PROVA SCRITTA DI ANALISI
(APPELLO DEL 27/5/2013)

1) Data la funzione
$$f(x,y) = \frac{x + \ln y}{\ln(x+y)} \sqrt{x\,(9 - x^2 - 9y^2)}$$

(a) studiare l'insieme di definizione di f ed il suo segno;

(b) rappresentare su grafico separato la linea di livello zero e la frontiera dell'insieme di definizione di f;

(c) studiare i limiti di f nei punti $\left(-\frac{3}{\sqrt{10}}, \frac{3}{\sqrt{10}}\right)$, $(2,0)$, $(0,2)$, $(3,0)$.

(d) Si calcoli, se esiste, la derivata direzionale della funzione
$$g(x,y) = \frac{x + \ln y}{\ln(x+y)}$$
nel punto $(1,1)$ secondo la direzione della retta di equazione $3x - 4y + 1 = 0$. Si scriva poi l'equazione del piano tangente al grafico di g in corrispondenza di tale punto.

2) Si calcoli l'integrale
$$\iint_{G \cup F} (xy + x + y + 1)\, dx\, dy$$
ove $G = \{(x,y) \in \mathbb{R}^2 \mid x^2 - 2x + y^2 \leq 0\}$

F = triangolo di vertici $(-2,0)$, $(2,2)$, $(0,2)$

3) (a) Si studi la convergenza puntuale ed uniforme della serie di funzioni
$$\sum_{n=1}^{+\infty} \left(\frac{\ln n}{n} + \frac{1}{n!}\right)(\ln x)^n$$

(b) Si determini la somma della serie di potenze
$$\sum_{n=1}^{\infty} \left(\frac{1}{n} + n(n+1)\right) x^n$$

4) Studiare l'esistenza di massimi e minimi della funzione
$$f(x,y) = \sqrt{x^2 - 2x + y}$$
nell'insieme $E = \{(x,y) \in \mathbb{R}^2 \mid y^2 - x^2 = 4\}$

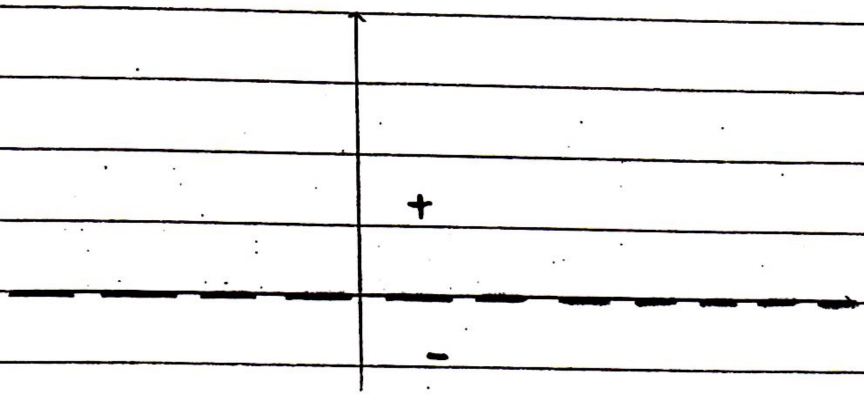
$\{(x,y) \in \mathbb{R}^2 \mid y^2 - x^2 \leq 4,\ x \leq 2,\ y \geq 0\}$

COMPITO del 27/05/2013

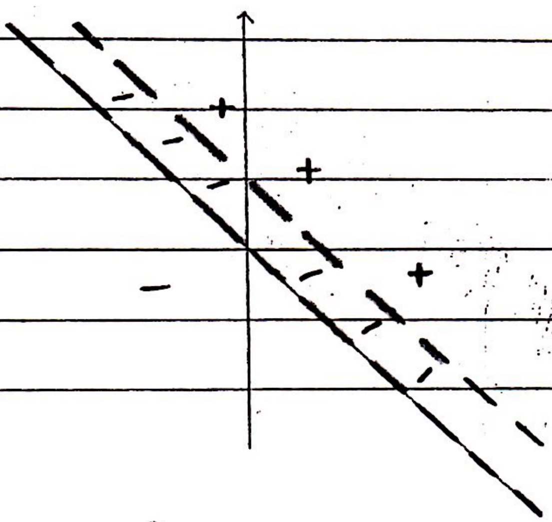
$\boxed{1}$ $f(x,y) = \dfrac{x + \ln y}{\ln(x+y)} \cdot \sqrt{x(9 - x^2 - 9y^2)}$

$A = \{ (x,y) \in \mathbb{R}^2 \mid y > 0,\ \ln(x+y) \neq 0,\ x + y > 0,\ x(9 - x^2 - 9y^2) \geq 0 \}$

- $f_1(x,y) = y$ $\quad f: y \to y \quad f \in \mathcal{C}(\mathbb{R})$ per il Th di COLLEGAMENTO
  $f_1 \in \mathcal{C}(\mathbb{R}^2)$
  $\Rightarrow$ MRC $\quad y = 0$



