Exam Automated Reasoning

Radboud University

13/01/2023

- This exam contains 9 questions.
- The maximum number of points is 100. For each exercise, the number of points is given in parentheses.
- Concise answers suffice.
- *Good luck!*

1 The DPLL algorithm (15)

1. Consider the CNF φ

   $$(\underbrace{\neg x_1 \vee x_3 \vee x_4 \vee x_5}_{c_1}) \wedge (\underbrace{\neg x_2 \vee \neg x_3 \vee x_4}_{c_2}) \wedge (\underbrace{x_1 \vee x_2 \vee x_5}_{c_3}) \wedge (\underbrace{x_1 \vee \neg x_2}_{c_4})$$

   and the partial assignment α: $\{x_1 \mapsto \texttt{false}, \quad x_2 \mapsto \texttt{true}\}$.

   For each clause, decide whether the clause is unassigned, satisfied, unsatisfiable or unit (multiple answers are possible).

2. Consider the CNF φ

   $$(\underbrace{\neg x_1 \vee x_3 \vee x_4 \vee x_5}_{c_1}) \wedge (\underbrace{\neg x_2 \vee \neg x_3 \vee \neg x_5}_{c_2}) \wedge (\underbrace{x_1 \vee \neg x_2 \vee x_5}_{c_3}) \wedge (\underbrace{x_1 \vee x_2}_{c_4})$$

   and the partial assignment α: $x_1 \mapsto \texttt{false}, \quad x_3 \mapsto \texttt{true}$

   Apply unit propagation repeatedly until you either

   - find a satisfiable assignment, or
   - find a conflict.

   Give the variables/assignments that you obtain and which clauses you use.

3. Consider CNF

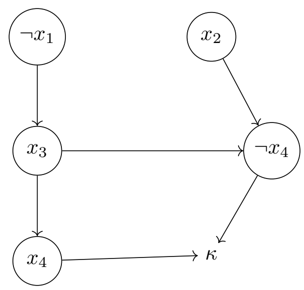
   $$(\underbrace{x_1 \vee x_3 \vee x_4 \vee \neg x_5}_{c_1}) \wedge (\underbrace{x_1 \vee \neg x_3 \vee x_4}_{c_2}) \wedge (\underbrace{x_1 \vee \neg x_2 \vee \neg x_4}_{c_3}) \wedge (\underbrace{x_1 \vee \neg x_2 \vee x_5}_{c_4})$$

and the following decisions

$$x_1 \mapsto \texttt{false}, \quad x_2 \mapsto \texttt{true}, \quad x_3 \mapsto \texttt{true}.$$

A CDCL solver detects a conflict. Give a conflict clause.

4. For the implication graph below:



   (a) Identify two conflict clauses.
   (b) Assume that x_1 was the last decision. Give an unique implication point (UIP).

2 Resolution rules (15)

1. Consider the CNF φ:

$$\underbrace{(x_1 \vee x_2)}_{c_1} \wedge \underbrace{(\neg x_1 \vee x_3)}_{c_2} \wedge \underbrace{(x_1 \vee \neg x_3)}_{c_3}$$

   (a) Apply **exhaustive resolution** on φ.
   (b) Argue whether φ is satisfiable. Use only(!) the outcome of the resolution.

2. Consider the predicate CNF:

$$\begin{array}{llr}
(& \neg\texttt{A}(x) \vee \texttt{A}(\texttt{f}(x)) &) \wedge \\
(& \texttt{B}(x) \vee \neg\texttt{A}(x) &) \wedge \\
(& \texttt{C}(\texttt{f}(x)) \vee \neg\texttt{B}(x) &) \wedge \\
(& \neg\texttt{C}(x) \vee \neg\texttt{A}(x) &) \wedge \\
(& \texttt{A}(\texttt{0}) &)
\end{array}$$

(Here, x is a variable, and all other symbols are predicates or function symbols.)

