi(4n^2-1)}$.

$$f(t) = C_0 + \sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} C_n e^{in2\pi t} = \frac{2A}{\pi} + \sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} \frac{-2A}{\pi(4n^2-1)} e^{in2\pi t}$$

$$\therefore\ f(t) = \frac{2A}{\pi} - \frac{2A}{\pi}\sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} \frac{e^{in2\pi t}}{(4n^2-1)}$$

b) CONVERTIR LA SERIE DE FOURIER COMPLEJA A LA SERIE DE FOURIER TRIGONOMÉTRICA. 63

Sol

$$C_0 = \frac{2A}{\pi} \; ; \; C_n = -\frac{2A}{\pi(4n^2-1)} \text{ y } \omega_0 = 2\pi$$

$$C_0 = \frac{a_0}{2} \Rightarrow a_0 = 2C_0 = 2\left(\frac{2A}{\pi}\right) = \frac{4A}{\pi} \quad \therefore a_0 = \frac{4A}{\pi}$$

$$a_n = C_n + C_n^* = -\frac{2A}{\pi(4n^2-1)} - \frac{2A}{\pi(4n^2-1)} = -\frac{4A}{\pi(4n^2-1)}$$

$$\therefore a_n = -\frac{4A}{\pi(4n^2-1)}$$

$$b_n = i\left[C_n - C_n^*\right] = i\left[\frac{-2A}{\pi(4n^2-1)} + \frac{2A}{\pi(4n^2-1)}\right] = i(0) = 0$$

$$\therefore b_n = 0$$

LA SERIE DE FOURIER TRIGONOMÉTRICA:

$$a_0 = \frac{4A}{\pi}, \; a_n = -\frac{4A}{\pi(4n^2-1)}, \; b_n = 0 \text{ y } \omega_0 = 2\pi$$

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left[a_n \cos n\omega_0 t + b_n \operatorname{sen} n\omega_0 t\right]$$

$$f(t) = \frac{2A}{\pi} + \sum_{n=1}^{\infty}\left[\left(\frac{-4A}{\pi(4n^2-1)}\right)\cos n2\pi t + \cancel{0 \operatorname{sen} n2\pi t}\right] = \frac{2A}{\pi} - \frac{4A}{\pi}\sum_{n=1}^{\infty}\frac{\cos n2\pi t}{(4n^2-1)}$$

TAREA 1. SERIES DE FOURIER. T-1-1

1 GRAFÍQUENSE LAS SIGUIENTES SEÑALES.

a $f(t)=|t|$, si $-1\leq t\leq 1$ y $f(t+2)=f(t)$, $T=2$.

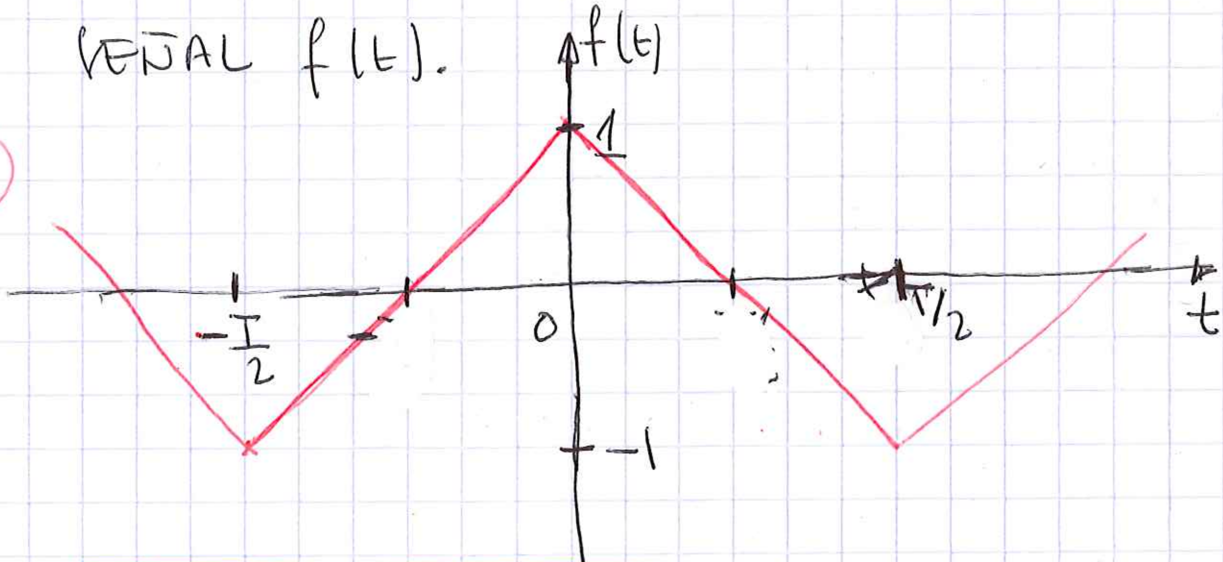
b $f(t)=t$, $-2<t<2$ y $f(t+4)=f(t)$; $T=4$.

c $$f(t)=\begin{cases}\pi & \text{si } -\pi<t<0\\ t & \text{si } 0<t<\pi.\end{cases} \qquad f(t+2\pi)=f(t)$$

d $$f(t)=\begin{cases}0 & \text{si } -\pi<t<0\\ \pi-t & \text{si } 0<t<\pi.\end{cases} \qquad f(t+2\pi)=f(t).$$

2 DE LAS SIGUIENTES GRÁFICAS ENCUENTRE LA SEÑAL $f(t)$.

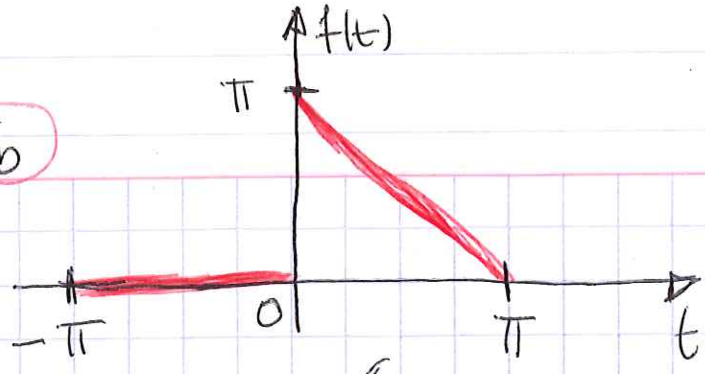
a



JUSTIFICAR LA RESPUESTA.

T-1-2

b



3 DÍGANSE CUÁLES DE LAS SIGUIENTES SEÑALES SON PARES O IMPARES (O NINGUNA DE LAS DOS):

a $f(t) = x^{4}\operatorname{sen} t$ b $f(t) = t + t^{4}$ c $f(t) = |t| \operatorname{sen} t$.

4 a GRAFICAR LA SEÑAL $f(t)$ Y DETERMINAR LA SERIE DE FOURIER TRIGONOMÉTRICA PARA:

$$f(t) = \begin{cases} 0, & -\pi < t < 0 \\ \operatorname{sen} t, & 0 < t < \pi. \end{cases} \quad \nexists \quad T = 2\pi.$$

R $$f(t) = \frac{1}{\pi} + \frac{\operatorname{sen} t}{2} + \sum_{n=2}^{\infty} \frac{1 + (-1)^{n}}{\pi(1-n^{2})} \cos nt$$

b ENCONTRAR LA SERIE DE FOURIER TRIGONOMÉTRICA PARA $f(t) = t$, $t \in [-\pi, \pi]$ CON $T = 2\pi$.

c APLICAR EL FENÓMENO DE GIBBS A $f(t) = t$.

5 HALLAR LA SERIE DE FOURIER PARA:

a $$f(t) = \begin{cases} 0, & -3 \le t \le 0 \\ t, & 0 \le t \le 3. \end{cases}$$

R $f(t) = \frac{3}{4} + \sum_{n=1}^{\infty}\left[\frac{6}{(2n-1)^2\pi^2}\cos\frac{(2n-1)\pi t}{3} - \frac{3(-1)^n}{n\pi}\operatorname{sen}\frac{n\pi t}{3}\right]$ T-1-3

b) GRAFICAR $f(t)$ Y DETERMINAR LA SERIE DE FOURIER.

$$f(t) = \begin{cases} -1, & -\frac{T}{2} < t < 0 \\ 1, & 0 < t < \frac{T}{2} \end{cases}$$

R $f(t) = \frac{4}{\pi}\sum_{n=1}^{\infty} \frac{\operatorname{sen}(2n-1)\omega_0 t}{(2n-1)}$

6) GRAFICAR LA SEÑAL $f(t)$ Y HALLAR LA SERIE DE FOURIER EN COSENOS (MEDIO RANGO).

$$f(t) = \begin{cases} 0, & 0 < t < \pi/2 \\ 1, & \pi/2 < t < \pi \end{cases}$$

R $f(t) = \frac{1}{2} - \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}\operatorname{sen}\frac{n\pi}{2}\cos nt$

7) SEA LA SEÑAL $f(t) = 5\cos 5t$, $0 < t < \pi$. HALLAR LA SERIE DE FOURIER EN SENOS (MEDIO RANGO).

R $f(t) = \frac{20}{\pi}\sum_{n=1}^{\infty}\frac{2n}{(2n)^2-25}\operatorname{sen} 2nt$

8 a HALLAR LA SERIE DE FOURIER COMPLEJA O EXPONENCIAL DE:

T-1-4

$$f(t) = \operatorname{sen}\left(\frac{t}{2}\right), \quad -\pi < t < \pi.$$

R
$$f(t) = \frac{2i}{\pi} \sum_{n=-\infty}^{\infty} \frac{n(-1)^{n+1}}{(1-4n^2)}.$$

b ENCONTRAR LA SERIE DE FOURIER COMPLEJA O EXPONENCIAL DE:

$$f(t) = \operatorname{senh} at, \quad a \neq 0, \quad -c < t < c$$

R
$$f(t) = i\pi \operatorname{senh}(ac) \sum_{n=-\infty}^{\infty} \frac{(-1)^n n}{a^2c^2 + n^2\pi^2} e^{in\pi t/c}$$

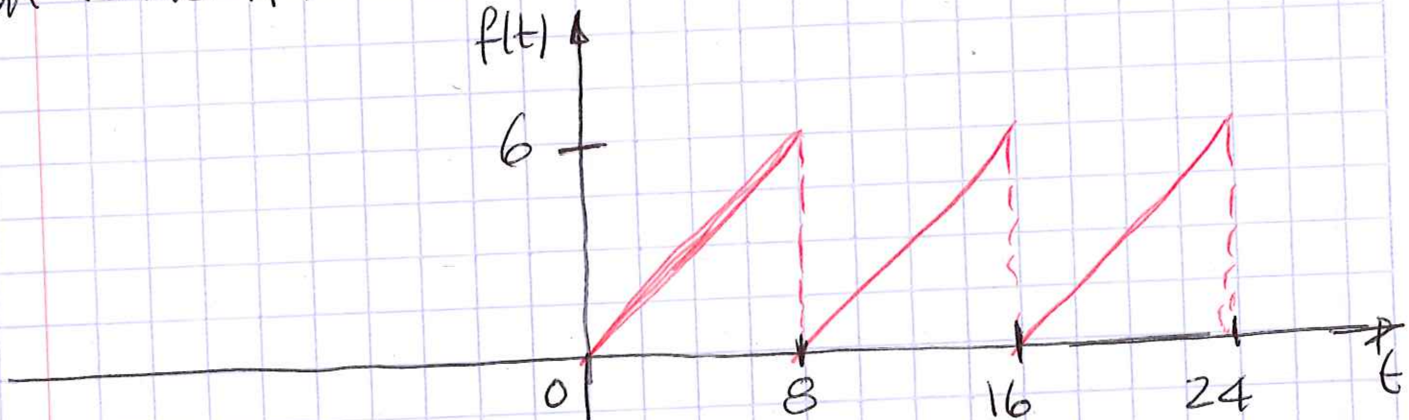
9 a DETERMINAR LA SERIE DE FOURIER COMPLEJA O EXPONENCIAL DE:

$$f(t) = 2t, \quad 0 \leq t < 3 \quad \text{con } T = 3$$

R
$$f(t) = 3 + \frac{3i}{\pi} \sum_{\substack{n=-\infty \\ n \neq 0}}^{\infty} \frac{1}{n} e^{in\frac{2\pi}{3}t}.$$

6) HALLAR LA SERIE DE FOURIER COMPLEJA O EXPONENCIAL DE LA FIGURA SIGUIENTE. ADEMAS DIBUJAR EL ESPECTRO DE FRECUENCIA DE LA SEÑAL SERRUCHO f DE PERIODO $T = 8$.

