
A New Theory for Matrix Completion

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Abstract

Prevalent matrix completion theories reply on an assumption that the locations of the missing data are distributed uniformly and randomly (i.e., uniform sampling). Nevertheless, the reason for observations being missing often depends on the unseen observations themselves, and thus the missing data in practice usually occurs in a nonuniform and deterministic fashion rather than randomly. To break through the limits of random sampling, this paper introduces a new hypothesis called *isomeric condition*, which is provably weaker than the assumption of uniform sampling and arguably holds even when the missing data is placed irregularly. Equipped with this new tool, we prove a series of theorems for missing data recovery and matrix completion. In particular, we prove that the exact solutions that identify the target matrix are included as critical points by the commonly used nonconvex programs. Unlike the existing theories for nonconvex matrix completion, which are built upon the same condition as convex programs, our theory shows that nonconvex programs have the potential to work with a much weaker condition. Comparing to the existing studies on nonuniform sampling, our setup is more general.

1 Introduction

Missing data is a common occurrence in modern applications such as computer vision and image processing, reducing significantly the representativeness of data samples and therefore distorting seriously the inferences about data. Given this pressing situation, it is crucial to study the problem of recovering the unseen data from a sampling of observations. Since the data in reality is often organized in matrix form, it is of considerable practical significance to study the well-known problem of *matrix completion* [1] which is to fill in the missing entries of a partially observed matrix.

Problem 1.1 (Matrix Completion). *Denote the (i, j) th entry of a matrix as $[\cdot]_{ij}$. Let $L_0 \in \mathbb{R}^{m \times n}$ be an unknown matrix of interest. In particular, the rank of L_0 is unknown either. Given a sampling of the entries in L_0 and a 2D index set $\Omega \subseteq \{1, 2, \dots, m\} \times \{1, 2, \dots, n\}$ consisting of the locations of the observed entries, i.e., given*

$$\{[L_0]_{ij} | (i, j) \in \Omega\} \quad \text{and} \quad \Omega,$$

can we restore the missing entries whose indices are not included in Ω , in an exact and scalable fashion? If so, under which conditions?

Due to its unique role in a broad range of applications, e.g., structure from motion and magnetic resonance imaging, matrix completion has received extensive attentions in the literatures, e.g., [2–13].

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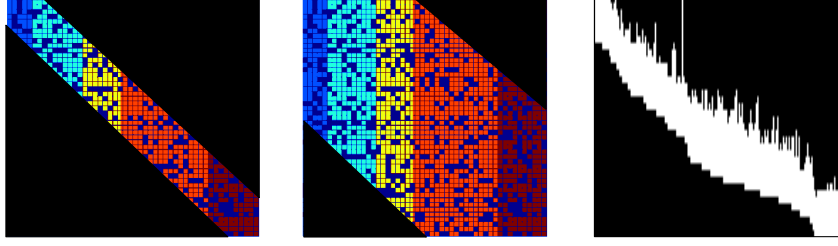


Figure 1: Left and Middle: Typical configurations for the locations of the observed entries. Right: A real example from the Oxford motion database. The black areas correspond to the missing entries.

In general, given no presumption about the nature of matrix entries, it is virtually impossible to restore L_0 as the missing entries can be of arbitrary values. That is, some assumptions are necessary for solving Problem 1.1. Based on the high-dimensional and massive essence of today’s data-driven community, it is arguable that the target matrix L_0 we wish to recover is often low rank [23]. Hence, one may perform matrix completion by seeking a matrix with the lowest rank that also satisfies the constraints given by the observed entries:

$$\min_L \text{rank}(L), \quad \text{s.t.} \quad [L]_{ij} = [L_0]_{ij}, \forall (i, j) \in \Omega. \quad (1)$$

Unfortunately, this idea is of little practical because the problem above is NP-hard and cannot be solved in polynomial time [15]. To achieve practical matrix completion, Candès and Recht [4] suggested to consider an alternative that minimizes instead the *nuclear norm* which is a convex envelope of the rank function [12]. Namely,

$$\min_L \|L\|_*, \quad \text{s.t.} \quad [L]_{ij} = [L_0]_{ij}, \forall (i, j) \in \Omega, \quad (2)$$

where $\|\cdot\|_*$ denotes the nuclear norm, i.e., the sum of the singular values of a matrix. Rather surprisingly, it is proved in [4] that the missing entries, with high probability, can be exactly restored by the convex program (2), as long as the target matrix L_0 is *low rank* and *incoherent* and the set Ω of locations corresponding to the observed entries is a set sampled *uniformly at random*. This pioneering work provides people several useful tools to investigate matrix completion and many other related problems. Those assumptions, including low-rankness, incoherence and uniform sampling, are now standard and widely used in the literatures, e.g., [14, 17, 22, 24, 28, 33, 34, 36]. In particular, the analyses in [17, 33, 36] show that, in terms of theoretical completeness, many nonconvex optimization based methods are as powerful as the convex program (2). Unfortunately, these theories still depend on the assumption of uniform sampling, and thus they cannot explain why there are many nonconvex methods which often do better than the convex program (2) in practice.

The missing data in practice, however, often occurs in a nonuniform and deterministic fashion instead of randomly. This is because the reason for an observation being missing usually depends on the unseen observations themselves. For example, in structure from motion and magnetic resonance imaging, typically the locations of the observed entries are concentrated around the main diagonal of a matrix⁴, as shown in Figure 1. Moreover, as pointed out by [19, 21, 23], the incoherence condition is indeed not so consistent with the mixture structure of multiple subspaces, which is also a ubiquitous phenomenon in practice. There has been sparse research in the direction of nonuniform sampling, e.g., [18, 25–27, 31]. In particular, Negahban and Wainwright [26] studied the case of weighted entrywise sampling, which is more general than the setup of uniform sampling but still a special form of random sampling. Király et al. [18] considered deterministic sampling and is most related to this work. However, they had only established conditions to decide whether a particular entry of the matrix can be restored. In other words, the setup of [18] may not handle well the dependence among the missing entries. In summary, matrix completion still starves for practical theories and methods, although has attained considerable improvements in these years.

