Formalizing Premium Calculation of Life Insurance

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Sompo Himawari Life Insurance Inc.

Disclaimer

1. The contents presented here are solely the speaker's opinions and do not reflect the views of Company.
2. There are some inaccuracies in explaining actuarial mathematics due to the priority on intuitive understanding.
3. This presentation is a progress report on the library "Actuarial Mathematics" in the Archive of Formal Proofs of Isabelle. The future version might be different from the one presented below.

Self Introduction (Professional Experience)

Sompo Himawari Life Insurance Inc., Dec. 2020 – Present.

- Aggregates the business results of life insurance products.

Meiji Yasuda Life Insurance Company, Apr. 2014 – Nov. 2020.

- Revised the reinsurance contracts.
- Determined the prices of life insurance products.
- Attended the approval negotiations with Financial Services Agency.
- Qualified as an actuary (Fellow of the Institute of Actuaries of Japan).
- Aggregated the business results of group life insurance.
- Calculated retirement benefit obligations of enterprises.
- Validated the financial soundness of Pension Plans.

Self Introduction (Education)

Nagoya University

- Master of Mathematical Sciences, Mar. 2014.

The University of Tokyo

- Bachelor of Science, Mathematics Course, Mar. 2012.

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What Is Actuarial Mathematics?

Actuarial mathematics is a branch of applied mathematics, which is used to evaluate financial risks of undesirable events.

It is related to

- calculus,
- probability theory,
- statistics,
- financial theory.

The traditional actuarial roles are considered as

- determining the prices of insurance products,
- estimating the liability of a company associating with the insurance contracts.

Recently, the risk management skill of actuaries is required in a wider range of businesses.

How We Determine the Premium of a Life Insurance

A **premium** (**保険料**) is paid to a life insurance company by the **policyholder** (**保険契約者**).

A **benefit** (**保険金**) is paid by a life insurance company according to the life insurance policy.

The **equivalence principle** (**収支相等の原則**) requires that the benefit outgo be equal to the premium income;

$$\begin{aligned}&\text{EPV (Expected Present Value) of the benefit outgo}\\ &\qquad = \text{EPV of the premium income.}\end{aligned}$$

Progress Report

I have sorted out the theory of the equivalence principle on paper in a rigorous, general and explicit manner.

- **Rigor**
  In order to formalize the equivalence principle, firstly we have to write it down rigorously.
- **Generality**
  The formalized principle should be applicable to all types of life insurance.
- **Explicitness**
  All the typical calculation formulas should be derived from the formalized equivalence principle.

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If we can easily earn 2% returns annually, \$100 at present is worth \$102 next year.

If an asset produces incomes C_t at the time t (in years), their **present value** is defined to be

$$\sum_t \frac{C_t}{(1+i)^t}$$

where i is an annual interest rate.

Pricing a Term Life Insurance (1-year policy)

"Term life insurance provides a death benefit for a specified period of time that pays the policyholder's beneficiaries" [2].

Assumption

- amount insured: \$10000
- policy period: 1 year
- time of payment: the end of the year
- annual mortality rate: 1%
- annual interest rate: 2%

The expected payment after 1 year is $\$10000 \times 1\% = \$100$. If the insurance company earns a 2% investment yield annually, the present value of this insurance is $\$100/(1+2\%) \approx \$98$.

Notations

T_x : future lifetime random variable of a person aged x

${}_tp_x := P(T_x > t \mid T_x > 0)$ (survival rate)

$p_x := {}_1p_x$

${}_tq_x := P(T_x \leq t \mid T_x > 0)$ (mortality rate)

$q_x := {}_1q_x$

$\mu_x := \lim_{\Delta t \searrow 0} \frac{{}_{\Delta t}q_x}{\Delta t}$ (force of mortality)

i : annual interest rate

$v := \frac{1}{1+i}$ (discount factor)

Pricing a Term Life Insurance

Definition

The present value of a term life insurance on a person aged x payable at the end of the year of death within n years is written as $A^1_{x:\overline{n}|}$ per unit insurance amount:

$$A^1_{x:\overline{n}|} = \sum_{k=1}^{n} v^k \cdot {}_{k-1}p_x \cdot q_{x+k-1}.$$

The present value means nothing other than the **single premium** (**一時払保険料**).

Pricing a Continuous Term Life Insurance

Definition

The present value of a term life insurance on a person aged x payable at the moment of death within n years is written as $\bar{A}^{1}_{x:\overline{n}|}$ per unit insurance amount:

$$\bar{A}^{1}_{x:\overline{n}|} = \int_{(0,n]} v^t \cdot {}_tp_x \cdot \mu_{x+t}\, dt.$$

Definition

The present value of a term life annuity on a person aged x payable at the beginning of each year within n years is written as $\ddot{a}_{x:\overline{n}|}$ per unit annual payment:

$$\ddot{a}_{x:\overline{n}|} = \sum_{k=1}^{n} v^{k-1} \cdot {}_{k-1}p_x.$$

Calculating Level Premiums

Level premiums (**平準払保険料**) are the premiums paid regularly whose amount is constant throughout the term.

