On the Shapley-Folkman Lemma

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Abstract

This report offers an exposition of the Shapley-Folkman Lemma and its extensions, a fundamental result in mathematics with significant applications in Economics, Optimization, and Probability Theory. It presents a proof of the original lemma and explores Starr's constructive algorithm, which generates a solution within the original space rather than simply demonstrating existence of a near-convex solution. The report also examines the lemma's role in economics and optimization, highlighting its significance for approximations in non-convex settings.

1 Introduction

The Shapley-Folkman Lemma was first published in Starr's 1969 paper investigating a pure exchange economy without assuming convex preferences common in prior studies [1]. Starr credits the lemma to Shapley and Folkman and a direct extension, the Shapley-Folkman Theorem, to one of their private correspondences and contributes an important corollary, the Shapley-Folkman-Starr Lemma. These results provide a way to quantify and upper-bound a notion of distance between the Minkowski Sum of a collection of sets and its convex hull, enabling us to make statements about the effects of convexifying nonconvex conditions. The lemma has applications in Economics, Optimization, and Probability Theory [2].

We state the original lemma and Starr's extensions formally in Section 2, providing a comprehensive proof for the original lemma. We also discuss Starr's approach in his later 1981 paper which demonstrates a constructive algorithm and proof of a variation of the Shapley-Folkman-Starr Lemma [3]. A high-level discussion about applications of the Shapley-Folkman Lemma is provided in Section 3. Finally, we conclude with an acknowledgement of recent works to improve Starr's bounds and possible directions for future work in Section 4.

2 The Shapley-Folkman Lemma

2.1 Preliminaries

We assume familiarity with common optimization concepts such as convex hulls and cones, and we outline some important results required to understand our exposition.

Theorem 2.1 (Carathéodory's Theorem [4]). *Let X be a non-empty subset of $\mathbb{R}^n$, then every nonzero vector from cone(X) can be represented as a positive combination of linearly independent vectors from X.*

In particular, any set of linearly independent vectors $X \subset \mathbb{R}^n$ can at most have size n [5]. So the corollary below follows.

Corollary 2.2 (Representation of Conic Elements in $\mathbb{R}^n$). *For a non-empty set $X \subset \mathbb{R}^n$, $\forall x \in cone(X)$, x can be represented as a positive combination of at most n linearly independent vectors from X.*

The following definitions are important for our discussion of Starr's constructive proof of the Shapley-Folkman-Starr Lemma. Below, we assume $S \subset \mathbb{R}^n$ and S compact.

Definition 2.3 (Circumradius[3]). *the circumradius, denote rad, of S is*

$$rad(S) := \inf_{x \in \mathbb{R}^n} \sup_{y \in S} |x - y|$$

We can think of this definition as fixing many points in $\mathbb{R}^n$ (indeed fixing all points in $\mathbb{R}^n$), and finding the smallest radius of a ball that would cover S from each fixed point. Then from these radii, we find the smallest radius. Thus, intuitively, the circumradius attempts to capture the size of the smallest ball centered anywhere that contains S. In contrast to the circumradius, we have the concept of inner radius.

Definition 2.4 (Inner Radius and Inner Diameter[3]). *The inner radius of S is*

$$r(S) = \sup_{x \in conv(S)} \inf_{T \subset S;\ T \text{ spans } x} rad(T)$$

where "T spans x" means in particular that some convex combination of points in T attains x. The inner diameter of S is

$$d(S) = 2r(S).$$

The inner radius fixes points x in the convex hull of S. By definition, x must be spanned by some set $T \subseteq S$. For each fixed x, the subset $T \subseteq S$ that spans x with the smallest circumradius is considered. The inner radius then picks the largest of these circumradii.

For any point in $s \in \text{conv}(S)$, there therefore must exist some point $x \in \mathbb{R}^n$ at which the ball $\mathbb{B}$ centered at x with radius $r(S)$ contains a subset of points $T \subseteq S$ that spans s. We know $s \in \mathbb{B}$ because $s \in \text{conv}(T) \subseteq \mathbb{B}$ by convexity of balls. Within $\mathbb{B}$, the furthest distance two points could assume is $d(S)$. Thus, the ball centered at s with radius $d(S)$ will include T which spans x.

