Exercises Category Theory and Coalgebra
Lecture 3

The items labelled with $(*)$ are optional. If you have any questions, email `mark.szeles@ru.nl`. The deadline is 18 February 23:59, CET. Solutions may only be submitted in Brightspace. In case you upload hand-written notes, please use a scanner app (instead of photos). Explain your answers! Think carefully about all the properties that you need to prove in each exercise.

1. (a) Recall the construction of the category $\mathbf{NonDet}$ of sets and non-deterministic functions from the lecture. Show that composition in $\mathbf{NonDet}$ is associative. This finishes the proof that $\mathbf{NonDet}$ is a category.
   (b) $(*)$ Prove that there is a functor $F : \mathbf{Rel} \to \mathbf{NonDet}$, which is an isomorphism of categories.

2. (a) Define carefully a category whose objects are sets and whose arrows are surjective functions. Show that your formulation indeed gives a category.
   (b) Prove that a function $f : X \to Y$ between sets X, Y is surjective if and only if there is a function $s\colon Y \to X$ such that $f \circ s = id$.
   (c) Let $F\colon \mathbf{Sets} \to \mathbf{Sets}$ be a functor. Show that, if f is surjective, then $F(f)$ is surjective as well.
   (d) $(*)$ How about injective functions? Can you prove something similar?

3. (a) Show that for every set X, the powerset forms a monoid $(\mathcal{P}(X), \cup, \emptyset)$, with union $\cup\colon \mathcal{P}(X) \times \mathcal{P}(X) \to \mathcal{P}(X)$ as binary operation and with $\emptyset \in \mathcal{P}(X)$ as identity element.
   (b) Recall the powerset functor $\mathcal{P} : \mathbf{Sets} \to \mathbf{Sets}$ from the lecture notes. Show in detail that this functor extends to a functor $\mathcal{P} : \mathbf{Sets} \to \mathbf{Mon}$ into the category of monoids and monoid homomorphisms.

4. (a) Consider the preorders $(\mathbb{N}, \leq)$ and $(\mathbb{N}, \geq)$ as categories. We claim that $(\mathbb{N}, \leq)^{op} = (\mathbb{N}, \geq)$. Clearly, they have the same objects. Show that they also have the same hom-sets.
   (b) Consider a monoid $(M, \cdot, u)$ and its variation $(M, *, u)$ where $x * y = y \cdot x$. Show that $(M, \cdot, u)^{op} = (M, *, u)$ as categories.

5. Recall that every monoid M can be viewed as a category with one object. Show that this assignment extends to a functor $\mathbf{Mon} \to \mathbf{Cat}$ to the category of (small) categories. See the lecture notes for the definition of $\mathbf{Cat}$.

6. $(*)$ Consider the functor F given on sets as the powerset: $F(X) = \mathcal{P}(X)$ but on a function $f\colon X \to Y$ by

$$\begin{aligned} F(f)\colon \mathcal{P}(X) &\to \mathcal{P}(Y) \\ U &\mapsto \{y \in Y \mid y \notin \{f(x) \mid x \in X \backslash U\}\} \end{aligned}$$

   Is this a functor? Justify your answer (give a proof or a counterexample).