Handout for lecture 12:

# Confluence and Completion

# 1. Overview

## 1.1 Properties of term rewriting systems

2

### Recall: some nice properties

- $\mathcal{R}$ is **weakly normalising** (WN):

  every term has a normal form

- $\mathcal{R}$ is **terminating** (= strongly normalising, SN):

  no infinite sequence of terms $t_1, t_2, t_3, \ldots$ exists such that $t_i \to_{\mathcal{R}} t_{i+1}$ for all $i$

- $\mathcal{R}$ is **confluent** (= Church-Rosser, CR):

  if $s \to_{\mathcal{R}}^* t$ and $s \to_{\mathcal{R}}^* q$ then a term $u$ exists satisfying $t \to_{\mathcal{R}}^* u$ and $q \to_{\mathcal{R}}^* u$

- $\mathcal{R}$ is **locally confluent** (= weak Church-Rosser, WCR):

  if $s \to_{\mathcal{R}} t$ and $s \to_{\mathcal{R}} q$ then a term $u$ exists satisfying $t \to_{\mathcal{R}}^* u$ and $q \to_{\mathcal{R}}^* u$

3

### Confluence implies uniqueness of normal forms

**Theorem**
If a TRS is confluent, then every term has at most one normal form.

**Proof:** Assume $t$ has two normal forms $u, u'$.
Then by confluence there is a $v$ such that $u \to_{\mathcal{R}}^* v$ and $u' \to_{\mathcal{R}}^* v$.
Since $u, u'$ are normal forms we have $u = v = u'$. $\square$

4

### Termination + confluence implies existence of unique normal forms

We conclude:

**Theorem**
If a TRS is terminating and confluent, then every term has exactly one normal form.

This normal form can be found just by rewriting the term until no further rewriting step is possible.

This is a very useful combination! After all, it allows us to essentially define a computable function using a term rewriting system: to calculate the result of the function on a given input, we simply reduce the term to normal form in whichever way we choose. We will later see that this combination is so useful in the context of equational logic.

---

5

# This week

- confluence versus local confluence
- critical pairs
- solving the **word problem** (using a combination of confluence and termination techniques)

# 2. Confluence

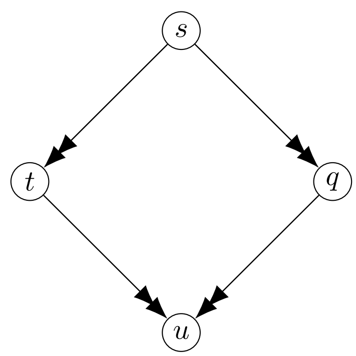
## 2.1 Confluence versus local confluence

---

6

### Confluence (Church-Rosser property)

If $s \to^* t$ and $s \to^* q$
then exists $u$ such that
$t \to^* u$ and $q \to^* u$.

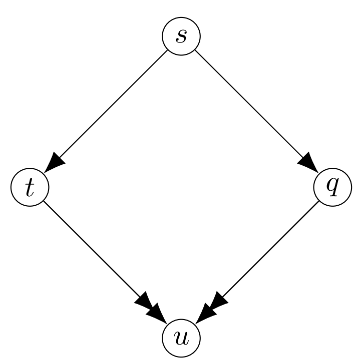



---

7

### Local confluence (Weak Church-Rosser property):

If $s \to t$ and $s \to q$
then exists $u$ such that
$t \to^* u$ and $q \to^* u$.




---

8

### Local versus general confluence

Clearly confluence implies local confluence.

However, local confluence does not imply confluence:

$$R = \{\quad \mathtt{a} \to \mathtt{b}, \quad \mathtt{b} \to \mathtt{a}, \quad \mathtt{a} \to \mathtt{c}, \quad \mathtt{b} \to \mathtt{d} \quad \}$$

This system can be presented in graph form as follows:

$$\begin{tikzcd} \mathtt{c} & \mathtt{a} \arrow[l] \arrow[r, bend left] & \mathtt{b} \arrow[l, bend left] \arrow[r] & \mathtt{d} \end{tikzcd}$$

This system is locally confluent:

> If $s \to_R t$ and $s \to_R q$ then either
>
> - $s = \mathtt{a}$, then choose $u = \mathtt{c}$, or
> - $s = \mathtt{b}$, then choose $u = \mathtt{d}$.
>
> In both cases we conclude $t \to_R^* u$ and $q \to_R^* u$.

