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Quiz

Given that there is at least one student, we know: for all lectures there is a student such that: if (s)he understands the lecture, then all students do.

1. Present this statement in predicate logic, using the relations:

   - **S** (arity 1): $\mathbf{S}(x)$ indicates that x is a student
   - **L** (arity 1): $\mathbf{L}(x)$ indicates that x is a lecture
   - **U** (arity 2): $\mathbf{U}(x, y)$ indicates that x understands y

   Answer:

   $$\exists x[\mathbf{S}(x)] \to \forall x[\mathbf{L}(x) \to \exists y[\mathbf{S}(y) \wedge (\mathbf{U}(y,x) \to \forall z[\mathbf{S}(z) \to \mathbf{U}(z,x)])]]$$

2. Find the negation of this formula, and bring it in Prenex normal form.

   Answer:

   $$\neg(\exists x[\mathbf{S}(x)] \to \forall x[\mathbf{L}(x) \to \exists y[\mathbf{S}(y) \wedge (\mathbf{U}(y,x) \to \forall z[\mathbf{S}(z) \to \mathbf{U}(z,x)])]])$$

   That is, removing all implications:

   $$\neg(\neg\exists x[\mathbf{S}(x)] \vee \forall x[\neg\mathbf{L}(x) \vee \exists y[\mathbf{S}(y) \wedge (\neg\mathbf{U}(y,x) \vee \forall z[\neg\mathbf{S}(z) \vee \mathbf{U}(z,x)])]])$$

   We pull all the negations on non-basic formulas inwards, which gives:

   $$\exists x[\mathbf{S}(x)] \wedge \exists x[\mathbf{L}(x) \wedge \forall y[\neg\mathbf{S}(y) \vee (\mathbf{U}(y,x) \wedge \exists z[\mathbf{S}(z) \wedge \neg\mathbf{U}(z,x)])]]$$

   Rename the first x to a fresh name, and then pull all the quantifiers to the left.

   $$\exists u[\exists x[\forall y[\exists z[\mathbf{S}(u) \wedge \mathbf{L}(x) \wedge (\neg\mathbf{S}(y) \vee (\mathbf{U}(y,x) \wedge \mathbf{S}(z) \wedge \neg\mathbf{U}(z,x)))]]]]$$

   By distribution we obtain:

   $$\exists u[\exists x[\forall y[\exists z[\mathbf{S}(u) \wedge \mathbf{L}(x) \wedge (\neg\mathbf{S}(y) \vee \mathbf{U}(y,x)) \wedge (\neg\mathbf{S}(y) \vee \mathbf{S}(z)) \wedge (\neg\mathbf{S}(y) \vee \neg\mathbf{U}(z,x))]]]]$$

   This predicate is in Prenex normal form.

3. Skolemize this negated formula.

   Answer: We replace u by $\mathbf{s}$, x by $\mathbf{l}$, and z by $\mathbf{f}(y)$. Omitting the universal quantification $\forall y$, this yields:

   $$\mathbf{S}(\mathbf{s}) \wedge \mathbf{L}(\mathbf{l}) \wedge (\neg\mathbf{S}(y) \vee \mathbf{U}(y,\mathbf{l})) \wedge (\neg\mathbf{S}(y) \vee \mathbf{S}(\mathbf{f}(y))) \wedge (\neg\mathbf{S}(y) \vee \neg\mathbf{U}(\mathbf{f}(y),\mathbf{l}))$$

   Or, placed in tabular notation:

1	$\mathbf{S}(\mathbf{s})$
2	$\mathbf{L}(\mathbf{l})$
3	$\neg\mathbf{S}(y) \vee \mathbf{U}(y,\mathbf{l})$
4	$\neg\mathbf{S}(y) \vee \mathbf{S}(\mathbf{f}(y))$
5	$\neg\mathbf{S}(y) \vee \neg\mathbf{U}(\mathbf{f}(y),\mathbf{l})$

4. Use resolution to derive $\perp$ from the resulting set of clauses.

Answer: There are multiple derivations possible; we use the following:

6	$\neg\mathbf{S}(\mathbf{f}(y)) \vee \neg\mathbf{S}(y)$	$(3,5)$	
7	$\neg\mathbf{S}(y)$	$(4,6)$	(because $\mathbf{S}(y) \vee \mathbf{S}(y) = \mathbf{S}(y)$)
8	$\perp$	$(1,7)$	