3

Model Checking

CTL Model Checking

[Baier & Katoen, Chapter 6.4]

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Topic

The CTL model-checking problem:

Given:

- A finite transition system TS
- CTL state-formula Φ

Decide whether $TS \models \Phi$, and if $TS \not\models \Phi$ provide a counterexample[1]

[1] CTL counterexamples are outside the scope of this course.

Overview

Overview

1. Recap: Computational Tree Logic
2. Existential Normal Form
3. Basic CTL Model-Checking Algorithm
4. Model Checking Existential Until and Always
5. Complexity Considerations
6. Summary

CTL Syntax

Definition: Syntax Computation Tree Logic

- CTL state-formulas with $a \in AP$ obey the grammar:

$$\Phi \ ::= \ \text{true} \ \Big| \ a \ \Big| \ \Phi_1 \wedge \Phi_2 \ \Big| \ \neg\Phi \ \Big| \ \exists\varphi \ \Big| \ \forall\varphi$$

- and φ is a path-formula formed by the grammar:

$$\varphi \ ::= \ \bigcirc\, \Phi \ \Big| \ \Phi_1 \,\mathsf{U}\, \Phi_2 .$$

Example CTL State-formulas

- $\forall\Box\exists\bigcirc\, a$
- $\exists(\forall\Box a) \,\mathsf{U}\, b$

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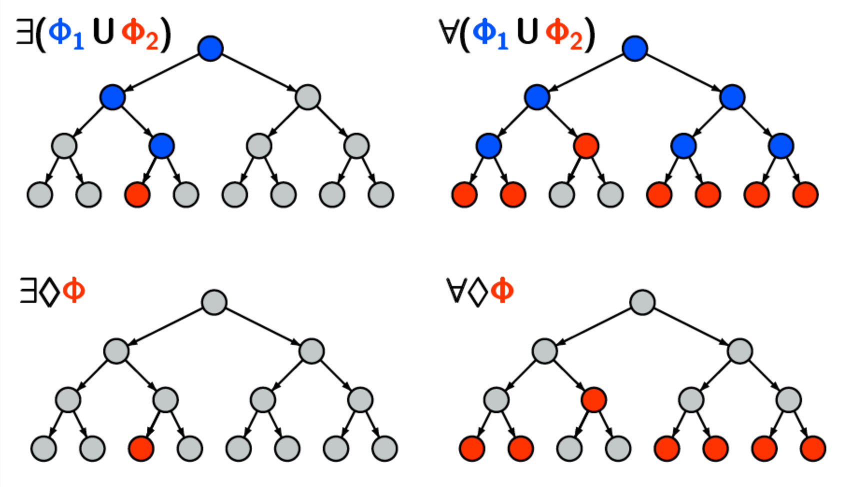
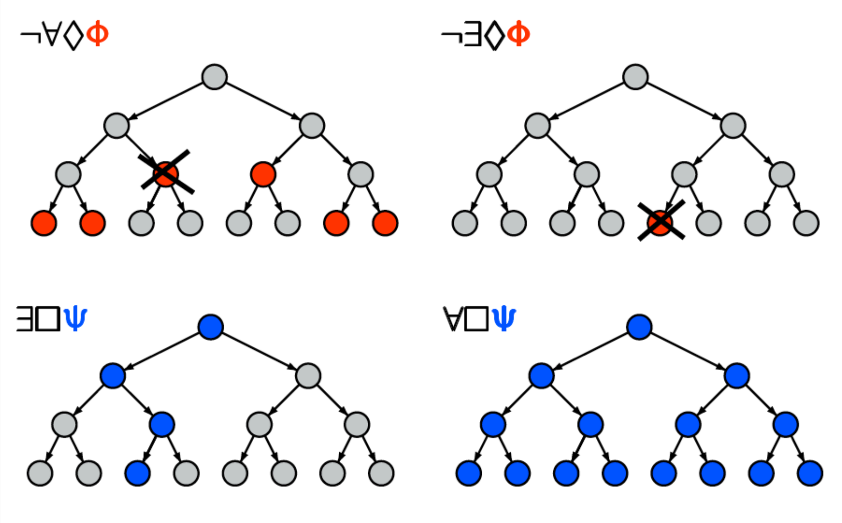
Intuition