However, it is not confluent:

> For $s = \mathtt{a}, t = \mathtt{c}, q = \mathtt{d}$ we have $s \to_R^* t$ and $s \to_R^* q$, but no $u$ exists satisfying $t \to_R^* u$ and $q \to_R^* u$.

This counterexample relies on the TRS being **non-terminating**.

---

9

# Newman's lemma (1942)

**Theorem**

For terminating TRSs the properties confluence and local confluence are equivalent.

For the proof of Newman's lemma we will use the principle of *well-founded induction*.

Note that SN($\to$), CR($\to$) and WCR($\to$) all can be defined for arbitrary binary relations $\to$. We will prove Newman's lemma in this general setting.

So SN($\to$) simply means the non-existence of an infinite sequence $t_1 \to t_2 \to t_3 \to \cdots$.

We write $\to^+$ for the transitive closure of $\to$: one or more steps.

---

10

# Principle of well-founded induction

**Theorem**

Let SN($\to$) and

$$\forall t[\underbrace{\forall u[t \to^+ u \Rightarrow P(u)]}_{\text{Induction Hypothesis}} \Rightarrow P(t)]$$

Then $P(t)$ holds for all $t$.

(Think of $t \to^+ u$ as $t > u$ as in well-known induction.)

---

11

# Proof of the principle of well-founded induction

Assume there exists $t$ such that $\neg P(t)$.

Then the induction hypothesis does not hold for this $t$, so $\neg \forall u[t \to^+ u \Rightarrow P(u)]$.

That is, $\exists u[t \to^+ u \wedge \neg P(u)]$.

Let $u$ be such a term, and repeat the argument for $u$, yielding a $v$, and so on, so yielding an infinite sequence

$$t \to^+ u \to^+ v \to^+ \cdots$$

This contradicts the assumption SN($\to$). $\square$

---

12

# Proof of Newman's Lemma

Assume SN($\to$) and WCR($\to$). We have to prove CR($\to$).

We apply the principle of well-founded induction for $P(s)$ being

$$\forall t, q[\text{if } s \to^* t \wedge s \to^* q \text{ then } \exists u[t \to^* u \wedge q \to^* u]]$$

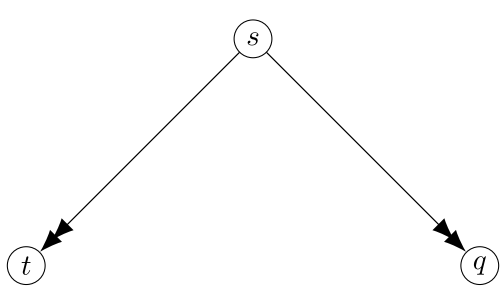
Suppose $s \to^* t$ and $s \to^* q$.

This is depicted in graph form as follows:

---

13

# Proof of Newman's Lemma

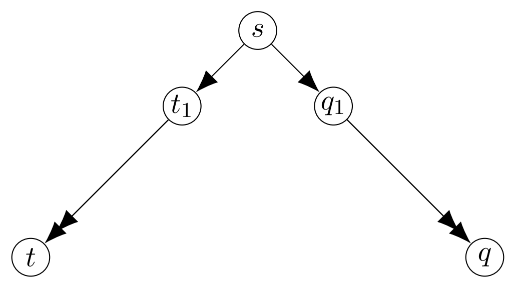


We must find $u$ such that $t \to^* u$ and $q \to^* u$.

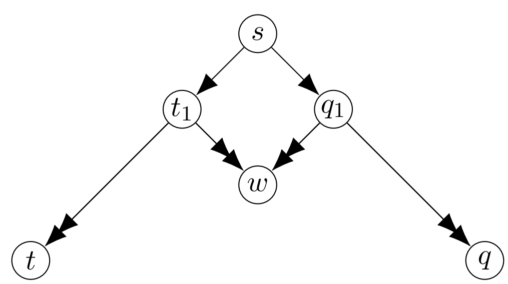
If $s = t$ we can choose $u := q$.

