Dynamic Choice on an Infinite Horizon

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0. Introduction

This is a replication of Adams, et al.(2015), Chapter 7.

1. Replication

Setup

```
# initialize
rm(list = ls())

# package
if(!require(pacman)) install.packages("packman")
pacman::p_load(
  tidyverse,
  tictoc
)
```

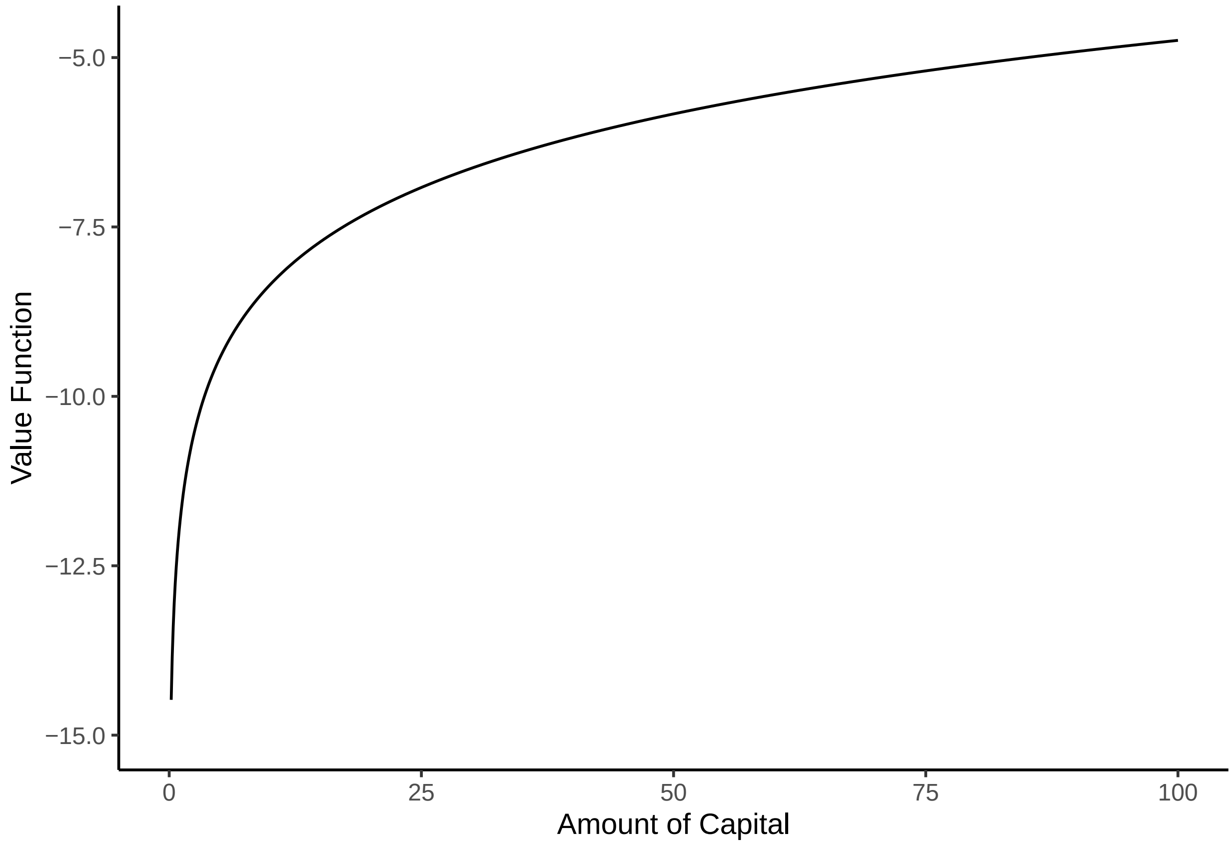
1-1. Value Function Iteration

```
# one iteration of the value function
IterateVF <- function(V, maxK){
  # basic parameters
  Alpha <- 0.65
  Beta <- 0.9
  Theta <- 1.2

  grid <- length(V)
  K <- seq(from = 1e-6, to = maxK, length.out = grid)
  TV <- rep(0, length(V))
  optK <- rep(0, length(V))

  # loop through and create new value function for each possible capital value
  for(k in 1:grid){
    c <- rep(Theta*(K[k]^Alpha), grid) - K
    c[c<=0] <- rep(0, sum(c<=0))
```

```
    u <- log(c)
    candid <- u + Beta*V
    TV[k] <- max(candid)
    optK[k] <- which(candid == max(candid))
  }

  # time consuming method
  # candid <- rep(NA, 1000)
  # c <- rep(NA, grid)
  # u <- rep(NA, grid)
  #
  # for(k in 1:grid){
  #   for(k_tilde in 1:grid){
  #     c[k_tilde] <- Theta*(K[k]^Alpha)-K[k_tilde]
  #     c[k_tilde] <- ifelse(c[k_tilde] > 0, c[k_tilde], 0)
  #     u[k_tilde] <- log(c[k_tilde])
  #     candid[k_tilde] <- u[k_tilde] + Beta*V[k_tilde]
  #   }
  #   TV[k] <- max(candid)
  #   optK[k] <- K[which(candid == max(candid))]
  # }

  sol <- matrix(c(TV, optK), nrow = length(V), ncol = 2, byrow = FALSE)
  return(sol)
}
```

