1

Quiz

1. Prove termination of the following TRS using a monotonic algebra to $\mathbb{N}$:

$$\begin{array}{rcl} append(nil, z) & \rightarrow & z \\ append(cons(x, y), z) & \rightarrow & cons(x, append(y, z)) \end{array}$$

   (a) give (linear) parametric interpretations for all function symbols

   Answer:

   - $[nil] = \underline{n_0}$
   - $[cons] = \lambda x, y.\underline{c_0} + \underline{c_1} * x + \underline{c_2} * y$
   - $[append] = \lambda xy.\underline{a_0} + \underline{a_1} * x + \underline{a_2} * y$

   (b) compute the requirements (monotonicity and rule orientation)

   Answer: first, we must impose the *monotonicity* requirements:

$$\begin{array}{rclcrcl} \underline{c_1} & \geq & 1 & \quad & \underline{a_1} & \geq & 1 \\ \underline{c_2} & \geq & 1 & \quad & \underline{a_2} & \geq & 1 \end{array}$$

   For the first rule, we must have:

$$\begin{array}{rl} & W(append(nil, z)) \\ = & \underline{a_0} + \underline{a_1} * W(nil) + \underline{a_2} * W(z) \\ = & \underline{a_0} + \underline{a_1} * \underline{n_0} + \underline{a_2} * z \\ > & z \\ = & W(z) \end{array}$$

   For the second rule, we must have:

$$W(append(cons(x, y), z)) > W(cons(x, append(y, z)))$$

   where:

$$\begin{array}{rl} & W(append(cons(x, y), z)) \\ = & \underline{a_0} + \underline{a_1} * W(cons(x, y)) + \underline{a_2} * W(z) \\ = & \underline{a_0} + \underline{a_1} * (\underline{c_0} + \underline{c_1} * x + \underline{c_2} * y) + \underline{a_2} * z \\ = & \underline{a_0} + \underline{a_1} * \underline{c_0} + \underline{a_1} * \underline{c_1} * x + \underline{a_1} * \underline{c_2} * y + \underline{a_2} * z \end{array}$$

   and

$$\begin{array}{rl} & W(cons(x, append(y, z))) \\ = & \underline{c_0} + \underline{c_1} * W(x) + \underline{c_2} * W(append(y, z)) \\ > & \underline{c_0} + \underline{c_1} * x + \underline{c_2} * (\underline{a_0} + \underline{a_1} * y + \underline{a_2} * z) \\ = & \underline{c_0} + \underline{c_1} * x + \underline{c_2} * \underline{a_0} + \underline{c_2} * \underline{a_1} * y + \underline{c_2} * \underline{a_2} * z \end{array}$$

(c) use absolute positiveness to find SMT requirements

Answer: The monotonicity requirements are already SMT requirements.
For the first rule, we must have:

$$(a_0 + a_1 * n_0) + a_2 * z > z = 0 + 1 * z$$

which by absolute positiveness yields the SMT requirements:

$$a_0 + a_1 * n_0 \;>\; 0 \qquad a_2 \;\geq\; 1$$

For the second rule, we must have:

$$\begin{array}{rl} & (a_0 + a_1 * c_0) + a_1 * c_1 * x + a_1 * c_2 * y + a_2 * z \\ > & (c_0 + c_2 * a_0) + c_1 * x + c_2 * a_1 * y + c_2 * a_2 * z \end{array}$$

which by absolute positiveness yields the SMT requirements:

$$\begin{array}{rcl@{\qquad}rcl} a_0 + a_1 * c_0 & > & c_0 + c_2 * a_0 & a_1 * c_1 & \geq & c_1 \\ a_1 * c_2 & \geq & c_2 * a_1 & a_2 & \geq & c_2 * a_2 \end{array}$$

Putting everything together, we thus end up with:

$$\begin{array}{rcl@{\qquad}rcl} c_1 & \geq & 1 & a_1 & \geq & 1 \\ c_2 & \geq & 1 & a_2 & \geq & 1 \\ a_0 + a_1 * n_0 & > & 0 & a_2 & \geq & 1 \\ a_0 + a_1 * c_0 & > & c_0 + c_2 * a_0 & a_1 * c_1 & \geq & c_1 \\ a_1 * c_2 & \geq & c_2 * a_1 & a_2 & \geq & c_2 * a_2 \end{array}$$

(d) solve them by hand and give the resulting inteprretation functions, and check your result!

Answer: We observe:

- $a_2 \geq 1$ is required twice
- $a_1 * c_2 \geq c_2 * a_1$ is always satisfied
- given that $a_1 \geq 1$, we also know that $a_1 * c_1 \geq c_1$ is satisfied
- given that a_2 cannot be 0, the requirement $a_2 \geq c_2 * a_2$ implies $c_2 \leq 1$; since we also have $c_2 \geq 1$, we know that $c_2 = 1$
- hence, the requirement $a_0 + a_1 * c_0 > c_0 + c_2 * a_0$ becomes $a_0 + a_1 * c_0 > c_0 + 1 * a_0$; removing a_0 on both sides, we end up with $a_1 * c_0 > c_0$
- this requirement then implies that c_0 must be at least 1 (since otherwise the requirement would simplify to $0 > 0$), and that a_1 must be larger than 1 (since $c_0 > c_0$ also does not hold).