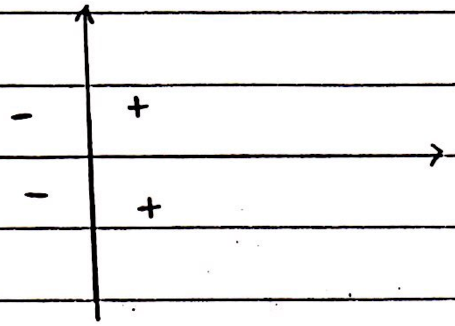
- $f_2(x,y) = \ln(x+y)$
  $f_3(x,y) = x + y$
  $f: x \to x \in \mathcal{C}(\mathbb{R})$ PER IL Th di COLLEGAMENTO $f(x,y) = x \in \mathcal{C}(\mathbb{R}^2)$
  $f: y \to y \in \mathcal{C}(\mathbb{R})$ PER IL Th di COLLEGAMENTO $f(x,y) = y \in \mathcal{C}(\mathbb{R}^2)$
  PER IL CRITERIO della SOMMA $f(x,y) = x + y \in \mathcal{C}(\mathbb{R}^2)$
  PER IL CRITERIO della COMPOSTA $\ln(x+y)$ È CONTINUA SUL SUO DOMINIO.
  $\Rightarrow$ MRC $\quad x + y = 0 \iff y = -x$
  $\ln(x+y) = 0 \iff x + y = 1 \iff y = -x + 1$



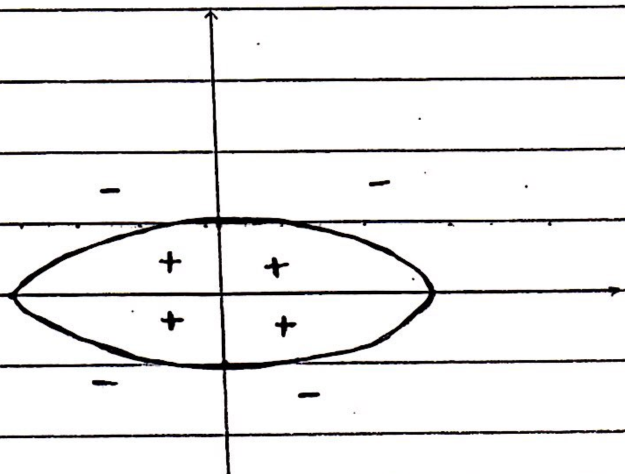
$f_3(-1;-1) = -2 < 0$
$f_3(2;0) = 2 > 0$

— segno di f_2

• $f_4(x,y) = x$ $\quad f_4 \in \mathcal{C}(\mathbb{R}^2)$ per MOTIVI ANALOGHI $\Longrightarrow$ MRC $\quad x = 0$



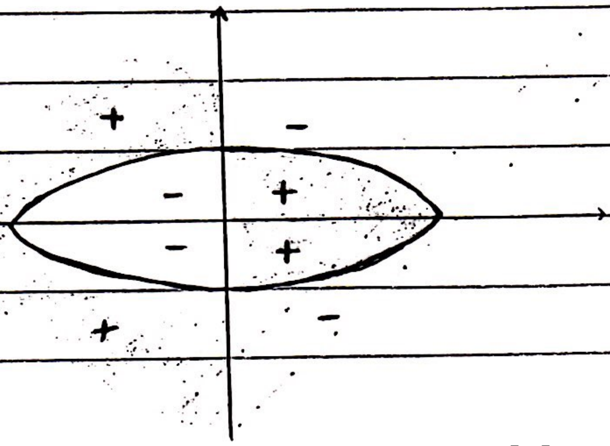
• $f_5 = 9 - x^2 - 9y^2$ $\quad f_5 \in \mathcal{C}(\mathbb{R}^2)$ per COLLEGAMENTO e SOMMA $\Longrightarrow$ MRC



