Linear gyrokinetic theory and computation of the gyrocenter motion based on the exact canonical variables for axisymmetric tokamaks

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Linear gyrokinetic theory based on the Lie-transform perturbation method is presented in terms of the exact canonical variables. In the linear drift approximation, it is shown that the gyrocenter equations of motion based on the canonical variables are equivalent to the usual guiding-center equations of motion. A numerical code is developed to advance the gyrocenter motion in terms of the exact canonical variables with arbitrary perturbations. It is found that a static magnetic island in a tokamak has little effect on the trapped particle orbits due to the conservation of the longitudinal invariant; and it induces the island structure of passing particle orbits due to the fact that the longitudinal invariant for the passing particles is broken by the asymmetric perturbation. © *2011 American Institute of Physics*. [doi:10.1063/1.3578366]

I. INTRODUCTION

Gyrokinetic theory plays an important role in theory and simulation of magnetic fusion plasmas. Its physical motivation is to decouple the gyromotion and the motion of gyrocenter (guiding-center in the perturbed electromagnetic field). The classical gyrokinetic theory[1–3] has been developed by the recursive method, while the modern gyrokinetic theory[4–12] has been developed by the Lie-transform perturbation method.[13–15] Most gyrokinetic theories have been developed in terms of the noncanonical variables.[4–7,9,11]

However, canonical variables may be of benefit to the simulation of magnetic fusion plasmas and simplify the derivation of gyrokinetic equations to some degree. Recently, Hamiltonian dynamics in the exact canonical variables[16–18] has been developed for charged particles in tokamaks. The exact canonical variables which can separate time scales of poloidal motion and toroidal drift may benefit the numerical computation.[17,19] Linear gyrokinetic theory using approximate[8] and exact[12] canonical variables has been developed recently.

Canonical theory of guiding-center orbits with the perturbation of electromagnetic fields has been investigated by White *et al.*[20–22] However, the perturbation they used is a special form, and the finite-Larmor-radius effect is ignored due to the drift approximation. Clearly, it is of interest to investigate the gyrocenter motion with arbitrary perturbations, in the context of gyrokinetic theory based on the exact canonical variables.

In this paper, we will present linear gyrokinetic theory in terms of the exact canonical variables, which is based on the Lie-transform perturbation method. The equivalence of the gyrocenter and guiding-center equations of motion is shown in the linear drift approximation. A numerical code based on the linear canonical gyrokinetic theory is developed to compute the gyrocenter orbits with arbitrary perturbations of electromagnetic fields.

The rest of the paper is organized as follows. In Sec. II, the unperturbed Hamiltonian dynamics in terms of the exact guiding-center canonical variables is briefly presented. In Sec. III, the Lie-transform perturbation method is introduced. In Sec. IV, gyrocenter equations of motion in terms of the exact canonical variables are presented. In Sec. V, the linear gyrokinetic Vlasov equation based on the exact canonical variables is presented. In Sec. VI, the numerical code gyrocenter canonical variables advance (GYCAVA) for advancing the gyrocenter motion is described; and as a numerical example of GYCAVA, the Poincare section plots of the gyrocenter motion in a tokamak with a static magnetic island are presented and discussed. In Sec. VII, the main results are summarized.

II. UNPERTURBED HAMILTONIAN DYNAMICS IN THE EXACT GUIDING-CENTER CANONICAL VARIABLES

For magnetized plasmas, there is a small parameter ϵ_B defined as the ratio of the particle gyroradius ρ_s and the background magnetic field scale L_B, that is, $\epsilon_B \equiv \rho_s/L_B \ll 1$. The guiding-center variables can be obtained by the guiding-center phase-space transformation.[23,24] In this section, the unperturbed guiding-center Hamiltonian dynamics in terms of the exact guiding-center canonical variables[17] is presented for an axisymmetric tokamak geometry. The equilibrium magnetic field in an axisymmetric torus can be written in a contravariant and a covariant form[25]

$$\boldsymbol{B} = q(\psi)\nabla\psi \times \nabla\theta + \nabla\zeta \times \nabla\psi \tag{1a}$$

$$= g(\psi)\nabla\zeta + I(\psi)\nabla\theta + g(\psi)\delta(\psi,\theta)\nabla\psi, \tag{1b}$$

where (ψ, θ, ζ) are the magnetic flux coordinates, with ψ the poloidal magnetic flux, θ the poloidal angle, and ζ the toroidal angle. $q(\psi)$ is the safety factor. Note that ζ is an

