PROVA SCRITTA DI ANALISI

(APPELLO DEL 4/11/2012)

1) Data la funzione

$$f(x,y) = \frac{\ln(x-y)}{x^2+y-1}\sqrt{(x-1)^2(x^2-4y^2+4)}$$

(a) studiare l'insieme di definizione ed il segno di f;

(b) rappresentare in grafico separato la linea di livello zero e la frontiera dell'insieme di definizione di f;

(c) Si calcoli la derivata direzionale della funzione $g(x,y) = \dfrac{e^{x-y}}{\ln(x+y+2)}$ nel punto $(0,0)$ lungo la direzione orientata che forma un angolo di $120°$ con l'asse delle x; scrivere l'equazione del piano tangente al grafico di g in corrispondenza di tal punto.

(2) Si calcoli l'integrale

$$\iint_{E\cup F} (e^{x} y + x + y)\,dx\,dy$$

ove $E = \{(x,y)\in\mathbb{R}^2 \mid x^2+y^2 \le 1,\ xy \ge 0\}$

F = triangolo di vertici $(1,0), (2,1), (2,-1)$

(3) Data la serie di funzioni

$$\sum_{n=1}^{+\infty}\left(\frac{1}{n e^n} - n\right)\left(\frac{x}{2x+3}\right)^n$$

(a) studiarne la convergenza puntuale ed uniforme; verificare se converge uniformemente nell'intervallo $]1,+\infty[$

(b) calcolarne la somma.

(4) Data la funzione $h(x,y) = \ln(x^2+y)$

(a) studiare i massimi e minimi di h nell'insieme

$$G = \{(x,y)\in\mathbb{R}^2 \mid 2x+y=3\}$$

(b) studiare i massimi e minimi assoluti di h nell'insieme

$$L = \{(x,y)\in\mathbb{R}^2 \mid 0 \le 2x+y \le 3,\ x \ge 0\}$$

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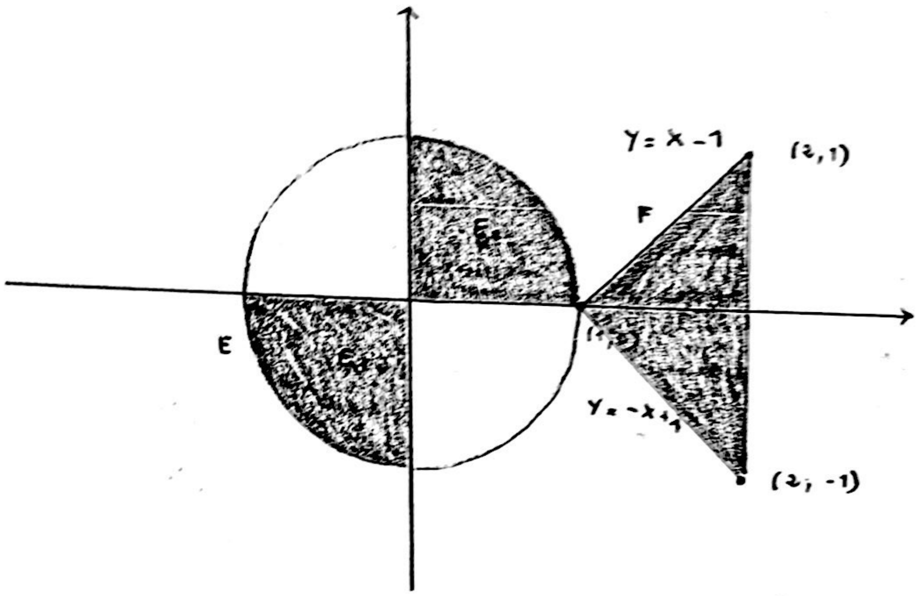
$$\iint_{E\cup F} (e^x y + x + y)\, dx\, dy = \boxed{[illegible]}$$

$$E = \{ (x,y) \in \mathbb{R}^2 \mid x^2 + y^2 \leq 1 ,\ xy \geq 0 \}$$

$F = \Delta \quad (1;0) \quad (2;1) \quad (2;-1)$



$$\iint_{E\cup F} h(x,y)\, dx\, dy = \iint_{E\cup F} e^x y\, dx\, dy + \iint_{E\cup F} x\, dx\, dy + \iint_{E\cup F} y\, dx\, dy =$$

$$= \underbrace{\iint_{E} e^x y\, dx\, dy}_{A} + \underbrace{\iint_{E\cup F} x\, dx\, dy}_{B} + \underbrace{\iint_{E} y\, dx\, dy}_{C}$$

