Exercises Category Theory and Coalgebra Lecture 4

The items labelled with $(*)$ are optional. If you have any questions, email `mark.szeles@ru.nl`. The deadline is 25 February 23:59, CET. Solutions may only be submitted in Brightspace. In case you upload hand-written notes, please use a scanner app (instead of photos). Explain your answers! Think carefully about all the properties that you need to prove in each exercise.

1. Give diagrammatic proofs of the coproduct equations

$$h \circ [f, g] = [h \circ f, h \circ g] \qquad \text{and} \qquad [\kappa_1, \kappa_2] = id.$$

2. Show that the category **PoSets** of partially ordered sets and monotone maps has finite products and coproducts.

   [They are essentially as in **Sets**, provided with appropriate order.]

3. Let $\mathbb{C}$ be a category with coproducts and fix an object $E \in \mathbb{C}$. Complete the definition of a category Exn of 'arrows with exceptions' where

   - objects are the objects X of $\mathbb{C}$
   - arrows $X \rightharpoonup Y$ in Exn are arrows $X \to Y + E$ in $\mathbb{C}$

   (a) Define identities and composition of Exn arrows.

   (b) Verify the category axioms.

   (c) Check that Exn has finite coproducts. Hint: the object part of the coproduct $X + Y$ is the same as in $\mathbb{C}$.

4. $(*)$ Let Inj denote the categories of sets and injective functions between them. Let Surj define the categories of sets and surjective functions between them. You do not need to prove that Inj and Surj are categories. Decide whether the following statements are true. Explain your answer briefly by either sketching a proof or providing a counterexample for each item.

   (a) Inj has all binary products.

   (b) Inj has all binary coproducts.

   (c) Surj has all binary products.

   (d) Surj has all binary coproducts.

5. $(*)$ The coproducts in **CMon**, the category of commutative monoids are deceptively simple. This is no longer true in the category **Mon** of monoids which are not assumed to be commutative. For this, consider the commutative monoid $(\mathbb{N}, +, 0)$ and recall the coproduct $\mathbb{N} \oplus \mathbb{N}$ in **CMon**. We show that $\mathbb{N} \oplus \mathbb{N}$ is no longer the coproduct in the category **CMon**.

(a) Let $M = \{l, r\}^*$ be the monoid of strings over the letters l, r with concatenation. Consider the diagram

$$\mathbb{N} \xrightarrow{i_1} M \xleftarrow{i_2} \mathbb{N} \tag{1}$$

where $i_1(n) = l^n$ and $i_2(n) = r^n$. Show that i_1, i_2 are monoid homomorphisms.

(b) Show that there exists no homomorphism h that would make the following diagram commute

$$\begin{tikzcd}
& M & \\
& \mathbb{N} \oplus \mathbb{N} \arrow[u, dotted, "h"'] & \\
\mathbb{N} \arrow[ur, "\kappa_1"'] \arrow[uur, bend left, "i_1"] & & \mathbb{N} \arrow[ul, "\kappa_2"] \arrow[uul, bend right, "i_2"']
\end{tikzcd}$$

(c) Show that M is in fact the coproduct $\mathbb{N} + \mathbb{N}$ in Mon, i.e. the diagram (1) is universal.