Formalizing Actuarial Mathematics in Isabelle/HOL

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Disclaimer

- The contents presented here are solely the speaker's opinions and do not reflect the views of Company.
- There are some inaccuracies in explaining actuarial mathematics due to the priority on intuitive understanding.
- This presentation is a progress report on the library "Actuarial Mathematics" in the Archive of Formal Proofs of Isabelle. The final edition might be different from the one presented below.

Self Introduction

Professional Experience

- Sompo Himawari Life Insurance Inc., December 2020 – Present.
  - Aggregates the business results of life insurance products.
- Meiji Yasuda Life Insurance Company, April 2014 – November 2020.
  - Revised the reinsurance contracts.
  - Determined the prices of life insurance products.
  - Attended the approval negotiations with Financial Services Agency.
  - Qualified as an actuary (Fellow of the Institute of Actuaries of Japan).
  - Aggregated the business results of group life insurance.
  - Calculated retirement benefit obligations of client enterprises.
  - Validated the financial soundness of Employees' Pension Plans.

Education

- Nagoya University
  - Master of Mathematical Sciences, March 2014.
- The University of Tokyo
  - Bachelor of Science, Mathematics Course, March 2012.

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What Is Actuarial Mathematics?

- Actuarial mathematics is a branch of applied mathematics, which is used to evaluate financial risks of undesirable events.
- It is related to
  - calculus,
  - probability theory,
  - statistics,
  - financial theory.
- The traditional actuarial roles are considered as
  - determining the prices of insurance products,
  - estimating the liability of a company associating with the insurance contracts.
- Recently, the risk management skill of actuaries is required in a wider range of businesses.

Pricing a Term Life Insurance I

Term Life Insurance: "a type of death benefit that pays the heirs of the policyholder throughout a specified period of time" [1]

Assumption

- *amount insured: \$10000*
- *entry age: 30 years old*
- *policy period: 1 year*
- *annual mortality rate: 1%*
- *annual interest rate: 2%*

- The expected payment after 1 year is $\$10000 \times 1\% = \$100$.
- If the insurance company earns a 2% investment yield annually, the required amount for this insurance can be discounted:

$$\frac{\$100}{(1+2\%)} \approx \$98.$$

Pricing a Term Life Insurance II

Definition

The present value of a term life insurance on a person aged x payable at the end of the year of death within n years is written as $A^{1}_{x:\overline{n}|}$ per unit insurance amount:

$$A^{1}_{x:\overline{n}|} := \sum_{k=1}^{n} \frac{{}_{k-1|}q_x}{(1+i)^k},$$

where

- ${}_{k-1|}q_x$ denotes the probability that the person aged x will die in the kth year,
- i denotes the annual interest rate.

- In the example above, $A^{1}_{30:\overline{1}|} \approx 0.0098$.

International Actuarial Notation I

- There are various actuarial symbols used worldwide since at least 20th century [3, Perryman 1949].

a_x = an annuity, first payment at the end of a year, to continue during the life of (x).

$\ddot{a}_x = 1 + a_x$ = an 'annuity-due' to continue during the life of (x), the first payment to be made at once.

A_x = an assurance payable at the end of the year of death of (x).

Note. $e_x = a_x$ at rate of interest $i = 0$.

A letter or number at the lower left corner of the principal symbol denotes the number of years involved in the probability or benefit in question. Thus:

${}_np_x$ = the probability that (x) will live n years.

${}_nq_x$ = the probability that (x) will die within n years.

Note. When $n = 1$ it is customary to omit it, as shown on page 2, provided no ambiguity is introduced.

${}_nE_x = v^n {}_np_x$ = the value of an endowment on (x) payable at the end of n years if (x) be then alive.

If the letter or number comes before a perpendicular bar it shows that a period of deferment is meant. Thus:

${}_{n|}q_x$ = the probability that (x) will die in a year, deferred n years; that is, that he will die in the $(n+1)$th year.

${}_{n|}a_x$ = an annuity on (x) deferred n years; that is, that the first payment is to be made at the end of $(n+1)$ years.

${}_{n|t}a_x$ = an intercepted or deferred temporary annuity on (x) deferred n years and, after that, to run for t years.