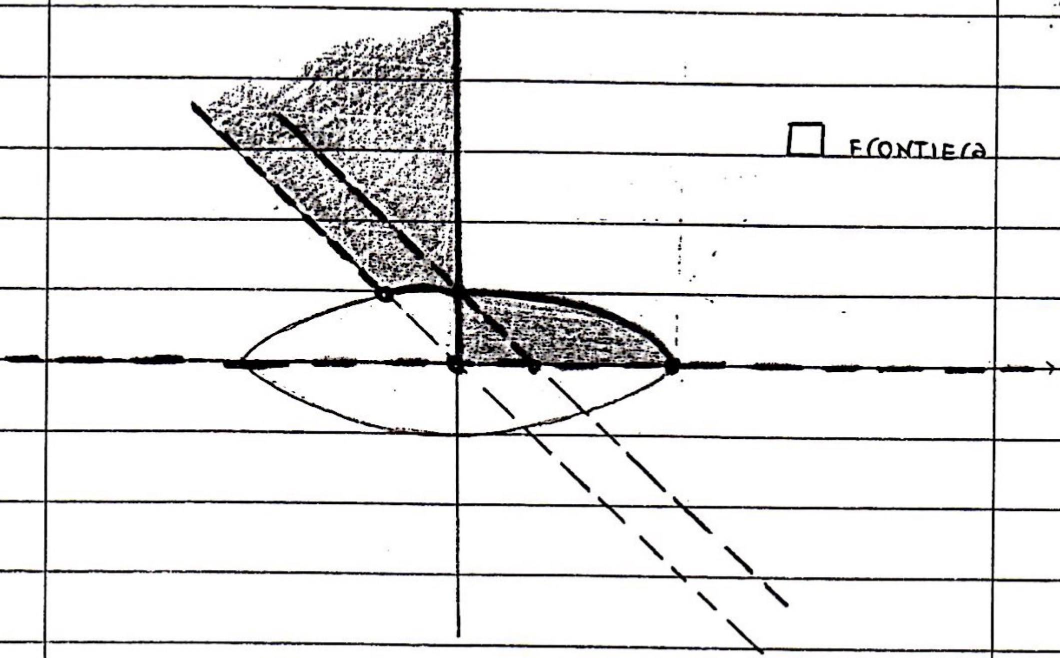
$f(0;0) = 9 > 0$
$f(4;0) = -7 < 0$

$9 - x^2 - 9y^2 = 0 \iff x^2 + 9y^2 = 9$

• $f_4 \cdot f_5 \geq 0$



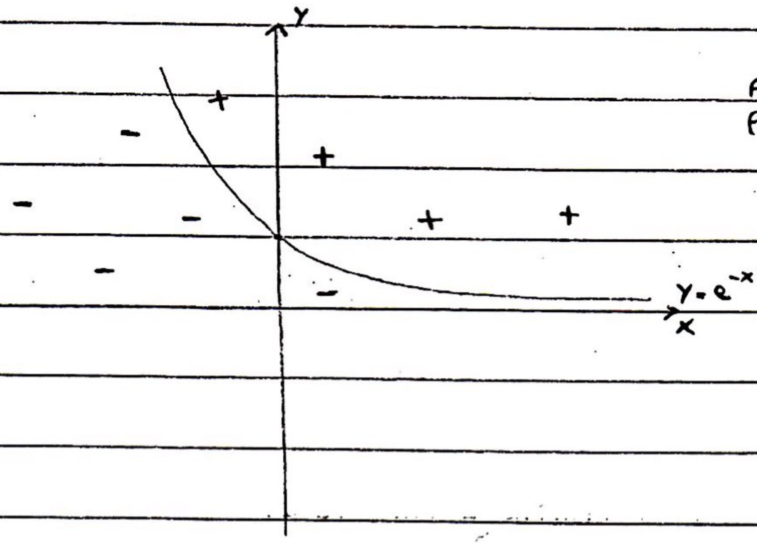
DOMINIO



• $f_6(x,y) = x + \ln y$

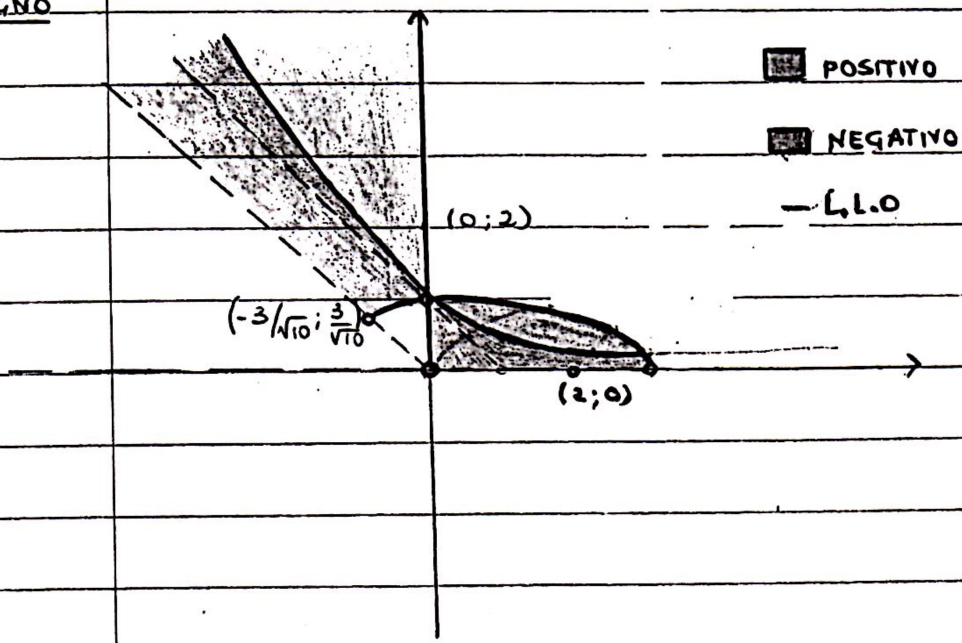
$f_6 \in C(\mathbb{R}^2)$ per criterio somma e collegamento $\Rightarrow$ MAC

