■ テーパ管のWebster方程式の解

```
Remove[f, p, u, S]
```

直径がd1->d2,長さLとする。半径の傾きtは

```
t = (d2 - d1) / (2 L)
```

$$\frac{-d1 + d2}{2\ L}$$

断面積函数は

```
S[x_] := Pi (t x + d1 / 2)^2 // Simplify
```

```
S[x]
```

$$\frac{Pi\ (d1\ (L - x) + d2\ x)^2}{4\ L^2}$$

Webster方程式は次のようになる。q[x]は速度ポテンシャル

```
eq = q''[x] + D[S[x], x] / S[x] q'[x] + k^2 q[x] == 0
```

$$k^2\ q[x] + \frac{2\ (-d1 + d2)\ q'[x]}{d1\ (L - x) + d2\ x} + q''[x] == 0$$

```
ss = DSolve[eq, q[x], x]
```

Power::infy: 無限式 $\frac{1}{0}$ が見つかりました.

$$\{\{q[x] \to \frac{C[1]}{E^{Sqrt[-k^2]\ x}\ (-(d1\ L) + d1\ x - d2\ x)} + \frac{E^{Sqrt[-k^2]\ x}\ C[2]}{2\ Sqrt[-k^2]\ (-(d1\ L) + d1\ x - d2\ x)}\}\}$$

```
f[x_] = q[x] /. ss /. (Sqrt[-k^2] → I k) // First // FullSimplify
```

$$\frac{-(2\ Sqrt[-k^2]\ C[1] + E^{2\ I\ k\ x}\ C[2])}{2\ E^{I\ k\ x}\ Sqrt[-k^2]\ (d1\ (L - x) + d2\ x)}$$

音圧をpで表せば、

```
p[x_] = f[x] I ω ρ
```

$$\frac{\frac{-I}{2}\ \rho\ \omega\ (2\ Sqrt[-k^2]\ C[1] + E^{2\ I\ k\ x}\ C[2])}{E^{I\ k\ x}\ Sqrt[-k^2]\ (d1\ (L - x) + d2\ x)}$$

体積速度は、

```
u[x_] = -D[f[x], x] S[x]
```

$$\left(\text{Pi}\,(\text{d1}\,(L-x)+\text{d2}\,x)^2\left(\frac{I\,E^{I\,k\,x}\,k\,C[2]}{\text{Sqrt}[-k^2]\,(\text{d1}\,(L-x)+\text{d2}\,x)}-\frac{(-\text{d1}+\text{d2})\,(2\,\text{Sqrt}[-k^2]\,C[1]+E^{2\,I\,k\,x}\,C[2])}{2\,E^{I\,k\,x}\,\text{Sqrt}[-k^2]\,(\text{d1}\,(L-x)+\text{d2}\,x)^2}-\frac{\frac{I}{2}\,k\,(2\,\text{Sqrt}[-k^2]\,C[1]+E^{2\,I\,k\,x}\,C[2])}{E^{I\,k\,x}\,\text{Sqrt}[-k^2]\,(\text{d1}\,(L-x)+\text{d2}\,x)}\right)\right)\Big/\left(4\,L^2\right)$$

出口での音圧及び体積速度をp2,u2とし、

```
eq1 = p[L] == p2
```

$$\frac{\frac{-I}{2}\,\rho\,\omega\,(2\,\text{Sqrt}[-k^2]\,C[1]+E^{2\,I\,k\,L}\,C[2])}{\text{d2}\,E^{I\,k\,L}\,\text{Sqrt}[-k^2]\,L} == \text{p2}$$

```
eq2 = u[L] == u2
```

$$\left(\text{d2}^2\,\text{Pi}\left(\frac{I\,E^{I\,k\,L}\,k\,C[2]}{\text{d2}\,\text{Sqrt}[-k^2]\,L}-\frac{(-\text{d1}+\text{d2})\,(2\,\text{Sqrt}[-k^2]\,C[1]+E^{2\,I\,k\,L}\,C[2])}{2\,\text{d2}^2\,E^{I\,k\,L}\,\text{Sqrt}[-k^2]\,L^2}-\frac{\frac{I}{2}\,k\,(2\,\text{Sqrt}[-k^2]\,C[1]+E^{2\,I\,k\,L}\,C[2])}{\text{d2}\,E^{I\,k\,L}\,\text{Sqrt}[-k^2]\,L}\right)\right)\Big/\,4 == \text{u2}$$

```
s1 = Solve[{eq1, eq2}, {C[1], C[2]}] // First
```