A:

$$\underbrace{\iint_{E_1} e^x y\, dx\, dy}_{1} + \underbrace{\iint_{E_2} e^x y\, dx\, dy}_{2} = \boxed{[illegible]}$$

(1) $\begin{cases} x = \rho\cos\theta \\ y = \rho\,\mathrm{sen}\,\theta \end{cases} \qquad \begin{array}{l} \rho \in [0;1] \\ \theta \in [\pi; 3/2\pi] \end{array}$

$$\iint_{E_1} \rho^2\,\mathrm{sen}\theta\, e^{\rho\cos\theta}\, d\rho\, d\theta = \int_0^1 \rho \left[\int_{\pi}^{3/2\pi} \rho\,\mathrm{sen}\theta\, e^{\rho\cos\theta}\, d\theta \right] d\rho$$

$$= \int_0^1 \rho \left[-e^{\rho\cos\theta} \right]_{\pi}^{3/2\pi} d\rho = \int_0^1 \rho \left[-1 + e^{-\rho} \right] d\rho =$$

$$= \int_0^1 -\rho \, d\rho + \int_0^1 \rho e^{-\rho} d\rho = \left[-\frac{\rho^2}{2} \right]_0^1 + \left[-\rho e^{-\rho} + \int e^{-\rho} d\rho \right]_0^1 =$$

$$= -\frac{1}{2} + \left[-\rho e^{-\rho} - e^{-\rho} \right]_0^1 = +\frac{1}{2} - 2e^{-1} = \boxed{\frac{1}{2} - \frac{2}{e}}$$

(2) $\begin{cases} x = \rho\cos\theta \\ y = \rho \operatorname{sen}\theta \end{cases}$ $\quad \rho \in [0;1]$, $\theta \in [0; \pi/2]$

$$\iint_{E_2} \rho^2 \operatorname{sen}\theta \, e^{+\rho\cos\theta} d\rho \, d\theta = \int_0^1 \rho \left[\int_0^{\pi/2} \rho \operatorname{sen}\theta \, e^{+\rho\cos\theta} d\theta \right] d\rho$$

$$= \int_0^1 \rho \left[-e^{\rho\cos\theta} \right]_0^{\pi/2} d\rho = \int_0^1 \rho \left(-1 + e^{\rho} \right) d\rho =$$

$$= \int_0^1 -\rho \, d\rho + \int_0^1 \rho e^{\rho} d\rho = \left[-\frac{\rho^2}{2} \right]_0^1 + \left[\rho e^{\rho} - \int e^{\rho} d\rho \right]_0^1 =$$

$$= -\frac{1}{2} + \left[\rho e^{\rho} - e^{\rho} \right]_0^1 = +\frac{1}{2} + e - e + 1 = \boxed{\frac{3}{2}}$$

B:

$$\underbrace{\iint_{E_1} x \, dx \, dy}_{1} + \underbrace{\iint_{E_2} x \, dx \, dy}_{2} + \underbrace{\iint_{F} x \, dx \, dy}_{3} = \boxed{\frac{5}{3}}$$

(1) $\begin{cases} x = \rho\cos\theta \\ y = \rho \operatorname{sen}\theta \end{cases}$ $\quad \rho \in [0;1]$, $\theta \in [\pi; 3/2\pi]$

$$\iint \rho^2 \cos\theta \, d\rho \, d\theta = \int_0^1 \rho^2 \left[\int_{\pi}^{3/2\pi} \cos\theta \, d\theta \right] d\rho =$$

$$\int_0^1 \rho^2 \left[\operatorname{sen}\theta \right]_{\pi}^{3/2\pi} d\rho = \int_0^1 \rho^2 (-1) \, d\rho = -\left[\frac{\rho^3}{3} \right]_0^1 = \boxed{-\frac{1}{3}}$$

(2) $\begin{cases} x = \rho\cos\theta \\ y = \rho \operatorname{sen}\theta \end{cases}$ $\quad \rho \in [0;1]$, $\theta \in [0; \pi/2]$

$$\iint \rho^2 \cos\theta \, d\rho \, d\theta = \int_0^1 \rho^2 \left[\int_0^{\pi/2} \cos\theta \, d\theta \right] d\rho = \int_0^1 \rho^2 \left[\operatorname{sen}\theta \right]_0^{\pi/2} d\rho$$

$$= \left[\frac{\rho^3}{3} \right]_0^1 = \boxed{\frac{1}{3}}$$

(3) $$\int_1^2 x \left[\int_{-x+1}^{x-1} dy \right] dx = \int_1^2 x \left[x - 1 + x - 1 \right] dx = \int_1^2 2x^2 - 2x \, dx =$$

$$\left[\frac{2x^3}{3} - x^2 \right]_1^2 = \boxed{\frac{5}{3}}$$

C: $\underbrace{\iint_{E_1} y\,dx\,dy}_{1} + \underbrace{\iint_{E_2} y\,dx\,dy}_{2} = \boxed{0}$

