Automated Reasoning

Week 2. Beyond the basics of SAT: integers and program verification

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Arithmetic in proposition logic

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Many practical problems use **integers**. But SAT only has booleans...

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Solution: use the binary representation!

$$a_n a_{n-1} \cdots a_1$$

Addition in primary school

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Given a and b, find d satisfying $a + b = d$.

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In decimal:

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Given a and b, find d satisfying $a + b = d$.

In decimal:

$$\begin{array}{cccccc} 1 & 3 & 7 & \leftarrow & & a \\ & 7 & 9 & \leftarrow & & b \\ \hline & & & & & + \end{array}$$

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Given a and b, find d satisfying $a + b = d$.

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$$\begin{array}{ccccc} 1 & 3 & 7 & \leftarrow & a \\ & 7 & 9 & \leftarrow & b \\ \hline & & & & + \end{array}$$

Addition in primary school

Given a and b, find d satisfying $a + b = d$.

In decimal:

$$\begin{array}{ccccc} 1 & 3 & 7 & \leftarrow & a \\ & 7 & 9 & \leftarrow & b \\ \hline & & \color{red}{6} & & + \end{array}$$

Addition in primary school

Given a and b, find d satisfying $a + b = d$.

In decimal:

$$
\begin{array}{ccccl}
 & \color{red}{1} & & \leftarrow & \text{carry} \\
1 & 3 & 7 & \leftarrow & a \\
 & 7 & 9 & \leftarrow & b \\
\hline
 & & 6 & & +
\end{array}
$$

Addition in primary school

Given a and b, find d satisfying $a + b = d$.

In decimal:

$$
\begin{array}{ccccc}
 & 1 & & \leftarrow & \text{carry} \\
1 & 3 & 7 & \leftarrow & a \\
 & 7 & 9 & \leftarrow & b \\
\hline
 & & 6 & & +
\end{array}
$$

Addition in primary school

Given a and b, find d satisfying $a + b = d$.

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$$\begin{array}{cccccc} \color{red}{1} & 1 & & \leftarrow & \text{carry} \\ 1 & 3 & 7 & \leftarrow & a \\ & 7 & 9 & \leftarrow & b \\ \hline & \color{red}{1} & 6 & & + \end{array}$$

Addition in primary school

Given a and b, find d satisfying $a + b = d$.

In decimal:

$$
\begin{array}{rrrcl}
\color{red}{1} & 1 & & \leftarrow & \text{carry} \\
\color{red}{1} & 3 & 7 & \leftarrow & a \\
 & 7 & 9 & \leftarrow & b \\
\hline
 & 1 & 6 & & +
\end{array}
$$

Addition in primary school

Given a and b, find d satisfying $a + b = d$.

In decimal:

$$
\begin{array}{cccclc}
0 & 1 & 1 & & \leftarrow & \text{carry} \\
 & 1 & 3 & 7 & \leftarrow & a \\
 & & 7 & 9 & \leftarrow & b \\
\hline
 & 2 & 1 & 6 & & +
\end{array}
$$

Addition in primary school

Given a and b, find d satisfying $a + b = d$.

In decimal:

$$
\begin{array}{cccccc}
0 & 1 & 1 & & \leftarrow & \text{carry} \\
 & 1 & 3 & 7 & \leftarrow & a \\
 & & 7 & 9 & \leftarrow & b \\
\hline
 & 2 & 1 & 6 & & +
\end{array}
$$

In binary: just the same!

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Given a and b, find d satisfying $a + b = d$.

In decimal:

$$
\begin{array}{cccccl}
0 & 1 & 1 & & \leftarrow & \text{carry} \\
 & 1 & 3 & 7 & \leftarrow & a \\
 & & 7 & 9 & \leftarrow & b \\
\hline
 & 2 & 1 & 6 & & +
\end{array}
$$

In binary: just the same!

$$
\begin{array}{cccccccl}
0 & 0 & 1 & 1 & 1 & & \leftarrow & \text{carry} \\
 & 0 & 0 & 1 & 1 & 1 & \leftarrow & a = 7 \\
 & 1 & 0 & 1 & 0 & 1 & \leftarrow & b = 21 \\
\hline
0 & 1 & 1 & 1 & 0 & 0 & \leftarrow & d = 7 + 21 = 28
\end{array}
$$

Adding two n-bit numbers

Goal: $a + b = d$, where a, b, d are all **5-bit numbers**.