If $s = q$ we can choose $u := t$.

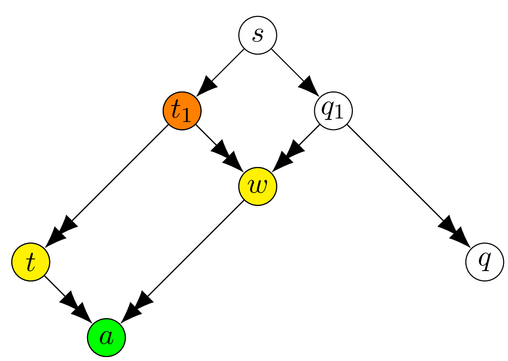
In the remaining case we have $s \to^+ t$ and $s \to^+ q$.



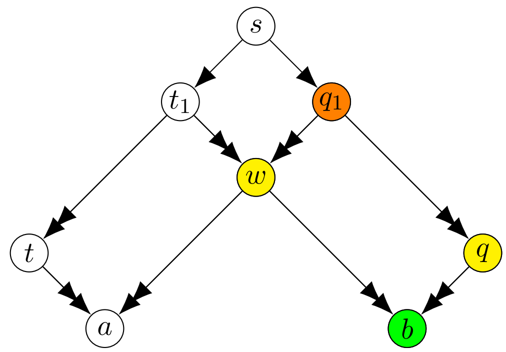

By WCR($\to$) we can find $w$ such that $t_1 \to^* w$ and $q_1 \to^* w$.



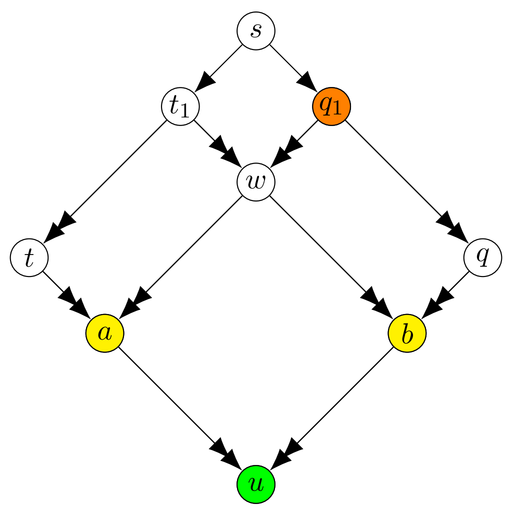

By the induction hypothesis on $t_1$, there is a common reduct for $t$ and $w$.




By the induction hypothesis on $q_1$, there is a common reduct for $w$ and $q$.




Again by the induction hypothesis on $q_1$, there is a common reduct for $a$ and $b$.




Hence, $t$ and $q$ indeed have a common reduct $u$!

In the literature, proofs of confluence given various conditions often use visual representations like the ones we used above.

## 2.2 Critical pairs

---

14

# Deciding confluence

**Bad news:** both confluence and local confluence are undecidable properties.

**Good news:** for *terminating* TRSs there is a simple decision procedure for local confluence, and hence for confluence too.

> Idea:
>
> analyze overlapping patterns in left hand sides of the rules, yielding **critical pairs**.

In our example for addition of natural numbers,

$$\begin{array}{rcl} \mathtt{add}(\mathtt{0}, y) & \to & y \\ \mathtt{add}(\mathtt{s}(x), y) & \to & \mathtt{s}(\mathtt{add}(x, y)) \end{array}$$

there is no overlap. Hence it is locally confluent.

Termination holds for instance by LPO. Hence, by Newman's lemma, it is confluent.   for instance by LPO.

---

15

# Critical pairs

**Some terminology:**

In term rewriting, a *context* is a term with a single hole $\square$. For instance $C = \mathtt{f}(x, \mathtt{g}(\square, z))$.

Then $C[t] = \mathtt{f}(x, \mathtt{g}(t, z))$.