2.2 The Shapley-Folkman Lemma

We first state the Lemma formally and demonstrate a proof of the Shapley-Folkman Lemma.

Theorem 2.5 (the Shapley-Folkman Lemma [6]). *For arbitrary, non-empty sets $D_1, ..., D_m \subseteq \mathbb{R}^n$,*

$$x \in conv \sum_{i=1}^{m} D_i \implies x = \sum_{i \in [1:m] \setminus I} D_i + \sum_{i \in I} conv(D_i)$$

for some index set I with cardinality $|I| \leq n$, $I \subset \mathbb{N}$.

Proof. We follow Bertsekas' approach [4, Page 229]. Consider any collection of m arbitrary, non-empty sets $D_1, D_2, ..., D_m \subseteq \mathbb{R}^n$ and an element $x \in \text{conv}\big(\sum_{i=1}^m D_i\big)$. We invoke a well-known result that the Minkowski Sum is well-behaved with respect to the convex hull operator, namely

$$\text{conv}\big(\sum_{i=1}^{m} D_i\big) = \sum_{i=1}^{m} \text{conv}(D_i),$$

to decompose x as the sum of some m elements $d_i \in \operatorname{conv}(D_i)$. In particular,

$$x = \sum_{i=1}^{m} d_i, \; d_i = \sum_{j=1}^{k_i} \alpha_{ij} d_{ij}$$

where $\alpha_{ij} > 0, \sum_{j=1}^{k_i} \alpha_{ij} = 1, k_i \in \mathbb{N}$. If $m \leq n$, then we choose $I = [1:m]$ and the statement of the theorem follows as required. Indeed for this I, $|I| \leq n$, $I \subset \mathbb{N}$, and

$$x = \sum_{i \in [1:m] \setminus [1:m]} D_i + \sum_{i \in I} \operatorname{conv}(D_i) = \sum_{i \in [1:m]} \operatorname{conv}(D_i),$$

which is trivially true since $x = \sum_{i=1}^{m} d_i$. So we assume $m > n$. We want to show that there can exist at most n of these d_i such that $d_i \in \operatorname{conv}(D_i) \setminus D_i$. We proceed by constructing the following vectors in $\mathbb{R}^{n+m}$

$$z = \begin{pmatrix} x \\ 1 \\ 1 \\ \ldots \\ 1 \end{pmatrix}, z_{1j_1} = \begin{pmatrix} d_{1j_1} \\ 1 \\ 0 \\ \ldots \\ 0 \end{pmatrix}, z_{2j_2} = \begin{pmatrix} d_{2j_2} \\ 0 \\ 1 \\ \ldots \\ 0 \end{pmatrix}, \ldots, z_{mj_m} = \begin{pmatrix} d_{mj_m} \\ 0 \\ 0 \\ \ldots \\ 1 \end{pmatrix}$$

for $j_1 \in \{1, 2, \ldots, k_1\}, j_2 \in \{1, 2, \ldots, k_2\}, \ldots, j_m \in \{1, 2, \ldots, k_m\}$. Clearly, we have

$$z = \sum_{i=1}^{m} \sum_{j=1}^{k_i} \alpha_{ij} z_{ij},$$

so z is a vector in the cone of the set $Z := \{z_{11}, z_{12}, \ldots, z_{1k_1}, z_{21}, z_{22}, \ldots, z_{2k_2}, \ldots, z_{mk_m}\}$. By **Corollary 2.3**, we know we can construct any vector $z \in \operatorname{cone}(Z)$ with a positive linear combination of $m+n$ linearly independent vectors in Z. Therefore we have

$$z = \sum_{i=1}^{m} \sum_{j=1}^{k_i} \beta_{ij} z_{ij}$$

for which at most $m+n$ coefficients β_{ij} are strictly positive and the rest are zero. Furthermore, $\forall l \in \{1, 2, \ldots, m\}$, we can exploit the $(n+l)$th entry of z to get

$$\Big(\sum_{i=1}^{m} \sum_{j=1}^{k_i} \beta_{ij} z_{ij}\Big)_{n+l} = \Big(\sum_{j=1}^{k_l} \beta_{lj} z_{lj}\Big)_{n+l} = \sum_{j=1}^{k_l} \beta_{lj} (z_{lj})_{n+l} = \sum_{j=1}^{k_l} \beta_{lj} = z_{n+l} = 1. \quad (1)$$