Analytical solution

```
# set parameters, plot analytical solution
Beta <- 0.9
Alpha <- 0.65
Theta <- 1.2
aB <- Alpha*Beta
K <- seq(from = 1e-6, to = 100, length.out = 1000)

E <- Alpha / (1 - aB)
f <- (1/(1-Beta))*(log(Theta*(1-aB))) + aB*log(aB*Theta)/((1-aB)*(1-Beta))
soln <- E*log(K) + f

ggplot() +
  geom_line(aes(x = K, y = soln)) +
  ylim(c(-15, NA)) +
  xlab("Amount of Capital") + ylab("Value Function") +
  theme_classic()
```



Iterated graph

```
tic()
n <- 15
# 10 iterations
TV <- matrix(rep(NA, 1000*n), ncol = n)
TV[, 1] <- rep(0, 1000)

for(iter in 1:n){
  cat("Iteration number:", iter, "\n")
  if(iter < n) TV[, iter+1] <- IterateVF(TV[, iter], 100)[, 1]
}
```

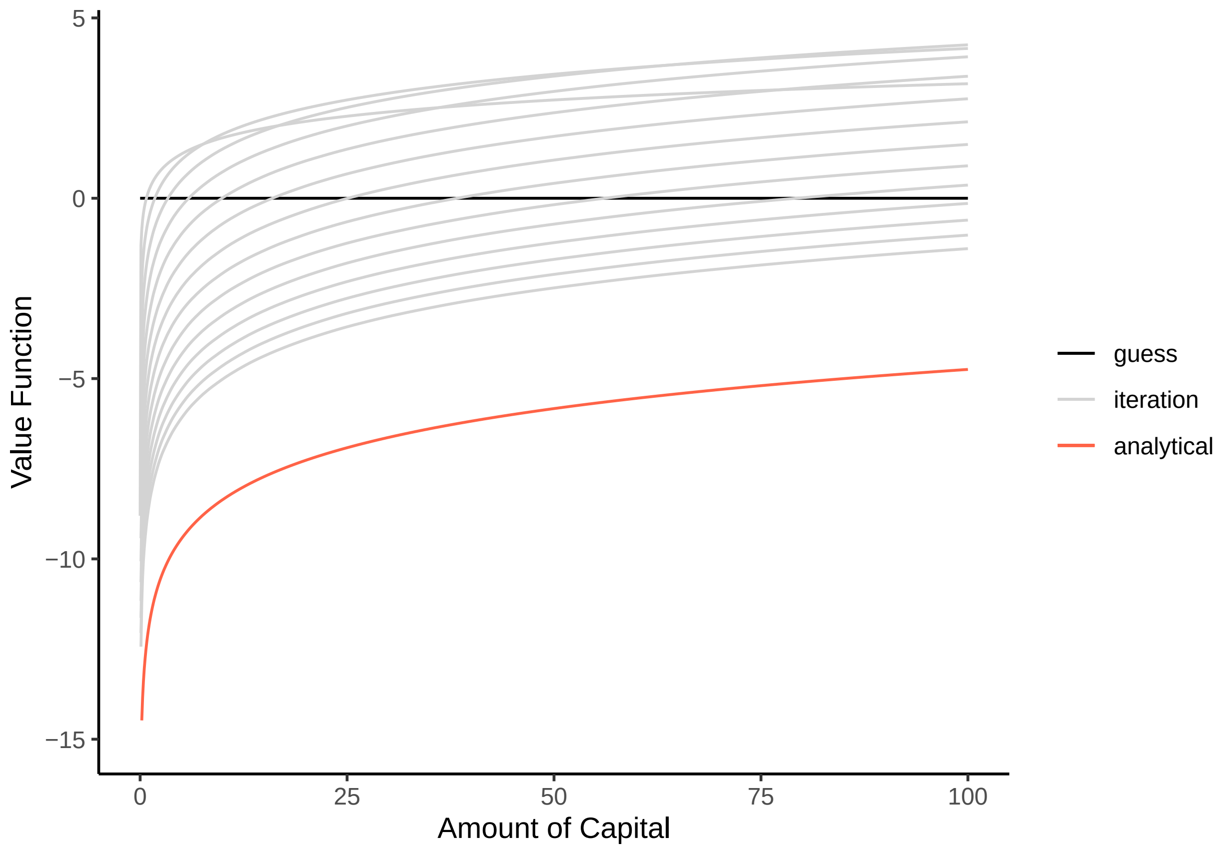
```
## Iteration number: 1
## Iteration number: 2
## Iteration number: 3
## Iteration number: 4
## Iteration number: 5
## Iteration number: 6
## Iteration number: 7
## Iteration number: 8
## Iteration number: 9
## Iteration number: 10
## Iteration number: 11
## Iteration number: 12
## Iteration number: 13
```

```
## Iteration number: 14
## Iteration number: 15
```

```r
toc()
```

```
## 0.416 sec elapsed
```

```r
# plot
tic()
p1 <- ggplot()
for(i in 1:n){
  df <- data.frame(k = K, tv = TV[, i]) # store TV in a data.frame to layer plots
  if(i == 1){
    p1 <- p1 + geom_line(data = df, aes(x = k, y = tv, color = 'guess'))
  } else{
    p1 <- p1 + geom_line(data = df, aes(x = k, y = tv, color = 'iteration'))
  }
}
p1 <- p1 +
  geom_line(data = df, aes(x = k, y = soln, color = "analytical")) +
  scale_color_manual(
    name = NULL,
    values = c("guess" = "black", "iteration" = "lightgray", "analytical" = "tomato"),
    labels = c("guess", "iteration", "analytical")
  ) +
  ylim(c(-15, NA)) +
  labs(x = "Amount of Capital", y = "Value Function") +
  theme_classic()
p1
```




```
toc()
```

```
## 0.187 sec elapsed
```

