# Automated Reasoning

## Week 10. Termination

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Fall 2024

# Recall: Term Rewriting Systems

$$
\begin{array}{rcl}
\mathtt{add}(x, \mathtt{s}(y)) & \Rightarrow & \mathtt{s}(\mathtt{add}(x, y)) \\
\mathtt{add}(x, \mathtt{p}(y)) & \Rightarrow & \mathtt{p}(\mathtt{add}(x, y)) \\
\mathtt{add}(x, \mathtt{0}) & \Rightarrow & x \\
\mathtt{s}(\mathtt{p}(x)) & \Rightarrow & x \\
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\end{array}
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\end{array}
$$

$$
\begin{array}{rcl}
\mathtt{rev}(\mathtt{nil}) & \Rightarrow & \mathtt{nil} \\
\mathtt{rev}(a \mathbin{\mathtt{:}} x) & \Rightarrow & \mathtt{conc}(\mathtt{rev}(x), a \mathbin{\mathtt{:}} \mathtt{nil}) \\
\mathtt{conc}(\mathtt{nil}, x) & \Rightarrow & x \\
\mathtt{conc}(a \mathbin{\mathtt{:}} x, y) & \Rightarrow & a \mathbin{\mathtt{:}} \mathtt{conc}(x, y)
\end{array}
$$

# Lecture plan

Last week: LPO

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- Simple
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This lecture: **monotonic algebras** and **dependency pairs**.

# Basic intuition of **monotonic algebras**

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*Find a weight function $W$ from terms to natural numbers in such a way that $W(u) > W(v)$ for all terms $u, v$ satisfying $u \Rightarrow_{\mathcal{R}} v$.*

# Example

Rules:

$$
\begin{array}{rcl}
\mathtt{add}(\mathtt{0}, y) & \rightarrow & y \\
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Define:

- $W(\texttt{0}) = 1$
- $W(\texttt{s}(t)) = W(t) + 1$
- $W(\texttt{add}(t, u)) = 2W(t) + W(u)$

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## Then:

$$
\begin{aligned}
W(\mathtt{add}(\mathtt{s}(\mathtt{0}), \mathtt{0})) &= 2 * W(\mathtt{s}(\mathtt{0})) + W(\mathtt{0}) = 2 * 2 + 1 = 5 \\
W(\mathtt{s}(\mathtt{add}(\mathtt{0}, \mathtt{0}))) &= W(\mathtt{add}(\mathtt{0}, \mathtt{0})) + 1 = (2 * 1 + 1) + 1 = 4 \\
W(\mathtt{s}(\mathtt{0})) &= W(\mathtt{0}) + 1 = 1 + 1 = 2
\end{aligned}
$$

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Problem: infinitely many terms!

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Solution: use a reduction ordering!

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- $\succ$ is **monotonic** (so preserved under contexts)

Then it suffices to prove $\ell \succ r$ for all the rules.

# Monotonic algebras

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Side bonus: no restriction to $\mathbb{N}$.

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Here **monotonic** means:

*if for all* $a_i, b_i \in \mathcal{A}$ *for* $i = 1, \ldots, n$ *with* $a_i > b_i$ *for some* $i$ *and* $a_j \geq b_j$ *for all* $j \neq i$ *then*

$$[\mathtt{f}](a_1, \ldots, a_n) \; > \; [\mathtt{f}](b_1, \ldots, b_n)$$

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$$[\mathtt{f}](a_1, \ldots, a_n) \;>\; [\mathtt{f}](b_1, \ldots, b_n)$$

**Examples:**

| *monotonic* | *not monotonic* |
|---|---|
| $\lambda x.\ x$ | $\lambda x.\ 2$ |
| $\lambda x.\ x + 1$ | $\lambda x, y.\ x + 1$ |
| $\lambda x.\ 2 * x$ | $\lambda x, y.\ x * y$ |
| $\lambda x, y.\ x + y$ | $\lambda x, y.\ \max(x, y)$ |
| $\lambda x, y.\ 2 * x + y + 1$ | |

