Introduction to Category Theory II
Structure in Categories

Bart Jacobs

iHub, Radboud University, Nijmegen, The Netherlands
bart@cs.ru.nl
February 19, 2025

Abstract. Finite products and coproducts are defined in an arbitrary category, and illustrated in several situations.

1 Finite products and coproducts

We start the investigation of certain structure within categories. We'll first show in this section how such fundamental notions as finite products and coproducts can be described abstractly. This means that we can recognise familiar (or maybe less familiar) constructs for sets, groups or posets as common instances of a general pattern. Products and coproducts are very basic in mathematics and occur throughout.

Definition 1. *An object* 1 *in a category* $\mathbb{C}$ *is called* **final object** *if for each object* $X \in \mathbb{C}$ *there is precisely one map* $X \to 1$.

Dually an **initial object** *in* $\mathbb{C}$ *is an object* 0 *such that for each* $X \in \mathbb{C}$ *there is precisely one map* $0 \to X$.

The unique arrow $X \to 1$ may be written as $!_X : X \to 1$, but often we don't name it at all. We also write $!_X : 0 \to X$. Notice that by uniqueness one has commuting diagrams,

$$\begin{tikzcd} X \arrow[rr, "f"] \arrow[dr] & & Y \arrow[dl] \\ & 1 & \end{tikzcd} \qquad \text{and} \qquad \begin{tikzcd} & 0 \arrow[dl] \arrow[dr] & \\ X \arrow[rr, "f"'] & & Y \end{tikzcd}$$

One has that $1 \in \mathbb{C}$ is final if and only if $1 \in \mathbb{C}^{op}$ is initial. Thus commutation of diagrams as above on the left, implies commutation as on the right. In general, by duality it suffices to prove things only for final objects (or only for initial objects).

For example: two final objects $1, 1' \in \mathbb{C}$ are necessarily isomorphic: there are (unique) maps

$$1' \longrightarrow 1 \qquad \text{and} \qquad 1 \longrightarrow 1'$$

for which the following diagrams commute.

$$\begin{tikzcd} 1 \arrow[rr] \arrow[dr, "id"'] & & 1' \arrow[dl] \\ & 1 & \end{tikzcd} \qquad \text{and} \qquad \begin{tikzcd} 1' \arrow[rr] \arrow[dr, "id"'] & & 1 \arrow[dl] \\ & 1' & \end{tikzcd}$$

Hence by duality, any two initial objects are isomorphic.

A final or initial object may or may not exist in a given category. Its existence is a property of the category, since there can be only one (up-to-isomorphism).

Example 2. 1. In the category $\mathbf{Sets}$ of sets and functions, any singleton set, say $1 = \{0\}$, is final. Indeed for a set X, there is a unique map:

$$X \longrightarrow 1 \qquad \text{by} \qquad x \mapsto 0.$$

Notice that maps $x\colon 1 \to X$ correspond to elements $x \in X$. Thus in the arrow-based formalism of category theory, one still has access to elements of sets.
The empty set $\emptyset$ is initial in $\mathbf{Sets}$: for any set X, there is a unique function $\emptyset \to X$, which does nothing (and there is only one way of doing nothing). If this sounds strange, look at a function $\emptyset \to X$ as a relation $R \subseteq \emptyset \times X$ which is single-valued and total. How many such relations are there?

2. In a preorder (or poset) $(X, \sqsubseteq)$, a final object is a top element $\top \in X$. It satisfies $x \sqsubseteq \top$ for all $x \in X$, as a final object should. And an initial object is a bottom element $\bot \in X$ (for which one has $\bot \leq x$ for all $x \in X$).
3. In the categories $\mathbf{Mon}$ and $\mathbf{Grp}$ of monoids and groups, a singleton $\{\emptyset\}$ (with trivial structure) is both final and initial. That it's final is clear (like in $\mathbf{Sets}$). It's initial because any map $\{\emptyset\} \to M$ must necessarily map $\emptyset$ to the unit $e \in M$, because $\emptyset$ is unit itself.
4. In the category $\mathbf{Cat}$ of categories and functors, one has a final object given by the category **1** consisting of one object and one arrow; the latter is then necessarily the identity. And the category **0** with no objects (and hence no arrows) is initial.