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$$
\begin{array}{cccccc}
a_5 & a_4 & a_3 & a_2 & a_1 & \\
b_5 & b_4 & b_3 & b_2 & b_1 & \\
\hline
d_5 & d_4 & d_3 & d_2 & d_1 & +
\end{array}
$$

Adding two n-bit numbers

Goal: $a + b = d$, where a, b, d are all **5-bit numbers.**

$$
\begin{array}{ccccccc}
c_5 & c_4 & c_3 & c_2 & c_1 & & \\
 & a_5 & a_4 & a_3 & a_2 & a_1 & \\
 & b_5 & b_4 & b_3 & b_2 & b_1 & \\
\hline
 & d_5 & d_4 & d_3 & d_2 & d_1 & +
\end{array}
$$

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Goal: $a + b = d$, where a, b, d are all **5-bit numbers.**

$$
\begin{array}{ccccccc}
c_5 & c_4 & c_3 & c_2 & c_1 & & \\
 & a_5 & a_4 & a_3 & a_2 & a_1 & \\
 & b_5 & b_4 & b_3 & b_2 & b_1 & \\
\hline
 & d_5 & d_4 & d_3 & d_2 & d_1 & +
\end{array}
$$

Requirements:

Adding two n-bit numbers

Goal: $a + b = d$, where a, b, d are all **5-bit numbers.**

$$
\begin{array}{cccccc}
c_5 & c_4 & c_3 & c_2 & c_1 & \\
 & a_5 & a_4 & a_3 & a_2 & a_1 \\
 & b_5 & b_4 & b_3 & b_2 & b_1 \\
\hline
 & d_5 & d_4 & d_3 & d_2 & d_1 \quad +
\end{array}
$$

Requirements:

- $d_1 = a_1$ XOR b_1
- for larger i: $d_i = (c_{i-1} + a_i + b_i)\%2$.

Adding two n-bit numbers

Goal: $a + b = d$, where a, b, d are all **5-bit numbers.**

$$
\begin{array}{cccccc}
c_5 & c_4 & c_3 & c_2 & c_1 & \color{red}{c_0} \\
 & a_5 & a_4 & a_3 & a_2 & a_1 \\
 & b_5 & b_4 & b_3 & b_2 & b_1 \\
\hline
 & d_5 & d_4 & d_3 & d_2 & d_1 & +
\end{array}
$$

Requirements:

- for all i: $d_i = (c_{i-1} + a_i + b_i)\%2$.

Adding two n-bit numbers

Goal: $a + b = d$, where a, b, d are all **5-bit numbers.**

$$
\begin{array}{cccccccc}
c_5 & c_4 & c_3 & c_2 & c_1 & c_0 & \\
 & a_5 & a_4 & a_3 & a_2 & a_1 & \\
 & b_5 & b_4 & b_3 & b_2 & b_1 & \\
\hline
 & d_5 & d_4 & d_3 & d_2 & d_1 & +
\end{array}
$$

Requirements:

- for all i: $d_i = (c_{i-1} + a_i + b_i)\%2$.

$$\bigwedge_{i=1}^{n-1} a_i \leftrightarrow b_i \leftrightarrow c_{i-1} \leftrightarrow d_i$$

Adding two n-bit numbers

Goal: $a + b = d$, where a, b, d are all **5-bit numbers.**

$$
\begin{array}{cccccc}
c_5 & c_4 & c_3 & c_2 & c_1 & c_0 \\
 & a_5 & a_4 & a_3 & a_2 & a_1 \\
 & b_5 & b_4 & b_3 & b_2 & b_1 \\
\hline
 & d_5 & d_4 & d_3 & d_2 & d_1 & +
\end{array}
$$

Requirements:

- for all i: $d_i = (c_{i-1} + a_i + b_i)\%2$.
- for all $i < n$: $c_i = 1$ if and only if $a_i + b_i + c_{i-1} > 1$.

