One-dimensional finite element method

BY YOUJUN HU

Institute of Plasma Physics, Chinese Academy of Sciences
Email: yjhu@ipp.cas.cn

Abstract

This note discusses the one-dimensional finite element method. Simple model problems are used as examples to illustrate the procedures involved to solve one-dimensional boundary value problems by using the finite element method.

1 Differential form

Any second-order linear ordinary differential equation can be put into the following Sturm-Louville differential equation (proof is given in Sec. 5)

$$\frac{d}{dx}\left[a(x)\frac{du}{dx}\right]+b(x)u=f(x). \tag{1}$$

where $a(x)$, $b(x)$, and $f(x)$ are known functions. In the following, we consider the case with the following boundary condtion:

$$u(0)=0, \quad u(1)=0. \tag{2}$$

2 Variational form

All the possible solutions to Eq. (1) with the boundary condition (2) belongs to the linear space defined by

$$\boldsymbol{V}=\{v\colon v(x)\,\text{is continuous on}\,[0,1],\,v'\,\text{is piecewise continuous on}\,[0,1],\,\text{and}\,v(0)=v(1)=0\}. \tag{3}$$

The equivalent variational form of Eq (1) (with the boundary condition (2)) is: Find $u\in\boldsymbol{V}$, so that, for any function $v\in\boldsymbol{V}$, the following equation is satisfied:

$$\left\langle v,\frac{d}{dx}\left[a(x)\frac{du}{dx}\right]+b(x)u\right\rangle=\langle v,f\rangle \tag{4}$$

where the inner product is defined by:

$$\langle v,u\rangle\equiv\int_0^1 v(x)u(x)dx \tag{5}$$

In the following, we will use the variational form Eq. (4) as a starting point to develop numerical schemes. By using integration by part, the first term on the left-hand side of Eq. (4) can be written as

$$\begin{aligned}\left\langle v,\frac{d}{dx}\left[a(x)\frac{du}{dx}\right]\right\rangle &= vau'|_0^1-\langle v',au'\rangle\\ &= -\langle v',au'\rangle.\end{aligned} \tag{6}$$

Using this, Eq. (4) is written as

$$-\langle v', au' \rangle + \langle v, bu \rangle = \langle v, f \rangle \tag{7}$$

2.1 Finite-dimensional variational form

Let $0 = x_0 < x_1 < x_2 .... < x_n < x_{n+1} = 1$, be a partition of the interval [0,1] into sub-interval $I_j = [x_{j-1}, x_j]$ of length $h_j = x_j - x_{j-1}$ for $j = 1, 2, ...n + 1$. Let $h = \max\{h_j\}$, then h is a measure of how fine the partition is. We will use $\boldsymbol{V}_h$, a finite-dimensional subspace of function space $\boldsymbol{V}$, to approximate infinite-dimensional space $\boldsymbol{V}$. Linear space $\boldsymbol{V}_h$ is defined as follows: $\boldsymbol{V}_h$ is a set of functions that are linear on the interval I_j, continuous at x_j with $j = 1, 2, ...n$, and $v(0) = v(1) = 0$. If h is sufficiently small (so n is big), the space $\boldsymbol{V}_h$ can be viewed as a good approximation of space $\boldsymbol{V}$. Therefore the problem in Eq. (7) can be approximately expressed as:

Find $u \in \boldsymbol{V}_h$, so that, for $\forall v_h \in \boldsymbol{V}_h$, the following equation is satisfied:

$$-\left\langle v_h', au' \right\rangle + \langle v, bu \rangle = \langle v_h, f \rangle \tag{8}$$

Define function $\varphi_j(x)$

$$\varphi_j(x) \equiv \begin{cases} \dfrac{x - x_{j-1}}{x_j - x_{j-1}} & \text{if} \quad x \in [x_{j-1}, x_j] \\ \dfrac{x_{j+1} - x}{x_{j+1} - x_j} & \text{if} \quad x \in [x_j, x_{j+1}] \\ 0 & \quad\text{otherwise} \end{cases}$$

then any function in $\boldsymbol{V}_h$ space can be expressed as the linear combination of φ_i, i.e.,

$$v_h(x) = \sum_1^n \xi_i \varphi_i(x), \tag{9}$$

where $\xi_i = v(x_i)$. Therefore, the linear space $\boldsymbol{V}_h$ is an n-dimensional space with φ_j being basis function. It is easy to show the problem in Eq. (8) is equivalent to the following problem:

