1. (a) Let $m_1, m_2, n_1, n_2 \in \mathbf{Z}$ such that $m_1 + n_1\sqrt{2} = m_2 + n_2\sqrt{2}$. I want to show f is one-to-one, namely $m_1 = m_2$ and $n_1 = n_2$. Rearranging the equation, we obtain $m_1 - m_2 = \sqrt{2}(n_2 - n_1)$. Therefore, if $m_1 - m_2 \neq 0$ we have a contradiction. Hence, $m_1 - m_2 = 0 \implies m_1 = m_2$ and thus $n_1 = n_2$.

(b) Note first that there exists an infinite number of perfect squares ($\forall n \in \mathbf{N}$, take n^2). Now, let $m \in \mathbf{N}$. Then $m\sqrt{2} = \sqrt{2m^2}$. Also, let s^2 denote the largest perfect square smaller than $2m^2$ (since $m^2 < 2m^2$ we know these squares exist so we are able to take the maximum one bounded by $2m^2$). Further let s'^2 denote the smallest perfect square larger than $2m^2$ (since there are an infinite number of squares, take the smallest $> 2m^2$ by the well ordering principle). Note additionally that $2m^2$ is not a perfect square (strict inequalities). Then:

$$s^2 < 2m^2 < s'^2$$

$$\implies s < \sqrt{2m^2} < s'$$

Notice s and s' must be consecutive integers. Since otherwise our choice of s^2 and s'^2 wouldn't have been the largest perfect square smaller than $2m^2$ and smallest perfect square larger than $2m^2$ respectively (**contradiction**). Therefore,

$$0 < -s + \sqrt{2m^2} < 1$$

This process can be repeated for all $m \in \mathbf{N}$ to generate infinite solutions.

(c) From b), we know that $\exists m_0, n_0 \in \mathbf{Z}, m_0 + n_0\sqrt{2} \in (0,1)$. Let $\epsilon \in \mathbf{R}, \epsilon > 0$. Note that by corollary in class there exists $n \in \mathbf{N}$ such that $1/n < \epsilon$. Denote the number of digits of n as $k \in \mathbf{N}$. Then $1/10^{k+1} < 1/n$. Now, notice that for $j \in \mathbf{N}$:

$$0 < m_0 + n_0\sqrt{2} < 1$$
$$\implies 0 < (m_0 + n_0\sqrt{2})^j < 1$$

By setting $(m_0 + n_0\sqrt{2})^j$ to be smaller than $1/10^{k+1}$, we get

$$m_0 + n_0\sqrt{2} < \frac{1}{10^{\frac{k+1}{j}}}$$

For this to hold true, we simply have to enforce $j > k+1$ since $m_0 + n_0\sqrt{2} < 1$. So,

$$0 < (m_0 + n_0\sqrt{2})^{k+2} < 1/10^{k+1} < \epsilon$$

Finally, by the binomial theorem, we know that $(m_0 + n_0\sqrt{2})^{k+2}$ will have form $m + n\sqrt{2}$ (since even powers of $\sqrt{2} \in \mathbf{Z}$ and odd powers will have form $c\sqrt{2}$ for $c \in \mathbf{Z}$). Therefore $(m_0 + n_0\sqrt{2})^{k+2} \in S \cap (0, \epsilon) \neq \emptyset$ as desired.

(d) Consider non empty intervals (a, b), namely, b > a.

Case 1 ($a > 0, b > 0$): Let $m, n \in \mathbf{R}$. Note by c) we can find a solution $0 < m + n\sqrt{2} < b - a$. Since $0 < m + n\sqrt{2}$, I propose there exists a finite number of "steps" with step length $m + n\sqrt{2}$ we can take such that:

$$a < k(m + n\sqrt{2})$$
$$\implies \frac{a}{k} < m + n\sqrt{2}$$
$$\implies \frac{a}{k} < b - a$$
$$\implies \frac{a}{b-a} < k$$

By Archimedes, $\exists k \in \mathbf{N}$ such that $k > \frac{a}{b-a}$. Thus the proposition is true. Since k is a non-empty set in $\mathbf{N}$, we can choose $k_0 \in \mathbf{N}$ to be the minimum k by the well ordering principle. Then we have:

$$a < k_0(m + n\sqrt{2})$$

Note that $(k_0 - 1)(m + n\sqrt{2}) \leq a$, since otherwise k_0 wouldn't be the smallest k such that $a < k_0(m + n\sqrt{2})$ (**contradiction**). Note also that since $a < k_0(m + n\sqrt{2})$, $a - (m + n\sqrt{2}) < (k_0 - 1)(m + n\sqrt{2})$. So:

$$a - (m + n\sqrt{2}) < (k_0 - 1)(m + n\sqrt{2}) \leq a$$

$$a < k_0(m + n\sqrt{2}) \leq a + (m + n\sqrt{2})$$
$$a < k_0(m + n\sqrt{2}) < a + (b - a)$$
$$a < k_0(m + n\sqrt{2}) < b$$

Thus any positive interval has $S \cap (a, b) \neq \emptyset$.