To break through the limits of the setup of random sampling, in this paper we introduce a new hypothesis called *isomeric condition*, which is a mixed concept that combines together the rank and coherence of L_0 with the locations and amount of the observed entries. In general, *isomerism* (noun

⁴This statement means that the observed entries are concentrated around the main diagonal after a permutation of the sampling pattern Ω .

of isomeric) is a very mild hypothesis and only a little bit more strict than the well-known *oracle assumption*; that is, the number of observed entries in each row and column of L_0 is not smaller than the rank of L_0 . It is arguable that the isomeric condition can hold even when the missing entries have irregular locations. In particular, it is provable that the widely used assumption of uniform sampling is *sufficient* to ensure isomerism, not necessary. Equipped with this new tool, isomerism, we prove a set of theorems pertaining to *missing data recovery* [35] and matrix completion. For example, we prove that, under the condition of isomerism, the exact solutions that identify the target matrix are included as critical points by the commonly used bilinear programs. This result helps to explain the widely observed phenomenon that there are many nonconvex methods performing better than the convex program (2) on real-world matrix completion tasks. In summary, the contributions of this paper mainly include:

- ◊ We invent a new hypothesis called isomeric condition, which provably holds given the standard assumptions of uniform sampling, low-rankness and incoherence. In addition, we also exemplify that the isomeric condition can hold even if the target matrix L_0 is not incoherent and the missing entries are placed irregularly. Comparing to the existing studies about nonuniform sampling, our setup is more general.
- ◊ Equipped with the isomeric condition, we prove that the exact solutions that identify L_0 are included as critical points by the commonly used bilinear programs. Comparing to the existing theories for nonconvex matrix completion, our theory is built upon a much weaker assumption and can therefore partially reveal the superiorities of nonconvex programs over the convex methods based on (2).
- ◊ We prove that the isomeric condition is *sufficient* and *necessary* for the column and row projectors of L_0 to be invertible given the sampling pattern Ω . This result implies that the isomeric condition is *necessary* for ensuring that the minimal rank solution to (1) can identify the target L_0 .

The rest of this paper is organized as follows. Section 2 summarizes the mathematical notations used in the paper. Section 3 introduces the proposed isomeric condition, along with some theorems for matrix completion. Section 4 shows some empirical results and Section 5 concludes this paper. The detailed proofs to all the proposed theorems are presented in the Supplementary Materials.

2 Notations

Capital and lowercase letters are used to represent matrices and vectors, respectively, except that the lowercase letters, $i, j, k, m, n, l, p, q, r, s$ and t , are used to denote some integers, e.g., the location of an observation, the rank of a matrix, etc. For a matrix M , $[M]_{ij}$ is its (i, j) th entry, $[M]_{i,:}$ is its i th row and $[M]_{:,j}$ is its j th column. Let ω_1 and ω_2 be two 1D index sets; namely, $\omega_1 = \{i_1, i_2, \dots, i_k\}$ and $\omega_2 = \{j_1, j_2, \dots, j_s\}$. Then $[M]_{\omega_1,:}$ denotes the submatrix of M obtained by selecting the rows with indices i_1, i_2, \dots, i_k , $[M]_{:, \omega_2}$ is the submatrix constructed by choosing the columns j_1, j_2, \dots, j_s , and similarly for $[M]_{\omega_1, \omega_2}$. For a 2D index set $\Omega \subseteq \{1, 2, \dots, m\} \times \{1, 2, \dots, n\}$, we imagine it as a sparse matrix and, accordingly, define its “rows”, “columns” and “transpose” as follows: The i th row $\Omega_i = \{j_1 | (i_1, j_1) \in \Omega, i_1 = i\}$, the j th column $\Omega^j = \{i_1 | (i_1, j_1) \in \Omega, j_1 = j\}$ and the transpose $\Omega^T = \{(j_1, i_1) | (i_1, j_1) \in \Omega\}$.

The special symbol $(\cdot)^+$ is reserved to denote the Moore-Penrose pseudo-inverse of a matrix. More precisely, for a matrix M with Singular Value Decomposition (SVD)⁵ $M = U_M \Sigma_M V_M^T$, its pseudo-inverse is given by $M^+ = V_M \Sigma_M^{-1} U_M^T$. For convenience, we adopt the conventions of using $\text{span}\{M\}$ to denote the linear space spanned by the columns of a matrix M , using $y \in \text{span}\{M\}$ to denote that a vector y belongs to the space $\text{span}\{M\}$, and using $Y \in \text{span}\{M\}$ to denote that all the column vectors of a matrix Y belong to $\text{span}\{M\}$.