$$\{C[1]\to-(E^{I\,k\,L}\,(-(\text{d1}\,\text{d2}\,\text{p2}\,\text{Pi})+\text{d2}^2\,\text{p2}\,\text{Pi}-I\,\text{d2}^2\,k\,L\,\text{p2}\,\text{Pi}-4\,I\,L\,\text{u2}\,\rho\,\omega))\,/\,(2\,\text{d2}\,k\,\text{Pi}\,\rho\,\omega),$$
$$C[2]\to-((k\,(-(\text{d1}\,\text{d2}\,\text{p2}\,\text{Pi})+\text{d2}^2\,\text{p2}\,\text{Pi}+I\,\text{d2}^2\,k\,L\,\text{p2}\,\text{Pi}-4\,I\,L\,\text{u2}\,\rho\,\omega))\,/\,(\text{d2}\,E^{I\,k\,L}\,\text{Sqrt}[-k^2]\,\text{Pi}\,\rho\,\omega))\}$$

入口でのそれをp1,u1とおく

```
p1 = p[0] /. s1
```

$$\Big(\frac{-I}{2}\,\rho\,\omega\,\Big(-((E^{I\,k\,L}\,\text{Sqrt}[-k^2]\,(-(\text{d1}\,\text{d2}\,\text{p2}\,\text{Pi})+\text{d2}^2\,\text{p2}\,\text{Pi}-I\,\text{d2}^2\,k\,L\,\text{p2}\,\text{Pi}-4\,I\,L\,\text{u2}\,\rho\,\omega))\,/\,(\text{d2}\,k\,\text{Pi}\,\rho\,\omega))-(k\,(-(\text{d1}\,\text{d2}\,\text{p2}\,\text{Pi})+\text{d2}^2\,\text{p2}\,\text{Pi}+I\,\text{d2}^2\,k\,L\,\text{p2}\,\text{Pi}-4\,I\,L\,\text{u2}\,\rho\,\omega))\,/\,(\text{d2}\,E^{I\,k\,L}\,\text{Sqrt}[-k^2]\,\text{Pi}\,\rho\,\omega)\Big)\Big)\,/\,(\text{d1}\,\text{Sqrt}[-k^2]\,L)$$

```
u1 = u[0] /. s1
```

$$
\begin{aligned}
&\Big(d1^2\,\mathrm{Pi}\,\Big(\big(I\,(-(d1\,d2\,p2\,\mathrm{Pi}) + d2^2\,p2\,\mathrm{Pi} + I\,d2^2\,k\,L\,p2\,\mathrm{Pi} - \\
&\qquad 4\,I\,L\,u2\,\rho\,\omega)\big) / \big(d1\,d2\,E^{I\,k\,L}\,L\,\mathrm{Pi}\,\rho\,\omega\big) - \\
&\quad \Big((-d1 + d2)\,\Big(-\Big(\big(E^{I\,k\,L}\,\mathrm{Sqrt}[-k^2] \\
&\qquad\qquad (-(d1\,d2\,p2\,\mathrm{Pi}) + d2^2\,p2\,\mathrm{Pi} - \\
&\qquad\qquad I\,d2^2\,k\,L\,p2\,\mathrm{Pi} - 4\,I\,L\,u2\,\rho\,\omega)\big) / \\
&\qquad\qquad (d2\,k\,\mathrm{Pi}\,\rho\,\omega)\Big) - \\
&\qquad \big(k\,(-(d1\,d2\,p2\,\mathrm{Pi}) + d2^2\,p2\,\mathrm{Pi} + I\,d2^2\,k\,L\,p2\,\mathrm{Pi} - \\
&\qquad\qquad 4\,I\,L\,u2\,\rho\,\omega)\big) / \big(d2\,E^{I\,k\,L}\,\mathrm{Sqrt}[-k^2]\,\mathrm{Pi}\,\rho\,\omega\big)\Big)\Big) \\
&\qquad / \big(2\,d1^2\,\mathrm{Sqrt}[-k^2]\,L^2\big) - \\
&\quad \Big(\frac{I}{2}\,k\,\Big(-\Big(\big(E^{I\,k\,L}\,\mathrm{Sqrt}[-k^2] \\
&\qquad\qquad (-(d1\,d2\,p2\,\mathrm{Pi}) + d2^2\,p2\,\mathrm{Pi} - \\
&\qquad\qquad I\,d2^2\,k\,L\,p2\,\mathrm{Pi} - 4\,I\,L\,u2\,\rho\,\omega)\big) / \\
&\qquad\qquad (d2\,k\,\mathrm{Pi}\,\rho\,\omega)\Big) - \\
&\qquad \big(k\,(-(d1\,d2\,p2\,\mathrm{Pi}) + d2^2\,p2\,\mathrm{Pi} + I\,d2^2\,k\,L\,p2\,\mathrm{Pi} - \\
&\qquad\qquad 4\,I\,L\,u2\,\rho\,\omega)\big) / \big(d2\,E^{I\,k\,L}\,\mathrm{Sqrt}[-k^2]\,\mathrm{Pi}\,\rho\,\omega\big)\Big)\Big) \\
&\qquad / \big(d1\,\mathrm{Sqrt}[-k^2]\,L\big)\Big)\Big) / 4
\end{aligned}
$$

```
m11 = Coefficient[p1, p2] // FullSimplify
```