Case 2 ($a < 0, b > 0$): Let $m, n \in \mathbf{R}$. Then by c) we can find $0 < m + n\sqrt{2} < b \implies a < m + n\sqrt{2} < b$.

Case 3 ($a < 0, b < 0$): Let $m, n \in \mathbf{R}$.Then using the process in case 1, find a solution $m + n\sqrt{2} \in \mathbf{S} \cap (-b, -a)$. Then $-m - n\sqrt{2} \in \mathbf{S} \cap (a, b)$.
Thus S is dense in $\mathbf{R}$ as desired.

2. Let $x \in \mathbf{R}$. For all x, I want to evaluate $\lim\limits_{n\to\infty} \dfrac{1}{1+nx}$.

Case 1 ($x = 0$): Then $\lim\limits_{n\to\infty} \dfrac{1}{1+nx} = \lim\limits_{n\to\infty} \dfrac{1}{1} = 1$.

Case 2 ($x > 0$): Given $\epsilon > 0$, choose an integer $N \geq \frac{\frac{1}{\epsilon}-1}{x}$. Then for all positive integers $n > N$, we have $n > \frac{\frac{1}{\epsilon}-1}{x} \implies \epsilon(xn+1) > 1$. Since $xn+1 > 0$, we get $\frac{1}{xn+1} < \epsilon$ and also $|\frac{1}{xn+1}| = \frac{1}{xn+1}$. Thus by limit definitions, $\lim\limits_{n\to\infty} \dfrac{1}{1+nx} = 0$.

Case 3 ($x < 0$): Given $\epsilon > 0$, choose an integer $N \geq \frac{\frac{1}{-\epsilon}-1}{x}$ sufficiently large such that $xN+1 < 0$. Then for all positive integers $n > N$, we have $n > \frac{\frac{1}{-\epsilon}-1}{x} \implies -\epsilon(xn+1) > 1$. We also have $n > N \implies nx+1 < Nx+1 < 0$. Thus, we get $\frac{1}{-(xn+1)} < \epsilon$ and also $|\frac{1}{xn+1}| = \frac{1}{-(xn+1)}$. Thus by limit definitions, $\lim\limits_{n\to\infty} \dfrac{1}{1+nx} = 0$.

(not used)

3. Let a_n a sequence. Assume a_n converges to A. I want to show that a_n^3 converges to A^3. Let $A \in \mathbf{R}$. First, note that:

$$|a_n^3 - A^3| = |(a_n - A)(a_n^2 + a_n A + A^2)|$$

By the Schwarz Inequality

$$|(a_n - A)(a_n^2 + a_n A + A^2)| \leq |a_n - A||a_n^2 + a_n A + A^2| \leq |a_n - A||a_n^2 + |a_n A| + A^2|$$

Since a_n converges to A, for some given $\epsilon_0 > 0, \exists N_0 \in \mathbf{N}$ such that for all integers $n > N_0$, $|a_n - A| < \epsilon_0 \implies A - \epsilon_0 < a_n < A + \epsilon_0$. Then $n > N_0$:

$$|a_n - A||a_n^2 + |a_n A| + A^2| \ < |a_n - A||(A + \epsilon_0)^2 + |A(A + \epsilon_0)| + A^2|$$

Notice that $(A + \epsilon_0)^2 > 0$ unless $A = -\epsilon_0$, in which case $A^2 = \epsilon^2 > 0$, so:

$$|(A + \epsilon_0)^2 + |A(A + \epsilon_0)| + A^2| > 0$$

Now, $\forall \epsilon > 0$, set $\eta = \dfrac{\epsilon}{|(A + \epsilon_0)^2 + |A(A + \epsilon_0)| + A^2|} > 0$. Then since a_n converges to A, we have that for each η, $\exists N \in \mathbf{N}$, and thus $N' = \max\{N, N_0\}$ such that for all integer $n > N'$:

$$|a_n - A| < \eta$$

$$\implies |a_n^3 - A^3| < |a_n - A||(A + \epsilon_0)^2 + |A(A + \epsilon_0)| + A^2| < \epsilon$$

Therefore by definition of convergence, a_n^3 converges to A^3 as desired.

