T2

Quiz 1

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INSTRUCTIONS: Solve the following problems. Justify your answers and show all your work.

Problem 1

A fair coin is toss three times.

1. Enumerate the sample space associated with this experiment and comopute the power set.
2. Compute a filtration for this random experiment.

Definition 1: Random Variable

Let $(\Omega, \mathscr{F}, P)$ be a probability space. A *random variable* X is a $(\mathcal{F}/\mathcal{B}(\mathbb{R}^n))$ measurable map $X : \Omega \to \mathbb{R}^n$, if for every Borel set $B \in \mathcal{B}(\mathbb{R}^n)$:

$$X^{-1}(B) = \{\omega \in \Omega : X(\omega) \in B\} \in \mathscr{F}.$$

Clearly,

- $X^{-1}(B) \in \mathscr{F} \quad \forall B \in \mathcal{B}(\mathbb{R}^n)$.

Problem 2

A σ-algebra generated by a random variable X, denoted by $\sigma(X)$, is the smallest σ-algebra for which X is measurable. For example, for $\Omega = \{\mathsf{HH}, \mathsf{HT}, \mathsf{TH}, \mathsf{TT}\}$.

$$X_1(\omega) = \begin{cases} 1, & \omega \in \{\mathsf{HH}, \mathsf{HT}\}, \\ 0, & \omega \in \{\mathsf{TH}, \mathsf{TT}\}. \end{cases} \quad X_2(\omega) = \begin{cases} 2, & \omega \in \{\mathsf{HH}\}, \\ 1, & \omega \in \{\mathsf{HT}\}, \\ -1, & \omega \in \{\mathsf{TH}\}, \\ -2, & \omega \in \{\mathsf{TT}\}. \end{cases}$$

$\sigma(X_1) = \{\emptyset, \{\mathsf{HH}, \mathsf{HT}\}, \{\mathsf{TH}, \mathsf{TT}\}, \Omega\}$ and $\sigma(X_2) = \mathscr{P}(\Omega)$. In particular, $\sigma(X_1) \subset \sigma(X_2) = \mathscr{P}(\Omega)$. What information is encoded in the algebras $\sigma(X_1)$ and $\sigma(X_2)$?

Problem 3

A fair coin is thrown four times.

1. Enumerate the sample space associated with this experiment and comopute the power set.
2. Define two random variables that encode the following properties of each outcome
   - X_1: Indicates whether the total number of heads in the four tosses is even.
   - X_2: Indicates whether exactly two tosses resulted in heads (H) and two in tails (T).
   - Compute the minimal σ-algebra for which X_1 is measureable; compute the minimal σ-algebra for which X_2 is measurable.

Problem 4

Consider the experiment in which we roll a three-faced dice: $\Omega = \{-2, 0, 2\}$ and $\mathscr{F}_1 = \{\emptyset, \Omega, \{-2, 2\}, \{0\}\}$.

1. Is $X_1(\omega) = \omega$ $\mathscr{F}_1/\mathcal{B}(\mathbb{R})$ measurable?
2. Is $X_2(\omega) = \omega^2$ is $\mathscr{F}_1/\mathcal{B}(\mathbb{R})$ measurable?
3. Is $X_2(\omega) = \omega^3$ is $\mathscr{F}_1/\mathcal{B}(\mathbb{R})$ measurable?

Problem 5

Let $\Omega = [0, 1]$, $\mathcal{F} = \mathcal{B}([0, 1])$, and define $X(\omega) = \omega^2$. Is X a random variable?. If so, is it continous or discrete? Notice that $X : [0, 1] \to \mathbb{R}$? Provide answers to the previous questions for $\Omega = [-1, 1]$.

Problem 6

Let $(\Omega, \mathscr{F})$ describe throwing two fair dice, i.e. $\Omega := \{(i, k) : 1 \leq i, k \leq 6\}$, $\mathscr{F} = \mathscr{P}(\Omega)$. The total number of points thrown $X : \Omega \to \{2, 3, \ldots, 12\}$, $X((i, j)) = i + j$ is a measurable map.

1. Compute $X^{-1}(6)$ and $X^{-1}(4)$.

Definition 2: Probability Distribution

Let $(\Omega, \mathscr{F}, \mu)$ be a measure space, and X a random variable $X : \Omega \to \mathbb{R}$, i.e. a measurable map. Then

$$P(X \in B) = \mu(X^{-1}(B)) = \mu(\{\omega : X(\omega) \in B\})$$

is a probability measure called the *law* or *distribution* of the random variable X. Recall that $P = \mu \circ X^{-1} : \mathcal{B}(\mathbb{R}) \to [0, 1]$.

Problem 7

Let $(\Omega, \mathscr{F})$ describe throwing two fair dice, i.e. $\Omega := \{(i, k) : 1 \leq i, k \leq 6\}$, $\mathscr{F} = \mathscr{P}(\Omega)$. The total number of points thrown $X : \Omega \to \{2, 3, \ldots, 12\}$, $U((i, j)) = i + j$ is a measurable map.

1. Compute the probability measure μ of the sample space $(\Omega, \mathscr{P}(\Omega), \mu)$.
2. Compute U's probability distribution $P = \mu \circ U^{-1}$.
3. Compute $P(U = 4) = P(U \in \{4\}) = \mu \circ U^{-1}(4)$.

Problem 8

A spacecraft's motor receives fuel from an injection system, denoted by A. If this injection system fails, a master alarm (B) is triggered on the commander's main dashboard. One possible cause of failure in the injection system is the occurrence of extreme vibrations, represented by C, which the spacecraft may experience during operation. However, it is important to note that extreme vibrations can erroneously trigger the master alarm even if the injection system itself is functioning correctly. Vibrations can also cause communication interference (D) with command control.

Compute the following. Note: For probability calculations use the data provided in the file prob.ipypnb. (This file will be provided by the instructor the day of the exam.)

- Compute the σ-algebra that encodes the following information: $\mathcal{G}_1$: "the injection system fails." $\mathcal{G}_2$: "the alarm is activated."
- What is the probability that the injection system fails?
- What is the probability that the injection system fails given that the alarm is activated?
- What is the probability that the injection system fails given that the alarm is activated and communication with command control is lost?

Ω's oucomes	DBAC	Probability
$D^cB^cA^cC^c$	0000	p_0=α_1
$D^cB^cA^cC$	0001	p_1=α_2
$D^cB^cAC^c$	0010	p_2=α_3
D^cB^cAC	0011	p_3=α_4
$D^cBA^cC^c$	0100	p_4=α_5
D^cBA^cC	0101	p_5=α_6
D^cBAC^c	0110	p_6=α_7
D^cBAC	0111	p_7=α_8
$DB^cA^cC^c$	1000	p_8=α_9
DB^cA^cC	1001	p_9=α_{10}
DB^cAC^c	1010	p_{10}=α_{11}
DB^cAC	1011	p_{11}=α_{12}
DBA^cC^c	1100	p_{12}=α_{13}
DBA^cC	1101	p_{13}=α_{14}
$DBAC^c$	1110	p_{14}=α_{15}
$DBAC$	1111	p_{15}=α_{16}