Variational principle

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This notes discuss the basic theory of the variational principle[1].

1 Euler-Lagrange equation

Consider a general functional

$$I = \int_a^b F(y, y', x) dx, \tag{1}$$

where the values of function y at the end points are fixed. We want to find the function y that minimizes or maximizes I (Of course y should satisfy the boundary condition specified above, i.e. the values of $y(a)$ and $y(b)$ are specified and fixed). This problem reduces to finding a function y that can make the variation in I be equal to zero, i.e.,

$$\delta I = 0. \tag{2}$$

We now derive a differential form equivalent to the variational form Eq. (2). The variation in I can be calculated as

$$\begin{aligned} \delta I &= \delta \int_a^b F(y, y', x) dx \\ &= \int_a^b \left(\frac{\partial F}{\partial y} \delta y + \frac{\partial F}{\partial y'} \delta y' \right) dx \end{aligned} \tag{3}$$

where $\delta y'$ is the variation of y', which can be further written as

$$\begin{aligned} \delta y' &= \delta \left(\frac{dy}{dx} \right) \\ &= \frac{d(\delta y)}{dx}. \end{aligned} \tag{4}$$

Then the second term on the right-hand side of Eq. (3) can be written as

$$\begin{aligned} \int_a^b \left(\frac{\partial F}{\partial y'} \delta y' \right) dx &= \int_a^b \left(\frac{\partial F}{\partial y'} \frac{d\delta y}{dx} \right) dx \\ &= \int_a^b \frac{\partial F}{\partial y'} d\delta y \\ &= \frac{\partial F}{\partial y'} \delta y \bigg|_a^b - \int_a^b \delta y d\left(\frac{\partial F}{\partial y'} \right) \end{aligned} \tag{5}$$

Since we require the values of y at the end points, a and b, be fixed, i.e., $\delta y = 0$ at the end points, the above equation is written as

$$\begin{aligned} \int_a^b \left(\frac{\partial F}{\partial y'} \delta y' \right) dx &= -\int_a^b \delta y d\left(\frac{\partial F}{\partial y'} \right) \\ &= -\int_a^b \delta y \frac{d}{dx} \left(\frac{\partial F}{\partial y'} \right) dx \end{aligned} \tag{6}$$

Using this, Eq. (3) is written as

$$\delta I = \int_a^b \delta y \left[\frac{\partial F}{\partial y} - \frac{d}{dx} \left(\frac{\partial F}{\partial y'} \right) \right] dx. \tag{7}$$

Thus $\delta I = 0$ is written as

$$\int_a^b \delta y \left[\frac{\partial F}{\partial y} - \frac{d}{dx} \left(\frac{\partial F}{\partial y'} \right) \right] dx = 0 \tag{8}$$

Noting that Eq. (8) must hold for arbitrary δy, the only way that can make this possible is

$$\frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right)-\frac{\partial F}{\partial y}=0 \tag{9}$$

Equation (9) is known as Euler-Lagrange equation, which is a differential equation for $y(x)$. The solution to the Euler-Lagrange equation gives the function that can maximize or minimize the functional I.

1.1 A simple application

Using Euler-Lagrange equation, we can easily prove that the shortest path through two points in a plane is a straight line. In this case the functional should be the length of curve through the two points,

$$\begin{aligned} l &= \int_a^b \sqrt{(dy)^2+(dx)^2} \\ &= \int_a^b \sqrt{(dy/dx)^2+1}dx \\ &= \int_a^b \sqrt{y'^2+1}dx. \end{aligned} \tag{10}$$

Thus, for this case, the F in Euler-Lagrange equation is given by

$$F=\sqrt{y'^2+1}, \tag{11}$$

which happens to be independent of y. Then Euler-Lagrange equation (9) is written as

$$\frac{d}{dx}\left(\frac{\partial\sqrt{y'^2+1}}{\partial y'}\right)=0, \tag{12}$$

which gives

$$\frac{\partial\sqrt{y'^2+1}}{\partial y'}=C, \tag{13}$$

where C is a constant. Equation (13) is written as

$$\frac{y'}{\sqrt{y'^2+1}}=C,$$

which gives

$$y'=\pm\frac{C}{\sqrt{1-C^2}}, \tag{14}$$

which indicates y is a straight line. The straight line makes the variation of l vanish, which corresponds to a maximum or minimum of l. It is easy to check that the case corresponds to a minimum value of l, i.e. the shortest path between two points in a plane.