# Definition (continued)

Define:

- $W(x) = x$ for a variable
- $W(\mathtt{f}(s_1, \ldots, s_n)) = [\mathtt{f}](W(s_1), \ldots, W(s_n))$

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**Theorem**

The relation $\succ$ defined by:

$$s \succ t \text{ if and only if } \forall \vec{x}[W(s) \succ W(t)],$$
$$\text{where } \{\vec{x}\} \text{ is the set of variables occurring in } s, t$$

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**Proof idea.** Stability follows from the $\forall \vec{x}$, monotonicity from monotonicity of all $[\mathtt{f}]$ interpretation functions, and well-foundedness from well-foundedness of $>$ in $\mathcal{A}$. $\square$

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and:

$$\begin{array}{rcl} W(\mathtt{add}(\mathtt{s}(x), y) & = & 2 * (x + 1) + y = 2 * x + y + 2 \\ & > & 2 * x + y + 1 = W(\mathtt{s}(\mathtt{add}(x, y))) \end{array}$$

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Hence proving termination of $\mathcal{R}$.

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$[\mathtt{g}]$ is not monotonic.

So monotonicity really is essential.

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Challenge: specify “an interpretation” in (basic) SMT!

# Parametric interpretations

Idea: assign to function symbol f of arity $n$ a *parametric interpretation function* of a specific shape; for instance

$$\begin{array}{rcl}[0] & = & \underline{n} \\ [s] & = & \lambda x.\underline{s_0} + \underline{s_1} * x \\ [add] & = & \lambda x, y.\underline{a_0} + \underline{a_1} * x + \underline{a_2} * y\end{array}$$

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\end{aligned}
$$

Compute the resulting requirements for the rules:

$$
\begin{aligned}
W(\mathtt{add}(\mathtt{0}, y)) &= \underline{a_0} + \underline{a_1} * \underline{n} + \underline{a_2} * y \\
W(y) &= y \\
W(\mathtt{add}(\mathtt{s}(x), y)) &= \underline{a_0} + \underline{a_1} * (\underline{s_0} + \underline{s_1} * x) + \underline{a_2} * y \\
&= \underline{a_0} + \underline{a_1} * \underline{s_0} + \underline{a_1} * \underline{s_1} * x + \underline{a_2} * y \\
W(\mathtt{s}(\mathtt{add}(x, y))) &= \underline{s_0} + \underline{s_1} * (\underline{a_0} + \underline{a_1} * x + \underline{a_2} * y) \\
&= \underline{s_0} + \underline{s_1} * \underline{a_0} + \underline{s_1} * \underline{a_1} * x + \underline{s_1} * \underline{a_2} * y
\end{aligned}
$$

# Inequalities with (universally quantified) variables

We now have to solve a problem of the shape:

*find parameters such that:*

- *all* $[f]$ *are monotonic functions, and*
- *for all rules* $\ell \to r$*, all* $\vec{x}$*:* $W(\ell) \succ W(r)$*.*

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Requiring monotonicity in our example is not hard:

$$\begin{array}{rcl} [\mathtt{0}] & = & \underline{n} \\ [\mathtt{s}] & = & \lambda x.\underline{s_0} + \underline{s_1} * x \\ [\mathtt{add}] & = & \lambda x, y.\underline{a_0} + \underline{a_1} * x + \underline{a_2} * y \end{array}$$

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We require that: $\underline{s_1} \geq 1$, $\underline{a_1} \geq 1$, $\underline{a_2} \geq 1$.

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We require that: $\underline{s_1} \geq 1$, $\underline{a_1} \geq 1$, $\underline{a_2} \geq 1$.

For the second requirement, we use **absolute positiveness**.