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ignorable coordinate. The Jacobian of the coordinate system (ψ,θ,ζ) is

$$J_{(\psi,\theta,\zeta)} = |\nabla\psi\times\nabla\theta\cdot\nabla\zeta|^{-1} = \frac{qg+I}{B^2}. \tag{2}$$

The coordinate system (ψ,θ,α) is used to separate the time scales of poloidal guiding-center motion and toroidal drift. The representation of $\boldsymbol{B}$ can be rewritten as

$$\boldsymbol{B} = \nabla\psi\times\nabla\alpha \tag{3a}$$

$$= -g(\psi)\nabla\alpha + [q(\psi)g(\psi)+I(\psi)]\nabla\theta + g(\psi)\left[\delta(\psi,\theta)+\frac{dq}{d\psi}\right]\nabla\psi \tag{3b}$$

$$\equiv -g(\psi)\nabla\alpha + h(\psi)\nabla\theta + g(\psi)\Delta(\psi,\theta)\nabla\psi, \tag{3c}$$

with α defined as

$$\alpha = -\zeta + q\theta. \tag{4}$$

The exact guiding-center canonical extended phase-space coordinates $Z^i \equiv (\theta,\alpha_c,\xi,t,P_\theta,P_\alpha,M,-U)$ are used, instead of the noncanonical variables. M and ξ are the guiding-center magnetic moment and the gyrophase angle, respectively. $(-U,t)$ are the canonical conjugate guiding-center energy-time variables, with U the total energy. The relations of the canonical variables $(P_\alpha,\alpha_c,P_\theta,\theta)$ and the noncanonical variables $(\psi,\theta,\zeta,\rho_\parallel g)$ are given below. Here, $\rho_\parallel = v_\parallel/B$. Throughout the paper, $e_s = m_s = 1$ is set to simplify the formulae, with e_s and m_s the electric charge and the mass of the particle species s, respectively. P_α, P_θ, and α_c are defined as

$$P_\alpha = \psi - \rho_\parallel g, \tag{5a}$$

$$P_\theta = \rho_\parallel g(q+I/g) - (\psi-\psi_0)Q(\psi_0,\theta) + [Q_I(\psi,\theta)-Q_I(\psi_0,\theta)] - \rho_\parallel g[Q(\psi,\theta)-Q(\psi_0,\theta)], \tag{5b}$$

$$\alpha_c = \alpha - \lambda(\psi,\theta,\psi_0) = -\zeta + q(\psi_0)\theta - \left[\tilde{\zeta}(\psi,\theta)-\tilde{\zeta}(\psi_0,\theta)\right], \tag{5c}$$

with ψ_0 chosen as the initial value of the poloidal magnetic flux or the toroidal angular momentum. $q_I(\psi)$, $\tilde{\zeta}(\psi,\theta)$, $\tilde{q}(\psi,\theta)$, $\tilde{q}_I(\psi,\theta)$, $Q(\psi,\theta)$, and $Q_I(\psi,\theta)$ are equilibrium functions, and they are defined as

$$q_I(\psi) \equiv \int_0^\psi q(\psi)d\psi, \tag{6a}$$

$$\tilde{\zeta}(\psi,\theta) \equiv \int_0^\psi \delta(\psi,\theta)d\psi, \tag{6b}$$

$$\tilde{q}(\psi,\theta) \equiv \partial_\theta\tilde{\zeta}(\psi,\theta), \tag{6c}$$

