Example Useful for the Project

Consider an undirected, connected, weighted graph given by Fig.1. The edge weights correspond to the distances between the nodes in km.

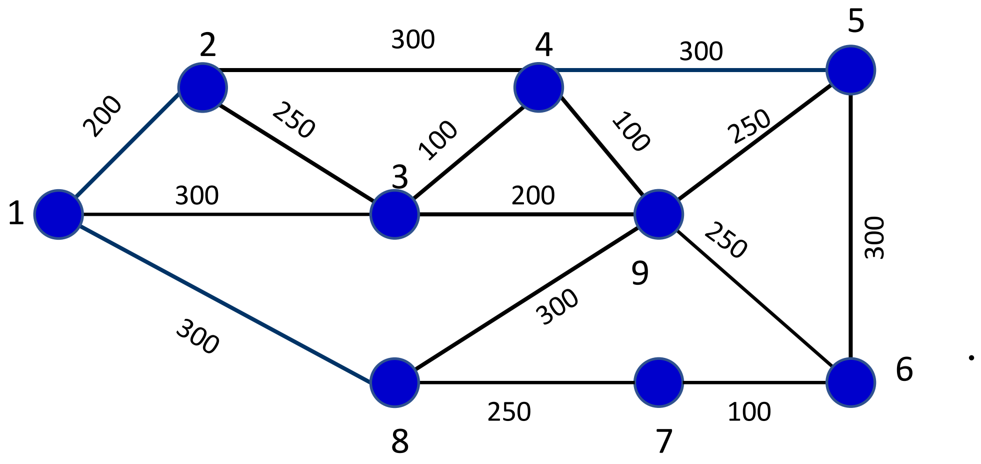

Fig. 1 Weighted graph with 9 nodes.

This graph is characterized for having $N = 9$ nodes and $L = 15$ links. So, the average node degree

$< \delta >= \sum_{i=1}^{N} \delta_i / N = 3.333$ and the variance of the degrees is $\sigma^2(\delta) = \sum_{i=1}^{N} (\delta_i - < \delta >)^2 / (N-1) = 0.75.$

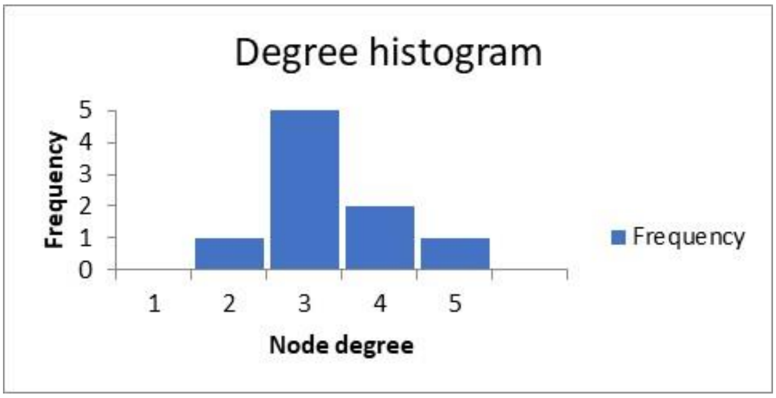

Fig. 2 Histogram of the node degree corresponding to the graph of Fig.1

Fig. 2 shows the histogram of the node degree of the graph of Fig. 1. This histogram was computed using the package Data Analysis of Excel.

The distance between node pairs $(v_i \,.\, v_j)$ in an unweighted graph $G(V, E)(\mathrm{dist}_u(v_i \,.\, v_j))$ is defined has the number of links of the shortest path connecting them. The distance can also be defined has the minimum number of hops $h_{i,j}$ corresponding to a demand between $(v_i \,.\, v_j)$. For example according to Fig. 1, the shortest path between 1 and 5 is

$$\pi_{1,5} = \{(1,2), (2,4), (4,5)\} \rightarrow \mathrm{dist}_u(1,5) = |\pi_{1,5}| = 3 \rightarrow h_{1,5} = 3$$

where $\mathrm{dist}_u(1{,}5)$ is the distance measured in the unweighted graph.