$$\frac{d2\,k\,L\,\mathrm{Cos}[k\,L] + d1\,\mathrm{Sin}[k\,L] - d2\,\mathrm{Sin}[k\,L]}{d1\,k\,L}$$

```
m12 = Coefficient[p1, u2] // FullSimplify
```

$$\frac{4\,I\,\rho\,\omega\,\mathrm{Sin}[k\,L]}{d1\,d2\,k\,\mathrm{Pi}}$$

```
m21 = Coefficient[u1, p2] // FullSimplify
```

$$
\begin{aligned}
&\Big(\frac{-I}{4}\,\mathrm{Pi}\,\big((d1 - d2)^2\,k\,L\,\mathrm{Cos}[k\,L] - \\
&\qquad ((d1 - d2)^2 + d1\,d2\,k^2\,L^2)\,\mathrm{Sin}[k\,L]\big)\Big) / (k\,L^2\,\rho\,\omega)
\end{aligned}
$$

```
m22 = Coefficient[u1, u2] // FullSimplify
```

$$\frac{d1\,k\,L\,\mathrm{Cos}[k\,L] - d1\,\mathrm{Sin}[k\,L] + d2\,\mathrm{Sin}[k\,L]}{d2\,k\,L}$$

伝達行列は

```
mm = {{m11, m12}, {m21, m22}}
```

$$\left\{\left\{\frac{\text{d2 k L Cos[k L] + d1 Sin[k L] - d2 Sin[k L]}}{\text{d1 k L}},\ \frac{4\ \text{I}\ \rho\ \omega\ \text{Sin[k L]}}{\text{d1 d2 k Pi}}\right\},\ \left\{\left(\frac{-\text{I}}{4}\ \text{Pi}\ ((\text{d1 - d2})^2\ \text{k L Cos[k L]} - ((\text{d1 - d2})^2 + \text{d1 d2 k}^2\ \text{L}^2)\ \text{Sin[k L]})\right) / (\text{k L}^2\ \rho\ \omega),\ \frac{\text{d1 k L Cos[k L] - d1 Sin[k L] + d2 Sin[k L]}}{\text{d2 k L}}\right\}\right\}$$

```
mm // MatrixForm // TraditionalForm
```

$$\begin{pmatrix} \frac{\mathrm{d2}\, k\, L \cos(k\, L)+\mathrm{d1} \sin(k\, L)-\mathrm{d2} \sin(k\, L)}{\mathrm{d1}\, k\, L} & \frac{4\, i\, \rho\, \omega \sin(k\, L)}{\mathrm{d1}\, \mathrm{d2}\, k\, \pi} \\ -\frac{i \pi\, ((\mathrm{d1}-\mathrm{d2})^2\, k\, L \cos(k\, L)-((\mathrm{d1}-\mathrm{d2})^2+\mathrm{d1}\, \mathrm{d2}\, k^2\, L^2) \sin(k\, L))}{4\, k\, L^2\, \rho\, \omega} & \frac{\mathrm{d1}\, k\, L \cos(k\, L)-\mathrm{d1} \sin(k\, L)+\mathrm{d2} \sin(k\, L)}{\mathrm{d2}\, k\, L} \end{pmatrix}$$

ストレート管の時は

```
mm /. {d1 → d, d2 → d} // Simplify // TraditionalForm
```

$$\begin{pmatrix} \cos(k\, L) & \frac{4\, i\, \rho\, \omega \sin(k\, L)}{d^2\, k\, \pi} \\ \frac{i\, d^2\, k\, \pi \sin(k\, L)}{4\, \rho\, \omega} & \cos(k\, L) \end{pmatrix}$$