Now, we focus on the first n components of z which give us

$$x = \sum_{i=1}^{m} \sum_{j=1}^{k_i} \beta_{ij} d_{ij}.$$

By equation (1), we know that for each $i \in \{1, 2, \ldots, m\}$, $\sum_{j=1}^{k_i} \beta_{ij} = 1 > 0$, so it follows that for each i that there must exist some $j \in \{1, 2, \ldots, k_i\}$ such that $\beta_{ij} > 0$. Since we have at most $m+n$ positive coefficients, at most n additional coefficients β_{ij} are strictly positive. Then there exists at least $m-n > 0$ sets in $D_1, D_2, \ldots, D_m$ for which in the decomposition of their associated d_i, only one coefficient $\beta_{ij} > 0$ for some $j \in \{1, \ldots, k_i\}$ and the rest are zero (in particular, $\beta_{ij} = 1$). We use the index of these sets as I, and the statement of the theorem follows. $\square$

Using this result, Starr provides upper bounds on the notion of distance between the sum of an arbitrary collection of non-empty sets $D = D_1 + D_2 + ... + D_m$ and its convex hull $\text{conv}(D)$, expressed in the Shapley-Folkman Theorem and the Shapley-Folkman-Starr Lemma [1].

Theorem 2.6 (the Shapley-Folkman Theorem [1]). *Given a collection of non-empty sets $D_1, ..., D_m$, let $x \in conv(\sum_{i=1}^{m} D_i)$. Then there exists $d_i \in D_i, i = 1, ..., m$ such that*

$$||x - \sum a_i||^2 \leq \sum_{i=1}^{m} rad\ ^2(D_i)$$

Theorem 2.7 (the Shapley-Folkman-Starr Lemma [1]). *Given a collection of non-empty sets $D_1, ..., D_m$, let $x \in conv(\sum_{i=1}^{m} D_i)$. Then there exists $d_i \in D_i, i = 1, ..., m$ such that*

$$||x - \sum a_i||^2 \leq \sum_{i=1}^{m} R^*$$

where R^* is the sum of the $\min(m, n)$ *largest $r^2(D_i)$.*

2.3 A Constructive Approach

Many proofs with different approaches have been shown since Starr's original proof in 1969, but most are non-constructive and simply indicate the existence of some point in the sum of the original sets that satisfy the distance boundary. Starr demonstrates an elementary, constructive approach of the Shapley-Star-Folkman Lemma in 1981 which iteratively finds a point in the sum of the original sets that is relatively near the convex hull. We showcase his approach here.

Lemma 2.8 (Starr's Lemma [3]). *Let $U \subset \mathbb{R}^n$, U compact, $v \in \mathbb{R}^n$, $y \in conv(U)$. Then there is some $x \in U$ such that $|x - y| \leq d(U)$ and $v \cdot (x - y) \leq 0$.*

Proof. We consider the subset of points in $T \subseteq U$ with the smallest circumradius whose convex combination makes up y. By Caratheodory's Theorem [4], we know $|T| \leq n + 1$. We denote $k = |T|$. So $\forall i \in \{1, ..., k\}$, let $x_i \in T$, $\alpha_i \geq 0$, $\sum_{i=1}^{k} \alpha_i = 1$ such that

$$\sum_{i=1}^{k} \alpha_i x_i = y.$$

Then

$$\sum_{i=1}^{k} \alpha_i x_i - y = 0$$

$$\iff \sum_{i=1}^{k} \alpha_i x_i - \sum_{i=1}^{k} \alpha_i y = 0$$

$$\iff \sum_{i=1}^{k} \alpha_i (x_i - y) = 0$$

$$\iff v^T \sum_{i=1}^{k} \alpha_i (x_i - y) = 0$$

$$\iff \sum_{i=1}^{k} \alpha_i v^T (x_i - y) = 0.$$

Since the above equation sums to 0, there must be some $j \in \{1, ..., k\}$ such that $v^T(x_j - y) \leq 0$. For this x_j, we also certainly have $|x_j - y| \leq d(U)$ since T was chosen to be the set of points that spans y ($y \in \text{conv}(T)$) with the smallest circumradius. $\square$

Now, we consider a collection of sets $S_1, ..., S_m$ such that $\forall i \in \{1, ..., m\}, d(S_i) \leq D$ is a constant upper bound on the inner diameter of each set in the collection. Starr proceeds with the following proposition.