Capital letters U, V, Ω and their variants (complements, subscripts, etc.) are reserved for left singular vectors, right singular vectors and index set, respectively. For convenience, we shall abuse the notation U (resp. V) to denote the linear space spanned by the columns of U (resp. V), i.e., the column space (resp. row space). The orthogonal projection onto the column space U , is denoted by \mathcal{P}_U and given by $\mathcal{P}_U(M) = U U^T M$, and similarly for the row space $\mathcal{P}_V(M) = M V V^T$. The same

⁵In this paper, SVD always refers to skinny SVD. For a rank- r matrix $M \in \mathbb{R}^{m \times n}$, its SVD is of the form $U_M \Sigma_M V_M^T$, where $U_M \in \mathbb{R}^{m \times r}$, $\Sigma_M \in \mathbb{R}^{r \times r}$ and $V_M \in \mathbb{R}^{n \times r}$.

notation is also used to represent a subspace of matrices (i.e., the image of an operator), e.g., we say that $M \in \mathcal{P}_U$ for any matrix M which satisfies $\mathcal{P}_U(M) = M$. We shall also abuse the notation Ω to denote the linear space of matrices supported on Ω . Then the symbol \mathcal{P}_Ω denotes the orthogonal projection onto Ω , namely,

$$[\mathcal{P}_\Omega(M)]_{ij} = \begin{cases} [M]_{ij}, & \text{if } (i, j) \in \Omega, \\ 0, & \text{otherwise.} \end{cases}$$

Similarly, the symbol \mathcal{P}_Ω^\perp denotes the orthogonal projection onto the complement space of Ω . That is, $\mathcal{P}_\Omega + \mathcal{P}_\Omega^\perp = \mathcal{I}$, where \mathcal{I} is the identity operator.

Three types of matrix norms are used in this paper, and they are all functions of the singular values: 1) The operator norm or 2-norm (i.e., largest singular value) denoted by $\|M\|$, 2) the Frobenius norm (i.e., square root of the sum of squared singular values) denoted by $\|M\|_F$ and 3) the nuclear norm or trace norm (i.e., sum of singular values) denoted by $\|M\|_*$. The only used vector norm is the ℓ_2 norm, which is denoted by $\|\cdot\|_2$. The symbol $|\cdot|$ is reserved for the cardinality of an index set.

3 Isomeric Condition and Matrix Completion

This section introduces the proposed isomeric condition and a set of theorems for matrix completion. But most of the detailed proofs are deferred until the Supplementary Materials.

3.1 Isomeric Condition

In general cases, as aforementioned, matrix completion is an ill-posed problem. Thus, some assumptions are necessary for studying Problem 1.1. To eliminate the disadvantages of the setup of random sampling, we define and investigate a so-called *isomeric condition*.

3.1.1 Definitions

For the ease of understanding, we shall begin with a concept called *k-isomerism* (or *k-isomeric* in adjective form), which could be regarded as an extension of low-rankness.

Definition 3.1 (*k-isomeric*). A matrix $M \in \mathbb{R}^{m \times l}$ is called *k-isomeric* if and only if any k rows of M can linearly represent all rows in M . That is,

$$\text{rank}([M]_{\omega,:}) = \text{rank}(M), \forall \omega \subseteq \{1, 2, \dots, m\}, |\omega| = k,$$

where $|\cdot|$ is the cardinality of an index set.

In general, *k-isomerism* is somewhat similar to *Spark* [37] which defines the smallest linearly dependent subset of the rows of a matrix. For a matrix M to be *k-isomeric*, it is necessary that $\text{rank}(M) \leq k$, not sufficient. In fact, *k-isomerism* is also somehow related to the concept of *coherence* [4, 21]. When the coherence of a matrix $M \in \mathbb{R}^{m \times l}$ is not too high, the rows of M will sufficiently spread, and thus M could be *k-isomeric* with a small k , e.g., $k = \text{rank}(M)$. Whenever the coherence of M is very high, one may need a large k to satisfy the *k-isomeric* property. For example, consider an extreme case where M is a rank-1 matrix with one row being 1 and everywhere else being 0. In this case, we need $k = m$ to ensure that M is *k-isomeric*.

While Definition 3.1 involves all 1D index sets of cardinality k , we often need the isomeric property to be associated with a certain 2D index set Ω . To this end, we define below a concept called *Ω -isomerism* (or *Ω -isomeric* in adjective form).

Definition 3.2 (*Ω -isomeric*). Let $M \in \mathbb{R}^{m \times l}$ and $\Omega \subseteq \{1, 2, \dots, m\} \times \{1, 2, \dots, n\}$. Suppose that $\Omega^j \neq \emptyset$ (empty set), $\forall 1 \leq j \leq n$. Then the matrix M is called *Ω -isomeric* if and only if

$$\text{rank}([M]_{\Omega^j,:}) = \text{rank}(M), \forall j = 1, 2, \dots, n.$$

Note here that only the number of rows in M is required to coincide with the row indices included in Ω , and thereby $l \neq n$ is allowable.

Generally, *Ω -isomerism* is less strict than *k-isomerism*. Provided that $|\Omega^j| \geq k, \forall 1 \leq j \leq n$, a matrix M is *k-isomeric* ensures that M is *Ω -isomeric* as well, but not vice versa. For the extreme example where M is nonzero at only one row, interestingly, M can be *Ω -isomeric* as long as the locations of the nonzero elements are included in Ω .

With the notation of $\Omega^T = \{(j_1, i_1) | (i_1, j_1) \in \Omega\}$, the isomeric property could be also defined on the column vectors of a matrix, as shown in the following definition.

Definition 3.3 (Ω/Ω^T -isomeric). Let $M \in \mathbb{R}^{m \times n}$ and $\Omega \subseteq \{1, 2, \dots, m\} \times \{1, 2, \dots, n\}$. Suppose $\Omega_i \neq \emptyset$ and $\Omega^j \neq \emptyset$, $\forall i = 1, \dots, m, j = 1, \dots, n$. Then the matrix M is called Ω/Ω^T -isomeric if and only if M is Ω -isomeric and M^T is Ω^T -isomeric as well.

