Fourier Analysis
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Any time-varying signal can be constructed by adding together sine waves of appropriate frequency, amplitude, and phase. Fourier analysis is a technique that is used to determine which sine waves a given signal is made of, i.e. to deconstruct the signal into its constituent sine waves. The result is expressed as sine wave amplitude as a function of frequency. If a frequency has a very small or zero amplitude associated with it, then it does not contribute to the signal. Thus, a signal consisting of a single sine wave will have a peak matching the frequency and amplitude of the sine wave, and zero contributions from all other frequencies. The relative phase of the constituent sine waves is rarely used in mechanical engineering applications. However, knowledge of the frequency content of a signal can be very useful. For example, if a piece of rotating machinery has an unwanted vibration, the frequency of the vibration, determined by Fourier analysis, can be used to determine the source of the vibration. The frequency content of turbulent flow in an automobile engine is used to improve engine designs. The frequency content of freeway noise is used to design noise abatement systems. The frequency content of human speech is used to design hearing aids.

The Fourier transform is used to perform the mapping between a signal as a function of time, and the constituent amplitudes as a function of frequency. In other words, a signal can be viewed in the *time domain* or in the *frequency domain*. The frequency domain representation is also called the *spectrum* of the signal. (Note, plural is *spectra.*) For certain signals, this can be performed analytically with calculus. For arbitrary signals, the signal must first be digitized, and a Discrete Fourier Transform (DFT) performed. The standard numerical algorithm used for the DFT is called the Fast Fourier Transform (FFT) or Discrete FFT (DFFT). Due to limitations inherent in digitization and numerical algorithms, the FFT will result in an approximation to the spectrum.

In this experiment we will study the frequency domain representations of a number of periodic and non-periodic signals for which there are analytic solutions, as well as FFT solutions. The limitations of the FFT will also be examined.

Fourier Series Representation of a Periodic Signal
A periodic function is any function for which

$$y(t) = y(t + nT), \qquad n = 0, \pm 1, \pm 2, ... \qquad (1)$$

for all t. The period T is the length of the time at which the function begins to repeat itself. Clearly the trigonometric functions $\sin wt$ and $\cos wt$ are periodic with period $T = 1/f = 2p/w$, where f is the frequency in cycles/s (Hz) and w is the circular (angular) frequency in radians/s. Figure 1 shows such a periodic function.

Any piecewise-continuous, integrable periodic function may be represented by a superposition of sine and cosine functions

$$y(t) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty}\left(a_n \cos n\omega_0 t + b_n \sin n\omega_0 t\right) \tag{2}$$

where w_0 is the fundamental frequency and $w_n = n w_0$ is the nth harmonic of the periodic function. Equation (2) is the Fourier series representation of the periodic function $y(t)$.
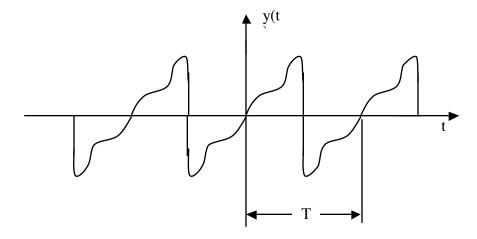

Figure 1. Example of a periodic function.

The orthogonality property of the sine and cosine functions gives the following expression for the Fourier coefficients a_n and b_n:

$$a_n = \frac{2}{T}\int_{-T/2}^{+T/2} y(t)\cos(n\omega_0 t)dt, \qquad n = 0,1,2,\ldots \tag{3}$$

$$b_n = \frac{2}{T}\int_{-T/2}^{+T/2} y(t)\sin(n\omega_0 t)dt, \qquad n = 0,1,2,\ldots \tag{4}$$

Example 1. Fourier Series of a Square Wave

Consider a square wave, as shown in figure 2a, described by the function

$$y(t) = \begin{cases} -1, & -T/2 < t < 0 \\ 1, & 0 < t < T/2 \end{cases}. \tag{5}$$

Evaluating the Fourier coefficients yields

$$a_n = 0 \qquad\qquad n = 1, 2, \dots \qquad\qquad (6)$$

$$b_n = \begin{cases} 0, & n \ even \\ \dfrac{4}{n\pi}, & n \ odd \end{cases}. \qquad\qquad (7)$$

Hence, the Fourier series is

$$y(t) = \frac{4}{\pi}\left(\sin\omega_0 t + \frac{1}{3}\sin 3\omega_0 t + \frac{1}{5}\sin 5\omega_0 t + \dots \right). \qquad\qquad (8)$$

This series representation can help us view y as a function of frequency. For this case the frequency spectrum is discrete, described by the Fourier coefficients (Equation 7). The spectrum has spikes at odd multiples of the fundamental frequency (odd harmonics) with a height equal to the value of the b_n. Thus, it is a series of spikes at the frequencies $w_0, 3w_0, 5w_0, \dots$ with relative amplitudes $1, 1/3, 1/5, \dots$ respectively.
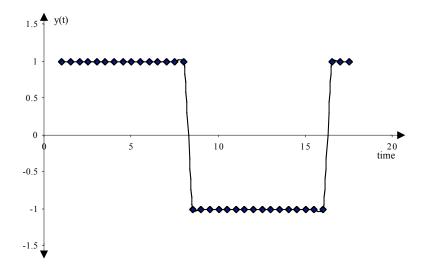