I also want to show that $a_n^{1/3}$ converges to $A^{1/3}$. Denote $|a_n^{2/3} + a_n^{1/3}A^{1/3} + A^{2/3}|$ as c_n. Note first that

$$|a_n^{1/3} - A^{1/3}| = \frac{|a_n - A|}{|a_n^{2/3} + a_n^{1/3}A^{1/3} + A^{2/3}|} = \frac{|a_n - A|}{c_n}$$

Since a_n converges to A, for some given ϵ_0, $\exists N_0 \in \mathbf{N}$ such that for all integers $n > N_0$:

$$A - \epsilon_0 < a_n < A + \epsilon_0$$

Then for all $n > N_0$, $c_n = |a_n^{2/3} + a_n^{1/3}A^{1/3} + A^{2/3}|$ is finite (since a_n is bounded). Let $n > N_0$.

Case 1($c_n = 0$): Then $|a_n - A| = |(a_n^{1/3} - A^{1/3}) \times c_n| = 0$ for $n > N_0$. Thus for all $n > N_0$, we have $A = a_n \implies A^{1/3} = a_n^{1/3}$. So $\forall \epsilon > 0$, and integer $n > N_0$ we have:

$$|a_n^{1/3} - c_n^{1/3}| = 0 < \epsilon$$

Thus by definition, $a_n^{1/3}$ converges to $A^{1/3}$ as desired.

Case 2($c_n > 0$): Since a_n converges, we have $\forall \epsilon > 0, \eta = \epsilon \times c_n > 0$, so $\exists N' \in \mathbf{N}$, and thus $N = \max\{N', N_0\}$ such that for all integers $n > N$, we have:

$$|a_n - A| < \eta$$

$$|a_n^{1/3} - c_n^{1/3}| = \frac{|a_n - A|}{c_n} < \epsilon$$

Thus by definition, $a_n^{1/3}$ converges to $A^{1/3}$ as desired.

(not used)

4. (a) Let $M, m, b \in \mathbf{R}$. Assume $M > m$. Pick $x > \frac{b}{M-m} = R$. Then we get $(M-m)x > b \implies Mx > mx + b$ as desired.

(b) Let y_n a sequence and suppose y_n/n converges to $M \in \mathbf{R}$. Since y_n/n converges to M, for some given $\epsilon_0 > 0$, $\exists N_1 \in \mathbf{N}$ such that for all integers $n > N_1$, we have:

$$|\frac{y_n}{n} - M| = |M - \frac{y_n}{n}| < \epsilon_0 \implies y_n - n\epsilon < Mn < y_n + n\epsilon$$

$\forall m < M$,

Case 1 ($M - m - \epsilon_0 > 0$): Pick $N_0 = \lceil \frac{b}{M-m-\epsilon_0} \rceil$. Then for all integers $n > N_0$:

$$n > \lceil \frac{b}{M-m-\epsilon_0} \rceil \implies n > \frac{b}{M-m-\epsilon_0}$$

$$\implies (M - m - \epsilon_0)n > b$$

$$\implies Mn > (m + \epsilon_0)n + b$$

Then for all integers n such that $n > N = \max\{N_0, N_1\}$ we have:

$$y_n + n\epsilon_0 > (m + \epsilon_0)n - b \implies y_n > mn + b$$

as desired.

Case 2 ($M - m - \epsilon_0 < 0$): Pick $N_0 = \lceil \frac{-b}{M-m-\epsilon_0} \rceil$. Then for all integers $n > N_0$:

$$\implies Mn > (m + \epsilon_0)n + b$$

Then the same result follows as in Case 1.

Case 3 ($M - m - \epsilon_0 = 0$): Then simply pick another $\epsilon_1 > 0$ such that $\epsilon_1 \neq \epsilon_0$ and repeat the proof. Then case 1 or 2 will follow.