Adding two n-bit numbers

Goal: $a + b = d$, where a, b, d are all **5-bit numbers.**

$$
\begin{array}{cccccc}
c_5 & c_4 & c_3 & c_2 & c_1 & c_0 \\
 & a_5 & a_4 & a_3 & a_2 & a_1 \\
 & b_5 & b_4 & b_3 & b_2 & b_1 \\
\hline
 & d_5 & d_4 & d_3 & d_2 & d_1 & +
\end{array}
$$

Requirements:

- for all i: $d_i = (c_{i-1} + a_i + b_i)\%2$.
- for all $i < n$: $c_i = 1$ if and only if $a_i + b_i + c_{i-1} > 1$.

$$\bigwedge_{i=1}^{n-1} c_i \leftrightarrow ((a_i \wedge b_i) \vee (a_i \wedge c_{i-1}) \vee (b_i \wedge c_{i-1}))$$

Adding two n-bit numbers

Goal: $a + b = d$, where a, b, d are all **5-bit numbers.**

$$\begin{array}{cccccccc}
c_5 & c_4 & c_3 & c_2 & c_1 & c_0 & \\
 & a_5 & a_4 & a_3 & a_2 & a_1 & \\
 & b_5 & b_4 & b_3 & b_2 & b_1 & \\
\hline
 & d_5 & d_4 & d_3 & d_2 & d_1 & +
\end{array}$$

Requirements:

- for all i: $d_i = (c_{i-1} + a_i + b_i)\%2$.
- for all $i < n$: $c_i = 1$ if and only if $a_i + b_i + c_{i-1} > 1$.
- $c_0 = 0$ (no initial carry) and $c_n = 0$ (no overflow).

Adding two n-bit numbers

Goal: $a + b = d$, where a, b, d are all **5-bit numbers.**

$$
\begin{array}{cccccccc}
c_5 & c_4 & c_3 & c_2 & c_1 & c_0 & \\
 & a_5 & a_4 & a_3 & a_2 & a_1 & \\
 & b_5 & b_4 & b_3 & b_2 & b_1 & \\
\hline
 & d_5 & d_4 & d_3 & d_2 & d_1 & +
\end{array}
$$

Requirements:

- for all i: $d_i = (c_{i-1} + a_i + b_i)\%2$.
- for all $i < n$: $c_i = 1$ if and only if $a_i + b_i + c_{i-1} > 1$.
- $c_0 = 0$ (no initial carry) and $c_n = 0$ (no overflow).

$$\neg c_0 \wedge \neg c_n$$

Making a SAT-solver add

Challenge: make a SAT-solver compute 17 + 11

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$$
\begin{array}{rcccccc}
17 & = & 1 & 0 & 0 & 0 & 1 \\
11 & = & 0 & 1 & 0 & 1 & 1
\end{array}
$$

Making a SAT-solver add

Challenge: make a SAT-solver compute 17 + 11

$$\begin{array}{ccccccc} 17 & = & 1 & 0 & 0 & 0 & 1 \\ 11 & = & 0 & 1 & 0 & 1 & 1 \end{array}$$

Solution:

$$\phi \ \wedge \ \underbrace{a_5 \wedge \neg a_4 \wedge \neg a_3 \wedge \neg a_2 \wedge a_1}_{\vec{a}=17} \ \wedge \ \underbrace{\neg b_5 \wedge b_4 \wedge \neg b_3 \wedge b_2 \wedge b_1}_{\vec{b}=11}$$

Making a SAT-solver add

Challenge: make a SAT-solver compute 17 + 11

$$\begin{array}{rcccccc} 17 & = & 1 & 0 & 0 & 0 & 1 \\ 11 & = & 0 & 1 & 0 & 1 & 1 \end{array}$$

Solution:

$$\phi \wedge \underbrace{a_5 \wedge \neg a_4 \wedge \neg a_3 \wedge \neg a_2 \wedge a_1}_{\vec{a}=17} \wedge \underbrace{\neg b_5 \wedge b_4 \wedge \neg b_3 \wedge b_2 \wedge b_1}_{\vec{b}=11}$$