Convergence of Value Function

```
tic()
# setting
K <- seq(from = 1e-6, to = 100, length.out = 1000) # grid
V <- rep(0, 1000) # guess
conv <- 100 # criterion for convergence
crit <- 1e-2 # stopping threshold
Iter <- 0 # numbering iteration

# for plot
df <- data.frame(K = K, V = V)
p2 <- ggplot() +
  geom_line(data = df, aes(x = K, y = V, color = 'guess'))

# iteration
while(conv>crit && Iter<1000){
  Iter <- Iter + 1
  if(Iter %/% 10 == Iter/10) cat("Iteration number:", Iter, "\n")

  # mapping
```

```r
  sol <- IterateVF(V, 100)
  TV <- sol[, 1]

  # distance between TV and V
  conv <- max(abs(TV-V))

  # for plot
  df$TV <- TV # store TV in a data.frame to layer plots
  p2 <- p2 + geom_line(data = df, aes(x = K, y = TV, color = 'iteration'))

  # pass TV to next iteration
  V <- TV
}
```

```
## Iteration number: 10
## Iteration number: 20
## Iteration number: 30
## Iteration number: 40
## Iteration number: 50
## Iteration number: 60
```

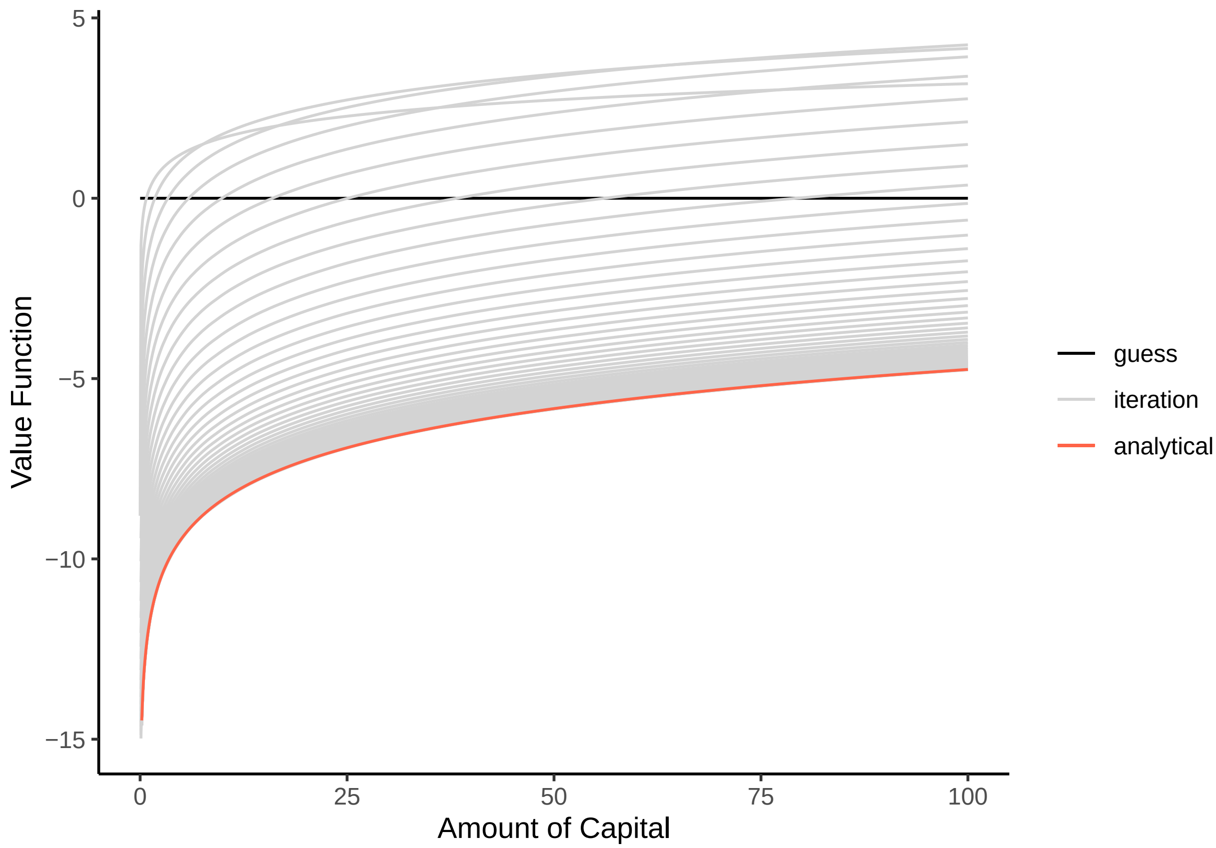
```r
toc()
```

```
## 1.994 sec elapsed
```

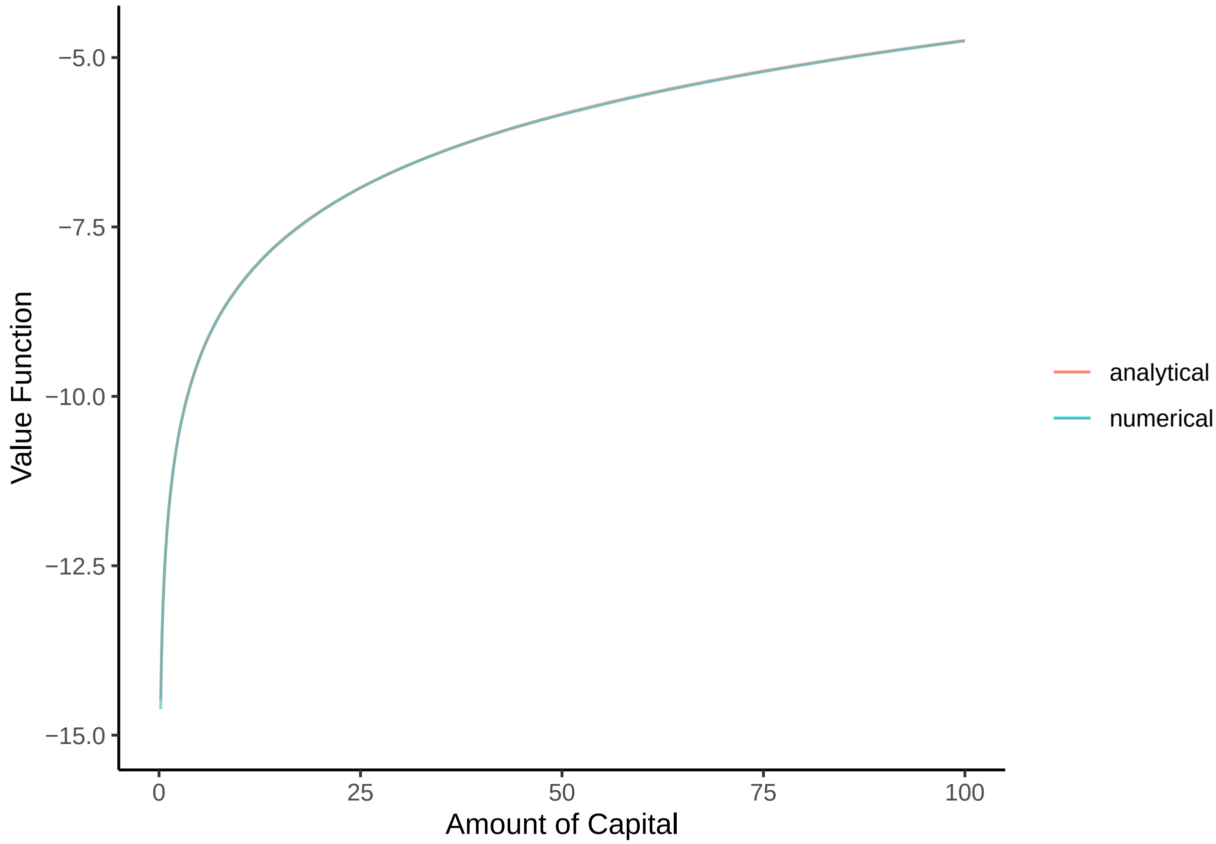
```r
cat("# of iterations:", Iter)
```

```
## # of iterations: 66
```

```r
# plot
p2 <- p2 +
  geom_line(aes(x = K, y = soln, color = 'analytical')) +
  scale_color_manual(
    name = NULL,
    values = c("guess" = "black", "iteration" = "lightgray", "analytical" = "tomato"),
    labels = c("guess", "iteration", "analytical")
  ) +
  ylim(c(-15, NA)) +
  labs(x = "Amount of Capital", y = "Value Function") +
  theme_classic()
p2
```