# Absolute positiveness

Goal: a *sufficient condition* to compare two polynomials

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$$a_0 + a_1 * x_1 + \cdots + a_m * x_m > b_0 + b_1 * x_1 + \cdots + b_m * x_m$$

certainly holds if:

- $a_0 > b_0$
- each $a_i \geq b_i$

# Example (continued)

We must show that:

$$
\begin{aligned}
W(\mathtt{add}(\mathtt{0}, y)) &= \underline{a_0} + \underline{a_1} * \underline{n} + \underline{a_2} * y \\
&> y \\
&= W(y) \\
W(\mathtt{add}(\mathtt{s}(x), y)) &= \underline{a_0} + \underline{a_1} * \underline{s_0} + \underline{a_1} * \underline{s_1} * x + \underline{a_2} * y \\
&> \underline{s_0} + \underline{s_1} * \underline{a_0} + \underline{s_1} * \underline{a_1} * x + \underline{s_1} * \underline{a_2} * y \\
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& > & \underline{s_0} + \underline{s_1} * \underline{a_0} + \underline{s_1} * \underline{a_1} * x + \underline{s_1} * \underline{a_2} * y \\
& = & (\mathtt{s}(\mathtt{add}(x, y)))
\end{array}$$

That is:

$$(\underline{a_0} + \underline{a_1} * \underline{n}) + \underline{a_2} * y > y$$

and

$$(\underline{a_0}+\underline{a_1}*\underline{s_0})+(\underline{a_1}*\underline{s_1})*x+\underline{a_2}*y > (\underline{s_0}+\underline{s_1}*\underline{a_0})+(\underline{s_1}*\underline{a_1})*x+(\underline{s_1}*\underline{a_2})*y$$

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Using absolute positiveness, it suffices if:

$$\begin{array}{rcl} \underline{a_0} + \underline{a_1} * \underline{n} & > & 0 \\ \\ \underline{a_2} & \geq & 1 \end{array} \qquad \begin{array}{rcl} \underline{a_0} + \underline{a_1} * \underline{s_0} & > & \underline{s_0} + \underline{s_1} * \underline{a_0} \\ \underline{a_1} * \underline{s_1} & \geq & \underline{s_1} * \underline{a_1} \\ \underline{a_2} & \geq & \underline{s_1} * \underline{a_2} \end{array}$$

# Completing the example

$$\begin{array}{rcl} \underline{a_0} + \underline{a_1} * \underline{n} & > & 0 \\ & & \\ \underline{a_2} & \geq & 1 \end{array} \qquad \begin{array}{rcl} \underline{a_0} + \underline{a_1} * \underline{s_0} & > & \underline{s_0} + \underline{s_1} * \underline{a_0} \\ \underline{a_1} * \underline{s_1} & \geq & \underline{s_1} * \underline{a_1} \\ \underline{a_2} & \geq & \underline{s_1} * \underline{a_2} \end{array} \qquad \begin{array}{rcl} \underline{s_1} & \geq & 1 \\ \underline{a_1} & \geq & 1 \\ \underline{a_2} & \geq & 1 \end{array}$$

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$$a_0 + a_1 * n > 0 \qquad a_2 \geq 1$$

$$\begin{array}{rcl} a_0 + a_1 * s_0 & > & s_0 + s_1 * a_0 \\ a_1 * s_1 & \geq & s_1 * a_1 \\ a_2 & \geq & s_1 * a_2 \end{array} \qquad \begin{array}{rcl} s_1 & \geq & 1 \\ a_1 & \geq & 1 \\ a_2 & \geq & 1 \end{array}$$

An SMT solver will for instance yield

$$n = 1,\ s_0 = 1,\ s_1 = 1,\ a_0 = 0,\ a_1 = 2,\ a_2 = 1$$

giving the same interpretations we had before:

$$\begin{array}{rcl} [\mathtt{0}] & = & n \\ [\mathtt{s}] & = & \lambda x. s_0 + s_1 * x \\ [\mathtt{add}] & = & \lambda x, y. a_0 + a_1 * x + a_2 * y \end{array} \Longrightarrow \begin{array}{rcl} [\mathtt{0}] & = & 1 \\ [\mathtt{s}] & = & \lambda x. 1 + x \\ [\mathtt{add}] & = & \lambda x, y. 2 * x + y \end{array}$$

