

# Probability Theory

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August 18, 2025

IN PROBABILITY THEORY, understanding how events relate to one another is essential for modeling uncertainty. The concept of independence describes situations where the occurrence of one event does not influence the likelihood of another. Building on this, conditional probability allows us to quantify the probability of an event given that another has occurred, forming the foundation for more complex reasoning. The chain rule extends this idea by expressing joint probabilities as a product of conditional probabilities, enabling the decomposition of multivariate distributions. Finally, Bayes' Theorem provides a powerful framework for updating beliefs in light of new evidence, reversing conditional probabilities to infer causes from observed outcomes. Together, these concepts form the backbone of probabilistic reasoning and inference.

## AGENDA:

- 1 Independence.
- 2 Conditional probability.
- 3 Chain rule.
- 4 Bayes Theorem.
- 5 Random Variables (Overview).

## Independent Events

$A_m, A_n \in \mathcal{F}$  are independent  $\implies P(A_m \cap A_n) = P(A_m)P(A_n)$

- $(A_n)_{n \in \mathbb{N}}$  are pair-wise independent  $\implies A_m$  and  $A_n$  are independent for any  $n \neq m, n, m \in \mathbb{N}$
- $(A_n)_{n \in I}$  are independent  $\implies P(\bigcap_{n \in I} A_n) = \prod_{n \in I} P(A_n)$

## Conditional probability

The conditional probability of event  $A$  conditioned to event  $B$  is defined as

$$P(A | B) = \frac{P(A \cap B)}{P(B)}$$

where  $P(B) > 0$ . A conditional probability is a *probability measure*. If  $A$  and  $B$  are independent  $P(A | B) = P(A)$  and  $P(B | A) = P(B)$ .

## Chain rule

Consider the events  $A_1, A_2, \dots, A_n$ . The chain rule is defined as follows.

$$P(A_1 \cap A_2 \cap \dots \cap A_n) =$$

$$P(A_1)P(A_2 | A_1)P(A_3 | A_1 \cap A_2) \dots P(A_n | A_1 \cap \dots \cap A_{n-1})$$

- Apply the chain rule to the following expression  $P(A_1 \cap A_2 \cap A_3 \cap A_4)$ .
- Is the following expression correct?  $P(A_1 \cap A_2 \cap A_3) = P(A_1 | A_2 \cap A_3)P(A_2 | A_3)P(A_3)$ .

*Bayes Theorem*

$$P(A | B) = \frac{P(B | A)P(A)}{P(B)}$$

where  $P(B) > 0$ . Recall that  $P(B) = \sum_{n \in \mathbb{N}} P(B \cap A_n)$  with  $\bigcup_{n \in \mathbb{N}} A_n = \Omega$ . Therefore:

$$\begin{aligned} P(A_m | B) &= \frac{P(B | A_m)P(A_m)}{P(B)} \\ &= \frac{P(B | A_m)P(A_m)}{\sum_{n \in \mathbb{N}} P(B \cap A_n)} = \frac{P(B | A_m)P(A_m)}{\sum_{n \in \mathbb{N}} P(B | A_n)P(A_n)} \end{aligned}$$

*Applying the Bayes Theorem*

- Mr. Holmes now lives in Los Angeles. One morning when Holmes leaves his house, he realizes that his grass is wet (H). Is it due to rain (R), or has he forgotten to turn off the sprinkler (S)? Next, he notices that the grass of his neighbor, Dr. Watson, is also wet (W).
  - What is the probability that Holmes's grass is wet (H) given that he forgot to turn the sprinkler off (S)?
  - What is the probability that Holmes's grass is wet (H) given that he forgot to turn the sprinkler off (S) and Dr. Watson's grass is also wet (W)?
- Consider the following events.
  - $A$  = "Holmes' grass is wet."
  - $A^c$  = "Holmes grass is dry. "
  - $B$  = "Holmes forgot to turn the sprinkler off."
  - $B'$  = "Holmes forgot to turn the sprinkler off and Watson's grass is wet."