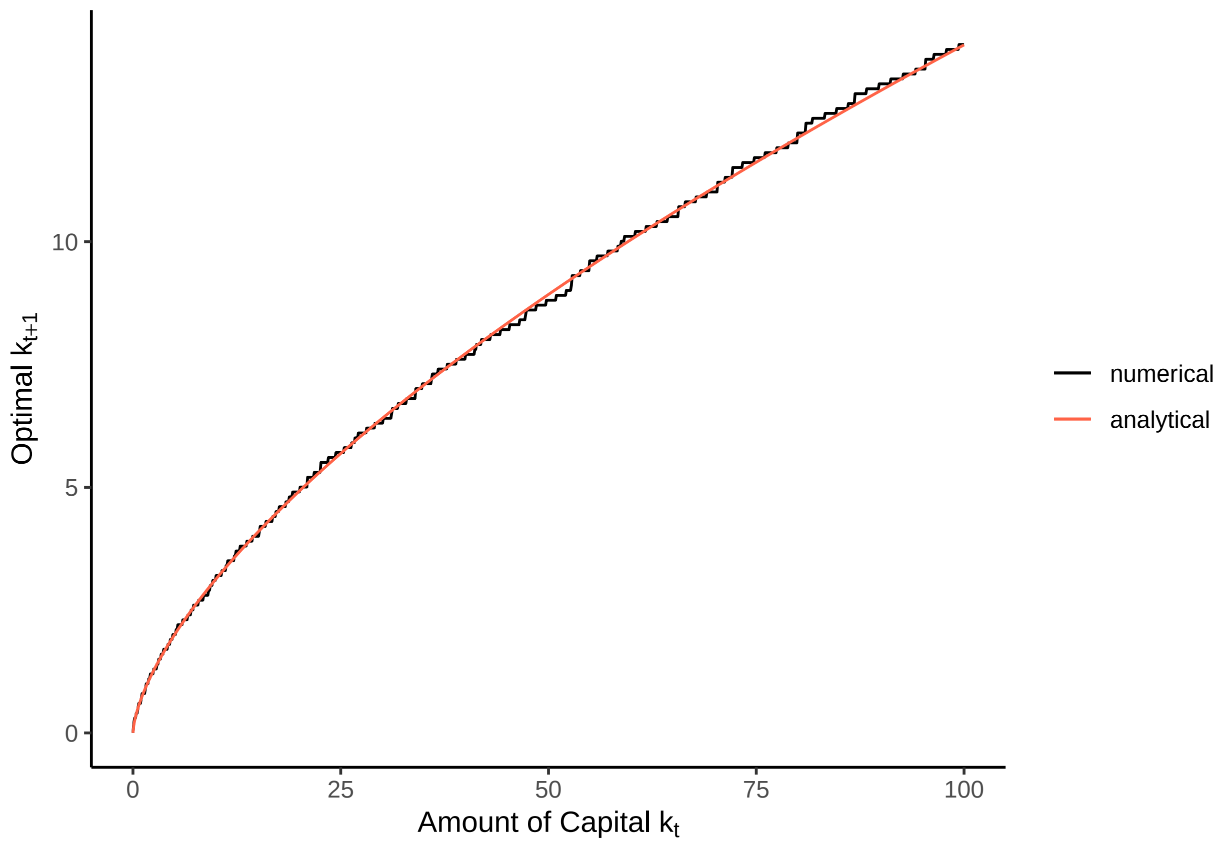



```r
# another plot
p3 <-
  ggplot() +
  geom_line(aes(x = K, y = soln, color = 'soln'), alpha = 0.5) +
  geom_line(aes(x = K, y = TV, color = 'TV'), alpha = 0.5) +
  scale_color_manual(
    name = NULL,
    values = c("soln" = "tomato", "TV" = "#00AFBB"),
    labels = c("analytical", "numerical")
  ) +
  ylim(c(-15, NA)) +
  labs(x = "Amount of Capital", y = "Value Function") +
  theme_classic()
p3 # seems overlapped!
```



