# Automated Reasoning

## Week 9. Term Rewriting

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Fall 2024

# Beyond predicate logic

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Recall: predicate logic

$$(\exists x[\mathtt{S}(x) \wedge \forall y[\mathtt{L}(y) \rightarrow \mathtt{A}(x,y)]]) \wedge$$

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In practice: equality is important!

$$\forall x[\forall y[\mathtt{suc}(x) = \mathtt{suc}(y) \rightarrow x = y]]$$

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Prerequisite: **term rewriting.**

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Challenge: analysing functional programs

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let rec conc xs ys =
    match xs with
        | [] ⇒ ys
        | h :: t ⇒ h :: (conc t ys)
;;
```

```
let rec rev xs =
    match xs with
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Given: an **arity** function, that maps each function symbol to a non-negative integer

- every variable is a term
- if $\mathtt{f} \in \Sigma$ and $arity(\mathtt{f}) = n$ and $s_1, \ldots, s_n$ are terms, then $\mathtt{f}(s_1, \ldots, s_n)$ is a term

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- $\mathtt{mul}(\mathtt{add}(\mathtt{0}, \mathtt{0}), \mathtt{s}(y))$

# Class exercise

Design a signature $(\Sigma, arity)$ that contains lists of natural numbers.

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Alter the signature to handle lists of integers.

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- $x \Rightarrow \mathtt{min}(\mathtt{min}(x))$
- $\mathtt{mul}(x, \mathtt{0}) \Rightarrow \mathtt{mul}(y, \mathtt{0})$

# Reducing terms

**Intuition:**

$\mathtt{min}(\mathtt{add}(\mathtt{s}(\mathtt{0}), \mathtt{s}(\mathtt{s}(\mathtt{0}))))$ **rewrites** to $\mathtt{min}(\mathtt{s}(\mathtt{add}(\mathtt{s}(\mathtt{0}), \mathtt{s}(\mathtt{0}))))$ using the rule $\mathtt{add}(x, \mathtt{s}(y)) \Rightarrow \mathtt{s}(\mathtt{add}(x, y))$.

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Ingredient: a **substitution** is a function from variables to terms.

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\begin{array}{rcl}
x\gamma & = & \gamma(x) \\
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Define:

- $\ell\gamma \Rightarrow_{\mathcal{R}} r\gamma$ for all $\ell \Rightarrow r \in \mathcal{R}$, all $\gamma$
- $\mathtt{f}(s_1, \dots, s_i, \dots, s_n) \Rightarrow_{\mathcal{R}} \mathtt{f}(s_1, \dots, t_i, \dots, s_n)$ if $s_i \Rightarrow_{\mathcal{R}} t_i$

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Example:

$$\begin{array}{ccc} \mathtt{min}(\underbrace{\mathtt{add}(\mathtt{s}(\mathtt{0}),\mathtt{s}(\mathtt{s}(\mathtt{0})))}) & \Rightarrow_{\mathcal{R}} & \mathtt{min}(\mathtt{s}(\mathtt{add}(\mathtt{s}(\mathtt{0}),\mathtt{s}(\mathtt{0})))) \\ \mathtt{add}(\;x\;,\mathtt{s}(\;y\;)) & \Rightarrow & \mathtt{s}(\mathtt{add}(\;x\;,\;y\;)) \end{array}$$

# Class exercise

Let

$$\mathcal{R} = \left\{ \begin{array}{rcl} \mathtt{add}(x, \mathtt{s}(y)) & \Rightarrow & \mathtt{s}(\mathtt{add}(x, y)) \\ \mathtt{add}(x, \mathtt{p}(y)) & \Rightarrow & \mathtt{p}(\mathtt{add}(x, y)) \\ \mathtt{add}(x, \mathtt{0}) & \Rightarrow & x \\ \mathtt{s}(\mathtt{p}(x)) & \Rightarrow & x \\ \mathtt{p}(\mathtt{s}(x)) & \Rightarrow & x \end{array} \right\}$$

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Question: what can we reduce the following term to?

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Question: what can we reduce the following term to?

$$\mathtt{s}(\mathtt{add}(\mathtt{0}, \mathtt{p}(\mathtt{s}(\mathtt{0}))))$$

Answer: two options!