# Completing the example

$$\underline{a_0} + \underline{a_1} * \underline{n} > 0 \qquad \underline{a_2} \geq 1$$

$$\begin{aligned} \underline{a_0} + \underline{a_1} * \underline{s_0} &> \underline{s_0} + \underline{s_1} * \underline{a_0} \\ \underline{a_1} * \underline{s_1} &\geq \underline{s_1} * \underline{a_1} \\ \underline{a_2} &\geq \underline{s_1} * \underline{a_2} \end{aligned}$$

$$\begin{aligned} \underline{s_1} &\geq 1 \\ \underline{a_1} &\geq 1 \\ \underline{a_2} &\geq 1 \end{aligned}$$

An SMT solver will for instance yield

$$\underline{n} = 1,\ \underline{s_0} = 1,\ \underline{s_1} = 1,\ \underline{a_0} = 0,\ \underline{a_1} = 2,\ \underline{a_2} = 1$$

giving the same interpretations we had before:

$$\begin{aligned} [\mathtt{0}] &= \underline{n} \\ [\mathtt{s}] &= \lambda x.\underline{s_0} + \underline{s_1} * x \\ [\mathtt{add}] &= \lambda x, y.\underline{a_0} + \underline{a_1} * x + \underline{a_2} * y \end{aligned} \quad \Longrightarrow \quad \begin{aligned} [\mathtt{0}] &= 1 \\ [\mathtt{s}] &= \lambda x.1 + x \\ [\mathtt{add}] &= \lambda x, y.2 * x + y \end{aligned}$$

Many other interpretations are also possible.

# Completing the example

$$\begin{array}{rcl} \underline{a_0} + \underline{a_1} * \underline{n} & > & 0 \\ \\ \underline{a_2} & \geq & 1 \end{array} \qquad \begin{array}{rcl} \underline{a_0} + \underline{a_1} * \underline{s_0} & > & \underline{s_0} + \underline{s_1} * \underline{a_0} \\ \underline{a_1} * \underline{s_1} & \geq & \underline{s_1} * \underline{a_1} \\ \underline{a_2} & \geq & \underline{s_1} * \underline{a_2} \end{array} \qquad \begin{array}{rcl} \underline{s_1} & \geq & 1 \\ \underline{a_1} & \geq & 1 \\ \underline{a_2} & \geq & 1 \end{array}$$

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Many other interpretations are also possible.

In practice: $\{0, 1, 2, 3\}$ usually suffice.

# Limitations

Absolute positiveness also works with combinations of variables, e.g.,

$$\underline{a} + \underline{b}y + \underline{c}xy + \underline{d}x^2y > \underline{e}x + \underline{d}y + \underline{f}xy$$

if $\underline{a} > 0 \wedge 0 \geq \underline{e} \wedge \underline{b} \geq \underline{d} \wedge \underline{c} \geq \underline{f} \wedge \underline{d} \geq 0$.

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Downside: more sophisticated shape = more complex SMT problem

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[\mathtt{p}] & = & \lambda \langle x_1, x_2 \rangle . \langle x_1, \max(x_2 - 1, 0) \rangle \\
[\mathtt{f}] & = & \lambda \langle x_1, x_2 \rangle . \langle x_1 + x_2, 0 \rangle
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This is also valuable in **complexity analysis** of programs.

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- sets of terms terminating under some different well-founded ordering $\succ$

# Subterm property

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The good:

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The bad:

- Not all TRSs can be ordered this way!