Calculated Best Path

```
ggplot() +
  geom_line(aes(x = K, y = K[sol[, 2]], color = "numerical")) +
  geom_line(aes(x = K, y = Theta*(K^Alpha)*(aB), color = "analytical")) +
  scale_color_manual(
    name = NULL,
    values = c("numerical" = "black", "analytical" = "tomato"),
    labels = c("numerical", "analytical")
  ) +
  labs(x = expression(paste("Amount of Capital ", k[t])),
       y = expression(paste("Optimal ", k[t+1]))) +
  theme_classic()
```



1-2. Policy Function Iteration

We are to solve the following functional equation:

$$V(k) = \max_{\tilde{k}} \{u(f(k) - \tilde{k}) + \beta V(\tilde{k})\}.$$

Note that

$$V_j = U_j + \beta Q_j V_j,$$

where V_j denotes a vector by which a value function is expressed, and Q_j denotes the binary transition matrix whose $i - l$ entry indicates whether each k_l is the optimal (or $\tilde{k}$) given $k_i = k$ or not. Here, U_j is a vactor whose each entry represents the maximized utility given today's capital k, i.e., each entry is $u(c_j(k))$. Solving this equation, we have

$$V_j = (I - \beta Q_j)^{-1} U_j.$$

This is the idea of the policy function iteration.

```
IteratePolicy <- function(V, maxK){
  # basic parameters
  Alpha <- 0.65
  Beta <- 0.9
  Theta <- 1.2

  grid <- length(V)
  K <- seq(from = 1e-6, to = maxK, length.out = grid)
  #TV <- rep(0, length(V))
```

```r
  opt <- rep(0, length(V))

  # loop through and create new value function for each possible capital value
  for(k in 1:grid){
    c <- rep(Theta*(K[k]^Alpha), grid) - K
    c[c<=0] <- rep(0, sum(c<=0))
    u <- log(c)
    candid <- u + Beta*V
    #TV[k] <- max(candid)
    opt[k] <- which(candid == max(candid))
  }

  kopt <- K[opt]
  c <- Theta*K^Alpha - kopt
  u <- log(c)
  Q <- matrix(rep(0, grid*grid), ncol = grid)

  # create the transition matrix
  for(k in 1:grid){
    Q[k, opt[k]] <- 1
  }

  TV <- solve(diag(grid)-Beta*Q)%*%u
  sol <- matrix(c(TV, opt), ncol = 2)
  V <- TV
  return(sol)
}
```

```r
tic()
# setting
K <- seq(from = 1e-6, to = 100, length.out = 1000) # grid
V <- rep(0, 1000) # guess
conv <- 100 # criterion for convergence
crit <- 1e-2 # stopping threshold
Iter <- 0 # numbering iteration


# for plot
df <- data.frame(K = K, V = V)
p4 <- ggplot()
# iteration
while(conv>crit && Iter<1000){
  Iter <- Iter + 1
  if(Iter %/% 10 == Iter/10) cat("Iteration number:", Iter, "\n")

  # mapping
  sol <- IteratePolicy(V, 100)
  TV <- sol[, 1]
  opt <- sol[, 2]

  # distance between TV and V
  conv<- max(abs(TV-V))
```