- $\mathtt{s}(\mathtt{add}(\mathtt{0}, \mathtt{0}))$
- $\mathtt{s}(\mathtt{p}(\mathtt{add}(\mathtt{0}, \mathtt{s}(\mathtt{0}))))$

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Example:

$$\mathtt{s}(\mathtt{add}(\mathtt{0}, \underline{\mathtt{p}(\mathtt{s}(\mathtt{0}))})) \quad \Rightarrow_{\mathcal{R}} \quad \mathtt{s}(\mathtt{add}(\mathtt{0}, \underline{\mathtt{0}}))$$

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Exercise: find a different reduction to normal form for $\mathtt{s}(\mathtt{add}(\mathtt{0}, \mathtt{p}(\mathtt{s}(\mathtt{0}))))$

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$$\begin{array}{rcl} \mathtt{add}(\mathtt{0}, y) & \Rightarrow & y \\ \mathtt{add}(\mathtt{s}(x), y) & \Rightarrow & \mathtt{s}(\mathtt{add}(x, y)) \end{array}$$

define the equations:

$$\begin{array}{lrcl} \forall y. & \mathtt{add}(\mathtt{0}, y) & = & y \\ \forall x \forall y. & \mathtt{add}(\mathtt{s}(x), y) & = & \mathtt{s}(\mathtt{add}(x, y)) \end{array}$$

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For any model $(M, [\![\cdot]\!]_\alpha)$ that makes the given equalities true:

If $s \Rightarrow_{\mathcal{R}} t$ then $[\![s]\!]_\alpha = [\![t]\!]_\alpha$ for all $\alpha$.

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For any model $(M, [\![\cdot]\!]_\alpha)$ that makes the given equalities true:

If $s \Rightarrow_{\mathcal{R}} t$ then $[\![s]\!]_\alpha = [\![t]\!]_\alpha$ for all $\alpha$.

Therefore, the computation
$\mathtt{add}(\mathtt{s}(\mathtt{s}(\mathtt{0})), \mathtt{s}(\mathtt{s}(\mathtt{0}))) \Rightarrow^*_{\mathcal{R}} \mathtt{s}(\mathtt{s}(\mathtt{s}(\mathtt{s}(\mathtt{0}))))$ **proves** that $2 + 2 = 4$!

# Term rewriting and Prover9

We can also reason about reduction in Prover9:

```
formulas(assumptions).
R(a(0,x),x).
R(a(s(x),y),s(a(x,y))).
R(x,y) -> R(a(x,z),a(y,z)).
R(x,y) -> R(a(z,x),a(z,y)).
R(x,y) -> R(s(x),s(y)).
RR(x,x).
(RR(x,y) & R(y,z)) -> RR(x,z).
end_of_list.
formulas(goals).
RR(a(s(s(0)),s(s(0))),s(s(s(s(0))))).
end_of_list.
```

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let rec conc xs ys =
    match xs with
        | [] ⇒ ys
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let rec rev xs =
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# Functional programming

Computation: reduction to normal form

```
rev nil        = nil
rev (a:x)      = conc (rev x (a:nil))
conc nil x     = x
conc (a:x) y   = a:(conc x y)
```

# Functional programming

Computation: reduction to normal form

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\begin{array}{lcl}
\texttt{rev nil} & = & \texttt{nil} \\
\texttt{rev } (a\texttt{:}x) & = & \texttt{conc (rev } x \texttt{ (}a\texttt{:nil))} \\
\texttt{conc nil } x & = & x \\
\texttt{conc } (a\texttt{:}x) \ y & = & a\texttt{:(conc } x \ y\texttt{)}
\end{array}
$$

Hence, this corresponds to the following term rewriting system:

$$
\begin{array}{rcl}
\texttt{rev}(\texttt{nil}) & \Rightarrow & \texttt{nil} \\
\texttt{rev}(a : \texttt{nil}) & \Rightarrow & \texttt{conc}(\texttt{rev}(x, a : \texttt{nil})) \\
\texttt{conc}(\texttt{nil}, x) & \Rightarrow & x \\
\texttt{conc}(a : x, y) & \Rightarrow & a : \texttt{conc}(x, y)
\end{array}
$$