REDUCIR EL RESULTADO ANTERIOR A LA FORMA DE LA SERIE TRIGONOMÉTRICA DE FOURIER.



R) $$f(t) = 3 + \frac{3i}{\pi} \sum_{\substack{n=-\infty \\ n \neq 0}}^{\infty} \frac{1}{n} e^{in\pi t/4}.$$

$$f(t) = 3 - \frac{6}{\pi} \sum_{n=1}^{\infty} \frac{1}{n} \operatorname{sen} n\omega_0 t \quad \text{si } \omega_0 = \frac{\pi}{4}.$$

T-1-6

10 SEA $f(t)=t$, $-\pi<t<\pi$ CON $T=2\pi$, CUYA SERIE DE FOURIER TRIGONOMÉTRICA ES:

$$f(t)=t=\sum_{n=1}^{\infty}\frac{2(-1)^{n+1}}{n}\operatorname{sen} nt$$

UTILIZANDO LA ECUACIÓN DE PARSEVAL.

$$\frac{1}{\pi}\int_{-\pi}^{\pi}\left[f(t)\right]^2 dt=\frac{a_0^2}{2}+\sum_{n=1}^{\infty}\left[a_n^2+b_n^2\right].$$

DEMOSTRAR QUE:

$$\sum_{n=1}^{\infty}\frac{1}{n^2}=\frac{\pi^2}{6}.$$

- TRANSFORMADA DE FOURIER.

INTRODUCCIÓN

SE HA VISTO QUE SI f ESTÁ DEFINIDA EN $[-c, c]$ O EN $[0, c]$ ALGUNAS VECES ES POSIBLE REPRESENTAR A f EN SERIE DE FOURIER. SI f ESTÁ DEFINIDA EN TODA LA RECTA REAL Y ES PERIÓDICA, AÚN PUEDE SER POSIBLE REPRESENTAR A f COMO UNA SERIE DE FOURIER.

AHORA EL OBJETIVO ES REPRESENTAR A UNA SEÑAL QUE ESTÁ DEFINIDA EN TODA LA RECTA REAL $[-\infty, \infty]$ O EN LA SEMIRRECTA $[0, \infty]$ PERO QUE NO ES NECESARIAMENTE PERIÓDICA. EN ESTAS CIRCUNSTANCIAS LA SERIE DE FOURIER SE REEMPLAZA POR UNA INTEGRAL DE FOURIER.

DEF. LA SEÑAL $F(\omega)$ DEFINIDA POR $F(\omega) = \int_{-\infty}^{\infty} f(t)\, e^{-i\omega t}\, dt$, SE CONOCE COMO LA TRANSFORMADA DE FOURIER DE $f(t)$, Y LA OPERACIÓN DE INTEGRACIÓN SE SIMBOLIZA ALGUNAS VECES POR $\mathcal{F}$; ES DECIR,

$$F(\omega) = \mathcal{F}[f(t)] = \int_{-\infty}^{\infty} f(t)\, e^{-i\omega t}\, dt. \quad (1)$$

TRANSFORMADA DE FOURIER.

INTRODUCCIÓN

SE HA VISTO QUE SI f ESTÁ DEFINIDA EN $[-C, C]$ O EN $[0, C]$ ALGUNAS VECES ES POSIBLE REPRESENTAR A f EN SERIE DE FOURIER. SI f ESTÁ DEFINIDA EN TODA LA RECTA REAL Y ES PERIÓDICA, AÚN PUEDE SER POSIBLE REPRESENTAR A f COMO UNA SERIE DE FOURIER. AHORA EL OBJETIVO ES REPRESENTAR A UNA SEÑAL QUE ESTÁ DEFINIDA EN TODA LA RECTA REAL $[-\infty, \infty]$ O EN LA SEMIRRECTA $[0, \infty]$ PERO QUE NO ES NECESARIAMENTE PERIÓDICA. EN ESTAS CIRCUNSTANCIAS LA SERIE DE FOURIER SE REEMPLAZA POR UNA INTEGRAL DE FOURIER.

DEF. LA SEÑAL $F(\omega)$ DEFINIDA POR $F(\omega) = \int_{-\infty}^{\infty} f(t) e^{-i\omega t} dt$ SE CONOCE COMO LA TRANSFORMADA DE FOURIER DE $f(t)$ Y LA OPERACIÓN DE INTEGRACIÓN SE SIMBOLIZA ALGUNAS VECES POR $\mathcal{F}$; ES DECIR,

$$F(\omega) = \mathcal{F}[f(t)] = \int_{-\infty}^{\infty} f(t) e^{-i\omega t} dt. \quad (1)$$

ANÁLOGAMENTE $\mathcal{F}^{-1}$ ES EL SÍMBOLO QUE SE UTILIZA PARA INDICAR LA OPERACIÓN INVERSA O SEA, OBTENER $f(t)$ CUANDO $F(\omega)$ ESTÁ DADO; ESTO ES,

$$f(t) = \mathcal{F}^{-1}[F(\omega)] = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\omega)\, e^{i\omega t}\, d\omega, \quad (2)$$

Y $f(t)$ SE DENOMINA TRANSFORMADA INVERSA DE FOURIER DE $F(\omega)$. LAS ECUACIONES (1) Y (2) SE CONOCEN COMO PAR DE TRANSFORMADAS DE FOURIER.

$\boxed{\text{COMENTARIO.}}$

SEA $f(t)$ UNA SEÑAL QUE SATISFACE LAS CONDICIONES DE DIRICHLET (VER PÁGINAS 15, 16 Y 17 APUNTES) Y TAL QUE:

$$\int_{-\infty}^{\infty} |f(t)|\, dt < \infty. \quad (\text{i})$$

ES DECIR LA SEÑAL ES ABSOLUTAMENTE INTEGRABLE O BIEN LA INTEGRAL DEL VALOR ABSOLUTO DE $f(t)$ DEBE SER FINITA O CONVERGENTE.

LA TRANSFORMADA DE FOURIER DE UNA SEÑAL $f(t)$, SE

DENOTA COMO $F(w) = \mathcal{F}[f(t)]$ Y SE DEFINE

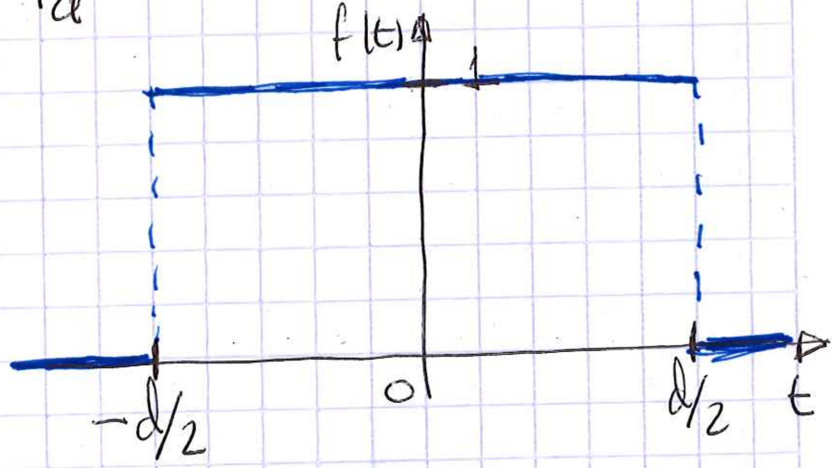
COMO:
$$F(w) = \mathcal{F}[f(t)] = \int_{-\infty}^{\infty} f(t)\, e^{-iwt}\, dt. \quad (ii)$$

LA CONDICIÓN ENUNCIADA EN (i) JUNTO CON LAS CONDICIONES DE DIRICHLET ASEGURA LA EXISTENCIA CONVERGENCIA DE LA INTEGRAL DADA POR (ii) Y POR TANTO LA EXISTENCIA DE LA TRANSFORMADA DE FOURIER PARA UNA SEÑAL QUE LAS SATISFACE.