(1) $\begin{cases} x = \rho\cos\theta \\ y = \rho\operatorname{sen}\theta \end{cases}$ $\quad \rho \in [0;1]$, $\theta \in [\pi;\ 3/2\,\pi]$

$$\iint_{E_1} \rho^2 \operatorname{sen}\theta\, d\rho\, d\theta = \int_0^1 \rho^2 \left[\int_\pi^{3/2\pi} \operatorname{sen}\theta\, d\theta\right] d\rho =$$

$$\int_0^1 \rho^2 \left[-\cos\theta\right]_\pi^{3/2\pi} d\rho = \left[-\frac{\rho^3}{3}\right]_0^1 = \boxed{-\frac{1}{3}}$$

(2) $\begin{cases} x = \rho\cos\theta \\ y = \rho\operatorname{sen}\theta \end{cases}$ $\quad \rho \in [0;1]$, $\theta \in [0;\ \pi/2]$

$$\int_0^1 \rho^2 \left[\int_0^{\pi/2} \operatorname{sen}\theta\, d\theta\right] d\rho = \int_0^1 \rho^2 \left[-\cos\theta\right]_0^{\pi/2} d\rho = \left[\frac{\rho^3}{3}\right]_0^1 = \boxed{\frac{1}{3}}$$

$\boxed{3}$

$$\sum_{m=1}^{+\infty} \left(\frac{1}{m e^m} - m\right)\left(\frac{x}{2x+3}\right)^m$$

$$t = \frac{x}{2x+3} \longrightarrow \sum_{m\geq 1} \left(\frac{1}{m e^m} - m\right) t^m$$

$$\lim_{m\to+\infty} \frac{a_{m+1}}{a_m} = \lim_{m\to+\infty} \frac{-(m+1)}{-m} = 1$$ per principio sostituzione infiniti

$\rho = 1 \longrightarrow (-1;\ 1) \subset \Gamma_t \subset [-1;\ 1]$

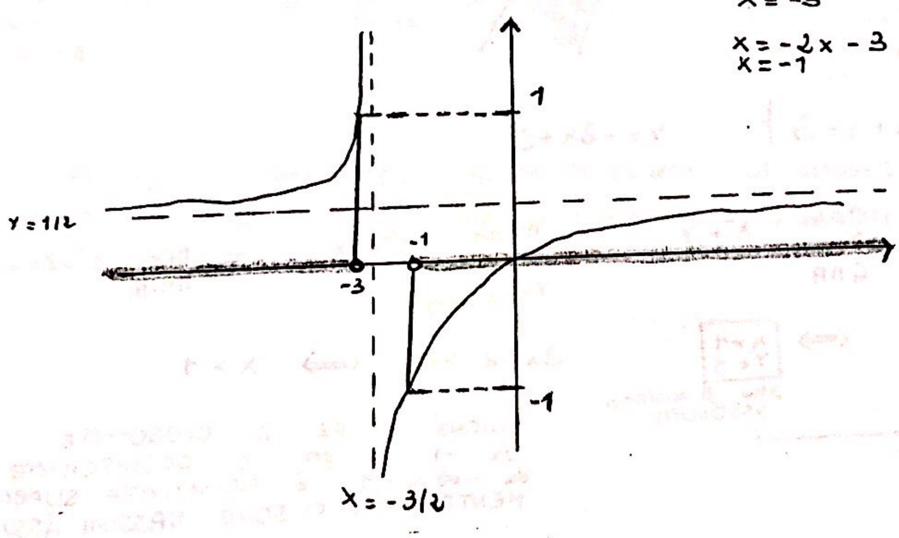
per $t=-1$, $t=1$ il termine m-esimo non è infinitesimo $\longrightarrow$ la serie non converge

$\Gamma_t = (-1;1)$

$x = 2x+3$
$x = -3$

$x = -2x - 3$
$x = -1$

$\Gamma_x = (-\infty;\ -3) \cup (-1;\ +\infty)$



la serie converge puntualmente in ogni $x \in \Gamma_x$.
converge uniformemente in ogni intervallo chiuso contenuto in Γ_x. Non converge uniformemente in $(-1;+\infty)$

$$\underbrace{\sum_{m\geq 1} \frac{t^m}{m e^m}}_{A} - \underbrace{\sum m t^m}_{B}$$