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$$
\begin{array}{rcl}
\mathtt{rev}(\mathtt{nil}) & \Rightarrow & \mathtt{nil} \\
\mathtt{rev}(a : \mathtt{nil}) & \Rightarrow & \mathtt{conc}(\mathtt{rev}(x, a : \mathtt{nil})) \\
\mathtt{conc}(\mathtt{nil}, x) & \Rightarrow & x \\
\mathtt{conc}(a : x, y) & \Rightarrow & a : \mathtt{conc}(x, y)
\end{array}
$$

Then we have a reduction to normal form:

$$
\begin{array}{ll}
\mathtt{rev}(x : y : \mathtt{nil}) & \Rightarrow \\
\mathtt{conc}(\mathtt{rev}(y : \mathtt{nil}), x : \mathtt{nil}) & \Rightarrow \\
\mathtt{conc}(\mathtt{conc}(\mathtt{rev}(\mathtt{nil}), y : \mathtt{nil}), x : \mathtt{nil}) & \Rightarrow \\
\mathtt{conc}(\mathtt{conc}(\mathtt{nil}, y : \mathtt{nil}), x : \mathtt{nil}) & \Rightarrow \\
\mathtt{conc}(y : \mathtt{nil}, x : \mathtt{nil}) & \Rightarrow \\
y : \mathtt{conc}(\mathtt{nil}, x : \mathtt{nil}) & \Rightarrow \\
y : x : \mathtt{nil} &
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- $\mathcal{R}$ is **locally confluent** (= weak Church-Rosser, WCR):
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The converse is not true: consider the TRS with rules:

$$\begin{array}{lcl} \mathtt{a} & \Rightarrow & \mathtt{a} \\ \mathtt{a} & \Rightarrow & \mathtt{b} \end{array}$$

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**Proof:** Assume $t$ has two normal forms $u, u'$.
Then by confluence there is a $v$ such that $u \Rightarrow^*_{\mathcal{R}} v$ and $u' \Rightarrow^*_{\mathcal{R}} v$.
Since $u, u'$ are normal forms we have $u = v = u'$. $\square$

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This is a very useful combination!

# Decidability of termination

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Nevertheless: often doable!

# Well-founded ordering

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There are many ways to generate such an ordering!

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Needed: $\texttt{add}(\texttt{0}, \texttt{0}) \succ \texttt{0}$, $\texttt{add}(\texttt{0}, \texttt{add}(x, y)) \succ \texttt{add}(x, y), \ldots$

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Such an ordering is called a **reduction ordering**.

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\begin{aligned}
\mathtt{add}(\mathtt{0}, y) &\Rightarrow y \\
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If $\ell \succ_{\mathrm{LPO}} r$ for all rules in $\mathcal{R}$, then the TRS with rules $\mathcal{R}$ is terminating.

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**Theorem**

If $\ell \succ_{\mathrm{LPO}} r$ for all rules in $\mathcal{R}$, then the TRS with rules $\mathcal{R}$ is terminating.

**Proof.** $\succ_{\mathrm{LPO}}$ is:

- stable: if $s \succ_{\mathrm{LPO}} t$ then $s\sigma \succ_{\mathrm{LPO}} t\sigma$
- monotonic: if $s \succ_{\mathrm{LPO}} t$ then $\mathtt{f}(\ldots, s, \ldots) \succ_{\mathrm{LPO}} \mathtt{f}(\ldots, t, \ldots)$

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- monotonic: if $s \succ_{\mathrm{LPO}} t$ then $\mathtt{f}(\dots, s, \dots) \succ_{\mathrm{LPO}} \mathtt{f}(\dots, t, \dots)$
- well-founded: there is no infinite decreasing sequence

# Ackermann example

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Consider the **Ackermann function:**

$$A(m,n) = \begin{cases} n+1 & \text{if } m = 0 \\ A(m-1,1) & \text{if } m > 0 \text{ and } n = 0 \\ A(m-1, A(m, n-1)) & \text{if } m > 0 \text{ and } n > 0 \end{cases}$$

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Looks simple, but $A(4,2)$ has 19,729 decimal digits.

