

Dynamic Programming

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Introduction

This is a replication of Adams, et al.(2015), mainly based on Chapter 7.

Overview

1. Replication
 1. Value Function Iteration
 2. Policy Function Iteration
 3. Finite Stochastic Dynamic Programming
2. Exercise

1. Replication

Setup

```
# initialize
rm(list = ls())

# package
if(!require(pacman)) install.packages("packman")
pacman::p_load(
  tidyverse,
  tictoc,
  RColorBrewer
)
```

1-1. Value Function Iteration

```
# one iteration of the value function
IterateVF <- function(V, maxK){
  # basic parameters
  Alpha <- 0.65
  Beta <- 0.9
```

```

Theta <- 1.2

grid <- length(V)
K <- seq(from = 1e-6, to = maxK, length.out = grid)
TV <- rep(0, length(V))
optK <- rep(0, length(V))

# loop through and create new value function for each possible capital value
for(k in 1:grid){
  c <- rep(Theta*(K[k]^Alpha), grid) - K
  c[c<=0] <- rep(0, sum(c<=0))
  u <- log(c)
  candid <- u + Beta*V
  TV[k] <- max(candid)
  optK[k] <- which(candid == max(candid))
}

# time consuming method
# candid <- rep(NA, 1000)
# c <- rep(NA, grid)
# u <- rep(NA, grid)
#
# for(k in 1:grid){
#   for(k_tilde in 1:grid){
#     c[k_tilde] <- Theta*(K[k]^Alpha)-K[k_tilde]
#     c[k_tilde] <- ifelse(c[k_tilde] > 0, c[k_tilde], 0)
#     u[k_tilde] <- log(c[k_tilde])
#     candid[k_tilde] <- u[k_tilde] + Beta*V[k_tilde]
#   }
#   TV[k] <- max(candid)
#   optK[k] <- K[which(candid == max(candid))]
# }

sol <- matrix(c(TV, optK), nrow = length(V), ncol = 2, byrow = FALSE)
return(sol)
}

```

Analytical solution

```

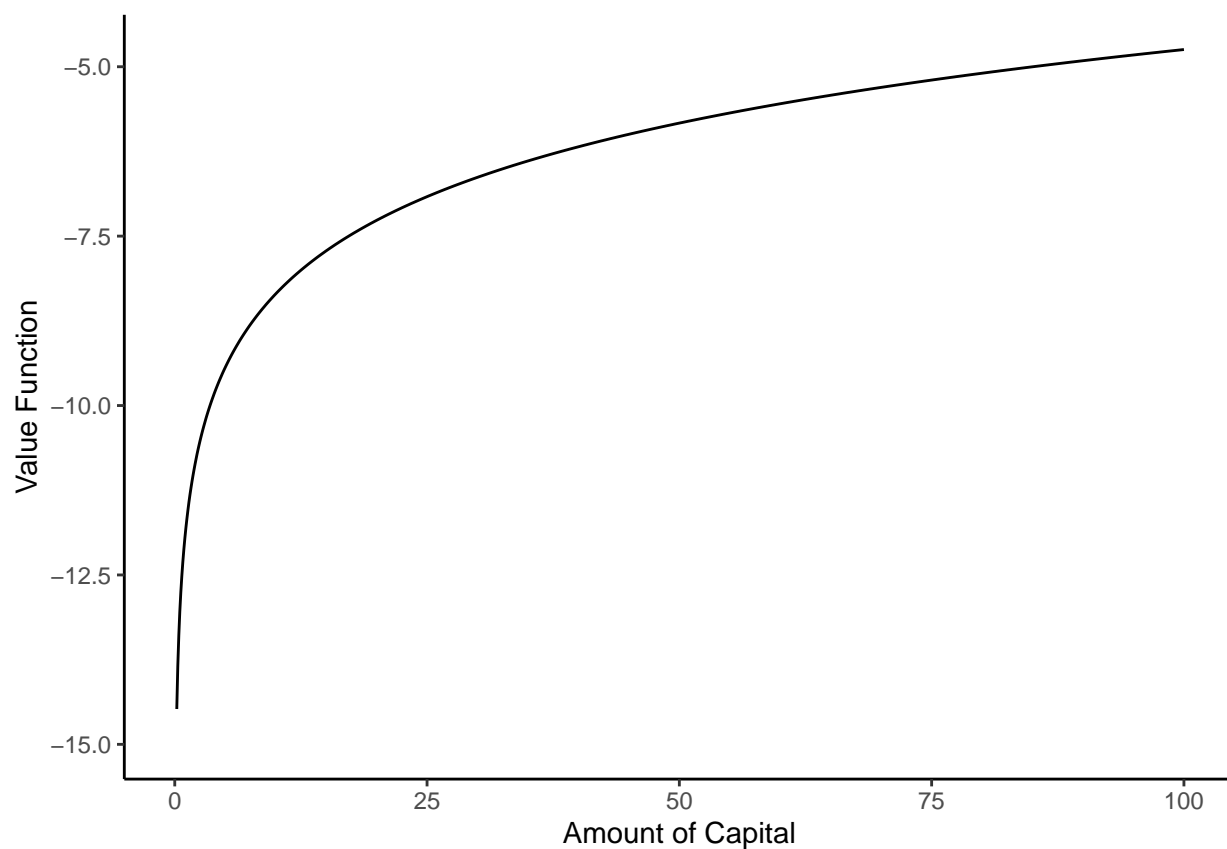
# set parameters, plot analytical solution
Beta <- 0.9
Alpha <- 0.65
Theta <- 1.2
aB <- Alpha*Beta
K <- seq(from = 1e-6, to = 100, length.out = 1000)

E <- Alpha / (1 - aB)
f <- (1/(1-Beta))*(log(Theta*(1-aB))) + aB*log(aB*Theta)/((1-aB)*(1-Beta))
soln <- E*log(K) + f

ggplot() +
  geom_line(aes(x = K, y = soln)) +
  ylim(c(-15, NA)) +