$$\tilde{q}_I(\psi,\theta) \equiv \int_0^\psi \tilde{q}(\psi)d\psi, \tag{6d}$$

$$Q(\psi,\theta) \equiv q(\psi) + \tilde{q}(\psi,\theta), \tag{6e}$$

$$Q_I(\psi,\theta) \equiv q_I(\psi) + \tilde{q}_I(\psi,\theta). \tag{6f}$$

And $\lambda(\psi,\theta,\psi_0)$ is defined as

$$\lambda(\psi,\theta,\psi_0) \equiv \int_{\psi_0}^{\psi}\Delta(\psi,\theta)d\psi = [q(\psi)-q(\psi_0)]\theta + \left[\tilde{\zeta}(\psi,\theta)-\tilde{\zeta}(\psi_0,\theta)\right]. \tag{7}$$

The components of the matrix of the transformation from the noncanonical variables $(\psi,\theta,\zeta,\rho_\parallel g)$ to the canonical variables $(P_\theta,\theta,P_\alpha,\alpha_c)$ are written as

$$\frac{\partial\psi}{\partial P_\theta} = \frac{\partial(\rho_\parallel g)}{\partial P_\theta} = \frac{g}{D}, \tag{8a}$$

$$\frac{\partial\psi}{\partial\theta} = \frac{\partial(\rho_\parallel g)}{\partial\theta} = \frac{g}{D}\Big\{\rho_\parallel g[\partial_\theta\tilde{q}(\psi,\theta)-\partial_\theta\tilde{q}(\psi_0,\theta)] - [\partial_\theta\tilde{q}_I(\psi,\theta) - \partial_\theta\tilde{q}_I(\psi_0,\theta)] + (\psi-\psi_0)\partial_\theta\tilde{q}(\psi_0,\theta)\Big\}, \tag{8b}$$

$$\frac{\partial\psi}{\partial P_\alpha} = \frac{\partial(\rho_\parallel g)}{\partial P_\alpha} + 1 = \frac{g}{D}\left\{q - [Q(\psi,\theta)-Q(\psi_0,\theta)] + \frac{I}{g}\right\}, \tag{8c}$$

$$\frac{\partial\theta}{\partial\theta} = -\frac{\partial\zeta}{\partial\alpha_c} = 1, \tag{8d}$$

$$\frac{\partial\zeta}{\partial P_\theta} = -\frac{\partial\psi}{\partial P_\theta}\delta, \tag{8e}$$

$$\frac{\partial\zeta}{\partial\theta} = -\frac{\partial\psi}{\partial\theta}\delta + q - [Q(\psi,\theta)-Q(\psi_0,\theta)], \tag{8f}$$

$$\frac{\partial\zeta}{\partial P_\alpha} = -\frac{\partial\psi}{\partial P_\alpha}\delta, \tag{8g}$$

where

$$D = qg + I + \rho_\parallel g(I' - g'I/g - g\partial_\theta\delta). \tag{9}$$

The components of the transformation matrix not listed are zero. Note that the canonical theory for the unperturbed guiding-center orbit is slightly different from Ref. 17. The initial value ψ_0 is used here in stead of the canonical toroidal angular momentum P_α, for the convenience of the further development of the canonical theory of the gyrocenter motion.

The unperturbed fundamental one-form and the Hamiltonian are written in terms of the canonical extended phase-space coordinates as

$$\Gamma_0 = \Gamma_{0i}dZ^i - h_0 d\tau = Md\xi + P_\alpha d\alpha_c + P_\theta d\theta - Udt - h_0 d\tau, \tag{10a}$$

$$h_0 = H_0(M,P_\theta,P_\alpha,\theta) - U \tag{10b}$$

$$= \frac{1}{2}\rho_\parallel^2 B^2 + MB + \phi - U. \tag{10c}$$

Here, H_0 is the unperturbed guiding-center Hamiltonian. τ is the independent parameter. The guiding-center motion is written in terms of the Hamiltonian and the Poisson bracket as

$$dZ^i/d\tau = \left\{Z^i, h_0\right\}_{gc} = J^{ij}\partial_j h_0, \tag{11}$$

where the canonical Poisson bracket and the canonical Poisson matrix $\boldsymbol{J}$ are defined as