- $s \models \forall\varphi$ if all paths starting in s fulfill φ
- $s \models \exists\varphi$ if some path starting in s fulfill φ

Intuitive CTL Semantics



Intuitive CTL Semantics



CTL Semantics

Define a satisfaction relation for CTL-formulas over AP for a given transition system TS without terminal states.

Two parts:

- Interpretation of state-formulas over states of TS
- Interpretation of path-formulas over paths of TS

CTL Semantics (1)

Notation
$TS, s \models \Phi$ if and only if state-formula Φ holds in state s of transition system TS. As TS is known from the context we simply write $s \models \Phi$.

Definition: Satisfaction relation for CTL state-formulas
The satisfaction relation $\models$ is defined for CTL state-formulas by:

$$
\begin{array}{lll}
s \models a & \text{iff} & a \in L(s) \\
s \models \neg \Phi & \text{iff} & \text{not } (s \models \Phi) \\
s \models \Phi \wedge \Psi & \text{iff} & (s \models \Phi) \text{ and } (s \models \Psi) \\
s \models \exists \varphi & \text{iff} & \text{there exists } \pi \in \mathit{Paths}(s).\ \pi \models \varphi \\
s \models \forall \varphi & \text{iff} & \text{for all } \pi \in \mathit{Paths}(s).\ \pi \models \varphi
\end{array}
$$

where the semantics of CTL path-formulas is defined on the next slide.

CTL Semantics (2)

Definition: satisfaction relation for CTL path-formulas
Given path π and CTL path-formula φ, the satisfaction relation $\models$ where $\pi \models \varphi$ if and only if path π satisfies φ is defined as follows:

$$\pi \models \bigcirc \Phi \qquad \text{iff } \pi[1] \models \Phi$$

$$\pi \models \Phi \cup \Psi \qquad \text{iff } (\exists\, j \geq 0.\, \pi[j] \models \Psi \text{ and } (\forall\, 0 \leq i < j.\, \pi[i] \models \Phi))$$

where $\pi[i]$ denotes the state s_i in the path $\pi = s_0\, s_1\, s_2 \ldots$.

Transition System Semantics

- For CTL-state-formula Φ, the satisfaction set $Sat(\Phi)$ is defined by:

$$Sat(\Phi) \ = \ \{ \, s \in S \mid s \models \Phi \, \}$$

- TS satisfies CTL-formula Φ iff Φ holds in all its initial states:

$$TS \models \Phi \quad \text{if and only if} \quad \forall s_0 \in I. \, s_0 \models \Phi$$

- Point of attention: $TS \not\models \Phi$ is not equivalent to $TS \models \neg\Phi$

  because of several initial states, e.g., $s_0 \models \exists\Box\Phi$ and $s_0' \not\models \exists\Box\Phi$

Overview

Existential Normal Form

Definition: existential normal form
A CTL formula is in existential normal form (ENF) if it is of the form:

$$\Phi \quad ::= \quad \text{true} \;\Big|\; a \;\Big|\; \Phi_1 \wedge \Phi_2 \;\Big|\; \neg\Phi \;\Big|\; \exists\bigcirc\Phi \;\Big|\; \exists(\Phi_1 \,\mathsf{U}\, \Phi_2) \;\Big|\; \exists\Box\Phi$$

Only existentially quantified temporal modalities $\bigcirc$, U and $\Box$.

For each CTL formula, there exists an equivalent CTL formula in ENF.

Existential Normal Form

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Only existentially quantified temporal modalities $\bigcirc$, U and $\Box$.