Expressed as a TRS:

$$\begin{array}{rcl} \mathtt{A}(\mathtt{0}, x) & \Rightarrow & \mathtt{s}(x) \\ \mathtt{A}(\mathtt{s}(x), \mathtt{0}) & \Rightarrow & \mathtt{A}(x, \mathtt{s}(\mathtt{0})) \\ \mathtt{A}(\mathtt{s}(x), \mathtt{s}(y)) & \Rightarrow & \mathtt{A}(x, \mathtt{A}(\mathtt{s}(x), y)) \end{array}$$

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For example, for the last rule:

$$\mathtt{A}(\mathtt{s}(x), \mathtt{s}(y)) \succ_{\mathrm{LPO}} \mathtt{A}(x, \mathtt{A}(\mathtt{s}(x), y))$$

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We conclude termination.

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Problem: how to know if rules are oriented by LPO? (And find $\rhd$?)

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Problem reformulation:

*Do there exist a symbol ordering $\rhd$,*
*and a sequence of proof steps*
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- require: $\langle \mathtt{f} \rhd \mathtt{g} \rangle \leftrightarrow \neg\langle \mathtt{g} \rhd \mathtt{f} \rangle$ for all $\mathtt{f}, \mathtt{g}$ with $\mathtt{f} \neq \mathtt{g}$

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- require: $\langle \mathtt{f} \rhd \mathtt{g} \rangle \wedge \langle \mathtt{g} \rhd \mathtt{h} \rangle \to \langle \mathtt{f} \rhd \mathtt{h} \rangle$ for all $\mathtt{f}, \mathtt{g}, \mathtt{h}$

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- require: $\langle \mathtt{f} \rhd \mathtt{g} \rangle \wedge \langle \mathtt{g} \rhd \mathtt{h} \rangle \rightarrow \langle \mathtt{f} \rhd \mathtt{h} \rangle$ for all $\mathtt{f}, \mathtt{g}, \mathtt{h}$
- encode the requirements for each inequality $\ell \succ_{\mathrm{LPO}} r$!

# Automation as one big formula

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$$\mathtt{f}(\mathtt{g}(x)) \succ_{\mathrm{LPO}} \mathtt{h}(\mathtt{f}(x))$$

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$$
\begin{array}{ll}
\mathtt{g}(x) \succeq_{\text{LPO}} \mathtt{h}(\mathtt{f}(x)) & \vee \\
(\ \langle \mathtt{f} \rhd \mathtt{h} \rangle \wedge \mathtt{f}(\mathtt{g}(x)) \succ_{\text{LPO}} \mathtt{f}(x)\ ) &
\end{array}
$$

# Automation as one big formula

$$
\begin{array}{ll}
x \succeq_{\text{LPO}} \mathtt{h}(\mathtt{f}(x)) & \vee \\
(\ \langle \mathtt{g} \rhd \mathtt{h} \rangle \wedge \mathtt{g}(x) \succ_{\text{LPO}} \mathtt{f}(x)\ ) & \vee \\
(\ \langle \mathtt{f} \rhd \mathtt{h} \rangle \wedge (\ \mathtt{g}(x) \succeq_{\text{LPO}} \mathtt{f}(x) \vee & \\
\qquad\qquad (\ \mathtt{f}(\mathtt{g}(x)) \succ_{\text{LPO}} x \wedge \mathtt{g}(x) \succ_{\text{LPO}} x\ )\ )\ ) &
\end{array}
$$

# Automation as one big formula

$$
\begin{array}{l}
\bot \\
( \langle \mathtt{g} \rhd \mathtt{h} \rangle \land \quad (x \succeq_{\mathrm{LPO}} \mathtt{f}(x) \lor (\langle \mathtt{g} \rhd \mathtt{f} \rangle \land \mathtt{g}(x) \succ_{\mathrm{LPO}} x)\ ) \\
( \langle \mathtt{f} \rhd \mathtt{h} \rangle \land \ ( \ (x \succeq_{\mathrm{LPO}} \mathtt{f}(x) \lor (\langle \mathtt{g} \rhd \mathtt{f} \rangle \land \mathtt{g}(x) \succ_{\mathrm{LPO}} x)\ ) \lor (\top \land \top)\ )\ )
\end{array}
$$