Find $u \in \boldsymbol{V}_h$, so that, for any φ_i, the following equation is satisfied:

$$-\left\langle \varphi_i', au' \right\rangle + \langle \varphi_i, bu \rangle = \langle \varphi_i, f \rangle, \quad \text{for } i = 1, 2, 3....n \tag{10}$$

2.2 Matrix form

Any function in the n-dimensional space $\boldsymbol{V}_h$ can be determined by its n coordinates, therefore we have

$$u(x) = \sum_1^n u_j \varphi_j(x), \tag{11}$$

$$f(x) = \sum_1^n f_j \varphi_j(x). \tag{12}$$

Using this expansion, the problem in Eq. (10) can be expressed in matrix form

$$A\boldsymbol{u} + B\boldsymbol{u} = M\boldsymbol{f}, \tag{13}$$

where $\boldsymbol{u}$ is the column vector $(u_1, u_2, ...u_j...., u_n)^T$; $\boldsymbol{f}$ is the column vector $(f_1, f_2, ..., f_j, ..., f_n)^T$; A is a $n \times n$ matrix with the matrix element $A_{ij} = -\left\langle \varphi_i', a\varphi_j' \right\rangle$, B is a $n \times n$ matrix with the matrix element $B_{ij} = \langle \varphi_i, b\varphi_j \rangle$, and M is a $n \times n$ matrix with the matrix element $M_{ij} = \langle \varphi_i, \varphi_j \rangle$.

2.3 Matrix elements

Due to the narrow support of the finite element basis functions, the integration, which results from the inner product, in the matrix elements can be evaluated analytically. The fact that the integration can be done by hand on paper, instead of by numerically integration on computer is important, because otherwise the finite elements method would involve more numerical steps and thus less efficient when compared with finite difference method. (Question: Is there the case where the integration needs to be evaluated numerically?)

Next, we try to perform analytically the integration in the elements of the matrix. Note that in the finite elment approximation, the coefficient $a(x)$ can be written as

$$a(x) = \sum_{1}^{n} a_j \varphi_j(x), \tag{14}$$

i.e., $a(x)$ is also assumed linear on the interval I_j and continuous at x_j with $j = 1, 2, ...n$. Using the above results, the matrix elements A_{ij} are written as

$$\begin{aligned}
\langle \varphi_i', a\varphi_i' \rangle &= \int_{x_{i-1}}^{x_i} a(x) \frac{1}{h_i^2} dx + \int_{x_i}^{x_{i+1}} a(x) \frac{1}{h_{i+1}^2} dx \\
&= \frac{1}{h_i^2} \int_{x_{i-1}}^{x_i} a(x) dx + \frac{1}{h_{i+1}^2} \int_{x_i}^{x_{i+1}} a(x) dx \\
&= \frac{1}{h_i^2} \left(\frac{a_i + a_{i-1}}{2} \right) h_i + \frac{1}{h_{i+1}^2} \left(\frac{a_{i+1} + a_i}{2} \right) h_{i+1} \\
&= \frac{(a_i + a_{i-1})}{2} \frac{1}{h_i} + \frac{(a_{i+1} + a_i)}{2} \frac{1}{h_{i+1}},
\end{aligned} \tag{15}$$

$$\langle \varphi_i', a\varphi_{i-1}' \rangle \approx - \int_{x_{i-1}}^{x_i} \left(\frac{a_i + a_{i-1}}{2} \right) \frac{1}{h_i^2} dx = - \frac{a_i + a_{i-1}}{2} \frac{1}{h_i}, \tag{16}$$

and

$$\langle \varphi_i', a\varphi_{i+1}' \rangle \approx - \int_{x_i}^{x_{i+1}} \left(\frac{a_{i+1} + a_i}{2} \right) \frac{1}{h_{i+1}^2} dx = - \frac{a_{i+1} + a_i}{2} \frac{1}{h_{i+1}}, \tag{17}$$