EJEMPLOS

1. HALLAR LA TRANSFORMADA DE FOURIER DEL PULSO RECTANGULAR $P_d(t)$ DEFINIDA POR:

$$P_d(t) = \begin{cases} 1, & |t| < d/2 \\ 0, & |t| > d/2 \end{cases}$$



SOL

$$\int_{-\infty}^{\infty} |f(t)|\,dt = \int_{-\infty}^{\infty} |P_d(t)|\,dt = \int_{-\infty}^{\infty} P_d(t)\,dt$$

$$= \int_{-\infty}^{-d/2} \cancelto{0}{P_d(t)}\,dt + \int_{-d/2}^{d/2} P_d(t)\,dt + \int_{d/2}^{\infty} \cancelto{0}{P_d(t)}\,dt = \int_{-d/2}^{d/2} (1)\,dt$$

$$= \int_{-d/2}^{d/2} dt = t \Big|_{-d/2}^{d/2} = d/2 - (-d/2) = d/2 + d/2 = \frac{2d}{2} = d$$ 67

$$\therefore \int_{-\infty}^{\infty} |P_d(t)| dt = d < \infty$$ CONVERGE

AHORA:

$$F(w) = \mathcal{F}\left[P_d(t)\right] = \int_{-\infty}^{\infty} P_d(t) e^{-iwt} dt$$

$$= \int_{-d/2}^{d/2} (1) e^{-iwt} dt = \int_{-d/2}^{d/2} e^{-iwt} dt = \frac{1}{-iw} e^{-iwt} \Big|_{-d/2}^{d/2}$$

$$= -\frac{1}{iw}\left[e^{-iwd/2} - e^{iwd/2}\right] = \frac{1}{iw}\left[-e^{-iwd/2} + e^{iwd/2}\right]$$

$$= \frac{1}{iw}\left[e^{iwd/2} - e^{-iwd/2}\right] \frac{2}{2} = \frac{2}{w}\left[\frac{e^{iwd/2} - e^{-iwd/2}}{2i}\right]$$

$$= \frac{2}{w} \operatorname{sen}\left(\frac{wd}{2}\right) = d \frac{\operatorname{sen}\left(\frac{wd}{2}\right)}{\left(\frac{wd}{2}\right)}$$

$$\therefore F(w) = d \frac{\operatorname{sen}\left(\frac{wd}{2}\right)}{\left(\frac{wd}{2}\right)}$$

RECUERDE QUE:

$$F(w) = d\,\frac{\operatorname{sen}\left(\frac{wd}{2}\right)}{\left(\frac{wd}{2}\right)} + i0 = \left(d\,\frac{\operatorname{sen}\left(\frac{wd}{2}\right)}{\left(\frac{wd}{2}\right)},\ 0\right)$$

• ESPECTRO DE AMPLITUD $|F(w)|$.

$$|F(w)| = \sqrt{\left(\frac{d\operatorname{sen}\left(\frac{wd}{2}\right)}{\left(\frac{wd}{2}\right)}\right)^2 + 0^2} = \frac{d\operatorname{sen}\left(\frac{wd}{2}\right)}{\left(\frac{wd}{2}\right)}$$

$$\therefore |F(w)| = \frac{d\operatorname{sen}\left(\frac{wd}{2}\right)}{\left(\frac{wd}{2}\right)}$$

PARA TABULAR LA ALTURA MAXIMA DE $|F(w)|$; ESTO ES,

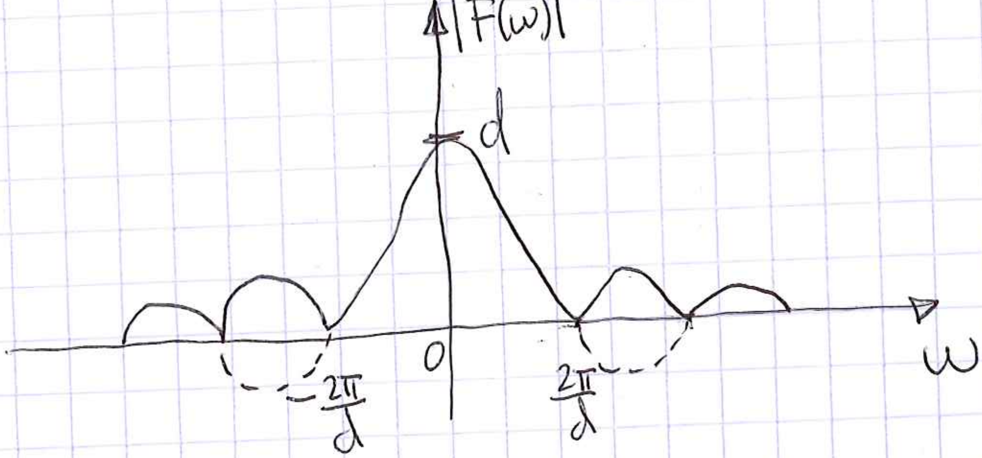
$$\text{Si } w = 0 \Rightarrow |F(0)| = \frac{d\cdot\operatorname{sen}0}{0} = \frac{0}{0} \text{ INDETERMINACIÓN}$$

POR HOSPITAL

$$|F'(w)| = \frac{d\cos\left(\frac{wd}{2}\right)\left(\frac{d}{2}\right)}{\left(\frac{d}{2}\right)} = d\cos\left(\frac{wd}{2}\right)$$

$$|F(0)| = d\cos 0° = d(1) = d \quad \therefore |F(0)| = d$$

LA LÍNEA CONTINUA ES EL ESPECTRO DE AMPLITUD $|F(\omega)|$ Y LA LÍNEA PUNTEADA ES $F(\omega)$.



② ENCONTRAR LA TRANSFORMADA DE FOURIER SI:

$$f(t) = \begin{cases} 0, & -\infty < t < 0 \\ e^{-at}, & 0 \le t < \infty. \end{cases}$$

SOL

$$\mathcal{F}[f(t)] = \int_{-\infty}^{\infty} f(t)\, e^{-i\omega t}\, dt = \cancel{\int_{-\infty}^{0} 0\, dt} + \int_{0}^{\infty} e^{-at} e^{-i\omega t}\, dt$$

$$= \int_{0}^{\infty} e^{-(a+i\omega)t}\, dt = -\frac{1}{a+i\omega}\, e^{-(a+i\omega)t}\Big|_{0}^{\infty}$$

$$= -\frac{1}{a+i\omega}\left[\cancel{e^{-\infty}} - e^{0}\right] = -\frac{1}{a+i\omega}\left[-1\right] = \frac{1}{a+i\omega}$$

$$\therefore\ \mathcal{F}[f(t)] = \frac{1}{a+i\omega} = F(\omega)$$

AHORA:

$$F(\omega) = \frac{1}{a+i\omega} = \frac{1}{a+i\omega}\,\frac{a-i\omega}{a-i\omega} = \frac{a-i\omega}{a^2-i^2\omega^2} = \frac{a-i\omega}{a^2+\omega^2}$$

$$F(\omega) = \frac{a}{a^2+\omega^2} - i\,\frac{\omega}{a^2+\omega^2} = \Big(\overbrace{\frac{a}{a^2+\omega^2}}^{x},\ \overbrace{-\frac{\omega}{a^2+\omega^2}}^{y}\Big).\ (*).$$

EL ESPECTRO DE AMPLITUD ES:

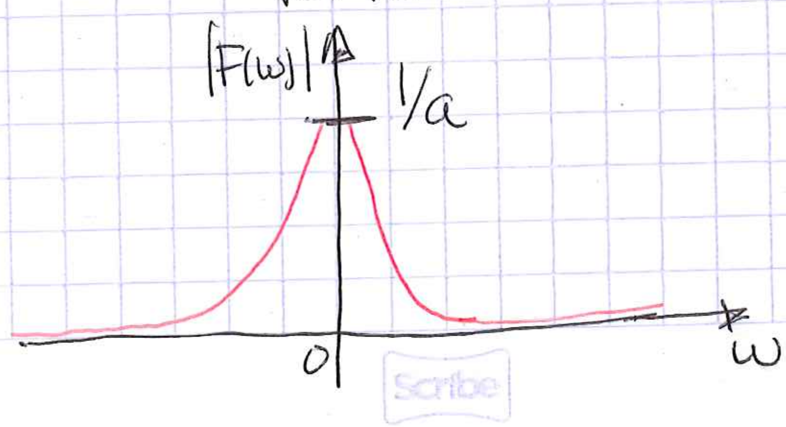
$$|F(\omega)| = \sqrt{\frac{a^2}{(a^2+\omega^2)^2} + \frac{\omega^2}{(a^2+\omega^2)^2}} = \sqrt{\frac{a^2+\omega^2}{(a^2+\omega^2)^2}}$$

$$= \sqrt{\frac{1}{(a^2+\omega^2)}} = \frac{1}{\sqrt{a^2+\omega^2}}$$

$$\therefore\ |F(\omega)| = \frac{1}{\sqrt{a^2+\omega^2}}$$

ALTURA MÁXIMA

- Si $\omega = 0 \Rightarrow |F(0)| = \dfrac{1}{\sqrt{a^2+0^2}} = \dfrac{1}{\sqrt{a^2}} = \dfrac{1}{a}$

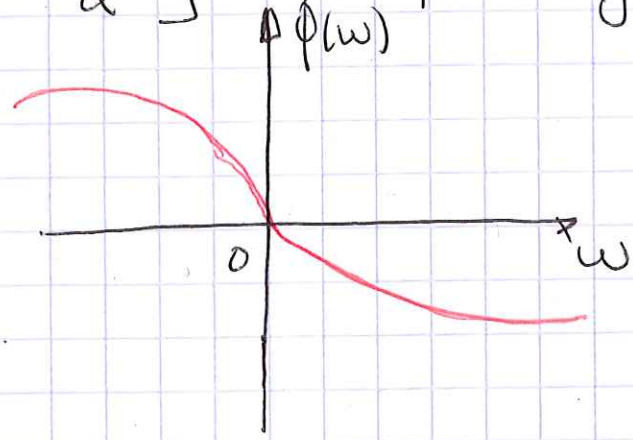


ESPECTRO DE FASE $\phi(\omega)$

DE (*):

$$\phi(\omega) = tg^{-1}\left[\frac{y}{x}\right] = tg^{-1}\left[\frac{-\frac{\omega}{a^2+\omega^2}}{\frac{a}{a^2+\omega^2}}\right]$$

$$= tg^{-1}\left[-\frac{\omega}{a}\right] \quad \therefore \phi(\omega) = -tg\left[\frac{\omega}{a}\right]$$

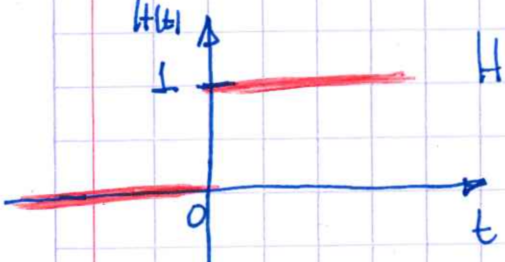


COMENTARIO

PARA JUSTIFICAR $f(t) = \begin{cases} 0, & -\infty < t < 0 \\ e^{-at}, & 0 \le t < \infty \end{cases}$ UTILIZA-

REMOS UNA SEÑAL AUXILIAR LLAMADA DE HEAVISIDE $H(t)$

DEFINIDA COMO:



$$H(t) = \begin{cases} 0, & t < 0 \\ 1, & t \ge 0 \end{cases} = \begin{cases} 0, & -\infty < t < 0 \\ 1, & 0 \le t < \infty \end{cases}$$

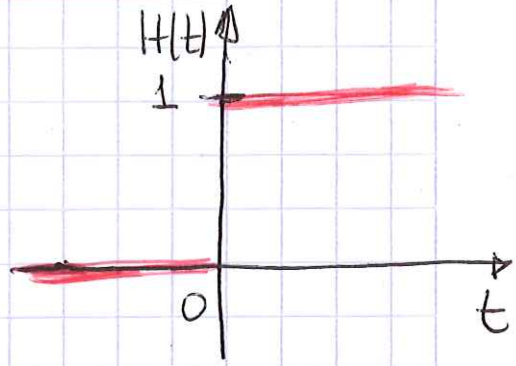
$$f(t) = e^{-at}H(t) = e^{at}\begin{cases} 0, & -\infty < t < 0 \\ 1, & 0 \le t < \infty \end{cases} = \begin{cases} 0, & -\infty < t < 0 \\ e^{-at}, & 0 \le t < \infty \end{cases}$$

ESTÁ JUSTIFICADO.

ALGUNAS SEÑALES.

