# Introduction to Category Theory III
# Natural Transformations

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**Abstract.** Natural transformations are introduced as "maps of functors"

## 1 Natural transformations

One of the things one learns in doing category theory, is always to ask oneself: what are appropriate morphisms between structures of a given kind. This also applies within category theory. We have seen functors as 'morphisms of categories', which preserve the structure. In a next step we describe natural transfomations as 'morphisms of functors'. Of course, one can go on, and ask what are 'morphisms of natural transformations', but we stop at this point.

**Definition 1.** *Consider two categories $\mathbb{C}, \mathbb{D}$, and two functors $F, G\colon \mathbb{C} \rightrightarrows \mathbb{D}$. A* **natural transformation** *$\alpha\colon F \to G$ consists of a collection of maps $\alpha_X\colon FX \to GX$ in $\mathbb{D}$, indexed by objects $X \in \mathbb{C}$ which satisfy the following naturality condition: for each map $f\colon X \to Y$ in $\mathbb{C}$, one has*

$$Gf \circ \alpha_X = \alpha_Y \circ Ff$$

*in $\mathbb{D}$. In a diagram:*

$$\begin{tikzcd} X \arrow[d, "f"'] \\ Y \end{tikzcd} \qquad \begin{tikzcd} F(X) \arrow[r, "\alpha_X"] \arrow[d, "F(f)"'] & G(X) \arrow[d, "G(f)"] \\ F(Y) \arrow[r, "\alpha_Y"] & G(Y) \end{tikzcd}$$

*The maps $\alpha_X$ are called the* **components** *of the natural transformation $\alpha$. Some authors use the notation $\alpha\colon F \Rightarrow G$ to indicate that $\alpha$ is a natural transformation from $F$ to $G$. In a diagram such a natural transformation $\alpha$ is written as a map between two functors:*

$$\begin{tikzcd} \mathbb{C} \arrow[r, bend left, "F"{name=U}] \arrow[r, bend right, "G"'{name=D}] & \mathbb{D} \arrow[from=U, to=D, phantom, "\Downarrow\alpha"] \end{tikzcd}$$

We describe several examples of natural transformations involving the list functor.

*Example 2.* We have seen the list functor $\mathcal{L} = (-)^\star \colon \mathbf{Sets} \to \mathbf{Sets}$, which assigns to a set $X$, the free monoid $\mathcal{L}(X) = X^\star$ of finite sequences of elements of $X$. For a function $f \colon X \to Y$ we have $\mathcal{L}(f) = f^\star \colon X^\star \to Y^\star$ by $\langle x_1, \ldots, x_n\rangle \mapsto \langle f(x_1), \ldots, f(x_n)\rangle$.

1. For every set there is a singleton-list function $X \to X^\star$. It is written as $\eta$, and defined as $\eta_X(x) = \langle x\rangle$, for $x \in X$. We claim that it is a natural transformation in a situation:

$$\begin{tikzcd} \mathbf{Sets} \arrow[r, bend left, "id"] \arrow[r, bend right, "\mathcal{L}"'] \arrow[r, phantom, "\Downarrow\eta"] & \mathbf{Sets} \end{tikzcd}$$

For a function $f \colon X \to Y$ we have to check that the following diagram commutes.

$$\begin{tikzcd} id(X) = X \arrow[r, "\eta_X"] \arrow[d, "id(f)=f"'] & X^\star \arrow[d, "f^\star"] \\ id(Y) = Y \arrow[r, "\eta_Y"] & Y^\star \end{tikzcd}$$

Commutation is obvious, since for $x \in X$,

$$\big(f^\star \circ \eta_X\big)(x) \;=\; f^\star\big(\langle x\rangle\big) \;=\; \langle f(x)\rangle \;=\; \eta_Y\big(f(x)\big) \;=\; \big(\eta_Y \circ f\big)(x).$$

