DISCRETE MATHEMATICS NOTES

Chapter 1: Logic and Proofs

Yoshita Aligina
Rutgers University
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1 Propositional Logic

1.1 Propositions

- A **proposition** is a declarative sentence that is either true or false, but not both.
- Examples of propositions:
  - "The sky is blue." (True)
  - "2 + 2 = 5." (False)
- Non-examples of propositions:
  - "What time is it?" (Not declarative)
  - "x + 2 = 4." (Not a proposition, as the truth value depends on x)

1.2 Logical Connectives

- The main logical connectives are:
  - **Negation** ($\neg p$): If p is true, then $\neg p$ is false, and vice versa.
  - **Conjunction** ($p \land q$): True if both p and q are true, otherwise false.
  - **Disjunction** ($p \lor q$): True if at least one of p or q is true, otherwise false.
  - **Implication** ($p \to q$): False only when p is true and q is false, otherwise true.
  - **Biconditional** ($p \leftrightarrow q$): True if p and q have the same truth value.

1.3 Truth Tables

- Truth tables provide a systematic way to determine the truth value of a proposition based on its components.
- Example: Truth table for $p \to q$.

p	q	$p \to q$
T	T	T
T	F	F
F	T	T
F	F	T

1.4 Equivalence of Propositions

- Two propositions are **logically equivalent** if they have the same truth value in all possible cases. We write $p \equiv q$ to denote this.
- Common equivalences:
  - De Morgan's Laws: $\neg(p \land q) \equiv \neg p \lor \neg q$, $\neg(p \lor q) \equiv \neg p \land \neg q$
  - Implication: $p \to q \equiv \neg p \lor q$

2 Predicate Logic

2.1 Predicates and Quantifiers

- A **predicate** is a function $P(x)$ that becomes a proposition when x is given a specific value.
- The two main quantifiers are:
  - **Universal Quantifier** ($\forall x\ P(x)$): True if $P(x)$ is true for every x in the domain.
  - **Existential Quantifier** ($\exists x\ P(x)$): True if $P(x)$ is true for at least one x in the domain.

2.2 Negating Quantified Statements

- Negating a universally quantified statement: $\neg(\forall x\ P(x)) \equiv \exists x\ \neg P(x)$
- Negating an existentially quantified statement: $\neg(\exists x\ P(x)) \equiv \forall x\ \neg P(x)$

3 Methods of Proof

3.1 Direct Proof

- A direct proof assumes the premise p is true and logically deduces that q must also be true.

3.2 Proof by Contraposition

- To prove $p \to q$, prove $\neg q \to \neg p$ instead, as these are logically equivalent.

3.3 Proof by Contradiction

- Assume p is true and q is false, then derive a contradiction.

3.4 Proof by Exhaustion

- Check all possible cases to confirm that the statement holds for all of them.

4 Common Mistakes in Proofs

- Assuming what is to be proven (begging the question).
- Confusing a statement and its converse.
- Misusing quantifiers (e.g., confusing $\forall x$ with $\exists x$).

5 Examples

Theorem 5.1. This is a theorem.

Proposition 5.2. This is a proposition.

Principle 5.3. This is a principle.

References