$$\{f,g\}_{gc} = \frac{\partial f}{\partial\theta}\frac{\partial g}{\partial P_\theta} - \frac{\partial f}{\partial P_\theta}\frac{\partial g}{\partial\theta} + \frac{\partial f}{\partial\alpha_c}\frac{\partial g}{\partial P_\alpha} - \frac{\partial f}{\partial P_\alpha}\frac{\partial g}{\partial\alpha_c} + \frac{\partial f}{\partial\xi}\frac{\partial g}{\partial\mu} - \frac{\partial f}{\partial\mu}\frac{\partial g}{\partial\xi} + \frac{\partial f}{\partial t}\frac{\partial g}{\partial(-U)} - \frac{\partial f}{\partial(-U)}\frac{\partial g}{\partial t}, \quad (12a)$$

$$\boldsymbol{J} = \begin{pmatrix} 0 & \boldsymbol{I} \\ -\boldsymbol{I} & 0 \end{pmatrix}, \quad (12b)$$

where $\boldsymbol{I}$ is the four-order unit matrix and 0 is the four-order matrix with all elements zero. Note that $\boldsymbol{J} = -\boldsymbol{J}^{-1}$. The guiding-center Hamilton canonical equations can be explicitly given as

$$\frac{d\xi}{d\tau} = \frac{\partial h_0}{\partial M} = \frac{\partial H_0}{\partial M}, \quad (13a)$$

$$\frac{dM}{d\tau} = -\frac{\partial h_0}{\partial\xi} = -\frac{\partial H_0}{\partial\xi} = 0, \quad (13b)$$

$$\frac{d\alpha_c}{d\tau} = \frac{\partial h_0}{\partial P_\alpha} = \frac{\partial H_0}{\partial P_\alpha}, \quad (13c)$$

$$\frac{dP_\alpha}{d\tau} = -\frac{\partial h_0}{\partial\alpha_c} = -\frac{\partial H_0}{\partial\alpha_c} = 0, \quad (13d)$$

$$\frac{d\theta}{d\tau} = \frac{\partial h_0}{\partial P_\theta} = \frac{\partial H_0}{\partial P_\theta}, \quad (13e)$$

$$\frac{dP_\theta}{d\tau} = -\frac{\partial h_0}{\partial\theta} = -\frac{\partial H_0}{\partial\theta}, \quad (13f)$$

$$\frac{dt}{d\tau} = \frac{\partial h_0}{\partial(-U)} = 1, \quad (13g)$$

$$\frac{d(-U)}{d\tau} = -\frac{\partial h_0}{\partial t} = -\frac{\partial H_0}{\partial t} = 0. \quad (13h)$$

From Eq. (13g), $\tau = t$ can be set for convenience. Eq. (13b) denotes the conservation of the guiding-center magnetic moment M, which means that the guiding-center motion and the gyromotion are decoupled.

III. LIE-TRANSFORM PERTURBATION THEORY

When a perturbation of electromagnetic fields is introduced, the guiding-center magnetic moment M is not conservative any more. The phase-space Lagrangian Lie-transform perturbation method[13–15] is applied for seeking a new conservative magnetic moment by the gyrocenter phase-space transformation. The gyrocenter phase-space transformation $\mathcal{T}_{gy}$ and its inverse $\mathcal{T}_{gy}^{-1}$ are defined as

$$\bar{Z}^i(Z;\epsilon_\delta) \equiv \mathcal{T}_{gy}Z^i, \quad (14a)$$

$$Z^i(\bar{Z};\epsilon_\delta) \equiv \mathcal{T}_{gy}^{-1}\bar{Z}^i. \quad (14b)$$

Here, ϵ_δ is a small parameter related to the perturbation of electromagnetic fields. A push-forward operator T_{gy}^{-1} and a pull-back operator T_{gy} induced by the transformation and its inverse are written as

$$\overline{\mathcal{F}} = T_{gy}^{-1}\mathcal{F}, \quad (15a)$$

$$\mathcal{F} = T_{gy}\overline{\mathcal{F}}. \quad (15b)$$

The scalar function $\mathcal{F}$ can be chosen as the Hamiltonian and the distribution function.

