Formalizing Actuarial Mathematics in Proof Assistants

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Disclaimer

1. The contents presented here are solely the speaker's opinions and do not reflect the views of any affiliations.
2. There are some inaccuracies in explaining actuarial mathematics due to the priority on intuitive understanding.
3. This presentation is a progress report on MathComp-Analysis in Rocq and the Archive of Formal Proofs in Isabelle. The future versions might be different from the ones presented below.

Self Introduction (Professional Experience)

Sompo Himawari Life Insurance Inc., Dec. 2020 – Present.

- Aggregates the business results of life insurance products.
- Improves business efficiency with standardization and automation.

Meiji Yasuda Life Insurance Company, Apr. 2014 – Nov. 2020.

- Revised the reinsurance contracts.
- Determined the prices of life insurance products.
- Attended the approval negotiations with Financial Services Agency.
- Aggregated the business results of group life insurance.
- Calculated retirement benefit obligations of enterprises.
- Validated the financial soundness of Pension Plans.

Self Introduction (Education)

Nagoya University

- Master of Mathematical Sciences, Mar. 2014.

The University of Tokyo

- Bachelor of Science, Mathematics Course, Mar. 2012.

Progress Report

Here are my contributions of this year.

1. MathComp-Analysis in Rocq (joint work with R. Affeldt and T. Saikawa)
   - Introduced the cumulative distribution function (cdf) and the complementary cumulative distribution function (ccdf).
   - Proved that a random variable induces a Lebesgue-Stieltjes measure, and vice versa.
   - Formalized the tail expectation formula.
2. Archive of Formal Proofs (AFP) in Isabelle
   - Created a new entry on the Lebesgue-Stieltjes integral, including formulas like

$$\int g(x)\, dF(x) = \int g(x) F'(x)\, dx.$$

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What Is Actuarial Mathematics?

Actuarial mathematics is a branch of applied mathematics, which is used to evaluate financial risks of undesirable events.
It is related to

1. calculus,
2. probability theory,
3. statistics,
4. financial theory.

The traditional actuarial roles are considered as

1. determining the prices of insurance products,
2. estimating the liability of a company associating with the insurance contracts.

Recently, the risk management skill of actuaries is required in a wider range of businesses.

Pricing a Term Life Insurance I

"Term life insurance provides a death benefit for a specified period of time that pays the policyholder's beneficiaries." [1]

Assumption

1. amount insured: \$10000
2. entry age: 30 years old
3. policy period: 1 year
4. annual mortality rate: 1%
5. annual interest rate: 2%

The expected payment after 1 year is $\$10000 \times 1\% = \$100$.
If the insurance company earns a 2% investment yield annually, the required amount for this insurance can be discounted:

$$\frac{\$100}{1 + 2\%} \approx \$98.$$

Pricing a Term Life Insurance II

Definition

The present value of a term life insurance on a person aged x payable at the end of the year of death within n years is denoted by $A^{1}_{x:\overline{n}|}$ per unit insurance amount:

$$A^{1}_{x:\overline{n}|} := \sum_{k=1}^{n} \frac{{}_{k-1|}q_x}{(1+i)^k},$$

where

1. i denotes the annual interest rate,
2. ${}_{k-1|}q_x$ denotes the probability that a person aged x dies in the k-th year.

In the example above, $A^{1}_{30:\overline{1}|} \approx 0.0098$.

International Actuarial Notation I

There are various actuarial symbols used worldwide since at least 20th century [4].

a_x = an annuity, first payment at the end of a year, to continue during the life of (x).

$\ddot{a}_x = 1 + a_x$ = an 'annuity-due' to continue during the life of (x), the first payment to be made at once.

A_x = an assurance payable at the end of the year of death of (x).

Note. $e_x = a_x$ at rate of interest $i = 0$.

A letter or number at the lower left corner of the principal symbol denotes the number of years involved in the probability or benefit in question. Thus:

${}_np_x$ = the probability that (x) will live n years.

${}_nq_x$ = the probability that (x) will die within n years.

Note. When $n = 1$ it is customary to omit it, as shown on page 2, provided no ambiguity is introduced.

${}_nE_x = v^n {}_np_x$ = the value of an endowment on (x) payable at the end of n years if (x) be then alive.

If the letter or number comes before a perpendicular bar it shows that a period of deferment is meant. Thus:

${}_{n|}q_x$ = the probability that (x) will die in a year, deferred n years; that is, that he will die in the $(n+1)$th year.