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$$
\begin{array}{l}
\bot \hfill \lor \\
(\ \langle \mathtt{g} \rhd \mathtt{h} \rangle \land \quad (\bot \lor (\langle \mathtt{g} \rhd \mathtt{f} \rangle \land \top)\ ) \hfill \lor \\
(\ \langle \mathtt{f} \rhd \mathtt{h} \rangle \land \ (\ (\bot \lor (\langle \mathtt{g} \rhd \mathtt{f} \rangle \land \top)\ ) \lor (\top \land \top)\ )\ )
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$$

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\begin{array}{l}
\bot \hfill \vee \\
(\langle \mathtt{g} \rhd \mathtt{h} \rangle \wedge \quad (\bot \vee (\langle \mathtt{g} \rhd \mathtt{f} \rangle \wedge \top))) \hfill \vee \\
(\langle \mathtt{f} \rhd \mathtt{h} \rangle \wedge \ ( \ (\bot \vee (\langle \mathtt{g} \rhd \mathtt{f} \rangle \wedge \top)) \vee (\top \wedge \top)))
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\begin{array}{l}
\bot \quad \vee \\
(\,\langle \mathtt{g} \rhd \mathtt{h}\rangle \wedge (\bot \vee (\langle \mathtt{g} \rhd \mathtt{f}\rangle \wedge \top))\,) \quad \vee \\
(\,\langle \mathtt{f} \rhd \mathtt{h}\rangle \wedge (\,(\bot \vee (\langle \mathtt{g} \rhd \mathtt{f}\rangle \wedge \top)) \vee (\top \wedge \top)\,)\,)
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This approach has some downsides:

- How to recover the proof that each $\ell \succ_{\mathrm{LPO}} r$?

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\bot \hspace{20em} \vee \\
(\, \langle \mathtt{g} \rhd \mathtt{h} \rangle \wedge \quad (\bot \vee (\langle \mathtt{g} \rhd \mathtt{f} \rangle \wedge \top)\,) \,) \hspace{6em} \vee \\
(\, \langle \mathtt{f} \rhd \mathtt{h} \rangle \wedge \; (\; (\bot \vee (\langle \mathtt{g} \rhd \mathtt{f} \rangle \wedge \top)\,) \vee (\top \wedge \top)\,)\,)
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\begin{array}{l}
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(\ \langle \mathtt{g} \rhd \mathtt{h} \rangle \wedge \qquad (x \succeq_{\mathrm{LPO}} \mathtt{f}(x) \vee (\langle \mathtt{g} \rhd \mathtt{f} \rangle \wedge \underline{\mathtt{g}(x) \succ_{\mathrm{LPO}} x})\ ) \\
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*Whenever* $a \succ_{\mathrm{LPO}} b$ *is used in the proof of* $s \succ_{\mathrm{LPO}} t$*:*
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Aside from the variables $\langle \texttt{f} \rhd \texttt{g} \rangle$, we introduce:

For every subterm $a$ of $s$;

for every subterm $b$ of $t$;

for every relation $\sharp$ in $\{\succ_{\mathrm{LPO}}, \succ_{\mathrm{LPO}}^{\text{sub}}, \succ_{\mathrm{LPO}}^{\text{copy}}, \succ_{\mathrm{LPO}}^{\text{lex}}\}$:

a variable $\langle a \sharp b \rangle$.

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Important observation:

> *Whenever $a \succ_{\text{LPO}} b$ is used in the proof of $s \succ_{\text{LPO}} t$: $a$ is a subterm of $s$, and $b$ is a subterm of $t$.*

Aside from the variables $\langle \mathtt{f} \rhd \mathtt{g} \rangle$, we introduce:

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a variable $\langle a \sharp b \rangle$.

Denote $\langle a \succeq_{\text{LPO}} b \rangle$ for either $\top$ (if $a = b$) or $\langle a \succ_{\text{LPO}} b \rangle$.

# Defining formulas per variable

For each variable $\langle a \sharp b \rangle$ we now require the **defining formula**.