Proposition 2.9 (Starr's Proposition [3]). *For $x \in conv(\sum_{i=1}^m S_i)$, there exists $x_i \in S_i$ so that*

$$|x - \sum_{i=1}^{m} x_i| \leq D\sqrt{m}$$

Finally, consider $y \in \sum_{i=1}^m \text{conv}(S_i)$ and decompose y as

$$y = \sum_{i=1}^{m} y_i$$

for $y_i \in \text{conv}(S_i)$. We first choose $x_1 \in S_1$ such that $|y_1 - x_1| \leq D$, which exists by **Proposition 2.9**. Then, we find $x_2 \in S_2$ such that $|y_2 - x_2| \leq D$ and $(y_1 - x_1)^T(y_2 - x_2) \leq 0$, which exists by **Lemma 2.8**. We once again employ **Proposition 2.9** to get

$$|\sum_{i=1}^{2} y_i - \sum_{i=1}^{2} x_i| \leq D\sqrt{2}.$$

By performing the same steps iteratively, we obtain

$$|\sum_{i=1}^{m} y_i - \sum_{i=1}^{m} x_i| = |y - \sum_{i=1}^{m} x_i| \leq D\sqrt{m}.$$

Using this approach, we can obtain $\sum_{i=1}^m x_i$ which is the point in $\sum_{i=1}^m S_i$ relatively close to y. An important observation is that the average discrepancy of the approximation, $D\sqrt{m}/m \to 0$ as $m \to 0$, suggesting that the average discrepancy of this method is reduced as we consider more sets.

3 Applications

3.1 Economics

The first and perhaps most notable application of the Shapley-Folkman-Starr Lemma was in Starr's 1969 term paper. In this work, Starr studies the behavior of an economy without convex consumer preferences by approximating the aggregate demand using the sum of the convex hull of the individual preferences [1]. By applying his lemma, Starr demonstrates that as the number of agents in the economy grew, an equilibrium for the convexified model can be used to closely approximate a pseudo-equilibrium, or "quasi-equilibria", for the original non-convex model.

Following his example, the Shapley-Folkman-Starr Lemma has since seen many adaptation into the study of non-convex economies.

3.2 Optimization

Many problems in optimization are concerned with non-convex spaces. Perhaps the most straightforward approach to give these problems convex structure is to optimize over the convex hull of the feasible space instead. The Shapley-Folkman Lemma enables us to bound the error between the solution we find from this approach and the solution to the original problem.

In particular, in additive optimization problems in which the objective $f(x)$ is separable into m components, we have

$$f(x) = \sum_{i=1}^{m} f_i(x_i)$$

in which x_i are the arguments for functions $f_i : \mathbb{R}^n \to \mathbb{R}$ not necessarily convex. Consider an optimal solution $x_{\min}$ that minimizes the convexified objective function $\sum_{i=1}^{m} \operatorname{conv}(f_i(x_i))$. Then this solution resides in the sum of the convex hulls of the epigraphs of the individual functions. Namely,

$$(x_{\min}, f(x_{\min})) \in \sum_{i=1}^{m} \operatorname{conv}(\operatorname{Epigraph}(f_i)).$$

The Shapley-Folkman Lemma tells us we can decompose the solution $(x_{\min}, f(x_{\min}))$ into a few points that belong to the convexified epigraphs and many points that belong to the original epigraphs of each component function for large m. This property also demonstrates why convex methods may approximate the solution to large-scale, non-convex additive optimization problems well [2].

4 Conclusion

This survey was limited by constraints in time and domain expertise. An improved analysis would perform a more in-depth and formal treatment of the Applications section, which would involve first studying the necessary preliminaries in the relevant area of interest, such as in Economics, Probability theory, and Optimization.

Another potential direction to explore in discussion is the more recent works that improved on the bound in the Shapley-Folkman-Starr theorem. These works argue that the Shapley-Folkman-Starr upper bound is rarely attainable, so a case-by-case upper bound could be more appropriate and practical for certain types of optimization of problems [7][8].

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