$x + \ln y = 0 \iff \ln y = -x \iff y = e^{-x}$



$f(-2;1) < 0$
$f(1;5) > 0$

SEGNO



LIMITI

• $\left(-\frac{3}{\sqrt{10}};\frac{3}{\sqrt{10}}\right) \in D(A)$ $\notin D(A)$ sono su $y=-x$ $\Rightarrow \nexists \lim$

$$\lim_{(x,y)\to\left(-\frac{3}{\sqrt{10}};\frac{3}{\sqrt{10}}\right)} \frac{\overbrace{x+\ln y}^{\to -3/\sqrt{10}+\ln(3(\sqrt{10})\ \text{PER CONTINUITÀ}}}{\underbrace{\ln(\underbrace{x+y}_{\to 0})}_{\to -\infty}} \underbrace{\sqrt{x(9-x^2-9y^2)}}_{\to 0} = 0$$

per criterio della COMPOSTA

per CONTINUITÀ e criterio della COMPOSTA

• $(2;0) \in D(A)$

$$\lim_{(x,y)\to(2;0)} \frac{\overbrace{x+\ln y}^{\to -\infty}}{\underbrace{\ln(x+y)}_{\to \ln(2)}} \underbrace{\sqrt{x(9-x^2-9y^2)}}_{\to \sqrt{10}} = -\infty$$

per criterio della COMPOSTA (F2 LIMITATA)

per criterio della COMPOSTA (F2 LIMITATA)

• $(0;2) \in D(A) \wedge f$ CONTINUA IN $(0;2)$

$$\lim_{(x,y)\to(0,2)} f(x,y) = f(0,2) = 0$$

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- $(0;0) \in D(A)$

$$\lim_{(x,y)\to 0} f(x,y) = 0$$

pto D

$$g(x,y) = \frac{x + \ln y}{\ln(x+y)} \qquad (1;1) \qquad 3x-4y+1=0$$

$$\parallel \quad 3x-4y=0 \qquad y = 3/4\,x$$

$$g \text{ DIFFERENZIABILE} \implies \exists \frac{\partial g}{\partial \underline{v}}(1;1) = \langle \nabla g(1;1), \underline{v} \rangle$$

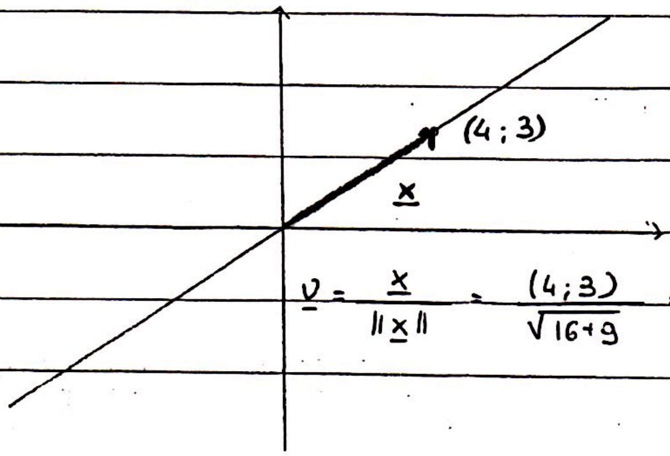
per il Th. del DIFFERENZIALE TOTALE AFFINCHÈ g SIA DIFFERENZIABILE le DERIVATE PARZIALI DEVONO ESISTERE IN UN INTORNO di (1;1) e DEVONO essere CONTINUE nel pto.

$$g'_x = \frac{\ln(x+y) - \frac{1}{x+y}(x+\ln y)}{\ln^2(x+y)} \in \mathcal{C}(A)$$

per criteri SOMMA/COMPOSTA/ COLLEGAMENTO

$$g'_y = \frac{\frac{1}{y}\ln(x+y) - \frac{1}{x+y}(x+\ln y)}{\ln^2(x+y)} \in \mathcal{C}(A)$$

$$\Rightarrow g \text{ è DIFFERENZIABILE} \qquad \nabla g(1;1) = \left(\frac{\ln 2 - 1/2}{\ln^2 2}, \frac{\ln 2 - 1/2}{\ln^2 2}\right)$$



$$\underline{v} = \frac{\underline{x}}{\|\underline{x}\|} = \frac{(4;3)}{\sqrt{16+9}} = \left(\frac{4}{5}; \frac{3}{5}\right)$$

$$\frac{\partial g}{\partial \underline{v}}(1;1) = \left\langle \left(\frac{\ln 2 - 1/2}{\ln^2 2}, \frac{\ln 2 - 1/2}{\ln^2 2} \right), (4/5, 3/5) \right\rangle$$

$$= \boxed{\frac{4}{5} \frac{\ln 2 - 1/2}{\ln^2 2} + \frac{3}{5} \frac{\ln 2 - 1/2}{\ln^2 2}}$$

PIANO TANGENTE

$$h(x,y) = \langle \nabla g(1,1), (x-1, y-1) \rangle =$$

$$\boxed{\frac{\ln 2 - 1/2}{\ln^2 2}(x-1) + \frac{\ln 2 - 1/2}{\ln^2 2}(y-1) = 0}$$