For each CTL formula, there exists an equivalent CTL formula in ENF.

Proof.
Universally quantified temporal modalities can be transformed as follows:

$$\begin{aligned} \forall\bigcirc\Phi &\equiv \neg\exists\bigcirc\neg\Phi \\ \forall(\Phi \, \mathsf{U} \, \Psi) &\equiv \neg\exists(\neg\Psi \, \mathsf{U} \, (\neg\Phi \wedge \neg\Psi)) \;\wedge\; \neg\exists\Box\neg\Psi \end{aligned}$$

□

Overview

Basic Idea

- How to check whether TS satisfies CTL formula Ψ?
  - convert the formula Ψ into the equivalent Φ in ENF
  - compute recursively the set $Sat(\Phi) = \{ s \in S \mid s \models \Phi \}$
  - $TS \models \Phi$ if and only if each initial state of TS belongs to $Sat(\Phi)$

Basic Idea

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  - $TS \models \Phi$ if and only if each initial state of TS belongs to $Sat(\Phi)$

$\Psi_1 \wedge \Psi_2$

$\wedge$ i

Ψ_1 Ψ_2 $i{+}1$

- Recursive bottom-up computation of $Sat(\Phi)$:
  - consider the parse tree of Φ
  - start to compute $Sat(a_i)$, for all leafs in the parse tree
  - then go one level up in the tree and determine $Sat(\cdot)$ for these nodes

$$\text{e.g.,:} \quad Sat(\underbrace{\Psi_1 \wedge \Psi_2}_{\text{node at level } i}) = Sat(\underbrace{\Psi_1}_{\substack{\text{node at} \\ \text{level } i+1}}) \cap Sat(\underbrace{\Psi_2}_{\substack{\text{node at} \\ \text{level } i+1}})$$

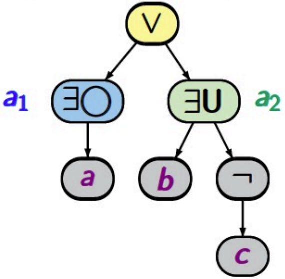
  - then go one level up and determine $Sat(\cdot)$ of these nodes
  - and so on....... until the root is treated, i.e., $Sat(\Phi)$ is computed

- Check whether $I \subseteq Sat(\Phi)$.

Basic Algorithm

$$\Phi = \underbrace{\exists \bigcirc a}_{\Phi_1} \vee \underbrace{\exists (b \, \mathsf{U} \, \neg c)}_{\Phi_2}$$

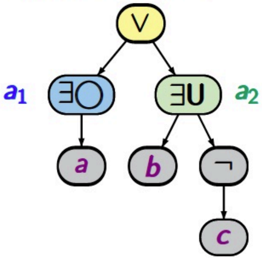
syntax tree for Φ



Basic Algorithm

$$\Phi = \underbrace{\exists\bigcirc a}_{\Phi_1} \vee \underbrace{\exists(b \,\mathsf{U}\, \neg c)}_{\Phi_2} \rightsquigarrow a_1 \vee a_2$$

syntax tree for Φ



processed in bottom-up fashion

compute $Sat(a), Sat(b), Sat(c)$

$Sat(\Phi_1) = \ldots = Sat(a_1)$

$Sat(\neg c) = S \setminus Sat(c)$

$Sat(\Phi_2) = \ldots = Sat(a_2)$

replace Φ_1 with a_1

replace Φ_2 with a_2

$Sat(\Phi) = Sat(a_1) \cup Sat(a_2)$

Basic Algorithm

$$
\begin{aligned}
Sat(\text{true}) &= S \\
Sat(a) &= \{\, s \in S \mid a \in L(s) \,\} \\
Sat(\neg\Phi) &= S \setminus Sat(\Phi) \\
Sat(\Phi \wedge \Psi) &= Sat(\Phi) \cap Sat(\Psi) \\
Sat(\exists \bigcirc \Phi) &= \{\, s \in S \mid Post(s) \cap Sat(\Phi) \neq \varnothing \,\} \\
Sat(\exists \Box \Phi) &= \ldots\ldots \\
Sat(\exists(\Phi \,\mathsf{U}\, \Psi)) &= \ldots\ldots
\end{aligned}
$$

