This note discusses what is runaway problem and how to define and calculate the runaway rate.

1 Relativistic Kinetic Equation

$$\frac{\partial f_e}{\partial t} + \boldsymbol{v}\cdot\nabla f_e + \frac{q_e}{m_e}\left(\boldsymbol{E} + \frac{\boldsymbol{v}\times\boldsymbol{B}}{c}\right)\cdot\nabla_u f_e = -\nabla_u\cdot\boldsymbol{S}_c, \tag{1}$$

where u is momentum-per-unit-rest-mass, m_e is rest-mass. Assuming f_e is spatial homogeneous and azimuthally symmetric about the magnetic field in velocity space. Then Eq. (1) reduces to

$$\frac{\partial f_e}{\partial t} = -\nabla_u\cdot(\boldsymbol{S}_c + \boldsymbol{S}_E), \tag{2}$$

where

$$\boldsymbol{S}_E = \frac{q_e\boldsymbol{E}}{m_e} f_e. \tag{3}$$

2 Runaway problem

The dynamics of electrons in plasmas is influenced by the coulomb collisions. The collision drag force is usually a decreasing function in electrons' velocity in the large velocity region, as shown in Fig. 1. If an electric field is applied to a plasma, a certain fraction of electrons will gain an energy such that the electric force on them exceeds the collision drag force and they will keep being accelerated. This process is called electron runaway[1].

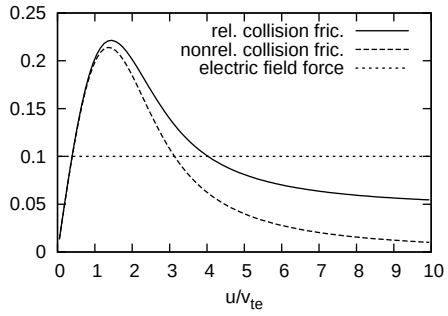

Figure 1. Relativistic and non-relativistic electron-electron collision friction coefficient. The electron temperature for the relativistic case is $T_e = 25\mathrm{keV}$.

We can define a critical velocity, v_c, for which collision drag force balance the electric force, i.e.,

$$F(v_c) = |qE|, \tag{4}$$

where $F(v)$ is the collision friction, E is the electric field and q is the electrical charge of the particle. Since for large velocity, $F(v)$ is a decreasing function in v, then for $v > v_c$, one has $F(v_c) < |qE|$. One can call the particle which has velocity larger than v_c "runway particle" or simply "runaway". More properly, we should choose another critical velocity, v_r, which is much larger than v_c (e.g. $v_r = 10v_c$) and call those particles with $v > v_r$ "runaway". In Tokamak discharge experiments, the particles that are considered as "runaways" usually have the energy at the order of MeV. (However, the birth energy (the energy of runaway particles when these runaway are created) may not be so large.)

The creation rate (or birth rate) of runaways due to a dc electric field, called runaway rate, is defined as the ratio of number of particles which runaway in a "collision time", $1/\nu_t$, to the number of particles which remain non-runaway. Runaway is a problem in velocity space. A runaway particle is created when it moves from the collision dominant region in velocity to the nearly collisionless region. For convenience, we use spherical coordinates, (v, θ), to describe velocity space (axial symmetry is assumed). The electron distribution is initially Maxwellian. A dc electric filed in the direction of the axis of symmetry is applied to the plasma at the initial time. We choose an arbitrary velocity, denoted as $v_{\max}$. Under the influence of the collision and electric field, some electrons whose velocity is initially larger than $v_{\max}$ may become smaller than $v_{\max}$, while some other electrons whose velocity is initially smaller than $v_{\max}$ may become larger than $v_{\max}$. This process generates a flux across the spherical surface $v = v_{\max}$. For a larger enough $v_{\max}$, e.g., $v_{\max} = 10v_c$, the collisional influence on the electrons may be neglected, the dominant process is the acceleration or deceleration by the electric field. In the vicinity of the $v = v_{\max}$ surface, the acceleration of electrons generates an out-flow flux across the surface while the deceleration generates an in-flow flux. The in-flow flux across the surface can be further neglected since there is few electrons initially in the region $v > v_{\max}$ (since $f(v)$ is initially Maxwellian and $v_{\max} \gg v_{te}$). (Note that, at later time, some electrons enter the region $v > v_{\max}$ through the acceleration by the electric field. However, since these particles are collisionless, they will keep being accelerated, with no chance to be decelerated to generate the in-flow flux.) In the next section, the out-going flux at the spherical surface, $v = v_{\max}$, will be used to define the runaway rate.

