1 Quiz

1. Give an example why local confluence does not imply confluence.

   Answer: note that any answer must be a TRS that satisfies the following properties:

   - all critical pairs converge (as we need this to have local confluence)
   - the TRS is non-terminating (as local confluence implies confluence for terminating TRSs)

   In addition, we want to find a term that reduces (in multiple steps) to terms that clearly do not converge. There are many ways to achieve this; for example:

$$
\begin{array}{lrcl}
1 & f(x) & \Rightarrow & g(f(x)) \\
2 & g(x) & \Rightarrow & f(g(x)) \\
3 & f(g(x)) & \Rightarrow & a(x) \\
4 & g(f(x)) & \Rightarrow & b(x) \\
5 & a(x) & \Rightarrow & a(g(x)) \\
6 & b(x) & \Rightarrow & b(f(x))
\end{array}
$$

   Critical pairs are:

   - 1 and 3: $\langle\ g(f(g(x)))\ ,\ a(x)\ \rangle$
     This converges because $g(f(g(x))) \Rightarrow f(g(f(g(x)))) \Rightarrow a(f(g(x)))$ and $a(x) \Rightarrow a(g(x)) \Rightarrow a(f(g(x)))$.
   - 1 and 4: $\langle\ g(g(f(x)))\ ,\ b(x)\ \rangle$
     This converges because $g(\underline{g(f(x))}) \Rightarrow g(f(g(f(x)))) \Rightarrow b(g(f(x)))$ and $b(x) \Rightarrow b(f(x)) \Rightarrow b(g(f(x)))$.
   - 2 and 3: $\langle\ f(f(g(x)))\ ,\ a(x)\ \rangle$
     This converges because $f(\underline{f(g(x))}) \Rightarrow f(g(f(g(x)))) \Rightarrow a(f(g(x)))$ and $a(x) \Rightarrow a(g(x)) \Rightarrow a(f(g(x)))$.
   - 2 and 4: $\langle f(g(f(x)))\ ,\ b(x)\ \rangle$
     This converges because $f(g(f(x))) \Rightarrow g(f(g(f(x)))) \Rightarrow b(g(f(x)))$ and $b(x) \Rightarrow b(f(x)) \Rightarrow b(g(f(x)))$

   All critical pairs converge, so local confluence holds. Confluence does *not* hold, since for instance $f(g(x)) \Rightarrow a(x)$ and $f(g(x)) \Rightarrow g(f(g(x))) \Rightarrow b(g(x))$, and terms $a(x)$ and $b(t)$ do not have a common reduct.

   (Note: this example could be significantly simplified, for example by not giving a and b arguments. I wanted to illustrate that non-confluence can arise also in cases where we don't end up reducing to a constant.)

2. Determine, using critical pairs, whether the following system is locally confluent:

$$\begin{array}{rrcl} 1 & f(g(x),g(b)) & \to & f(x,x) \\ 2 & g(a) & \to & b \\ 3 & b & \to & a \end{array}$$

Answer: critical pairs are:

- 1 and 2: $\langle\ f(b,g(b))\ ,\ f(a,a)\ \rangle$
- 1 and 3: $\langle\ f(g(x),g(a))\ ,\ f(x,x)\ \rangle$

Now, $f(x,x)$ does not reduce, while $f(g(x),g(a))$ has only two reducts: $f(g(x),b)$ and $f(g(x),a)$. Hence, we see that the second critical pair does not converge, so this system is not locally confluent.

3. Use Knuth-Bendix completion to find a complete TRS with the same $\leftrightarrow_R$ relation as the above TRS.

Answer: as a reduction ordering, we choose LPO with $g \rhd b \rhd a$[1]

$$\mathcal{E} = \left\{ \begin{array}{rcl} f(g(x),g(b)) & \approx & f(x,x) \\ g(a) & \approx & b \\ b & \approx & a \end{array} \right\} \qquad \mathcal{R} = \emptyset$$

We first take $f(g(x),g(b)) \approx f(x,x)$, orient it from left to right, and thus add $f(g(x),g(b)) \to f(x,x)$ into $\mathcal{R}$. This rule has no critical pairs with itself (or anything else in $\mathcal{R}$), so no further steps are required. We proceed with:

$$\mathcal{E} = \left\{ \begin{array}{rcl} g(a) & \approx & b \\ b & \approx & a \end{array} \right\} \qquad \mathcal{R} = \left\{\ f(g(x),g(b)) \ \to\ f(x,x)\ \right\}$$