Challenge: make a SATs-solver compute 17 + 18

Making a SAT-solver add

Challenge: make a SAT-solver compute 17 + 11

$$\begin{array}{rcccccc} 17 & = & 1 & 0 & 0 & 0 & 1 \\ 11 & = & 0 & 1 & 0 & 1 & 1 \end{array}$$

Solution:

$$\phi \wedge \underbrace{a_5 \wedge \neg a_4 \wedge \neg a_3 \wedge \neg a_2 \wedge a_1}_{\vec{a}=17} \wedge \underbrace{\neg b_5 \wedge b_4 \wedge \neg b_3 \wedge b_2 \wedge b_1}_{\vec{b}=11}$$

Challenge: make a SATs-solver compute 17 + 18

Solution: just add a leading 0 to a and b and use 6-bit addition!

Making a SAT-solver add and subtract

Challenge: make a SAT-solver compute 17 + 11

$$\begin{array}{rcccccc} 17 & = & 1 & 0 & 0 & 0 & 1 \\ 11 & = & 0 & 1 & 0 & 1 & 1 \end{array}$$

Solution:

$$\phi \;\wedge\; \underbrace{a_5 \wedge \neg a_4 \wedge \neg a_3 \wedge \neg a_2 \wedge a_1}_{\vec{a}=17} \;\wedge\; \underbrace{\neg b_5 \wedge b_4 \wedge \neg b_3 \wedge b_2 \wedge b_1}_{\vec{b}=11}$$

Challenge: compute 17 - 11.

Making a SAT-solver add and subtract

Challenge: make a SAT-solver compute 17 + 11

$$\begin{array}{ccccccc} 17 & = & 1 & 0 & 0 & 0 & 1 \\ 11 & = & 0 & 1 & 0 & 1 & 1 \end{array}$$

Solution:

$$\phi \wedge \underbrace{a_5 \wedge \neg a_4 \wedge \neg a_3 \wedge \neg a_2 \wedge a_1}_{\vec{a}=17} \wedge \underbrace{\neg b_5 \wedge b_4 \wedge \neg b_3 \wedge b_2 \wedge b_1}_{\vec{b}=11}$$

Challenge: compute 17 - 11.

Solution:

$$\phi \wedge \underbrace{d_5 \wedge \neg d_4 \wedge \neg d_3 \wedge \neg d_2 \wedge d_1}_{\vec{d}=17} \wedge \underbrace{\neg a_5 \wedge a_4 \wedge \neg a_3 \wedge a_2 \wedge a_1}_{\vec{a}=11}$$

Multiplication in primary school

Given a and b, find d satisfying $a * b = d$.

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Given a and b, find d satisfying $a * b = d$.

In decimal:

$$
\begin{array}{ccc}
1 & 2 & 3 \\
4 & 5 & 6 \\
\hline
\\
\hline
\end{array}
$$

Multiplication in primary school

Given a and b, find d satisfying $a * b = d$.

In decimal:

$$
\begin{array}{ccccl}
 & 1 & 2 & 3 & \\
 & 4 & 5 & \color{red}{6} & \\
\hline
 & 7 & 3 & 8 & = 6 * 123 \\
 & & & & \\
\hline
\end{array}
$$

Multiplication in primary school

Given a and b, find d satisfying $a * b = d$.

In decimal:

$$
\begin{array}{rrrrl}
 & 1 & 2 & 3 & \\
 & 4 & 5 & 6 & \\
\hline
 & 7 & 3 & 8 & = 6 * 123 \\
6 & 1 & 5 & & = 5 * 123 \\
 & & & & \\
\hline
\end{array}
$$

Multiplication in primary school

Given a and b, find d satisfying $a * b = d$.

In decimal:

$$
\begin{array}{cccccc}
 & & 1 & 2 & 3 & \\
 & & \color{red}{4} & 5 & 6 & \\
\hline
 & & 7 & 3 & 8 & = 6 * 123 \\
 & 6 & 1 & 5 & & = 5 * 123 \\
4 & 9 & 2 & & & = 4 * 123 \\
\hline
\end{array}
$$

Multiplication in primary school

Given a and b, find d satisfying $a * b = d$.