3 Runaway rate

Electron runaway rate is defined as

$$\lambda = -\frac{1}{n_e}\frac{\partial n_e}{\partial t} \tag{5}$$

where n_e is the density in the volume $v < v_{\max}$. Note that the distribution satisfies the equation

$$\frac{\partial f_e}{\partial t} = -\nabla \cdot \boldsymbol{S}, \tag{6}$$

where $\boldsymbol{S}$ is the flux in velocity space due to the collision and electric field. Integrating Eq. (6) in the velocity space, $v \leqslant v_{\max}$, one gets

$$\begin{aligned} \frac{\partial}{\partial t}\int_V f_e\, d^3\boldsymbol{v} &= -\int_V (\nabla \cdot \boldsymbol{S}) d^3\boldsymbol{v} \\ &= -\int_A \boldsymbol{S} \cdot d^2\boldsymbol{\sigma}, \end{aligned} \tag{7}$$

where A is the spherical surface, $v = v_{\max}$. Note that the left-hand side of Eq.(7) is $\partial n_e/\partial t$, then we get

$$\frac{\partial n_e}{\partial t} = -\int_A \boldsymbol{S} \cdot d^2\boldsymbol{\sigma} \tag{8}$$

Using this in the definition of runaway rate, Eq. (5), gives

$$\lambda = \frac{1}{n_e}\int_A \boldsymbol{S} \cdot d^2\boldsymbol{\sigma} \tag{9}$$

This form is used in the numerical code to calculate the runaway rate. Time will be normalized to ν_t^{-1}, so the runaway rate can be interpreted as the fraction of particles which runaway in a "collision time", $1/\nu_t$.

4 Boundary condition

We choose $v = v_{\max}$ as the boundary of the computational region. As discussed in the above, if $v_{\max}$ is large enough, the collision can be neglected in the vicinity of this surface. Thus the flux across the boundary surface is solely due to the electric field. Recall that the electric-field-induced flux is given by

$$\boldsymbol{S}_E = \frac{q_e \boldsymbol{E}}{m_e} f_e. \tag{10}$$

For a dc electric field in the direction of the axis of symmetry, $\boldsymbol{E} = E\hat{\boldsymbol{z}} = E\cos\theta\hat{\boldsymbol{v}} - E\sin\theta\hat{\boldsymbol{\theta}}$, the components of the flux are given respectively by

$$S_{E\theta} = -\frac{q_e E}{m_e}\sin\theta f_e. \tag{11}$$

$$S_{Ev} = \frac{q_e E}{m_e}\cos\theta f_e, \tag{12}$$

The value of $S_{E\theta}$ at the boundary $v = v_{\max}$ is not needed in the difference scheme, thus is not discussed here. The v component of the flux, S_{Ev}, is needed in the scheme, so we must specify its value. One may think that since f_e at the boundary is usually small, Eq.(12) tells that S_{Ev} can be set to zero. In fact, this is wrong. In the initial time, f_e is indeed small on the boundary since f_e is Maxwellian and $v_{\max} \gg v_{te}$. At the later time, some electrons may reach this boundary, so f_e at $v = v_{\max}$ may not be small. Now we describe how to set a value for S_{Ev} at the boundary. The friction coefficient in the v direction due to the electric field is given by

$$F_{Ev} = \frac{q_e E}{m_e}\cos\theta \tag{13}$$

Note that

$$\frac{\partial f_e}{\partial t} + \nabla \cdot \boldsymbol{S} = 0 \tag{14}$$

In the vicinity of the boundary, the above equation reduces to

$$\frac{\partial f_e}{\partial t} + \frac{F_{Ev}}{v^2}\frac{\partial v^2 f_e}{\partial v} + ... = 0 \tag{15}$$