Next, we take $b \approx a$ from $\mathcal{E}$, and orient it from left to right (since $b \rhd a$). We add it to $\mathcal{R}$, but observe that it has a critical pair with the rule that's already in $\mathcal{R}$: $\langle\ f(g(x),g(a))\ ,\ f(x,x)\ \rangle$. Both sides are already in normal form with respect to the rules currently in $\mathcal{R}$, we so we add the pair to $\mathcal{E}$ as an equation. We proceed with:

$$\mathcal{E} = \left\{ \begin{array}{rcl} g(a) & \approx & b \\ f(g(x),g(a)) & \approx & f(x,x) \end{array} \right\} \qquad \mathcal{R} = \left\{ \begin{array}{rcl} f(g(x),g(b)) & \to & f(x,x) \\ b & \to & a \end{array} \right\}$$

Next, we take $g(a) \approx b$ from $\mathcal{E}$, and orient it from left to right (since $g \rhd b$). We add it to $\mathcal{R}$, but observe that it has a critical pair with the first rule in $\mathcal{R}$: $\langle\ f(b,g(b))\ ,\ f(a,a)\ \rangle$. The right-hand side of the CP is already in normal form with respect to $\mathcal{R}$, but the left-hand side is not; we reduce it: $f(b,g(b)) \Rightarrow f(a,g(b)) \Rightarrow f(a,g(a)) \Rightarrow f(a,b) \Rightarrow f(a,a)$. Thus, this critical pair converges, and we do not need to add it to $\mathcal{E}$. We proceed with:

$$\mathcal{E} = \left\{\ f(g(x),g(a)) \ \approx\ f(x,x)\ \right\} \qquad \mathcal{R} = \left\{ \begin{array}{rcl} f(g(x),g(b)) & \to & f(x,x) \\ b & \to & a \\ g(a) & \to & b \end{array} \right\}$$

[1] Note that we could also have chosen a different ordering, but with this ordering we will be able to orient the original rules as they are, and thus reuse the critical pairs we computed in the previous question.

Next, we take $f(g(x), g(a)) \approx f(x,x)$ from $\mathcal{E}$, and orient it from left to right (since $g(x) \succ x$ by the (sub) rule). We add it to $\mathcal{R}$, and find two critical pairs, both with the third rule in there: $\langle\ f(b, g(a))\ ,\ f(a,a)\ \rangle$ and $\langle\ f(g(x), b)\ ,\ f(x,x)\ \rangle$.

For the first, we normalise: $f(b, g(a)) \Rightarrow f(a, g(a)) \Rightarrow f(a, b) \Rightarrow f(a, a)$, so this critical pair converges and does not need to be added to $\mathcal{E}$.

For the second, we normalise: $f(g(x), b) \Rightarrow f(g(x), a)$. Thus, we do need to add $f(g(x), a) \approx f(x,x)$ to $\mathcal{E}$. We proceed with:

$$\mathcal{E} = \{\ f(g(x), a) \quad \approx \quad f(x,x)\ \} \qquad \mathcal{R} = \left\{ \begin{array}{rcl} f(g(x), g(b)) & \rightarrow & f(x,x) \\ b & \rightarrow & a \\ g(a) & \rightarrow & b \\ f(g(x), g(a)) & \rightarrow & f(x,x) \end{array} \right\}$$

Next, we take the one remaining equation $f(g(x), a) \approx f(x,x)$ from $\mathcal{E}$, orient it from left to right, and add it to $\mathcal{R}$. It has a critical pair with the third rule of $\mathcal{R}$: $\langle\ f(b, a)\ ,\ f(a,a)\ \rangle$. However, this critical pair converges: $f(b, a) \Rightarrow f(a, a)$. So it does not need to be added into $\mathcal{E}$.

We end with:

$$\mathcal{E} = \{\} \qquad \mathcal{R} = \left\{ \begin{array}{rcl} f(g(x), g(b)) & \rightarrow & f(x,x) \\ b & \rightarrow & a \\ g(a) & \rightarrow & b \\ f(g(x), g(a)) & \rightarrow & f(x,x) \\ f(g(x), a) & \rightarrow & f(x,x) \end{array} \right\}$$

This TRS is terminating (as all rules are captured by LPO) and locally confluent (as all critical pairs converge with $\mathcal{R}$), and therefore is complete.