$\boxed{3}$

$$\sum_{m=1}^{+\infty} \left(\frac{\ln m}{m} + \frac{1}{m!} \right) (\ln x)^m$$

$$t = \ln x \implies \sum_{m=1}^{+\infty} \left(\frac{\ln m}{m} + \frac{1}{m!} \right) t^m$$

$$\lim_{m \to +\infty} \frac{\frac{\ln(m+1)}{m+1} + \frac{1}{(m+1)!}}{\frac{\ln(m)}{m} + \frac{1}{m!}} = \lim_{m \to +\infty} \frac{m! \ln(m+1) + 1}{(m+1)\cancel{m!}} \cdot \frac{\cancel{m!}}{(m-1)! \ln(m)+1}$$

$$= \lim_{m \to +\infty} \frac{\ln(m+1)}{\ln(m)} \cdot \frac{m}{m+1} = 1 \qquad \rho = 1$$

$$(-1;1) \subset \Gamma_t \subset [-1;1]$$

per $t = -1$

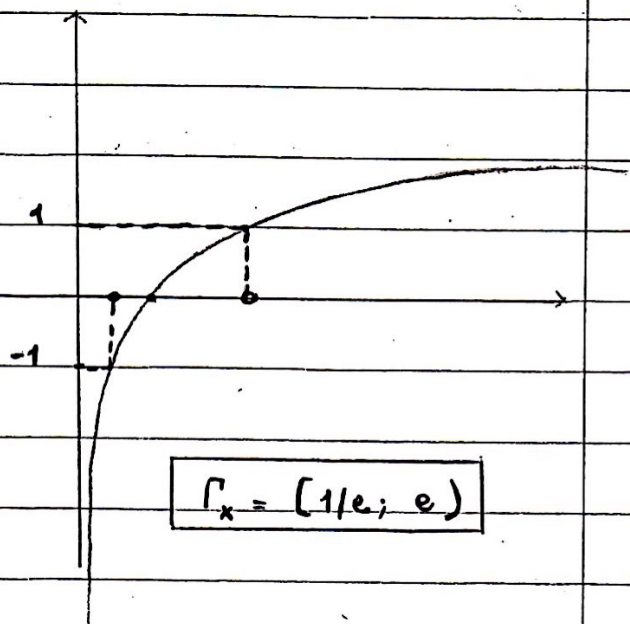
TERMINE m-ESIMO INFINITESIMO
TERMINI di SEGNO ALTERNATO
TERMINI DECRESCENTI

per LEIBNITZ la SERIE CONVERGE
$\{-1\} \in \Gamma_t$

per $t=-1$ $\quad \sum \frac{\ln m}{m} + \frac{1}{m!} = \underbrace{\sum \frac{\ln m}{m}}_{\text{DIVERGE}} + \underbrace{\sum \frac{1}{m!}}_{\text{CONVERGE}}$ $\quad \{1\} \notin \Gamma_t$

$\Gamma_t = [-1;1)$

$\ln x = -1 \iff x = 1/e$
$\ln x = 1 \iff x = e$



$$\boxed{\Gamma_x = [1/e;\ e)}$$

la serie converge puntualmente in ogni pto di Γ_x

la serie converge uniformemente in ogni intervallo chiuso e limitato $[a;b]$ tale che $1/e \le a < b < e$

SOMMA:

$$\sum_{m=1}^{+\infty}\left[\frac{1}{2m} + m(m+1)\right]x^m = \underbrace{\frac{1}{2}\sum_{m\ge1}\frac{x^m}{m}}_{A} + \underbrace{x\sum_{m\ge1} m(m+1)\,x^{m-1}}_{B}$$

A: $-\frac{1}{2}\ln(1-x)$

B: $x\sum_{m\ge1}\frac{d^2}{dx}(x^{m+1}) = x\,\frac{d''}{dx}\sum_{m\ge1}x^{m+1} = x\cdot\frac{d''}{dx}\left(\frac{1}{1-x}-1-x\right)$

$= \frac{2x}{(1-x)^3}$

$$\iint_{G\cup F} f(x,y)\,dx\,dy = \iint_{G\cup F} xy\,dx\,dy + \iint_{G\cup F} x\,dx\,dy + \iint_{G\cup F} y\,dx\,dy$$

$$+ \underbrace{\iint_{G\cup F} dx\,dy}_{m(G\cup F)} = \frac{10}{3} + \frac{3}{4}\pi - \frac{1}{3} - \frac{5}{12} + \frac{3\pi}{4} + \frac{7}{2} =$$

[redacted]

(1) $$\iint_{G\cup F} y\,dx\,dy = \iint_{G} y\,dx\,dy + \iint_{F-G} y\,dx\,dy = \boxed{\frac{10}{3}}$$

G SIMMETRICO RISPETTO ASSE X ; $f(-y) = -f(y)$ DISPARI

$$\Longrightarrow \iint_G y\,dx\,dy = 0$$

$$\iint_{F-G} y\,dx\,dy = \underbrace{\iint_{F_1} y\,dx\,dy}_{A} + \underbrace{\iint_{D} y\,dx\,dy}_{B} - \underbrace{\iint_{\Delta} y\,dx\,dy}_{C}$$

$\underline{\underline{A}}$: $$\int_0^2 y\left[\int_{y-2}^0 dx\right] dy = \int_0^2 y(2-y)\,dy = \left[y^2 - \frac{y^3}{3}\right]_0^2 = \frac{4}{3}$$