```

```
xlab("Amount of Capital") + ylab("Value Function") +
theme_classic()
```



Iterated graph

```
tic()
n <- 15
# 10 iterations
TV <- matrix(rep(NA, 1000*n), ncol = n)
TV[, 1] <- rep(0, 1000)

for(iter in 1:n){
  cat("Iteration number:", iter, "\n")
  if(iter < n) TV[, iter+1] <- IterateVF(TV[, iter], 100)[, 1]
}
```

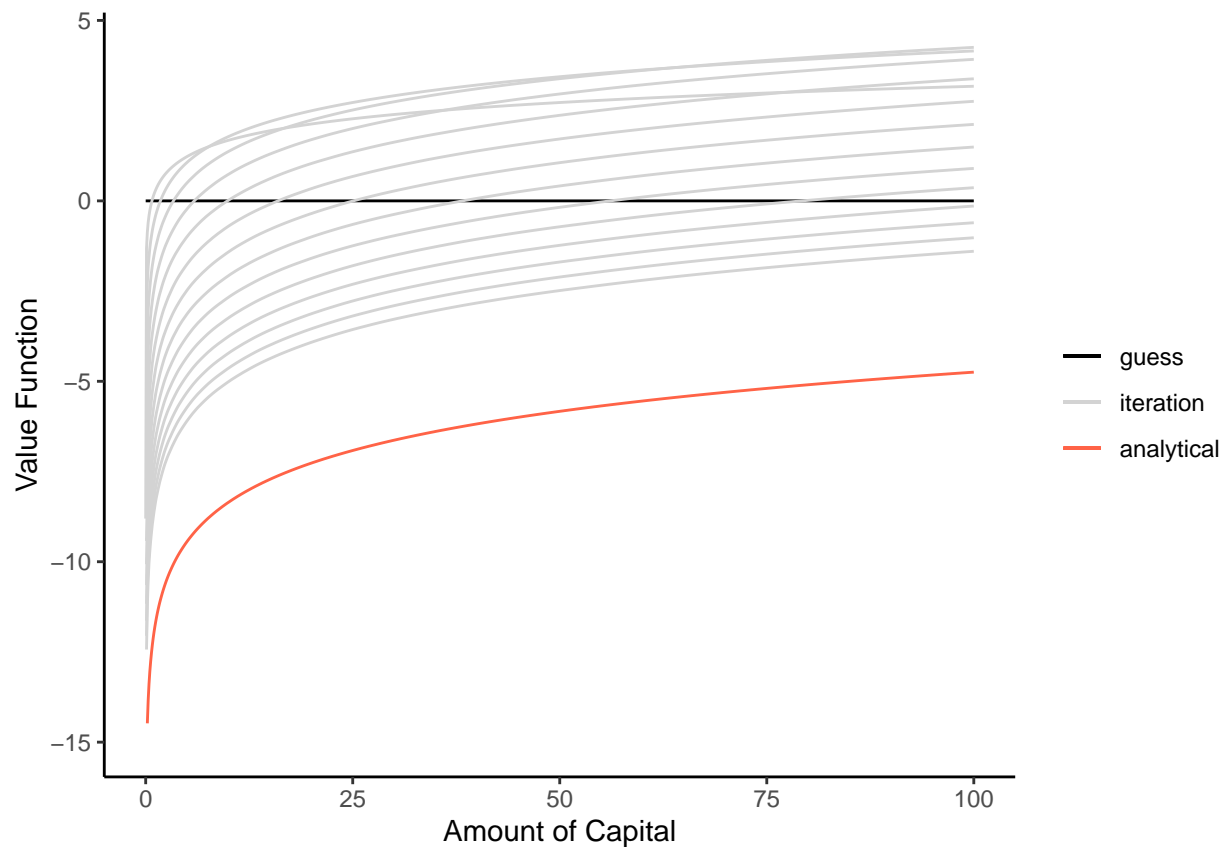
```
## Iteration number: 1
## Iteration number: 2
## Iteration number: 3
## Iteration number: 4
## Iteration number: 5
## Iteration number: 6
## Iteration number: 7
## Iteration number: 8
## Iteration number: 9
```

```
## Iteration number: 10
## Iteration number: 11
## Iteration number: 12
## Iteration number: 13
## Iteration number: 14
## Iteration number: 15
```

```
toc()
```

```
## 0.44 sec elapsed
```

```
# plot
tic()
p1 <- ggplot()
for(i in 1:n){
  df <- data.frame(k = K, tv = TV[, i]) # store TV in a data.frame to layer plots
  if(i == 1){
    p1 <- p1 + geom_line(data = df, aes(x = k, y = tv, color = 'guess'))
  } else{
    p1 <- p1 + geom_line(data = df, aes(x = k, y = tv, color = 'iteration'))
  }
}
p1 <- p1 +
  geom_line(data = df, aes(x = k, y = soln, color = "analytical")) +
  scale_color_manual(
    name = NULL,
    values = c("guess" = "black", "iteration" = "lightgray", "analytical" = "tomato"),
    labels = c("guess", "iteration", "analytical")
  ) +
  ylim(c(-15, NA)) +
  labs(x = "Amount of Capital", y = "Value Function") +
  theme_classic()
p1
```



```
toc()
```

```
## 0.191 sec elapsed
```