**Definition**

Let $\ell_1 \to r_1$ and $\ell_2 \to r_2$ be two (possibly equal) rewrite rules, which have no variables in common. (Possibly through renaming.)

Suppose $\ell_2 = C[t]$, where $C$ is a (possibly trivial) context, and $t$ not a variable.

Suppose $t$ and $\ell_1$ unify with mgu $\sigma$: $t\sigma = \ell_1\sigma$.

Then $\langle C[r_1]\sigma, r_2\sigma\rangle$ is a **critical pair** of $\ell_1 \to r_1$ and $\ell_2 \to r_2$.

Note that $\ell_2\sigma = C[t]\sigma = C\sigma[\ell_1\sigma]$ can be rewritten in two ways:

- with $\ell_2 \to r_2$: $\ell_2\sigma \to r_2\sigma$
- with $\ell_1 \to r_1$: $C[\ell_1]\sigma \to C[r_1]\sigma$

Hence, a critical pair expresses two possible one-step reducts of the same term.

---

16

# Example

Assume we have rules for arithmetic including:

$$\begin{array}{rcl} x \mathbin{\mathtt{-}} x & \to & \mathtt{0} \\ \mathtt{s}(x) \mathbin{\mathtt{-}} y & \to & \mathtt{s}(x \mathbin{\mathtt{-}} y) \end{array}$$

Then $\mathtt{s}(x) \mathbin{\mathtt{-}} \mathtt{s}(x)$ can be rewritten in two ways:

- to $\mathtt{0}$ by the first rule;
- to $\mathtt{s}(x \mathbin{\mathtt{-}} \mathtt{s}(x))$ by the second rule.

Now $\langle \mathtt{0}, \mathtt{s}(x \mathbin{\mathtt{-}} \mathtt{s}(x)) \rangle$ is a **critical pair**.

More precisely, in the above notation we choose:

- $\ell_1 \to r_1$ to be the rule $z \mathbin{\mathtt{-}} z \to \mathtt{0}$
- $\ell_2 \to r_2$ to be the rule $\mathtt{s}(x) \mathbin{\mathtt{-}} y \to \mathtt{s}(x \mathbin{\mathtt{-}} y)$
- $C$ to be the trivial context $\square$
- $t = \ell_2 = \mathtt{s}(x) \mathbin{\mathtt{-}} y$

Indeed $t, \ell_1$ unify, with mgu $\sigma$:

$$\sigma(x) = x, \quad \sigma(y) = \sigma(z) = \mathtt{s}(x)$$

---

17
# Example

Let $R$ consist of the single rule

$$\mathtt{f}(\mathtt{f}(x)) \to \mathtt{g}(x)$$

By choosing

- $\ell_1 \to r_1$ to be the rule $\mathtt{f}(\mathtt{f}(x)) \to \mathtt{g}(x)$
- $\ell_2 \to r_2$ to be the rule $\mathtt{f}(\mathtt{f}(y)) \to \mathtt{g}(y)$
- $C = \mathtt{f}(\square)$
- $t = \mathtt{f}(y)$

we see that $t, \ell_1$ unify, with mgu $\sigma$:

$$\sigma(x) = x, \quad \sigma(y) = \mathtt{f}(x)$$

This yields the critical pair $\langle \mathtt{f}(\mathtt{g}(x)), \mathtt{g}(\mathtt{f}(x)) \rangle$.

---

18
# Convergence

A critical pair $\langle t, q \rangle$ is said to **converge** if there is a term $u$ such that $t \to_R^* u$ and $q \to_R^* u$.

**Example**

$$x \mathbin{\mathtt{+}} \mathtt{s}(y) \quad \to \quad \mathtt{s}(x \mathbin{\mathtt{+}} y) \qquad\qquad \mathtt{s}(x) \mathbin{\mathtt{+}} y \quad \to \quad \mathtt{s}(x \mathbin{\mathtt{+}} y)$$

Critical pair: $\langle \mathtt{s}(x \mathbin{\mathtt{+}} \mathtt{s}(y)), \mathtt{s}(\mathtt{s}(x) \mathbin{\mathtt{+}} y) \rangle$

We have: $\mathtt{s}(x \mathbin{\mathtt{+}} \mathtt{s}(y)) \to_{\mathcal{R}} \mathtt{s}(\mathtt{s}(x \mathbin{\mathtt{+}} y)) \leftarrow_{\mathcal{R}} \mathtt{s}(\mathtt{s}(x) \mathbin{\mathtt{+}} y)$.