$\underline{\underline{A}}$: $\sum_{m\geq 1} \frac{1}{m}\left(\frac{t}{e}\right)^m = -\ln\left(1-\frac{t}{e}\right)$

$\underline{\underline{B}}$: $\sum_{m\geq 1} m t^m = t \sum_{m\geq 1} m t^{m-1} = t \sum_{m\geq 1} \frac{d}{dt}(t^m) = t \frac{d}{dt} \sum_{m\geq 1} t^m$

$$= t \cdot \frac{d}{dt}\left(\frac{1}{1-t} - 1\right) = \frac{t}{(1-t)^2}$$

$$\boxed{S_t = -\ln\left(1-\frac{t}{e}\right) - \frac{t}{(1-t)^2}}$$

per trovare la SOMMA IN X SOSTITUIRE $t = \frac{x}{2x+3}$

$\boxed{4}$

$h(x,y) = \ln(x^2+y)$

$A = \{(x,y) \in \mathbb{R}^2 \mid x^2+y > 0\}$

$x^2+y \in \mathcal{C}(\mathbb{R}^2)$ per criterio collegamento-somma

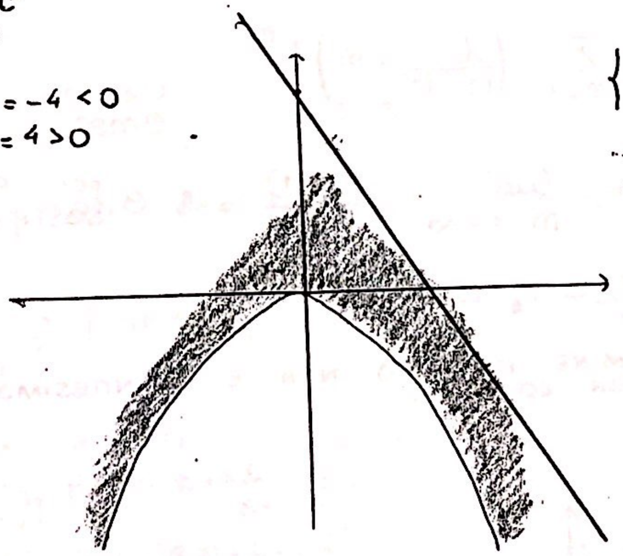
$\longrightarrow$ MRC

$x^2+y=0$

$y=-x^2$

$f(0;-4) = -4 < 0$

$f(0;4) = 4 > 0$



$\begin{cases} y = -x^2 \\ y = -2x+3 \end{cases}$

$x^2-2x+3=0$

$\iff x_{1,2} = \frac{2 \pm \sqrt{4-12}}{2}$ NON SI INTERSECANO MAI

$G = \{(x,y) \in \mathbb{R}^2 \mid 2x+y=3\}$ $\quad y = -2x+3$

$\max\limits_{G}\min h(x,y) \equiv \max\limits_{G\cap A}\min x^2+y \equiv \max\limits_{y=-2x+3}\min x^2+y \equiv \max\min x^2-2x+3$

$\longrightarrow 2x-2=0 \iff \boxed{\begin{matrix} x=1 \\ y=1 \end{matrix}}$ pto di MINIMO ASSOLUTO $\quad 2x-2>0 \iff x>1$

POICHÈ la Fz è crescente da 1 in poi e decrescente da $-\infty$ a 1, è illimitata superiormente NON CI SONO MASSIMI ASSOLUTI