$\underline{\underline{B}}$: $$\int_0^1 \left[\int_0^{-x+2} y\,dy\right] dx = \int_0^1 \frac{(-x+2)^2}{2}\,dx = \frac{1}{2}\left[\frac{x^3}{3} - 2x^2 + 4x\right]_0^1 = \frac{7}{3}$$

$\underline{\underline{C}}$: $$\begin{cases} x = \rho\cos\theta + 1 \\ y = \rho\,\mathrm{sen}\,\theta \end{cases} \qquad \begin{matrix} \rho \in [0;1] \\ \theta \in [\pi/2;\pi] \end{matrix}$$

$$\iint_{\Delta} \rho^2\,\mathrm{sen}\,\theta\,d\rho\,d\theta = \int_0^1 \rho^2\left[\int_{\pi/2}^{\pi} \mathrm{sen}\,\theta\,d\theta\right] d\rho = \int_0^1 \rho^2\left[-\cos\theta\right]_{\pi/2}^{\pi} d\rho$$

$$= \left[\frac{\rho^3}{3}\right]_0^1 = 1/3$$

(2) $\iint_{G\cup F} x\,dx\,dy = \iint_F x\,dx\,dy + \iint_{G-F} x\,dx\,dy$

F: SIMMETRICO RISPETTO ASSE Y; $f(-x) = -f(x)$ DISPARI

$\longrightarrow \iint_F x\,dx\,dy = 0$

$$\iint_{G-F} x\,dx\,dy = \underbrace{\iint_{G'} x\,dx\,dy}_{A} + \underbrace{\iint_{D} x\,dx\,dy}_{B} - \underbrace{\iint_{\triangle} x\,dx\,dy}_{C} = \boxed{\frac{3}{4}\pi - \frac{1}{3}}$$

A: $\begin{cases} x-1 = \rho\cos\theta \\ y = \rho\sin\theta \end{cases}$ $\quad \rho \in [0;1]$, $\theta \in [\pi; 2\pi]$

$$\iint_{G'} (1+\rho\cos\theta)\rho\,d\rho\,d\theta = \int_0^1 \rho\left[\int_\pi^{2\pi} d\theta\right] d\rho + \int_0^1 \rho^2\left[\int_\pi^{2\pi} \cos\theta\,d\theta\right] d\rho =$$

$$\pi\left[\frac{\rho^2}{2}\right]_0^1 = \frac{\pi}{2}$$

B: $\begin{cases} x-1 = \rho\cos\theta \\ y = \rho\sin\theta \end{cases}$ $\quad \rho \in [0;1]$, $\theta \in [0; \pi/2]$

$$\iint_D (1+\rho\cos\theta)\rho\,d\rho\,d\theta = \int_0^1 \rho\left[\int_0^{\pi/2} d\theta\right] d\rho + \int_0^1 \rho^2\left[\int_0^{\pi/2} \cos\theta\,d\theta\right] d\rho =$$

$$= \frac{\pi}{4} + \left[\frac{\rho^3}{3}\right]_0^1 = \frac{\pi}{4} + \frac{1}{3}$$

C: $\displaystyle \int_1^2 x\left[\int_0^{2-x} dy\right] dx = \int_1^2 x(2-x)\,dx = \left| x^2 - \frac{x^3}{3}\right|_1^2 = \frac{2}{3}$

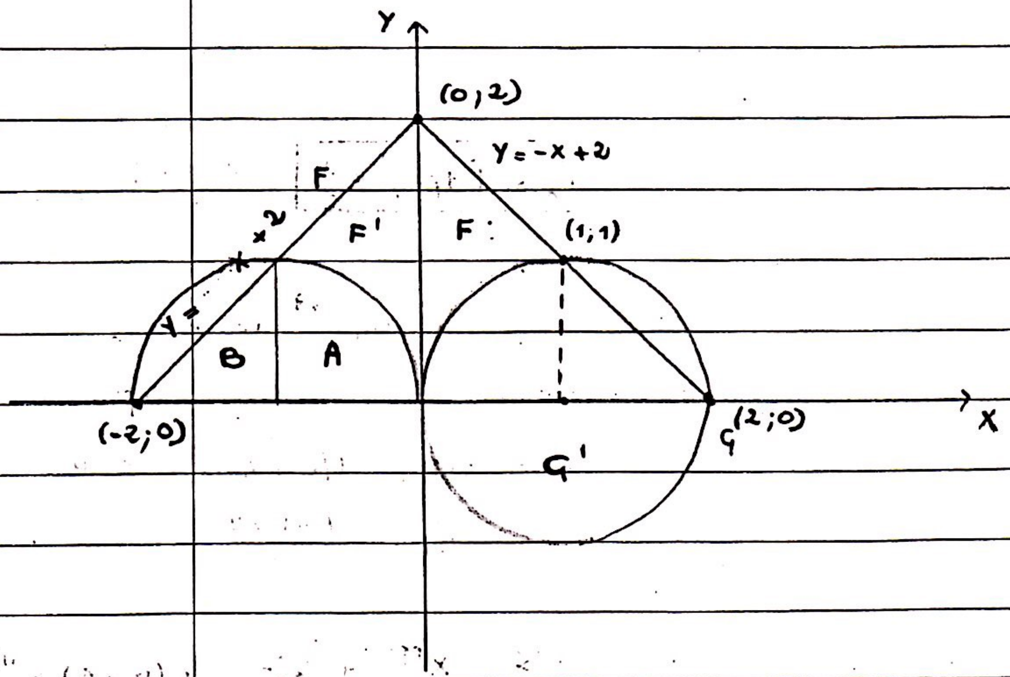
$$\boxed{S_x = \frac{2x}{(1-x)^3} - \frac{1}{2}\ln(1-x)}$$