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For example, if $a = \mathtt{f}(x, s, y)$ and $b = \mathtt{f}(x, t, u)$ with $s \neq t$:

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- $\langle a \succ_{\mathrm{LPO}}^{\mathsf{sub}} b \rangle \to \langle x \succeq_{\mathrm{LPO}} b \rangle \vee \langle s \succeq_{\mathrm{LPO}} b \rangle \vee \langle y \succeq_{\mathrm{LPO}} b \rangle$

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- $\langle a \succ_{\mathrm{LPO}} b \rangle \to \langle a \succ_{\mathrm{LPO}}^{\text{sub}} b \rangle \vee \langle a \succ_{\mathrm{LPO}}^{\text{copy}} b \rangle \vee \langle a \succ_{\mathrm{LPO}}^{\text{lex}} b \rangle$
- $\langle a \succ_{\mathrm{LPO}}^{\text{sub}} b \rangle \to \langle x \succeq_{\mathrm{LPO}} b \rangle \vee \langle s \succeq_{\mathrm{LPO}} b \rangle \vee \langle y \succeq_{\mathrm{LPO}} b \rangle$
- $\neg \langle a \succ_{\mathrm{LPO}}^{\text{copy}} b \rangle$ (since $\mathtt{f} \rhd \mathtt{f}$ cannot hold)

# Defining formulas per variable

For each variable $\langle a \sharp b \rangle$ we now require the **defining formula.**

For example, if $a = \mathtt{f}(x, s, y)$ and $b = \mathtt{f}(x, t, u)$ with $s \neq t$:

- $\langle a \succ_{\mathrm{LPO}} b \rangle \to \langle a \succ_{\mathrm{LPO}}^{\textsf{sub}} b \rangle \vee \langle a \succ_{\mathrm{LPO}}^{\textsf{copy}} b \rangle \vee \langle a \succ_{\mathrm{LPO}}^{\textsf{lex}} b \rangle$
- $\langle a \succ_{\mathrm{LPO}}^{\textsf{sub}} b \rangle \to \langle x \succeq_{\mathrm{LPO}} b \rangle \vee \langle s \succeq_{\mathrm{LPO}} b \rangle \vee \langle y \succeq_{\mathrm{LPO}} b \rangle$
- $\neg \langle a \succ_{\mathrm{LPO}}^{\textsf{copy}} b \rangle$ (since $\mathtt{f} \rhd \mathtt{f}$ cannot hold)
- $\langle a \succ_{\mathrm{LPO}}^{\textsf{lex}} b \rangle \to \langle a \succ_{\mathrm{LPO}} x \rangle \wedge \langle a \succ_{\mathrm{LPO}} t \rangle \wedge \langle a \succ_{\mathrm{LPO}} u \rangle \wedge \langle s \succ_{\mathrm{LPO}} t \rangle$

# Defining formulas: formally

For all subterms $a$ of $s$ and $b$ of $t$, add the following formulas:

# Defining formulas: formally

For all subterms $a$ of $s$ and $b$ of $t$, add the following formulas:

