Automated Reasoning

Week 12. Confluence and Completion

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Fall 2024

Recall: some nice properties

- $\mathcal{R}$ is **weakly normalising** (WN):
  every term has a normal form

- $\mathcal{R}$ is **terminating** ($=$ strongly normalising, SN):
  no infinite sequence of terms $t_1, t_2, t_3, \ldots$ *exists such that* $t_i \to_{\mathcal{R}} t_{i+1}$ *for all* i

- $\mathcal{R}$ is **confluent** ($=$ Church-Rosser, CR):
  if $s \to_{\mathcal{R}}^* t$ *and* $s \to_{\mathcal{R}}^* q$ *then a term* u *exists satisfying* $t \to_{\mathcal{R}}^* u$ *and* $q \to_{\mathcal{R}}^* u$

- $\mathcal{R}$ is **locally confluent** ($=$ weak Church-Rosser, WCR):
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Proof: Assume t has two normal forms u, u'.
Then by confluence there is a v such that $u \to_{\mathcal{R}}^{*} v$ and $u' \to_{\mathcal{R}}^{*} v$.
Since u, u' are normal forms we have $u = v = u'$. $\square$

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This is a very useful combination!

This week

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- solving the **word problem**

Confluence (Church-Rosser property)

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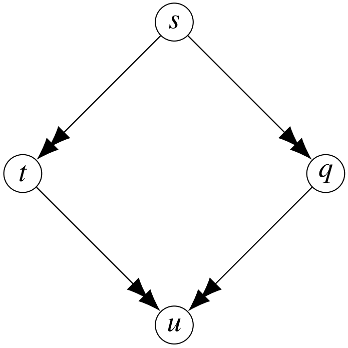
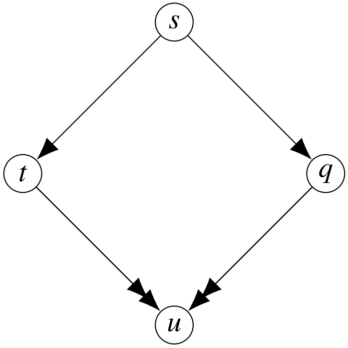


Local confluence (Weak Church-Rosser property):

If $s \to t$ *and* $s \to q$
then exists u *such that*
$t \to^* u$ *and* $q \to^* u$*.*



Local versus general confluence

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$$R = \{ \quad \mathtt{a} \to \mathtt{b}, \quad \mathtt{b} \to \mathtt{a}, \quad \mathtt{a} \to \mathtt{c}, \quad \mathtt{b} \to \mathtt{d} \quad \}$$

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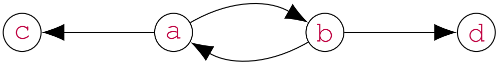

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$$\begin{tikzcd}
\mathtt{c} & \mathtt{a} \arrow[l] \arrow[r, bend left] & \mathtt{b} \arrow[l, bend left] \arrow[r] & \mathtt{d}
\end{tikzcd}$$

This counterexample relies on the TRS being **non-terminating**.

Newman's lemma (1942)

Theorem

For terminating TRSs the properties confluence and local confluence are equivalent.

Principle of well-founded induction

Theorem

Let SN($\rightarrow$) and

$$\forall t[\ \underbrace{\forall u[t \rightarrow^{+} u \Rightarrow P(u)]}_{\text{Induction Hypothesis}}\ \Rightarrow P(t)]$$

Then $P(t)$ holds for all t.

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Proof: by contradiction! Assume $\neg P(t)$

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Suppose $s \rightarrow^* t$ and $s \rightarrow^* q$.

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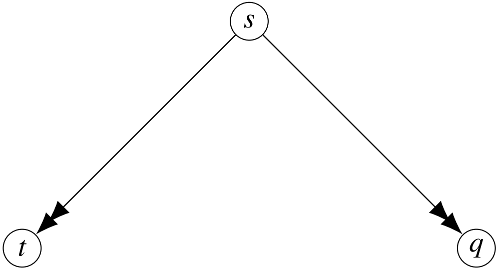

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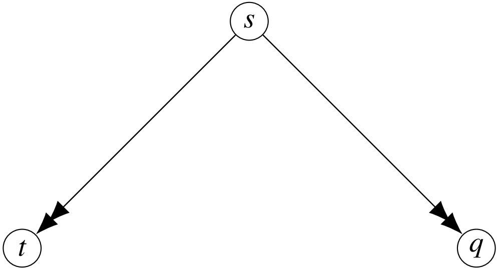