2. For every set $X$ there is a 'reverse' function $rev_X \colon X^\star \to X^\star$ which reverses the order of elements in a list:

$$rev_X(\langle x_1, \ldots, x_n\rangle) = \langle x_n, \ldots, x_1\rangle$$

One sees that the action of $rev_X$ doesn't really depend on the set $X$: it works in the same way for every other set $Y$. This uniformity of reversal is expressed by naturality. The maps $rev_X$ form components of a natural transformation $rev \colon (-)^\star \to (-)^\star$, in a situation:

$$\begin{tikzcd} \mathbf{Sets} \arrow[r, bend left, "\mathcal{L}"] \arrow[r, bend right, "\mathcal{L}"'] \arrow[r, phantom, "\Downarrow rev"] & \mathbf{Sets} \end{tikzcd}$$

For a function $f \colon X \to Y$ we have a naturality diagram,

$$\begin{tikzcd} X^\star \arrow[r, "rev_X"] \arrow[d, "f^\star"'] & X^\star \arrow[d, "f^\star"] \\ Y^\star \arrow[r, "rev_Y"] & Y^\star \end{tikzcd}$$

as is shown by a short computation:

$$\begin{aligned} \big(f^\star \circ rev_X\big)(\langle x_1, \ldots, x_n\rangle) &= f^\star(\langle x_n, \ldots, x_1\rangle) \\ &= \langle f(x_n), \ldots, f(x_1)\rangle \\ &= rev_Y(\langle f(x_1), \ldots, f(x_n)\rangle) \\ &= \big(rev_Y \circ f^\star\big)(\langle x_1, \ldots, x_n\rangle). \end{aligned}$$

Computer scientists use the term 'polymorphism' for this uniformity in $X$ of reversal $rev_X$. For them, polymorphic functions are important because their code does not depend on the type of the input: it means that they don't have to write separate reversal programs for lists of integers, or lists of booleans, or of characters, etcetera.

3. There is a function $\mu\colon \mathcal{L}\big(\mathcal{L}(X)\big) \to \mathcal{L}(X)$ that 'flattens' a list of lists to a single list by removing inner brackets. Thus, for $x_{i,j} \in X$,

$$\mu\Big(\langle\langle x_{1,1},\ldots,x_{1,n_1}\rangle,\ldots,\langle x_{m,1},\ldots x_{m,n_m}\rangle\rangle\Big) \;=\; \langle x_{1,1},\ldots,x_{m,n_m}\rangle.$$

This flattening works in the same way for each set $X$ and thus forms a natural transformation, of the form:

$$\begin{tikzcd}
\mathbf{Sets} \arrow[r, bend left, "\mathcal{L}^2"] \arrow[r, bend right, "\mathcal{L}"'] \arrow[r, phantom, "\Downarrow\mu"] & \mathbf{Sets}
\end{tikzcd}$$

The relevant naturality equation, for $f\colon X \to Y$ takes the form:

$$\mathcal{L}(f) \circ \mu_X \;=\; \mu_Y \circ \mathcal{L}\big(\mathcal{L}(f)\big).$$

Prove this equation yourself.

If we view natural transformations as morphisms between functors, we better establish that we get a category in this way.

**Lemma 3.** *Identity maps $FX \to FX$ form an identity natural transformation $id\colon F \to F$. And if $\alpha\colon F \Rightarrow G$ and $\beta\colon G \Rightarrow H$ are natural transformations, then componentwise composition $\beta_X \circ \alpha_X$ forms a composite natural transformation $\beta \circ \alpha\colon F \Rightarrow H$.*

*Thus for categories $\mathbb{C}, \mathbb{D}$, there is a category $\mathbb{D}^{\mathbb{C}}$ of functors $\mathbb{C} \to \mathbb{D}$ and natural transformations between them. It is a* **functor category**. □