In this analysis it is necessary to compute the shortest paths between all the pairs of nodes of the graph. Firstly, using for example the Dijkstra algorithm one computes the shortest path between one node and the other $N-1$ nodes. Then the process is repeated sequentially by starting with each one of the remaining $N-1$ nodes. As an example, let´s consider that node 1 is the starting node. Then

$\pi_{1,2} = \{(1{,}2)\} \to \mathrm{dist}_u(1{,}2) = 1; \pi_{1,3} = \{(1{,}2),(2.3)\} \to \mathrm{dist}_u(1{,}3) = 2\ ; \pi_{1,4} = \{(1{,}2),(2.4)\} \to \mathrm{dist}_u(1{,}4) = 2;\ \pi_{1,5} = \{(1{,}2),(2{,}4),(4{,}5)\} \to \mathrm{dist}_u(1{,}5) = 3\ ;\ ;\ \pi_{1,6} = \{(1{,}3),(3{,}9),(9{,}6)\} \to \mathrm{dist}_u(1{,}6) = 3; \pi_{1,7} = \{(1{,}8),(8.7)\} \to \mathrm{dist}_u(1{,}7) = 2,\ \pi_{1,8} = \{(1{,}8)\} \to \mathrm{dist}_u(1{,}8) = 1; \pi_{1,9} = \{(1{,}3),(3{,}9)\} \to \mathrm{dist}_u\ (1{,}9)$=2.

In the next step the starting point is node 2, leading to $\pi_{2,1} = \{(2{,}1)\} \to \mathrm{dist}_u(2{,}1) = 1; \pi_{2,3} = \{(2.3)\} \to \mathrm{dist}_u(2{,}3) = 1\ ; \pi_{2,4} = \{(2.4)\} \to \mathrm{dist}_u(2{,}4) = 1;\ \pi_{2,5} = \{\ (2{,}4),(4{,}5)\} \to \mathrm{dist}_u(2{,}5) = 2\ ;\ ;\ \pi_{2,6} = \{(2{,}4),(4{,}5),(5{,}6)\} \to \mathrm{dist}_u(2{,}6) = 3; \pi_{2,7} = \{(2{,}1),(1{,}8),(8.7)\} \to \mathrm{dist}_u(2{,}7) = 3$, $\pi_{2,8} = \{(2{,}1),(1{,}8)\} \to \mathrm{dist}_u(2{,}8) = 2; \pi_{2,9} = \{(1{,}3),(3{,}9)\} \to \mathrm{dist}_u\ (2{,}9)$=2.

Therefore the starting point is node3 and we continue the analysis in the same way. As the graph under consideration is an undirected graph we have $\mathrm{dist}_u(1{,}2) = \mathrm{dist}_u(2{,}1)$, so it is not necessary to consider the path to $\pi_{2,1}$. In a synthetic way in this study we only consider the paths in one direction, since the distance in the other direction is identical.

Figure 3 shows a histogram of the values of the distances previously computed. The number of shortest paths between *N* nodes considering only one direction is $N(N-1)/2$, which gives 36. . According with the histogram of Fig.3 we have 14(1 hop)+16(2 hops)+ 6(3 hops)=36 shortest paths , as expected. One can associate each shortest-path with a traffic demand. So we can conclude that the number of bidirectional demands is equal to $D_2 = N(N-1)/2 = 9 \times 8/2 = 36$. An important definition is the average value of the minimum number of hops, computed for a set of D demands which is given by

$$< h >= \frac{\sum_{i<j}^{D} h_{i,j}}{D}$$

with $D = D_2$. Using the data given we have $< h >= (14 \times 1 + 16 \times 2 + 6 \times 3)/36$ =1.777(7). A possible semi-empirical estimation for $< h >$ is

$$< h >\approx \sqrt{\frac{(N-2)}{(< \delta > -1)}}$$

Using the semi-empirical estimator one arrives to

$$< h >\approx \sqrt{\frac{(9-2)}{(3.333-1)}} = 1.732$$

which is quite closer to the rigorous value computed before.

The diameter of a graph, $\mathrm{diam}\ (\mathrm{G})$ is the greatest distance between any two nodes or the length of the longest shortest. For the graph of Fig. 1 we have, $\mathrm{diam}(G) = 3$.