Definition 3. *In a category $\mathbb{C}$, a* **product** *of two object $X, Y \in \mathbb{C}$ consists of a diagram of the form,*

$$X \xleftarrow{\pi_1} X \times Y \xrightarrow{\pi_2} Y$$

which is **universal** *in the following sense: for any other diagram,*

$$X \xleftarrow{f} Z \xrightarrow{g} Y$$

there is a unique map $Z \to X \times Y$, usually written $\langle f, g \rangle$, in,

$$\begin{tikzcd}
X & X \times Y \arrow[l, "\pi_1"'] \arrow[r, "\pi_2"] & Y \\
& Z \arrow[ul, "f"] \arrow[u, dashed, "\langle f,g \rangle"'] \arrow[ur, "g"'] &
\end{tikzcd}$$

Dually, a **coproduct** *of X, Y is a product in $\mathbb{C}^{op}$. That is, a universal diagram,*

$$X \xrightarrow{\kappa_1} X + Y \xleftarrow{\kappa_2} Y$$

such that for any

$$X \xrightarrow{f} Z \xleftarrow{g} Y$$

there is a unique $[f, g]\colon X + Y \to Z$ *in*

$$\begin{tikzcd}
X \arrow[r, "\kappa_1"] \arrow[dr, "f"'] & X + Y \arrow[d, dashed, "{[f,g]}"] & Y \arrow[l, "\kappa_2"'] \arrow[dl, "g"] \\
& Z &
\end{tikzcd}$$

The maps π, π' *are called* **projections** *and* $\langle f, g\rangle$ *is the* **tuple** *of* f *and* g*. Similarly,* κ_1, κ_2 *are* **coprojections** *and* $[f, g]$ *is the* **cotuple***.*

The above definition gives (co)products in terms of universal properties of certain diagrams. Such definitions are typical in category theory. We'll see many more examples. The general shape is

$$\forall x.\, \exists! y.\, \ldots$$

where $\exists! y.\, \ldots$ means: there is a unique y such that ... The uniqueness allows us to do diagrams chases. For example, one has for

$$W \xrightarrow{h} Z, \qquad Z \xrightarrow{f} X, \qquad Z \xrightarrow{g} Y$$

that $\langle f, g\rangle \circ h = \langle f \circ h, g \circ h\rangle$ in

$$\begin{tikzcd}
& & W \arrow[dl, bend right, "h"'] \arrow[dd, dashed, "{\langle f\circ h, g\circ h\rangle}"] \arrow[ddll, bend right, "f\circ h"'] \arrow[ddrr, bend left, "g\circ h"] & & \\
& Z \arrow[dr, dashed, "{\langle f,g\rangle}"] & & & \\
X & & X \times Y \arrow[ll, "\pi_1"] \arrow[rr, "\pi_2"'] & & Y
\end{tikzcd}$$

because the dashed arrow $\langle f \circ h, g \circ h\rangle$ is *unique* with $\pi_1 \circ \langle f \circ h, g \circ h\rangle = f \circ h$ and $\pi_2 \circ \langle f \circ h, g \circ h\rangle = g \circ h$. But one also has,

$$\pi_1 \circ \big(\langle f, g\rangle \circ h\big) = f \circ h \qquad \text{and} \qquad \pi_2 \circ \big(\langle f, g\rangle \circ h\big) = g \circ h$$

Hence we can conclude $\langle f, g\rangle \circ h = \langle f \circ h, g \circ h\rangle$.

With a similar diagram chase argument one can show that $\langle \pi_1, \pi_2\rangle = id$.

These equations

$$\begin{array}{rclcrcl}
\pi_1 \circ \langle f, g\rangle & = & f & \qquad & \langle f, g\rangle \circ h & = & \langle f \circ h, g \circ h\rangle \\
\pi_2 \circ \langle f, g\rangle & = & g & & \langle \pi_1, \pi_2\rangle & = & id
\end{array}$$

characterise products: they yield that the tuple $\langle f, g\rangle\colon Z \to X \times Y$ is unique with $\pi_1 \circ \langle f, g\rangle = f$ and $\pi_2 \circ \langle f, g\rangle = g$. Indeed if $k\colon Z \to X \times Y$ also satisfies $\pi_1 \circ k = f$ and $\pi_2 \circ k = g$, then

$$k = \langle \pi_1, \pi_2\rangle \circ k = \langle \pi \circ k, \pi_2 \circ k\rangle = \langle f, g\rangle.$$

Dually coproducts are characterised by the equations

$$\begin{aligned} [f,g] \circ \kappa_1 &= f \qquad\qquad & h \circ [f,g] &= [h \circ f, h \circ g] \\ [f,g] \circ \kappa_2 &= g & [\kappa_1, \kappa_2] &= id. \end{aligned}$$

It is good to memorise (and recognise) these equations since they are often needed for equational proofs.