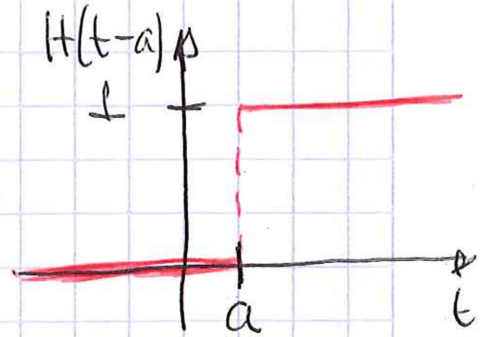
SEÑAL HEAVISIDE

$$H(t)=\begin{cases}0, & t<0\\ 1, & t\geq 0\end{cases}=\begin{cases}0, & -\infty<t<0\\ 1, & 0\leq t<\infty.\end{cases}$$

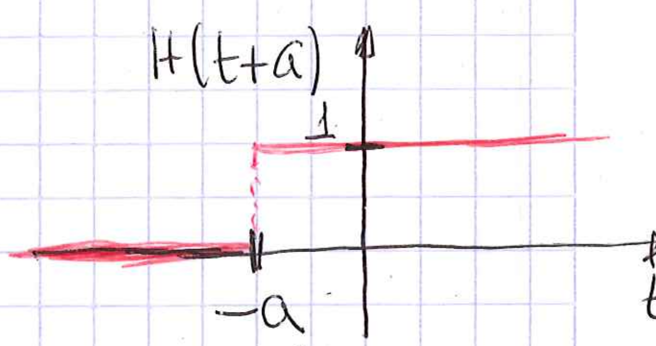


HEAVISIDE CON DESPLAZAMIENTO.

$$H(t-a)=\begin{cases}0, & t<a\\ 1, & t\geq a.\end{cases}$$

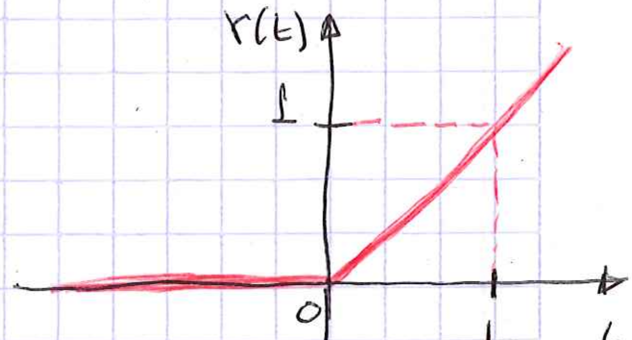


$$H(t+a)=\begin{cases}0, & t<-a\\ 1, & t\geq -a.\end{cases}$$



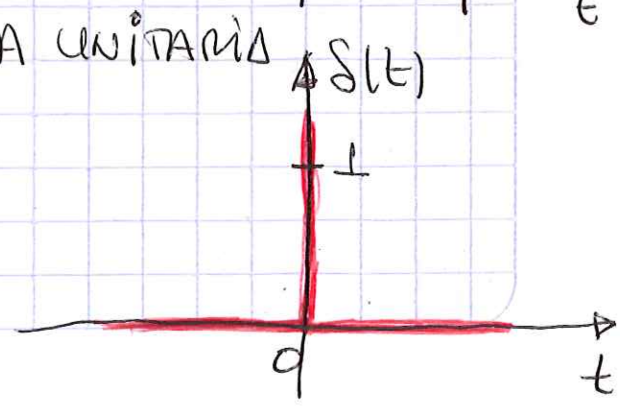
SEÑAL RAMPA UNITARIA.

$$r(t)=\begin{cases}0, & -\infty<t<0\\ t, & 0\leq t<\infty.\end{cases}$$

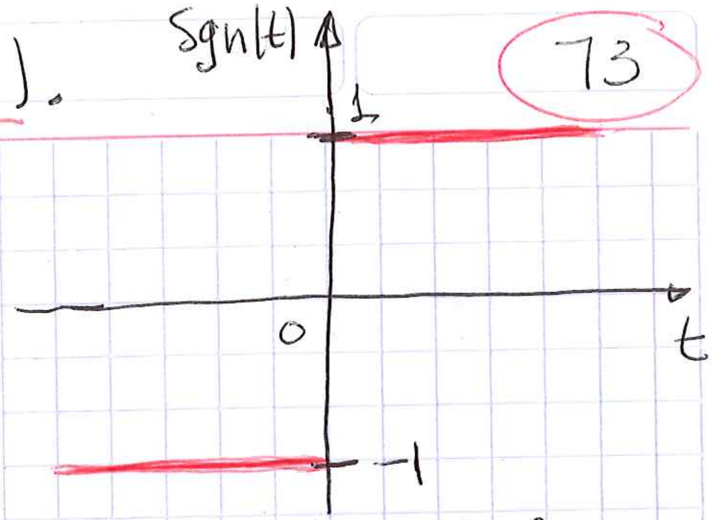


SEÑAL IMPULSO UNITARIO O DELTA UNITARIA

$$\delta(t)=\begin{cases}0 \text{ si } t\neq 0\\ \infty \text{ si } t=0.\end{cases}$$



• SEÑAL SIGNO $\mathrm{Sgn}(t)$.



$$\mathrm{Sgn}(t) = \begin{cases} -1, & t<0 \\ 1, & 0<t \end{cases}$$

• PROPIEDADES DE LA TRANSFORMADA DE FOURIER.

LINEALIDAD.

TEOREMA 1: Si α y $\beta \in \mathbb{R}$:

$$\mathcal{F}[\alpha f(t) + \beta g(t)] = \mathcal{F}[\alpha f(t)] + \mathcal{F}[\beta g(t)]$$

$$= \alpha \mathcal{F}[f(t)] + \beta \mathcal{F}[g(t)] = \alpha F(\omega) + \beta G(\omega).$$

SUPONGA QUE LAS TRANSFORMADAS DE FOURIER f Y g EXISTEN.

POR OTRO LADO:

$$\mathcal{F}^{-1}[\alpha F(\omega) + \beta G(\omega)] = \alpha \mathcal{F}^{-1}[F(\omega)] + \beta \mathcal{F}^{-1}[G(\omega)]$$

$$= \alpha f(t) + \beta g(t).$$

CORRIMIENTO DEL TIEMPO.

TEOREMA 2: Si $\mathcal{F}[f(t)] = F(\omega) \Rightarrow$

$$\mathcal{F}[f(t-t_0)] = e^{-i\omega_0 t_0} F(\omega).$$

- CORRIMIENTO O DESPLAZAMIENTO DE FRECUENCIA.

TEOREMA 3. Si $\mathfrak{F}[f(t)] = F(\omega) \Rightarrow \mathfrak{F}[e^{i\omega_0 t} f(t)] = F(\omega - \omega_0)$, TAL QUE ω_0 ES UNA CONSTANTE REAL.

- ESCALA.

TEOREMA 4. Si $\mathfrak{F}[f(t)] = F(\omega)$ Y a ES UNA CONSTANTE REAL DISTINTA DE CERO.

$$\mathfrak{F}[f(at)] = \frac{1}{|a|} F\left(\frac{\omega}{a}\right).$$

- INVERSIÓN DEL TIEMPO.

TEOREMA 5. Si $\mathfrak{F}[f(t)] = F(\omega) \Rightarrow \mathfrak{F}[f(-t)] = F(-\omega)$.

- SIMETRÍA.

TEOREMA 6. Si $\mathfrak{F}[f(t)] = F(\omega) \Rightarrow \mathfrak{F}[f(t)] = 2\pi f(-\omega)$.

- DIFERENCIACIÓN EN EL TIEMPO.

TEOREMA 7. Si $\mathfrak{F}[f(t)] = F(\omega)$ y $f(t) \to 0$ CUANDO $t \to \pm\infty$ ENTONCES $\mathfrak{F}[f'(t)] = i\omega F(\omega) = i\omega\, \mathfrak{F}[f(t)]$.

• DIFERENCIACIÓN EN LA FRECUENCIA

TEOREMA 8 Si $F(w) = \mathcal{F}[f(t)] \Rightarrow \mathcal{F}[-itf(t)] = \dfrac{dF(w)}{dw}$.

• DEMOSTRAR EL TEOREMA 2 (CORRIMIENTO DEL TIEMPO).

Si $\mathcal{F}[f(t)] = F(w) \Rightarrow \mathcal{F}[f(t-t_0)] = e^{-iw_0t_0}F(w)$.

DEMOSTRACIÓN.

HECHO: $\mathcal{F}[f(t)] = \int_{-\infty}^{\infty} f(t)\, e^{-iwt}\, dt$.

CAMBIO DE VARIABLE.

$x = t - t_0 \,;\; t = x + t_0 \Rightarrow dt = dx$

SUSTITUYENDO EN EL HECHO.

$$\mathcal{F}[f(t-t_0)] = \int_{-\infty}^{\infty} f(t-t_0)\, e^{-iwt}\, dt = \int_{-\infty}^{\infty} f(x)\, e^{-iw(x+t_0)}\, dx$$

$$= \int_{-\infty}^{\infty} e^{-iwx}\, e^{-iwt_0}\, dx = e^{-iwt_0}\int_{-\infty}^{\infty} f(x)\, e^{-iwx}\, dx$$

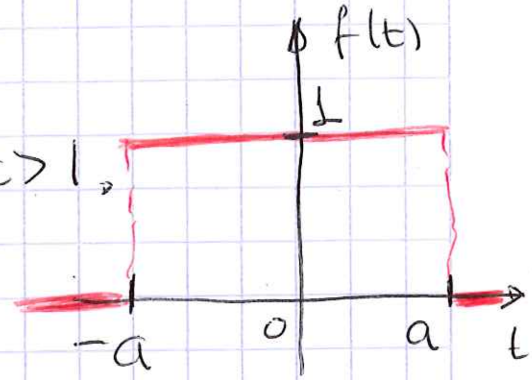
$$= e^{-iwt_0}\int_{-\infty}^{\infty} f(x)\, e^{-iwx}\, dx = e^{-iwt_0}F(w)$$

$$\therefore \mathcal{F}[f(t-t_0)] = e^{-iwt_0}F(w).$$

1) HALLAR LA TRANSFORMADA DE FOURIER PARA:

a)

$$f(t) = \begin{cases} 1, & |t|<a \;;\; -a<t<a \\ 0, & |t|>a \;;\; t<-1 \text{ o } t>1. \end{cases}$$



SOL

$$F(w) = \int_{-\infty}^{\infty} f(t)\, e^{-iwt}\, dt = \int_{-a}^{a} (1)\, e^{-iwt}\, dt$$

$$= \int_{-a}^{a} e^{-iwt}\, dt = -\frac{1}{iw}\, e^{-iwt}\Big|_{-a}^{a} = -\frac{1}{iw}\left[e^{-iwa} - e^{iwa}\right]$$

$$= \frac{-e^{-iwa} + e^{iwa}}{iw} = \frac{e^{iw} - e^{-iwa}}{iw}$$

$$= \frac{\cos wa + i\,\mathrm{sen}\, wa - \cos wa + i\,\mathrm{sen}\, wa}{iw} = \frac{2i\,\mathrm{sen}\, wa}{iw}$$

$$= \frac{2\,\mathrm{sen}\, wa}{w} \qquad \therefore F(w) = \frac{2\,\mathrm{sen}\, wa}{w} \text{ si } w \neq 0$$

b) CALCULAR $|F(w)|$.