(c) I want to show $\forall \epsilon > 0, \exists N \in \mathbf{N}$ such that for all integers $n > N$, we have:

$$|y_n - Mn| < \epsilon \implies Mn - \epsilon < y_n < Mn + \epsilon \tag{1}$$

Consider $\epsilon > 0$. Then Since y_n/n converges to M, given $\epsilon_0 > 0$ we have:

$$(M - \epsilon_0)n < y_n < (M + \epsilon_0)n$$

Then pick $m = M - \epsilon_0 < M$ and $b = -\epsilon \in \mathbf{R}$. Then by b), there exists $N_1 \in \mathbf{N}$ such that for $n > N_1$:

$$mn + b = Mn - \epsilon < y_n$$

Note that since y_n/n converges to M, $\forall \eta > 0, \exists N \in \mathbf{N}$ such that $n > N \implies |\frac{y_n}{n} - M| = |\frac{-y_n}{n} + M| < \eta$. Therefore by definition, $-y_n/n$ converges to $-M$. Let sequence $x_n = -y_n$. So we have:

$$(-M - \epsilon_0)n < x_n < (-M + \epsilon_0)n$$

Then pick $m' = -M - \epsilon_0 < -M$ and $b' = -\epsilon \in \mathbf{R}$. Then by b), there exists $N_2 \in \mathbf{N}$ such that for $n > N_2$:

$$m'n + b' = -Mn - \epsilon < x_n = -y_n$$

$$m'n + b' = Mn + \epsilon > y_n$$

Therefore, $\forall \epsilon > 0, \exists N = \max\{N_1, N_2\}$ (note N_1, N_2 can be different for each ϵ) such that for all integers $n > N$, we have:

$$Mn - \epsilon < y_n < Mn + \epsilon$$

$$|y_n - Mn| < \epsilon$$

So by definition, $|y_n - Mn|$ converges to 0 as desired.

5. Let $\alpha, \beta \in \mathbf{R}$. If $\alpha = \beta$, we obtain:

$$\lim_{n\to\infty}(\alpha^n + \beta^n)^{1/n} = \lim_{n\to\infty}(2\alpha^n)^{1/n} = \lim_{n\to\infty} 2^{1/n} \cdot \lim_{n\to\infty} \alpha = \alpha = \max\{\alpha, \beta\}$$

as shown in class as desired. Next, without loss of generality, assume $\alpha > \beta$. Then

$$\lim_{n\to\infty}(\alpha^n + \beta^n)^{1/n} = \lim_{n\to\infty}(\alpha^n(1 + (\frac{\beta}{\alpha})^n))^{1/n} = \alpha \lim_{n\to\infty}(1 + (\frac{\beta}{\alpha})^n)^{1/n}$$

Note,

$$0 < \frac{\beta}{\alpha} < 1$$
$$\Rightarrow 0 < (\frac{\beta}{\alpha})^n < 1$$
$$\Rightarrow 1 < 1 + (\frac{\beta}{\alpha})^n < 2$$
$$\Rightarrow 1^{1/n} < (1 + (\frac{\beta}{\alpha})^n)^{1/n} < 2^{1/n}$$

for all $n \in \mathbf{N}$

Let $(a_n), (b_n)$ be two sequences such that $(a_n) = 1^{1/n}, (b_n) = 2^{1/n}$. Then $\lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n = 1$ as shown in class. So by the Squeeze theorem, $\lim_{n\to\infty}(1 + (\frac{\beta}{\alpha})^n)^{1/n} = 1$. Thus

$$\alpha \lim_{n\to\infty}(1 + (\frac{\beta}{\alpha})^n)^{1/n} = \alpha = \max\{a, b\}$$

as desired.

(not used)

6. (a) Assume for (x_n) a sequence of positive real numbers that $\lim_{n\to\infty} \frac{x_n+1}{x_n} < 1$. Namely, there exists $c \in \mathbf{R}, 0 < c < 1$, such that $\lim_{n\to\infty} \frac{x_{n+1}}{x_n} = c$. By definition, $\forall \epsilon > 0, \exists N \in \mathbf{N}$ such that $n > N \implies |\frac{x_{n+1}}{x_n} - c| < \epsilon$. Choose $0 < \epsilon_0 < c$, then $\exists N_0 \in \mathbf{N}$ such that $n > N_0 \implies |\frac{x_{n+1}}{x_n} - c| < \epsilon_0$, then we have:

$$-\epsilon_0 < \frac{x_{n+1}}{x_n} - c < \epsilon_0$$

$$x_n(-\epsilon_0 + c) < x_{n+1} < x_n(\epsilon_0 + c) \tag{1}$$

Notice that we can using (1) recursively by setting $n = N_0 + 1$, $n = N_0 + 2$, ..., $n = N_0 + j$ for $j \in \mathbf{N}$:

