Exercises Category Theory and Coalgebra
Lecture 2

If you have any questions, email `jrot@cs.ru.nl`. The deadline is 11 February 23:59 PM, CET. Please hand in your work via brightspace. In case you upload hand-written notes, please use a scanner app (instead of photos). Explain your answers! In all exercises, let A be an arbitrary set unless otherwise specified. Given $a \in A$, a^ω denotes the constant stream $(a, a, a, \ldots)$. The last question, marked by **(*)** is an extra hard bonus question.

1. We define the maps $g, h \colon A^\omega \to A^\omega$ by stream differential equations:

$$\begin{aligned} g(\sigma)(0) &= \sigma(0) \qquad & h(\sigma)(0) &= \sigma(0) \\ g(\sigma)' &= h(\sigma) \qquad & h(\sigma)' &= g(\sigma') \end{aligned}$$

   Prove, by defining suitable bisimulations, that for all $\sigma, \tau \in A^\omega$:

$$g(\sigma) = \mathit{zip}(\sigma, \sigma)\,.$$

2. Given functions $f, g \colon A \to A$, we define $\varphi_{f,g} \colon A^\omega \to A^\omega$ by

$$\varphi_{f,g}(\sigma) = (f(\sigma(0)), g(\sigma(1)), f(\sigma(2)), g(\sigma(3)), \ldots)\,.$$

   (a) Characterise $\varphi_{f,g}$ in terms of stream differential equations (initial value and derivative).

   (b) Characterise the map $\mathsf{alt} \colon A^\omega \times A^\omega \to A^\omega$, given by $\mathsf{alt}(\sigma, \tau) = (\sigma(0), \tau(1), \sigma(2), \tau(3), \ldots)$, by stream differential equations.

   (c) Show that for any functions $f, g \colon A \to A$ and streams $\sigma, \tau \in A^\omega$ we have

$$\varphi_{f,g}(\mathsf{alt}(\sigma, \tau)) = \mathsf{alt}(\varphi_{f,f}(\sigma), \varphi_{g,g}(\tau))$$

   using bisimulations.

3. Define a stream system $\langle o, f\rangle \colon A^\omega \times A^\omega \to A \times A^\omega \times A^\omega$ such that the unique homomorphism from $(A^\omega \times A^\omega, \langle o, f\rangle)$ to the final stream system over A coincides with $\mathsf{alt} \colon A^\omega \times A^\omega \to A^\omega$ (see the second exercise). Explain your answer.

4. In the lecture, we only defined bisimulations on the final stream system; we now generalise this to arbitrary stream systems. For a stream system $(X, \langle o, f\rangle)$, we say $R \subseteq X \times X$ is a bisimulation if

   (a) $o(x) = o(y)$, and

   (b) $(f(x), f(y)) \in R$.

   Now suppose $(X, \langle o, f\rangle)$ and $(Y, \langle p, g\rangle)$ are stream systems and $h \colon X \to Y$ is a homomorphism between them. Show that:

   - if $R \subseteq X \times X$ is a bisimulation, then so is $\{(h(x), h(y) \mid (x, y) \in R\}$.
   - if $S \subseteq Y \times Y$ is a bisimulation, then so is $\{(x, y) \mid (h(x), h(y)) \in S\}$.

5. Finish the proof from the lecture that $(A^\omega, \langle i, d\rangle)$ is a final stream system, by showing that if k, l are homomorphisms from an arbitrary stream system $(X, \langle o, t\rangle)$ to $(A^\omega, \langle i, d\rangle)$ then $k = l$.

6. **(*)** Show that the stream system $(A^\omega \times A^\omega, \langle o, t\rangle)$, defined by $o((\sigma, \tau)) = \sigma(0)$ and $t(\sigma, \tau) = (\tau, \sigma')$, for all $(\sigma, \tau) \in A^\omega \times A^\omega$, is final.