Definition 4. *A category $\mathbb{C}$ has (binary)* **products** *(or* **coproducts***) if for each pair of objects $X, Y \in \mathbb{C}$ there is a product (or coproduct) diagram in $\mathbb{C}$. And $\mathbb{C}$ has* **finite** *products (or coproducts) if it has binary products $\times$ plus a final object 1 (or binary coproducts $+$ plus an initial object 0).*

It's time to look at some examples of categories with finite (co)products.

Example 5. 1. The category $\mathbf{Sets}$ has products and coproducts,

$$\begin{tikzcd} & X \times Y = \{(x,y) \mid x \in X \text{ and } y \in Y\} \arrow[dl, "\pi_1"'] \arrow[dr, "\pi_2"] & \\ X & & Y \end{tikzcd}$$

and

$$\begin{tikzcd} X \arrow[dr, "\kappa_1"'] & & Y \arrow[dl, "\kappa_2"] \\ & X + Y = \{(0,x) \mid x \in X\} \cup \{(1,y) \mid y \in Y\} & \end{tikzcd}$$

where the latter is disjoint union $X + Y$ of the sets X, Y.
For functions $f \colon Z \to X$ and $g \colon Z \to Y$ we have an obvious tuple

$$\langle f, g \rangle \colon Z \longrightarrow X \times Y \qquad \text{by} \qquad z \mapsto (f(z), g(z))$$

and for $f \colon X \to Z$ and $g \colon Y \to Z$ there is the cotuple

$$[f,g] \colon X + Y \longrightarrow Z \qquad \text{by} \qquad \begin{cases} (0,x) \mapsto f(x) \\ (1,y) \mapsto g(y) \end{cases}$$

This cotuple is like an "if" or "match" in programming: it can be described as:

$$\begin{aligned} [f,g](w) = {} & \textit{if } w = (0,x) \textit{ then } f(x) \\ & \textit{elif } w = (1,y) \textit{ then } g(y). \end{aligned}$$

2. In a preorder or poset $(X, \sqsubseteq)$ a product of $x, y \in X$ is the meet $x \wedge y \in X$. Indeed the meet has projection maps

$$x \sqsupseteq x \wedge y \sqsubseteq y$$

and if there is an element $z \in X$ with maps

$$x \ \sqsupseteq \ z \ \sqsubseteq \ y$$

then there is a (necessarily unique) map

$$z \ \sqsubseteq x \wedge y$$

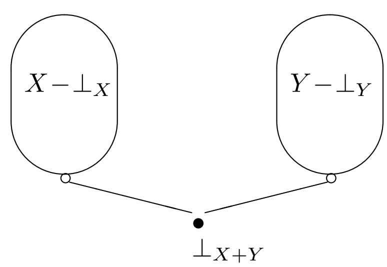
This says that $x \wedge y$ is the **greatest lower bound** 'glb' (or **meet** or **infimum**) of x, y.
Dually the coproduct of x, y—if it exists—is the **join** (or **least upper bound** 'lub' or **supremum**) $x \vee y \in X$.

3. The product category $\mathbb{C} \times \mathbb{D}$ from with its projection functors $\mathbb{C} \leftarrow \mathbb{C} \times \mathbb{D} \rightarrow \mathbb{D}$ forms a product in the category $\mathbf{Cat}$ of categories and functors. The coproduct $\mathbb{C} + \mathbb{D}$ is obtained by taking the disjoint unions of objects and arrows,

$$\mathit{Obj}(\mathbb{C}) + \mathit{Obj}(\mathbb{D}) \qquad \text{and} \qquad \mathit{Arr}(\mathbb{C}) + \mathit{Arr}(\mathbb{D})$$

with domain and codomain in the way that one expects: that is, one only has morphisms between objects that come from the same category.

4. In the category $\mathbf{Sets}_\perp$ of pointed sets, the product of X, Y is the product $X \times Y$ of the underlying sets with the pair $(\perp_X, \perp_Y)$ as base point and obvious projections. The coproduct is more subtle: one takes the disjoint union of $X - \perp_X$ and $Y - \perp_Y$ and adds a new base point $\perp_{X+Y}$, as in



with obvious coprojections.