Apply **ordered resolution** to prove that this CNF is not satisfiable, using the ordering $\texttt{C}(s) > \texttt{B}(t) > \texttt{A}(u)$ for all s, t, u (for instance $\texttt{B}(s)$ and $\texttt{B}(t)$ are incomparable if $s \neq t$).

(If you cannot recall ordered resolution, just apply resolution for partial points.)

3 SMT (8)

1. For the following SMT formula, provide the Boolean skeleton for use in a lazy SMT solver.

$$\Big((a \geq 0 \wedge a + b \leq 0) \rightarrow c \geq 0\Big) \wedge (a \leq 5 \vee a + c \leq 0)$$

2. Put the following conjunction of constraints into a simplex tableaux as presented in the lecture. *Hint: Give some intermediate steps!*

$$x_1 + x_2 \leq 4 \wedge 3x_1 - x_3 \geq 0 \wedge x_2 - 3 \geq 2x_3$$

3. Consider the constraints

$$2x + y \geq 1 \wedge 3x + y \leq 2$$

over the integer variables. The simplex algorithm will return as a solution

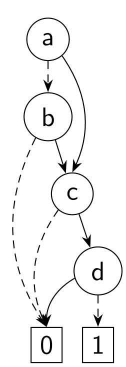
$$\alpha = \{x \mapsto \frac{1}{2}, \quad y \mapsto 0\}$$

This assignment is not a satisfying assignment to the original problem as $\alpha(x)$ is not integral. Branch-and-bound will now be recursively invoked on two formulas. Give both.

4 BDDs (10)

1. For $y \vee (x \wedge z)$ and variable order $x < y < z$, give the ROBDD.

2. For the following BDD representing propositional formula f.
Recall that dashed lines represent negative assignments and solid lines positive assignments.



   (a) Give the cofactor $f_{|a=0,b=1}$ represented as a BDD.

   (b) Give the cofactor $f_{|c=1,b=1,a=1}$ represented as a propositional formula.

   (c) Give *all* satisfying assignments. *Hint: these are less than 5.*

5 Formula normal forms (10)

1. Translate $x \wedge (\neg(y \vee \neg z))$ into CNF using Tseitin's Transformation. Make it clear how you got to this CNF (for instance by indicating the formulas $A \leftrightarrow \varphi$ that you are using).
2. Bring the following predicates to prenex normal form:
   (a) $\forall x[\mathtt{P}(x)] \vee \exists x[\neg(\mathtt{R}(x) \wedge \neg\mathtt{Q})]$
   (b) $\exists x[\mathtt{Q}(x)] \rightarrow \mathtt{P}$
3. Apply skolemization on the predicate

$$\exists x \forall y \forall z \exists u \forall q[\mathtt{P}(x,y) \wedge (\mathtt{Q}(u) \vee \neg\mathtt{R}(q,x,z))]$$

6 Rewriting terms (10)

1. Consider the term rewriting system with the following rules:

$$\begin{array}{rcl} \mathtt{add}(\mathtt{s}(x), y) & \rightarrow & \mathtt{s}(\mathtt{add}(y, x)) \\ \mathtt{add}(x, \mathtt{s}(y)) & \rightarrow & \mathtt{add}(\mathtt{s}(y), x) \\ \mathtt{add}(x, \mathtt{0}) & \rightarrow & x \end{array}$$

   Rewrite the term $\mathtt{add}(\mathtt{s}(x), \mathtt{add}(\mathtt{0}, y))$ to normal form. Show every step.