SOL

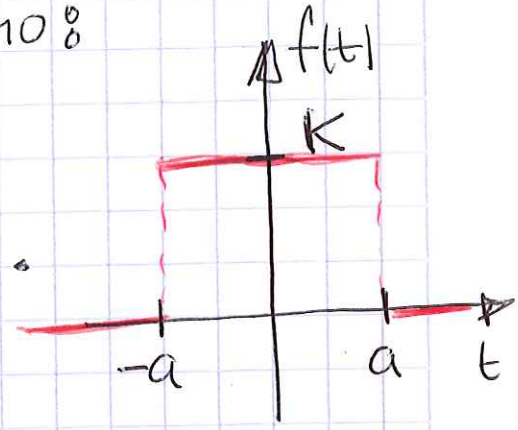
$$|F(w)| = \sqrt{\left(\frac{2\,\mathrm{sen}\, wa}{w}\right)^2 + 0^2} = \frac{2\,\mathrm{Sen}\, wa}{w}$$

$$\therefore |F(w)| = \frac{2\,\mathrm{sen}\, wa}{w}$$

POR HOSPITAL SI $w=0 \Rightarrow |F(w)| = 2a.$

2 EXPRESAR LA SEÑAL $f(t)$ EN TÉRMINOS DE LA SEÑAL DE HEAVISIDE $H(t)$ COMO:

$$f(t) = K\begin{cases} 1, & -a \le t < a \\ 0, & t < -a \text{ y } t \ge a. \end{cases}$$

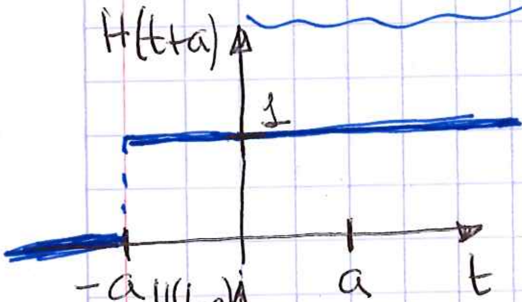


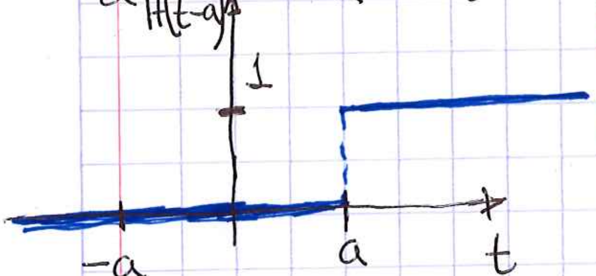
$$f(t) = K\left[H(t+a) - H(t-a)\right]$$

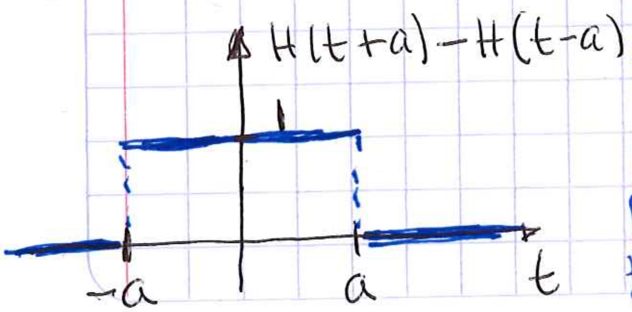
10L

HECHO:

$$H(t+a) = \begin{cases} 0, & t < -a \\ 1, & t \ge -a \end{cases} \quad y \quad H(t-a) = \begin{cases} 0, & t < a \\ 1, & t \ge 1. \end{cases}$$







$$H(t+a) - H(t-a) = \begin{cases} 0-0=0; & t < -a \\ 1-0=1; & -a < t < a \\ 1-1=0; & t > a \end{cases}$$

$$H(t+a) - H(t-a) = \begin{cases} 1, & -a \le t < a \\ 0, & t < -a \text{ y } t \ge a \end{cases}$$

FINALMENTE:

$$f(t) = K\left[H(t+a) - H(t-a)\right] = K\begin{cases} 1, & -a \le t < a \\ 0, & t < -a \text{ y } t \ge a \end{cases}$$

• RESOLVIENDO LA SIGUIENTE ECUACIÓN DIFEREN-CIAL, UTILIZANDO EL MÉTODO DE LA TRANSFORMADA DE FOURIER. 78

$$y'' + 7y' + 6y = \delta(t)$$

HECHOS:

$$\mathcal{F}[y'] = i\omega F(\omega) = i\omega \Psi(\omega)$$

$$\mathcal{F}[y''] = -\omega^2 F(\omega) = -\omega^2 \Psi(\omega)$$

$$\mathcal{F}[y] = \Psi(\omega).$$

SOL:

$$y'' + 7y' + 6y = \delta(t)$$

$$\mathcal{F}[y''] + 7\,\mathcal{F}[y'] + 6\,\mathcal{F}[y] = \mathcal{F}[\delta(t)]$$

POR TABLAS: $\mathcal{F}[\delta(t)] = 1.$

APLICANDO EL HECHO.

$$-\omega^2 \Psi(\omega) + 7i\,\Psi(\omega) + 6\Psi(\omega) = 1$$

$$-\omega^2 \Psi(\omega) + 7i\,\Psi(\omega) + 6\Psi(\omega) = 1$$

$$\Psi(\omega)\,(-\omega^2 + 7i\omega + 6) = 1$$

$$\Psi(\omega) = \frac{1}{-\omega^2 + 7i\omega + 6} = \frac{1}{(6+i\omega)(1+i\omega)} = \frac{A}{(6+i\omega)} + \frac{B}{(1+i\omega)}$$

$$Y(w) = \frac{A}{6+iw} + \frac{B}{1+iw} \quad (*)$$

$$A = (6+iw)\left.\frac{1}{(6+iw)(1+iw)}\right|_{iw=-6} = \left.\frac{1}{1+iw}\right|_{iw=-6} = \frac{1}{1-6} = -\frac{1}{5}$$

$$\therefore A = -\frac{1}{5}$$

$$B = (1+iw)\left.\frac{1}{(6+iw)(1+iw)}\right|_{iw=-1} = \left.\frac{1}{6+iw}\right|_{iw=-1} = \frac{1}{6-1} = \frac{1}{5}$$

$$\therefore B = \frac{1}{5}$$

SUSTITUYENDO A y B EN (*):

$$Y(w) = -\frac{1}{5}\frac{1}{(6+iw)} + \frac{1}{5}\frac{1}{1+iw}$$

$$y(t) = \frac{1}{5}\left\{ \mathcal{F}^{-1}\left[-\frac{1}{6+iw}\right] + \mathcal{F}^{-1}\left[\frac{1}{1+iw}\right] \right\}$$

$$\boxed{\text{POR TABLAS: } e^{-\alpha t}H(t) \longleftrightarrow \frac{1}{\alpha+iw}}$$

$$y(t) = \frac{1}{5}\left[-e^{-6t}H(t) + e^{-t}H(t)\right] = \frac{1}{5}\left[-e^{-6t} + e^{-t}\right]H(t)$$

$$\therefore y(t) = \frac{1}{5}\left[-e^{-6t} + e^{-t}\right]H(t)$$

PROPIEDADES DE LA TRANSFORMADA DE FOURIER

E
APENDICE

Las funciones son periódicas con período T, $a > 0$; b, t_o y $\omega_o = 2\pi/T$, son constantes reales, con $n = 1, 2, \cdots$.

$f(t)$	$F(\omega)$
$a_1 f_1(t) + a_2 f_2(t)$	$a_1 F_1(\omega) + a_2 F_2(\omega)$
$f(at)$	$\frac{1}{\lvert a\rvert} F\left(\frac{\omega}{a}\right)$
$f(-t)$	$F(-\omega)$
$f(t - t_0)$	$F(\omega)\, e^{-j\omega t_0}$
$f(t)\, e^{j\omega_0 t}$	$F(\omega - \omega_0)$
$f(t) \cos \omega_0 t$	$\frac{1}{2} F(\omega - \omega_0) + \frac{1}{2} F(\omega + \omega_0)$
$f(t) \operatorname{sen} \omega_0 t$	$\frac{1}{2j} F(\omega - \omega_0) - \frac{1}{2j} F(\omega + \omega_0)$
$f_e(t) = \frac{1}{2}[f(t) + f(-t)]$	$R(\omega)$
$f_o(t) = \frac{1}{2}[f(t) - f(-t)]$	$jX(\omega)$
$f(t) = f_e(t) + f_o(t)$	$F(\omega) = R(\omega) + jX(\omega)$
$F(t)$	$2\pi f(-\omega)$
$f'(t)$	$j\omega F(\omega)$
$f^{(n)}(t)$	$(j\omega)^n F(\omega)$
$\int_{-\infty}^{t} f(x)\, dx$	$\frac{1}{j\omega} F(\omega) + \pi F(0)\delta(\omega)$
$-jt f(t)$	$F'(\omega)$
$(-jt)^n f(t)$	$F^{(n)}(\omega)$
$f_1(t) * f_2(t) = \int_{-\infty}^{\infty} f_1(x) f_2(t-x)\, dx$	$F_1(\omega) F_2(\omega)$

$f(t)$	$F(\omega)$
$f_1(t)\, f_2(t)$	$\dfrac{1}{2\pi} F_1(\omega) * F_2(\omega) = \dfrac{1}{2\pi}\displaystyle\int_{-\infty}^{\infty} F_1(y) F_2(\omega - y)\, dy$
$e^{-at}u(t)$	$\dfrac{1}{j\omega + a}$
$e^{-a\|t\|}$	$\dfrac{2a}{a^2 + \omega^2}$
e^{-at^2}	$\sqrt{\dfrac{\pi}{a}}\, e^{-\omega^2/(4a)}$
$p_a(t) = \begin{cases} 1 & \text{para } \|t\| < a/2 \\ 0 & \text{para } \|t\| > a/2 \end{cases}$	$a \dfrac{\operatorname{sen}\left(\dfrac{\omega a}{2}\right)}{\left(\dfrac{\omega a}{2}\right)}$
$\dfrac{\operatorname{sen} at}{\pi t}$	$p_{2a}(\omega)$
$te^{-at}u(t)$	$\dfrac{1}{(j\omega + a)^2}$
$\dfrac{t^{n-1}}{(n-1)!} e^{-at}u(t)$	$\dfrac{1}{(j\omega + a)^n}$
$e^{-at} \operatorname{sen} bt\, u(t)$	$\dfrac{b}{(j\omega + a)^2 + b^2}$
$e^{-at} \cos bt\, u(t)$	$\dfrac{j\omega + a}{(j\omega + a)^2 + b^2}$
$\dfrac{1}{a^2 + t^2}$	$\dfrac{\pi}{a} e^{-a\|\omega\|}$
$\dfrac{\cos bt}{a^2 + t^2}$	$\dfrac{\pi}{2a}\left[e^{-a\|\omega-b\|} + e^{-a\|\omega+b\|}\right]$
$\dfrac{\operatorname{sen} bt}{a^2 + b^2}$	$\dfrac{\pi}{2aj}\left[e^{-a\|\omega-b\|} - e^{-a\|\omega+b\|}\right]$
$\delta(t)$	1
$\delta(t - t_0)$	$e^{-j\omega t_0}$
$\delta'(t)$	$j\omega$
$\delta^{(n)}(t)$	$(j\omega)^n$
$u(t)$	$\pi\delta(\omega) + \dfrac{1}{j\omega}$
$u(t - t_0)$	$\pi\delta(\omega) + \dfrac{1}{j\omega} e^{-j\omega t_0}$
1	$2\pi\delta(\omega)$
t	$2\pi j\, \delta'(\omega)$
t^n	$2\pi j^n \delta^{(n)}(\omega)$

