Exercises Category Theory and Coalgebra
Lecture 5

The items labelled with $(*)$ are optional. If you have any questions, email `mark.szeles@ru.nl`. The deadline is 04 March 23:59, CET. Solutions may only be submitted in Brightspace. In case you upload hand-written notes, please use a scanner app (instead of photos). Explain your answers! Think carefully about all the properties that you need to prove in each exercise.

1. If $\mathbb{C}$ has coproducts, show that we can extend the assignment $(X, Y) \mapsto X + Y$ to a functor $(+) : \mathbb{C} \times \mathbb{C} \to \mathbb{C}$:
   (a) Define explicitly the functorial action on morphisms $(f, g) \mapsto f + g$
   (b) Verify carefully the functor axioms (preservations of identities and composition) using the categorical structure of $\mathbb{C} \times \mathbb{C}$

2. Let $F : \mathbb{C} \to \mathbb{C}$ be a functor.
   (a) Show that if $\mathbb{C}$ has products, then so does $\mathrm{Alg}(F)$.
   (b) Show that if $\mathbb{C}$ has coproducts, then so does $\mathrm{CoAlg}(F)$.

3. Let $F : \mathbf{Sets} \to \mathbf{Sets}$ be the functor $F(X) = X + 1$.
   (a) Write out the functorial action of $F(f)$ on morphisms $f : X \to Y$
   (b) The natural numbers $\mathbb{N} = \{0, 1, \ldots\}$ have the structure of an F-algebra by means of the map $\alpha : F(\mathbb{N}) \to \mathbb{N}$ where $\alpha(n) = n + 1$ for $n \in \mathbb{N}$, and $\alpha(\star) = 0$. Show that $(\mathbb{N}, \alpha)$ is an initial F-algebra.
   (c) By Lambek's Lemma, α must be an isomorphism (in $\mathbf{Sets}$). In this exercise, we will see that the converse of Lambek's lemma does not hold in general: Consider the map $\beta : F(\mathbb{N}) \to \mathbb{N}$ defined by

$$\beta(*) = 0, \qquad \beta(n) = \begin{cases} n + 2, & \text{if } n \text{ even} \\ n, & \text{if } n \text{ odd} \end{cases}$$

   Show that β is an isomorphism, but the F-algebra $(\mathbb{N}, \beta)$ is *not* initial.

4. Show that the powerset functor $\mathcal{P} : \mathbf{Sets} \to \mathbf{Sets}$ cannot admit initial algebras or final coalgebras.