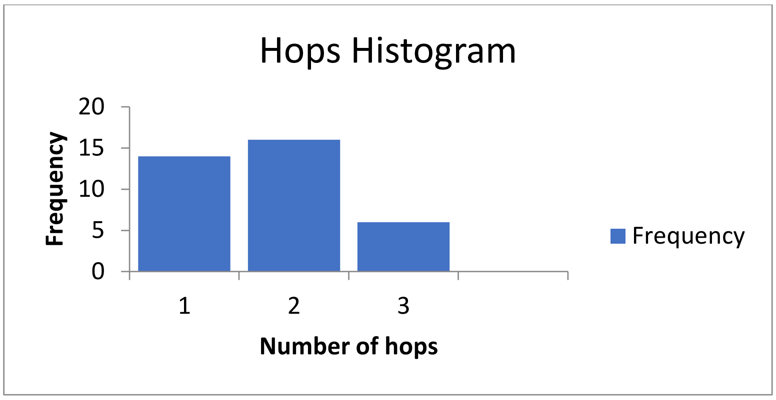

Fig.3 Histogram of the number of hops.

The next step in the analysis is to compute the distances $\mathrm{dist}_w(s,d)$ for all the shortest-paths. We proceed as before in order to calculate all the shortest paths, but now the distances are calculated by adding the weights of all the edges that are traversed by the path. For example for the shortest-path between 1 and 5 we have

$$\pi_{1,5} = \{(1,3),(3,9),(9,5)\} \rightarrow \mathrm{dist}_w(1,5) = 300 + 200 + 250 = 750$$

The results obtained permit us to draw the histogram shown in Fig. 4. As seen the maximum values are within the bin 800, and the average value is equal 398.6 km.

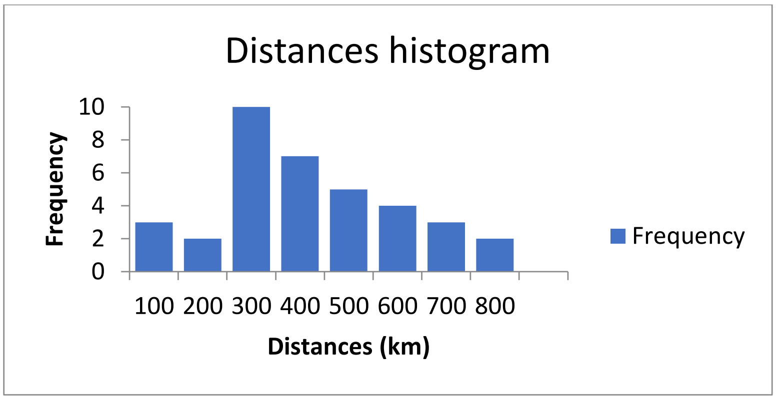

Fig. 4 Histogram of the node distances

The node and edge connectivity of the graph are other parameters that is worth analysing. As known the node connectivity is the minimum number of nodes that must be removed in order to disconnect the graph or make it trivial. As seen from Fig. 5 $\kappa(G) = 2$ (nodes 6 and 8 removed).

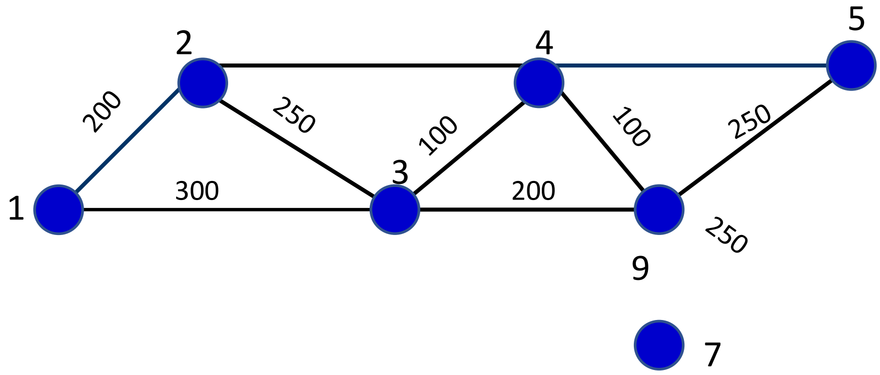

Fig.5 Sub-graphs obtained by a node cut

The minimum node cut set is $V^{\prime} = \{6,8\}$, and $S_1 = \{7\}$, and $S_2 = \{1,2,3,4,5,9\}$, (see Fig. 5). The edge connectivity is the minimum number of edges that can be removed from the graph to disconnect it. This number is equal to the minimum edge-cut set which is given by $E^{\prime} = \{(6,7), (7,8\}$, so $\gamma(G) = 2$. Note that in this case so $\kappa(G) = \gamma(G) = \delta(G) = 2$.