Convergence of Value Function

```
tic()
# setting
K <- seq(from = 1e-6, to = 100, length.out = 1000) # grid
V <- rep(0, 1000) # guess
conv <- 100 # criterion for convergence
crit <- 1e-2 # stopping threshold
Iter <- 0 # numbering iteration

# for plot
df <- data.frame(K = K, V = V)
p2 <- ggplot() +
  geom_line(data = df, aes(x = K, y = V, color = 'guess'))

# iteration
while(conv > crit && Iter < 1000){
  Iter <- Iter + 1
  if(Iter %/% 10 == Iter/10) cat("Iteration number:", Iter, "\n")

  # mapping
```

```

sol <- IterateVF(V, 100)
TV <- sol[, 1]

# distance between TV and V
conv <- max(abs(TV-V))

# for plot
df$TV <- TV # store TV in a data.frame to layer plots
p2 <- p2 + geom_line(data = df, aes(x = K, y = TV, color = 'iteration'))

# pass TV to next iteration
V <- TV
}

```

```

## Iteration number: 10
## Iteration number: 20
## Iteration number: 30
## Iteration number: 40
## Iteration number: 50
## Iteration number: 60

```

```

toc()

```

```

## 2.013 sec elapsed

```

```

cat("# of iterations:", Iter)