To solve Problem 1.1 without the imperfect assumption of missing at random, as will be shown later, we need to assume that L_0 is Ω/Ω^T -isomeric. This condition has excluded the unidentifiable cases where any rows or columns of L_0 are wholly missing. In fact, whenever L_0 is Ω/Ω^T -isomeric, the number of observed entries in each row and column of L_0 has to be greater than or equal to the rank of L_0 ; this is consistent with the results in [20]. Moreover, Ω/Ω^T -isomerism has actually well treated the cases where L_0 is of high coherence. For example, consider an extreme case where L_0 is 1 at only one element and 0 everywhere else. In this case, L_0 cannot be Ω/Ω^T -isomeric unless the nonzero element is observed. So, generally, it is possible to restore the missing entries of a highly coherent matrix, as long as the Ω/Ω^T -isomeric condition is obeyed.

3.1.2 Basic Properties

While its definitions are associated with a certain matrix, the isomeric condition is actually characterizing some properties of a space, as shown in the lemma below.

Lemma 3.1. Let $L_0 \in \mathbb{R}^{m \times n}$ and $\Omega \subseteq \{1, 2, \dots, m\} \times \{1, 2, \dots, n\}$. Denote the SVD of L_0 as $U_0 \Sigma_0 V_0^T$. Then we have:

1. L_0 is Ω -isomeric if and only if U_0 is Ω -isomeric.
2. L_0^T is Ω^T -isomeric if and only if V_0 is Ω^T -isomeric.

Proof. It could be manipulated that

$$[L_0]_{\Omega^j, \cdot} = ([U_0]_{\Omega^j, \cdot}) \Sigma_0 V_0^T, \forall j = 1, \dots, n.$$

Since $\Sigma_0 V_0^T$ is row-wisely full rank, we have

$$\text{rank}([L_0]_{\Omega^j, \cdot}) = \text{rank}([U_0]_{\Omega^j, \cdot}), \forall j = 1, \dots, n.$$

As a result, L_0 is Ω -isomeric is equivalent to that U_0 is Ω -isomeric. In a similar way, the second claim is proved. \square

It is easy to see that the above lemma is still valid even when the condition of Ω -isomerism is replaced by k -isomerism. Thus, hereafter, we may say that a space is isomeric (k -isomeric, Ω -isomeric or Ω^T -isomeric) as long as its basis matrix is isomeric. In addition, the isomeric property is subspace successive, as shown in the next lemma.

Lemma 3.2. Let $\Omega \subseteq \{1, 2, \dots, m\} \times \{1, 2, \dots, n\}$ and $U_0 \in \mathbb{R}^{m \times r}$ be the basis matrix of a Euclidean subspace embedded in \mathbb{R}^m . Suppose that U is a subspace of U_0 , i.e., $U = U_0 U_0^T U$. If U_0 is Ω -isomeric then U is Ω -isomeric as well.

Proof. By $U = U_0 U_0^T U$ and U_0 is Ω -isomeric,

$$\begin{aligned} \text{rank}([U]_{\Omega^j, \cdot}) &= \text{rank}([U_0]_{\Omega^j, \cdot} U_0^T U) = \text{rank}(U_0^T U) \\ &= \text{rank}(U_0 U_0^T U) = \text{rank}(U), \forall 1 \leq j \leq n. \end{aligned}$$

\square

The above lemma states that, in one word, the subspace of an isomeric space is isomeric as well.

3.1.3 Important Properties

As aforementioned, the isometric condition is actually necessary for ensuring that the minimal rank solution to (1) can identify L_0 . To see why, let's assume that $U_0 \cap \Omega^\perp \neq \{0\}$, where we denote by $U_0 \Sigma_0 V_0^T$ the SVD of L_0 . Then one could construct a nonzero perturbation, denoted as $\Delta \in U_0 \cap \Omega^\perp$, and accordingly, obtain a feasible solution $\tilde{L}_0 = L_0 + \Delta$ to the problem in (1). Since $\Delta \in U_0$, we have $\text{rank}(\tilde{L}_0) \leq \text{rank}(L_0)$. Even more, it is entirely possible that $\text{rank}(\tilde{L}_0) < \text{rank}(L_0)$. Such a case is unidentifiable in nature, as the global optimum to problem (1) cannot identify L_0 . Thus,

to ensure that the global minimum to (1) can identify L_0 , it is essentially necessary to show that $U_0 \cap \Omega^\perp = \{0\}$ (resp. $V_0 \cap \Omega^\perp = \{0\}$), which is equivalent to that the operator $\mathcal{P}_{U_0} \mathcal{P}_\Omega \mathcal{P}_{U_0}$ (resp. $\mathcal{P}_{V_0} \mathcal{P}_\Omega \mathcal{P}_{V_0}$) is invertible (see Lemma 6.8 of the Supplementary Materials). Interestingly, the isomeric condition is indeed a *sufficient* and *necessary* condition for the operators $\mathcal{P}_{U_0} \mathcal{P}_\Omega \mathcal{P}_{U_0}$ and $\mathcal{P}_{V_0} \mathcal{P}_\Omega \mathcal{P}_{V_0}$ to be invertible, as shown in the following theorem.

Theorem 3.1. *Let $L_0 \in \mathbb{R}^{m \times n}$ and $\Omega \subseteq \{1, 2, \dots, m\} \times \{1, 2, \dots, n\}$. Let the SVD of L_0 be $U_0 \Sigma_0 V_0^T$. Denote $\mathcal{P}_{U_0}(\cdot) = U_0 U_0^T(\cdot)$ and $\mathcal{P}_{V_0}(\cdot) = (\cdot) V_0 V_0^T$. Then we have the following:*

1. *The linear operator $\mathcal{P}_{U_0} \mathcal{P}_\Omega \mathcal{P}_{U_0}$ is invertible if and only if U_0 is Ω -isomeric.*
2. *The linear operator $\mathcal{P}_{V_0} \mathcal{P}_\Omega \mathcal{P}_{V_0}$ is invertible if and only if V_0 is Ω^T -isomeric.*

The necessity stated above implies that the isomeric condition is actually a very mild hypothesis. In general, there are numerous reasons for the target matrix L_0 to be isomeric. Particularly, the widely used assumptions of low-rankness, incoherence and uniform sampling are indeed *sufficient* (but not necessary) to ensure isomerism, as shown in the following theorem.