2. **Not relevant in 23/24**.
   Consider the following higher-order term rewriting system:

$$\begin{array}{rcl} \mathtt{filter}\ F\ \mathtt{nil} & \rightarrow & \mathtt{nil} \\ \mathtt{filter}\ F\ (\mathtt{cons}\ x\ y) & \rightarrow & \mathtt{consif}\ (F\ x)\ x\ (\mathtt{filter}\ F\ y) \\ \mathtt{consif}\ \mathtt{true}\ x\ y & \rightarrow & \mathtt{cons}\ x\ y \\ \mathtt{consif}\ \mathtt{false}\ x\ y & \rightarrow & y \\ \mathtt{isnul}\ \mathtt{0} & \rightarrow & \mathtt{true} \\ \mathtt{isnul}\ (\mathtt{s}\ x) & \rightarrow & \mathtt{false} \end{array}$$

   Rewrite the term $\mathtt{filter}\ (\lambda x.\mathtt{isnul}\ x)\ (\mathtt{cons}\ \mathtt{0}\ \mathtt{nil})$ to normal form. Show every step.

7 Confluence (12)

1. In the TRS of the previous question,

$$\begin{array}{rcl} \mathtt{add}(\mathtt{s}(x), y) & \rightarrow & \mathtt{s}(\mathtt{add}(y, x)) \\ \mathtt{add}(x, \mathtt{s}(y)) & \rightarrow & \mathtt{add}(\mathtt{s}(y), x) \\ \mathtt{add}(x, \mathtt{0}) & \rightarrow & x \end{array}$$

   calculate the critical pairs.

2. Is the system above locally confluent? Explain your answer. (If you do not know how to compute the critical pairs, explain how you would determine local confluence knowing the critical pairs.)

8 Termination (12)

1. Consider the TRS with rules:

$$\begin{array}{rcl} \mathtt{f}(\mathtt{a}(x), y) & \to & \mathtt{a}(\mathtt{f}(x, \mathtt{g}(y))) \\ \mathtt{g}(x) & \to & \mathtt{a}(x) \end{array}$$

The goal is to find a weight function to prove termination of this TRS using monotonic algebras. We will do this in a number of steps. It may be helpful to read all the steps before you start.

Your interpretations should have the following shape:

$$\begin{array}{rcl} [\mathtt{f}](x, y) & = & c * x + d * y + e \\ [\mathtt{a}](x) & = & i * x + j \\ [\mathtt{g}](x) & = & n * x + m \end{array}$$

where c, d, e, i, j, n, m are replaced by natural numbers.

(a) Determine the integer requirements (e.g., $a \geq 1, b \leq 2$) on c, d, e, i, j, n, m needed to satisfy the *monotonicity requirements* to apply this method.

(b) Determine the polynomial requirements (e.g., $a * b * x + a * y \geq 3 * y$) on c, d, e, i, j, n, m to guarantee $W(\ell) > W(r)$ for all rules.

(c) Use absolute positiveness to determine integer requirements over b, d, e, i, j, n, m. (That is, remove the x and y from the formulas you found in question 1b.)

(If you did not answer question 1b, you can instead find the requirements for $a*b*x+a*y+b*y+i*j+m > a*x+i*y+m*x*y+j$.)

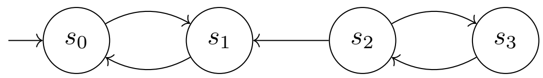
(d) **[BONUS EXERCISE]**[1] Using your previous answers, give a weight function on this TRS that orients both rules.

2. What are the dependency pairs of the TRS above?

[1] Since we do not expect you to be an SMT-solver, you can get full marks for the exercise without this question. But, if you have time left, you can earn some extra points by completing the termination proof.

9 Reachability (8)

1. Consider the transition system below. Give *all* inductive state sets that include the initial state s_0.



2. Consider a symbolic transition system

$$\left\langle V = \{x, y, z\}, I = \neg x \wedge y, T = \neg x \wedge (y \vee x') \wedge z' \wedge (z \oplus z') \right\rangle.$$

   Give a formula such that the satisfying assignments describe the states reachable via a path of lenght up to 2. *Hint: Do not draw a graph or even think about the explicit graph, but construct the correct answer from the definition of the symbolic transition system.*

This is the last page.