International Actuarial Notation II

- In life insurance mathematics, the relations between the actuarial symbols are well examined, e.g.

$$A_{x:\overline{n}|} = 1 - d\ddot{a}_{x:\overline{n}|},$$

where

$$v := \frac{1}{1+i} \quad \text{(the present value of 1 to be paid 1 year from now)},$$

$$d := 1 - v \quad \text{(the annual effective discount rate)},$$

$$\ddot{a}_{x:\overline{n}|} := \sum_{k=0}^{n-1} {}_k p_x v^k \quad \text{(the present value of a life annuity due)},$$

$$A_{x:\overline{n}|} := A^{1}_{x:\overline{n}|} + {}_n p_x v^n \quad \text{(the present value of an endowment)}.$$

- Actuaries use these symbols and formulas efficiently to calculate prices of products, reserves of the company, etc.

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What Is Life Table?

- A life table, a.k.a. mortality table, "shows the rate of deaths occurring in a defined population during a selected time interval, or survival rates from birth to death" [1].
- For example, IAJ (The Institute of Actuaries of Japan) produced Standard Mortality Table 2018 [2].

Age x	Lives l_x	Deaths d_x	Mortality rate q_x	Average life expectancy $\mathring{e}_x$
0	100, 000	81	0. 00081	80. 77
1	99, 919	56	0. 00056	79. 84
2	99, 863	36	0. 00036	78. 88
3	99, 827	22	0. 00022	77. 91
4	99, 805	14	0. 00014	76. 92
5	99, 791	10	0. 00010	75. 94
6	99, 781	9	0. 00009	74. 94
7	99, 772	9	0. 00009	73. 95
8	99, 763	9	0. 00009	72. 96
9	99, 754	9	0. 00009	71. 96

How to Use Life Table

- We can calculate many kinds of statistics relative to life and death such as death rates, life expectancies, etc.

$$p_x = \frac{l_{x+1}}{l_x}, \qquad q_x = \frac{d_x}{l_x},$$

$$_n p_x = \frac{l_{x+n}}{l_x}, \qquad {}_{f|}q_x = \frac{d_{x+f}}{l_x},$$

$$\mu_x = -\frac{1}{l_x}\frac{dl_x}{dx}, \qquad \mathring{e}_0 = \frac{1}{l_0}\int_0^\infty x l_x \mu_x \, dx.$$

Age x	Lives l_x	Deaths d_x	Mortality rate q_x	Average life expectancy $\mathring{e}_x$
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Previous Work: Formalization in Coq I

- I released `coq-actuary` in 2021.
  `https://github.com/Yosuke-Ito-345/Actuary/`
- In `coq-actuary`, I introduced the life table axiomatically and defined the statistics as above.

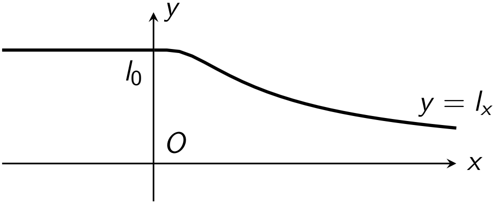
Definition (Life Table in Coq)

The function l from $\mathbb{R}$ to $\mathbb{R}$ is said to be a *life table* if it obeys the following properties.

1. $l_0 > 0$,
2. $l_x = l_0$ if $x < 0$,
3. $\lim_{x \to \infty} l_x = 0$,
4. l is non-increasing.

Previous Work: Formalization in Coq II

- The Coq formalization presented above has several drawbacks.
  1. When we need to differentiate the life table, we have to assume that it is continuously differentiable at *all real numbers*. (This restriction arises from the `Coquelicot` library.)
  2. The life table is not implemented by a distribution function in probability theory. (This is because the formalization of probability theory in Coq is not completed.)



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Life Table to Tail Distribution I

Definition (in probability theory)

Assume that X is a real-valued random variable. We define
distribution function $F_X(x) := P(X \leq x)$,
tail distribution $\overline{F}_X(x) := P(X > x)$.