**Example**

$$\mathtt{not}(\top) \;\to\; \bot \qquad \mathtt{not}(\bot) \;\to\; \top \qquad \mathtt{not}(\mathtt{not}(x)) \;\to\; x$$

Critical pair: $\langle \mathtt{not}(\bot),\ \top\rangle$

We have: $\mathtt{not}(\bot) \to \top \leftarrow_R^* \top$.

---

19

# Critical pair lemma

**Theorem**

A TRS $R$ is locally confluent if and only if all its critical pairs converge.

**Example**

The single rewrite rule $\mathtt{f}(\mathtt{f}(x)) \to \mathtt{g}(x)$ is not locally confluent, so also not confluent, since its critical pair $\langle \mathtt{f}(\mathtt{g}(x)), \mathtt{g}(\mathtt{f}(x))\rangle$ consists of two distinct normal forms.

**Proof idea:** if $s \to t$ and $s \to q$ there are three cases:

- the reductions happen at distinct places in $s$
  For example, $s = \mathtt{f}(s_1, s_2)$ and $t = \mathtt{f}(s_1', s_2)$ and $q = \mathtt{f}(s_1, s_2')$. We can just choose $u := \mathtt{f}(s_1', s_2')$.
- one reduction occurs above the other, but without an overlap
  For example, $s = \mathtt{f}(s)$ and $t = \mathtt{g}(s, s)$ with rule $\mathtt{f}(x) \to \mathtt{g}(x, x)$ and $q = \mathtt{f}(s')$ with a reduction inside $s$. Then we can choose $u := \mathtt{g}(s', s')$.
- the two reductions are an instance of a critical pair

---

20

# Decision procedure

Using the theorem, for a finite terminating TRS $\mathcal{R}$ we indeed have an algorithm to decide whether WCR($\mathcal{R}$) holds:

- Compute all critical pairs $\langle t, q\rangle$

  (They are found by unification of left hand sides with subterms of left hand sides: there are finitely many of them.)

- For all critical pairs $\langle t, q\rangle$ compute

$$\{v \mid t \to^* v\} \cap \{u \mid q \to^* u\}$$

- If one of these sets is empty then WCR($\mathcal{R}$) does not hold.

- If all of these sets are non-empty then $\mathrm{WCR}(\mathcal{R})$ does hold.

---

21
# Non-overlapping systems

A TRS is said to have **no overlap** if there are only **trivial** critical pairs, i.e., the critical pairs obtained by unifying a left hand side with itself.

A trivial critical pair always converges since it has a shape $\langle t, t\rangle$.

As a consequence, every TRS having no overlap is locally confluent.

It is not the case that every TRS having no overlap is confluent:

$$\begin{array}{rcl} \mathtt{d}(x,x) & \to & \mathtt{b} \\ \mathtt{c}(x) & \to & \mathtt{d}(x,\mathtt{c}(x)) \\ \mathtt{a} & \to & \mathtt{c}(\mathtt{a}) \end{array}$$

has no overlap but is not confluent:

$$\mathtt{c}(\mathtt{a}) \to \mathtt{d}(\mathtt{a},\mathtt{c}(\mathtt{a})) \to \mathtt{d}(\mathtt{c}(\mathtt{a}),\mathtt{c}(\mathtt{a})) \to \mathtt{b}$$

$$\mathtt{c}(\mathtt{a}) \to \mathtt{c}(\mathtt{c}(\mathtt{a})) \to^{+} \mathtt{c}(\mathtt{b})$$

while $\mathtt{b}$ and $\mathtt{c}(\mathtt{b})$ have no common reduct.