We write down the outcomes including these events and their corresponding probabilities in the tables shown below:

$\Omega$	$P(\omega)$
$\omega_1 = H^c W^c S^c R^c$	$\alpha_1$
$\omega_2 = H^c W^c S^c R$	$\alpha_2$
$\omega_3 = H^c W^c S R^c$	$\alpha_3$
$\omega_4 = H^c W^c S R$	$\alpha_4$
$\omega_5 = H^c W S^c R^c$	$\alpha_5$
$\omega_6 = H^c W S^c R$	$\alpha_6$
$\omega_7 = H^c W S R^c$	$\alpha_7$
$\omega_8 = H^c W S R$	$\alpha_8$
$\omega_9 = H W^c S^c R^c$	$\alpha_9$
$\omega_{10} = H W^c S^c R$	$\alpha_{10}$
$\omega_{11} = H W^c S R^c$	$\alpha_{11}$
$\omega_{12} = H W^c S R$	$\alpha_{12}$
$\omega_{13} = H W S^c R^c$	$\alpha_{13}$
$\omega_{14} = H W S^c R$	$\alpha_{14}$
$\omega_{15} = H W S R^c$	$\alpha_{15}$
$\omega_{16} = H W S R$	$\alpha_{16}$

$A^c$	$P(\omega)$
$\omega_1 = H^c W^c S^c R^c$	$\alpha_1$
$\omega_2 = H^c W^c S^c R$	$\alpha_2$
$\omega_3 = H^c W^c S R^c$	$\alpha_3$
$\omega_4 = H^c W^c S R$	$\alpha_4$
$\omega_5 = H^c W S^c R^c$	$\alpha_5$
$\omega_6 = H^c W S^c R$	$\alpha_6$
$\omega_7 = H^c W S R^c$	$\alpha_7$
$\omega_8 = H^c W S R$	$\alpha_8$

$B'$	$P(\omega)$
$\omega_7 = H^c W S R^c$	$\alpha_7$
$\omega_8 = H^c W S R$	$\alpha_8$
$\omega_{15} = H W S R^c$	$\alpha_{15}$
$\omega_{16} = H W S R$	$\alpha_{16}$

$A$	$P(\omega)$
$\omega_9 = H W^c S^c R^c$	$\alpha_9$
$\omega_{10} = H W^c S^c R$	$\alpha_{10}$
$\omega_{11} = H W^c S R^c$	$\alpha_{11}$
$\omega_{12} = H W^c S R$	$\alpha_{12}$
$\omega_{13} = H W S^c R^c$	$\alpha_{13}$
$\omega_{14} = H W S^c R$	$\alpha_{14}$
$\omega_{15} = H W S R^c$	$\alpha_{15}$
$\omega_{16} = H W S R$	$\alpha_{16}$

$B$	$P(\omega)$
$\omega_3 = H^c W^c S R^c$	$\alpha_3$
$\omega_4 = H^c W^c S R$	$\alpha_4$
$\omega_7 = H^c W S R^c$	$\alpha_7$
$\omega_8 = H^c W S R$	$\alpha_8$
$\omega_{11} = H W^c S R^c$	$\alpha_{11}$
$\omega_{12} = H W^c S R$	$\alpha_{12}$
$\omega_{15} = H W S R^c$	$\alpha_{15}$
$\omega_{16} = H W S R$	$\alpha_{16}$

### Box 1: Example solutions

We use conditional probability and the Bayes theorem to compute the requested probabilities.

$$P(B | A) = \frac{P(B \cap A)}{P(A)} = \frac{\alpha_{11} + \alpha_{12} + \alpha_{15} + \alpha_{16}}{\sum_{k=9}^{16} \alpha_k}.$$

$$P(A | B) = \frac{P(B | A)P(A)}{P(B)} = \frac{\alpha_{11} + \alpha_{12} + \alpha_{15} + \alpha_{16}}{\alpha_3 + \alpha_4 + \alpha_7 + \alpha_8 + \alpha_{11} + \alpha_{12} + \alpha_{15} + \alpha_{16}}.$$

$$P(B' | A) = \frac{P(B' \cap A)}{P(A)} = \frac{\alpha_{15} + \alpha_{16}}{\sum_{k=9}^{16} \alpha_k}.$$

$$P(A | B') = \frac{P(B' | A)P(A)}{P(B')} = \frac{\alpha_{15} + \alpha_{16}}{\alpha_7 + \alpha_8 + \alpha_{15} + \alpha_{16}}.$$

