Introduction to Category Theory III Natural Transformations

Bart Jacobs

iHub, Radboud University, Nijmegen, The Netherlands bart@cs.ru.nl February 12, 2025

Abstract. Natural transformations are introduced as "maps of functors"

1 Natural transformations

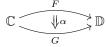
One of the things one learns in doing category theory, is always to ask oneself: what are appropriate morphisms between structures of a given kind. This also applies within category theory. We have seen functors as 'morphisms of categories', which preserve the structure. In a next step we describe natural transfomations as 'morphisms of functors'. Of course, one can go on, and ask what are 'morphisms of natural transformations', but we stop at this point.

Definition 1. Consider two categories \mathbb{C}, \mathbb{D} , and two functors $F, G: \mathbb{C} \rightrightarrows \mathbb{D}$. A **natural transformation** $\alpha: F \to G$ consists of a collection of maps $\alpha_X: FX \to GX$ in \mathbb{D} , indexed by objects $X \in \mathbb{C}$ which satisfy the following naturality condition: for each map $f: X \to Y$ in \mathbb{C} , one has

$$Gf \circ \alpha_X = \alpha_Y \circ Ff$$

in \mathbb{D} . In a diagram:

The maps α_X are called the **components** of the natural transformation α . Some authors use the notation $\alpha \colon F \Rightarrow G$ to indicate that α is a natural transformation from F to G. In a diagram such a natural transformation α is written as a map between two functors:



We describe several examples of natural transformations involving the list functor.

Example 2. We have seen the list functor $\mathcal{L} = (-)^* \colon \mathbf{Sets} \to \mathbf{Sets}$, which assigns to a set X, the free monoid $\mathcal{L}(X) = X^*$ of finite sequences of elements of X. For a function $f \colon X \to Y$ we have $\mathcal{L}(f) = f^* \colon X^* \to Y^*$ by $\langle x_1, \ldots, x_n \rangle \mapsto \langle f(x_1), \ldots, f(x_n) \rangle$.

1. For every set there is a singleton-list function $X \to X^*$. It is written as η , and defined as $\eta_X(x) = \langle x \rangle$, for $x \in X$. We claim that it is a natural transformation in a situation:

$$\mathbf{Sets} \xrightarrow{id} \mathbf{Sets}$$

For a function $f: X \to Y$ we have to check that the following diagram commutes.

$$\begin{split} id\left(X\right) &= X & \xrightarrow{\qquad \eta_X \qquad} X^\star \\ & id\left(f\right) &= f \bigg| & & \bigg| f^\star \\ & id\left(Y\right) &= Y & \xrightarrow{\qquad \eta_Y \qquad} Y^\star \end{split}$$

Commutation is obvious, since for $x \in X$,

$$(f^* \circ \eta_X)(x) = f^*(\langle x \rangle) = \langle f(x) \rangle = \eta_Y(f(x)) = (\eta_Y \circ f)(x).$$

2. For every set X there is a 'reverse' function $rev_X \colon X^* \to X^*$ which reverses the order of elements in a list:

$$rev_X(\langle x_1, \dots, x_n \rangle) = \langle x_n, \dots, x_1 \rangle$$

One sees that the action of rev_X doesn't really depend on the set X: it works in the same way for every other set Y. This uniformity of reversal is expressed by naturality. The maps rev_X form components of a natural transformation $rev: (-)^* \to (-)^*$, in a situation:

$$\mathbf{Sets} \xrightarrow{\mathcal{L}} \mathbf{Sets}$$

For a function $f \colon X \to Y$ we have a naturality diagram,

$$\begin{array}{ccc}
X^{\star} & \xrightarrow{rev_X} & X^{\star} \\
f^{\star} & & \downarrow f^{\star} \\
Y^{\star} & \xrightarrow{rev_Y} & Y^{\star}
\end{array}$$

as is shown by a short computation:

$$\begin{split} \big(f^{\star} \circ \operatorname{rev}_{X}\big)(\langle x_{1}, \dots, x_{n} \rangle) &= f^{\star}(\langle x_{n}, \dots, x_{1} \rangle) \\ &= \langle f(x_{n}), \dots, f(x_{1}) \rangle \\ &= \operatorname{rev}_{Y}(\langle f(x_{1}), \dots, f(x_{n}) \rangle) \\ &= \big(\operatorname{rev}_{Y} \circ f^{\star}\big)(\langle x_{1}, \dots, x_{n} \rangle). \end{split}$$

Computer scientists use the term 'polymorphism' for this uniformity in X of reversal rev_X . For them, polymorphic functions are important because their code does not depend on the type of the input: it means that they don't have to write separate reversal programs for lists of integers, or lists of booleans, or of characters, etcetera.