---

22
# Orthogonality

A rule $\ell \to r$ is **left-linear** if no variable occurs more than once in $\ell$.
For example, $\mathtt{d}(x,x) \to \mathtt{b}$ is not left-linear because $x$ occurs twice, but $\mathtt{add}(\mathtt{s}(x),y) \to \mathtt{s}(\mathtt{add}(x,y))$ is.

A TRS is **left-linear** if all its rules are.

A TRS is **orthogonal** if it is left-linear and has no overlap.

**Theorem**
Any orthogonal TRS is confluent.

**Proof idea:** this uses that if $s \twoheadrightarrow\!\!\!\!\!\!+\, t$ and $s \twoheadrightarrow\!\!\!\!\!\!+\, q$, there exists $u$ such that $t \twoheadrightarrow\!\!\!\!\!\!+\, u$ and $q \twoheadrightarrow\!\!\!\!\!\!+\, u$ (where $\twoheadrightarrow\!\!\!\!\!\!+$ indicates a *parallel* move).

A TRS is **weakly orthogonal** if it is left-linear and all its critical pairs have a form $\langle t, t\rangle$.

**Theorem**
Any weakly orthogonal TRS is confluent.

The proof of this theorem is a bit more complex.

# 3. Completion

## 3.1 The Word Problem

23

### The Word Problem

Write $\leftrightarrow_{\mathcal{R}}^{*}$ for the reflexive symmetric transitive closure of $\to_{\mathcal{R}}$, i.e., $s \leftrightarrow_{\mathcal{R}}^{*} t$ holds if and only if terms $u_1, \ldots, u_n$ exist for $n \geq 1$ such that

- $u_1 = s$
- $u_n = t$
- For every $i = 1, \ldots, n-1$ either $u_i \to_{\mathcal{R}} u_{i+1}$ or $u_{i+1} \to_{\mathcal{R}} u_i$ holds.

A general question is: given $\mathcal{R}, s, t$, does $s \leftrightarrow_{\mathcal{R}}^{*} t$ hold?

This is called the **word problem**.

In general the word problem is undecidable.

However, in case $\mathcal{R}$ is terminating and confluent then the word problem is decidable and admits a simple algorithm.

A terminating and confluent TRS is called **complete**.

Now we give a decision procedure for the word problem for complete TRSs.

**Theorem**

If $\mathcal{R}$ is a complete TRS
and $s', t'$ are normal forms of $s, t$,
then $s \leftrightarrow_{\mathcal{R}}^{*} t$ if and only if $s' = t'$.

24

### Proof (deciding the word lemma)

For the proof we need a lemma that is easily proved by induction on the length of the path corresponding to $s \leftrightarrow_{\mathcal{R}}^{*} t$:

**Lemma:**

If $\mathcal{R}$ is confluent and $s \leftrightarrow_{\mathcal{R}}^{*} t$
then a term $q$ exists such that $s \to_{\mathcal{R}}^{*} q$ and $t \to_{\mathcal{R}}^{*} q$.

**Proof** of the theorem:

If $s' = t'$ then $s \to_{\mathcal{R}}^{*} s' = t' \leftarrow_{\mathcal{R}}^{*} t$, hence $s \leftrightarrow_{\mathcal{R}}^{*} t$.

Conversely assume $s \leftrightarrow^*_{\mathcal{R}} t$.

Then $s' \leftarrow^*_{\mathcal{R}} s \leftrightarrow^*_{\mathcal{R}} t \rightarrow^*_{\mathcal{R}} t'$, hence $s' \leftrightarrow^*_{\mathcal{R}} t'$.

According to the lemma a term $q$ exists such that $s' \rightarrow^*_{\mathcal{R}} q$ and $t' \rightarrow^*_{\mathcal{R}} q$.

Since $s', t'$ are normal forms we have $s' = q = t'$. $\square$

---

25

# Example

$\mathcal{R}$ consists of the rule $\mathtt{f}(\mathtt{f}(\mathtt{f}(x))) \rightarrow x$

Does $\mathtt{f}^{17}(\mathtt{0}) \leftrightarrow^*_{\mathcal{R}} \mathtt{f}^{10}(\mathtt{0})$ hold?