${}_{n|}a_x$ = an annuity on (x) deferred n years; that is, that the first payment is to be made at the end of $(n+1)$ years.

${}_{n|t}a_x$ = an intercepted or deferred temporary annuity on (x) deferred n years and, after that, to run for t years.

International Actuarial Notation II

1. In life insurance mathematics, the relations between the actuarial symbols are well examined, e.g.

$$A_{x:\overline{n}|} = 1 - d\,\ddot{a}_{x:\overline{n}|},$$

   where

$$v := \frac{1}{1+i} \quad \text{(the present value of 1 to be paid 1 year from now)},$$
$$d := 1 - v \quad \text{(the annual effective discount rate)},$$
$$\ddot{a}_{x:\overline{n}|} := \sum_{k=0}^{n-1} {}_{k}p_{x} v^{k} \quad \text{(the present value of a life annuity due)},$$
$$A_{x:\overline{n}|} := A^{1}_{x:\overline{n}|} + {}_{n}p_{x} v^{n} \quad \text{(the present value of an endowment)}.$$

2. Actuaries use these symbols and formulas efficiently to calculate prices of products, reserves of the company, etc.

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Notation of Lebesgue-Stieltjes Measure

Let $F : \mathbb{R} \to \mathbb{R}$ be non-decreasing and right-continuous.

Notation (Lebesgue-Stieltjes measure)

1. Let μ_F denote the Lebesgue-Stieltjes measure relative to F;

$$\mu_F\big((a, b]\big) = F(b) - F(a) \quad \text{for } a \leq b.$$

2. When $F = \mathrm{id}$, we simply write μ instead of μ_{id}. This is called the Lebesgue measure on $\mathbb{R}$.

Notation (Lebesgue-Stieltjes integral)

1. The Lebesgue integral relative to μ_F is called the Lebesgue-Stieltjes integral and denoted by $\displaystyle\int f \, d\mu_F$ or $\displaystyle\int f(r) \, dF(r)$.
2. When $F = \mathrm{id}$, the Lebesgue-Stieltjes integral is also denoted by $\displaystyle\int f(r) \, dr$.

Definitions of cdf and ccdf

Let X be a real-valued random variable.

Definition

1. The function $\mathrm{cdf}_X(r) := \Pr(X \leq r)$ is called the cumulative distribution function (cdf) of X.
2. The function $\mathrm{ccdf}_X(r) := \Pr(X > r)$ is called the complementary cumulative distribution function (ccdf) of X.

Lemma

$$\mathrm{cdf}_X(r) + \mathrm{ccdf}_X(r) = 1.$$

Random Variable Induces Lebesgue-Stieltjes Measure

Let X be a real-valued random variable.

Lemma

1. cdf_X *is non-decreasing and right-continuous (called cumulative in MathComp-Analysis).*
2. $\lim_{r \to -\infty} \mathrm{cdf}_X(r) = 0.$
3. $\lim_{r \to +\infty} \mathrm{cdf}_X(r) = 1.$

Hence we can define the Lebesgue-Stieltjes measure μ_{cdf_X}.

Proposition

If $X = \mathrm{id}_{\mathbb{R}}$, *then we have* $\mu_{\mathrm{cdf}_X} = \mathrm{Pr}$.

Lebesgue-Stieltjes Measure Induces Random Variable

Proposition

Let $F : \mathbb{R} \to \mathbb{R}$ satisfy the following properties:

1. *F is cumulative,*
2. $\lim_{r \to -\infty} F(r) = 0$, *and*
3. $\lim_{r \to +\infty} F(r) = 1$.

There exist a probability space Ω and a random variable $X : \Omega \to \mathbb{R}$ such that $\operatorname{cdf}_X = F$.

In fact, we can take

1. Ω : the Borel space with the Lebesgue-Stieltjes measure relative to F,
2. X : $\mathrm{id}_{\mathbb{R}}$.

Tail Expectation Formula

Proposition

For a non-negative random variable X, we have

$$\mathrm{E}[X] = \int_0^\infty \mathrm{ccdf}_X(r)\, dr.$$

Example

When X represents a lifespan, we can calculate the average life expectancy $\mathrm{E}[X]$ as $\int_0^\infty \Pr(X > r)\, dr$.