The above equation tells that, in the direction of v, it is a hyperbolic equation. According to the theory of characteristic line, a up-wind difference scheme should be used to discrete the above equation. Specifically, when $F_{Ev} > 0$, a left-direction difference scheme should be used. Otherwise, a right-direction difference scheme should be used. Then, for the first case, only the value of distribution in the inner region $v < v_{\max}$ is needed to construct the difference scheme. So we do not need to impose any boundary condition for this case. (This corresponds to the case that the characteristic line is leaving the computational region; no boundary condition is needed.)

If F_{Ev} is negative, the character is entering the computational region, i.e., something out of the computational region will influence the solution in the computational region, so some boundary conditions must be imposed to reflect this influence. We note that the in-flow flux across the surface is small if $v_{\max} \gg v_{te}$ (The reason is discussed in the first section). Thus we use $S_v = 0$ as the boundary condition where F_{Ev} is negative.

5 Collision operator

If $f(\boldsymbol{u}) \approx f_m(u)$ for $u \leqslant u_t$, the following linearized collision operator, $C^l(f)$, is a good approximation to the full collision operator $C(f, f)$.

$$C(f, f) \approx C(f, f_m) + C(f_m, f) \equiv C^l(f) \tag{16}$$

In the numerical calculation, the second term of the linearized collision operator is difficult to deal with because the collision coefficient provided by the two-dimension distribution, f, is difficult to calculate. In crude approximation, the second term of the linearized collision term is simply ignored, giving a simple collision operator

$$C(f, f) \approx C(f, f_m) \equiv C_{\max}(f), \tag{17}$$

which is easy to be numerically calculated.

In order to make the linearized collision operator have the property of momentum-conservation, the second term of the linearized collision operator must be retained. It can be proved that it is only the first Legendre harmonic of this term that is responsible for the parallel momentum-conservation[6]. Therefore, to make momentum conserve, and at the same time to make the collision operator simple, we can retain only the $l = 0$ and $l = 1$ Legendre harmonics of this term. In practice, the $l = 0$ term is usually discarded. The purpose in doing this is to artificially remove the Ohm heating from the system. (This procedure is obviously not-physical and unreasonable. I do not like this, so I prefer the approximation in Eq. (17) although it may be inaccurate.)

The second term of the linearized collision operator is modified to include only the first Legendre harmonic, i.e. the collision operator actually used in the numerical calculation is

$$C_{\text{num}}(f) \equiv C(f, f_m) + C(f_m, f_1(u)P_1(\cos\theta)) \tag{18}$$

where

$$f_1(u) = \frac{3}{2}\int_0^\pi f(u, \theta)\, P_1(\cos\theta)\sin\theta d\theta, \tag{19}$$

is the first coefficient of the Legendre harmonics decomposition.

The collision operator defined in Eq. (18) is widely used in numerical codes dealing with noninductive current drive by waves[6].

6 Effective temperature

The effective temperature is defined as

$$T_{\text{eff}} = \frac{2}{3}\frac{\int \frac{1}{2}mv^2 f(\boldsymbol{v})d^3\boldsymbol{v}}{\int f(\boldsymbol{v})d^3\boldsymbol{v}} \tag{20}$$

We can ask the question why there is a factor of 2/3 in the definition.

7 Normalization

Electric field is normalized to E_0 given by

$$E_0 = \frac{m_e v_{te}\nu_{te}}{q_e}, \tag{21}$$

which is twice the Dreicer electric field defined in Ref. [1].

Time is normalized to $1/\nu_{te}$. Velocity is normalized to v_{te}.