### Random Variables

A random variable is also called a measurable function. Suppose you have two measurable spaces. One could be the pair of sample space and event space  $(\Omega, \mathcal{F})$ , and the other one could be some other arbitrary pair of a set and of its  $\sigma$ -algebra  $(E, \mathcal{E})$ , although usually, we choose  $E = \mathbb{R}^n$  and  $\mathcal{E} = \mathcal{B}(\mathbb{R}^n)$ ,  $n = 1, 2, \dots$ . Then a measurable function is a function  $X$  that maps elements in  $\Omega$  to elements in  $E$  with some additional properties. Notice how this function maps outcomes to elements of  $E$ , it does not map events (Figure 20).

The mapping  $X$  guarantees a correspondence between the events in our original event space and our transformed event space. For this reason, we require the random variable  $X : \Omega \rightarrow E$  ( $X : \Omega \rightarrow \mathbb{R}$ ) is such that to be such that the preimage  $X^{-1}(B)$  of any  $\mathcal{E}$ -measurable set  $B \in \mathcal{B}$  is a  $\mathcal{F}$  measurable set.

$$X^{-1}(B) = \{\omega \in \Omega : X(\omega) \in B\} \in \mathcal{F} \quad \forall B \in \mathcal{E}.$$

In the diagram below we can see how the set  $B$ , which is an element of  $\mathcal{E}$  has a pre-image,  $X^{-1}(B)$ , which is an element of  $\mathcal{F}$ . From here on, we use  $E = \mathbb{R}^n$ , and  $\mathcal{E} = \mathcal{B}(\mathbb{R}^n)$  to define a random variable.

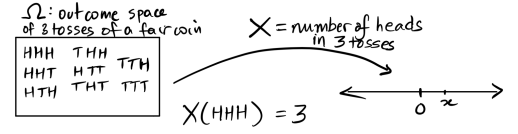
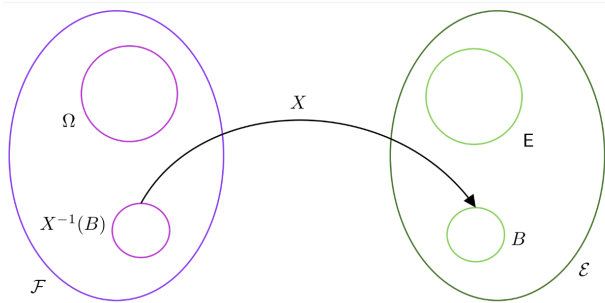


Figure 1: Random variable example defined on the real line  $\mathbb{R}$ . Notice that the inverse mapping  $X^{-1}$  maps real numbers into measurable events. For example  $\{HHH\} = X^{-1}(3)$  and  $\{\{HHT\}, \{HTH\}, \{THH\}\} = X^{-1}(2)$  belong to the  $\sigma$ -algebra  $\mathcal{F} = \mathcal{P}(\Omega)$ .

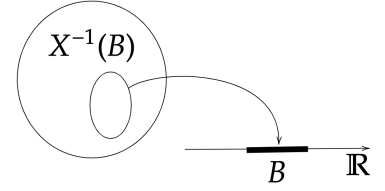


Figure 2: Random variable definition.

Figure 3: Random variable definition.

**Definition 1: Random Variable**

Let  $(\Omega, \mathcal{F}, \mu)$  be a probability space. A *random variable*  $X$  is a  $(\mathcal{F}/\mathcal{B}(\mathbb{R}^n))$  measurable map  $X : \Omega \rightarrow \mathbb{R}^n$ , if for every Borel set  $B \in \mathcal{B}(\mathbb{R}^n)$ :

$$X^{-1}(B) = \{\omega \in \Omega : X(\omega) \in B\} \in \mathcal{F}.$$

Clearly,

- $X^{-1}(B) \in \mathcal{F} \quad \forall B \in \mathcal{B}(\mathbb{R}^n)$ .