In decimal:

$$
\begin{array}{ccccccl}
 & & 1 & 2 & 3 & & \\
 & & 4 & 5 & 6 & & \\
\hline
 & & 7 & 3 & 8 & & = 6 * 123 \\
 & 6 & 1 & 5 & & & = 5 * 123 \\
4 & 9 & 2 & & & & = 4 * 123 \\
\hline
5 & 6 & 0 & 8 & 8 & + &
\end{array}
$$

Multiplication in primary school

Given a and b, find d satisfying $a * b = d$.

In decimal:

$$
\begin{array}{cccccl}
 & & 1 & 2 & 3 & \\
 & & 4 & 5 & 6 & \\
\hline
 & & 7 & 3 & 8 & = 6 * 123 \\
 & 6 & 1 & 5 & & = 5 * 123 \\
4 & 9 & 2 & & & = 4 * 123 \\
\hline
5 & 6 & 0 & 8 & 8 & +
\end{array}
$$

Proposed algorithm:

Multiplication in primary school

Given a and b, find d satisfying $a * b = d$.

In decimal:

$$
\begin{array}{ccccc|l}
 & & 1 & 2 & 3 & \\
 & & 4 & 5 & 6 & \\
\hline
 & & 7 & 3 & 8 & = 6 * 123 \\
 & 6 & 1 & 5 & & = 5 * 123 \\
4 & 9 & 2 & & & = 4 * 123 \\
\hline
5 & 6 & 0 & 8 & 8 & +
\end{array}
$$

Proposed algorithm:

d := 0

Multiplication in primary school

Given a and b, find d satisfying $a * b = d$.

In decimal:

$$
\begin{array}{ccccccl}
 & & 1 & 2 & 3 & & \\
 & & 4 & 5 & 6 & & \\
\hline
 & & 7 & 3 & 8 & & = 6 * 123 \\
 & 6 & 1 & 5 & & & = 5 * 123 \\
4 & 9 & 2 & & & & = 4 * 123 \\
\hline
5 & 6 & 0 & 8 & 8 & + &
\end{array}
$$

Proposed algorithm:

d := 0

for i := n downto 1 do:

Multiplication in primary school

Given a and b, find d satisfying $a * b = d$.

In decimal:

$$\begin{array}{ccccc|l}
 & & 1 & 2 & 3 & \\
 & & 4 & 5 & 6 & \\
\hline
 & & 7 & 3 & 8 & = 6 * 123 \\
 & 6 & 1 & 5 & & = 5 * 123 \\
4 & 9 & 2 & & & = 4 * 123 \\
\hline
5 & 6 & 0 & 8 & 8 & +
\end{array}$$

Proposed algorithm:

d := 0

for i := n downto 1 do:

$d := 10 * d + b_i * a$

Binary multiplication

Decimal algorithm:

d := 0

for i := n downto 1 do:

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Binary multiplication

Decimal algorithm:

d := 0
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Binary multiplication

Decimal algorithm:

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for i := n downto 1 do:
$d := 10 * d + b_i * a$

Binary algorithm:

d := 0
for i := n downto 1 do:
if b_i **then** $d := 2 * d + a$
else $d := 2 * d$

Binary multiplication

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d := 0

for i := n downto 1 do:

if b_i **then** $d := 2 * d + a$

else $d := 2 * d$

Example: 9 * 11 (01011)

Binary multiplication

Binary algorithm:

d := 0

for i := n downto 1 do:

if b_i **then** $d := 2 * d + a$

else $d := 2 * d$

Example: 9 * 11 (01011)

- $d = 0$

Binary multiplication

Binary algorithm:

d := 0

for i := n downto 1 do:

if b_i **then** $d := 2 * d + a$

else $d := 2 * d$

Example: 9 * 11 (01011)

- $d = 0$
- $i = 5$ and $d = 0$ $(0 * 2)$

Binary multiplication

Binary algorithm:

d := 0

for i := n downto 1 do:

if b_i **then** $d := 2 * d + a$

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Example: 9 * 11 (01011)