For resilience/survivability purpose it is also useful to compute the minimum 1-5 (and 1-6) node cut-set, as well as the minimum 1-5 (and 1-6) edge cut-set. In this type of analysis will be paramount to analyse the worst-case scenarios, which in the present graph corresponds to the paths with the greatest distance. Let´s start with the path $\pi_{1,5}$: The minimum node cut -set is $V^{\prime} = \{3,4\ 8\}$, and $S_1 = \{1,2\}$, and and $S_2 = \{5,6,7,,9\}$ as exemplified in the Fig. 6. So $|V^{\prime}| = 3$.

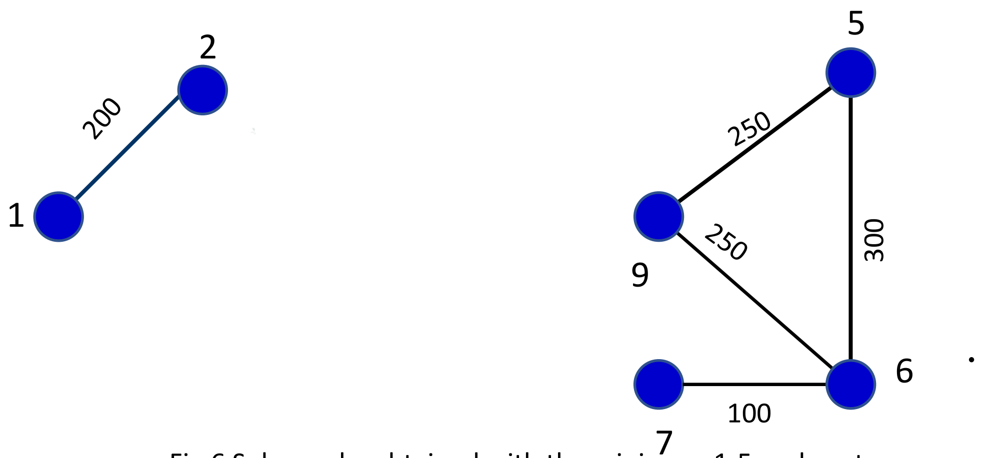

Fig.6 Sub-graphs obtained with the minimum 1-5 node cut

The minimum edge cut set is $E^{\prime} = \{(4,5), (5,9), (5,6)\}$ or $E^{\prime} = \{(1,2), (1,3), (1,8)\}$. Fig. 7 represents the scenario for the first case. As seen $S_1 = \{1,2,3,4,6,7,8,9\}$ and $S_2 = \{5\}$. So $|E^{\prime}| = 3$. The analysis for the second case is identical and the solutions is left to the reader.