Theorem 3.2. *Let $L_0 \in \mathbb{R}^{m \times n}$ and $\Omega \subseteq \{1, 2, \dots, m\} \times \{1, 2, \dots, n\}$. Denote $n_1 = \max(m, n)$ and $n_2 = \min(m, n)$. Suppose that L_0 is incoherent and Ω is a 2D index set sampled uniformly at random, namely $\Pr((i, j) \in \Omega) = \rho_0$ and $\Pr((i, j) \notin \Omega) = 1 - \rho_0$. For any $\delta > 0$, if $\rho_0 > \delta$ is obeyed and $\text{rank}(L_0) < \delta n_2 / (c \log n_1)$ holds for some numerical constant c then, with high probability at least $1 - n_1^{-10}$, L_0 is Ω/Ω^T -isomeric.*

It is worth noting that the isomeric condition can be obeyed in numerous circumstances other than the case of uniform sampling *plus* incoherence. For example,

$$\Omega = \{(1, 1), (1, 2), (1, 3), (2, 1), (3, 1)\} \text{ and } L_0 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

where L_0 is a 3×3 matrix with 1 at $(1, 1)$ and 0 everywhere else. In this example, L_0 is not incoherent and the sampling is not uniform either, but it could be verified that L_0 is Ω/Ω^T -isomeric.

3.2 Results

In this subsection, we shall show how the isomeric condition can take effect in the context of nonuniform sampling, establishing some theorems pertaining to *missing data recovery* [35] as well as matrix completion.

3.2.1 Missing Data Recovery

Before exploring the matrix completion problem, for the ease of understanding, we would like to consider a *missing data recovery* problem studied by Zhang [35], which could be described as follows: Let $y_0 \in \mathbb{R}^m$ be a data vector drawn from some low-dimensional subspace, denoted as $y_0 \in \mathcal{S}_0 \subset \mathbb{R}^m$. Suppose that y_0 contains some available observations in $y_b \in \mathbb{R}^k$ and some missing entries in $y_u \in \mathbb{R}^{m-k}$. Namely, after a permutation,

$$y_0 = \begin{bmatrix} y_b \\ y_u \end{bmatrix}, y_b \in \mathbb{R}^k, y_u \in \mathbb{R}^{m-k}. \quad (3)$$

Given the observations in y_b , we seek to restore the unseen entries in y_u . To do this, we consider the prevalent idea that represents a data vector as a linear combination of the bases in a given dictionary:

$$y_0 = A x_0, \quad (4)$$

where $A \in \mathbb{R}^{m \times p}$ is a dictionary constructed in advance and $x_0 \in \mathbb{R}^p$ is the representation of y_0 . Utilizing the same permutation used in (3), we can partition the rows of A into two parts according to the indices of the observed and missing entries, respectively:

$$A = \begin{bmatrix} A_b \\ A_u \end{bmatrix}, A_b \in \mathbb{R}^{k \times p}, A_u \in \mathbb{R}^{(m-k) \times p}. \quad (5)$$

In this way, the equation in (4) gives that

$$y_b = A_b x_0 \quad \text{and} \quad y_u = A_u x_0.$$

As we now can see, the unseen data y_u could be restored, as long as the representation x_0 is retrieved by only accessing the available observations in y_b . In general cases, there are infinitely many representations that satisfy $y_0 = Ax_0$, e.g., $x_0 = A^+y_0$, where $(\cdot)^+$ is the pseudo-inverse of a matrix. Since A^+y_0 is the representation of minimal ℓ_2 norm, we revisit the traditional ℓ_2 program:

$$\min_x \frac{1}{2} \|x\|_2^2, \quad \text{s.t.} \quad y_b = A_b x, \quad (6)$$

where $\|\cdot\|_2$ is the ℓ_2 norm of a vector. Under some verifiable conditions, the above ℓ_2 program is indeed *consistently successful* in a sense as in the following: For any $y_0 \in \mathcal{S}_0$ with an arbitrary partition $y_0 = [y_b; y_u]$ (i.e., arbitrarily missing), the desired representation $x_0 = A^+y_0$ is the unique minimizer to the problem in (6). That is, the unseen data y_u is exactly recovered by firstly computing the minimizer x^* to problem (6) and then calculating $y_u = A_u x^*$.

Theorem 3.3. *Let $y_0 = [y_b; y_u] \in \mathbb{R}^m$ be an authentic sample drawn from some low-dimensional subspace \mathcal{S}_0 embedded in \mathbb{R}^m , $A \in \mathbb{R}^{m \times p}$ be a given dictionary and k be the number of available observations in y_b . Then the convex program (6) is consistently successful, provided that $\mathcal{S}_0 \subseteq \text{span}\{A\}$ and the dictionary A is k -isomeric.*

Unlike the theory in [35], the condition of which is unverifiable, our k -isomeric condition could be verified in finite time. Notice, that the problem of missing data recovery is closely related to matrix completion, which is actually to restore the missing entries in multiple data vectors simultaneously. Hence, Theorem 3.3 can be naturally generalized to the case of matrix completion, as will be shown in the next subsection.