- $d = 0$
- $i = 5$ and $d = 0$ $(0 * 2)$
- $i = 4$ and $d = 9$ $(0 * 2 + 9)$

Binary multiplication

Binary algorithm:

d := 0

for i := n downto 1 do:

if b_i **then** $d := 2 * d + a$

else $d := 2 * d$

Example: 9 * 11 (01011)

- $d = 0$
- $i = 5$ and $d = 0$ $(0 * 2)$
- $i = 4$ and $d = 9$ $(0 * 2 + 9)$
- $i = 3$ and $d = 18$ $(9 * 2)$

Binary multiplication

Binary algorithm:

d := 0

for i := n downto 1 do:

if b_i **then** $d := 2 * d + a$

else $d := 2 * d$

Example: 9 * 11 (01011)

- $d = 0$
- $i = 5$ and $d = 0$ $(0 * 2)$
- $i = 4$ and $d = 9$ $(0 * 2 + 9)$
- $i = 3$ and $d = 18$ $(9 * 2)$
- $i = 2$ and $d = 45$ $(18 * 2 + 9)$

Binary multiplication

Binary algorithm:

d := 0

for i := n downto 1 do:

if b_i **then** $d := 2 * d + a$

else $d := 2 * d$

Example: 9 * 11 (01011)

- $d = 0$
- $i = 5$ and $d = 0$ $(0 * 2)$
- $i = 4$ and $d = 9$ $(0 * 2 + 9)$
- $i = 3$ and $d = 18$ $(9 * 2)$
- $i = 2$ and $d = 45$ $(18 * 2 + 9)$
- $i = 1$ and $d = 99$ $(45 * 2 + 9)$

Binary multiplication

d := 0
for i := n downto 1 do:
 if b_i then $d := 2 * d + a$
 else $d := 2 * d$

Invariant: $d = [b_n \cdots b_i] * a$, so at the end $d = b * a$

Binary multiplication

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Goal: $\vec{x} = 2\vec{a}$

Binary multiplication

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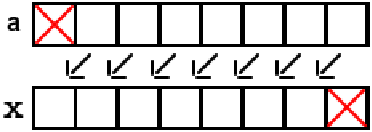
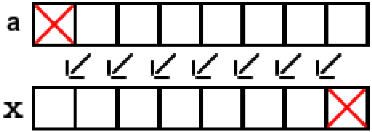


Binary multiplication

d := 0
for i := n downto 1 do:
  if b_i then $d := 2 * d + a$
  else $d := 2 * d$

Invariant: $d = [b_n \cdots b_i] * a$, so at the end $d = b * a$

Goal: $\vec{x} = 2\vec{a}$



Solution:

$$\mathrm{dup}(\vec{a}, \vec{x}) = \neg a_n \wedge \neg x_1 \wedge \bigwedge_{i=1}^{n-1} (x_{i+1} \leftrightarrow a_i)$$

Encoding a variable that changes over time

d := 0
for i := n downto 1 do:
    if b_i then $d := 2 * d + a$
    else $d := 2 * d$

In every step, d changes!

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for i := n downto 1 do:
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Solution: introduce boolean variables $\vec{r}_i$ for $i \in \{0, \ldots, n\}$.

Encoding a variable that changes over time

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for i := n downto 1 do:
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In every step, d changes!

Solution: introduce boolean variables $\vec{r}_i$ for $i \in \{0, \ldots, n\}$.

That is: for $i \in \{0, \ldots, n\}, j \in \{1, \ldots, n\}$, we introduce a boolean variable $r_{i,j}$.

Encoding a variable that changes over time

d := 0
for i := n downto 1 do:
    if b_i then $d := 2 * d + a$
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In every step, d changes!

Solution: introduce boolean variables $\vec{r}_i$ for $i \in \{0, \ldots, n\}$.

That is: for $i \in \{0, \ldots, n\}, j \in \{1, \ldots, n\}$, we introduce a boolean variable $r_{i,j}$.

Also introduce $\vec{s}_i$ for $i \in \{1, \ldots, n\}$.