we obtain

$$A_{ij} \equiv - \langle \varphi_i', a\varphi_j' \rangle \approx \begin{cases} - \frac{(a_i + a_{i-1})}{2} \frac{1}{h_i} - \frac{(a_{i+1} + a_i)}{2} \frac{1}{h_{i+1}} & \text{if} \quad j = i \\ \frac{a_i + a_{i-1}}{2} \frac{1}{h_i} & \text{if} \quad j = i - 1 \\ \frac{a_{i+1} + a_i}{2} \frac{1}{h_{i+1}} & \text{if} \quad j = i + 1 \\ 0 & \text{otherwise} \end{cases} \tag{18}$$

For $j = 1, 2, ..., n$

$$\begin{aligned}
\langle \varphi_j, b(x) \varphi_j \rangle &= \int_{x_{j-1}}^{x_j} b(x) \left(\frac{x - x_{j-1}}{x_j - x_{j-1}} \right)^2 dx + \int_{x_j}^{x_{j+1}} b(x) \left(\frac{x_{j+1} - x}{x_{j+1} - x_j} \right)^2 dx \\
&= \frac{1}{h_j^2} \int_{x_{j-1}}^{x_j} b(x)(x - x_{j-1})^2 dx + \frac{1}{h_{j+1}^2} \int_{x_j}^{x_{j+1}} b(x)(x_{j+1} - x)^2 dx \\
&= \frac{1}{h_j^2} \int_{x_{j-1}}^{x_j} \frac{b_{j-1} + b_j}{2} (x - x_{j-1})^2 dx + \frac{1}{h_{j+1}^2} \int_{x_j}^{x_{j+1}} \frac{b_j + b_{j+1}}{2} (x_{j+1} - x)^2 dx \\
&= \frac{b_{j-1} + b_j}{2} \frac{1}{h_j^2} \frac{\xi^3}{3} \Big|_0^{x_j - x_{j-1}} + \frac{b_j + b_{j+1}}{2} \frac{1}{h_{j+1}^2} \frac{\xi^3}{3} \Big|_{x_j - x_{j+1}}^0 \\
&= \frac{b_{j-1} + b_j}{2} \frac{1}{h_j^2} \frac{h_j^3}{3} + \frac{b_j + b_{j+1}}{2} \frac{1}{h_{j+1}^2} \frac{h_{j+1}^3}{3} \\
&= \frac{b_{j-1} + b_j}{2} \frac{h_j}{3} + \frac{b_j + b_{j+1}}{2} \frac{h_{j+1}}{3}
\end{aligned} \tag{19}$$

For $j=2,3,....,n$

$$\begin{aligned}
\langle \varphi_j, b(x)\varphi_{j-1}\rangle &= \int_{x_{j-1}}^{x_j} b(x)\left(\frac{x-x_{j-1}}{x_j-x_{j-1}}\right)\left(\frac{x_j-x}{x_j-x_{j-1}}\right)dx \\
&\approx \frac{1}{h_j^2}\int_{x_{j-1}}^{x_j} \frac{b_{j-1}+b_j}{2}(x-x_{j-1})(x_j-x)dx \\
&= \frac{b_{j-1}+b_j}{2}\frac{1}{h_j^2}\int_{x_{j-1}}^{x_j} \left[-x^2+(x_{j-1}+x_j)x-x_{j-1}x_j\right]dx \\
&= \frac{b_{j-1}+b_j}{2}\frac{1}{h_j^2}\left[-\frac{x_j^3-x_{j-1}^3}{3}+(x_{j-1}+x_j)\frac{x_j^2-x_{j-1}^2}{2}-x_{j-1}x_jh_j\right] \\
&= \frac{b_{j-1}+b_j}{2}\frac{1}{h_j}\left[-\frac{x_j^2+x_jx_{j-1}+x_{j-1}^2}{3}+\frac{(x_{j-1}+x_j)^2}{2}-x_{j-1}x_j\right] \\
&= \frac{b_{j-1}+b_j}{2}\frac{1}{h_j}\left[-\frac{2x_j^2+2x_jx_{j-1}+2x_{j-1}^2}{6}+\frac{3(x_{j-1}+x_j)^2-6x_{j-1}x_j}{6}\right] \\
&= \frac{b_{j-1}+b_j}{2}\frac{1}{h_j}\left[-\frac{2x_jx_{j-1}}{6}+\frac{x_{j-1}^2+x_j^2}{6}\right] \\
&= \frac{b_{j-1}+b_j}{2}\frac{1}{h_j}\left[\frac{(x_j-x_{j-1})^2}{6}\right] \\
&= \frac{b_{j-1}+b_j}{2}\frac{h_j}{6}
\end{aligned}$$