3.2.2 Matrix Completion

The spirits of the ℓ_2 program (6) can be easily transferred to the case of matrix completion. Following (6), one may consider Frobenius norm minimization for matrix completion:

$$\min_X \frac{1}{2} \|X\|_F^2, \quad \text{s.t.} \quad \mathcal{P}_\Omega(AX - L_0) = 0, \quad (7)$$

where $A \in \mathbb{R}^{m \times p}$ is a dictionary assumed to be given. As one can see, the problem in (7) is equivalent to (6) if L_0 is consisting of only one column vector. The same as (6), the convex program (7) can also exactly recover the desired representation matrix A^+L_0 , as shown in the theorem below. The difference is that we here require Ω -isomerism instead of k -isomerism.

Theorem 3.4. *Let $L_0 \in \mathbb{R}^{m \times n}$ and $\Omega \subseteq \{1, 2, \dots, m\} \times \{1, 2, \dots, n\}$. Suppose that $A \in \mathbb{R}^{m \times p}$ is a given dictionary. Provided that $L_0 \in \text{span}\{A\}$ and A is Ω -isomeric, the desired representation $X_0 = A^+L_0$ is the unique minimizer to the problem in (7).*

Theorem 3.4 tells us that, in general, even when the locations of the missing entries are interrelated and nonuniformly distributed, the target matrix L_0 can be restored as long as we have found a proper dictionary A . This motivates us to consider the commonly used bilinear program that seeks both A and X simultaneously:

$$\min_{A, X} \frac{1}{2} \|A\|_F^2 + \frac{1}{2} \|X\|_F^2, \quad \text{s.t.} \quad \mathcal{P}_\Omega(AX - L_0) = 0, \quad (8)$$

where $A \in \mathbb{R}^{m \times p}$ and $X \in \mathbb{R}^{p \times n}$. The problem above is bilinear and therefore nonconvex. So, it would be hard to obtain a strong performance guarantee as done in the convex programs, e.g., [4, 21]. Interestingly, under a very mild condition, the problem in (8) is proved to include the exact solutions that identify the target matrix L_0 as the critical points.

Theorem 3.5. *Let $L_0 \in \mathbb{R}^{m \times n}$ and $\Omega \subseteq \{1, 2, \dots, m\} \times \{1, 2, \dots, n\}$. Denote the rank and SVD of L_0 as r_0 and $U_0 \Sigma_0 V_0^T$, respectively. If L_0 is Ω/Ω^T -isomeric then the exact solution, denoted by (A_0, X_0) and given by*

$$A_0 = U_0 \Sigma_0^{\frac{1}{2}} Q^T, \quad X_0 = Q \Sigma_0^{\frac{1}{2}} V_0^T, \quad \forall Q \in \mathbb{R}^{p \times r_0}, \quad Q^T Q = \mathbf{I},$$

is a critical point to the problem in (8).

To exhibit the power of program (8), however, the parameter p , which indicates the number of columns in the dictionary matrix A , must be close to the true rank of the target matrix L_0 . This is

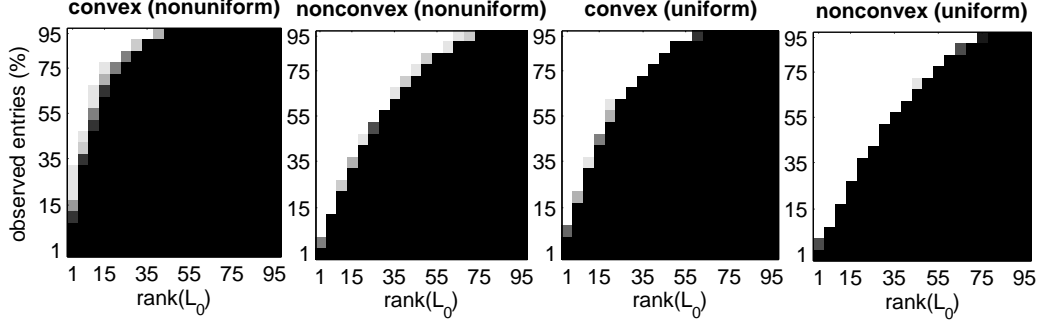


Figure 2: Comparing the bilinear program (9) ($p = m$) with the convex method (2). The numbers plotted on the above figures are the success rates within 20 random trials. The white and black points mean “succeed” and “fail”, respectively. Here the success is in a sense that $\text{PSNR} \geq 40\text{dB}$, where PSNR standing for peak signal-to-noise ratio.

impractical in the cases where the rank of L_0 is unknown. Notice, that the Ω -isomeric condition imposed on A requires

$$\text{rank}(A) \leq |\Omega^j|, \forall j = 1, 2, \dots, n.$$

This, together with the condition of $L_0 \in \text{span}\{A\}$, essentially need us to solve a *low rank matrix recovery* problem [14]. Hence, we suggest to combine the formulation (7) with the popular idea of nuclear norm minimization, resulting in a bilinear program that jointly estimates both the dictionary matrix A and the representation matrix X by

$$\min_{A, X} \|A\|_* + \frac{1}{2} \|X\|_F^2, \text{ s.t. } \mathcal{P}_\Omega(AX - L_0) = 0, \quad (9)$$

which, by coincidence, has been mentioned in a paper about optimization [32]. Similar to (8), the program in (9) has the following theorem to guarantee its performance.

Theorem 3.6. *Let $L_0 \in \mathbb{R}^{m \times n}$ and $\Omega \subseteq \{1, 2, \dots, m\} \times \{1, 2, \dots, n\}$. Denote the rank and SVD of L_0 as r_0 and $U_0 \Sigma_0 V_0^T$, respectively. If L_0 is Ω/Ω^T -isomeric then the exact solution, denoted by (A_0, X_0) and given by*

$$A_0 = U_0 \Sigma_0^{\frac{2}{3}} Q^T, X_0 = Q \Sigma_0^{\frac{1}{3}} V_0^T, \forall Q \in \mathbb{R}^{p \times r_0}, Q^T Q = I,$$

is a critical point to the problem in (9).