$$x_{N_0+1}(-\epsilon_0 + c) < x_{N_0+2} < x_{N_0+1}(\epsilon_0 + c)$$

$$x_{N_0+1}(-\epsilon_0 + c)^2 < x_{N_0+2}(-\epsilon_0 + c) < x_{N_0+3} < x_{N_0+2}(\epsilon_0 + c) < x_{N_0+1}(\epsilon_0 + c)^2$$

$$x_{N_0+1}(-\epsilon_0 + c)^j < x_{N_0+1+j} < x_{N_0+1}(\epsilon_0 + c)^j$$

Since $0 < \epsilon_0 < c$,

$$0 < x_{N_0+1+j} < x_{N_0+1}(\epsilon_0 + c)^j$$

Let $C = x_{N_0+1}, r = (\epsilon_0 + c)$. Then for $n > N_0$, we have

$$0 < x_n < x_{N_0+1}(\epsilon_0 + c)^{n-N_0-1}$$

(b) Let y_n, x_n two sequences such that $x_n \to 0$ as $n \to \infty$. Suppose $y_n = 1/x_n$ converges to some $L \in \mathbf{R}$. Then by definition, we can choose some real number $\epsilon > |L|$ and obtain an $N \in \mathbf{N}$ such that whenever $n > N$, we have:

$$|\frac{1}{x_n} - L| < \epsilon$$

$$-\epsilon + L < \frac{1}{x_n} < \epsilon + L$$

Consider the upper bound. Then $\epsilon + L > |L| + L \implies \epsilon + L > 0$. By taking the reciprocal we get

$$x_n > \frac{1}{\epsilon + L} > 0$$

Thus, for $n > N$, we have that x_n is bounded below by some positive real number (**contradiction** by definition since $x_n \to 0$ as $n \to \infty$).

(c) **1.** Note that for $n \in \mathbf{N}$, $\frac{10^n}{n!} > 0$. Also note that

$$\frac{10^{n+1}}{n+1!} / \frac{10^n}{n!} = \frac{10}{n+1}$$

Since $\frac{10}{n+1} < 1$ for sufficiently large n ($n > 10$), by a) we know that $\frac{10^n}{n!}$ converges to 0.

2. Let $x_n = \frac{n}{2^n}$. Note that x_n is a positive sequence. Then using limit theorems and example from class, I obtain

$$\lim_{n\to\infty} \frac{\frac{n+1}{2^{n+1}}}{\frac{n}{2^n}} = \lim_{n\to\infty} \frac{1}{2} + \frac{1}{2n} = \frac{1}{2} + \lim_{n\to\infty} \frac{1}{2n} = \frac{1}{2} < 1$$

Thus, by a), we know that x_n converges to 0 and by b), we know that $1/x_n = \frac{2^n}{n}$ cannot converge (diverges).

3. Note that for $n \in \mathbf{N}$, $\frac{2^{3n}}{3^{2n}} > 0$. Also note that

$$\frac{2^{3n+3}}{3^{2n+2}} / \frac{2^{3n}}{3^{2n}} = \frac{8}{9} < 1$$

Thus, by a), we obtain that $\frac{2^{3n}}{3^{2n}}$ converges to 0.

(not used)

7. Let x_n, y_n two sequences such that $(x_n y_n)$ converges (to some real L). Also assume y_n converges to $+\infty$. Then given some real $\epsilon_0 > 0$, we know that $\exists N_0$ such that whenever $n > N_0$:

$$-\epsilon_0 + L < x_n y_n < \epsilon_0 + L$$

$$\frac{-\epsilon_0 + L}{y_n} < x_n < \frac{\epsilon_0 + L}{y_n}$$

Case 1 ($L \geq 0$): Then

$$\frac{-\epsilon_0 - L}{y_n} < \frac{-\epsilon_0 + L}{y_n} < x_n < \frac{\epsilon_0 + L}{y_n}$$

$$|x_n| < \frac{\epsilon_0 + L}{|y_n|}$$

Since $y_n \to \infty$ as $n \to \infty$, we obtain that $\forall \eta > 0$, we have $\frac{\epsilon_0 + L}{\eta} > 0$, so $\exists N$ and additionally $\exists N_2 = \max\{N_0, N\}$, such that $\forall n > N_2$, $y_n \geq \frac{\epsilon_0 + L}{\eta} > 0$. So for $\forall \eta > 0, \exists N_2$ such that $\forall n > N_2$:

$$|x_n| < \frac{\epsilon_0 + L}{|y_n|} \leq \eta \implies |x_n| < \eta$$

So by definition, x_n converges to 0.