Summarizing the above results, we obtain

$$B_{ij}=\langle\varphi_i, b(x)\varphi_j\rangle \approx \begin{cases} \frac{b_{i-1}+b_i}{2}\frac{h_i}{3}+\frac{b_i+b_{i+1}}{2}\frac{h_{i+1}}{3} & \text{if} \quad j=i \\ \frac{b_{i-1}+b_i}{2}\frac{h_i}{6} & \text{if} \quad j=i-1 \\ \frac{b_i+b_{i+1}}{2}\frac{h_{i+1}}{6} & \text{if} \quad j=i+1 \\ 0 & \text{otherwise} \end{cases}$$

3 Numerical results

Consider the Sturm-Liouville eigenvalue problem

$$\frac{d}{dx}\left(x^3\frac{dy}{dx}\right)+\lambda x y=0. \tag{20}$$

The eigenvalue and eigenfunction of equation (20) with boundary condition $y(1)=0$ and $y(2)=0$ can be obtained analytically, which are given respectively by

$$\lambda_n=1+\left(\frac{n\pi}{\ln 2}\right)^2, n=1,2,3..., \tag{21}$$

and

$$y_n(x)=\frac{1}{x}\sin\left(\frac{n\pi}{\ln 2}\ln(x)\right). \tag{22}$$

Figure 1 plots the numerical result obtained by using the finite element method.

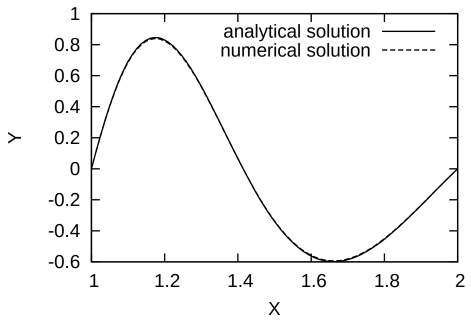

Figure 1. Numerical eigenfunction obtained by the finite element method. The eigenvalue obtained is $\lambda = 83.22459$, which corresponds to $n = 2.00067$ in Eq. (21). Also plotted in the figure is the analytical eigenfunction given by Eq. (22) with $n = 2$. Note that the curves of the numerical solution and the analytical one agree with each other so well that they are indistinguishable at this scale. The numerical eigenfunction is obtained through the singular value decomposition method, and we have scaled the numerical results by a factor 4.46 to make it match the analytical solution (this is justified since an eigenfunction multiplied by any constant is still an eigenfunction).

Consider the Hermite differential equation

$$\begin{cases} y_{xx} + (\lambda - x^2) y = 0 \\ y(+\infty) = 0, y(-\infty) = 0 \end{cases}$$

The eigenvalue of the above equation is $\lambda = 2n + 1$, $n = 0, 1, 2, 3\ldots$. The corresponding eigenfunction is $y_n(x) = h_n(x)$, where

$$h_n(x) = \frac{1}{\pi^{1/4}\sqrt{2^n n!}} \exp\left(-\frac{x^2}{2}\right) H_n(x), \tag{23}$$

where

$$H_n(x) = (-1)^n \exp(x^2) \left(\frac{d}{dx}\right)^n \exp(-x^2) \tag{24}$$

Figure 2 plots the numerical result obtained by using the finite element method.