5. We already saw that in the category of monoids and groups, final and initial object coincide. Also product and coproduct coincide in the sense that they have the same underlying sets. Given two monoids (or groups) $(M, \cdot, u)$ and $(N, \bullet, v)$ we can form the cartesian product of the underlying sets and endow it with a monoids (or group) structure with

multiplication	$(x, y) \star (x', y') = (x \cdot x', y \bullet y')$
unit	(u, v)
[inverse	$(x, y)^{-1} = (x^{-1}, y^{-1})$]

The projection functions $M \xleftarrow{\pi_1} M \times N \xrightarrow{\pi_2} N$ are then homomorphisms and form an appropriate product (in $\mathbf{Mon}$ or $\mathbf{Grp}$). But there are also coprojections $M \xrightarrow{\kappa_1} M \times N \xleftarrow{\kappa_2} N$ involving the unit elements:

$$\kappa_1(x) = (x, v) \qquad \text{and} \qquad \kappa_1(y) = (u, y)$$

These are homomorphisms; for example,

$$\kappa_1(x \cdot x') = (x \cdot x', v) = (x \cdot x', v \bullet v) = (x, v) \star (x, v) = \kappa_1(x) \star \kappa_1(x').$$

And if we have homorphisms $f\colon M \to K$, $g\colon N \to K$, then we can define a cotuple homomorphism $[f, g]\colon M \times N \to K$ by $(x, y) \mapsto f(x) + g(y)$ where $+$ is the multiplication in K. For uniqueness, let $h\colon M \times N \to K$ satisfy $h \circ \kappa_1 = f$ and $h \circ \kappa_2 = g$. Then:

$$h(x, y) \;=\; h((x, v) \star (u, y)) \;=\; h(x, v) + h(u, y) \;=\; f(x) + g(y) \;=\; [f, g](x, y).$$

However, proving that the function $[f, g]\colon M \times N \to K$ is a homorphism of monoids requires commutatitivity of the monoid K. We go through the steps carefully:

$$\begin{array}{rcll}
[f,g]\big((x,y) \star (x',y')\big) & = & [f,g]\big(x \cdot x', y \bullet y'\big) & \\
& = & f(x \cdot x') + g(y \bullet y') & \\
& = & \big(f(x) + f(x')\big) + \big(g(y) + g(y')\big) & \\
& = & \big(f(x) + g(y)\big) + \big(f(x') + g(y')\big) & \text{by commutativity} \\
& = & [f,g](x,y) + [f,g](x',y'). &
\end{array}$$

Hence, things are subtle: the product $M \times N$ is also a coproduct in the category $\mathbf{CMon}$ of *commutative* monoids. In $\mathbf{Mon}$ the coproduct is more complicated, and is given by what is called the “free product”.

In the remainder of this section we’ll consider some abstract properties of products and coproducts. For a start, they are unique up-to-isomorphism. Let’s do this for coproducts; by duality we then get the same result for products. Assume for objects X, Y *two* coproduct diagrams exist:

$$X \xrightarrow{\kappa_1} X + Y \xleftarrow{\kappa_2} Y \qquad \text{and} \qquad X \xrightarrow{\kappa} X \amalg Y \xleftarrow{\kappa'} Y$$

Then we get mediating cotuples

$$X + Y \xrightarrow{[\kappa, \kappa']} X \amalg Y \qquad \text{and} \qquad X \amalg Y \xrightarrow{[\kappa_1, \kappa_2]} X + Y$$

for which the composites

$$X + Y \longrightarrow X \amalg Y \longrightarrow X + Y \qquad \text{and} \qquad X \amalg Y \longrightarrow X + Y \longrightarrow X \amalg Y$$

are identities by uniqueness. Hence the maps $[\kappa, \kappa']$ and $[\kappa_1, \kappa_2]$ are each other’s inverses in an isomorphism $X + Y \cong X \amalg Y$.