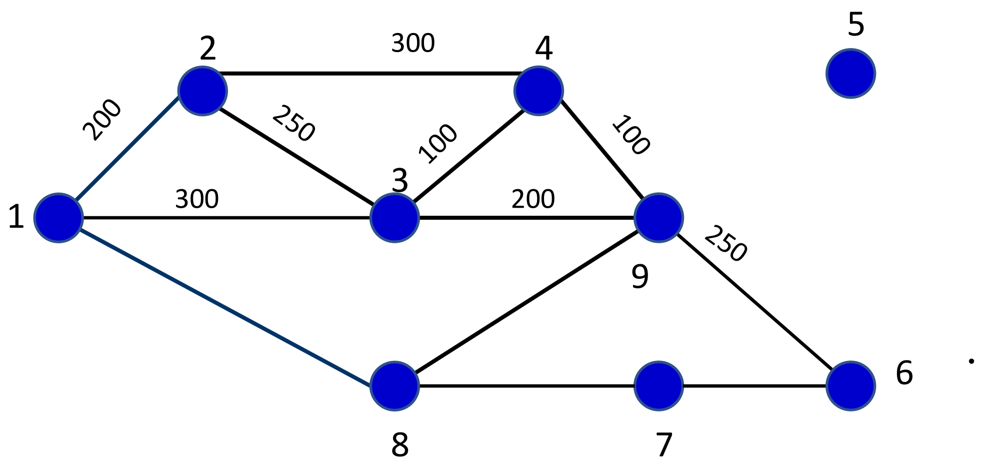

Fig.7 Sub-graphs obtained with the minimum 1-5 edge cut

As the size of the minimum 1-5 node cut set is equal to 3 one can conclude that according to the Merger´s theorem the maximum number of internally disjoint paths from 1 to 5 is also 3. Similarly one can conclude that the maximum number of edge independent paths is equal to 3. In terms of resilience/survivability analysis these results permit us to conclude that a working path between 1 and 5 can be protected by two internally link disjoint/or edge disjoint paths as exemplified in the Figure 5. As seen the working path is given by $\pi_{1,5}^{W} = \{(1,3),(3,9),(9,5)\}$ and has a length of 750 km. The shortest backup path is the backup #1, which is described by $\pi_{1,5}^{b1} = \{(1,2),(3,4),(4,5)\}$ and has a length of 800 km. Finally the longest backup path is the backup #2, which is described by $\pi_{1,5}^{b2} = \{(1,8),(8,7),(7,6),(6,5)\}$ and has a length of 950 km.

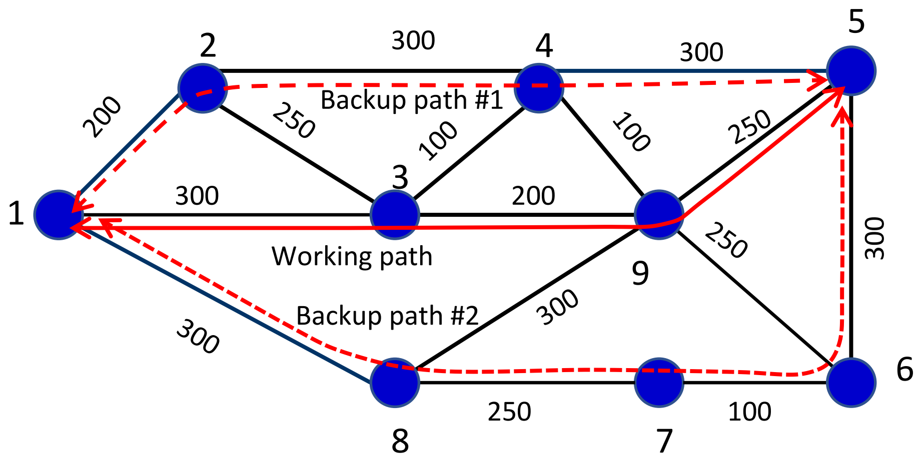

Fig.8 Working path and backup path

The network survivability/ resilience behaviour depends on the level of network redundancy used. Network redundancy is achieved through the addition of alternate networks paths, so that when the primary (working) path fails the alternate path can be instantly deployed to ensure the continuity of networks services. The minimum requirement to introduce redundancy consists in adding a single backup path, this is denoted as $N+1$ redundancy. A more robust solution consists in relying on two separate backups paths ($N+2$ reduncancy). In this scenario the network can still continue operational even if a single backup fails. In networks with very high level of survivability even more backup resources are required. These are referred as $N+X$, where X indicates the number of backup resources used. As seen the network under study provides $N+2$ redundancy since the working path is protected by two backup paths.