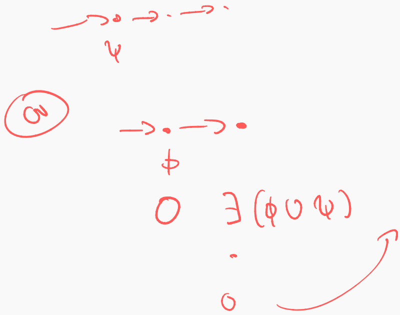
Treatment of $\exists \Box \Phi$ and $\exists(\Phi \,\mathsf{U}\, \Psi)$: via a fixed-point computation

Overview

Expansion law:

$$\exists(\Phi \,\mathsf{U}\, \Psi) \;\equiv\; \Psi \;\vee\; \big(\Phi \;\wedge\; \exists\bigcirc\exists(\Phi \,\mathsf{U}\, \Psi)\big)$$

In fact, $\exists(\Phi \,\mathsf{U}\, \Psi)$ is the smallest solution of this recursive equation



Characterization of *Sat* for $\exists U$

Expansion law:

$$\exists(\Phi \, U \, \Psi) \;\equiv\; \Psi \;\vee\; \big(\Phi \;\wedge\; \exists\bigcirc\exists(\Phi \, U \, \Psi)\big)$$

In fact, $\exists(\Phi \, U \, \Psi)$ is the smallest solution of this recursive equation

$Sat(\exists(\Phi \, U \, \Psi))$ is the smallest subset T of S, such that:

(1) $Sat(\Psi) \subseteq T$ and (2) $\big(s \in Sat(\Phi)$ and $Post(s) \cap T \neq \varnothing\big) \;\Rightarrow\; s \in T$.

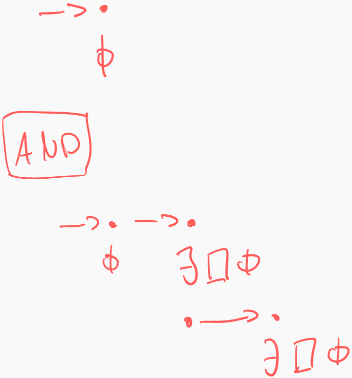
That is, $T = Sat(\exists(\Phi \, U \, \Psi))$ is the smallest fixed point of the (higher-order) function $\Omega : 2^S \to 2^S$ given by:

$$\Omega(T) \;=\; Sat(\Psi) \cup \{\, s \in Sat(\Phi) \mid Post(s) \cap T \neq \varnothing \,\}$$

Expansion law:

$$\exists\Box\Phi \equiv \Phi \wedge \exists\bigcirc\exists\Box\Phi$$

In fact, $\exists\Box\Phi$ is the largest solution of this recursive equation

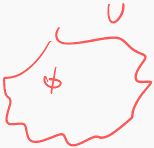


Characterization of *Sat* for $\exists\square$

Expansion law:

$$\exists\square\Phi \;\equiv\; \Phi \;\wedge\; \exists\bigcirc\,\exists\square\Phi$$

In fact, $\exists\square\Phi$ is the largest solution of this recursive equation



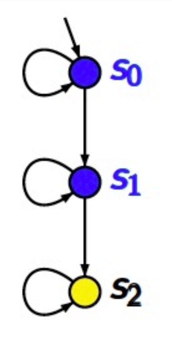
$Sat(\exists\square\Phi)$ is the largest subset V of S, such that:

$$(1)\ V \subseteq Sat(\Phi) \quad \text{and} \quad (2)\ s \in V \text{ implies } Post(s) \cap V \neq \varnothing.$$