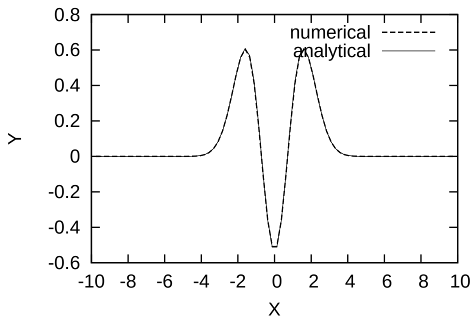

Figure 2. Numerical eigenfunction obtained by the finite element method for the Hermite equation with the boundary condition $y(-10) = 0$ and $y(10) = 0$. The eigenvalue obtained is $\lambda = 5.0696$. Also plotted in the figure is the analytical eigenfunction given by Eq. (23) with $n = 2$. Note that the curves of the numerical solution and the analytical one agree with each other so well that they are almost indistinguishable at this scale. The numerical eigenfunction is obtained through the singular value decomposition method, and the numerical results habe been scaled by a factor of two to make it match the analytical solution (this is justified since an eigenfunction multiplied by any constant is still an eigenfunction).

Next we consider a singular Sturm-Liouville equation:

$$\frac{d}{dx}\left[(\omega - x)\frac{dy}{dx}\right] + (\omega - x)\, y = 0, \tag{25}$$

with the boundary condition $y(0)=0$ and $y(1)=0$. An Sturm-Liouville equation is called "singular" if the coefficients have singularities. In our case, the singularity refers to that the coefficients before the highest order derivative is zero at the location $x=\omega$. For singular Sturm-Liouville equation , the eigenfrequency will have continuous spectrum. Eigenfunction corresponding every frequency in the continuous spectrum will have singularity in the location where the coefficients are zero. An example is plotted in Fig. 3.

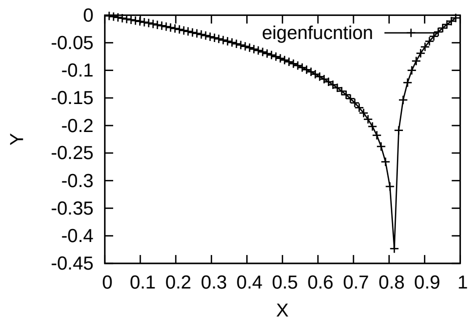

Figure 3. Eigenfunction corresponding eigenfrequency $\omega=0.816736$. Note that the eigenfunction has singularity at the location $x=0.814$ which approximately corresponds to the zero points of the coefficients $\omega-x$.

4 Model Eigenvalue Problems

$$\begin{cases} y_{xx}-\lambda y=0 \\ y(0)=0, y(1)=0 \end{cases} \tag{26}$$

The eigenvalue of the above problem is $\lambda=-n^2\pi^2$ with $n=1,2,3\ldots$. The corresponding eigenfunction is $y_n(x)=\sin(n\pi x)$.

4.1 Analytic method to obtain the eigenvalue of Eq. (26)

When $\lambda>0$, general solution to the Equation is given by

$$y=C_1\exp(\sqrt{\lambda}x)+C_2\exp(-\sqrt{\lambda}x) \tag{27}$$

Boundary conditions requires that

$$\begin{array}{l} C_1+C_2=0 \\ C_1\exp(\sqrt{\lambda})+C_2\exp(-\sqrt{\lambda})=0 \end{array} \tag{28}$$

To get a nonzero solution, the determination of the above linear equation system should be zero, i.e.,

$$\left|\begin{array}{ll} 1 & 1 \\ \exp(\sqrt{\lambda}) & \exp(-\sqrt{\lambda}) \end{array}\right|=0, \tag{29}$$

which requires $\lambda=0$. Substituting this into Eq. (27) gives a zero solution. Thus λ with $\lambda>0$ can not be an eigenvalue of Eq. (26).

When $\lambda=0$ it is obvious that only zero solution exists. Thus $\lambda=0$ is not an eigenvalue.