**Proposition 4.** *Let $F, G\colon \mathbb{C} \to \mathbb{C}$ be two endofunctors with a natural transformation $\alpha\colon F \Rightarrow G$ between them, in a situation:*

$$\begin{tikzcd}
\mathbb{C} \arrow[r, bend left, "F"] \arrow[r, bend right, "G"'] \arrow[r, phantom, "\Downarrow\alpha"] & \mathbb{C}
\end{tikzcd}$$

*This $\alpha$ gives rise to two functors between categories of (co)algebras, of the form:*

$$\begin{array}{ccc}
\mathit{CoAlg}(F) & \xrightarrow{\mathit{CoAlg}(\alpha)} & \mathit{CoAlg}(G) \\
\Big(X \xrightarrow{c} F(X)\Big) & \longmapsto & \Big(X \xrightarrow{\alpha_X \circ c} G(X)\Big) \\
f & \longmapsto & f
\end{array}
\qquad
\begin{array}{ccc}
\mathit{Alg}(G) & \xrightarrow{\mathit{Alg}(\alpha)} & \mathit{Alg}(F) \\
\Big(G(X) \xrightarrow{z} X\Big) & \longmapsto & \Big(F(X) \xrightarrow{a \circ \alpha_X} X\Big) \\
f & \longmapsto & f
\end{array}$$

*Proof.* We shall do the case of coalgebras. What we have to check is that $f$ is a map of $F$-coalgebras, as on the left below, then it is also a map of $G$-coalgebras, as on the right.

$$\begin{tikzcd} F(X) \arrow[r, "F(f)"] & F(Y) \\ X \arrow[u, "c"] \arrow[r, "f"] & Y \arrow[u, "d"'] \end{tikzcd} \qquad \begin{tikzcd} G(X) \arrow[r, "G(f)"] & G(Y) \\ F(X) \arrow[u, "\alpha_X"] & F(Y) \arrow[u, "\alpha_Y"'] \\ X \arrow[u, "c"] \arrow[r, "f"] & Y \arrow[u, "d"'] \end{tikzcd}$$

This clearly works via a simple diagram chase, by including the map $F(f)$ on the right, and using naturality of $\alpha$. $\square$

We conclude this short section with the notion of equivalence of categories. Two categories $\mathbb{C}, \mathbb{D}$ are 'isomorphic', if they are isomorphic as objects of $\mathbf{Cat}$. This means that there are functors $F: \mathbb{C} \to \mathbb{D}$ and $G: \mathbb{D} \to \mathbb{C}$ satisfying $FG = id$ and $GF = id$. In this case we write $\mathbb{C} \cong \mathbb{D}$. There is a more general and often more useful way of considering categories as identical, namely 'equivalence' of categories. Then one replaces the identities $FG = id$ and $GF = id$ by 'natural isomorphisms'. We first investigate these.

**Lemma 5.** *A natural transformation $\alpha: F \to G$ between functors $F, G: \mathbb{C} \rightrightarrows \mathbb{D}$ is an isomorphism in the functor category $\mathbb{D}^{\mathbb{C}}$ if and only if each component $\alpha_X: FX \to GX$ is an isomorphism in $\mathbb{D}$.*

*Such an invertible natural transformation is called a* **natural isomorphism***.*

Thus the natural isomorphisms are the isomorphisms in functor categories.

*Proof.* The (only if)-part is obvious. And for (if), we have to check that the inverses $\alpha_X^{-1}$ are components of a natural transformation. This is easy: for $f: X \to Y$ we have $Gf \circ \alpha_X = \alpha_Y \circ Ff$, and hence $\alpha_Y^{-1} \circ Gf = Ff \circ \alpha_X^{-1}$. $\square$

**Definition 6.** *Two categories $\mathbb{C}, \mathbb{D}$ are* **equivalent***, if there are functors $F: \mathbb{C} \to \mathbb{D}$ and $G: \mathbb{D} \to \mathbb{C}$ with natural isomorphisms $FG \cong id$ and $GF \cong id$. We write $\mathbb{C} \simeq \mathbb{D}$ for equivalence.*