$L = \{(x,y) \in \mathbb{R}^2 \mid 0 \le 2x+y \le 3 \ , \ x \ge 0\}$

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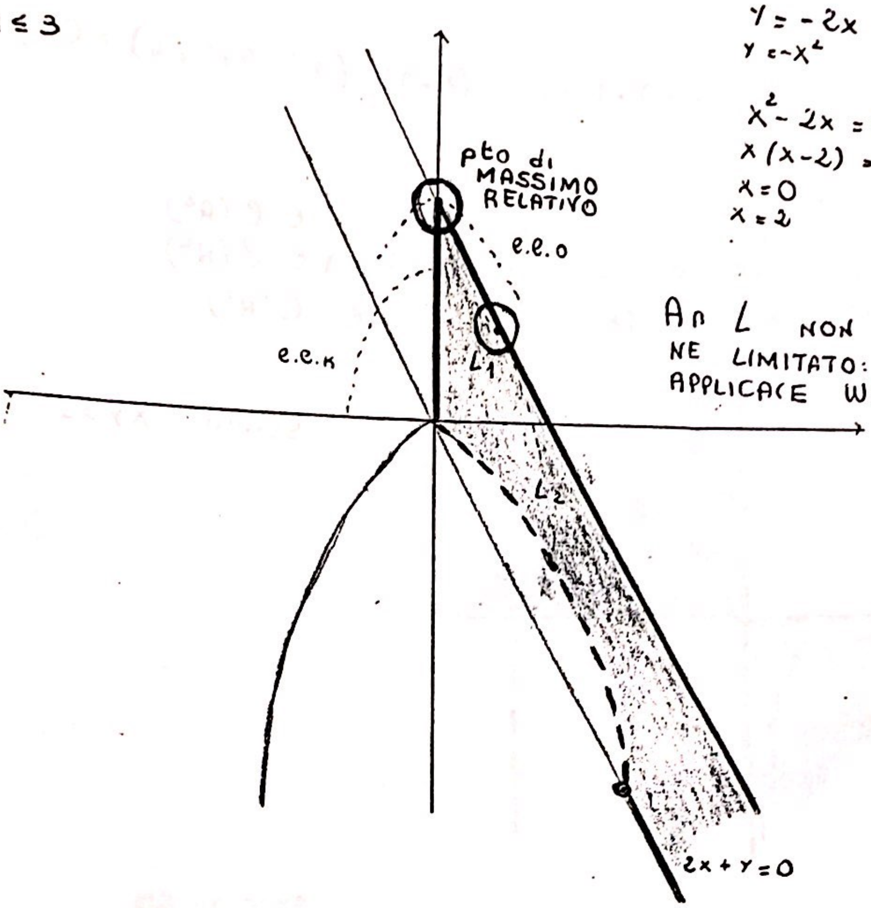
$2x+y \ge 0$
MRC
$y = -2x$

$2x+y \le 3$
MRC

$y = -2x$
$y = -x^2$

$x^2 - 2x = 0$
$x(x-2) = 0$
$x = 0$
$x = 2$



$A \cap L$ NON È NÈ CHIUSO NE LIMITATO: NON SI PUÒ APPLICARE WEIERSTRASS

pti INTERNI: $g'_x = 2x = 0 \iff x = 0$

$g'_y = 1 \ne 0 \iff$ IMP. NON CI SONO pti di OTTIMO INTERNI

Frontiera:

$\underset{x=0}{\max / \min} \ x^2 + y \equiv \max / \min \ y \longrightarrow 1 = 0$ IMP

$\underset{y=-2x}{\max / \min} \ x^2 + y \equiv \max / \min \ x^2 - 2x \longrightarrow 2x - 2 = 0$
$\iff \boxed{x = 1 \\ y = -2}$ N.A.

$\underset{y=-2x+3}{\max / \min} \ x^2 + y \equiv \max / \min \ x^2 - 2x + 3 \longrightarrow 2x - 2 = 0$
$\iff \boxed{x = 1 \\ y = 1}$ NON è pto di OTTIMO

$h(x,y)$ è ILLIMITATA SUPERIORMENTE ed INFERIORMENTE SULL'INSIEME: NON CI SONO pti di MASSIMO/MINIMO ASSOLUTI

1

SVOLTO $f(x,y) = \frac{\ln(xy)}{x^2+y-1} \sqrt{(x-1)^2(x^2-4y^2+4)}$

$A = \{(x,y) \in \mathbb{R}^2 \mid xy>0,\ x^2+y-1 \neq 0,\ (x-1)^2(x^2-4y^2+4) \geq 0\}$

$f_1(x,y) = xy$

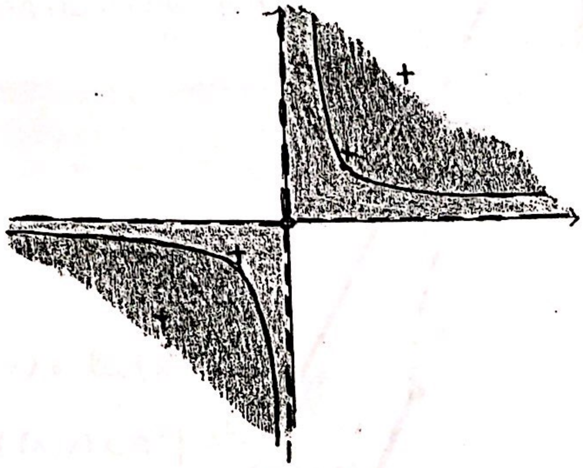
$f: x \to x \in \mathcal{C}(\mathbb{R})$ per criterio collegamento $f \in \mathcal{C}(\mathbb{R}^2)$