Lemma 6. *1. In a category with finite products $(1, \times)$ one has, up-to-isomorphism, that 1 is neutral element for $\times$ and $\times$ is associative:*

$$1 \times X \cong X \qquad \textit{and} \qquad (X \times Y) \times Z \cong X \times (Y \times Z).$$

2. *Dually, in a category with finite coproducts one has*

$$0 + X \cong X \qquad \textit{and} \qquad (X+Y)+Z \cong X+(Y+Z).$$

Proof. We only do $1 \times X \cong X$ and leave the rest to the reader. There are maps

$$1 \times X \xrightarrow{\pi_2} X \qquad \text{and} \qquad X \xrightarrow{\langle !, id \rangle} 1 \times X$$

which are each other's inverses:

$$\pi_2 \circ \langle !, id \rangle = id \ : \ X \longrightarrow X$$

and

$$\langle !, id \rangle \circ \pi_2 = \langle ! \circ \pi_2, \pi_2 \rangle \overset{(*)}{=} \langle \pi_1, \pi_2 \rangle = id \ : \ 1 \times X \longrightarrow 1 \times X$$

where the equality $(*)$ holds because the following diagrams commutes

$$\begin{tikzcd} 1 \times X \arrow[r, "\pi_2"] \arrow[dr, "\pi_1"'] & X \arrow[d, "!"] \\ & 1 \end{tikzcd}$$

since 1 is final object. $\square$

This lemma shows that $(1, \times)$ and $(0, +)$ behave almost like a monoid. The proper notion here is that of a 'monoidal category'. It generalizes the notion of monoid; we refer to the literature for details.

Next we show that chosing products and coproducts is functorial.

Lemma 7. *If $\mathbb{C}$ is a category with products $\times$, then the assignment $(X, Y) \mapsto X \times Y$ extends to a* **product functor** $\times \colon \mathbb{C} \times \mathbb{C} \to \mathbb{C}$.

And if $\mathbb{C}$ has coproducts $+$, then we get a **coproduct functor** $+ \colon \mathbb{C} \times \mathbb{C} \to \mathbb{C}$.

Proof. For maps $X \xrightarrow{f} U$ and $Y \xrightarrow{g} V$, we define $\times$ on morphisms as

$$f \times g = \langle f \circ \pi_1, g \circ \pi_2 \rangle \ : \ X \times Y \longrightarrow U \times V$$

We get a functor in this way because,

$$id \times id = \langle id \circ \pi, id \circ \pi_2 \rangle = \langle \pi, \pi_2 \rangle = id.$$

and

$$\begin{aligned} (f \circ h) \times (g \circ k) &= \langle f \circ h \circ \pi_1, g \circ k \circ \pi_2 \rangle \\ &= \langle f \circ \pi_1 \circ \langle h \circ \pi_1, k \circ \pi_2 \rangle, \ g \circ \pi_2 \circ \langle h \circ \pi_1, k \circ \pi_2 \rangle \rangle \\ &= \langle f \circ \pi_1, g \circ \pi_2 \rangle \ \circ \ \langle h \circ \pi_1, k \circ \pi_2 \rangle \\ &= (f \times g) \circ h \times k). \end{aligned}$$

Similarly, for $X \xrightarrow{f} U$ and $Y \xrightarrow{g} V$ one puts

$$f + g \coloneqq [\kappa_1 \circ f, \kappa_2 \circ g] \ : \ X + Y \longrightarrow U + V.$$

❏

It is time to introduce coalgebras and algebras.

Definition 8. *Let $\mathbb{C}$ be a category with an (endo)functor $F\colon \mathbb{C} \to \mathbb{C}$.*

1. *An F-**coalgebra** is a morphism in $\mathbb{C}$ of the form $c\colon X \to F(X)$. The object X is often called the **carrier** of the coalgebra and the map c is the **transition map**.*
   *A **map of coalgebras**, from $c\colon X \to F(X)$ to $d\colon Y \to F(Y)$ is a morphism $f\colon X \to Y$ in $\mathbb{C}$ between the carriers that commutes with the transition maps. The latter means that the following diagram commutes.*

$$\begin{tikzcd} F(X) \arrow[r, "F(f)"] & F(Y) \\ X \arrow[u, "c"] \arrow[r, "f"'] & Y \arrow[u, "d"'] \end{tikzcd}$$

   We write $\mathrm{CoAlg}(F)$ for the category of coalgebras of F and their maps (or homomorphisms).
2. *An F-**algebra** is a morphism in $\mathbb{C}$ of the form $a\colon F(X) \to X$, where X is again called the **carrier** and a the **structure map**. A **map of algebras**, from $a\colon F(X) \to X$ to $b\colon F(Y) \to Y$ is a morphism $f\colon X \to Y$ in $\mathbb{C}$ with:*

$$\begin{tikzcd} F(X) \arrow[r, "F(f)"] \arrow[d, "a"'] & F(Y) \arrow[d, "b"] \\ X \arrow[r, "f"'] & Y \end{tikzcd}$$

   The resulting category will be written as $\mathrm{Alg}(F)$.