8 Numerical results

8.1 Steady-state of distribution

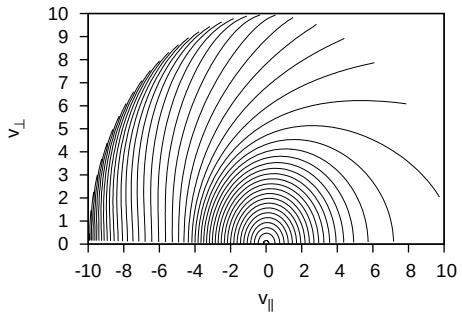

Figure 2. Contour of the steady-state of the distribution function in the presence of a dc electric field. The parameters are $Z_i = 1$, $E/E_0 = 0.06$, $M = N = 100$, $v_{\max} = 10$ and electron-electron collision is calculated using non-relativistic $C^{e/e}_{\rm Max}$. The contour levels are $f = (2\pi)^{-3/2}\exp\big(-0.5(j/5)^2\big)$ for $j = 0, 1, 2, ...50$. This figure is a re-plot of Fig. 10 in Karney's paper[2].

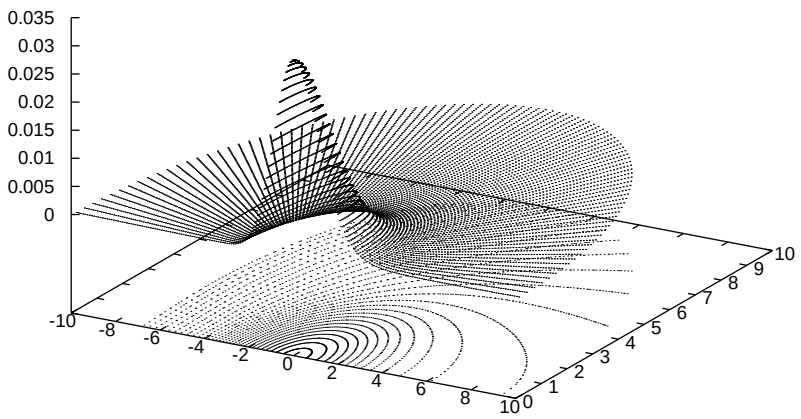

Figure 3. Steady-state of the distribution function in the presence of a dc electric field. The electron-electron collision is calculated using non-relativistic $C^{e/e}_{\rm Max}$.

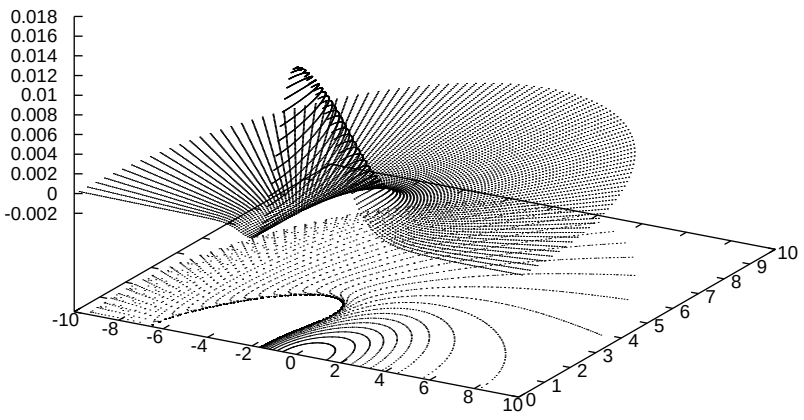

Figure 4. Steady-state of the distribution function in the presence of a dc electric field. The electron-electron collision is calculated using non-relativistic $C^{e/e}_{\rm num}$.

8.2 Runaway rate

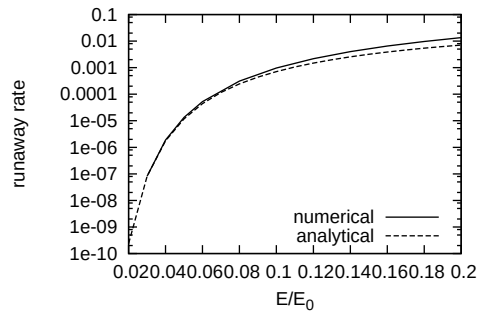

Figure 5. Runaway rate as a function of E/E_0 for $Z_i = 1$. The analytical result is from Eq. (2b) of Ref. [3].