3. There is a function $\mu \colon \mathcal{L}(\mathcal{L}(X)) \to \mathcal{L}(X)$ that 'flattens' a list of lists to a single list by removing inner brackets. Thus, for $x_{i,j} \in X$,

$$\mu\Big(\langle\langle x_{1,1},\ldots,x_{1,n_1}\rangle,\ldots,\langle x_{m,1},\ldots x_{m,n_m}\rangle\rangle\Big) = \langle x_{1,1},\ldots,x_{m,n_m}\rangle.$$

This flattening works in the same way for each set X and thus forms a natural transformation, of the form:

$$\mathbf{Sets} \xrightarrow{\mathcal{L}^2} \mathbf{Sets}$$

The relevant naturality equation, for $f: X \to Y$ takes the form:

$$\mathcal{L}(f) \circ \mu_X = \mu_Y \circ \mathcal{L}(\mathcal{L}(f)).$$

Prove this equation yourself.

If we view natural transformations as morphisms between functors, we better establish that we get a category in this way.

Lemma 3. Identity maps $FX \to FX$ form an identity natural transformation $id \colon F \to F$. And if $\alpha \colon F \Rightarrow G$ and $\beta \colon G \Rightarrow H$ are natural transformations, then componentwise composition $\beta_X \circ \alpha_X$ forms a composite natural transformation $\beta \circ \alpha \colon F \Rightarrow H$.

Thus for categories \mathbb{C}, \mathbb{D} , there is a category $\mathbb{D}^{\mathbb{C}}$ of functors $\mathbb{C} \to \mathbb{D}$ and natural transformations between them. It is a **functor category**.

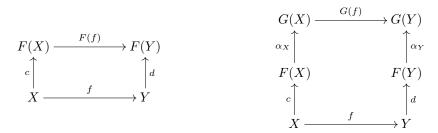
Proposition 4. Let $F, G: \mathbb{C} \to \mathbb{C}$ be two endofunctors with a natural transformation $\alpha: F \Rightarrow G$ between them, in a situation:

$$\mathbb{C} \xrightarrow{F} \mathbb{C}$$

This α gives rise to two functors between categories of (co)algebras, of the form:

$$\begin{array}{cccc} \operatorname{CoAlg}(F) & \xrightarrow{\operatorname{CoAlg}(\alpha)} & \operatorname{CoAlg}(G) & \operatorname{Alg}(G) & \xrightarrow{\operatorname{Alg}(\alpha)} & \operatorname{Alg}(F) \\ \left(X \xrightarrow{c} F(X)\right) & \longmapsto \left(X \xrightarrow{\alpha_X \circ c} G(X)\right) & \left(G(X) \xrightarrow{z} X\right) & \longmapsto \left(F(X) \xrightarrow{a \circ \alpha_X} X\right) \\ f & \longmapsto & f & f & \vdots \\ \end{array}$$

Proof. We shall do the case of coalgebras. What we have to check is that f is a map of F-coalgebras, as on the left below, then it is also a map of G-coalgebras, as on the right.



This clearly works via a simple diagram chase, by including the map F(f) on the right, and using naturality of α .

We conclude this short section with the notion of equivalence of categories. Two categories \mathbb{C},\mathbb{D} are 'isomorphic', if they are isomorphic as objects of \mathbf{Cat} . This means that there are functors $F\colon \mathbb{C}\to \mathbb{D}$ and $G\colon \mathbb{D}\to \mathbb{C}$ satisfying FG=id and GF=id. In this case we write $\mathbb{C}\cong \mathbb{D}$. There is a more general and often more useful way of considering categories as identical, namely 'equivalence' of categories. Then one replaces the identities FG=id and GF=id by 'natural isomorphisms'. We first investigate these.

Lemma 5. A natural transformation $\alpha \colon F \to G$ between functors $F, G \colon \mathbb{C} \rightrightarrows \mathbb{D}$ is an isomorphism in the functor category $\mathbb{D}^{\mathbb{C}}$ if and only if each component $\alpha_X \colon FX \to GX$ is an isomorphism in \mathbb{D} .

Such an invertible natural transformation is called a natural isomorphism.

Thus the natural isomorphisms are the isomorphisms in functor categories.

Proof. The (only if)-part is obvious. And for (if), we have to check that the inverses α_X^{-1} are components of a natural transformation. This is easy: for $f\colon X\to Y$ we have $Gf\circ\alpha_X=\alpha_Y\circ Ff$, and hence $\alpha_Y^{-1}\circ Gf=Ff\circ\alpha_X^{-1}$.

Definition 6. Two categories \mathbb{C}, \mathbb{D} are **equivalent**, if there are functors $F : \mathbb{C} \to \mathbb{D}$ and $G : \mathbb{D} \to \mathbb{C}$ with natural isomorphisms $FG \cong id$ and $GF \cong id$. We write $\mathbb{C} \simeq \mathbb{D}$ for equivalence.