```

```

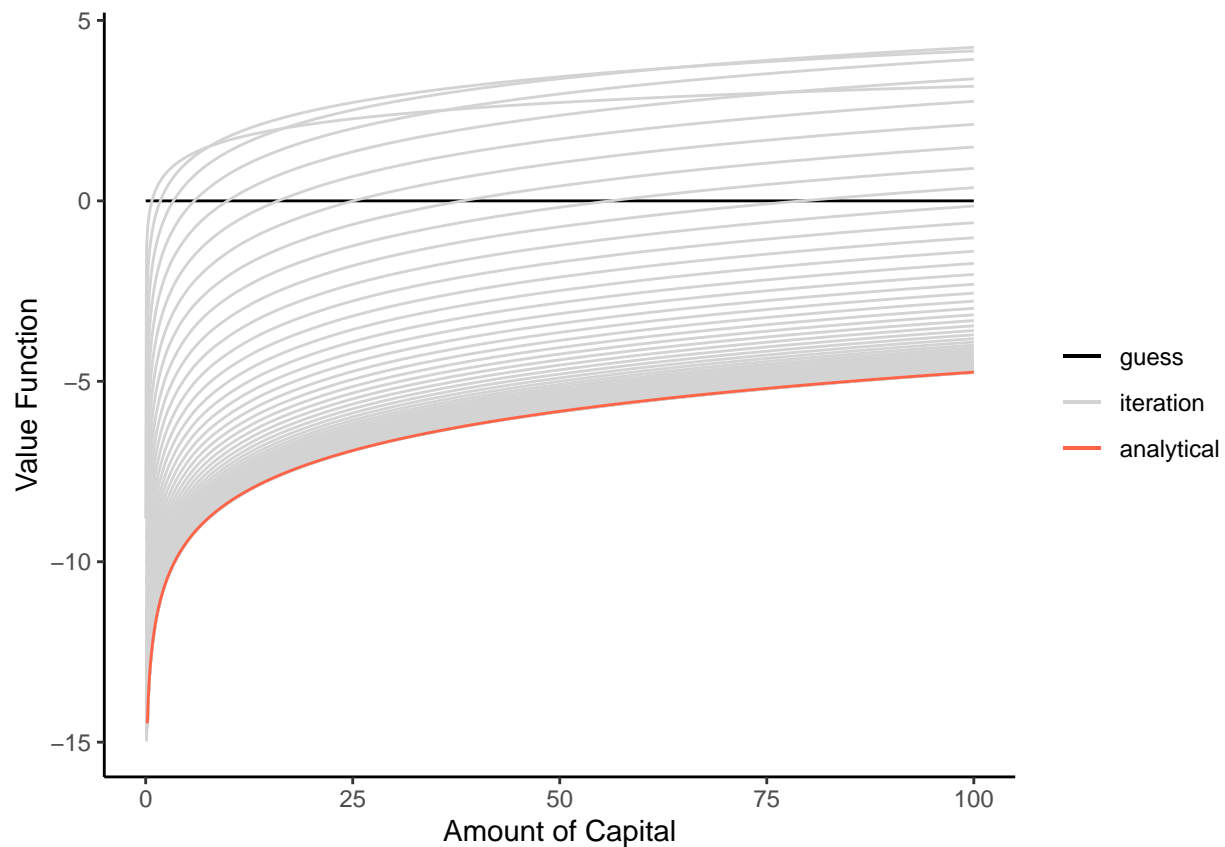
## # of iterations: 66

```

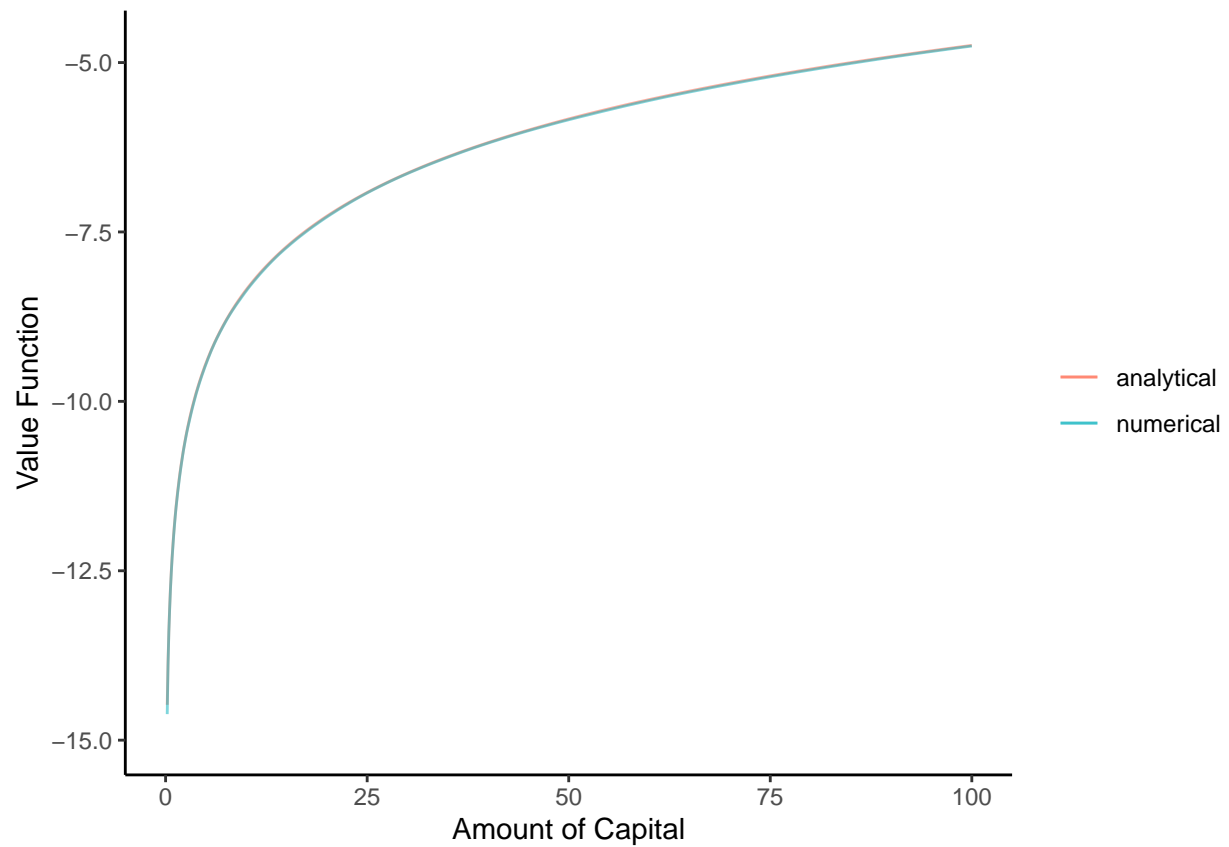
```

# plot
p2 <- p2 +
  geom_line(aes(x = K, y = soln, color = 'analytical')) +
  scale_color_manual(
    name = NULL,
    values = c("guess" = "black", "iteration" = "lightgray", "analytical" = "tomato"),
    labels = c("guess", "iteration", "analytical")
  ) +
  ylim(c(-15, NA)) +
  labs(x = "Amount of Capital", y = "Value Function") +
  theme_classic()
p2

```

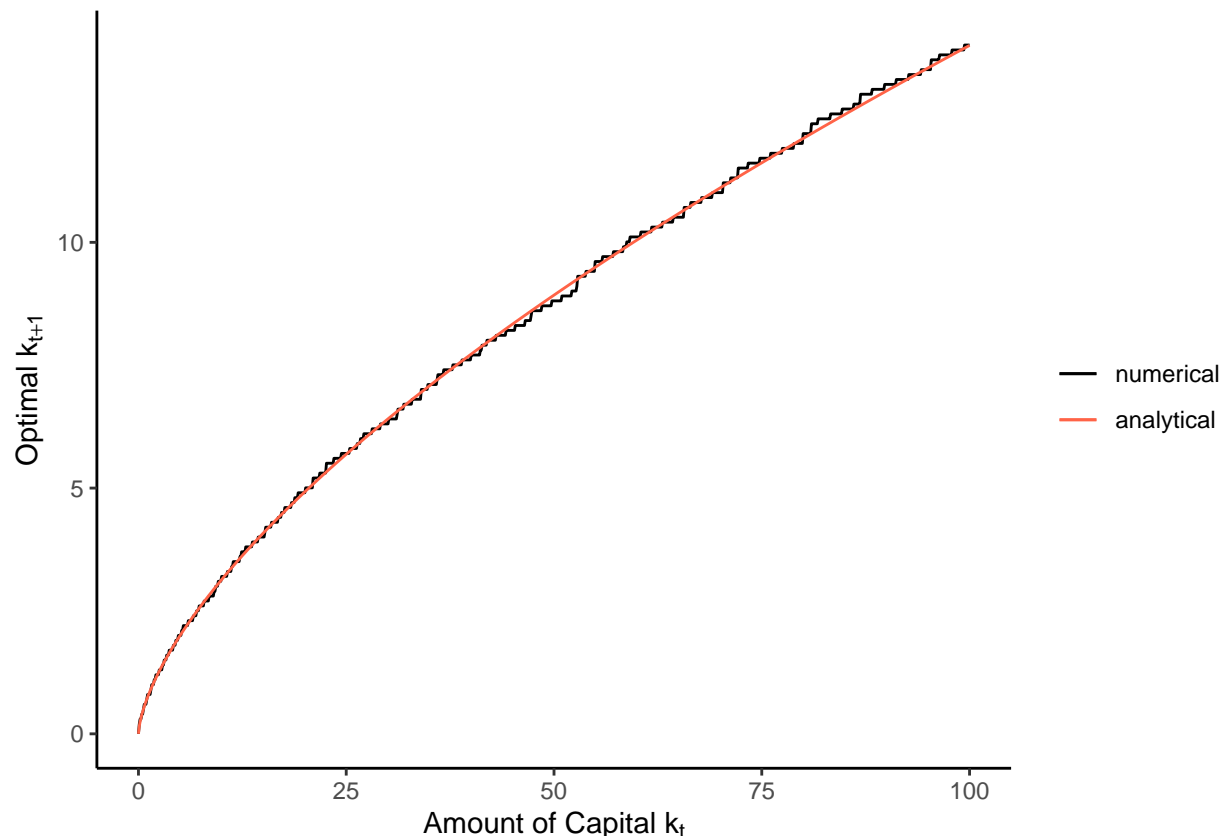


```
# another plot
p3 <-
  ggplot() +
    geom_line(aes(x = K, y = soln, color = 'soln'), alpha = 0.5) +
    geom_line(aes(x = K, y = TV, color = 'TV'), alpha = 0.5) +
    scale_color_manual(
      name = NULL,
      values = c("soln" = "tomato", "TV" = "#00AFBB"),
      labels = c("analytical", "numerical")
    ) +
    ylim(c(-15, NA)) +
    labs(x = "Amount of Capital", y = "Value Function") +
    theme_classic()
p3 # seems overlapped!
```