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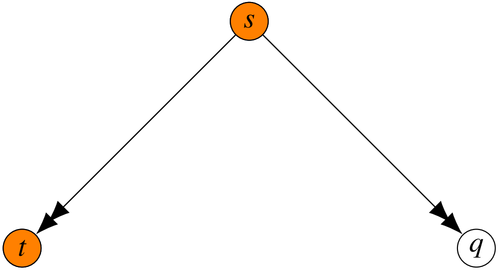

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If $s = t$ we can choose $u := q$.

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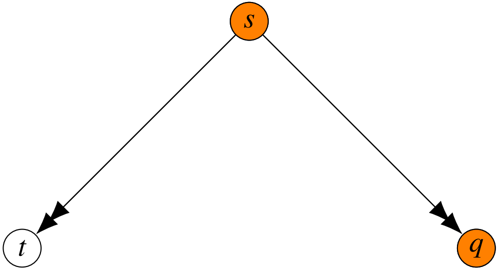

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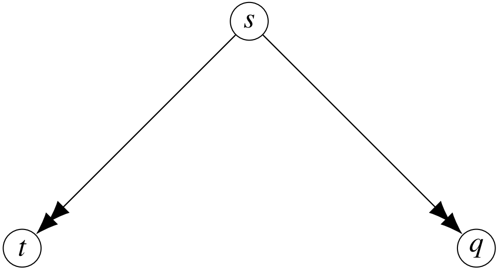

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In the remaining case we have $s \to^+ t$ and $s \to^+ q$.

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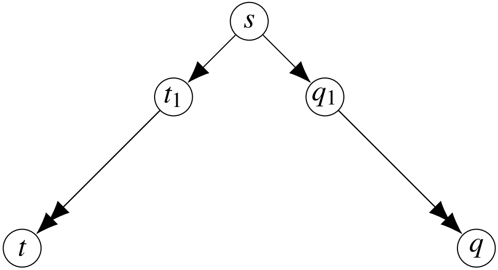

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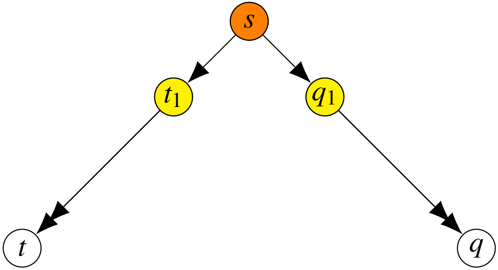
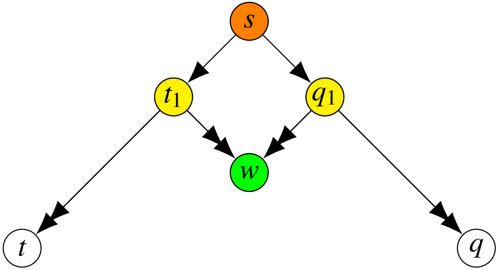


By WCR($\to$) we can find w such that $t_1 \to^* w$ and $q_1 \to^* w$.

Proof of Newman's Lemma



Proof of Newman's Lemma

$$
\begin{tikzcd}[column sep=small]
 & & s \arrow[dl] \arrow[dr] & & \\
 & t_1 \arrow[dl] \arrow[dr] & & q_1 \arrow[dl] \arrow[dr] & \\
t & & w & & q
\end{tikzcd}
$$

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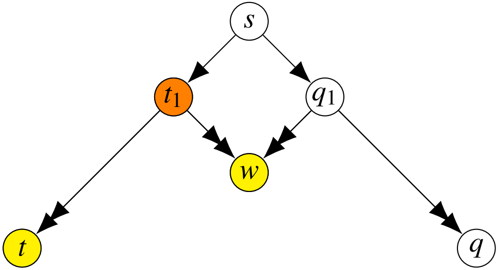

By the induction hypothesis on t_1, there is a common reduct for t and w.