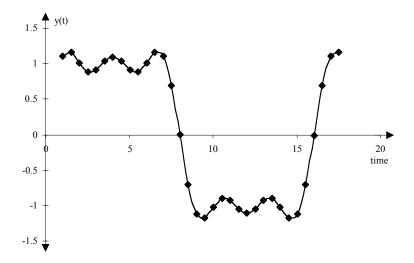**Figure 2a. Square wave in the time domain.**

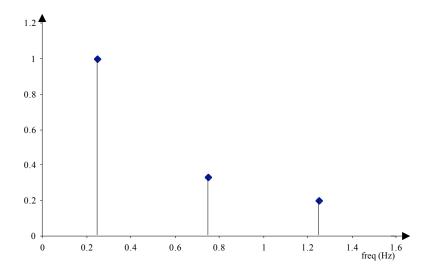**Figure 2b. Fourier series representation of square wave (3 terms in the series)**.

Figure 2c. Frequency spectrum of square wave (3 terms, fundamental freq = 0.25 Hz)

Example 2. Fourier Series of a Triangle Wave

Consider the triangle wave, described by the function

$$y(t) = \begin{cases} 1 + 4\dfrac{t}{T}, & -T/2 < t < 0 \\ 1 - 4\dfrac{t}{T}, & 0 < t < T/2 \end{cases} \tag{9}$$

Evaluating the Fourier coefficients yields

$$a_0 = 0 \tag{10}$$

$$a_n = \begin{cases} 0, & n \ even \\ \dfrac{8}{n^2 \pi^2}, & n \ odd \end{cases} \tag{11}$$

$$b_n = 0 \qquad\qquad n = 1, 2, \ldots \tag{12}$$

Hence, the Fourier series is

$$y(t) = \frac{8}{\pi^2} \left(\cos \omega_0 t + \frac{1}{3^2} \cos 3\omega_0 t + \frac{1}{5^2} \cos 5\omega_0 t + \ldots \right). \tag{13}$$

Time Domain

Frequency Domain

(a) Sine Wave

$f = \dfrac{1}{T}$

(b) Square Wave

$\dfrac{1}{T}$ $\dfrac{3}{T}$ $\dfrac{5}{T}$

(c) Damped Sine Wave

(d) Impulse

(e) Pulse Train

$\dfrac{1}{\tau}$ $\dfrac{2}{\tau}$

(f) Single Pulse

$\dfrac{1}{\tau}$ $\dfrac{2}{\tau}$

Figure 5. Analytic frequency spectrum examples

Figure 5 shows the representation of several common functions in the time and frequency domains, as determined by the analytic (exact) Fourier transform.

Discrete Fourier Transform

The digital representation of a continuous signal y(t) in the time domain is the series

$$y_j = y(t_j) = y(j\delta t), \qquad j = 0, 1, 2, ..., N \qquad (32)$$

where $dt = 1/S_{rate}$ is the sampling interval, S_{rate} is the sampling rate or sampling frequency, and N is the number of samples.

The DFT (Discrete Fourier Transform) of the discrete series y_j is given by

$$Y_k = Y(f_k) = \delta t \sum_{j=0}^{N-1} y_j e^{-i2\pi jk/N}, \qquad k = 0, 1, 2, ..., N-1 \qquad (33)$$

where

$$f_k = \frac{k}{N\delta t}. \qquad (34)$$

Equation (33) performs the numerical integration corresponding to the continuous integration in the definition of the Fourier transform. The values of $T(f_k)$ represent $k = N/2$ discrete amplitudes spaced at discrete frequency intervals having a resolution of S_{rate}/N. Note, **the maximum frequency of the spectrum obtained from the DFT is $S_{rate}/2$,** i.e. the Nyquist frequency; thus there is no information on frequencies above one half of the sampling rate. If the signal has content at frequencies above this value, aliasing will occur. To avoid this, signals are often filtered to remove frequency content above the Nyquist before they are sampled. Note that applying a digital filter after sampling will be ineffective, since aliasing will have already occurred.

From Eq. (34), it can be seen that reducing the sampling frequency S_{rate} leads to improved frequency resolution. However, in order to avoid aliasing, the sample frequency must not be reduced to less than what is required by the Nyquist criterion. The frequency resolution ma be improved without changing the sample rate by increasing the number of samples taken, but this is not always possible or practical.

The most widely used algorithm for computing the discrete Fourier transform is the FFT (Fast Fourier Transform). This algorithm is implemented in LabView, Excel, Matlab, Mathcad, Mathematica, etc. The size of the data array sent to the FFT, N, must be a power of two (otherwise you have a Slow Fourier Transform). Many implementations return an array of equal size, N points, but only the first N/2 are valid. The second half will simply be a mirror image of the first half. If only N/2 points are returned, they are all valid. In addition, the frequencies are usually not returned along with the amplitude information. The frequencies can be computed knowing that the bandwidth of the spectrum is from 0 to $S_{rate}/2$ Hz, and there will be N/2 equally spaced frequencies.

Another characteristic of various implementations of the FFT is the arbitrary scale for amplitude. One technique is to *normalize* the spectrum: divide all amplitudes by the value of the highest peak. Then the highest peak will have a value of 1.0 and all other values will be a fraction of that.

References

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3. Hewlett-Packard Product Note 54600-4, *Using the Fast Fourier Transform in HP 54600 Series Oscilloscopes*.
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