# Motivating example

$$\begin{array}{rcl}
\mathtt{minus}(x, \mathtt{0}) & \Rightarrow & x \\
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If $\mathtt{minus}(x, y) \succ y$ and $\mathtt{s}(a) \succ a$, then:

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Hence, if the last rule is oriented with $\succ$, then by stability:

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Hence, this system **cannot** be ordered using any recursive path ordering, or an interpretation to $\mathbb{N}$.

# Secondary motivation

Program analysis: often hundreds or thousands of rules.

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Wish: split a termination problem into multiple smaller problems.

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The core idea of dependency pairs is to look at **function calls**.

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The core idea of dependency pairs is to look at **function calls.**

To start, split functions in `constructors` and `defined symbols`.

# Identifying function calls

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We isolate the calls from one defined symbol to another:

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# Dependency pair chains

We can prove the following result:

**Theorem**

For a given set of rules $R$, let $\mathtt{DP}$ be the corresponding set of dependency pairs.

$\Rightarrow_{\mathcal{R}}$ is terminating if and only if there is no infinite **$(\mathtt{DP}, R)$-chain**: a reduction $s_1 \Rightarrow_{\mathtt{DP}} \Rightarrow_{\mathcal{R}}^* s_2 \Rightarrow_{\mathtt{DP}} \Rightarrow_{\mathcal{R}}^* s_3 \dots$

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So $\Rightarrow_{\mathcal{R}}$ is terminating iff there is no infinite sequence where:

- the steps using $\Rightarrow_{\mathtt{DP}}$ occur at the root of the term;
- the steps using $\Rightarrow_{\mathcal{R}}$ do not occur at the root of the term;
- there are infinitely many steps using $\Rightarrow_{\mathtt{DP}}$.

# Example: an infinite DP chain

$$R = \left\{ \begin{array}{rcl} \mathtt{f}(\mathtt{0}, x) & \to & \mathtt{g}(\mathtt{f}(\mathtt{g}(x), x)) \\ \mathtt{g}(x) & \to & x \end{array} \right\}$$

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**Infinite chain:**

$$\underline{\mathtt{f}^\sharp(\mathtt{0}, \mathtt{0})} \Rightarrow_{\mathtt{DP}} \mathtt{f}^\sharp(\underline{\mathtt{g}(\mathtt{0})}, \mathtt{0}) \Rightarrow_{\mathcal{R}} \underline{\mathtt{f}^\sharp(\mathtt{0}, \mathtt{0})} \Rightarrow_{\mathtt{DP}} \mathtt{f}^\sharp(\underline{\mathtt{g}(\mathtt{0})}, \mathtt{0}) \Rightarrow_{\mathcal{R}} \ldots$$

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- $\ell \succeq r$ for all rules $\ell \to r \in R$,
- $\succ$ is **well-founded** and **stable** (but not necessarily monotonic),
- $\succeq$ is **stable** and **monotonic** (but not necessarily well-founded),
- and $s \succ t \succeq u$ implies $s \succ u$,

# Building block: reduction pairs

How do you prove that there is no infinite chain?

One possibility: using a variant of a reduction ordering: a **reduction pair**.

**Theorem**

If:

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- $\succeq$ is **stable** and **monotonic**,
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then there is no infinite $(\mathtt{DP}, R)$-chain!

**Proof idea.** By these requirements:

- If $s \Rightarrow_{\mathtt{DP}} t$ (by a step at the root), then $s \succ t$.
- If $s \Rightarrow_{\mathcal{R}} t$ then $s \succeq t$.
- Hence, if $s \Rightarrow_{\mathtt{DP}} \cdot \Rightarrow_{\mathcal{R}}^{*} t$, then $s \succ \cdot \succeq^{*} t$, and therefore $s \succ t$.