$g: y \to y \in \mathcal{C}(\mathbb{R})$ per criterio collegamento $g \in \mathcal{C}(\mathbb{R}^2)$

per criterio prodotto $f_1(x,y) : (x,y) \longrightarrow xy \in \mathcal{C}(\mathbb{R}^2)$

$\Longrightarrow$ MRC

SEGNO: $xy > 1$



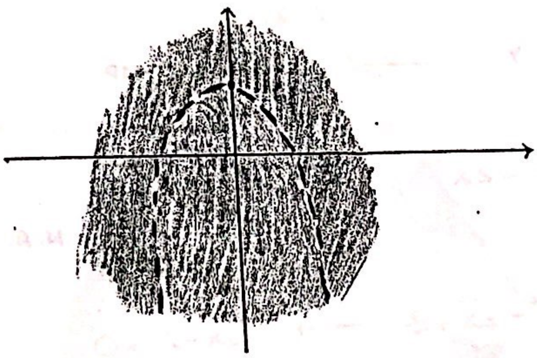
$f_2(x,y) = x^2+y-1$

$f_2 \in \mathcal{C}(\mathbb{R}^2)$ per criterio collegamento e somma $\longrightarrow$ MRC

$x^2+y-1=0 \iff y=-x^2+1$

$f(0;0) = -1$

$f(0;2) = 1$



$f_3(x,y) = (x-1)^2$

$f_3 \in \mathcal{C}(\mathbb{R}^2)$ per criterio collegamento e prodotto $\longrightarrow$ MRC

$(x-1)^2=0 \iff x=1$

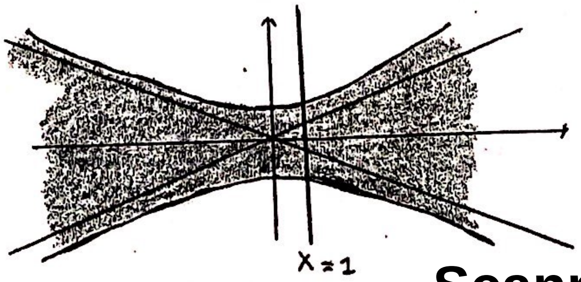
$f_4(x,y) = x^2-4y^2+4$

$f_4 \in \mathcal{C}(\mathbb{R}^2)$ per criterio collegamento e somma $\longrightarrow$ MRC