Formalization in MathComp-Analysis

The definitions and lemmas presented before are formalized in `probability.v` in MathComp-Analysis:

- `https://github.com/math-comp/analysis/blob/master/theories/probability.v`

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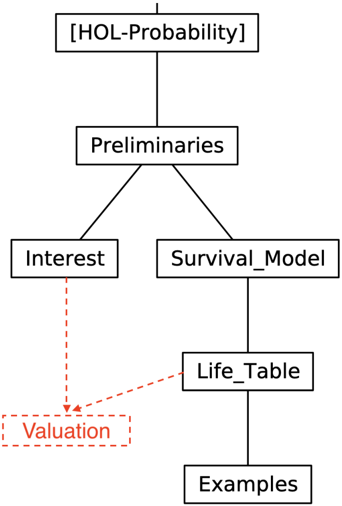
Current Status of “Actuarial Mathematics” in AFP

https://www.isa-afp.org/entries/Actuarial_Mathematics.html



By combining `Interest` and `Life_Table`, we can formalize the valuation theory of life insurance products.

Calculating Lebesgue-Stieltjes Integral

Assumption

1. $F : \mathbb{R} \to \mathbb{R}$ is non-decreasing and right-continuous.
2. $g : \mathbb{R} \to \mathbb{R}$ is Borel measurable.

Lemma

1. *If F is constant, then μ_F is the trivial null measure.*
2. $\mu_F\big(\{r\}\big) = F(r) - \lim\limits_{x \to r-0} F(x)$ *for any* $r \in \mathbb{R}$.
3. $\displaystyle\int g(x)\,d(F+c)(x) = \int g(x)\,dF(x)$ *for any* $c \in \mathbb{R}$.
4. *If F is piecewise-differentiable, then* $\displaystyle\int g(x)\,dF(x) = \int g(x)F'(x)\,dx$.

These formulas are essential to calculating a Lebesgue-Stieltjes integral.

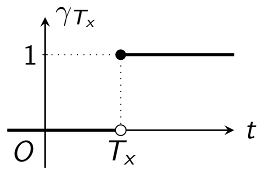
Usage in Actuarial Mathematics I

Notation

1. x : the entry age
2. v : the present value of 1 to be paid 1 year from now
3. T_x : the future lifetime random variable
4. $\gamma_{T_x}(t)$: the total amount of an insurance paid until the time t

The present value of a whole life insurance ($\gamma_{T_x} = \mathbf{1}_{[T_x,\infty)}$) is calculated as

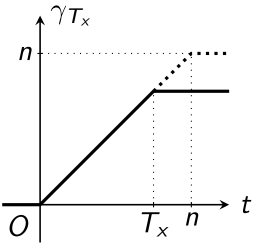
$$\int_0^\infty v^t \, d\gamma_{T_x}(t) = v^{T_x}.$$



Usage in Actuarial Mathematics II

The present value of an n-year term life annuity ($\gamma_{T_x}(t) = t$ if $0 \le t \le \min\{n, T_x\}$) is calculated as

$$\int_0^\infty v^t \, d\gamma_{T_x}(t) = \int_0^{\min\{n,T_x\}} v^t \, d\gamma_{T_x}(t) = \int_0^{\min\{n,T_x\}} v^t \frac{dt}{dt} \, dt$$
$$= \frac{v^{\min\{n,T_x\}} - 1}{\ln v}.$$



Overview of “Lebesgue-Stieltjes Integral” in AFP

Here is the outline of the entry “Lebesgue-Stieltjes Integral” (under review).

1. Basic Calculations
2. Changing the Underlying Function
3. Restricting the Integral
4. Calculation by the Derivative

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Future Work

For Rocq, I plan to

1. further develop `probability.v` in MathComp-Analysis,
2. formalize the theory of a life table, etc.

For Isabelle, I plan to

1. formalize the valuation theory of life insurance products,
2. add general lemmas to the Isabelle distribution, etc.

Future Applications

1. Prevent errors in examination papers of actuarial mathematics.
   - The qualification examinations of actuaries are held in various countries including Japan.
   - If we can formally verify the solutions of the examination problems, any errors will be eliminated in advance.
2. Detect errors in actuarial documents, e.g. statement of calculation procedures.
   - When we produce a new insurance product, we have to get approval from the Financial Services Agency in Japan.
   - Formalized actuarial theories could be useful to eliminate errors in the documents before we submit them to the agency.
3. Formally verify programs in actuarial software.
   - We sometimes use software specialized for complicated actuarial calculations.
   - Proof assistants may be suitable for verifying the software or the programming codes we write.