$f(t)$	$F(\omega)$
$e^{j\omega_0 t}$	$2\pi\delta(\omega-\omega_0)$
$\cos\omega_0 t$	$\pi[\delta(\omega-\omega_0)+\delta(\omega+\omega_0)]$
$\operatorname{sen}\omega_0 t$	$-j\pi[\delta(\omega-\omega_0)-\delta(\omega+\omega_0)]$
$\operatorname{sen}\omega_0 t\, u(t)$	$\dfrac{\omega_0}{\omega_0^2-\omega^2}+\dfrac{\pi}{2j}[\delta(\omega-\omega_0)-\delta(\omega+\omega_0)]$
$\cos\omega_0 t\, u(t)$	$\dfrac{j\omega}{\omega_0^2-\omega^2}+\dfrac{\pi}{2}[\delta(\omega-\omega_0)+\delta(\omega+\omega_0)]$
$t\, u(t)$	$j\pi\,\delta'(\omega)-\dfrac{1}{\omega^2}$
$\dfrac{1}{t}$	$\pi j-2\pi j\; u(\omega)$
$\dfrac{1}{t^n}$	$\dfrac{(-j\omega)^{n-1}}{(n-1)!}[\pi j-2\pi j\; u(\omega)]$
$\operatorname{sgn} t$	$\dfrac{2}{j\omega}$
$\delta_T(t)=\sum_{n=-\infty}^{\infty}\delta(t-nT)$	$\omega_0\delta_{\omega_0}(\omega)=\omega_0\sum_{n=-\infty}^{\infty}\delta(\omega-n\omega_0)$

Otras propiedades:

$$\int_{-\infty}^{\infty} f_1(t)\, f_2(t)\, dt=\frac{1}{2\pi}\int_{-\infty}^{\infty} F_1(\omega)\, F_2^*(\omega)\, d\omega,$$

$$\int_{-\infty}^{\infty} |f(t)|^2\, dt=\frac{1}{2\pi}\int_{-\infty}^{\infty} |F(\omega)|^2\, d\omega,$$

$$\int_{-\infty}^{\infty} f(x)\, G(x)\, dx=\int_{-\infty}^{\infty} F(x)\, g(x)\, dx.$$

Tabla 18-1 Algunas parejas familiares de transformada de Fourier

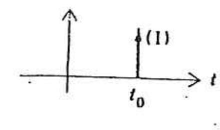
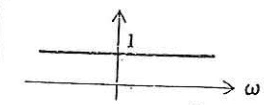
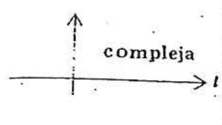
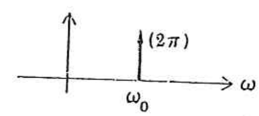
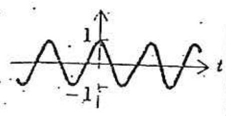
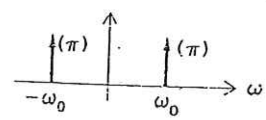
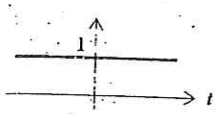
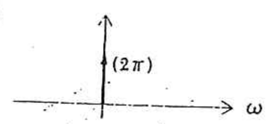
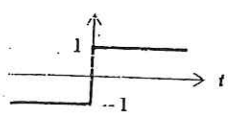
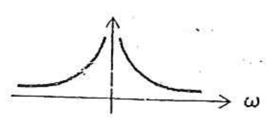
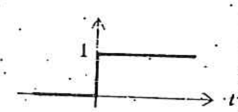
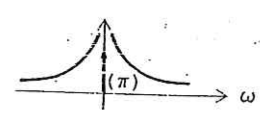
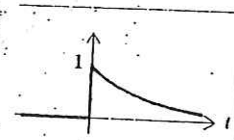
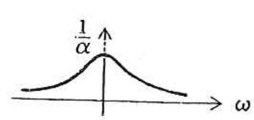
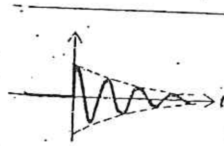
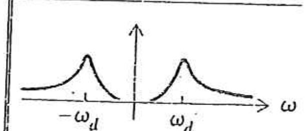
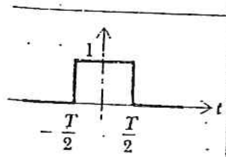
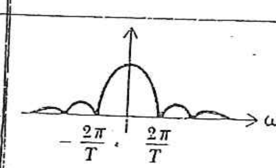
$f(t)$	$f(t)$	$\mathfrak{F}\{f(t)\} = \mathbf{F}(j\omega)$	$\lvert\mathbf{F}(j\omega)\rvert$
	$\delta(t - t_0)$	$e^{-j\omega t_0}$	
	$e^{j\omega_0 t}$	$2\pi\delta(\omega - \omega_0)$	
	$\cos \omega_0 t$	$\pi[\delta(\omega + \omega_0) + \delta(\omega - \omega_0)]$	
	1	$2\pi\,\delta(\omega)$	
	$\mathrm{sgn}(t)$	$\dfrac{2}{j\omega}$	
	$u(t)$	$\pi\,\delta(\omega) + \dfrac{1}{j\omega}$	
	$e^{-\alpha t}u(t)$	$\dfrac{1}{\alpha + j\omega}$	
	$e^{-\alpha t}\cos \omega_d t \cdot u(t)$	$\dfrac{\alpha + j\omega}{(\alpha + j\omega)^2 + \omega_d^2}$	
	$u(t + \frac{1}{2}T) - u(t - \frac{1}{2}T)$	$T\dfrac{\operatorname{sen}\frac{\omega T}{2}}{\frac{\omega T}{2}}$	

TABLA 18.2 Transformadas finitas de Fourier en cosenos

$f(x)$	$C_n\{f(x)\} = F_C(n)$
1	$\begin{cases} 0 & \text{si } n = 1, 2, 3, \ldots \\ \pi & \text{si } n = 0 \end{cases}$
$\begin{cases} 1, & \text{si } 0 \le x \le k \\ -1, & \text{si } k < x < \pi \end{cases}$	$\begin{cases} \dfrac{2}{n}\operatorname{sen}(nk) & \text{si } n = 1, 2, 3, \ldots \\ 2k - \pi & \text{si } n = 0 \end{cases}$
x	$\begin{cases} \dfrac{-[1-(-1)^n]}{n^2} & \text{si } n = 1, 2, 3, \ldots \\ \dfrac{\pi^2}{2} & \text{si } n = 0 \end{cases}$
$\dfrac{x^2}{2\pi}$	$\begin{cases} \dfrac{(-1)^n}{n} & \text{si } n = 1, 2, 3, \ldots \\ \dfrac{\pi^2}{6} & \text{si } n = 0 \end{cases}$
x^3	$\begin{cases} 3\pi^2\dfrac{(-1)^n}{n^2} + 6\dfrac{[1-(-1)^n]}{n^4} & \text{si } n = 1, 2, 3, \ldots \\ \dfrac{\pi^2}{4} & \text{si } n = 0 \end{cases}$
e^{ax}	$\left(\dfrac{(-1)^n e^{a\pi} - 1}{n^2 + a^2}\right)a$
$\operatorname{sen}(ax), \quad a \neq \text{entero}$	$\left(\dfrac{(-1)^n \cos(a\pi) - 1}{n^2 - a^2}\right)a$
$\operatorname{sen}(kx), \quad k = 1, 2, \ldots$	$\begin{cases} \dfrac{(-1)^{n+k} - 1}{n^2 - k^2}\,k & \text{si } n \neq k \\ 0 & \text{si } n = k \end{cases}$
$\cos(kx), \quad k = 1, 2, 3, \ldots$	$\begin{cases} 0 & \text{si } n \neq k \\ \dfrac{\pi}{2} & \text{si } n = k \end{cases}$
$f(\pi - x)$	$(-1)^n F_C(n)$
$\dfrac{1}{\pi}\dfrac{1-k^2}{1+k^2-2k\cos(x)}$	$k^n \quad (\lvert k\rvert < 1)$
$-\dfrac{1}{\pi}\ln[1+k^2-2k\cos(x)]$	$\dfrac{1}{n}k^n \quad (\lvert k\rvert < 1)$
$\dfrac{\cosh[k(\pi - x)]}{k\operatorname{senh}(k\pi)}$	$\dfrac{1}{k^2+n^2} \quad (k \neq 0)$
$\dfrac{1}{\pi}\dfrac{\operatorname{senh}(y)}{\cosh(y) - \cos(x)}$	$e^{-ny} \quad (y > 0)$
$\dfrac{1}{\pi}[y - \ln(2\cosh(y) - 2\cos(x))]$	$\dfrac{1}{n}e^{-ny} \quad (y > 0)$

, 2, 3, . . .