*Types of Random Variables*

In practice, we seldom bother working with the abstract concept of a probability space  $(\Omega, \mathcal{F}, P)$ , but rather just focus on the distributional properties of a random variable  $X$  representing the random phenomenon. We are interested in random variables that are *discrete* and *continuous*.

- A random variable  $X$  is discrete if it only takes values on a countable set  $\Omega = \{\omega_1, \omega_2, \omega_3, \dots\}$ , which is called the support of  $X$ .
- A random variable  $X$  is a continuous random variable if it only takes values on a non-countable set  $\Omega$ .
- Random variables can also be mixed.

*Examples of Discrete Random Variables*

- Consider the experiment in which we roll a die:  $\Omega = \{1, 2, 3, 4, 5, 6\}$ .
  - Let  $\mathcal{F} = \mathcal{P}(\Omega)$ . The random variable  $X_1(\omega) = \omega$ , is  $\mathcal{F}/\mathcal{B}(\mathbb{R})$  measurable.  $X_1$  gives the exact outcome of the roll.
  - Let  $\mathcal{F}_1 = \{\emptyset, \Omega, \{1, 3, 5\}, \{2, 4, 6\}\}$ . The random variable

$$X_2(\omega) = \begin{cases} 1, & \omega \in \{1, 3, 5\} \\ 0, & \omega \in \{2, 4, 6\} \end{cases}$$

is  $\mathcal{F}_1/\mathcal{B}(\mathbb{R})$  measurable; its value depends on whether the roll is odd or even.

- If we only have information on whether the roll is odd or even, we can determine the value of  $X_2$  but not the value of  $X_1$ .
- The random variable  $X_1$ , is not  $\mathcal{F}_1/\mathcal{B}(\mathbb{R})$  measurable. For instance,  $X_1^{-1}(1) = \{1\} \notin \mathcal{F}_1$ .
- Consider the experiment in which we roll a three-faced dice:  $\Omega = \{-1, 0, 1\}$  and  $\mathcal{F}_1 = \{\emptyset, \Omega, \{-1, 1\}, \{0\}\}$ .
  - $X_1(\omega) = \omega$  is not  $\mathcal{F}_1/\mathcal{B}(\mathbb{R})$  measurable. For example,  $X^{-1}(1) = \{1\} \notin \mathcal{F}_1$ .
  - $X_2(\omega) = \omega^2$  is  $\mathcal{F}_1/\mathcal{B}(\mathbb{R})$  measurable.

- Let  $(\Omega, \mathcal{F})$  describe throwing two fair dice, i.e.  $\Omega := \{(i, k) : 1 \leq i, k \leq 6\}$ ,  $\mathcal{F} = \mathcal{P}(\Omega)$ . The total number of points thrown  $X : \Omega \rightarrow \{2, 3, \dots, 12\}$ ,  $U((i, j)) = i + j$  is a measurable map.
- A  $\sigma$ -algebra generated by a random variable  $U$ , denoted by  $\sigma(U)$ , is the smallest  $\sigma$ -algebra for which  $U$  is measurable. For example, for  $\Omega = \{HH, HT, TH, TT\}$ .

$$X_1(\omega) = \begin{cases} 2, & \omega \in \{HH\}, \\ 1, & \omega \in \{HH, HT\}, \\ 0, & \omega \in \{TH, TT\}. \end{cases} \quad X_2(\omega) = \begin{cases} 1, & \omega \in \{HT\}, \\ -1, & \omega \in \{TH\}, \\ -2, & \omega \in \{TT\}. \end{cases}$$

$\sigma(X_1) = \{\emptyset, \{HH, HT\}, \{TH, TT\}, \Omega\}$  and  $\sigma(X_2) = \mathcal{P}(\Omega)$ . In particular,  $\sigma(X_1) \subset \sigma(X_2) = \mathcal{P}(\Omega)$ .