References

[1] Julia Kagan.
Term life insurance: What it is, different types, pros and cons.
https://www.investopedia.com/terms/t/termlife.asp, 2024.

[2] Julia Kagan.
Mortality table: Definition, types, and uses.
https://www.investopedia.com/terms/m/mortality-table.asp, 2025.

[3] The Institute of Actuaries of Japan.
Development process of Standard Mortality Table 2018.
https://www.actuaries.jp/english/lib/Development_Process_of_Standard_Mortality_Table_2018.pdf, 2017.

[4] Francis S. Perryman.
International actuarial notation.
In *Proceedings of the Casualty Actuarial Society*, Vol. 36, pp. 123–131, 1949.
Also available as https://www.casact.org/sites/default/files/database/proceed_proceed49_49123.pdf.

Q&A

Any questions or comments? (Both in Japanese and English.)
You can also reach me at GitHub:
`https://github.com/Yosuke-Ito-345`

Contents

What Is Life Table?

- A life table, a.k.a. mortality table, “shows the rate of deaths occurring in a defined population during a selected time interval, or survival rates from birth to death” [2].
- For example, IAJ (The Institute of Actuaries of Japan) produced Standard Mortality Table 2018 [3].

Age x	Lives l_x	Deaths d_x	Mortality rate q_x	Average life expectancy $\mathring{e}_x$
0	100, 000	81	0. 00081	80. 77
1	99, 919	56	0. 00056	79. 84
2	99, 863	36	0. 00036	78. 88
3	99, 827	22	0. 00022	77. 91
4	99, 805	14	0. 00014	76. 92
5	99, 791	10	0. 00010	75. 94
6	99, 781	9	0. 00009	74. 94
7	99, 772	9	0. 00009	73. 95
8	99, 763	9	0. 00009	72. 96
9	99, 754	9	0. 00009	71. 96

How to Use Life Table

- We can calculate many kinds of statistics relative to life and death such as death rates, life expectancies, etc.

$$p_x = \frac{l_{x+1}}{l_x}, \qquad q_x = \frac{d_x}{l_x},$$

$$_n p_x = \frac{l_{x+n}}{l_x}, \qquad {}_{f|}q_x = \frac{d_{x+f}}{l_x},$$

$$\mu_x = -\frac{1}{l_x}\frac{dl_x}{dx}, \qquad \mathring{e}_0 = \frac{1}{l_0}\int_0^{\infty} x l_x \mu_x \, dx.$$

Age x	Lives l_x	Deaths d_x	Mortality rate q_x	Average life expectancy $\mathring{e}_x$
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4	99, 805	14	0. 00014	76. 92

Previous Work: Formalization in Coq I

- I released `coq-actuary` in 2021.
  `https://github.com/Yosuke-Ito-345/Actuary/`
- In `coq-actuary`, I introduced the life table axiomatically and defined the statistics as above.

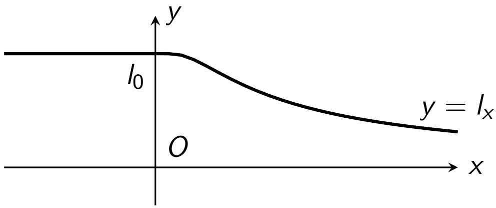
Definition (Life Table in Coq)

The function l from $\mathbb{R}$ to $\mathbb{R}$ is said to be a *life table* if it obeys the following properties.

1. $l_0 > 0$,
2. $l_x = l_0$ if $x < 0$,
3. $\lim_{x \to \infty} l_x = 0$,
4. l is non-increasing.

Previous Work: Formalization in Coq II

- The Coq formalization presented above had several drawbacks.
  1. When we needed to differentiate the life table, we had to assume that it was continuously differentiable at *all real numbers*. (This restriction arose from the `Coquelicot` library.)
  2. The life table was not implemented by a distribution function in probability theory. (This is because the formalization of probability theory in Coq was not completed.)



Features of Rocq vs Isabelle/HOL

My Personal Impression of Rocq and Isabelle/HOL

Perspectives	Rocq	Isabelle/HOL
logic	intuitionistic	classical
axiom of choice	optional	embedded
style	procedural	structured
distribution	big	very large
community	active	less active
tutorials	enough	not sufficient
ATP	implemented	fully used
tactical	well-developed	implemented