TABLA 18.3 Transformadas de Fourier en senos

$f(x)$	$\mathscr{F}_S\{f(x)\} = F_S(\omega)$
$\dfrac{1}{x}$	$\begin{cases} \dfrac{\pi}{2} & \text{si} \quad \omega > 0 \\ -\dfrac{\pi}{2} & \text{si} \quad \omega < 0 \end{cases}$
$x^{r-1} \quad (0 < r < 1)$	$\Gamma(r)\operatorname{sen}\left(\dfrac{\pi r}{2}\right)\omega^{-r}$
$\dfrac{1}{\sqrt{x}}$	$\sqrt{\dfrac{\pi}{2}}\,\omega^{-1/2}$
$e^{-ax} \quad (a > 0)$	$\dfrac{\omega}{a^2 + \omega^2}$
$xe^{-ax} \quad (a > 0)$	$\dfrac{2a\omega}{(a^2 + \omega^2)^2}$
$xe^{-a^2x^2} \quad (a > 0)$	$\dfrac{\sqrt{\pi}}{4}\,a^{-3}\omega e^{-\omega^2/4a^2}$
$x^{-1}e^{-ax} \quad (a > 0)$	$\tan^{-1}\left(\dfrac{\omega}{a}\right)$
$\dfrac{x}{a^2 + x^2} \quad (a > 0)$	$\dfrac{\pi}{2}\,e^{-a\omega}$
$\dfrac{x}{(a^2 + x^2)^2} \quad (a > 0)$	$2^{-3/2}a^{-1}\omega e^{-a\omega}$
$\dfrac{1}{x(a^2 + x^2)} \quad (a > 0)$	$\dfrac{\pi}{2}\,a^{-2}(1 - e^{-a\omega})$
$e^{-x/\sqrt{2}}\operatorname{sen}\left(\dfrac{x}{\sqrt{2}}\right)$	$\dfrac{\omega}{1 + \omega^4}$
$\dfrac{2}{\pi}\,\dfrac{x}{a^2 + x^2}$	$e^{-a\omega}$
$\dfrac{2}{\pi}\tan^{-1}\left(\dfrac{a}{x}\right)$	$\dfrac{1}{\omega}(1 - e^{-a\omega}) \quad (a > 0)$
$\operatorname{erfc}\left(\dfrac{x}{2\sqrt{a}}\right)$	$\dfrac{1}{\omega}(1 - e^{-a\omega^2})$
$\dfrac{4}{\pi}\,\dfrac{x}{4 + x^4}$	$e^{-\omega}\operatorname{sen}(\omega)$
$\sqrt{\dfrac{2}{\pi x}}$	$\dfrac{1}{\sqrt{\omega}}$

TABLA 18.4 Transformadas de Fourier en cosenos

$f(x)$	$\mathscr{F}_C\{f(x)\} = F_C(\omega)$
$x^{r-1} \quad (0 < r < 1)$	$\Gamma(r)\cos\left(\frac{\pi r}{2}\right)\omega^{-r}$
$e^{-ax} \quad (a > 0)$	$\frac{a}{a^2 + \omega^2}$
$xe^{-ax} \quad (a > 0)$	$\frac{a^2 - \omega^2}{(a^2 + \omega^2)^2}$
$e^{-a^2x^2} \quad (a > 0)$	$\frac{\sqrt{\pi}}{2} a^{-1}\omega e^{-\omega^2/4a^2}$
$\frac{1}{a^2 + x^2} \quad (a > 0)$	$\frac{\pi}{2}\frac{1}{a} e^{-a\omega}$
$\frac{1}{(a^2 + x^2)^2} \quad (a > 0)$	$\frac{\pi}{4} a^{-3} e^{-a\omega}(1 + a\omega)$
$\cos\left(\frac{x^2}{2}\right)$	$\frac{\sqrt{\pi}}{2}\left[\cos\left(\frac{\omega^2}{2}\right) + \operatorname{sen}\left(\frac{\omega^2}{2}\right)\right]$
$\operatorname{sen}\left(\frac{x^2}{2}\right)$	$\frac{\sqrt{\pi}}{2}\left[\cos\left(\frac{\omega^2}{2}\right) - \operatorname{sen}\left(\frac{\omega^2}{2}\right)\right]$
$\frac{1}{2}(1 + x)e^{-x}$	$\frac{1}{(1 + \omega^2)^2}$
$\sqrt{\frac{2}{\pi x}}$	$\frac{1}{\sqrt{\omega}}$
$e^{-x/\sqrt{2}}\operatorname{sen}\left(\frac{\pi}{4} + \frac{x}{\sqrt{2}}\right)$	$\frac{1}{1 + \omega^2}$
$e^{-x/\sqrt{2}}\cos\left(\frac{\pi}{4} + \frac{x}{\sqrt{2}}\right)$	$\frac{\omega^2}{1 + \omega^4}$
$\frac{2}{x} e^{-x}\operatorname{sen}(x)$	$\tan^{-1}\left(\frac{2}{\omega^2}\right)$
$H(x) - H(x - a)$	$\frac{1}{\omega}\operatorname{sen}(a\omega)$

TABLA 18.5 Transformadas de Fourier

$f(x)$	$\mathscr{F}\{f(x)\} = F(\omega)$
1	$2\pi\delta(\omega)$
$\dfrac{1}{x}$	$\begin{cases} i & \text{si } \omega > 0 \\ 0 & \text{si } \omega = 0 \\ -i & \text{si } \omega < 0 \end{cases}$
$e^{-a\|x\|} \quad (a > 0)$	$\dfrac{2a}{a^2 + \omega^2}$
$xe^{-a\|x\|} \quad (a > 0)$	$\dfrac{4ai}{(a^2 + \omega^2)^2}$
$\|x\|e^{-a\|x\|} \quad (a > 0)$	$\dfrac{2(a^2 - \omega^2)}{(a^2 + \omega^2)^2}$
$e^{-a^2x^2} \quad (a > 0)$	$\dfrac{\sqrt{\pi}}{a} e^{-\omega^2/4a^2}$
$\dfrac{1}{a^2 + x^2} \quad (a > 0)$	$\dfrac{\pi}{a} e^{-a\|\omega\|}$
$\dfrac{x}{a^2 + x^2} \quad (a > 0)$	$-\dfrac{i}{2}\dfrac{\pi}{a} \omega e^{-a\|\omega\|}$
$H(x + a) - H(x - a)$	$\dfrac{2\,\text{sen}(a\omega)}{\omega}$

TABLA 18.1 Transformadas finitas de Fourier en senos

$f(x)$	$S_n\{f(x)\} = F_S(n)$
$\dfrac{\pi - x}{\pi}$	$\dfrac{1}{n}$
$\dfrac{x}{\pi}$	$\dfrac{(-1)^{n+1}}{n}$
1	$\dfrac{1-(-1)^n}{n}$
$\dfrac{x(\pi^2 - x^2)}{6}$	$\dfrac{(-1)^{n+1}}{n^3}$
$\dfrac{x(\pi - x)}{2}$	$\dfrac{1-(-1)^n}{n^3}$
x^2	$\dfrac{\pi^2(-1)^{n+1}}{n} - \dfrac{2[1-(-1)^n]}{n^3}$
x^3	$\pi(-1)^n\left(\dfrac{6}{n^3} - \dfrac{\pi^2}{n}\right)$
e^{ax}	$\dfrac{n}{n^2 + a^2}[1 - (-1)^n e^{a\pi}]$
$\operatorname{sen}(kx)$, $k = 1, 2, 3, \ldots$	$\begin{cases} 0 & \text{si } n \neq k \\ \pi/2 & \text{si } n = k \end{cases}$
$\cos(ax)$, $a \neq$ entero	$\dfrac{n}{n^2 - a^2}[1 - (-1)^n \cos(a\pi)]$
$\cos(kx)$, $k = 1, 2, 3, \ldots$	$\begin{cases} \dfrac{n}{n^2 - a^2}[1 - (-1)^{n+k}], & n \neq k \\ \dfrac{\pi}{2}, & n = k \end{cases}$

TABLA 18.1 Transformadas finitas de Fourier en senos (cont.)

$f(x)$	$S_n\{f(x)\} = F_S(n)$
$\frac{2}{\pi}\tan^{-1}\left(\frac{2a\operatorname{sen}(x)}{1-a^2}\right) \quad (\lvert a\rvert < 1)$	$\frac{1-(-1)^n}{n}a^n$
$f(\pi - x)$	$(-1)^n F_S(n)$
$\frac{x}{n}(\pi - x)(2\pi - x)$	$\frac{6}{n^3}$
$\cosh(kx)$	$\frac{n}{n^2+k^2}[1-(-1)^n\cosh(k\pi)]$
$\frac{\operatorname{sen}[k(\pi - x)]}{\operatorname{sen}(k\pi)}$	$\frac{n}{n^2-k^2} \quad (k \neq \text{entero})$
$\frac{\operatorname{senh}[k(\pi - x)]}{\operatorname{senh}(k\pi)}$	$\frac{n}{n^2+k^2} \quad (k \neq 0)$
$\frac{2}{\pi}\frac{k\operatorname{sen}(x)}{1+k^2-2k\cos(x)}$	$k^n \quad (\lvert k\rvert < 1)$
$\frac{2}{\pi}\tan^{-1}\left[\frac{k\operatorname{sen}(x)}{1-k\cos(x)}\right]$	$\frac{1}{n}k^n \quad (\lvert k\rvert < 1,\ n \neq 0)$
$\frac{1}{\pi}\frac{\operatorname{sen}(x)}{\cosh(y)-\cos(x)}$	$e^{-ny} \quad (y > 0)$
$\frac{1}{\pi}\tan^{-1}\left[\frac{\operatorname{sen}(x)}{e^y-\cos(x)}\right]$	$\frac{1}{n}e^{-ny} \quad (y > 0)$

k

k

TABLA
TRANSFORMADA DE FOURIER DE SENOS

$f(t)$		$F_s[f(t)]=F_s(w)$
$\begin{cases}1 & \text{si } 0<t<a\\ 0 & \text{si } t>a\end{cases}$		$\dfrac{1-\cos aw}{w}$
t^{a-1}		$\dfrac{\Gamma(a)}{w^a}\operatorname{sen}\left(\dfrac{aw}{2}\right)$
$\dfrac{1}{t}$		$\begin{cases}\pi/2 & \text{si } w>0\\ -\pi/2 & \text{si } w<0\end{cases}$
$\dfrac{1}{\sqrt{t}}$		$\sqrt{\dfrac{\pi}{2w}}$
$\dfrac{t}{a^2+t^2}$	$(a>0)$	$\dfrac{\pi}{2e^{aw}}$
$\dfrac{t}{(a^2+t^2)^2}$	$(a>0)$	$\dfrac{\pi w}{4ae^{aw}}$
e^{-at}	$(a>0)$	$\dfrac{w}{a^2+w^2}$
te^{-at}	$(a>0)$	$\dfrac{2aw}{(a^2+w^2)^2}$
$te^{-a^2t^2}$	$(a>0)$	$\dfrac{\sqrt{\pi}}{4a^3}we^{-\frac{w^2}{4a^2}}$

TABLA
TRANSFORMADA DE FOURIER DE SENOS

$f(t)$		$F_s[f(t)]=F_s(w)$
$t^{-1}e^{-at}$	$(a>0)$	$\tan^{-1}\left(\dfrac{w}{a}\right)$
$\sqrt{\dfrac{2}{\pi t}}$		$\dfrac{1}{\sqrt{w}}$
$\dfrac{\operatorname{sen} at}{t}$		$\dfrac{1}{2}\ln\left(\dfrac{w+a}{w-a}\right)$
$\dfrac{\operatorname{sen} at}{t^2}$		$\begin{cases}\dfrac{\pi w}{2} & w<a\\ \dfrac{\pi a}{2} & w>a\end{cases}$
$\dfrac{\cos at}{t}$		$\begin{cases}0 & w<a\\ \pi/4 & w=a\\ \pi/2 & w>a\end{cases}$
$\tan^{-1}\left(\dfrac{t}{a}\right)$		$\dfrac{\pi}{2we^{aw}}$
$\csc at$		$\dfrac{\pi}{2a}\tanh\dfrac{\pi w}{2a}$
$e^{-\frac{t}{\sqrt{2}}}\operatorname{sen}\left(\dfrac{t}{\sqrt{2}}\right)$		$\dfrac{w}{1+w^4}$