That is, $V = Sat(\exists\square\Phi)$ is the largest fixed point of the (higher-order) function $\Omega : 2^S \to 2^S$ given by:

$$\Omega(V) \;=\; \{\, s \in Sat(\Phi) \mid Post(s) \;\cap\; V \neq \varnothing \,\}$$

Example for $\exists\Box a$

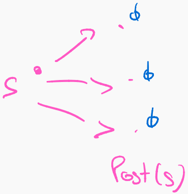


$V = \{ s_0 \}$ satisfies the condition:

$V \subseteq \{ s \in Sat(a) \mid Post(s) \cap V \neq \varnothing \}$

But $V \subsetneq Sat(\exists\Box a) = \{ s_0, s_1 \}$

Universally Quantified Formulas



- $Sat(\forall\bigcirc\,\Phi) \;=\; \{\, s \in S \mid Post(s) \subseteq Sat(\Phi)\,\}$

- $Sat(\forall\Box\Phi)$ equals the largest set T of states such that:

$$T \;\subseteq\; \{\, s \in Sat(\Phi) \mid Post(s) \subseteq T\,\}$$

- $Sat(\forall(\Phi \,\mathsf{U}\, \Psi))$ is the smallest set T of states such that:

$$Sat(\Psi) \cup \{\, s \in Sat(\Phi) \mid Post(s) \subseteq T\,\} \;\subseteq\; T$$

Model Checking ∃U

$Sat(\exists(\Phi \,\mathsf{U}\, \Psi))$ is the smallest subset T of S, such that:

$Sat(\phi)$ $Sat(\psi)$ T

(1) $Sat(\Psi) \subseteq T$ and (2) $\big(s \in Sat(\Phi) \text{ and } Post(s) \cap T \neq \varnothing\big) \;\Rightarrow\; s \in T$.

- This suggests to compute $Sat(\exists(\Phi \,\mathsf{U}\, \Psi))$ iteratively:

  $$T_0 \;=\; Sat(\Psi) \quad\text{and}\quad T_{i+1} \;=\; T_i \;\cup\; \{\, s \in Sat(\Phi) \mid Post(s) \cap T_i \neq \varnothing \,\}$$

- T_i = states that can reach a Ψ-state in at most i steps via Φ states

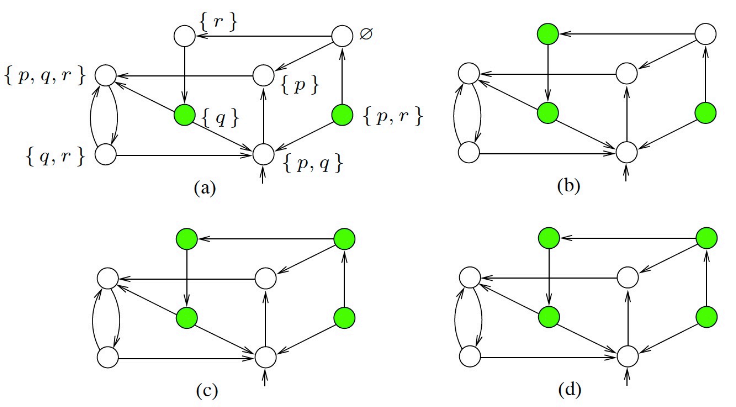
- By induction it follows:

  $$T_0 \subsetneq T_1 \subsetneq \ldots \subsetneq T_j = T_{j+1} = Sat(\exists(\Phi \,\mathsf{U}\, \Psi))$$

- As TS is finite, we have $T_{j+1} = T_j = Sat(\exists(\Phi \,\mathsf{U}\, \Psi))$ for some j.

