MAT3300 Homework 1 Report

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Below is an SIR model with births and deaths for infectious disease. $S(t)$ is the number of susceptible individuals, and $I(t)$ is the number of the infected individuals. It is assumed that susceptible individuals can produce an uninfected offspring with rate b, that they become infected when they come in contact with infected people at rate β, and that the infected individuals die at rate k. This is described by the following equations:

$$\frac{dS}{dt} = bS - \beta IS$$
$$\frac{dI}{dt} = \beta IS - kI$$

1. Find all equilibriums and their stability.
2. Solve S in terms of a function in I analytically (it is OK to leave a free constant). Are the orbits $(S(t); I(t)), t \geq 0$ periodic in the $S - I$ plane?
3. Sketch several typical phase lines in $S - I$ plane.
4. Solve the ODE system numerically with $b = \beta = k = 3$. Try initial conditions $S(0) = 1$ and $I(0)$ equal to either 0.1, 0.5 or 1.3. Vary the time step size as $\Delta = 0.1$, 0.01, 0.001. Do you observe the convergence?

Solution

Given a system of ODE as follows.

$$\frac{dS}{dt} = bS - \beta IS \tag{1}$$
$$\frac{dI}{dt} = \beta IS - kI \tag{2}$$

It could be defined two functions, $f(S, I) = bS - \beta IS$ and $g(S, I) = \beta IS - kI$, such that

$$\frac{dS}{dt} = f(S, I) \tag{3}$$
$$\frac{dI}{dt} = g(S, I) \tag{4}$$

By definition, the equilibrium points are point (S, I) such that $f(S, I) = g(S, I) = 0$. $(S, I) = (0, 0)$ obviously satisfies the condition and therefore an equilibrium point. In order to find the other one, consider when both $(S, I) = (0, 0)$. Hence, it is equivalent to solving the following system of equations.

$$(b - \beta I)S = 0 \tag{5}$$
$$(\beta S - k)I = 0 \tag{6}$$

which is equivalent to solving

$$b - \beta I = 0$$
$$\beta S - k = 0$$

Hence, the other equilibrium point is $(S, I) = (\frac{k}{\beta}, \frac{b}{\beta})$. Thus, there are two equilibrium points, $(0, 0)$ and $(\frac{k}{\beta}, \frac{b}{\beta})$.

For the stability, point $(0, 0)$ is **unstable** because entrance to this equilibrium point is along the line $S = 0$, which over time will make I to decline to 0. This makes sense because the infected individuals cannot exists with the prior absence of susceptible individuals.
On the other hand, point $(\frac{k}{\beta}, \frac{b}{\beta})$ is **stable** because the trajectory that passed this point will always stay close as $t \to \infty$.

By diving the two autonomous ODE, then

$$\frac{dS}{dI} = \frac{(b - \beta I)S}{(\beta S - k)I}$$

Then, by seperation of variables,

$$\left(\beta - \frac{k}{S}\right) dS = \left(\frac{b}{I} - \beta\right) dI$$

and take the integration of both sides.

$$\int \left(\beta - \frac{k}{S}\right) dS = \int \left(\frac{b}{I} - \beta\right) dI$$

the result is as follows.

$$\beta S - k\ln(S) = b\ln(I) - \beta I + k$$

for some constant k. Thus,

$$\frac{e^{\beta S+\beta I}}{S^k I^b} = K$$

for some constant K. This equation defines the solution trajectories in the phase plane. Based on the the $S-I$ plane, the orbits $(S(t), I(t))$ is periodic.

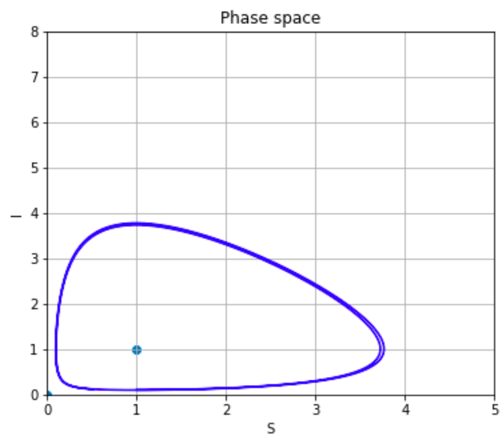

Figure 1. *Sketch for the S-I plane*

The image above was is a closely accurate result to the true plot. However, it is important to note that for different time step, it might affect heavily on the plot that was programmed. As we can see in the following images, especially for those with time step $\Delta = 0.1$ and $\Delta = 0.01$, they are not either not periodic or end up being unstable due to the big gap each time takes.

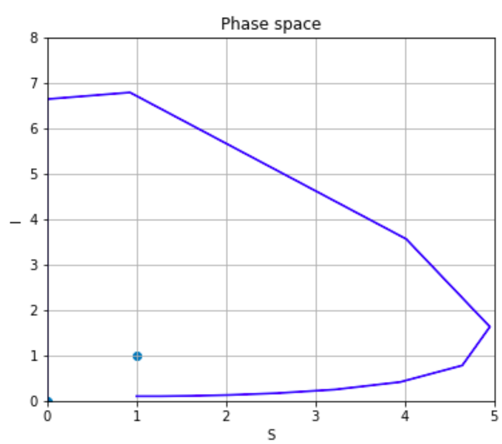

Figure 2. *Sketch for the S-I plane when* $\Delta = 0.1$

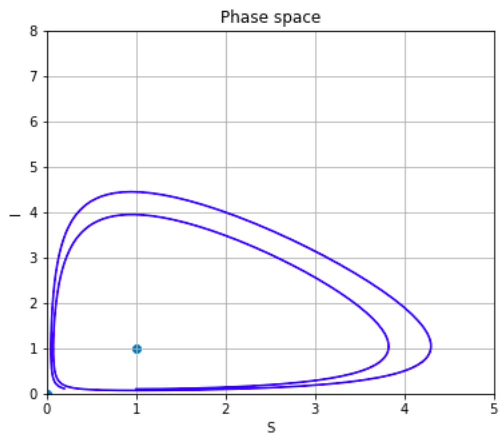

Figure 3. *Sketch for the S-I plane when* $\Delta = 0.01$

Other more specific plot can be found in the Jupyter Notebook, that will be attached alongside the document.

Conclusion

Thus, based on the phase plane generated by the Python program (*written in Jupyter Notebook*), only the graph that has $\Delta = 0.001$ will converge. Otherwise, other results are representing inaccurate representation of the system of ODE.

Another interesting thing to point out is that, from the result in Jupyter Notebook, the higher the intiial value is, the "nearer" its trajectory to the equilibrium point $(\frac{k}{\beta}, \frac{b}{\beta})$.

In addition to the plot of $S - I$ phase plane, another plot, which pictures the evolution of number of susceptible individuals and infected individuals. which shows fluctuation over time, meaning that the numbers of susceptible individuals and infected individuals will take turns in reaching its maximum value and dropping to its minimum value. And, accordingly to the model interpretation, the susceptible individuals must exist initially or have higher number before the infected individuals outnumbered the susceptible ones. It is important to note that similar to the $S - I$ plane, for $\Delta = 0.1$ and $\Delta = 0.01$, the plot have some inaccuracies.

GitHub Repository

- https://github.com/YosephKS/Infectious-Disease-SIR-Mathematical-Model