Calculated Best Path

```
ggplot() +
  geom_line(aes(x = K, y = K[sol[, 2]], color = "numerical")) +
  geom_line(aes(x = K, y = Theta*(K^Alpha)*(aB), color = "analytical")) +
  scale_color_manual(
    name = NULL,
    values = c("numerical" = "black", "analytical" = "tomato"),
    labels = c("numerical", "analytical")
  ) +
  labs(x = expression(paste("Amount of Capital ", k[t])),
       y = expression(paste("Optimal ", k[t+1]))) +
  theme_classic()
```

1-2. Policy Function Iteration

We are to solve the following functional equation:

$$V(k) = \max_{\tilde{k}} \{u(f(k) - \tilde{k}) + \beta V(\tilde{k})\}.$$

Note that

$$V_j = U_j + \beta Q_j V_j,$$

where V_j denotes a vector by which a value function is expressed, and Q_j denotes the binary transition matrix whose $i - l$ entry indicates whether each k_l is the optimal (or \tilde{k}) given $k_i = k$ or not. Here, U_j is a vector whose each entry represents the maximized utility given today's capital k , i.e., each entry is $u(c_j(k))$. Solving this equation, we have

$$V_j = (I - \beta Q_j)^{-1} U_j.$$

This is the idea of the policy function iteration.

```
IteratePolicy <- function(V, maxK){
  # basic parameters
  Alpha <- 0.65
  Beta <- 0.9
  Theta <- 1.2

  grid <- length(V)
  K <- seq(from = 1e-6, to = maxK, length.out = grid)
  #TV <- rep(0, length(V))
```

```

opt <- rep(0, length(V))

# loop through and create new value function for each possible capital value
for(k in 1:grid){
  c <- rep(Theta*(K[k]^Alpha), grid) - K
  c[c<=0] <- rep(0, sum(c<=0))
  u <- log(c)
  candid <- u + Beta*V
  #TV[k] <- max(candid)
  opt[k] <- which(candid == max(candid))
}

kopt <- K[opt]
c <- Theta*K^Alpha - kopt
u <- log(c)
Q <- matrix(rep(0, grid*grid), ncol = grid)

# create the transition matrix
for(k in 1:grid){
  Q[k, opt[k]] <- 1
}

TV <- solve(diag(grid)-Beta*Q)%*%u
sol <- matrix(c(TV, opt), ncol = 2)
V <- TV
return(sol)
}

```

```

tic()
# setting
K <- seq(from = 1e-6, to = 100, length.out = 1000) # grid
V <- rep(0, 1000) # guess
conv <- 100 # criterion for convergence
crit <- 1e-2 # stopping threshold
Iter <- 0 # numbering iteration

# for plot
df <- data.frame(K = K, V = V)
p4 <- ggplot()
# iteration
while(conv>crit && Iter<1000){
  Iter <- Iter + 1
  if(Iter %/% 10 == Iter/10) cat("Iteration number:", Iter, "\n")

  # mapping
  sol <- IteratePolicy(V, 100)
  TV <- sol[, 1]
  opt <- sol[, 2]

  # distance between TV and V
  conv<- max(abs(TV-V))
}

```

```

# for plot
df$K_opt <- K[opt]
p4 <- p4 + geom_line(data = df, aes(x = K, y = K_opt, color = 'iteration'))

# pass TV to next iteration
V <- TV
}
toc()