$\boxed{2}$

$$\iint_{G\cup F} (xy+x+y+1)\,dx\,dy$$

$G = \{(x,y)\in\mathbb{R}^2 \mid x^2-2x+y^2 \le 0\}$

$F = \Delta \quad (-2;0),\ (2;0),\ (0;2)$



$$x^2-2x+y^2 = x^2-2x+1+y^2-1 = (x-1)^2+y^2-1$$

$$\Longrightarrow (x-1)^2+y^2=1$$

(3) $\iint_{G\cup F} xy\,dx\,dy = \iint_{A\cup B} xy\,dx\,dy = \underbrace{\iint_A xy\,dx\,dy}_{A} + \underbrace{\iint_B xy\,dx\,dy}_{B} = \boxed{\frac{-5}{12}}$

A: $\begin{cases} x+1 = \rho\cos\theta \\ y = \rho\,\text{sem}\theta \end{cases}$ $\quad \rho \in [0;1]$, $\theta \in [0;\pi/2]$

$$\iint_A (\rho\cos\theta - 1)\,\rho^2 \text{sem}\theta\, d\rho\, d\theta =$$

$$\iint_A \rho^3 \cos\theta\,\text{sem}\theta\, d\rho\, d\theta - \iint_A \rho^2\,\text{sem}\theta\, d\rho\, d\theta =$$

$$\int_0^1 \rho^3 \left[\int_0^{\pi/2} \cos\theta\,\text{sem}\theta\, d\theta\right] d\rho - \int_0^1 \rho^2 \left[\int_0^{\pi/2} \text{sem}\theta\, d\theta\right] d\rho =$$

$$\int_0^1 \rho^3 \left[\frac{\text{sem}^2\theta}{2}\right]_0^{\pi/2} d\rho - \int_0^1 \rho^2 \left[-\cos\theta\right]_0^{\pi/2} d\rho =$$

$$\frac{1}{2}\left[\frac{\rho^4}{4}\right]_0^1 - \left[\frac{\rho^3}{3}\right]_0^1 = -\frac{5}{24}$$

B: $\displaystyle\int_{-2}^{-1} x \left[\int_0^{x+2} y\, dy\right] dx = \int_{-2}^{-1} x\, \frac{x^2+4x+4}{2}\, dx =$

$$\frac{1}{2}\left[\frac{x^4}{4} + \frac{4x^3}{3} + 2x^2\right]_{-2}^{-1} = \frac{-5}{24}$$

(4) $m(F\cup G) = \boxed{\pi + 2 + \frac{3}{2} - \frac{\pi}{4}}$

$\boxed{4}$ $h(x,y) = \sqrt{x^2 - 2x + y^2}$

$h(x,y) \in \mathcal{C}(A)$

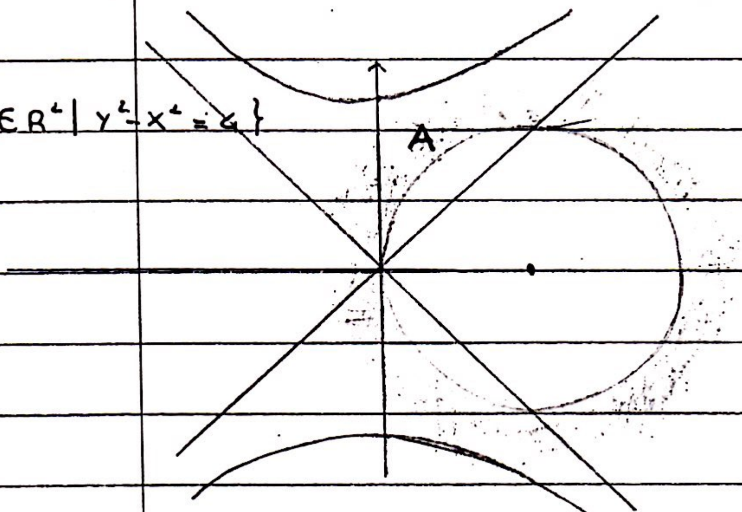
$x^2 - 2x + y^2 \in \mathcal{C}(\mathbb{R}^2) \Longrightarrow$ MRC

$x^2 - 2x + y^2 = (x-1)^2 + y^2 - 1 = 0 \iff (x-1)^2 + y^2 = 1$

$E = \{(x,y) \in \mathbb{R}^2 \mid y^2 - x^2 = 4\}$

$f(1;0) < 0$

$f(3;0) = 3 > 0$



(a) $\max\limits_{\substack{\min\\E}} h(x,y) \equiv \max\limits_{\substack{\min\\E\cap A}} x^2 - 2x + y^2 \equiv \max\limits_{\substack{\min\\y^2 = 4 + x^2}} x^2 - 2x + y^2$