Unlike (8), which possesses superior performance only if p is close to $\text{rank}(L_0)$ and the initial solution is chosen carefully, the bilinear program in (9) can work well by simply choosing $p = m$ and using $A = I$ as the initial solution. To see why, one essentially needs to figure out the conditions under which a specific optimization procedure can produce an optimal solution that meets an exact solution. This requires extensive justifications and we leave it as future work.

4 Simulations

To verify the superiorities of the nonconvex matrix completion methods over the convex program (2), we would like to experiment with randomly generated matrices. We generate a collection of $m \times n$ ($m = n = 100$) target matrices according to the model of $L_0 = BC$, where $B \in \mathbb{R}^{m \times r_0}$ and $C \in \mathbb{R}^{r_0 \times n}$ are $\mathcal{N}(0, 1)$ matrices. The rank of L_0 , i.e., r_0 , is configured as $r_0 = 1, 5, 10, \dots, 90, 95$. Regarding the index set Ω consisting of the locations of the observed entries, we consider two settings: One is to create Ω by using a Bernoulli model to randomly sample a subset from $\{1, \dots, m\} \times \{1, \dots, n\}$ (referred to as “uniform”), the other is as in Figure 1 that makes the locations of the observed entries be concentrated around the main diagonal of a matrix (referred to as “nonuniform”). The observation fraction is set to be $|\Omega|/(mn) = 0.01, 0.05, \dots, 0.9, 0.95$. For each pair of $(r_0, |\Omega|/(mn))$, we run 20 trials, resulting in 8000 simulations in total.

When $p = m$ and the identity matrix is used to initialize the dictionary A , we have empirically found that program (8) has the same performance as (2). This is not strange, because it has been proven in [16] that $\|L\|_* = \min_{A, X} \frac{1}{2} (\|A\|_F^2 + \|X\|_F^2)$, s.t. $L = AX$. Figure 2 compares the bilinear

program (9) to the convex method (2). It can be seen that (9) works distinctly better than (2). Namely, while handling the nonuniformly missing data, the number of matrices successfully restored by the bilinear program (9) is 102% more than the convex program (2). Even for dealing with the missing entries chosen uniformly at random, in terms of the number of successfully restored matrices, the bilinear program (9) can still outperform the convex method (2) by 44%. These results illustrate that, even in the cases where the rank of L_0 is unknown, the bilinear program (9) can do much better than the convex optimization based method (2).

5 Conclusion and Future Work

This work studied the problem of matrix completion with nonuniform sampling, a significant setting not extensively studied before. To figure out the conditions under which exact recovery is possible, we proposed a so-called isomeric condition, which provably holds when the standard assumptions of low-rankness, incoherence and uniform sampling arise. In addition, we also exemplified that the isomeric condition can be obeyed in the other cases beyond the setting of uniform sampling. Even more, our theory implies that the isomeric condition is indeed necessary for making sure that the minimal rank completion can identify the target matrix L_0 . Equipped with the isomeric condition, finally, we mathematically proved that the widely used bilinear programs can include the exact solutions that recover the target matrix L_0 as the critical points; this guarantees the recovery performance of bilinear programs to some extent.

However, there still remain several problems for future work. In particular, it is unknown under which conditions a specific optimization procedure for (9) can produce an optimal solution that exactly restores the target matrix L_0 . To do this, one needs to analyze the convergence property as well as the recovery performance. Moreover, it is unknown either whether the isomeric condition suffices for ensuring that the minimal rank completion can identify the target L_0 . These require extensive justifications and we leave them as future work.

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References

- [1] Emmanuel Candès and Terence Tao. The power of convex relaxation: Near-optimal matrix completion. *IEEE Transactions on Information Theory*, 56(5):2053–2080, 2009.
- [2] Emmanuel Candès and Yaniv Plan. Matrix completion with noise. In *IEEE Proceeding*, volume 98, pages 925–936, 2010.
- [3] William E. Bishop and Byron M. Yu. Deterministic symmetric positive semidefinite matrix completion. In *Neural Information Processing Systems*, pages 2762–2770, 2014.
- [4] Emmanuel Candès and Benjamin Recht. Exact matrix completion via convex optimization. *Foundations of Computational Mathematics*, 9(6):717–772, 2009.
- [5] Eyal Heiman, Gideon Schechtman, and Adi Shraibman. Deterministic algorithms for matrix completion. *Random Structures and Algorithms*, 45(2):306–317, 2014.
- [6] Raghunandan H. Keshavan, Andrea Montanari, and Sewoong Oh. Matrix completion from a few entries. *IEEE Transactions on Information Theory*, 56(6):2980–2998, 2010.
- [7] Raghunandan H. Keshavan, Andrea Montanari, and Sewoong Oh. Matrix completion from noisy entries. *Journal of Machine Learning Research*, 11:2057–2078, 2010.
- [8] Akshay Krishnamurthy and Aarti Singh. Low-rank matrix and tensor completion via adaptive sampling. In *Neural Information Processing Systems*, pages 836–844, 2013.
- [9] Troy Lee and Adi Shraibman. Matrix completion from any given set of observations. In *Neural Information Processing Systems*, pages 1781–1787, 2013.
- [10] Rahul Mazumder, Trevor Hastie, and Robert Tibshirani. Spectral regularization algorithms for learning large incomplete matrices. *Journal of Machine Learning Research*, 11:2287–2322, 2010.
- [11] Karthik Mohan and Maryam Fazel. New restricted isometry results for noisy low-rank recovery. In *IEEE International Symposium on Information Theory*, pages 1573–1577, 2010.