When $\lambda<0$, general solution to the equation is given by

$$y=C_1\sin(\sqrt{-\lambda}x)+C_2\cos(\sqrt{-\lambda}x) \tag{30}$$

Boundary conditions require that

$$\begin{array}{l} C_2=0 \\ C_1\sin(\sqrt{-\lambda})+C_2\cos(\sqrt{-\lambda})=0 \end{array} \tag{31}$$

To get a nonzero solution, the determination of the above linear equation system should be zero, i.e.,

$$\begin{vmatrix} 0 & 1 \\ \sin\sqrt{-\lambda} & \cos\sqrt{-\lambda} \end{vmatrix} = 0, \tag{32}$$

which reduces to

$$\sqrt{-\lambda} = n\pi \Rightarrow \lambda = -n^2\pi^2, \tag{33}$$

with $n = 1, 2, \ldots..$ Substituting this into Eq. (30) gives the solution

$$y = C_1 \sin(\sqrt{-\lambda} x) = C_1 \sin(n\pi x), \tag{34}$$

which is a nontrivial solution, thus, is an eigenfunction of Eq. (26).

4.2 Numerical method to get the eigenvalue of Eq. (26)—Finite difference method

Using the center difference scheme for the second order derivative, the discrete form of Eq. (26) is written as

$$\frac{y_{i-1} - 2y_i + y_{i+1}}{h^2} - \lambda y_i = 0, \tag{35}$$

which can be arranged in the form

$$y_{i-1} + (-\lambda h^2 - 2) y_i + y_{i+1} = 0. \tag{36}$$

The matrix form of Eq. (36) is written as

$$\mathbf{M} y = 0, \tag{37}$$

where

$$\mathbf{M} = \begin{pmatrix} -\lambda h^2 - 2 & 1 & 0 & 0 & 0 \\ 1 & -\lambda h^2 - 2 & 1 & 0 & 0 \\ 0 & 1 & -\lambda h^2 - 2 & 1 & 0 \\ 0 & 0 & 1 & -\lambda h^2 - 2 & 1 \\ 0 & 0 & 0 & 1 & -\lambda h^2 - 2 \end{pmatrix} \tag{38}$$

Define

$$\mathbf{A} = \begin{pmatrix} -2 & 1 & 0 & 0 & 0 \\ 1 & -2 & 1 & 0 & 0 \\ 0 & 1 & -2 & 1 & 0 \\ 0 & 0 & 1 & -2 & 1 \\ 0 & 0 & 0 & 1 & -2 \end{pmatrix}, \tag{39}$$

and

$$\mathbf{B} = \begin{pmatrix} -\lambda h^2 & 0 & 0 & 0 & 0 \\ 0 & -\lambda h^2 & 0 & 0 & 0 \\ 0 & 0 & -\lambda h^2 & 0 & 0 \\ 0 & 0 & 0 & -\lambda h^2 & 0 \\ 0 & 0 & 0 & 0 & -\lambda h^2 \end{pmatrix} \tag{40}$$

then the matrix $\boldsymbol{M}$ is written as

$$\mathbf{M} = \mathbf{A} + \mathbf{B} \tag{41}$$

Thus Eq. (37) can be expressed as

$$\mathbf{A} y = -\mathbf{B} y, \tag{42}$$

i.e.,

$$\mathbf{A} y = \lambda h^2 y, \tag{43}$$

which is an eigenvalue problem for matrix $\boldsymbol{A}$. My numerical codes solving the eigenvalue problem are located at /home/yj/project_new/matrix_eigenvalue.

4.3 Numerical method to get the eigenvalue of Eq. (26)—finite element method

As discussed in Sec. (2).

5 Sturm-Liouville differential equation

Sturm-Liouville equation is a second-order linear ordinary differential equation of the form

$$-\frac{d}{dx}\left[p(x)\frac{dy}{dx}\right]+[\lambda w(x)-q(x)]y=0. \tag{44}$$

In general, any second-order ordinary differential equation

$$P(x)y''+Q(x)y'+R(x)y=0, \tag{45}$$

can be put into the Sturm-Liouville form by dividing by $P(x)$, multiplying by the integrating factor $e^{\int Q(x)/P(x)dx}$ and then collecting terms to give the form of Eq. (44). For example, consider the differential equation

