Excercises category theory week 2 The

1 Let relation R, s.t. $R = \{(g(\sigma), zip(\sigma,\sigma)) \mid \sigma \in A^\omega\}$.

- $g(\sigma)(0) = \sigma(0) = zip(\sigma,\sigma)(0)$ ✓
- $g(\sigma)' = h(\sigma)\ R'\ zip(\sigma,\sigma)$ ✓

add $R' = R \cup \{(h(\sigma), zip(\sigma,\sigma)) \mid \sigma \in A^\omega\}$

- $h(\sigma)(0) = \sigma(0) = zip(\sigma,\sigma)(0)$ ✓
- $h(\sigma)' = g(\sigma)\ R'\ zip(\sigma,\sigma)$ ✓

We have shown that R' is a bisimulation. It follows from the principle of co-induction that $g(\sigma) = zip(\sigma,\sigma)$, $\forall \sigma \in A^\omega$.

N.B. $\tau = T$ haha

2a $\ell_{f,g}(\sigma)(0) = f(\sigma(0))$, where $\ell_{g,f}(\sigma)(0) = g(\sigma(0))$

$\ell_{f,g}(\sigma)' = \ell_{g,f}(\sigma')$ $\qquad \ell_{g,f}(\sigma)' = \ell_{f,g}(\sigma')$

Works because $\ell_{f,g}(\sigma)(0) = f(\sigma(0)) \cdot \ell_{g,f}(\sigma') = f(\sigma(0)) \cdot g(\sigma(1)) \cdot \ell_{f,g}(\sigma'') = f(\sigma(0)) \cdot g(\sigma(1)) \cdot f(\sigma(2)) \ldots$

b $alt(\sigma,\tau)(0) = \sigma(0)$

$alt(\sigma,\tau)' = alt(\tau,\sigma')$ Works because $alt(\sigma,\tau) = \sigma(0) \cdot alt(\tau,\sigma') = \sigma(0) \cdot \tau(1) \cdot alt(\sigma'\tau') = \sigma(0) \cdot \tau(1) \cdot \sigma(1) \ldots$

c Let $R = \{(\ell_{f,g}(alt(\sigma,\tau)), alt(\ell_{f,f}(\sigma), \ell_{g,g}(\tau))) \mid \sigma,\tau \in A^\omega \wedge f,g : A^\omega \to A^\omega\}$

Prove that R is bisimilar:

- $\ell_{f,g}(alt(\sigma,\tau))(0)$
$$= f(alt(\sigma,\tau)(0)) = f(\sigma(0)) = \ell_{f,f}(\sigma)(0) = alt(\ell_{f,f}(\sigma), \ell_{g,g}(\sigma))$$
- $\ell_{f,g}(alt(\sigma,\tau))'$
$$= \ell_{g,f}(alt(\sigma,\tau)') = \ell_{g,f}(alt(\tau,\sigma'))\ R\ alt(\ell_{g,g}(\tau), \ell_{f,f}(\sigma')) = alt(\ell_{g,g}(\tau), \ell_{f,f}(\sigma)') = alt(\ell_{f,f}(\sigma), \ell_{g,g}(\tau))$$

We have shown that R is a bisimulation. Then it follows from the principle of coinduction that $\forall \sigma \in A^\omega$, $\ell_{f,g}(alt(\sigma,\tau)) = alt(\ell_{f,f}(\sigma), \ell_{g,g}(\tau))$

~~$zip(\sigma,\rho)' = zip(\rho,\sigma')' = \sigma(0).\,zip(\sigma',\rho')'$~~

~~$= \sigma(0)\, zip(\sigma',\rho')$~~

~~$alt(t_{f,f}(\sigma), t_{g,g}(\tau))' = alt(t_{g,g}(\tau), t_{f,f}(\sigma'))$~~ ~~$\neq$~~

3 $X = A^\omega \times A^\omega$

Recall, final stream system:

$i(\sigma) = \sigma(0)$

$d(\sigma) = \sigma'$

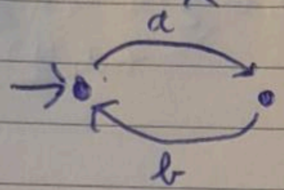
$\underbrace{A^\omega \times A^\omega}_{X} \to A \times \underbrace{A^\omega \times A^\omega}_{X}$

$X \to A \times X$

$h : \underbrace{A^\omega \times A^\omega}_{X} \to A^\omega$

$X \to A^\omega$

Let stream system $\langle o, t\rangle$ be defined as:



$o(\sigma,\tau) = \sigma(0)$

$t(\sigma,\tau) = (\tau,\sigma)$

$$\begin{tikzcd}
A^\omega \times A^\omega \arrow[r, "h"] \arrow[d, "\langle o, t\rangle"'] & A^\omega \arrow[d, "\langle i, d\rangle"] \\
A \times A^\omega \times A^\omega \arrow[r, "id \times h"'] & A \times A^\omega
\end{tikzcd}$$

Now show that alt is a homomorphism

1. $o(\sigma,\tau) = \sigma(0) = alt(\sigma,\tau)(0) = i(alt(\sigma,\tau))$ ✓

2. $alt(t(\sigma,\tau)) = alt(\tau,\sigma) = d(alt(\sigma,\tau)) = alt'(\sigma,\tau)$

Then it follows from the definition of stream systems and 1 and 2, that alt is a homomorphism to the final stream system. It follows from Theorem 3.2 (lecture notes) that alt is the unique homomorphism from $\langle o,t\rangle$ to $\langle i,d\rangle$.

4 a (I)

a. $p(h(x)) \overset{*}{=} o(x) \overset{+}{=} o(y) \overset{*}{=} p(h(y))$

b. $g(h(x)) \overset{*}{=} h(f(x)) \overset{2}{R} h(f(y)) \overset{*}{=} g(h(y))$

$*$: by definition of homomorphism (in terms of equations)
$+$: by definition of bisimulation, and the fact that R is a bisimulation, and $(x, y) \in R$.
2: by definition of ~~$\{(h(x), h(y)) \mid (x, y) \in R\}$~~ R.

Now we have shown that conditions a and b are satisfied according to the definition of bisimulation. Then it follows that $\{(h(x), h(y)) \mid (x, y) \in R\}$ is a bisimulation.

(II)

a. $o(x) \overset{*}{=} p(h(x)) \overset{+}{=} p(h(y)) \overset{*}{=} o(y)$

b. $f(h(x)) \overset{*}{=} h(g(x)) \overset{2}{S} h(g(y)) \overset{*}{=} f(h(y))$

$*$: by definition of homomorphism (1),
$+$: by def. of bisim and the fact that S is a bisim, and $(h(x), h(y)) \in S$.
2: by def. of S.

Then a and b are satisfied, hence again $\{(x, y) \mid (h(x), h(y)) \in S\}$ is a bisimulation.

5 To show that $h = l$, show that

$R = \{(h(x), \cancel{\ }(x)) \mid x \in X\}$ is a bisimulation.

a. $i(h(x)) \overset{*}{=} o(x) \overset{*}{=} i(l(x))$

b. $d(h(x)) \overset{*}{=} h(x)' \overset{*}{=} d(l(x))$

(Oh oops ehm. let $h = k$ ...)

*: by definition of homomorphism in terms of equations.

We have now shown that R is a bisimulation (according to the definition in exercise 4 and a, b). Then it follows from the coinduction principle that $h(x) = l(x)$ $\forall x \in X$, hence $h = l$. :)