One needs to check a few things to be sure that *CoAlg*(F) and *Alg*(F) are categories. They come with obvious forgetful functors:

$$\begin{tikzcd} \mathit{CoAlg}(F) \arrow[dr] & & \mathit{Alg}(F) \arrow[dl] \\ & \mathbb{C} & \end{tikzcd}$$

We often simply write algebra or coalgebra, instead of F-algebra or F-coalgebra, when the functor F is clear from the context.

We shall see many examples of (co)algebras. For now, consider a monoid $(M, \cdot, u)$. Its operations can be combined into an algebra of the form $1 + (M \times M) \to M$, for the functor $F(X) = 1 + (X \times X)$ on $\mathbf{Sets}$. Check yourself what an algebra map is.

Similarly, one can have a coalgebra $A \to \mathbb{N} \times A$ for streams, using the functor $F(X) = \mathbb{N} \times X$. Check again what the maps of coalgebras are.

In these categories of algebras and coalgebras one can study initial and final objects. We shall in particular be interested in algebras in *Alg*(F) which are initial, among all algebras, and in coalgebras in *CoAlg*(F), which are final among all coalgebras.

We shall see that the initial algebra of the functor $X \mapsto 1 + X$ on $\mathbf{Sets}$ has the natural numbers as carrier, with the cotuple of zero and successor as algebra map. In terms of these (co-)algebras one can describe many more data types than just natural numbers.

But first we mention the following basic result, known as Lambek's Lemma, after Jim Lambek.

Lemma 9. *For an algebra $a\colon F(A) \to A$ which is initial in $\mathrm{Alg}(F)$, the structure map a isomorphism $a\colon F(A) \xrightarrow{\cong} A$.*

Similarly, a final coalgebra has an isomorphism as transition map.

Thus an initial algebra is a fixed point $F(A) \xrightarrow{\cong} A$ of the functor F. Similarly, a final coalgebra is a fixed point $Z \xrightarrow{\cong} F(Z)$.

Proof. We shall do the proof for coalgebras. Let $\zeta\colon Z \to F(Z)$ be final. We can form a new coalgebra $F(\zeta)\colon F(Z) \to F(F(Z))$, with $F(Z)$ as carrier. The morphism ζ is then a map of coalgebras, since the following diagram obviously commutes.

$$\begin{tikzcd}
F(Z) \arrow[r, "F(\zeta)"] & F(F(Z)) \\
Z \arrow[u, "\zeta"] \arrow[r, "\zeta"'] & F(Z) \arrow[u, "F(\zeta)"']
\end{tikzcd}$$

In the other direction, by finality in $\mathit{CoAlg}(F)$, there is a unique map of coalgebras $f\colon F(Z) \to Z$ in:

$$\begin{tikzcd}
F(F(Z)) \arrow[r, "F(f)"] & F(Z) \\
F(Z) \arrow[u, "F(\zeta)"] \arrow[r, "f"'] & Z \arrow[u, "\zeta"']
\end{tikzcd}$$

We claim that f is the inverse of ζ in $\mathbb{C}$. We note that $f \circ \zeta\colon Z \to Z$ is a map of coalgebras, via an easy diagram chase in:

$$\begin{tikzcd}
F(Z) \arrow[r, "F(\zeta)"] & F(F(Z)) \arrow[r, "F(f)"] & F(Z) \\
Z \arrow[u, "\zeta"] \arrow[r, "\zeta"'] & F(Z) \arrow[u, "F(\zeta)"] \arrow[r, "f"'] & Z \arrow[u, "\zeta"']
\end{tikzcd}$$

Commutation of the outer diagram is obtained by pasting the previous two diagrams together. We thus have that $f \circ \zeta\colon \big(Z \xrightarrow{\zeta} F(Z)\big) \to \big(Z \xrightarrow{\zeta} F(Z)\big)$ is a map of coalgebras. There is another such map of coalgebras, namely the identity on Z. Since ζ is final, there is precisely one coalgebra map $\big(Z \xrightarrow{\zeta} F(Z)\big) \to \big(Z \xrightarrow{\zeta} F(Z)\big)$. We conclude that $f \circ \zeta = id_Z$. For the other direction we use the first diagram above, plus the fact that F is a functor, in:

$$\zeta \circ f = F(f) \circ F(\zeta) = F(f \circ \zeta) = F(id_Z) = id_{F(Z)}. \qquad ❑$$

We include a classic illustration, namely of streams as final coalgebra.