We can establish fully automatically that this is not the case:

- check that $\mathcal{R}$ is terminating
- check that $\mathcal{R}$ is locally confluent
- compute the normal form $\mathtt{f}(\mathtt{f}(\mathtt{0}))$ of $\mathtt{f}^{17}(\mathtt{0})$
- compute the normal form $\mathtt{f}(\mathtt{0})$ of $\mathtt{f}^{10}(\mathtt{0})$
- these are different, hence the answer is **No**.

## 3.2 Knuth-Bendix Completion

---

26

# Completing a TRS

Often a TRS $\mathcal{R}$ is not complete, but a complete TRS $\mathcal{R}'$ satisfying

$$\leftrightarrow^*_{\mathcal{R}'} \; = \; \leftrightarrow^*_{\mathcal{R}}$$

can be found in a systematic way.

Finding such a complete TRS is called **(Knuth-Bendix) completion**.

The new complete TRS can be used for the word problem.

Often, instead of an original TRS we only have a set of (unoriented) equations.

---

27

# Knuth-Bendix Completion: ingredients

**Reduction order:** fix a reduction order $\succ$ on terms, i.e.,

- $\succ$ is well-founded
- $\succ$ is stable (if $s \succ t$ then $s\sigma \succ t\sigma$)
- $\succ$ is monotonic (if $s \succ t$ then $\mathtt{f}(\dots, s, \dots) \succ \mathtt{f}(\dots, t, \dots)$)

Then, as we saw before, we have:

If $\ell \succ r$ for every rule $\ell \succ r$ in $\mathcal{R}$, then $\mathrm{SN}(\mathcal{R})$.

Implementations of this algorithm often use recursive path orderings.

**Equations and rules:**

- Initially, $E$ contains the equations we want to complete.
- Initially, $\mathcal{R} = \emptyset$.

---

28

# Knuth-Bendix Completion algorithm

**Repeat** until $E$ is empty:

**1.** Remove an equation $s \mathbin{\texttt{=}} t$ from $E$, and

- add $s \to t$ to $\mathcal{R}$ if $s \succ t$
- add $t \to s$ to $\mathcal{R}$ if $t \succ s$
- give up otherwise

**2.** After adding a new rule $\ell \to r$ to $\mathcal{R}$:

- compute all critical pairs between it and existing rules of $\mathcal{R}$
- compute all critical pairs between the new rule and itself

**3.** For every such critical pair $\langle u, v\rangle$:

- $\mathcal{R}$-rewrite $u$ to a normal form $u'$
- $\mathcal{R}$-rewrite $v$ to a normal form $v'$
- if $u' \neq v'$, then add $u' \mathbin{\texttt{=}} v'$ as an equation to the set $E$

---

29

# Three cases

What can happen in this Knuth-Bendix procedure?

- it fails due to an equation $s = t$ in $E$ for which neither $s \succ t$ nor $t \succ s$ holds;
- it fails since the procedure goes on forever: $E$ gets larger and is never empty;
- it ends with $E$ being empty.

---

30

# Completing the procedure

**Suppose:** the procedure ends with $E$ empty.

Then we really have success:

- $\mathcal{R}$ is terminating since it only contains rules $\ell \to r$ satisfying $\ell \succ r$.
- $\mathcal{R}$ is locally confluent since all critical pairs converge, so $\mathcal{R}$ is complete.
- Convertibility $\leftrightarrow^*_{\mathcal{R}}$ of the resulting $\mathcal{R}$ is equivalent to convertibility of the original $E$ since in the whole procedure $\leftrightarrow^*_{\mathcal{R}\cup E}$ remains invariant.

---

31

# Example

Let $E$ consist of the single equation

$$\mathtt{f}(\mathtt{f}(x)) = \mathtt{g}(x)$$

We want to know: does it follow that $\mathtt{f}(\mathtt{g}(\mathtt{h}(\mathtt{f}(\mathtt{f}((x)))))) = \mathtt{g}(\mathtt{f}(\mathtt{h}(\mathtt{g}(x))))$?