Encoding a variable that changes over time

d := 0
for i := n downto 1 do:
    if b_i then $d := 2 * d + a$
    else $d := 2 * d$

In every step, d changes!

Solution: introduce boolean variables $\vec{r}_i$ for $i \in \{0, \ldots, n\}$.

That is: for $i \in \{0, \ldots, n\}, j \in \{1, \ldots, n\}$, we introduce a boolean variable $r_{i,j}$.

Also introduce $\vec{s}_i$ for $i \in \{1, \ldots, n\}$.

We will use $\vec{s}$ to represent $2 * \vec{r}$.

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Also introduce $\vec{s}_i$ for $i \in \{1, \ldots, n\}$.

We will use $\vec{s}$ to represent $2 * \vec{r}$.

Updated algorithm:

$\vec{r}_n = 0$

for i := n downto 1 do:

$\vec{s}_i = 2 * \vec{r}_i$

if b_i then $\vec{r}_{i-1} = \vec{s}_i + \vec{a}$

else $\vec{r}_{i-1} = \vec{s}_i$

Bringing it all together: multiplication

The requirement

$$\vec{a} * \vec{b} = \vec{r}_0$$

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Bringing it all together: multiplication

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mul$(\vec{a}, \vec{b}, \vec{r}_0) \quad =$

$$\bigwedge_{j=1}^{n} \neg r_{nj}$$

$\vec{r}_n := 0$

$$\wedge$$

$$\bigwedge_{i=1}^{n} \left(\begin{array}{c} \text{dup}(\vec{r}_i, \vec{s}_i) \\ \wedge \\ b_i \rightarrow \text{plus}(\vec{a}, \vec{s}_i, \vec{r}_{i-1})) \\ \wedge \\ \neg b_i \rightarrow \bigwedge_{j=1}^{n} (s_{ij} \leftrightarrow r_{(i-1)j}) \end{array} \right)$$

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- fac(1234567897) is satisfiable, yielding

$$1234567897 = 1241 \times 994817$$

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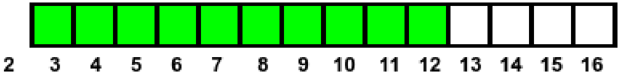

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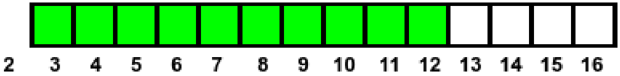

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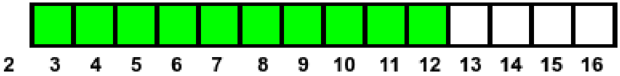

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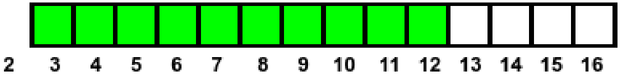

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Well-definedness condition:

$$\bigwedge_{k=i+2}^{j} x_k \rightarrow x_{k-1}$$

Unary addition

Given: $\vec{a} \in \{a_{\text{min}}..a_{\text{max}}\}$ and $\vec{b} \in \{b_{\text{min}}..b_{\text{max}}\}$.
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Numbers like this occur in many practical problems.

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- For-loops `for i := 1 to m do X` are treated as m copies of `X`.

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For proving $\{P\}S\{Q\}$, add the formula

$$P_0 \quad \wedge \quad \neg Q_m$$

and prove that the resulting formula is unsatisfiable.

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Negation of the postcondition: $\neg(a_{1,m-1} \leftrightarrow a_{m,m-1})$

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$$\bigwedge_{j=1}^{n} \neg a_{0,j} \quad \wedge \quad \bigwedge_{i=0}^{m-1} \text{plus}(\vec{a}_i, \vec{k}, \vec{a}_{i+1}) \quad \wedge$$

$$\neg\text{mul}([\vec{m}], \vec{k}, \vec{a}_m)$$

where $[\vec{m}]$ is the binary encoding of number m.