- if $a = b$, then $\neg\langle a \sharp b\rangle$ for all $\sharp \in \{\succ_{\mathrm{LPO}}, \succ^{\text{sub}}_{\mathrm{LPO}}, \succ^{\text{copy}}_{\mathrm{LPO}}, \succ^{\text{lex}}_{\mathrm{LPO}}\}$
- otherwise, if $a$ is a variable: $\neg\langle a \sharp b\rangle$ for all $\sharp \in \{\succ_{\mathrm{LPO}}, \succ^{\text{sub}}_{\mathrm{LPO}}, \succ^{\text{copy}}_{\mathrm{LPO}}, \succ^{\text{lex}}_{\mathrm{LPO}}\}$
- otherwise, if $a = \mathtt{f}(a_1, \dots, a_n)$:
  - $\langle a \succ_{\mathrm{LPO}} b\rangle \rightarrow \langle a \succ^{\text{sub}}_{\mathrm{LPO}} b\rangle \vee \langle a \succ^{\text{copy}}_{\mathrm{LPO}} b\rangle \vee \langle a \succ^{\text{lex}}_{\mathrm{LPO}} b\rangle$
  - $\langle a \succ^{\text{sub}}_{\mathrm{LPO}} b\rangle \rightarrow \langle a_1 \succeq_{\mathrm{LPO}} b\rangle \vee \dots \vee \langle a_n \succeq_{\mathrm{LPO}} b\rangle$
  - if $b = \mathtt{f}(b_1, \dots, b_n)$, and $i$ is the lowest index such that $a_i \neq b_i$, then:
    - $\neg\langle a \succ^{\text{copy}}_{\mathrm{LPO}} b\rangle$
    - $\langle a \succ^{\text{lex}}_{\mathrm{LPO}} b\rangle \rightarrow \langle a \succ_{\mathrm{LPO}} b_1\rangle \wedge \dots \wedge \langle a \succ_{\mathrm{LPO}} b_n\rangle \wedge \langle a_i \succ_{\mathrm{LPO}} b_i\rangle$
  - otherwise, if $b = \mathtt{g}(b_1, \dots, b_m)$ with $\mathtt{f} \neq \mathtt{g}$ then:
    - $\langle a \succ^{\text{copy}}_{\mathrm{LPO}} b\rangle \rightarrow \langle \mathtt{f} \rhd \mathtt{g}\rangle \wedge \langle a \succ_{\mathrm{LPO}} b_1\rangle \wedge \dots \wedge \langle a \succ_{\mathrm{LPO}} b_m\rangle$
    - $\neg\langle a \succ^{\text{lex}}_{\mathrm{LPO}} b\rangle$
  - otherwise, $\neg\langle a \sharp b\rangle$ for $\sharp \in \{\succ^{\text{copy}}_{\mathrm{LPO}}, \succ^{\text{lex}}_{\mathrm{LPO}}\}$

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\langle \mathtt{f}(\mathtt{g}(x)) \succ_{\mathrm{LPO}} \mathtt{h}(\mathtt{f}(x)) \rangle & \to & \langle \mathtt{f}(\mathtt{g}(x)) \succ^{\text{sub}}_{\mathrm{LPO}} \mathtt{h}(\mathtt{f}(x)) \rangle \vee \\
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\langle \mathtt{f}(\mathtt{g}(x)) \succ^{\text{sub}}_{\mathrm{LPO}} \mathtt{h}(\mathtt{f}(x)) \rangle & \to & \langle \mathtt{g}(x) \succ_{\mathrm{LPO}} \mathtt{h}(\mathtt{f}(x)) \rangle \\
\langle \mathtt{f}(\mathtt{g}(x)) \succ^{\text{copy}}_{\mathrm{LPO}} \mathtt{h}(\mathtt{f}(x)) \rangle & \to & \langle \mathtt{f} \rhd \mathtt{h} \rangle \wedge \mathtt{f}(\mathtt{g}(x)) \succ_{\mathrm{LPO}} \mathtt{f}(x) \rangle \\
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\langle \mathtt{g}(x) \succ^{\text{copy}}_{\mathrm{LPO}} \mathtt{f}(x) \rangle & \to & \langle \mathtt{g} \rhd \mathtt{f} \rangle \wedge \mathtt{g}(x) \succ_{\mathrm{LPO}} x \rangle \\
\langle \mathtt{g}(x) \succ_{\mathrm{LPO}} x \rangle & \to & \langle \mathtt{g}(x) \succ^{\text{sub}}_{\mathrm{LPO}} x \rangle \vee \langle \mathtt{g}(x) \succ^{\text{copy}}_{\mathrm{LPO}} x \rangle \vee \langle \mathtt{g}(x) \succ \\
\langle \mathtt{g}(x) \succ^{\text{sub}}_{\mathrm{LPO}} x \rangle & \to & \top \\
 & \ldots & 
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# Finishing up the SAT encoding

We also require:

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These formulas are small, and easily converted to CNF by the Tseitin transformation.

A satisfying assignment immediately allows us to read off the proof!