TABLA
TRANSFORMADAS DE FOURIER DE COSENOS

$f(t)$		$F_c[f(t)] = F_c(w)$
$\begin{cases} 1 & \text{si } 0 < t < a \\ 0 & \text{si } t > a \end{cases}$		$\dfrac{\text{sen}\, aw}{w}$
t^{a-1}	$0 < a < 1$	$\dfrac{\Gamma(a)}{w^a}\cos\dfrac{aw}{2}$
e^{-at}	$a > 0$	$\dfrac{a}{a^2+w^2}$
$e^{-\frac{t^2}{2}}$		$e^{-\frac{w^2}{2}}$
e^{-at^2}	$a > 0$	$\dfrac{1}{2}\sqrt{\dfrac{\pi}{a}}\, e^{-\frac{w^2}{4a}}$
$e^{-a^2t^2}$	$a > 0$	$\dfrac{\sqrt{\pi}}{2a}\, w\, e^{-\frac{w^2}{4a^2}}$
$t^{-\frac{1}{2}}$		$\sqrt{\dfrac{\pi}{2w}}$
t^{-n}		$\dfrac{\pi w^{n-1}\sec\frac{n\pi}{2}}{2\Gamma(n)}$
$\dfrac{1}{a^2+t^2}$	$a > 0$	$\dfrac{\pi e^{-aw}}{2a}$

TABLA
TRANSFORMADAS DE FOURIER DE COSENOS

$f(t)$		$F_c[f(t)] = F_c(w)$
$\dfrac{1}{(a^2+t^2)^2}$	$a > 0$	$\dfrac{\pi}{4a^3} e^{-aw}(1+aw)$
$\dfrac{\text{sen}\, at}{t}$		$\begin{cases} \pi/2 & w < a \\ \pi/4 & w = a \\ 0 & w > a \end{cases}$
$\text{sen}\, at^2$		$\sqrt{\dfrac{\pi}{8a}}\left(\cos\dfrac{w^2}{4a} - \text{sen}\dfrac{w^2}{4a}\right)$
$\cos at^2$		$\sqrt{\dfrac{\pi}{8a}}\left(\cos\dfrac{w^2}{4a} + \text{sen}\dfrac{w^2}{4a}\right)$
$\text{sen}\left(\dfrac{t^2}{2}\right)$		$\dfrac{\sqrt{\pi}}{2}\left[\cos\left(\dfrac{w^2}{2}\right) - \text{sen}\left(\dfrac{w^2}{2}\right)\right]$
$\sqrt{\dfrac{2}{\pi t}}$		$\dfrac{1}{\sqrt{w}}$
$e^{-\frac{t}{\sqrt{2}}}\,\text{sen}\left(\dfrac{\pi}{4} + \dfrac{t}{\sqrt{2}}\right)$		$\dfrac{1}{1+w^2}$
$\dfrac{(1+t)}{2e^t}$		$\dfrac{1}{(1+w^2)^2}$

TABLA
PARES DE TRANSFORMADAS DE FOURIER

$f(t)$		$F[f(t)] = F(w)$
$\begin{cases} 1 & \text{si} \quad \lvert t\rvert < a \\ 0 & \text{si} \quad \lvert t\rvert > a \end{cases}$		$\dfrac{2\operatorname{sen} aw}{w}$
$\dfrac{1}{t}$		$\begin{cases} i & \text{si} \quad w > 0 \\ 0 & \text{si} \quad w = 0 \\ -i & \text{si} \quad w < 0 \end{cases}$
$e^{-a\lvert t\rvert}$	$(a > 0)$	$\dfrac{2a}{a^2 + w^2}$
$t e^{-a\lvert t\rvert}$	$(a > 0)$	$-\dfrac{4aiw}{(a^2 + w^2)^2}$
$\lvert t\rvert e^{-a\lvert t\rvert}$	$(a > 0)$	$\dfrac{2(a^2 - w^2)}{(a^2 + w^2)^2}$
$e^{-a^2 t^2}$	$(a > 0)$	$\dfrac{\sqrt{\pi}}{a} e^{-\frac{w^2}{4a^2}}$
$\dfrac{1}{a^2 + t^2}$	$(a > 0)$	$\dfrac{\pi}{a e^{a\lvert w\rvert}}$
$\dfrac{t}{a^2 + t^2}$	$(a > 0)$	$-\dfrac{\pi i w}{2a e^{a\lvert w\rvert}}$
$\delta(t)$		1

TABLA 4.1 PROPIEDADES DE LA TRANSFORMADA DE FOURIER

Sección	Propiedad	Señal aperiódica	Transformada de Fourier
		$x(t)$	$X(j\omega)$
		$y(t)$	$Y(j\omega)$
4.3.1	Linealidad	$ax(t) + by(t)$	$aX(j\omega) + bY(j\omega)$
4.3.2	Desplazamiento de tiempo	$x(t - t_0)$	$e^{-j\omega t_0}X(j\omega)$
4.3.6	Desplazamiento de frecuencia	$e^{j\omega_0 t}x(t)$	$X(j(\omega - \omega_0))$
4.3.3	Conjugación	$x^*(t)$	$X^*(-j\omega)$
4.3.5	Inversión de tiempo	$x(-t)$	$X(-j\omega)$
4.3.5	Escalamiento de tiempo y de frecuencia	$x(at)$	$\frac{1}{\lvert a\rvert}X\left(\frac{j\omega}{a}\right)$
4.4	Convolución	$x(t) * y(t)$	$X(j\omega)\,Y(j\omega)$
4.5	Multiplicación	$x(t)y(t)$	$\frac{1}{2\pi}X(j\omega) * Y(j\omega)$
4.3.4	Diferenciación en tiempo	$\frac{d}{dt}x(t)$	$j\omega X(j\omega)$
4.3.4	Integración	$\int_{-\infty}^{t} x(t)dt$	$\frac{1}{j\omega}X(j\omega) + \pi X(0)\delta(\omega)$
4.3.6	Diferenciación en frecuencia	$tx(t)$	$j\frac{d}{d\omega}X(j\omega)$
4.3.3	Simetría conjugada para señales reales	$x(t)$ real	$\begin{cases} X(j\omega) = X^*(-j\omega) \\ \mathcal{R}_e\{X(j\omega)\} = \mathcal{R}_e\{X(-j\omega)\} \\ \mathcal{I}_m\{X(j\omega)\} = -\mathcal{I}_m\{X(-j\omega)\} \\ \lvert X(j\omega)\rvert = \lvert X(-j\omega)\rvert \\ \sphericalangle X(j\omega) = -\sphericalangle X(-j\omega) \end{cases}$
4.3.3	Simetría para señales real y par	$x(t)$ real y par	$X(j\omega)$ real y par
4.3.3	Simetría para señales real e impar	$x(t)$ real e impar	$X(j\omega)$ puramente imaginaria e impar
4.3.3	Descomposición par-impar de señales reales	$x_e(t) = \mathcal{E}v\{x(t)\}$ [$x(t)$ real] $x_o(t) = \mathcal{O}d\{x(t)\}$ [$x(t)$ real]	$\mathcal{R}_e\{X(j\omega)\}$ $j\mathcal{I}_m\{X(j\omega)\}$

4.3.7 Relación de Parseval para señales aperiódicas

$$\int_{-\infty}^{+\infty} |x(t)|^2 dt = \frac{1}{2\pi}\int_{-\infty}^{+\infty} |X(j\omega)|^2 d\omega$$

TAREA 2 TRANSFORMADA DE FOURIER T-2-1

1 HALLAR LA TRANSFORMADA DE FOURIER SI:

$$f(t) = \cos 2t\left[H(t) - H(t-\pi)\right].$$

R $$\mathcal{F}[f(t)] = \frac{i\omega}{4-\omega^2}\left[1 - e^{-i\omega\pi}\right].$$

2 RESUELVA LA ECUACIÓN DIFERENCIAL UTILIZANDO LA TRANSFORMADA DE FOURIER.

$$y'' + 3y' + 2y = H(t).$$

DONDE: $\mathcal{F}[y'] = i\omega F(\omega) = i\omega \Psi(\omega).$

$\mathcal{F}[y''] = -\omega^2 F(\omega) = -\omega^2 \Psi(\omega).$

$\mathcal{F}[y] = \Psi(\omega).$

$\omega\delta(\omega) = 0. \Rightarrow$ POR TABLAS $\mathcal{F}[H(t)] = \pi\delta(\omega) + \dfrac{1}{i\omega}.$

R $$y(t) = \frac{1}{2}\left[1 + e^{-2t} - 2e^{-t}\right]H(t).$$

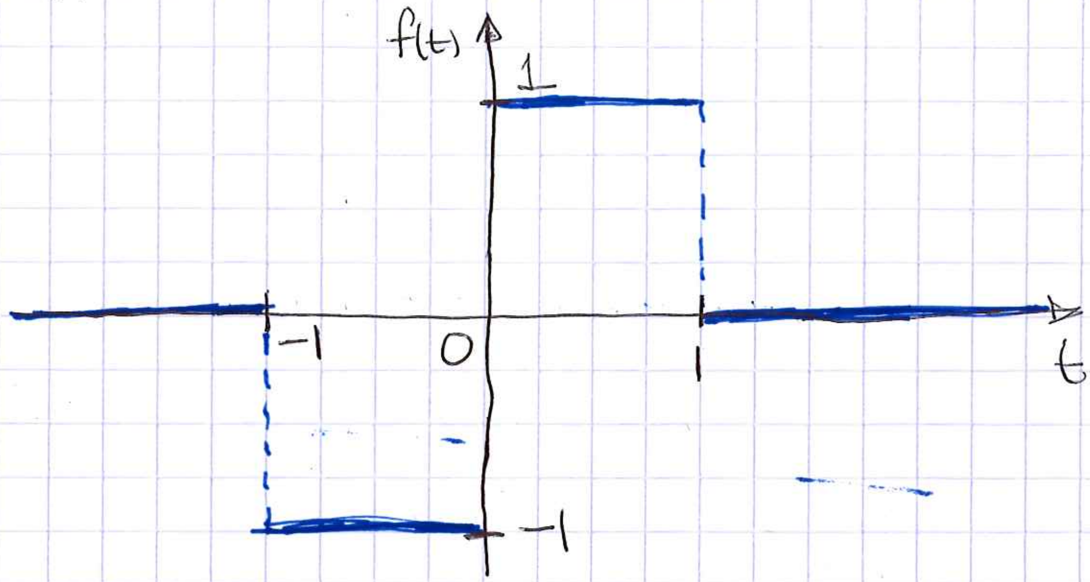
3 a OBTENER LA TRANSFORMADA DE FOURIER PARA:

$$f(t) = -t + 1$$

R $$F(\omega) = \frac{e^{-i\omega}}{i\omega} + \frac{e^{-i\omega}}{\omega^2}.$$

ⓑ DE LA FIGURA, OBTENER LA TRANSFORMADA DE FOURIER.

T-2-2



ⓡ $F(\omega) = \frac{2}{\omega}\left[\cos\omega - 1\right]$.