### Probability Distributions

A probability distribution is also called a *push-forward* ( $P_X = P_*X = P \circ X^{-1}$ ) measure of the *probability measure*  $P$ , via the random variable  $X$ . Suppose we have a probability space  $(\Omega, \mathcal{F}, P)$ . This means we can assign probabilities to events in  $\mathcal{F}$ . Now suppose we have a measurable space  $(E, \mathcal{E})$  but we don't yet have a probability measure to measure events in it. How can we go about measuring sets in  $\mathcal{E}$ ?

The key idea is that, given a set  $B$  in  $\mathcal{E}$ , we can use a random variable  $X$  to find the pre-image of such set in the event space  $\mathcal{F}$  and then measure this set via the probability measure  $P$ . This will then be our probability measurement for  $B$ , as shown in the figure below.

The *probability distribution*, is defined as  $P_X = P_*X = P \circ X^{-1} : \mathcal{E} \rightarrow [0, 1]$ . The probability distribution is therefore a function mapping sets in  $\mathcal{E}$  into  $[0, 1]$ . Here we use  $(E, \mathcal{E}) = (\mathbb{R}^n, \mathcal{B}(\mathbb{R}^n))$ , that is,  $P \circ X^{-1} : \mathcal{B}(\mathbb{R}^n) \rightarrow [0, 1]$ .

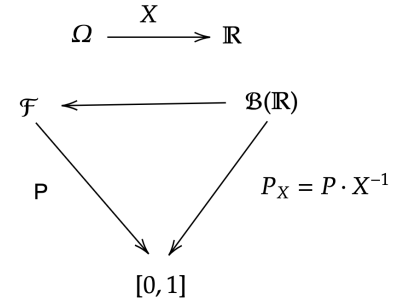
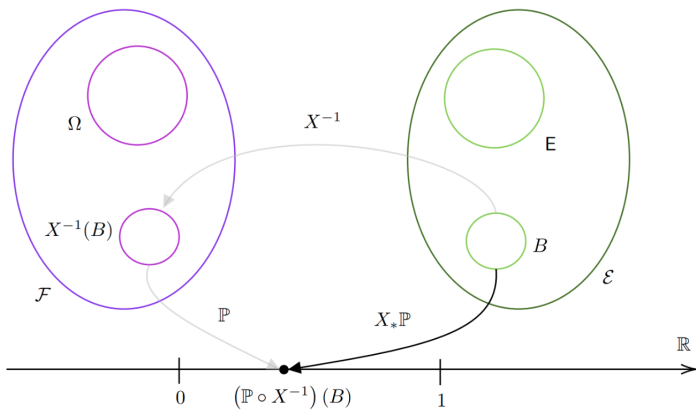


Figure 4: Probability distribution defined on the real line  $\mathbb{R}$ .

Figure 5: Probability distribution definition.

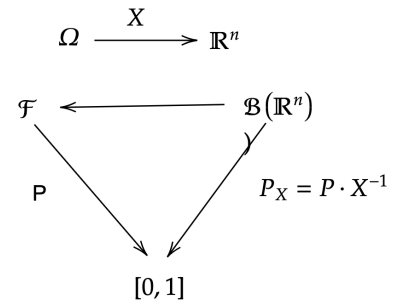


Figure 6: Probability distribution defined on the  $n$ -dimensional real space  $\mathbb{R}^n$ .

**Definition 2: Probability Distribution**

Let  $(\Omega, \mathcal{F}, \mu)$  be a measure space, and  $X$  a random variable  $X : \Omega \rightarrow \mathbb{R}$ , i.e. a measurable map. Then

$$P(X \in B) = \mu(X^{-1}(B)) = \mu(\{\omega : X(\omega) \in B\})$$

is a probability measure called the *law* or *distribution* of the random variable  $X$ . Recall that  $P = \mu \circ X^{-1} : \mathcal{B}(\mathbb{R}) \rightarrow [0, 1]$ .