In the analytical theory of Cohen *et al.*[4, 3, 5], the runaway rate λ is expressed as a function of E/E_0 and Z_i,

$$\lambda = K(Z_i)\left(\frac{E}{E_0}\right)^{-3(Z_i+1)/16} \exp\left(-\frac{1}{4E/E_0} - \sqrt{\frac{Z_i+1}{E/E_0}}\right), \tag{22}$$

where $K(Z_i)$ is weak function of Z_i, whose form can not be given in the analytical theory and must be determined by the numerical solution to the Fokker-Planck equation. Ref. [3] gives $K(1) = 0.32$, $K(2) = 0.43$.

Task: Using numerically determined $K(Z_i)$, I can interpolate the numerical results to get an analytical expression for $K(Z_i)$. Plot the figure of $K(Z_i)$ and write down the analytic expression.

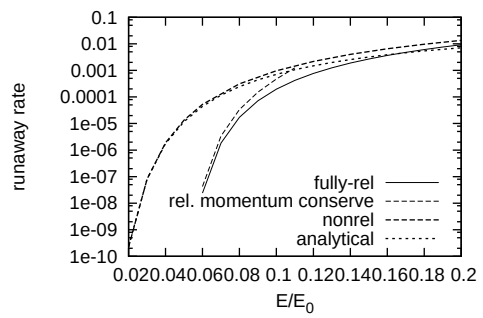

Figure 6. Runaway rate as a function of E/E_0 for $Z_i = 1$. The relativistic case is for $T_e = 25$keV.

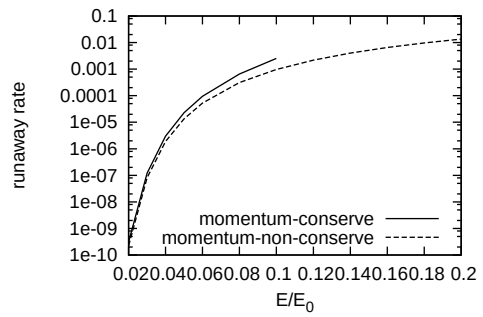

Figure 7. Runaway rate as a function of E/E_0 for $Z_i = 1$.

9 Manuscript

The distribution function of electrons, $f(\boldsymbol{v}, t)$, satisfies the following equation

$$\frac{\partial f}{\partial t} + A f = 0, \tag{23}$$

where A is a matrix resulting from the difference scheme. If $f'(\boldsymbol{v}, t)$ is related to $f(\boldsymbol{v}, t)$ by

$$f(\boldsymbol{v}, t) = f'(\boldsymbol{v}, t)\exp\left(-\int_0^t \gamma(t')dt'\right), \tag{24}$$

Then Eq.(23) tells that f' satisfies the following equation

$$\frac{\partial f'}{\partial t} + (A - \gamma(t))\, f' = 0 \tag{25}$$

$$f = f_m + \delta f$$

$$C(f, f) \;\approx\; C(\delta f, f_m) + C(f_m, \delta f) \equiv C^l(\delta f) \tag{26}$$

If $\delta f \ll f_m$, the above linearized collision operator $C^l(\delta f)$ is a good approximation to the full collision operator $C(f, f)$.

In order to artificially remove the Ohm heating from the system, the second term of the linearized collision operator is modified to include only the first Legendre harmonic, i.e. the collision operator actually used in the numerical calculation is

$$C_{\rm num}(\delta f) \equiv C(\delta f, f_m) + C(f_m, (\delta f)_1 P_1(\cos\theta)) \tag{27}$$

where

$$(\delta f)_1 = \frac{3}{2}\int_0^\pi \delta f\, P_1(\cos\theta)\sin\theta d\theta, \tag{28}$$

is the first coefficient of the Legendre harmonics decomposition.

Bibliography

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