```

```
## 1.354 sec elapsed
```

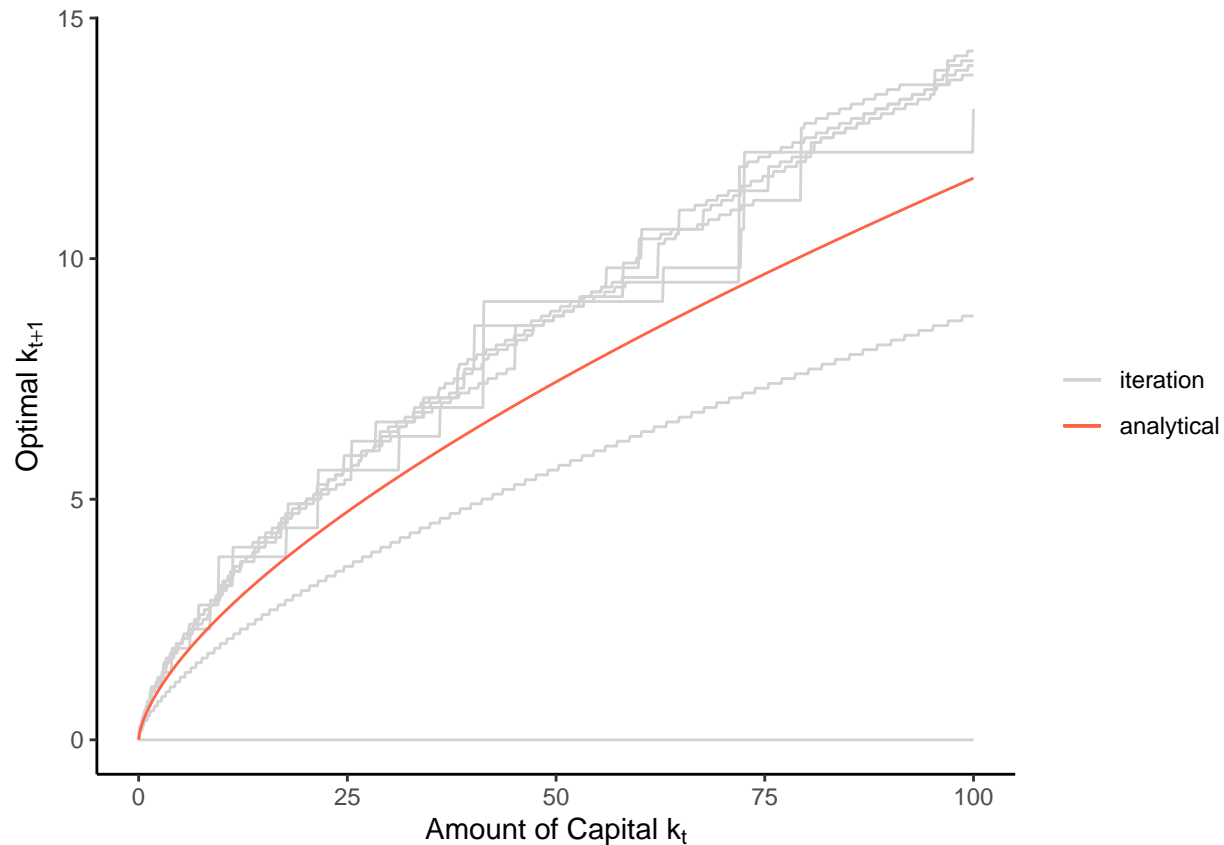
```
cat("# of outer iterations:", Iter)
```

```
## # of outer iterations: 7
```

```

# plot
p4 <- p4 +
  geom_line(aes(x = K, y = aB*(K^Alpha), color = 'analytical')) +
  scale_color_manual(
    name = NULL,
    values = c("iteration" = "lightgray", "analytical" = "tomato"),
    labels = c("iteration", "analytical")
  ) +
  labs(x = expression(paste("Amount of Capital ", k[t])),
    y = expression(paste("Optimal ", k[t+1])) +
  theme_classic()
p4

```



1-3. Finite Stochastic Dynamic Programming

This part is based on Chapter 6, “6.4 Stochastic Dynamic Programming”.

```
tic()
# setup parameters
e <- c(-2, 2)
PI <- c(0.5, 0.5)
Beta <- 0.9
Theta <- 1.2
Alpha <- 0.98
Kl <- 100
grid <- 0.1
t <- 10
K <- seq(from = 0, to = Kl + max(e), by = grid)
V <- cbind(matrix(rep(NA, length(K)*t), ncol = t), matrix(rep(0, length(K)), ncol = 1))
aux <- array(rep(NA, length(K)*length(K)*t), dim = c(length(K), length(K), t))

# loop over preriodes
for(t_iter in seq(from = t, to = 1, by = -1)){
  cat("Currently in period", t_iter, "\n")

  # loop over k_{t}
  for(inK in 1:length(K)){
    # for(inK in 1:length(seq(from = 0, to = Kl, by = grid))){
```

```

# loop over k_{t + 1}
for(outK in 1:inK){
  c <- K[inK] - (K[outK]/Theta)^(1/Alpha) # note this is scaler
  nextKl <- Theta*(K[inK] - c)^Alpha + e[1]
  nextKh <- Theta*(K[inK] - c)^Alpha + e[2]
  nextKl[nextKl < 0] <- rep(0, sum(nextKl < 0))

  EnextV <-
    PI[1] * V[ifelse((round(nextKl/grid) + 1) <= length(K),
                     (round(nextKl/grid) + 1), length(K)), t_iter+1] +
    PI[2] * V[ifelse((round(nextKh/grid) + 1) <= length(K),
                     (round(nextKh/grid) + 1), length(K)), t_iter+1]
    #PI[1] * V[(round(nextKl/grid) + 1), t_iter+1] +
    #PI[2] * V[(round(nextKh/grid) + 1), t_iter+1]
    #c <- ifelse(c <= 0, 1e-100, c)
  aux[inK, outK, t_iter] <- log(c) + Beta*EnextV
}
}
V[, t_iter] <-
  apply(
    X = as.matrix(aux[, , t_iter]), MARGIN = 1, FUN = max, na.rm = TRUE
  )
}

```

```

## Currently in period 10
## Currently in period 9
## Currently in period 8
## Currently in period 7
## Currently in period 6
## Currently in period 5
## Currently in period 4
## Currently in period 3
## Currently in period 2
## Currently in period 1

```

```

toc()

```

```

## 15.982 sec elapsed

```

Simulation

```

tic()
# setup parameters and simulate shocks
set.seed(2022)
people <- 100
epsilon <-
  ifelse(purrr::rbernoulli(people*(t+1), 0.5), 2, -2) %>%
  matrix(ncol = t+1)
vf <-
  rep(NA, people*t) %>%
  matrix(ncol = t)
kap <- # capital