We will use Knuth-Bendix completion using the lexicographic path order defined by $\mathtt{f} \rhd \mathtt{g}$.

Since

$$\mathtt{f}(\mathtt{f}(x)) \succ_{\mathtt{RPO}} \mathtt{g}(x)$$

we add the rule $\mathtt{f}(\mathtt{f}(x)) \to \mathtt{g}(x)$ to the empty TRS $\mathcal{R}$.

Now the critical pair $\langle \mathtt{f}(\mathtt{g}(x)), \mathtt{g}(\mathtt{f}(x)) \rangle$ gives rise to the new equation $\mathtt{f}(\mathtt{g}(x)) = \mathtt{g}(\mathtt{f}(x))$ in $E$.

---

32

Since

$$\mathtt{f}(\mathtt{g}(x)) \succ_{\mathtt{RPO}} \mathtt{g}(\mathtt{f}(x))$$

we add the rule $\mathtt{f}(\mathtt{g}(x)) \to \mathtt{g}(\mathtt{f}(x))$ to the TRS $\mathcal{R}$.

Together with the older rule $\mathtt{f}(\mathtt{f}(x)) \to \mathtt{g}(x)$ we get the critical pair $\langle \mathtt{f}(\mathtt{g}(\mathtt{f}(x))), \mathtt{g}(\mathtt{g}(x)) \rangle$.

Since $\mathtt{g}(\mathtt{g}(x))$ is a normal form and

$$\mathtt{f}(\mathtt{g}(\mathtt{f}(x))) \to \mathtt{g}(\mathtt{f}(\mathtt{f}(x))) \to \mathtt{g}(\mathtt{g}(x))$$

no new equation is added to $E$, and $E$ is empty.

So we end up in the complete TRS $\mathcal{R}$ consisting of the two rules

$$\mathtt{f}(\mathtt{f}(x)) \to \mathtt{g}(x), \quad \mathtt{f}(\mathtt{g}(x)) \to \mathtt{g}(\mathtt{f}(x))$$

having the same convertibility relation as the original equation $\mathtt{f}(\mathtt{f}(x)) = \mathtt{g}(x)$.

---

33

By the rules

$$\mathtt{f}(\mathtt{f}(x)) \to \mathtt{g}(x), \quad \mathtt{f}(\mathtt{g}(x)) \to \mathtt{g}(\mathtt{f}(x))$$

we have:

- $\mathtt{f}(\mathtt{g}(\mathtt{h}(\mathtt{f}(\mathtt{f}((x))))))$
  $\to \mathtt{g}(\mathtt{f}(\mathtt{h}(\mathtt{f}(\mathtt{f}((x))))))$
  $\to \mathtt{g}(\mathtt{f}(\mathtt{h}(\mathtt{g}((x)))))$, a normal form

- $\mathtt{g}(\mathtt{f}(\mathtt{h}(\mathtt{g}(x))))$ is already in normal form

As the two normal forms are the same, the equality $\mathtt{f}(\mathtt{g}(\mathtt{h}(\mathtt{f}(\mathtt{f}((x)))))) = \mathtt{g}(\mathtt{f}(\mathtt{h}(\mathtt{g}(x))))$ indeed follows from $\mathtt{f}(\mathtt{f}(x)) = \mathtt{g}(x)$.

---

34

# Observation

The relation $\leftrightarrow^*_{\mathcal{R}}$ is an equivalence relation.

In a complete TRS, the normal form is a unique representation for the corresponding equivalence class.

This is used for instance in **superposition** (next week).

# 4. Quiz

35

## Quiz

1. Give an example why local confluence does not imply confluence.

2. Determine, using critical pairs, whether the following system is locally confluent:

$$\begin{aligned} \mathtt{f}(\mathtt{g}(x), \mathtt{g}(\mathtt{b})) &\to \mathtt{f}(x, x) \\ \mathtt{g}(\mathtt{a}) &\to \mathtt{b} \\ \mathtt{b} &\to \mathtt{a} \end{aligned}$$

3. Use Knuth-Bendix completion to find a complete TRS with the same $\leftrightarrow_R$ relation as the above TRS.