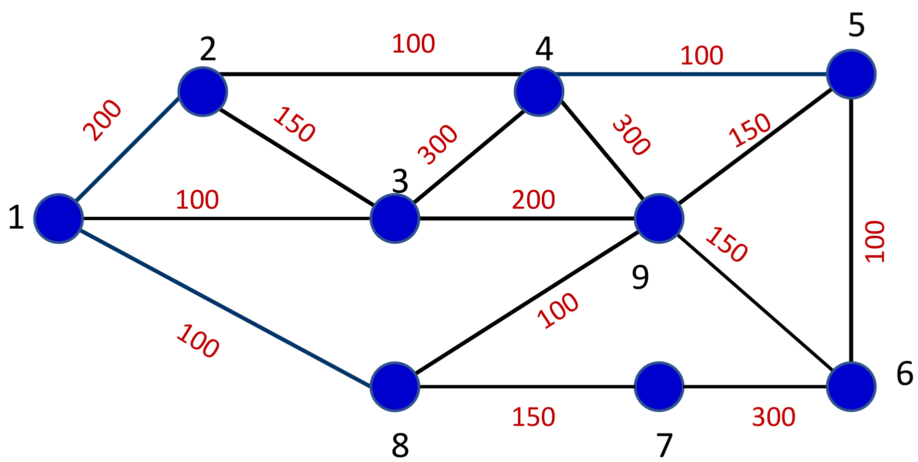

Fig. 9 Network showing the link capacities

Capacity is another important network property. The network capacity depends on the capacity of its links, which is defined as the maximum amount of traffic that it can transport. The capacity of link (i, j) is denoted $u(i, j)$ while the network capacity is referred as $U = \{u(i, j), (i.j) \in E\}$. In this study we assume that $u(i, j)$ is related with the link length $l(i, j)$ in the following way $u(i, j) = -l(i, j) + 400$. Furthermore, we assume that the links capacities are given in Gb/s.

Another important concept is path capacity, which is defined as being the minimum value of the capacities of all the edges traversed by the graph. For example the path $\pi_{1,5}^1 = \{(1,2), (2,4), (4,5)\}$ has the capacity $Cap(\pi_{1,5}^1) = \min[u(1,2), u(2,4), u(4,5)] = \min[200, 100, 100] = 100$.

Another relevant topic is the computation of the maximum amount flow (traffic) going from the node 1 (source node) to node 5 (sink node). According to the Max-Flow Min-Cut theorem this amount is equal to the capacity (summation of the weights of all edges) of the minimum edge cut. The different edge cuts are shown in Fig. 10.

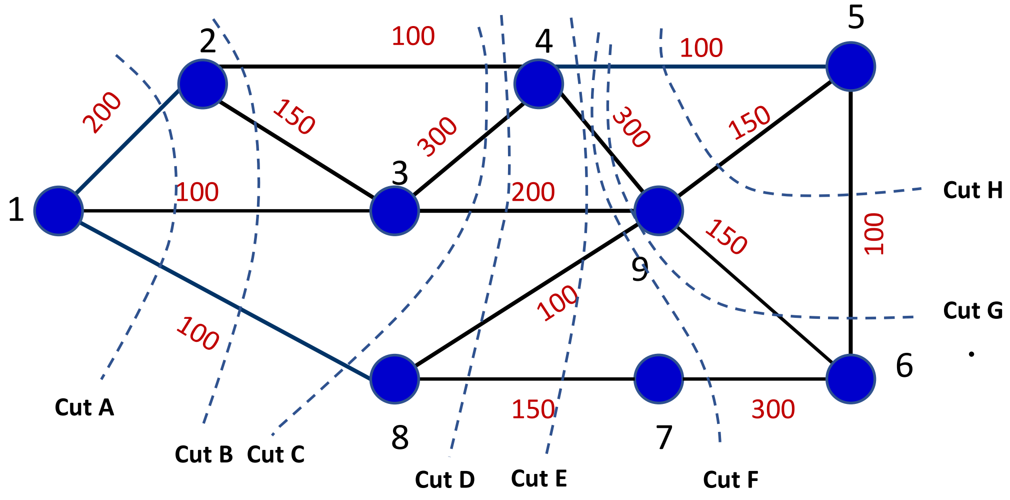

Figure 10 Network showing different link cuts

The capacity of the different cuts is the following:

$$\text{Cut A} = 200 + 100 + 100 = 400$$

$$\text{Cut B} = 100 + 150 + 100 + 100 = 450$$

$$\text{Cut C} = 100 + 300 + 200 + 100 = 700$$

$$\text{Cut D} = 100 + 300 + 200 + 100 + 100 = 800$$

$$\text{Cut E} = 100 + 300 + 200 + 100 + 150 = 850$$

$$\text{Cut F} = 100 + 300 + 200 + 100 + 300 = 1000$$

$$\text{Cut G} = 100 + 300 + 200 + 150 + 100 = 850$$

$$\text{Cut G} = 100 + 150 + 100 = 350$$

Taking into account that the $\text{maximum flow} = \text{minimum cut}$, one concludes that the maximum flow between 1 and 5 is 350 Gb/s, since the cut with the smallest capacity is cut G with 350 Gb/s.