Example

Computing true $\mathsf{U}\big((p{=}r) \wedge (p \neq q)\big)$ over $AP = \{p, q, r\}$:



Algorithm for $\exists(\Phi_1 \, \mathsf{U} \, \Phi_2)$

Compute $Sat\big(\exists(\Phi_1 \, \mathsf{U} \, \Phi_2)\big)$ by a linear-time enumerative backward search

$T := Sat(\Phi_2)$ ⟵ $\boxed{\text{collects all states } s \models \exists(\Phi_1 \, \mathsf{U} \, \Phi_2)}$

$E := Sat(\Phi_2)$ ⟵ $\boxed{\text{set of states still to be expanded}}$

```
WHILE E ≠ ∅ DO
    select a state s' ∈ E and remove s' from E
    FOR ALL s ∈ Pre(s') DO
        IF s ∈ Sat(Φ1) \ T THEN add s to T and E FI
    OD
OD
return T
```

Model Checking $\exists\Box$

$Sat(\exists\Box\Phi)$ is the largest subset V of S, such that:

$$(1)\ V \subseteq Sat(\Phi) \quad \text{and} \quad (2)\ s \in V \text{ implies } Post(s) \cap V \neq \varnothing.$$

- This suggests to compute $Sat(\exists\Box\Phi)$ iteratively:

$$V_0 \ = \ Sat(\Phi) \quad \text{and} \quad V_{i+1} \ = \ \{\, s \in V_i \mid Post(s) \ \cap \ V_i \neq \varnothing \,\}$$

- V_i = states that have some Φ-path of at least i transitions

- By induction it follows:

$$V_0 \supsetneq V_1 \supsetneq \ldots \supsetneq V_j = V_{j+1} = Sat(\exists\Box\Phi)$$

- As TS is finite, we have $V_{j+1} = V_j = Sat(\exists\Box\Phi)$ for some j.

Algorithm for $\exists\Box\Phi$

Compute $Sat(\exists\Box\Phi)$ by an enumerative backward search

$T := Sat(\Phi) \leftarrow$ organizes the candidates for $s \models \exists\Box\Phi$

$E := S \setminus T \leftarrow$ set of states to be expanded

```
WHILE E ≠ ∅ DO
   pick a state s' ∈ E and remove s' from E
   FOR ALL s ∈ Pre(s') DO
    IF s ∈ T and Post(s) ∩ T = ∅ THEN
          remove s from T and add s to E
    FI
OD
return T
```

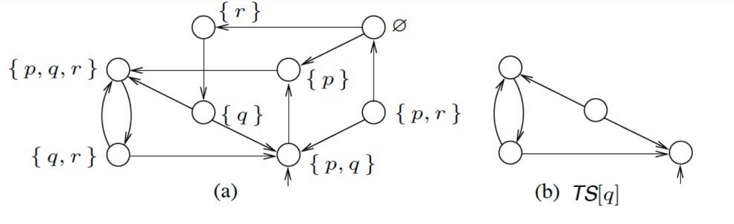
naïve implementation:
quadratic time complexity

An Alternative SCC-Based Algorithm

An SCC-based algorithm for determining $Sat(\exists\square\Phi)$:

1. Eliminate all states $s \notin Sat(\Phi)$:
   - determine $TS[\Phi] = (S', Act, \rightarrow', I', AP, L')$ with $S' = Sat(\Phi)$, $\rightarrow' = \rightarrow \cap (S' \times Act \times S')$, $I' = I \cap S'$, and $L'(s) = L(s)$ for $s \in S'$
   - Why? all removed states refute $\exists\square\Phi$ and thus can be safely removed

Example: $Sat(\exists\square q)$



(a) (b) $TS[q]$

An Alternative SCC-Based Algorithm

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   - Why? all removed states refute $\exists\Box\Phi$ and thus can be safely removed