A more complicated program correctness example

```
ret := 0
for i := 20 to 30 do
  if i < x then
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  x := x + 1
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Claim: if $\underbrace{x < 20}_{\text{pre-condition}}$ at the start of the program,

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(1)             ret := 0
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Total steps: 23

A more complicated program correctness example

$\text{smaller}(\vec{x}_0, [\vec{20}])$	$\wedge$	pre-condition
$\neg\text{equal}(\vec{r}_{23}, [\vec{0}])$	$\wedge$	post-condition
$\text{equal}(\vec{r}_1, [\vec{0}])$	$\wedge$	(1) `ret := 0`
$\text{equal}(\vec{x}_1, \vec{x}_0)$	$\wedge$	
$\bigwedge_{i=20}^{30} \text{smaller}([\vec{i}], \vec{x}_{2(i-20)+1}) \rightarrow$		`(2(i-19)) ret :=`
$\text{plus}(r_{2(i-20)+1}, [\vec{i}], r_{2(i-19)})$	$\wedge$	`ret + i`
$\bigwedge_{i=20}^{30} \text{smaller}([\vec{i}], \vec{x}_{2(i-20)+1}) \rightarrow$		
$\text{equal}(x_{2(i-20)+1}, x_{2(i-19)})$	$\wedge$	
$\bigwedge_{i=20}^{30} \neg\text{smaller}([\vec{i}], \vec{x}_{2(i-20)+1}) \rightarrow$		`(2(i-19)) skip`
$\text{equal}(r_{2(i-20)+1}, r_{2(i-19)})$	$\wedge$	
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$\text{equal}(x_{2(i-20)+1}, x_{2(i-19)})$	$\wedge$	
$\bigwedge_{i=20}^{30} \text{plus}(\vec{x}_{2(i-19)}, [\vec{1}], \vec{x}_{2(i-19)+1})$		`(2(i-19)+1) x := x + 1`

Program correctness summary

Overall: a rich class of imperative programs is covered.

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SAT versus SMT: bounded or unbounded integers.

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every 0 in the truth table yields a clause
proposition $\equiv$ *conjunction of these clauses*

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So far:

- SAT solvers can handle **CNF** as input format.
- However: a lot of $\leftrightarrow$ in this lecture...

Need: transform arbitrary boolean formulas to CNF.

Straightforward approach: transform the proposition to a logically equivalent CNF.

every 0 in the truth table yields a clause
proposition $\equiv$ *conjunction of these clauses*

However: this is worst-case exponential.

Formulas whose CNF is always large

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$\Longrightarrow$ Any CNF B equivalent to A has size exponential in $|A|$.

Alternatives to an equivalent CNF

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This will result in the **Tseitin transformation**.

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Proof sketch:

$\Leftarrow$ A satisfying assignment for $T(A)$ restricting to the variables from A yields a satisfying assignment for A.

$\Rightarrow$ A satisfying assignment for A is extended to a satisfying assignment for $T(A)$ by giving n_D the value of D obtained from the satisfying assignment for A. $\square$

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- $T(A)$ is a **3-CNF**.
- To determine satisfiability of A: run a SAT-solver on $T(A)$.

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$$PF_n = C_n \wedge R_n$$

Intuition for the pigeonhole formulas

$$\begin{matrix} P_{11} & P_{12} & \cdots & P_{1,n+1} \\ P_{21} & P_{22} & \cdots & P_{2,n+1} \\ \vdots & \vdots & & \vdots \\ P_{n1} & P_{n2} & \cdots & P_{n,n+1} \end{matrix}$$

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PF_n and modifications are a good test case for implementations of methods for SAT.

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(Internally, Tseitin Transformation and perhaps bitblasting are done.)

Quiz

1. Provide a SAT encoding that expresses that $a + b = c$, where a, b, c are all two-bit binary numbers.
2. Provide a SAT encoding that expresses $a > b$ when $a, b \in \{0, \ldots, 3\}$ are encoded as unary numbers.
3. Why is it sometimes useful to use the unary encoding instead of binary?
4. Use Tseitin's Transformation to give a CNF whose satisfiability is equivalent to:

$$x \leftrightarrow ((y \wedge \neg x) \wedge (z \rightarrow w))$$

5. Why are pigeonhole formulas a good testcase for SAT solvers?