$x^2-4y^2+4=0$

$f(0;0) = 4>0$

$f(0;2) = -12<0$



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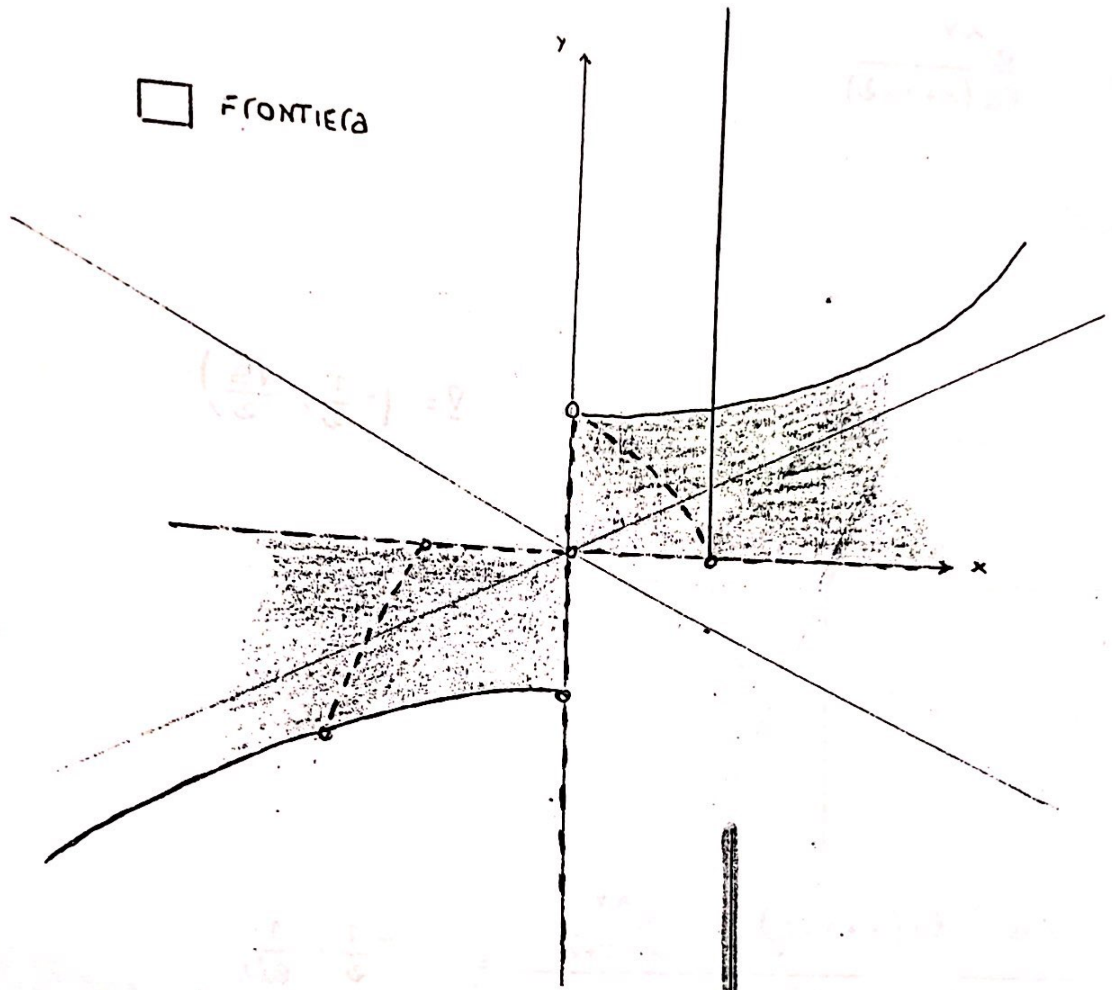