The policyholder has to keep paying the level premium as long as the **insured** (**被保険者**) is alive.

The EPV (Expected Present Value) of the annually-payable level premium P of an n-year life insurance is $P \cdot \ddot{a}_{x:\overline{n}|}$.

Equivalence Principle

It is required that

$$\text{EPV of the benefit outgo} = \text{EPV of the premium income.}$$

Example

The level premium P of an n-year continuous term life insurance should satisfy $\bar{A}^{1}_{x:\overline{n}|} = P \cdot \ddot{a}_{x:\overline{n}|}$. Therefore,

$$P = \frac{\bar{A}^{1}_{x:\overline{n}|}}{\ddot{a}_{x:\overline{n}|}}.$$

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Formulating EPV (Expected Present Value) i

The goal is to express the equivalence principle by a mathematical formula.

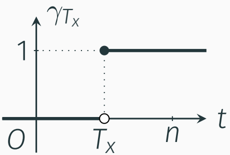
It suffices to uniformly define the EPV for any type of life insurance.

We introduce a function $\gamma(t)$ representing the **total** amount of benefits to be paid until the time t.

The function $\gamma(t)$ may depend on T_x. We henceforth write it as $\gamma_{T_x}(t)$.

For a continuous term life insurance,

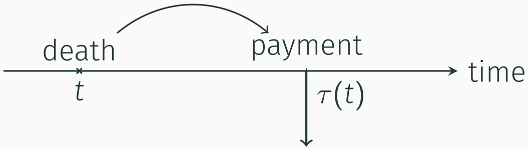
$$\gamma_{T_x}(t) := \begin{cases} 1, & \text{if } t \geq T_x \text{ and } T_x \leq n, \\ 0, & \text{otherwise.} \end{cases}$$



The EPV of this life insurance can be expressed as

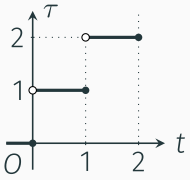
$$E\left[\int_{\mathbb{R}} v^t \, d\gamma_{T_x}(t) \;\middle|\; T_x > 0\right].$$

We also introduce a function $\tau(t)$ representing the actual time when the benefit is paid whose obligation is incurred at the time t.



Consider a term life insurance payable at the end of each year. Let $\gamma_{T_x}(t)$ be the same as the previous one, and

$$\tau(t) := \lceil t \rceil \quad \text{(ceil function)}.$$



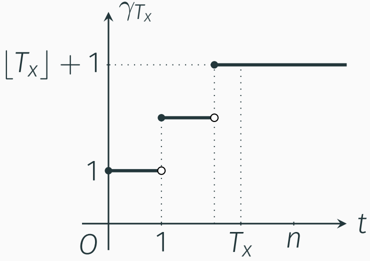
Then the EPV of this life insurance can be expressed as

$$E\left[\int_{\mathbb{R}} v^{\tau(t)}\, d\gamma_{T_x}(t) \;\middle|\; T_x > 0 \right].$$

This framework is applicable to a term life annuity-due:

$$\gamma_{T_x}(t) := \begin{cases} \min\{n, \max\{0, \lfloor t \rfloor + 1\}\}, & \text{if } t < T_x, \\ \lim\limits_{s \nearrow T_x} \gamma_{T_x}(s), & \text{if } t \geq T_x, \end{cases}$$

$$\tau(t) := t.$$



Formulating EPV (Expected Present Value) vi

The EPV of the life annuity can be expressed in the same formula as the life insurance:

$$E\left[\int_{\mathbb{R}} v^{\tau(t)}\, d\gamma_{T_x}(t) \,\middle|\, T_x > 0\right].$$

Axioms of the Framework of EPV

The specification of life insurance is determined by $\gamma : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ and $\tau : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$.

These functions should satisfy the following axioms.

1. For any $\theta \in \mathbb{R}$, $\gamma(\theta, \cdot) : \mathbb{R} \to \mathbb{R}$ is right-continuous and nondecreasing.
2. For any $\theta \in \mathbb{R}$, $\tau(\theta, \cdot) : \mathbb{R} \to \mathbb{R}$ is Borel measurable.
3. The function $\mathbb{R} \to \mathbb{R}; \theta \mapsto \displaystyle\int_{\mathbb{R}} v^{\tau(\theta,t)}\, d\gamma_\theta(t)$ is Borel measurable.
4. For any θ and $t \in \mathbb{R}$, $t < 0$ implies $\gamma(\theta, t) = 0$.

These are sufficient to derive calculation formulas such as

$$\ddot{a}_{x:\overline{n}|} = \sum_{k=1}^{n} v^{k-1} \cdot {}_{k-1}p_x.$$

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Formulating the Equivalence Principle

Let (γ^B, τ^B) denote the specification of the life insurance benefits. Let (γ^P, τ^P) denote the specification of the premium payments.