This leaves us with:

$$\begin{array}{rcl@{\qquad}rcl} c_1 & \geq & 1 & a_1 & \geq & 2 \\ c_2 & = & 1 & a_2 & \geq & 1 \\ a_0 + a_1 * n_0 & > & 0 & c_0 & \geq & 1 \end{array}$$

So now we can simply choose $c_1 := 1, c_2 := 1, a_1 := 2, a_2 := 1, c_0 := 1$ and $a_0 := 1$, ane leave $n_0 := 0$ and $c_0 := 0$ (though we could also have chosen $a_0 := 0$ and $n_0 := 1$). This gives the interpretation functions:

- $[nil] = 0$
- $[cons] = \lambda x, y.1 + x + y$
- $[append] = \lambda xy.1 + 2 * x + y$

2. Determine the dependency pairs of:

$$\begin{array}{rcl} f(h(x), y) & \to & g(x, f(x, h(y))) \\ g(x, h(y)) & \to & g(h(x), y) \end{array}$$

   Answer: The defined symbols are f and g (as these occur at the root of the left-hand sides of rules), not h. Thus, the DPs for the first rule are:

$$\begin{array}{lrcl} \text{A.} & f^\sharp(h(x), y) & \to & g^\sharp(x, f(x, h(y))) \\ \text{B.} & f^\sharp(h(x), y) & \to & f^\sharp(x, h(y)) \end{array}$$

   And the dependency pair for the second rule is:

$$\begin{array}{lrcl} \text{C.} & g^\sharp(x, h(y)) & \to & g^\sharp(h(x), y) \end{array}$$

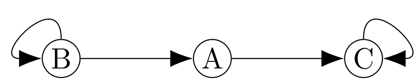
3. Split these dependency pairs up into one or more groups of DPs that can be analysed separately.

   Answer option 1: In an infinite chain $s_1 \Rightarrow_{\mathtt{DP}} t_1 \Rightarrow_{\mathcal{R}}^* s_2 \Rightarrow_{\mathtt{DP}} t_2 \Rightarrow_{\mathcal{R}}^* s_3 \ldots$,

   - due to root symbols, if the step $s_i \Rightarrow_{\mathtt{DP}} t_i$ uses C, then so does the step $s_{i+1} \Rightarrow_{\mathtt{DP}} t_{i+1}$
   - similarly, if the step $s_i \Rightarrow_{\mathtt{DP}} t_i$ uses A, then step $s_{i+1} \Rightarrow_{\mathtt{DP}} t_{i+1}$ uses C as well
   - hence, an infinite chain either uses only dependency pair B, or it has an infinite tail using only dependency pair C

   Therefore, the groups $\{B\}$ and $\{C\}$ can be analysed separately: if neither admits an infinite chain, then the original system is terminating.

   Answer option 2: We use the dependency graph processor, using the following graph approximation (based on the root symbol of each DP):



   The strongly connected components of this graph are $\{B\}$ and $\{C\}$, so these can be analysed separately.

4. Prove termination of the above TRS.

   Answer option 1: It suffices to find a reduction pair such that:

$$\begin{array}{rcl} f(h(x), y) & \succeq & g(x, f(x, h(y))) \\ g(x, h(y)) & \succeq & g(h(x), y) \\ f^\sharp(h(x), y) & \succ & g^\sharp(x, f(x, h(y))) \\ f^\sharp(h(x), y) & \succ & f^\sharp(x, h(y)) \\ g^\sharp(x, h(y)) & \succ & g^\sharp(h(x), y) \end{array}$$

   We can do this for instance using a weakly monotonic algebra with:

- $[f] = \lambda x, y.y$
- $[g] = \lambda x, y.0$
- $[h] = \lambda x.x + 1$
- $[f^\sharp] = \lambda x, y.2 * x + y$
- $[g^\sharp] = \lambda x, y.y$

Then the requirements evaluate to:

$$\begin{array}{rcl} y & \geq & 0 \\ 0 & \geq & 0 \\ 2 * x + y + 2 & > & y + 1 \\ 2 * x + y + 2 & > & 2 * x + y + 1 \\ y + 1 & > & y \end{array}$$

Answer option 2: As reasoned above, $\{B\}$ and $\{C\}$ can be analysed separately. First, we handle $\{B\}$. This can be done using the following weakly monotonic algebra:

$$\begin{array}{rclcrcl} [f] & = & \lambda x, y.0 & \quad & [g] & = & \lambda x, y.0 \\ [f^\sharp] & = & \lambda x, y.x & \quad & [h] & = & \lambda x.x + 1 \end{array}$$

Since this gives:

$$\begin{array}{rcccccc} W(f(h(x), y)) & = & 0 & \geq & 0 & = & W(g(x, f(x, h(y)))) \\ W(g(x, h(y))) & = & 0 & \geq & 0 & = & W(g(h(x), y)) \\ W(f^\sharp(h(x), y)) & = & x + 1 & > & x & = & W(f^\sharp(x, h(y))) \end{array}$$

Next, we handle $\{C\}$. This can be done using the following weakly monotonic algebra:

$$\begin{array}{rclcrcl} [f] & = & \lambda x, y.0 & \quad & [g] & = & \lambda x, y.0 \\ [h] & = & \lambda x.x + 1 & \quad & [g^\sharp] & = & \lambda x, y.y \end{array}$$

Since this orients the rules as before, and additionally gives:

$$W(g^\sharp(x, h(y))) \quad = \quad y + 1 \quad > \quad y \quad = \quad W(g^\sharp(h(x), y))$$

Answer option 3:

As reasoned above, $\{B\}$ and $\{C\}$ can be analysed separately. Both are handled with the subterm criterion:

- B using projection function $\nu(f^\sharp) = 1$, since x is a strict subterm of $h(x)$
- C using projection function $\nu(g^\sharp) = 2$, since y is a strict subterm of $h(y)$