$$x^3y''-xy'+2y=0. \tag{46}$$

Dividing the above equation by x^3 gives

$$y''-\frac{1}{x^2}y'+\frac{2y}{x^3}=0. \tag{47}$$

Multiplying the above eqution by an integrating factor

$$e^{\int -1/x^2dx}=e^{1/x}, \tag{48}$$

gives

$$e^{1/x}y''-e^{1/x}\frac{1}{x^2}y'+e^{1/x}\frac{2y}{x^3}=0, \tag{49}$$

The first two terms of above equation can be collected to form a perfect derivative, giving

$$\frac{d}{dx}\left[e^{1/x}\frac{dy}{dx}\right]+e^{1/x}\frac{2y}{x^3}=0, \tag{50}$$

which is in the Sturm-Liouville form.

Consider another example

$$x^2y''+3xy'+\lambda y=0 \tag{51}$$

$$\Rightarrow y''+\frac{3}{x}y'+\frac{\lambda}{x^2}y=0 \tag{52}$$

Multiplying the above equation by an integrating factor $e^{\int 3/x dx}=e^{3\ln x}=x^3$ gives

$$x^3y''+3x^2y'+\lambda xy. \tag{53}$$

$$\Rightarrow \frac{d}{dx}\left(x^3\frac{dy}{dx}\right)+\lambda xy=0, \tag{54}$$

which is in the Sturm-Liouville form.

[By the way, the eigenvalue and eigenfunction of equation (54) with boundary condition $y(1)=0$ and $y(2)=0$ can be obtained analytically, which are given respectively by

$$\lambda_n=1+\left(\frac{n\pi}{\ln 2}\right)^2, n=1,2,3..., \tag{55}$$

and

$$y_n(x)=\frac{1}{x}\sin\left(\frac{n\pi}{\ln 2}\ln(x)\right). \tag{56}$$

]

6 ddd

$$\begin{aligned} &y'' + \lambda y = 0 \\ &y(-1) = 0, y(1) = 0 \end{aligned} \tag{57}$$

The eigenvalue of the above equation is $\lambda = (n\pi/2)^2$, $n = 1, 2, 3....$ The eigenfunction of the above equation is

$$\begin{aligned} y_n(x) &= C\cos(\frac{n\pi}{2}x) \qquad n = 1, 3, 5, ... \\ y_n(x) &= C\sin(\frac{n\pi}{2}x) \qquad n = 2, 4, 6, ... \end{aligned} \tag{58}$$

Proof: When $\lambda < 0$, general solution to the Equation is given by

$$y = C_1\exp(\sqrt{-\lambda}x) + C_2\exp(-\sqrt{-\lambda}x) \tag{59}$$

Boundary conditions demand:

$$\begin{aligned} C_1\exp(-\sqrt{-\lambda}) + C_2\exp(\sqrt{-\lambda}) = 0 \\ C_1\exp(\sqrt{-\lambda}) + C_2\exp(-\sqrt{-\lambda}) = 0 \end{aligned} \tag{60}$$

to get a nonzero solution ,

$$\left| \begin{array}{ll} \exp(-\sqrt{-\lambda}) & \exp(\sqrt{-\lambda}) \\ \exp(\sqrt{-\lambda}) & \exp(-\sqrt{-\lambda}) \end{array} \right| = 0 \tag{61}$$

No solution exists for the above equation. So , in this case Eq. (57) has only zero solution.

When $\lambda = 0$ it is obvious that only zero solution exists.

When $\lambda > 0$, general solution to the equation is given by

$$y = C_1\sin(\sqrt{\lambda}x) + C_2\cos(\sqrt{\lambda}x) \tag{62}$$

Boundary conditions demand:

$$\begin{aligned} C_1\sin(-\sqrt{\lambda}) + C_2\cos(-\sqrt{\lambda}) = 0 \\ C_1\sin(\sqrt{\lambda}) + C_2\cos(\sqrt{\lambda}) = 0 \end{aligned} \tag{63}$$

to get nonzero solution

$$\left| \begin{array}{ll} \sin(-\sqrt{\lambda}) & \cos(-\sqrt{\lambda}) \\ \sin(\sqrt{\lambda}) & \cos(\sqrt{\lambda}) \end{array} \right| = 0 \tag{64}$$

that is

$$\sin(2\sqrt{\lambda}) = 0$$

that is

$$2\sqrt{\lambda} = n\pi \Rightarrow \lambda = \left(\frac{n\pi}{2}\right)^2 \tag{65}$$