The (level) premium of the life insurance satisfying the equivalence principle is

$$\frac{E\left[\int_{\mathbb{R}} v^{\tau^B(T_x,t)}\, d\gamma^B_{T_x}(t) \;\middle|\; T_x > 0\right]}{E\left[\int_{\mathbb{R}} v^{\tau^P(T_x,t)}\, d\gamma^P_{T_x}(t) \;\middle|\; T_x > 0\right]}.$$

Implementing the Equivalence Principle in Isabelle

I plan to implement the equivalence principle by `locale` and `sublocale`.

Cf. Snapshot of Valuation.thy.

Future Work

1. Complete the implementation of the equivalence principle in Isabelle/HOL.
    - The current theory of actuarial mathematics is in the AFP: `https://www.isa-afp.org/entries/Actuarial_Mathematics.html`
2. Simultaneously proceed with Coq (Rocq) formalization.
    - The probability-free elementary formalization of actuarial mathematics can be found in `coq-actuary`: `https://github.com/Yosuke-Ito-345/Actuary`
    - This is based on Coquelicot, but I am upgrading it based on MathComp-Analysis.

Future Applications

1. Prevent errors in examination papers of actuarial mathematics.
2. Detect errors in actuarial documents, e.g. statement of calculation procedures.
3. Formally verify programs in actuarial software.

References

[1] Insuranceopedia Inc.
Insuranceopedia.
https://www.insuranceopedia.com/, 2024.

[2] Julia Kagan.
Term life insurance: What it is, different types, pros and cons.
https://www.investopedia.com/terms/t/termlife.asp, 2024.

Q&A

Any questions or comments? (Both in Japanese and English.)

You can also reach me at GitHub:
`https://github.com/Yosuke-Ito-345`

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Terms in Life Insurance Contract

An **insurer** (**保険者**) “is a party that agrees to compensate people, companies or other organizations for specific financial losses” [1].

The **policyholder** (**保険契約者**) “is the person or entity that has purchased the policy and has the authority to exercise the rights stated in the insurance policy contract” [1].

An **insured** (**被保険者**) “is a party protected against specific perils, either as the holder of an insurance policy or through other coverage” [1].

A **beneficiary** (**保険金受取人**) “is any person or legal entity who is entitled to the benefits, proceeds, and/or earnings of a life or health insurance policy” [1].

Calculation Formula of Continuous Term Life Insurance

Suppose $\gamma_{T_x}(t) = \mathrm{I}_{[T_x,\infty)}(t) \cdot \mathrm{I}_{(-\infty,n]}(T_x)$ and $\tau(t) = t$. If the cumulative distribution F_{T_x} is differentiable, then

$$
\begin{aligned}
E\left[\int_{\mathbb{R}} v^{\tau(t)}\, d\gamma_{T_x}(t) \;\middle|\; T_x > 0\right] &= E\left[v^{T_x} \cdot \mathrm{I}_{(0,n]}(T_x) \;\middle|\; T_x > 0\right] \\
&= \int_{(0,\infty)} v^t \cdot \mathrm{I}_{(0,n]}(t) \cdot F'_{T_x}(t)\, dt \\
&= \int_{(0,n]} v^t \cdot {}_t p_x \cdot \mu_{x+t}\, dt.
\end{aligned}
$$

Calculation Formula of Term Life Annuity-Due i

Let $\gamma_\infty(t) := \min\{n, \max\{0, \lfloor t \rfloor + 1\}\}$, and suppose

$$\gamma_{T_x}(t) := \begin{cases} \gamma_\infty(t), & \text{if } t < T_x, \\ \lim\limits_{s \nearrow T_x} \gamma_\infty(s), & \text{if } t \geq T_x, \end{cases}$$

$$\tau(t) := t.$$

Calculation Formula of Term Life Annuity-Due ii

Using Fubini's Theorem, we have

$$
\begin{aligned}
E\left[\int_{\mathbb{R}} v^{\tau(t)}\, d\gamma_{T_x}(t) \,\middle|\, T_x > 0\right]
&= E\left[\int_{\mathbb{R}} \mathrm{I}_{(-\infty,T_x)}(t)\cdot v^t\, d\gamma_\infty(t) \,\middle|\, T_x > 0\right] \\
&= \int_{\mathbb{R}} \left(\int_{\mathbb{R}} \mathrm{I}_{(-\infty,s)}(t)\cdot v^t\, d\gamma_\infty(t)\right) dF_{T_x}(s) \\
&= \int_{\mathbb{R}} \left(\int_{\mathbb{R}} \mathrm{I}_{(t,\infty)}(s)\cdot v^t\, dF_{T_x}(s)\right) d\gamma_\infty(t) \\
&= \int_{\mathbb{R}} v^t \left(\int_{\mathbb{R}} \mathrm{I}_{(t,\infty)}(s)\, dF_{T_x}(s)\right) d\gamma_\infty(t) \\
&= \sum_{k=1}^{n} v^{k-1}\cdot {}_{k-1}p_x.
\end{aligned}
$$