□

# Quot/minus: ordering requirements

$$
\begin{array}{rcl}
\mathtt{minus}(x, \mathtt{0}) & \succeq & x \\
\mathtt{minus}(\mathtt{s}(x), \mathtt{s}(y)) & \succeq & \mathtt{minus}(x, y) \\
\mathtt{quot}(\mathtt{0}, \mathtt{s}(y)) & \succeq & \mathtt{0} \\
\mathtt{quot}(\mathtt{s}(x), \mathtt{s}(y)) & \succeq & \mathtt{s}(\mathtt{quot}(\mathtt{minus}(x, y), \mathtt{s}(y))) \\
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Like monotonic algebras, but $[\mathtt{f}]$ only needs to be **weakly monotonic.**

$$\text{if } s \geq t \text{, then } [\mathtt{f}](\dots, s, \dots) \geq [\mathtt{f}](\dots, t, \dots).$$

Many functions that are not monotonic, are still weakly monotonic; for example:

- functions that ignore arguments: $\lambda x, y.\ x$
- $\mathsf{min}\,/\,\mathsf{max}$ functions: $\lambda x, y.\ \mathsf{min}(x, y)$ or $\lambda x.\ \mathsf{max}(x - 1, 0)$

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Back to our example!

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This is satisfied by choosing:

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\begin{array}{rclrclrcl}
[\mathtt{0}] & = & 0 & [\mathtt{minus}] & = & \lambda x, y.\ x & [\mathtt{minus}^\sharp] & = & \lambda x, y.\ x \\
[\mathtt{s}] & = & \lambda x.\ x + 1 & [\mathtt{quot}] & = & \lambda x, y.\ x & [\mathtt{quot}^\sharp] & = & \lambda x, y.\ x
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Back to our example!

$$\begin{array}{rcl} x & \geq & x \\ x+1 & \geq & x \\ 0 & \geq & 0 \\ x+1 & \geq & x+1 \\ x+1 & > & x \\ x+1 & > & x \\ x+1 & > & x \end{array}$$

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Hence, we clearly gained something!

# Step-by-step proofs

Challenge: dealing with **large** systems.

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Idea: split large problems into smaller sub-problems.

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\textbf{DPs:} & \mathrm{A} & \mathtt{minus}^\sharp(\mathtt{s}(x), \mathtt{s}(y)) & \Rightarrow & \mathtt{minus}^\sharp(x, y) \\
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Question: what does an infinite chain look like?

We conclude: split $\mathrm{DP}$ into $\{A\}$ and $\{C\}$!

# Mandatory material ends here

The material before this slide is expected knowledge, and you should be able to use monotonic algebras, and dependency pairs with weakly monotonic algebras, on the exam. You should also know the overall idea of the DP framework.

The following slides are optional material. However, you are *allowed* to use it on the exam; for example to answer a question “prove termination of this system”, which can often be done much faster using the dependency pair framework with the graph and subterm criterion (see subsequent slides).

# Using a graph

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Idea: Graph with DPs as nodes, edges between nodes if one can follow another in a chain.

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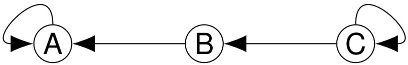

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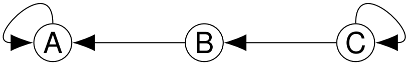


Observation: each **strongly connected component** may be considered separately.

# Dependency graph processor – another example

$$R = \{\mathtt{f}(\mathtt{f}(x)) \to \mathtt{f}(\mathtt{g}(\mathtt{f}(x)))\}$$

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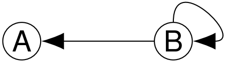

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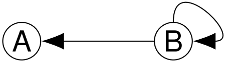


Hence, we can remove dependency pair A.

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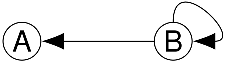


Hence, we can remove dependency pair A.

Note: not always easy to see if one DP can follow another, but we can use approximations.

# Dependency graph processor

Formally:

# Dependency graph processor

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**Definition**

Let $(\mathcal{D}, R)$ be a DP problem, and $G$ a graph whose nodes are the elements of $\mathcal{D}$, and which has an edge from $\rho$ to $\mu$ if it is possible for $\rho$ to be followed by $\mu$ in a $(\mathcal{D}, R)$-chain (there may be more edges than this).