# Reading off the proof

$$
\begin{array}{rcl}
\underline{\langle \mathtt{f}(\mathtt{g}(x)) \succ_{\mathrm{LPO}} \mathtt{h}(\mathtt{f}(x))\rangle} & \rightarrow & \underline{\langle \mathtt{f}(\mathtt{g}(x)) \succ^{\textsf{sub}}_{\mathrm{LPO}} \mathtt{h}(\mathtt{f}(x))\rangle} \vee \\
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\langle \mathtt{f}(\mathtt{g}(x)) \succ^{\textsf{copy}}_{\mathrm{LPO}} \mathtt{h}(\mathtt{f}(x))\rangle & \rightarrow & \langle \mathtt{f} \rhd \mathtt{h}\rangle \wedge \mathtt{f}(\mathtt{g}(x)) \succ_{\mathrm{LPO}} \mathtt{f}(x)\rangle \\
\neg\langle \mathtt{f}(\mathtt{g}(x)) \succ^{\textsf{lex}}_{\mathrm{LPO}} \mathtt{h}(\mathtt{f}(x))\rangle & & \\
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\underline{\langle \mathtt{g}(x) \succ^{\textsf{copy}}_{\mathrm{LPO}} \mathtt{h}(\mathtt{f}(x))\rangle} & \rightarrow & \underline{\langle \mathtt{g} \rhd \mathtt{h}\rangle} \wedge \underline{\langle \mathtt{g}(x) \succ_{\mathrm{LPO}} \mathtt{f}(x)\rangle} \\
\underline{\langle \mathtt{g}(x) \succ_{\mathrm{LPO}} \mathtt{f}(x)\rangle} & \rightarrow & \langle \mathtt{g}(x) \succ^{\textsf{sub}}_{\mathrm{LPO}} \mathtt{f}(x)\rangle \vee \underline{\langle \mathtt{g}(x) \succ^{\textsf{copy}}_{\mathrm{LPO}} \mathtt{f}(x)\rangle} \vee \\
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\underline{\langle \mathtt{g}(x) \succ_{\mathrm{LPO}} x\rangle} & \rightarrow & \underline{\langle \mathtt{g}(x) \succ^{\textsf{sub}}_{\mathrm{LPO}} x\rangle} \vee \langle \mathtt{g}(x) \succ^{\textsf{copy}}_{\mathrm{LPO}} x\rangle \vee \langle \mathtt{g}(x) \succ \\
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\end{array}
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  - either lexicographic or multiset, depending on the function symbol
- using a **quasi-ordering** for the symbol comparison $\rhd$
- if $\mathsf{c}$ is the smallest symbol in $\rhd$ and has arity $0$, letting $x \succeq_{\mathrm{LPO}} \mathsf{c}$ also for variables

# Subterm property

The lexicographic path ordering (and all variations) has the following property:

**Theorem**

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**Proof.** This easily follows from rule **(sub)**. $\square$

# Quiz

1. What is the difference between weak and strong normalisation?
2. What is the difference between local confluence and general confluence?
3. Use the lexicographic path ordering (by hand) to prove termination of:

$$\begin{aligned} \mathtt{f}(\mathtt{g}(x), \mathtt{g}(\mathtt{b})) &\Rightarrow \mathtt{f}(x, x) \\ \mathtt{g}(\mathtt{a}) &\Rightarrow \mathtt{b} \\ \mathtt{b} &\Rightarrow \mathtt{a} \end{aligned}$$

4. What properties should a relation $\succ$ satisfy to be a reduction order?

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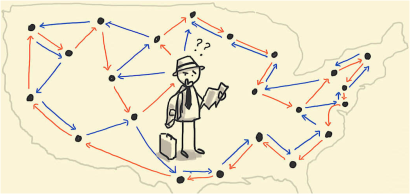
Suppose: you have to find a solution for the Traveling Salesman Problem.



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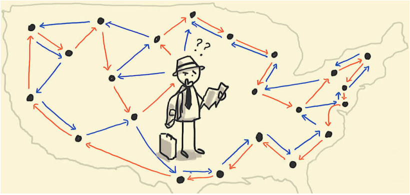


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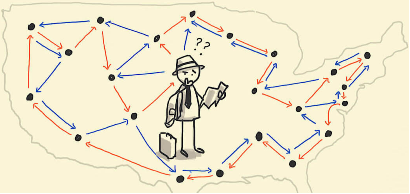


Problem: find **the shortest** route visiting a number of cities

Solution:

- encode, for given $N$, the problem "find a route $\leq N$" into SMT
- use binary search to find the smallest $N$ for which this is satisfiable