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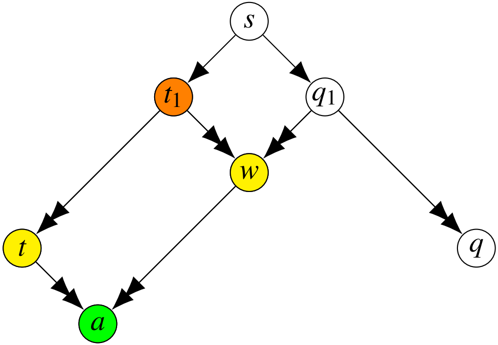

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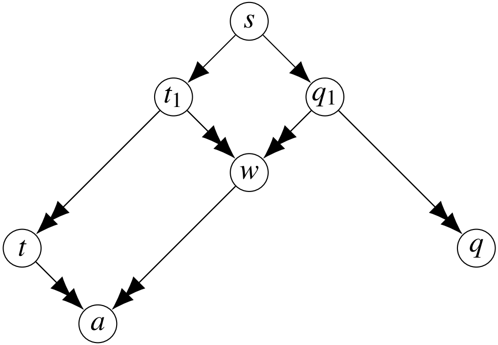

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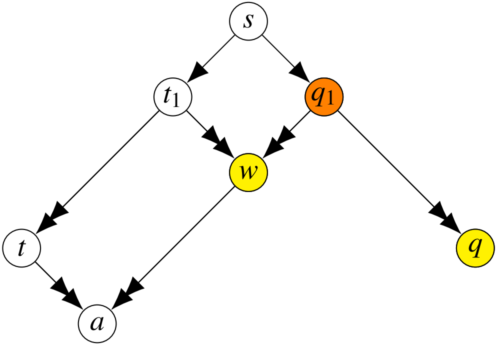

By the induction hypothesis on q_1, there is a common reduct for w and q.

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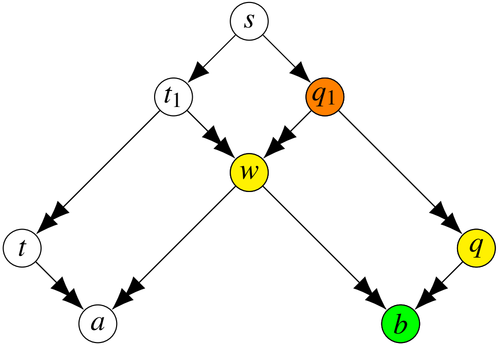

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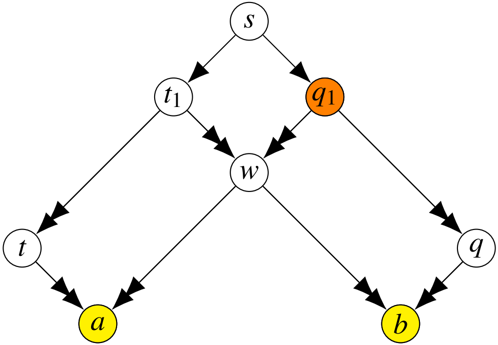

Again by the induction hypothesis on q_1, there is a common reduct for a and b.

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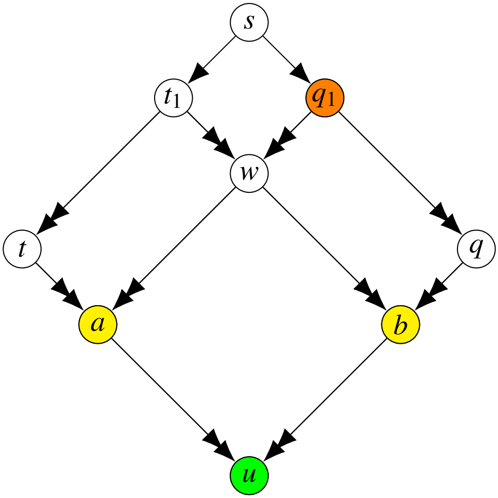

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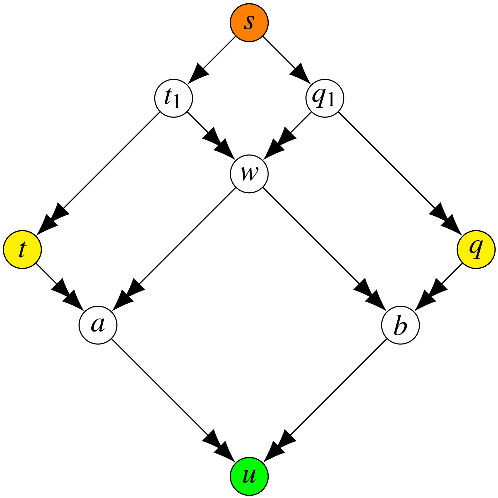

Hence, t and q indeed have a common reduct u!