Suppose $A_1, \ldots, A_n$ are the **strongly connected components** of $G$.

Then the dependency graph processor maps $(\mathcal{D}, R)$ to $\{(A_1, R), \ldots, (A_n, R)\}$.

# The subterm criterion

A. $\mathtt{exp}^\sharp(\mathtt{s}(x), y) \Rightarrow \mathtt{double}^\sharp(x, y, \mathtt{0})$

B. $\mathtt{double}^\sharp(x, \mathtt{0}, z) \Rightarrow \mathtt{exp}^\sharp(x, z)$

C. $\mathtt{double}^\sharp(x, \mathtt{0}, z) \Rightarrow \mathtt{double}^\sharp(x, y, \mathtt{s}(\mathtt{s}(z)))$

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$$\begin{array}{llll} \text{A.} & \mathtt{exp}^\sharp(\mathtt{s}(x), y) & \Rightarrow & \mathtt{double}^\sharp(x, y, \mathtt{0}) \\ \text{B.} & \mathtt{double}^\sharp(x, \mathtt{0}, z) & \Rightarrow & \mathtt{exp}^\sharp(x, z) \\ \text{C.} & \mathtt{double}^\sharp(x, \mathtt{0}, z) & \Rightarrow & \mathtt{double}^\sharp(x, y, \mathtt{s}(\mathtt{s}(z))) \end{array}$$

Idea: consider the **first argument** of each side of the dependency pairs.

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\end{array}
$$

Idea: consider the **first argument** of each side of the dependency pairs.

Then we can remove the dependency pairs where the chosen argument becomes smaller (in this case A).

# The subterm criterion processor

**Definition**

For all marked symbols $\mathtt{f}^\sharp$, let $\nu(\mathtt{f}^\sharp) \in \{1, \dots, arity(\mathtt{f})\}$.

Let $(\mathcal{D}, R)$ be a DP problem, and write $\mathcal{D} = \mathcal{D}_1 \cup \mathcal{D}_2$.

Suppose:

- $\ell_{\nu(\mathtt{f}^\sharp)} = r_{\nu(\mathtt{g}^\sharp)}$ for all $\mathtt{f}^\sharp(\ell_1, \dots, \ell_n) \Rightarrow \mathtt{g}^\sharp(r_1, \dots, r_m) \in \mathcal{D}_1$
- $r_{\nu(\mathtt{f}^\sharp)}$ is a subterm of $\ell_{\nu(\mathtt{g}^\sharp)}$ for all $\mathtt{f}^\sharp(\ell_1, \dots, \ell_n) \Rightarrow \mathtt{g}^\sharp(r_1, \dots, r_m) \in \mathcal{D}_2$

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Then the subterm criterion processor maps $(\mathcal{D}, R)$ to $\{(\mathcal{D}_1, R)\}$.

Implementation: a simple SMT implementation using integer variables $\nu(\mathtt{f}^\sharp)$, and boolean variables $\mathtt{strict}_\rho$.

# Graph + subterm criterion

Class exercise:

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\begin{array}{rcl}
\mathtt{exp}(\mathtt{0}, y) & \Rightarrow & y \\
\mathtt{exp}(\mathtt{s}(x), y) & \Rightarrow & \mathtt{double}(\mathtt{0}, x, y) \\
\mathtt{double}(r, x, \mathtt{0}) & \Rightarrow & \mathtt{exp}(x, r) \\
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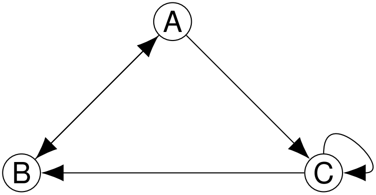

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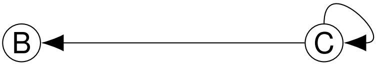

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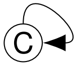

# Reduction pair processor

We can also reformulate reduction pairs as a processor:

**Definition**

Let $\succ$ be a well-founded, stable ordering and $\succeq$ a stable monotonic quasi-ordering on terms, such that $\succ \succeq \subseteq \succ$.