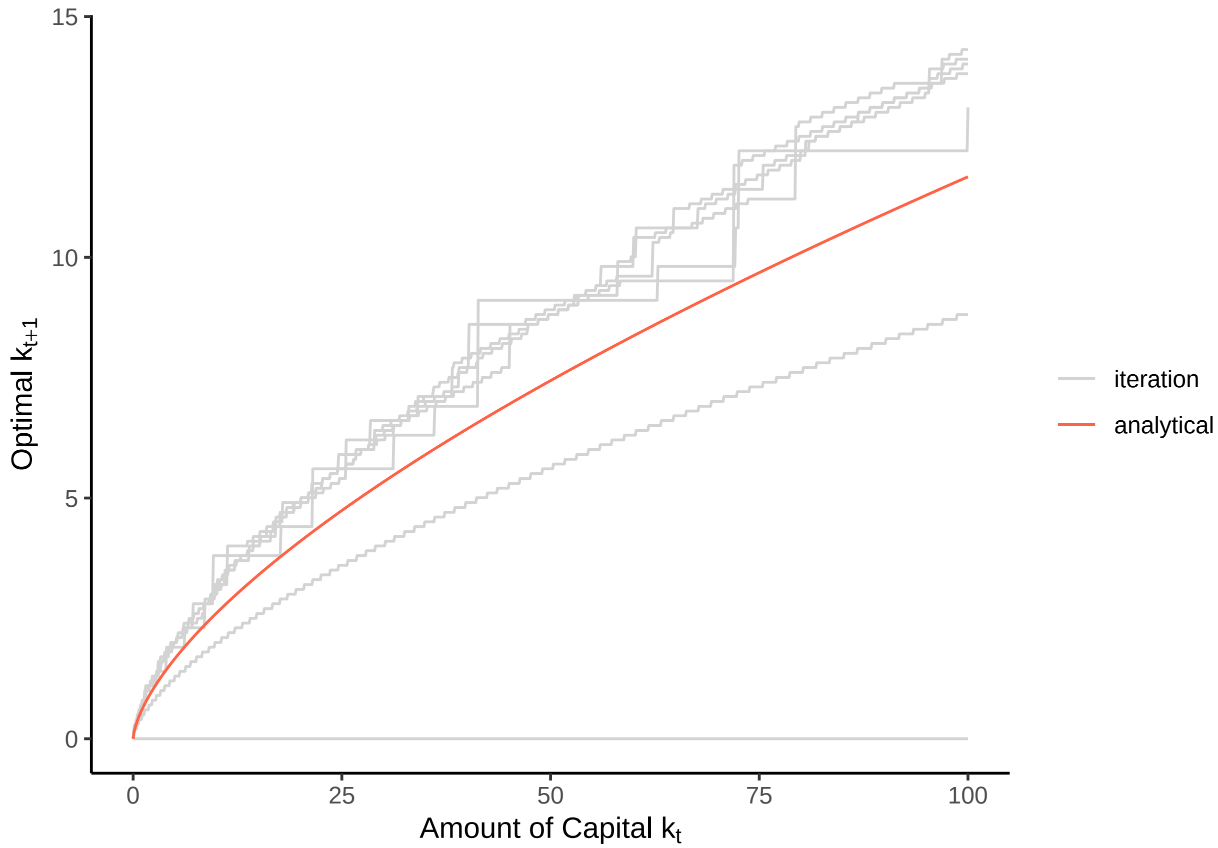
```r
  # for plot
  df$K_opt <- K[opt]
  p4 <- p4 + geom_line(data = df, aes(x = K, y = K_opt, color = 'iteration'))

  # pass TV to next iteration
  V <- TV
}
toc()
```

```
## 1.236 sec elapsed
```

```r
cat("# of outer iterations:", Iter)
```

```
## # of outer iterations: 7
```

```r
# plot
p4 <- p4 +
  geom_line(aes(x = K, y = aB*(K^Alpha), color = 'analytical')) +
  scale_color_manual(
    name = NULL,
    values = c("iteration" = "lightgray", "analytical" = "tomato"),
    labels = c("iteration", "analytical")
  ) +
  labs(x = expression(paste("Amount of Capital ", k[t])),
       y = expression(paste("Optimal ", k[t+1]))) +
  theme_classic()
p4
```



2. Exercise

(i)