```

```

cbind(
  K1*(rep(1, people)), #  $k_{0}$ 
  matrix(rep(NA, people*t), ncol = t)
)
con <- # consumption
rep(NA, people*t) %>%
matrix(ncol = t)

for(p in 1:people){
  for(t_iter in 1:t){
    position <- round(kap[p, t_iter]/grid + 1)
    vf[p, t_iter] <- V[position, t_iter]
    kap[p, t_iter + 1] <- K[which(aux[position, , t_iter] == vf[p, t_iter])]
    con[p, t_iter] <- kap[p, t_iter] - (kap[p, t_iter+1]/Theta)^(1/Alpha)
    kap[p, t_iter + 1] <- kap[p, t_iter + 1] + epsilon[p + t_iter + 1]
  }
}
toc()

```

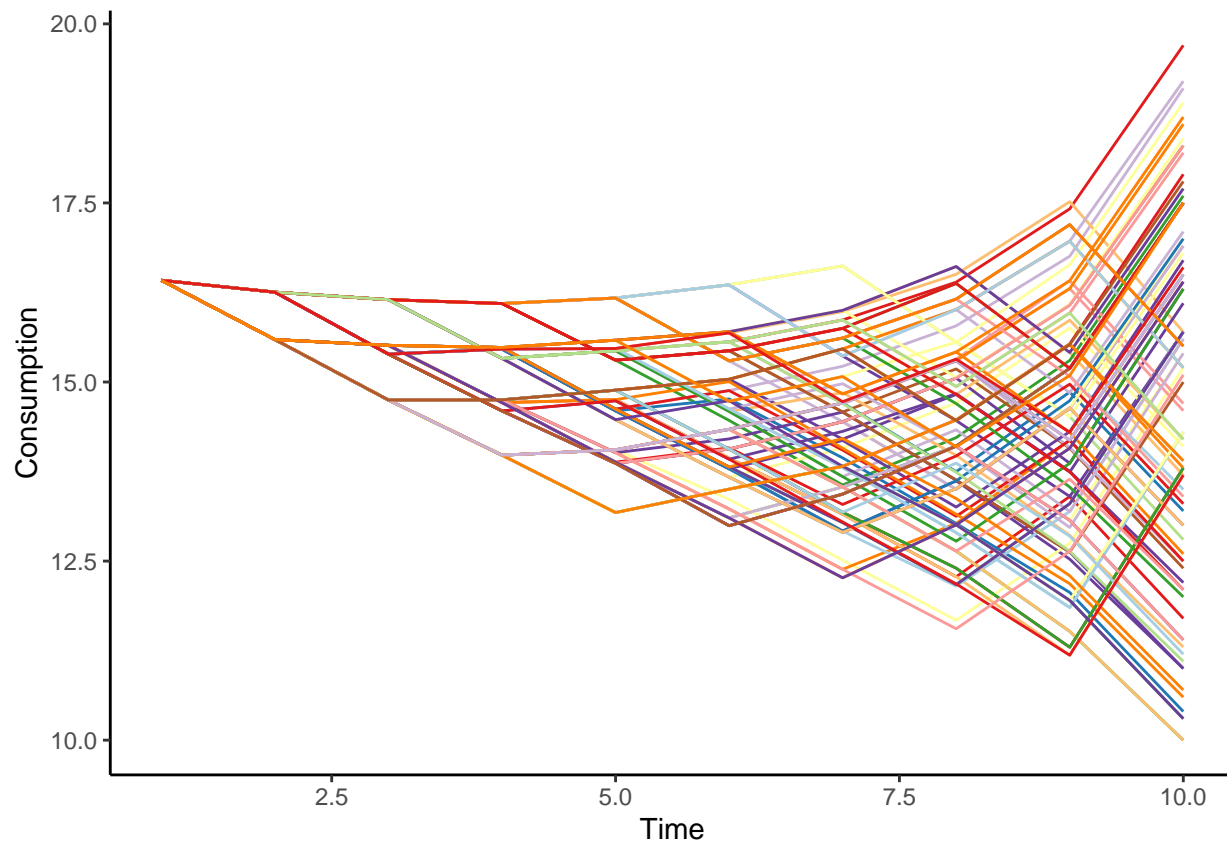
0.035 sec elapsed

plot

```

p_5 <- ggplot()
for(i in 1:people){
  df <- data.frame(time = 1:t, consumption = con[i, ])
  p_5 <- p_5 +
    geom_line(
      data = df, aes(x = time, y = consumption),
      color = sample(brewer.pal(12, "Paired"), 1)
    )
}
p_5 <- p_5 +
  labs(x = "Time", y = "Consumption") +
  theme_classic()
p_5

```



2. Exercise

(i)

We have

$$c_t = \theta k_t^\alpha - k_{t+1}.$$

```

Beta <- 0.9
Alpha <- 0.65
Theta <- 1.2
aB <- Alpha*Beta

K <- seq(from = 1e-6, to = 100, length.out = 1000) # grid
V <- rep(0, 1000) # guess
conv <- 100 # criterion for convergence
crit <- 1e-2 # stopping threshold
Iter <- 0 # numbering iteration

# for plot
df <- data.frame(K = K, V = V)

# iteration
while(conv>crit && Iter<1000){

```

```

Iter <- Iter + 1
if(Iter %/% 10 == Iter/10) cat("Iteration number:", Iter, "\n")

# mapping
sol <- IterateVF(V, 100)
TV <- sol[, 1]

# distance between TV and V
conv <- max(abs(TV-V))

# for plot
df$TV <- TV # store TV in a data.frame to layer plots

# pass TV to next iteration
V <- TV
}