Nonzero solution is

$$\begin{aligned} y &= C\cos(\frac{n\pi}{2}x) \qquad n = 1, 3, 5, ... \\ y &= C\sin(\frac{n\pi}{2}x) \qquad n = 2, 4, 6, ... \end{aligned} \tag{66}$$

7 Comparison with finite difference method

For the case of constant coefficients ($a = 1$, $b = 1$) and uniform grid, the matrix form of the model equation corresponds to the following difference form:

$$\frac{u_{i-1} - 2u_i + u_{i+1}}{h^2} = \frac{1}{6}f_{i-1} + \frac{2}{3}f_i + \frac{1}{6}f_{i+1} \tag{67}$$

This is a generalization of the following difference form:

$$\frac{u_{i-1} - 2u_i + u_{i+1}}{h^2} = f_i \tag{68}$$

8 tmp—to be deleted

If the original differential equation contains the first order derivative, then we will encounter term $\langle \varphi_i, \varphi_j' \rangle$. (Actually, we do not need the following expression because any second order ODE can be written in the ST form which does not contain a first-order derivative.)

For $i = 2, 3, \ldots., n$

$$\begin{aligned}
\langle \varphi_i, \varphi_i' \rangle &= \int_{x_{i-1}}^{x_i} \frac{1}{h_i}\left(\frac{x - x_{i-1}}{x_i - x_{i-1}}\right) dx - \int_{x_i}^{x_{i+1}} \frac{1}{h_{i+1}}\left(\frac{x_{i+1} - x}{x_{i+1} - x_i}\right) dx \\
&= \frac{1}{h_i^2}\int_{x_{i-1}}^{x_i} (x - x_{i-1}) dx - \frac{1}{h_{i+1}^2}\int_{x_i}^{x_{i+1}} (x_{i+1} - x) dx \\
&= \frac{1}{h_i^2}\frac{(x - x_{i-1})^2}{2}|_{x_{i-1}}^{x_i} + \frac{1}{h_{i+1}^2}\cdot\frac{(x - x_{i+1})^2}{2}|_{x_i}^{x_{i+1}} \\
&= \frac{1}{h_i^2}\frac{h_i^2}{2} + \frac{1}{h_{i+1}^2}(0 - \frac{h_{i+1}^2}{2}) \\
&= 0
\end{aligned}$$

$$\begin{aligned}
\langle \varphi_i, \varphi_{i-1}' \rangle &= \int_{x_{i-1}}^{x_i} \left(-\frac{1}{h_i}\right)\left(\frac{x - x_{i-1}}{x_i - x_{i-1}}\right) dx \\
&= -\frac{1}{h_i^2}\frac{(x - x_{i-1})^2}{2}|_{x_{i-1}}^{x_i} \\
&= -\frac{1}{2}
\end{aligned}$$

$$\begin{aligned}
\langle \varphi_i, \varphi_{i+1}' \rangle &= \int_{x_i}^{x_{i+1}} \frac{1}{h_{i+1}}\left(\frac{x_{i+1} - x}{x_{i+1} - x_i}\right) dx \\
&= -\frac{1}{h_{i+1}^2}\frac{(x - x_{i+1})^2}{2}|_{x_i}^{x_{i+1}} \\
&= -\frac{1}{h_{i+1}^2}(0 - \frac{h_{i+1}^2}{2}) \\
&= \frac{1}{2}
\end{aligned}$$

Summarizing the above results, we obtain

$$\langle \varphi_i, \varphi_j' \rangle = \begin{cases} -\frac{1}{2} & \text{if} \quad j = i - 1 \\ \frac{1}{2} & \text{if} \quad j = i + 1 \\ 0 & \text{otherwise} \end{cases}$$

$$\langle \varphi_i, \varphi_j'' \rangle = -\langle \varphi_i', \varphi_j' \rangle$$