Deciding confluence

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In our example for addition of natural numbers,

$$\begin{array}{rcl} \mathtt{add}(\mathtt{0}, y) & \to & y \\ \mathtt{add}(\mathtt{s}(x), y) & \to & \mathtt{s}(\mathtt{add}(x, y)) \end{array}$$

there is no overlap. Hence it is locally confluent.

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Termination: for instance by LPO.

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Definition

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Note that $\ell_2\sigma = C[t]\sigma = C\sigma[\ell_1\sigma]$ can be rewritten in two ways:

- with $\ell_2 \to r_2$
- with $\ell_1 \to r_1$

Example

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Assume we have rules for arithmetic including:

$$
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x - x & \rightarrow & \mathtt{0} \\
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Notation:

- $\ell_1 \to r_1$ to be the rule $z - z \to 0$
- $\ell_2 \to r_2$ to be the rule $\mathtt{s}(x) - y \to \mathtt{s}(x - y)$
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Indeed t, ℓ_1 unify, with mgu σ:$\sigma(x) = x, \;\; \sigma(y) = \sigma(z) = \mathtt{s}(x)$

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we see that t, ℓ_1 unify, with mgu σ:

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This yields the critical pair $\langle \mathtt{f}(\mathtt{g}(x)), \mathtt{g}(\mathtt{f}(x)) \rangle$.

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Critical pair: $\langle \mathtt{s}(x + \mathtt{s}(y)), \mathtt{s}(\mathtt{s}(x) + y)\rangle$

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Example

$$\mathtt{not}(\top) \;\to\; \bot \qquad \mathtt{not}(\bot) \;\to\; \top \qquad \mathtt{not}(\mathtt{not}(x)) \;\to\; x$$

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$$\mathtt{not}(\top) \;\to\; \bot \qquad \mathtt{not}(\bot) \;\to\; \top \qquad \mathtt{not}(\mathtt{not}(x)) \;\to\; x$$

Critical pair: $\langle \mathtt{not}(\bot),\ \top\rangle$

Convergence

A critical pair $\langle t, q \rangle$ **converges** if $t \to_R^* u$ and $q \to_R^* u$ for some u.

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Theorem

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- the two reductions are an instance of a critical pair

Decision procedure

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If $\mathcal{R}$ is a complete TRS
and s', t' are normal forms of s, t,
then $s \leftrightarrow^*_{\mathcal{R}} t$ if and only if $s' = t'$.

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Since s', t' are normal forms we have $s' = q = t'$. $\Box$

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- these are different, hence the answer is **No**.

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Equations and rules:

- Initially, E contains the equations we want to complete.
- Initially, $\mathcal{R} = \emptyset$.

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1. Remove an equation $s = t$ from E, and
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3. For every such critical pair $\langle u, v \rangle$:
   - $\mathcal{R}$-rewrite u to a normal form u'
   - $\mathcal{R}$-rewrite v to a normal form v'
   - if $u' \neq v'$, then add $u' = v'$ as an equation to the set E

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- $\mathcal{R}$ is locally confluent since all critical pairs converge, so $\mathcal{R}$ is complete.
- Convertibility $\leftrightarrow^*_{\mathcal{R}}$ of the resulting $\mathcal{R}$ is equivalent to convertibility of the original E since in the whole procedure $\leftrightarrow^*_{\mathcal{R}\cup E}$ remains invariant.

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This is used for instance in **superposition** (next week).

Quiz

1. Give an example why local confluence does not imply confluence.
2. Determine, using critical pairs, whether the following system is locally confluent:

$$\begin{array}{rcl} \mathtt{f}(\mathtt{g}(x), \mathtt{g}(\mathtt{b})) & \to & \mathtt{f}(x, x) \\ \mathtt{g}(\mathtt{a}) & \to & \mathtt{b} \\ \mathtt{b} & \to & \mathtt{a} \end{array}$$

3. Use Knuth-Bendix completion to find a complete TRS with the same $\leftrightarrow_R$ relation as the above TRS.