2 Euler-Lagrange equation for higher derivatives case

Consider a functional that contains higher derivatives

$$I=\int_a^b F(x,y,y',y'',\ldots,y^{(n)})dx. \tag{15}$$

Following similar procedures as discussed above, we can obtain a general Euler-Lagrange equation for $\delta I=0$,

$$\frac{\partial F}{\partial y}-\frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right)+\frac{d^2}{dx^2}\left(\frac{\partial F}{\partial y''}\right)-\ldots+(-1)^n\frac{d^n}{dx^n}\left(\frac{\partial F}{\partial y^{(n)}}\right)=0 \tag{16}$$

(Ref.: http://en.wikipedia.org/wiki/Euler%E2%80%93Lagrange_equation.)

3 Variation for self-adjoint operator

In general, we construct a functional of the form

$$I=\int_a^b y\,[L(y)-2\,g(x)]\,dx. \tag{17}$$

Then the variation of I is written as

$$\begin{aligned}\delta I &= \delta\int_a^b y\,[L(y)-2\,g]\,dx\\ &= \int_a^b [\delta y\,L(y)+y\,L(\delta y)-2\,g\delta y]\,dx.\end{aligned} \tag{18}$$

If the linear operator L is self-adjoint, i.e., $\int_a^b \delta y\,L(y)\,dx=\int_a^b y\,L(\delta y)dx$, then Eq. (18) is written as

$$\delta I=2\int_a^b \delta y[\,L(y)-g\,]\,dx. \tag{19}$$

Thus $\delta I=0$ is equivalent to

$$L(y)-g=0. \tag{20}$$

And it is obvious that this equation is the Euler-Lagrange equation for $\delta I=0$.

4 Constraint variation

Suppose we now want to find the maximum or minimum of the following functional

$$I=\int_a^b F(y,y',x)dx, \tag{21}$$

and we further require that $y(x)$ is subject to the constraint that the value of

$$J=\int_a^b G(y,y',x)dx, \tag{22}$$

remains constant. How to sovle this problem? Recalling the method we learn in the above, we may consider solving this problem by two steps: (1) first find the solutions to $\delta I=0$, (the solution contanis undetermined coefficients) then (2) require the solution to $\delta I=0$ to satisfy Eq. (22). (this step can determine some undertermined coefficients).

Define a new functional

$$K=I+\lambda(x)J=\int_a^b F(y,y',x)dx+\lambda(x)\int_a^b G(y,y',x)dx \tag{23}$$

where $\lambda(x)$ is a unknown function, then $\delta K=0$ is equivalent to the following Lagrange-Euler equation: (proof needed)

$$\frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right)-\frac{\partial F}{\partial y}+\frac{d}{dx}\left(\frac{\partial\lambda G}{\partial y'}\right)-\frac{\partial\lambda G}{\partial y}=0 \tag{24}$$

As an example, we consider the well-knonw catinary problem. We want to minimise the functional

$$U=-\rho g\int_{-a/2}^{a/2} y\sqrt{y'^2+1}dx, \tag{25}$$

and, at the same time, to make the value of

$$l=\int_{-a/2}^{a/2}\sqrt{y'^2+1}dx, \tag{26}$$

remain constant.

$$K=-\rho g y\sqrt{y'^2+1}+\lambda\sqrt{y'^2+1} \tag{27}$$

$$y'\frac{\partial K}{\partial y'} - K = k \tag{28}$$

$$y'\left(-\rho g y\frac{y'}{\sqrt{y'^2+1}} + \lambda\frac{y'}{\sqrt{y'^2+1}}\right) + \rho g y\sqrt{y'^2+1} - \lambda\sqrt{y'^2+1} = k$$

$$\frac{y'^2}{\sqrt{y'^2+1}}(\lambda - \rho g y) - (\lambda - \rho g y)\sqrt{y'^2+1} = k, \tag{29}$$

where k is a constant. The above equation reduces to

$$y'^2(\lambda - \rho g y) - (\lambda - \rho g y)\left(y'^2+1\right) = k\sqrt{y'^2+1}, \tag{30}$$

$$-(\lambda - \rho g y) = k\sqrt{y'^2+1}, \tag{31}$$

$$y'^2 = \left(\lambda' + \frac{y}{h}\right)^2 - 1 \tag{32}$$

where

$$\lambda' = \frac{\lambda}{k}$$

$$h = -\frac{k}{\rho g}$$

Let

$$\lambda' + \frac{y}{h} = -\cosh z. \tag{33}$$

$$\frac{1}{h}y' = -\sinh z\,\frac{dz}{dx}$$

Making this substitution, Eq. (32) yields

$$h^2\sinh^2 z\left(\frac{dz}{dx}\right)^2 = \cosh^2 z - 1$$

$$\Rightarrow h^2\sinh^2 z\left(\frac{dz}{dx}\right)^2 = \sinh^2 z$$

$$\Rightarrow \frac{dz}{dx} = \pm\frac{1}{h} \tag{34}$$

Refere to Fitzpatrick's book[1] for the rest of the derivation.

Bibliography

[1] *Analytical Classical Dynamics*. Richard Fitzpatrick, 2004.