Case 2 ($L < 0$): Then let $S \in \mathbf{R}$ such that $S = -L > 0$.

$$\frac{-\epsilon_0 + L}{y_n} < x_n < \frac{\epsilon_0 + L}{y_n}$$

$$\implies \frac{-\epsilon_0 - S}{y_n} < x_n < \frac{\epsilon_0 - S}{y_n} < \frac{\epsilon_0 + S}{y_n}$$

$$|x_n| < \frac{\epsilon_0 + S}{|y_n|}$$

The rest of the proof follows identically from case 1.

(not used)

8. Let a_n, s_n be two sequences such that $s_n = \frac{a_1+a_2+...+a_n}{n}$.

(a) Assume a_n converges to a. $\forall \epsilon > 0$, pick $\eta = \epsilon/2 > 0$. Then since a_n converges to a, $\exists N_0 \in \mathbf{N}$ such that for all integers $n > N_0$, we get:

$$|a_n - a| < \eta \implies -\eta < a_n - a < \eta$$

Let $n > N_0$. Also let a constant $r = (a_1 - a) + ... + (a_N - a) \in \mathbf{R}$. Then:

$$\left|\frac{a_1 + a_2 + ... + a_N + a_{N+1} + ... + a_n}{n} - a\right|$$

$$= \left|\frac{(a_1 - a) + (a_2 - a) + ... + (a_N - a)}{n} + \frac{(a_{N+1} - a) + ... + (a_n - a)}{n}\right|$$

$$< \left|\frac{r}{n} + \frac{(n-N)\eta}{n}\right| < \left|\frac{r}{n}\right| + \eta \tag{*}$$

From Piazza, we know that $\frac{r}{n} \to 0$ for $r \in \mathbf{R}$. Thus, $\forall \epsilon/2 > 0$, since a_n converges to a, $\exists N_1 \in \mathbf{N}$ such that for all integers $n > N_1$, we get:

$$|\frac{r}{n}| < \epsilon/2$$

Thus we have for $n > \max\{N_0, N_1\}$, we use (*) to get:

$$\left|\frac{r}{n}\right| + \eta < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$$

Therefore,

$$\left|\frac{a_1 + a_2 + ... + a_N + a_{N+1} + ... + a_n}{n} - a\right| < \epsilon$$

So s_n converges to a as well by definition.

(b) A counter example is $a_n = (-1)^n$. $\frac{0}{n} \leq s_n = \frac{-1+1-1+...+(-1)^n}{n} = \frac{(-1+1)+(-1+1)+...+(-1)^n}{n} \leq \frac{1}{n}$ which converges to 0 by squeeze theorem since $1/n$ converges to 0 by example on Piazza. However, $(-1)^n$ does not converge as shown in class.

(a) Assume $a_n \to \infty$. $\forall J > 0$, choose $M > J > 0$. Then since $a_n \to \infty$, we know $\exists N \in \mathbf{N}$ such that for all integers $n > N$, we get:

$$a_n \geq M$$

Let $n > N$. Also let a constant $r = a_1 + ... + a_N \in \mathbf{R}$. Then:

$$s_n = \frac{r + a_{N+1} + ... + a_n}{n} \geq \frac{r + (n-N)M}{n}$$

Also note that for integers $n > \frac{MN-r}{M-J}$, we get:

$$n > \frac{MN - r}{M - J}$$

$$Mn - Jn > MN - r$$

$$Mn - MN + r > Jn$$

$$\frac{M(n-N) + r}{n} > J$$

Thus, for $n > \max\{\frac{MN-r}{M-J}, N\}$, we obtain:

$$s_n \geq \frac{r + (n-N)M}{n} > J$$

So by definition, $s_n \to \infty$.

(b) Assume $s_n \to \infty$. Then $\forall M > 0$, $\exists N \in \mathbf{N}$ such that for all integer $n > N$, we get

$$s_n = \frac{a_1 + a_2 + ... + a_n}{n} \geq M$$

$$s_n = M + \frac{a_{N+1} + ... + a_n}{n} \geq M$$

$$\frac{a_{N+1} + ... + a_n}{n} \geq 0$$

$$a_{N+1} + ... + a_n \geq 0$$