$\equiv \max\limits_{\min} x^2 - 2x + x^2 + 4 \equiv \max\limits_{\min} 2x^2 - 2x + 4 \longrightarrow 4x - 2 = 0$

$\iff x = 1/2$

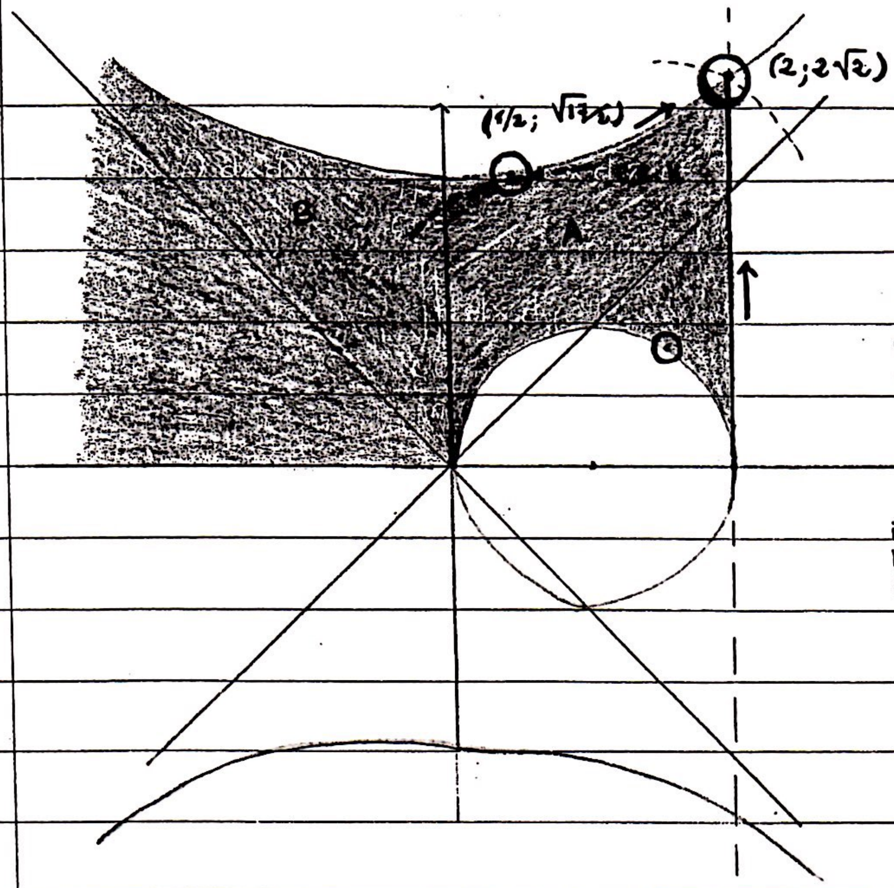
$y = \pm \frac{\sqrt{17}}{2}$

1/2

$(1/2; \sqrt{17}/2)$
$(1/2; -\sqrt{17}/2)$ pti di MINIMO ASSOLUTI

f è ILLIMITATA SUPERIORMENTE sulla restrizione $\Longrightarrow$ NON ESISTONO pti di MASSIMO ASSOLUTO

(b)



H∩A NON è NÈ CHIUSO NÈ LIMITATO ⟹ NON POSSO APPLICARE WEIERSTRASS GLOBALMENTE.

IN H C'È UN MINIMO e UN MASSIMO ASSOLUTO.

$\max_{H\cap A}$ / $\min_{H\cap A}$ x^2-2x+y^2

pti INTERNI: g'_x $2x-2=0 \iff x=1$ (1;0) NON ACCETTABILE

g'_y $2y=0 \iff y=0$

pti FRONTIERA: $\boxed{\text{i pti sulla CRF SONO pti di MINIMO ASSOLUTO POICHÈ IN OGNI INTORNO } f(x_0)\le f(x)}$

$\max_{x=2}$ / $\min_{x=2}$ $x^2-2x+y^2 \equiv \max/\min\ y^2 \longrightarrow 2y=0 \iff \boxed{y=0,\ x=2}$ MINIMO ASSOLUTO

IL PTO $\boxed{(2;\ 2\sqrt{2})}$ è pto di MASSIMO RELATIVO per H∩A

$\max_{x=0}$ / $\min_{x=0}$ $x^2-2x+y^2 \Longrightarrow \boxed{x=0,\ y=0}$ MINIMO ASSOLUTO

$\max$ / $\min$ ($y^2-x^2=4$, $0\le x\le 2$) x^2-2x+y^2 $\boxed{x=1/2,\ y=+\sqrt{17}/2}$ NON è pto di OTTIMO

$\max_{y=0}$ / $\min_{y=0}$ $x^2-2x+y^2 = \max/\min\ x^2-2x \longrightarrow 2x-2=0$ $\boxed{x=1,\ y=0}$ NON ACC.

la FZ SU H∩A è ILLIMITATA ⟶ NON C'È MASSIMO ASSOLUTO