Quiz 1

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INSTRUCTIONS: Solve the following problems. Justify your answers and show all your work.

Problem 1

A fair coin is toss three times.
1. Enumerate the sample space associated with this experiment and comopute the power set.
2. Compute a filtration for this random experiment.

Definition 1: Random Variable

Let $(\Omega, \mathscr{F}, P)$ be a probability space. A *random variable X* is a $(\mathcal{F}/\mathcal{B}(\mathbb{R}^n))$ measurable map $X : \Omega \to \mathbb{R}^n$, if for every Borel set $B \in \mathcal{B}(\mathbb{R}^n)$:

$$X^{-1}(B) = \{\omega \in \Omega : X(\omega) \in B\} \in \mathscr{F}.$$

Clearly,
- $X^{-1}(B) \in \mathscr{F} \quad \forall B \in \mathcal{B}(\mathbb{R}^n).$

Problem 2

A σ-algebra generated by a random variable X, denoted by $\sigma(X)$, is the smallest σ-algebra for which X is measurable. For example, for $\Omega = \{HH, HT, TH, TT\}$.

$$X_1(\omega) = \begin{cases} 1, & \omega \in \{HH, HT\}, \\ 0, & \omega \in \{TH, TT\}. \end{cases} \qquad X_2(\omega) = \begin{cases} 2, & \omega \in \{HH\}, \\ 1, & \omega \in \{HT\}, \\ -1, & \omega \in \{TH\}, \\ -2, & \omega \in \{TT\}. \end{cases}$$

$\sigma(X_1) = \{\emptyset, \{HH, HT\}, \{TH, TT\}, \Omega\}$ and $\sigma(X_2) = \mathscr{P}(\Omega)$. In particular, $\sigma(X_1) \subset \sigma(X_2) = \mathscr{P}(\Omega)$. What information is encoded in the algebras $\sigma(X_1)$ and $\sigma(X_2)$?

Problem 3

A fair coin is thrown four times.
1. Enumerate the sample space associated with this experiment and comopute the power set.
2. Define two random variables that encode the following properties of each outcome
   - X_1: Indicates whether the total number of heads in the four tosses is even.
   - X_2: Indicates whether exactly two tosses resulted in heads (H) and two in tails (T).
   - Compute the minimal σ-algebra for which X_1 is measureable; compute the minimal σ-algebra for which X_2 is measurable.

Problem 4

Consider the experiment in which we roll a three-faced dice: $\Omega = \{-2,0,2\}$ and $\mathscr{F}_1 = \{\varnothing, \Omega, \{-2,2\}, \{0\}\}$.
1. Is $X_1(\omega) = \omega$ $\mathscr{F}_1/\mathcal{B}(\mathbb{R})$ measurable?
2. Is $X_2(\omega) = \omega^2$ is $\mathscr{F}_1/\mathcal{B}(\mathbb{R})$ measurable?
3. Is $X_2(\omega) = \omega^3$ is $\mathscr{F}_1/\mathcal{B}(\mathbb{R})$ measurable?

Problem 5

Let $\Omega = [0,1]$, $\mathcal{F} = \mathcal{B}([0,1])$, and define $X(\omega) = \omega^2$. Is X a random variable?. If so, is it continous or discrete? Notice that $X : [0,1] \to \mathbb{R}$? Provide answers to the previous questions for $\Omega = [-1,1]$.

Problem 6

Let $(\Omega, \mathscr{F})$ describe throwing two fair dice, i.e. $\Omega := \{(i,k) : 1 \le i, k \le 6\}$, $\mathscr{F} = \mathscr{P}(\Omega)$. The total number of points thrown $X : \Omega \to \{2,3,\ldots,12\}$, $X((i,j)) = i+j$ is a measurable map.
1. Compute $X^{-1}(6)$ and $X^{-1}(4)$.

> **Definition 2: Probability Distribution**
>
> Let $(\Omega, \mathscr{F}, \mu)$ be a measure space, and X a random variable $X : \Omega \to \mathbb{R}$, i.e. a measurable map. Then
>
> $$P(X \in B) = \mu(X^{-1}(B)) = \mu(\{\omega : X(\omega) \in B\})$$
>
> is a probability measure called the *law* or *distribution* of the random variable X. Recall that $P = \mu \circ X^{-1} : \mathcal{B}(\mathbb{R}) \to [0,1]$.

> **Problem 7**
>
> Let $(\Omega, \mathscr{F})$ describe throwing two fair dice, i.e. $\Omega := \{(i,k) : 1 \leq i, k \leq 6\}$, $\mathscr{F} = \mathscr{P}(\Omega)$. The total number of points thrown $X : \Omega \to \{2, 3, \dots, 12\}$, $U((i,j)) = i + j$ is a measurable map.
> 1. Compute the probability measure μ of the sample space $(\Omega, \mathscr{P}(\Omega), \mu)$.
> 2. Compute U's probability distribution $P = \mu \circ U^{-1}$.
> 3. Compute $P(U = 4) = P(U \in \{4\}) = \mu \circ U^{-1}(4)$.

Problem 8

A spacecraft's motor receives fuel from an injection system, denoted by A. If this injection system fails, a master alarm (B) is triggered on the commander's main dashboard. One possible cause of failure in the injection system is the occurrence of extreme vibrations, represented by C, which the spacecraft may experience during operation. However, it is important to note that extreme vibrations can erroneously trigger the master alarm even if the injection system itself is functioning correctly. Vibrations can also cause communication interference (D) with command control.

Compute the following. Note: For probability calculation use the data provided in the file prob.ipypnb. (This file will be provided by the instructor the day of the exam.)

- Compute the σ-algebra that encodes the following information: $\mathcal{G}_1$: "the injection system fails." $\mathcal{G}_2$: "the alarm is activated."
- What is the probability that the injection system fails?
- What is the probability that the injection system fails given that the alarm is activated?
- What is the probability that the injection system fails given that the alarm is activated and communication with command control is lost?

Ω's oucomes	DBAC	Probability
$D^cB^cA^cC^c$	0000	$p_0=\alpha_1$
$D^cB^cA^cC$	0001	$p_1=\alpha_2$
$D^cB^cAC^c$	0010	$p_2=\alpha_3$
D^cB^cAC	0011	$p_3=\alpha_4$
$D^cBA^cC^c$	0100	$p_4=\alpha_5$
D^cBA^cC	0101	$p_5=\alpha_6$
D^cBAC^c	0110	$p_6=\alpha_7$
D^cBAC	0111	$p_7=\alpha_8$
$DB^cA^cC^c$	1000	$p_8=\alpha_9$
DB^cA^cC	1001	$p_9=\alpha_{10}$
DB^cAC^c	1010	$p_{10}=\alpha_{11}$
DB^cAC	1011	$p_{11}=\alpha_{12}$
DBA^cC^c	1100	$p_{12}=\alpha_{13}$
DBA^cC	1101	$p_{13}=\alpha_{14}$
$DBAC^c$	1110	$p_{14}=\alpha_{15}$
$DBAC$	1111	$p_{15}=\alpha_{16}$