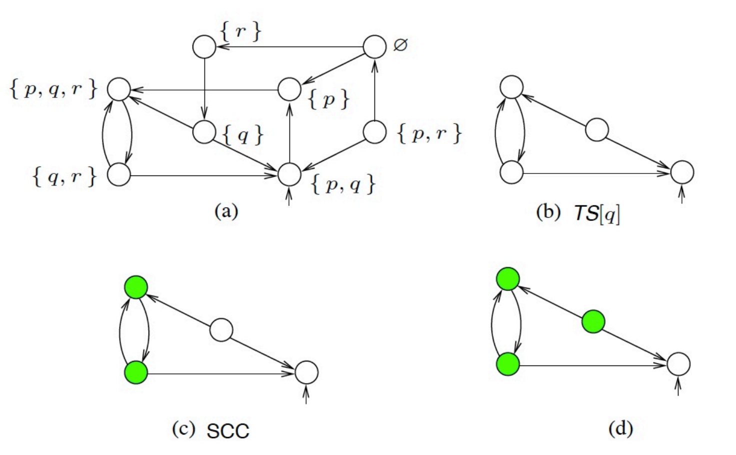
2. Determine all non-trivial strongly connected components in $TS[\Phi]$
   - non-trivial SCC = maximal, connected sub-graph with > 0 transition

   $\Rightarrow$ any state in such SCC satisfies $\exists\Box\Phi$

3. $s \models \exists\Box\Phi$ is equivalent to "an SCC in $TS[\Phi]$ is reachable from s"
   - this search can be done in a backward manner in linear time

Example

Determining $Sat(\exists\Box q)$ using the SCC-based algorithm



CTL Model-Checking Algorithm

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\begin{array}{rcl}
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Sat(\exists\bigcirc \Phi) & = & \{\, s \in S \mid Post(s) \cap Sat(\Phi) \neq \varnothing \,\} \\
Sat(\exists\Box\Phi) & = & \bigcap_{n\geq 0} V_n \text{ where} \\
 & & V_0 = Sat(\Phi) \\
 & & V_{n+1} = \{\, s \in V_n \mid Post(s) \cap V_n \neq \varnothing \,\} \\
Sat(\exists(\Phi \,\mathsf{U}\, \Psi)) & = & \bigcup_{n\geq 0} V_n \text{ where} \\
 & & T_0 = Sat(\Psi) \\
 & & T_{n+1} = T_n \cup \{\, s \in Sat(\Phi) \mid Post(s) \cap T_n \neq \varnothing \,\}
\end{array}
$$

Overview

Time Complexity

The CTL model-checking problem can be solved in $O(|\Phi| \cdot |TS|)$.

Proof.

1. The parse tree of Φ has size $O(|\Phi|)$
2. The time complexity at a node of the parse tree is in $O(|TS|)$
3. This holds in particular for computing $Sat(\exists\,\mathsf{U})$ and $Sat(\exists\Box\ldots)$
4. The entire time complexity is thus in $O(|\Phi| \cdot |TS|)$

$\square$

Complexity of CTL Model-Checking Problem

The CTL model-checking problem is PTIME-complete.

Proof.

Containment: Our algorithm runs in polynomial time

Hardness: Reduction from (monotone) Boolean circuit value problem

$\square$

Complexity of CTL Model-Checking Problem

The CTL model-checking problem is PTIME-complete.

Proof.

Containment: Our algorithm runs in polynomial time

Hardness: Reduction from (monotone) Boolean circuit value problem

□

(Monotone) Boolean Circuit Value Problem

Given: Circuit consisting of $\wedge, \vee, 1$, and 0 gates organised in layers

Question: Does the circuit evaluate to 1?

- This problem is well-known to be PTIME-hard

Overview

Summary

- CTL model checking determines *Sat*(Φ) by a recursive descent over Φ

  → parse tree of the CTL formula

- *Sat*($\exists(\Phi \mathsf{U} \Psi)$) is approximated from below by a backward search from Ψ-states

- *Sat*($\exists\Box\Phi$) is approximated from above by a backward search from Φ-states

- The CTL model-checking problem is PTIME-complete