Let $(\mathcal{D}, R)$ be a DP problem, and write $\mathcal{D} = \mathcal{D}_1 \cup \mathcal{D}_2$.

Suppose:

- $\ell \succeq r$ for all $\ell \to r \in R$
- $\ell \succeq r$ for all $\ell \Rightarrow r \in \mathcal{D}_1$, and
- $\ell \succ r$ for all $\ell \Rightarrow r \in \mathcal{D}_2$.

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Then the reduction pair processor maps $(\mathcal{D}, R)$ to $\{(\mathcal{D}_1, R)\}$.

That is, we can use a reduction pair, and remove all dependency pairs that were ordered with $\succ$.

# Example: using a reduction pair processor

$$
\begin{array}{rcl}
\mathtt{append}(\mathtt{nil}, z) & \to & z \\
\mathtt{append}(\mathtt{cons}(x, y), z) & \to & \mathtt{cons}(x, \mathtt{append}(y, z)) \\
\mathtt{rev}(\mathtt{nil}) & \to & \mathtt{nil} \\
\mathtt{rev}(\mathtt{cons}(x, y)) & \to & \mathtt{append}(\mathtt{rev}(y), \mathtt{cons}(x, \mathtt{nil})) \\
\mathtt{append}^\sharp(\mathtt{cons}(x, y), z) & \Rightarrow & \mathtt{append}^\sharp(y, z) \\
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We choose:

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\begin{array}{rcl}
[\mathtt{nil}] & = & 0 \\
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\begin{array}{rcll}
[\mathtt{append}] & = & \lambda x, y. & x + y \\
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[\mathtt{append}^\sharp] & = & \lambda x, y. & x + y \\
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$$\begin{array}{rcl} z & \geq & z \\ (y+1)+z & \geq & (y+z)+1 \\ 0 & \geq & 0 \\ y+1 & \geq & y+(0+1) \\ (y+1)+z & > & y+z \\ y+1 & > & y \\ y+1 & \geq & y+(0+1) \end{array}$$

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Hence, we can remove all but the last dependency pair, and continue with:

$$(\{\mathtt{rev}^\sharp(\mathtt{cons}(x,y)) \Rightarrow \mathtt{append}^\sharp(\mathtt{rev}(y),\mathtt{cons}(x,\mathtt{nil}))\},R)$$

# Argument filterings

Idea: remove arguments before using a reduction ordering

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Searching a filter can be included in the SAT encoding of LPO.

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We do get two extra requirements, $\mathtt{c}(x,y) \succeq x$ and $\mathtt{c}(x,y) \succeq y$.

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Finding usable rules is a very simple reachability algorithm.

More complex (but powerful!): combining usable rules with an argument filtering and a reduction pair.

# Quiz

1. Prove termination of the following TRS using a monotonic algebra to $\mathbb{N}$:

$$\begin{array}{rcl} \mathtt{append}(\mathtt{nil}, z) & \to & z \\ \mathtt{append}(\mathtt{cons}(x, y), z) & \to & \mathtt{cons}(x, \mathtt{append}(y, z)) \end{array}$$

   - give (linear) parametric interpretations for all symbols
   - compute the requirements (monotonicity and rule orientation)
   - use absolute positiveness to find SMT requirements
   - solve them by hand and give the resulting inteprrretation functions, and check your result!

2. Determine the dependency pairs of:

$$\begin{array}{rcl} \mathtt{f}(\mathtt{h}(x), y) & \to & \mathtt{g}(x, \mathtt{f}(x, \mathtt{h}(y))) \\ \mathtt{g}(x, \mathtt{h}(y)) & \to & \mathtt{g}(\mathtt{h}(x), y) \end{array}$$

3. Split these dependency pairs up into one or more groups of DPs that can be analysed separately.