# mini hw-7

We have a directed, no multiple edges, no self-loop graph $G = (V, E)$, which has the property that if $(u, v) \in E$, $(v, u) \notin E$.

Consider we run DFS algorithm on $G$ to build a DFS forest consisting of several root trees. Please provide informal proof that there exists a cross edge from vertex $u$ to vertex $v$, $u$ and $v$ belongs to the same SCC(strongly connected component) **only if** their lowest common ancestor $w$ is also in the same SCC.

Note: The lowest common ancestor of $u$ and $v$ is the deepest vertex in a tree that has both $u$ and $v$ as descendants.

Your proof should be written briefly in pure text. If you need to write math formulas, please use COOL's math tool to format your answer, or TA may not understand it.

Hint:
1. Use the White Path Theorem!
2. You can review the definition of cross edge through the slide of week 8.

Because there exists a cross edge from $u$ to $v$, so they must in different subtree under a tree node.  If $u$ and $v$ belong to the same SCC, it means that there exist some paths that lead $u$ to $v$ and also $v$ to $u$.

1. There exists a cross edge from $u$ to $v$, so $u$ can go to $v$.
2. To prove $v$ can go to $u$ only if their lowest common ancestor $w$ is also in the same SCC, we can prove via contradiction.

   Suppose their lowest common ancestor $w$ is not in the same SCC. There must have no path go from $v$ to $w$, which also implies we can go from $u$ to $w$ since 1. . In a SCC, there must exist a back edge in its spanning tree. So if $u$ and $v$ are in the same SCC, then there must exist a back edge from $v$ to one of their common ancestors $k$. Then, it contructs a path from $v$ (to $k$ then )to $w$ and then $u$, $v$, and $w$ are in same SCC, which contradicts our assumption. Thus, their lowest common ancestor $w$ is also in the same SCC.