- [12] B. Recht, W. Xu, and B. Hassibi. Necessary and sufficient conditions for success of the nuclear norm heuristic for rank minimization. Technical report, CalTech, 2008.
- [13] Markus Weimer, Alexandros Karatzoglou, Quoc V. Le, and Alex J. Smola. Cofi rank - maximum margin matrix factorization for collaborative ranking. In *Neural Information Processing Systems*, 2007.
- [14] Emmanuel J. Candès, Xiaodong Li, Yi Ma, and John Wright. Robust principal component analysis? *Journal of the ACM*, 58(3):1–37, 2011.
- [15] Alexander L. Chistov and Dima Grigoriev. Complexity of quantifier elimination in the theory of algebraically closed fields. In *Proceedings of the Mathematical Foundations of Computer Science*, pages 17–31, 1984.
- [16] Maryam Fazel, Haitham Hindi, and Stephen P. Boyd. A rank minimization heuristic with application to minimum order system approximation. In *American Control Conference*, pages 4734–4739, 2001.
- [17] Rong Ge, Jason D. Lee, and Tengyu Ma. Matrix completion has no spurious local minimum. In *Neural Information Processing Systems*, pages 2973–2981, 2016.
- [18] Franz J. Király, Louis Theran, and Ryota Tomioka. The algebraic combinatorial approach for low-rank matrix completion. *J. Mach. Learn. Res.*, 16(1):1391–1436, January 2015.
- [19] Guangcan Liu and Ping Li. Recovery of coherent data via low-rank dictionary pursuit. In *Neural Information Processing Systems*, pages 1206–1214, 2014.
- [20] Daniel L. Pimentel-Alarcón and Robert D. Nowak. The Information-theoretic requirements of subspace clustering with missing data. In *International Conference on Machine Learning*, 48:802–810, 2016.
- [21] Guangcan Liu and Ping Li. Low-rank matrix completion in the presence of high coherence. *IEEE Transactions on Signal Processing*, 64(21):5623–5633, 2016.
- [22] Guangcan Liu, Zhouchen Lin, Shuicheng Yan, Ju Sun, Yong Yu, and Yi Ma. Robust recovery of subspace structures by low-rank representation. *IEEE Transactions on Pattern Recognition and Machine Intelligence*, 35(1):171–184, 2013.
- [23] Guangcan Liu, Qingshan Liu, and Ping Li. Blessing of dimensionality: Recovering mixture data via dictionary pursuit. *IEEE Transactions on Pattern Recognition and Machine Intelligence*, 39(1):47–60, 2017.
- [24] Guangcan Liu, Huan Xu, Jinhui Tang, Qingshan Liu, and Shuicheng Yan. A deterministic analysis for LRR. *IEEE Transactions on Pattern Recognition and Machine Intelligence*, 38(3):417–430, 2016.
- [25] Raghu Meka, Prateek Jain, and Inderjit S. Dhillon. Matrix completion from power-law distributed samples. In *Neural Information Processing Systems*, pages 1258–1266, 2009.
- [26] Sahand Negahban and Martin J. Wainwright. Restricted strong convexity and weighted matrix completion: Optimal bounds with noise. *Journal of Machine Learning Research*, 13:1665–1697, 2012.
- [27] Yudong Chen, Srinadh Bhojanapalli, Sujay Sanghavi, and Rachel Ward. Completing any low-rank matrix, provably. *Journal of Machine Learning Research*, 16: 2999–3034, 2015.
- [28] Praneeth Netrapalli, U. N. Niranjan, Sujay Sanghavi, Animashree Anandkumar, and Prateek Jain. Non-convex robust PCA. In *Neural Information Processing Systems*, pages 1107–1115, 2014.
- [29] Yuzhao Ni, Ju Sun, Xiaotong Yuan, Shuicheng Yan, and Loong-Fah Cheong. Robust low-rank subspace segmentation with semidefinite guarantees. In *International Conference on Data Mining Workshops*, pages 1179–1188, 2013.
- [30] R. Rockafellar. *Convex Analysis*. Princeton University Press, Princeton, NJ, USA, 1970.
- [31] Ruslan Salakhutdinov and Nathan Srebro. Collaborative filtering in a non-uniform world: Learning with the weighted trace norm. In *Neural Information Processing Systems*, pages 2056–2064, 2010.
- [32] Fanhua Shang, Yuanyuan Liu, and James Cheng. Scalable algorithms for tractable Schatten quasi-norm minimization. In *AAAI Conference on Artificial Intelligence*, pages 2016–2022, 2016.
- [33] Ruoyu Sun and Zhi-Quan Luo. Guaranteed matrix completion via non-convex factorization. *IEEE Transactions on Information Theory*, 62(11):6535–6579, 2016.
- [34] Huan Xu, Constantine Caramanis, and Sujay Sanghavi. Robust PCA via outlier pursuit. *IEEE Transactions on Information Theory*, 58(5):3047–3064, 2012.
- [35] Yin Zhang. When is missing data recoverable? *CAAM Technical Report TR06-15*, 2006.
- [36] Tuo Zhao, Zhaoran Wang, and Han Liu. A nonconvex optimization framework for low rank matrix estimation. In *Neural Information Processing Systems*, pages 559–567, 2015.
- [37] David L. Donoho and Michael Elad. Optimally sparse representation in general (nonorthogonal) dictionaries via ℓ_1 minimization. *Proceedings of the National Academy of Sciences*, 100(5): 2197–2202, 2003.