Example 10. For an arbitrary set B consider the functor $F(X) = B \times X$ on the category $\mathbf{Sets}$. We write $B^{\mathbb{N}}$ for the set of streams (infinite sequences) of elements from the set B. Such a stream can be described equivalently as a function $s\colon \mathbb{N} \to B$, as a collection $(b_n)_{n\in\mathbb{N}}$, or as a equence $(b_0, b_1, b_2, \ldots)$. This set of streams $B^{\mathbb{N}}$ carries an F-coalgebra given by a tuple of head and tail operations:

$$B^{\mathbb{N}} \xrightarrow{\langle \mathit{hd}, \mathit{tl}\rangle} B \times B^{\mathbb{N}} \qquad \text{where} \qquad \begin{cases} \mathit{hd}\big(b_0, b_1, \ldots\big) = b_0 \\ \mathit{tl}\big(b_0, b_1, \ldots\big) = (b_1, b_2, \ldots) \end{cases}$$

We claim that this coalgebra is final, that is, a final object in the category $\mathit{CoAlg}(F)$.

To see this, consider an arbitrary coalgebra $c\colon X \to B \times X$. It is a map into a product, so this coalgebra c can be written as tuple. Let's write $h = \pi_1 \circ c$ and $t = \pi_2 \circ c$, so that $c = \langle h, t\rangle$. We need to construct a (unique) coalgebra map $f\colon X \to B^{\mathbb{N}}$. For an arbitrary element $x \in X$ we take:

$$f(x) \coloneqq \big(h(x),\, h(t(x)),\, h(t(t(x))),\, \ldots\,\big).$$

Thus, as a function $f(x)\colon \mathbb{N} \to B$ we can write $f(x)(n) = t\big(h^n(x)\big)$, where $h^n = h \circ \cdots \circ h$, with $h^0 = \mathit{id}$.

First we need to show that $f\colon X \to B^{\mathbb{N}}$ is a map of coalgebra, that is, that it satisfies $F(f) \circ \langle h, t\rangle = \langle \mathit{hd}, \mathit{tl}\rangle \circ f$, where $F(f) = \mathit{id}_B \times f$. This amounts to two equations:

$$\mathit{hd} \circ f = f \qquad \text{and} \qquad \mathit{tl} \circ f = f \circ t.$$

The first one obviously holds. We elaborate the second one:

$$\begin{aligned} \big(\mathit{tl} \circ f\big)(x) &= \mathit{tl}\big(f(x)\big) \\ &= \mathit{tl}\big(h(x),\, h(t(x)),\, h(t(t(x))),\, \ldots\,\big) \\ &= \big(h(t(x)),\, h(t(t(x))),\, \ldots\,\big) \\ &= f\big(t(x)\big) \\ &= \big(f \circ t\big)(x). \end{aligned}$$

We still need to prove uniqueness, of $f\colon X \to B^{\mathbb{N}}$ as coalgebra map. Suppose that $g\colon X \to B^{\mathbb{N}}$ is also a map of coalgebras, satisfying, like in the above two equations $\mathit{hd} \circ g = h$ and $\mathit{tl} \circ g = g \circ t$. Then one can show by induction on $n \in \mathbb{N}$ that $g(x)(n) = f(x)(n)$, for each $x \in X$. Indeed:

$$\begin{aligned} g(x)(0) &= \mathit{hd}\big(g(x)\big) \\ &= t(x) \\ &= f(x)(0). \end{aligned} \qquad\qquad \begin{aligned} g(x)(n+1) &= \mathit{tl}\big(g(x)\big)(n) \\ &= g\big(t(x)\big)(n) \\ &\overset{\text{(IH)}}{=} f\big(t(x)\big)(n) \\ &= f(x)(n+1). \end{aligned}$$

This shows that $g = f$, and thus that f is unique.

We conclude that the map $\langle \mathit{hd}, \mathit{tl}\rangle\colon B^{\mathbb{N}} \to B \times B^{\mathbb{N}}$ is a final coalgebra. The above lemma of Lambek then says that it is an isomorphism. This is easy to see, with inverse adding an element at the front of a stream.