```

```

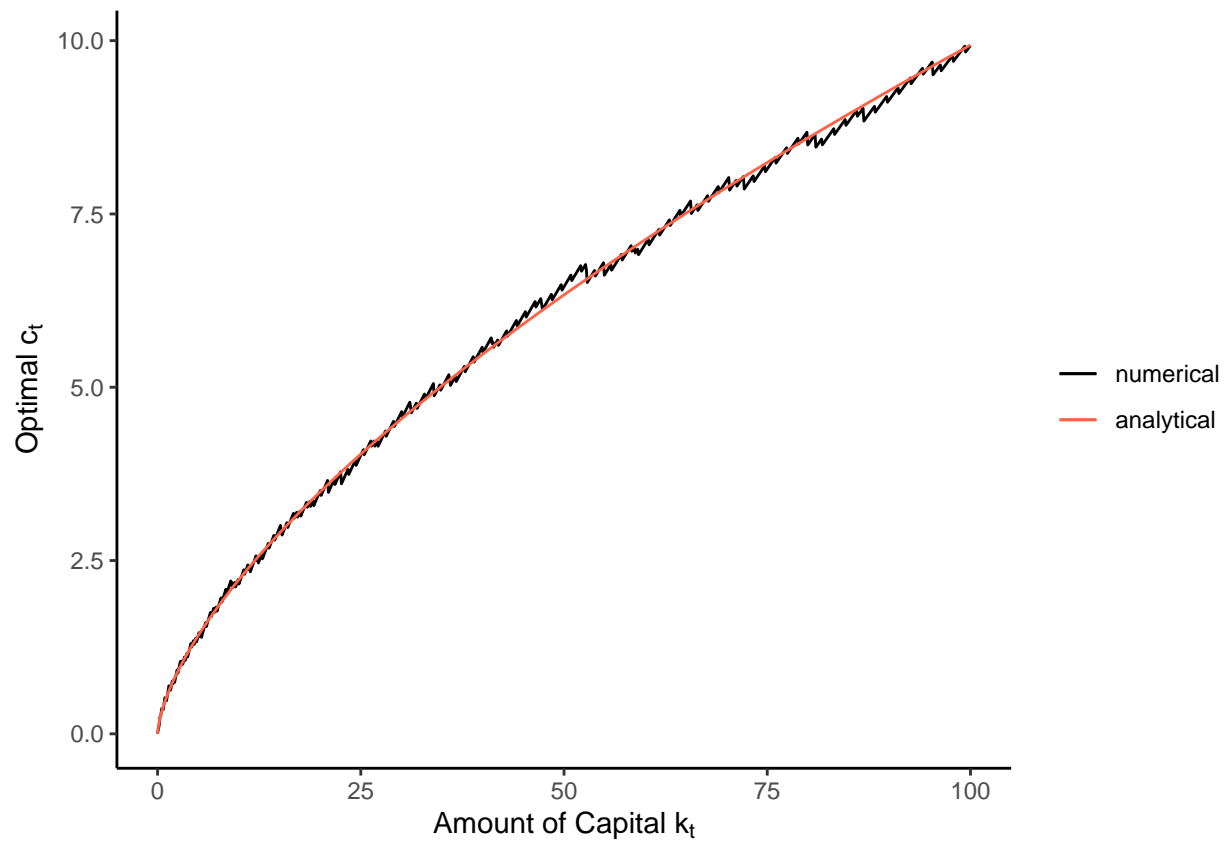
## Iteration number: 10
## Iteration number: 20
## Iteration number: 30
## Iteration number: 40
## Iteration number: 50
## Iteration number: 60

```

```

C_opt <- Theta*K^Alpha - K[sol[, 2]]
ggplot() +
  geom_line(aes(x = K, y = C_opt, color = "numerical")) +
  geom_line(aes(x = K, y = Theta*(K^Alpha)*(1-aB), color = "analytical")) +
  scale_color_manual(
    name = NULL,
    values = c("numerical" = "black", "analytical" = "tomato"),
    labels = c("numerical", "analytical")
  ) +
  labs(x = expression(paste("Amount of Capital ", k[t])),
       y = expression(paste("Optimal ", c[t]))) +
  theme_classic()

```

(ii)

Note that the transition equation is

$$k_{t+1} = f(k_t) - c_t + \varepsilon_{t+1}.$$

The value function is written as

$$V(k_t) = \max_{k_{t+1}} \{u(c_t) + \beta E_t[V(k_{t+1} - \varepsilon_{t+1})]\}.$$

(ii)-(a)

```
# one iteration of the value function
IterateStochastic <- function(V, maxK){
  # basic parameters
  Alpha <- 0.65
  Beta <- 0.9
  Theta <- 1.2

  grid <- length(V)
  K <- seq(from = 1e-6, to = maxK, length.out = grid)
  TV <- rep(0, length(V))
  optK <- rep(0, length(V))
}
```

```

# loop through and create new value function for each possible capital value
for(k in 1:grid){
  c <- rep(Theta*(K[k]^Alpha), grid) - K
  c[c<=0] <- rep(0, sum(c<=0))
  u <- log(c)
  candid <- u + Beta*V
  TV[k] <- max(candid)
  optK[k] <- which(candid == max(candid))
}

sol <- matrix(c(TV, optK), nrow = length(V), ncol = 2, byrow = FALSE)
return(sol)
}

```

(ii)-(b)

(iii)

Reference

- Adams, A., D. Clarke, and S. Quinn. Microeconometrics and MATLAB. Oxford University Press, 2015.