We have

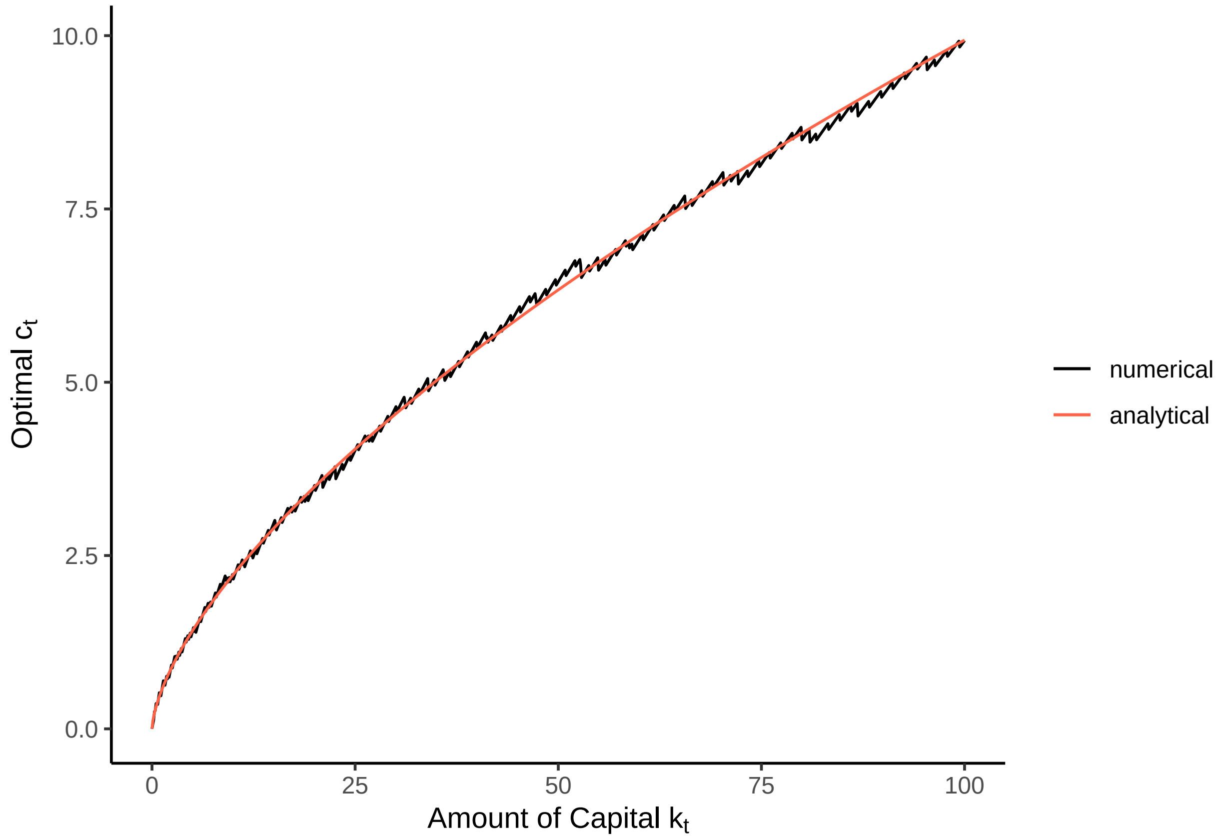
$$c_t = \theta k_t^{\alpha} - k_{t+1}.$$

```
Beta <- 0.9
Alpha <- 0.65
Theta <- 1.2
aB <- Alpha*Beta


K <- seq(from = 1e-6, to = 100, length.out = 1000) # grid
V <- rep(0, 1000) # guess
conv <- 100 # criterion for convergence
crit <- 1e-2 # stopping threshold
Iter <- 0 # numbering iteration

# for plot
df <- data.frame(K = K, V = V)

# iteration
while(conv>crit && Iter<1000){
```

```r
  Iter <- Iter + 1
  if(Iter %/% 10 == Iter/10) cat("Iteration number:", Iter, "\n")

  # mapping
  sol <- IterateVF(V, 100)
  TV <- sol[, 1]

  # distance between TV and V
  conv <- max(abs(TV-V))

  # for plot
  df$TV <- TV # store TV in a data.frame to layer plots

  # pass TV to next iteration
  V <- TV
}
```

```
## Iteration number: 10
## Iteration number: 20
## Iteration number: 30
## Iteration number: 40
## Iteration number: 50
## Iteration number: 60
```

```r
C_opt <- Theta*K^Alpha - K[sol[, 2]]
ggplot() +
  geom_line(aes(x = K, y = C_opt, color = "numerical")) +
  geom_line(aes(x = K, y = Theta*(K^Alpha)*(1-aB), color = "analytical")) +
  scale_color_manual(
    name = NULL,
    values = c("numerical" = "black", "analytical" = "tomato"),
    labels = c("numerical", "analytical")
  ) +
  labs(x = expression(paste("Amount of Capital ", k[t])),
       y = expression(paste("Optimal ", c[t]))) +
  theme_classic()
```



(ii)

The value function is written as

$$V(k) = \max_{\tilde{k}} \{u(f(k) - \tilde{k}) + \beta E_t[V(\tilde{k})]\}$$

(ii)-(a)

(ii)-(b)

(iii)

Reference

- Adams, A., D. Clarke, and S. Quinn. Microeconometrics and MATLAB. Oxford University Press, 2015.