- If a life table (in Coq) l is right-continuous, the function $\tilde{l}$ defined by $\tilde{l}_x := l_x / l_0$ is a tail distribution of some random variable X:

$$\tilde{l}_x = \overline{F}_X(x) = P(X > x).$$

  Here, the random variable X represents *the age at death*.
- Since Isabelle/HOL has a large library `HOL-Probability`, actuarial mathematics can be effectively formalized by using probability theory.

Life Table to Tail Distribution II

- When X is a random variable representing the age at death, actuarial statistics can be written as follows:

$$p_x = P(X > x + 1 \mid X > x), \quad q_x = P(X \le x + 1 \mid X > x),$$
$${}_n p_x = P(X > x + n \mid X > x), \quad {}_{f|}q_x = P(x + f < X \le x + f + 1 \mid X > x),$$
$$\mu_x = \lim_{h \to +0} \frac{P(x < X \le x + h \mid X > x)}{h}, \quad \mathring{e}_0 = E[X].$$

- Therefore, actuarial mathematics can be formalized in the following order.

  survival model Introduce the age-at-death random variable X, and define actuarial statistics in the context of probability theory. Many important formulas are proved without a life table.

  life table Define a life table axiomatically, and reduce it to the survival model. Formulas including the life table are derived by easy translation from the survival model.

Supplementary Lemmas of the Probability Library

Newly Formalized Lemmas on Probability Library

1. theory of tail distribution

```
definition ccdf :: "real measure ⇒ real ⇒ real"
  where "ccdf M ≡ λx. measure M {x<..}"
    — ‹complementary cumulative distribution function (tail distribution)›
```

2. tail expectation formula: for any non-negative random variable X,

$$E[X] = \int_0^\infty P(X > x)\,dx.$$

```
lemma expectation_nonneg_tail:
  assumes [measurable]: "random_variable borel X"
    and X_nonneg: "⋀x. x ∈ space M ⟹ X x ≥ 0"
  defines "F u ≡ cdf (distr M borel X) u"
  shows "(∫⁺x. ennreal (X x) ∂M) = (∫⁺u∈{0..}. ennreal (1 - F u) ∂lborel)"
```

3. theory of *hazard rate*: the generalized notion of μ_x.

```
definition hazard_rate :: "('a ⇒ real) ⇒ real ⇒ real"
  where "hazard_rate X t ≡
    Lim (at_right 0) (λdt. 𝒫(x in M. t < X x ∧ X x ≤ t + dt ¦ X x > t) / dt)"
```

Implementation of Survival Model and Life Table

- The survival model and the life table are formalized by using locale.

```
locale prob_space_actuary = MM_PS: prob_space 𝔐 for 𝔐
locale survival_model = prob_space_actuary +
  fixes X :: "'a ⇒ real"
  assumes X_RV[simp]: "MM_PS.random_variable (borel :: real measure) X"
    and X_pos_ae[simp]: "AE ξ in 𝔐. X ξ > 0"

locale life_table =
  fixes l :: "real ⇒ real" ("$l'_" [101] 200)
  assumes l_0_pos: "0 < l 0"
    and l_neg_nil: "⋀x. x ≤ 0 ⟹ l x = l 0"
    and l_PInfty_0: "(l ⟶ 0) at_top "
    and l_antimono: "antimono l"
    and l_right_continuous: "⋀x. continuous (at_right x) l"
```

- After I complete formalization, I will upload these theories to the Archive of Formal Proofs.
  `https://www.isa-afp.org/entries/Actuarial_Mathematics.html`

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Future Applications and Obstacles

Possible Future Applications

- Prevent errors in examination papers of actuarial mathematics.
- Detect errors in actuarial documents, e.g. statement of calculation procedures.
- Formally verify programs in actuarial software, e.g. MG-ALFA.

Obstacles

- There are few actuaries familiar with proof assistants.
- Formal verification requires enormous cost and time.
- Application range is limited.

Features of Isabelle/HOL vs Coq

My Personal Impression of Isabelle/HOL and Coq

Perspectives	Isabelle/HOL	Coq
logic	classical	intuitionistic
axiom of choice	embedded	optional
style	structured	procedural
distribution	very large	big
community	less active	active
tutorials	not sufficient	enough
ATP	full use	trying to use
tactical	implemented	well-developed

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