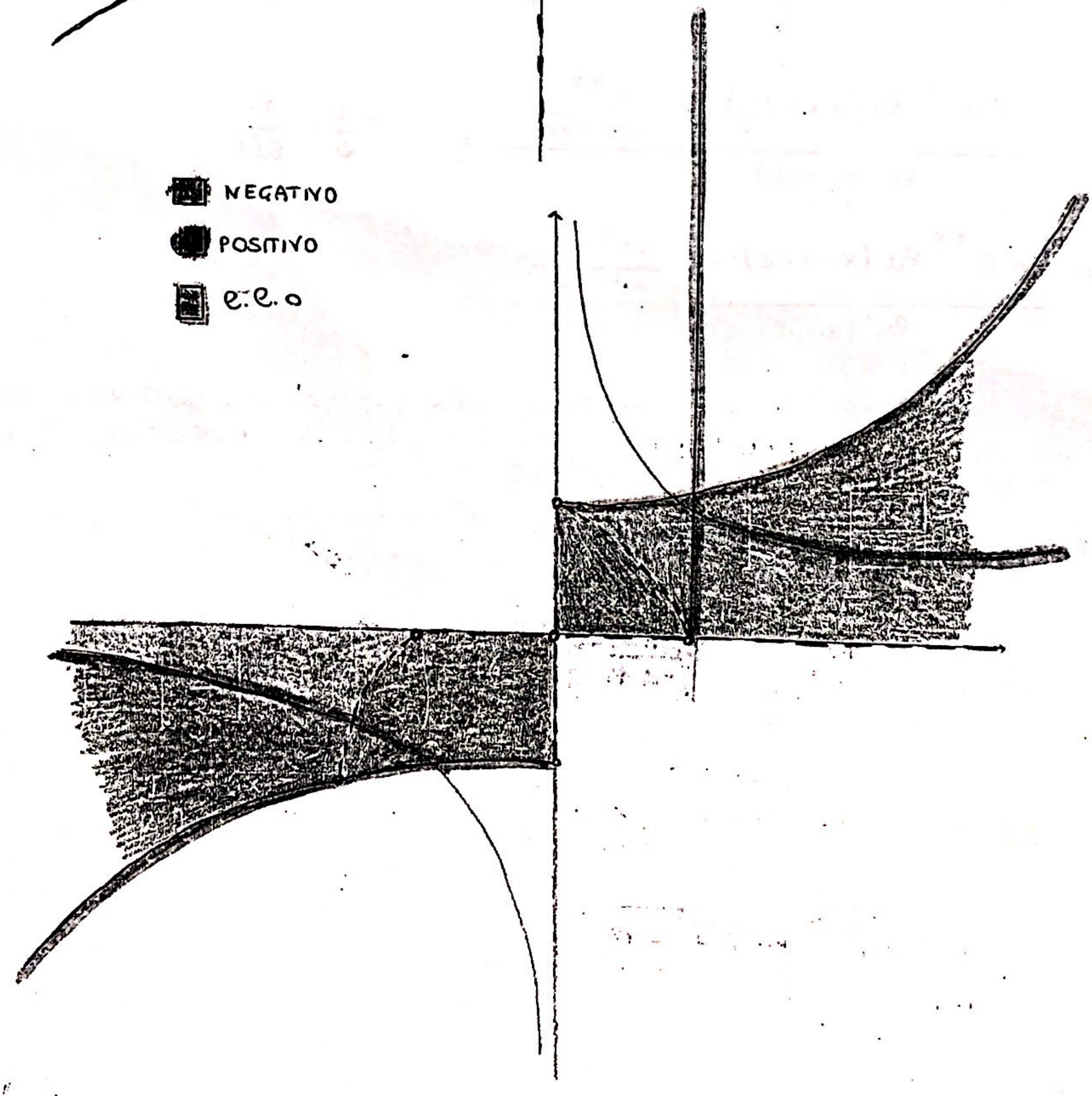

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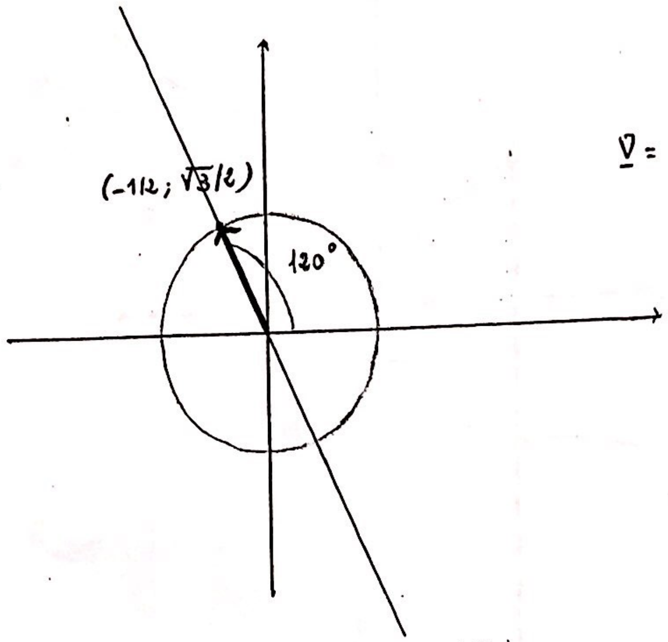
$$g(x,y) = \frac{e^{xy}}{\ln(x+y+2)}$$

$(0;0)$

$\alpha = 120°$



$\underline{v} = \left(-\frac{1}{2} ; \frac{\sqrt{3}}{2}\right)$

$$g'_x = \frac{y e^{xy} \cdot \ln(x+y+2) - \frac{e^{xy}}{x+y+2}}{\ln^2(x+y+2)} = -\frac{1}{2} \cdot \frac{1}{\ln^2 2}$$

$$g'_y = \frac{x e^{xy} \cdot \ln(x+y+2) - \frac{e^{xy}}{x+y+2}}{\ln^2(x+y+2)} = -\frac{1}{2} \cdot \frac{1}{\ln^2 2}$$

g'_x e g'_y ESISTONO IN UN INTORNO di $(0;0)$ e SONO CONTINUE nel pto. PER IL Th del DIFFERENZIALE TOTALE g è DIFFERENZIABILE IN $(0;0)$. ALLORA

$$\exists \frac{\partial g}{\partial \underline{v}}(0,0) = \langle \nabla g(0;0), \underline{v} \rangle = \left\langle \left(\frac{-1}{2\ln 2}, \frac{-1}{2\ln 2}\right), \left(-\frac{1}{2} ; \frac{\sqrt{3}}{2}\right) \right\rangle$$

$$= \frac{1}{4\ln^2 2} - \frac{\sqrt{3}}{4\ln^2 2} = \boxed{\frac{1-\sqrt{3}}{4\ln^2 2}}$$

$$z = \langle \nabla g(0;0), (x,y) \rangle = \frac{-x}{2\ln^2 2} - \frac{y}{2\ln^2 2}$$

$$\boxed{x + y + 2\ln^2 2\, z = 0}$$

PIANO TANGENTE