The transformed fundamental one-form can be written as

$$\bar{\Gamma} = T_{gy}^{-1}\Gamma + dS, \quad (16)$$

where S is a gauge function used to simplify the equations of motion, that is, to seeking a new conservative magnetic moment.

IV. GYROCENTER EQUATIONS OF MOTION IN TERMS OF THE EXACT CANONICAL VARIABLES

In this section, the gyrocenter equations of motion in terms of the exact canonical variables are presented. A general electromagnetic perturbation $(\delta\varphi, \delta\boldsymbol{A})$ is introduced as

$$\delta\boldsymbol{A}(\boldsymbol{r},t) = \delta A_\psi(\boldsymbol{r},t)\nabla\psi + \delta A_\theta(\boldsymbol{r},t)\nabla\theta + \delta A_\zeta(\boldsymbol{r},t)\nabla\zeta, \quad (17a)$$

$$\delta\phi = \delta\phi(\boldsymbol{r},t). \quad (17b)$$

Then, the fundamental one-form is written as

$$\Gamma \equiv \Gamma_0 + \Gamma_1, \quad (18)$$

where the perturbed part of the one-form is written as[8]

$$\begin{aligned}\Gamma_1 &= \delta\boldsymbol{A}(\boldsymbol{X}+\boldsymbol{\rho},t)\cdot d(\boldsymbol{X}+\boldsymbol{\rho}) - \delta\phi(\boldsymbol{X}+\boldsymbol{\rho},t)dt \\ &= \delta\boldsymbol{A}_\psi d\psi + \delta\boldsymbol{A}_\theta d\theta + \delta\boldsymbol{A}_\zeta d\zeta + (\delta\boldsymbol{A}\cdot\partial_M\boldsymbol{\rho})dM \\ &\quad + (d\boldsymbol{A}\cdot\partial_\xi\boldsymbol{\rho})d\xi - \delta\phi dt, \end{aligned} \quad (19)$$

where $\boldsymbol{X}$ is the guiding-center coordinates.

The gyrocenter phase-space transformation Eq. (14a) can be expanded in powers of ϵ_δ up to $O(\epsilon_\delta)$, written as

$$\bar{Z}^i = Z^i + \epsilon_\delta G_1^i, \quad (20)$$

where G_1 is the first-order generating vector field function.

The transformation of the extended phase-space Lagrangian Eq. (16) can be expanded in powers of ϵ_δ up to $O(\epsilon_\delta)$, written as

$$\bar{\Gamma}_{0i} = \Gamma_{0i}, \quad (21a)$$

$$\bar{\Gamma}_{1i} = \Gamma_{1i} - G_1^j\omega_{0ji} + \partial_i S_1, \quad (21b)$$

where $\omega_{0ji} = \partial_j\Gamma_{0i} - \partial_i\Gamma_{0j}, \omega_{1ji} = \partial_j\Gamma_{1i} - \partial_i\Gamma_{1j}$, and the conditions $S_0 = 0, \bar{\Gamma}_{1i} = 0$ are used. Here, the term $d(-G_1^i\Gamma_{0i})$ is absorbed in the gauge term dS_1. From Eq. (21b), the first-order generating vector field function $\boldsymbol{G}_1$ is written as

$$G_1^i = (\partial_j S_1 + \Gamma_{1j})J^{ji}. \quad (22)$$

The components of $\boldsymbol{G}_1$ can be explicitly written as

$$G_1^\theta = -\{[\delta A_\psi - \delta A_\zeta\delta(\psi,\theta)]\partial_{P_\theta}\psi + \partial_{P_\theta}S_1\}, \quad (23a)$$

$$G_1^{\alpha_c} = -\left\{\left[\delta A_\psi - \delta A_\zeta \delta(\psi,\theta)\right]\partial_{P_\alpha}\psi + \partial_{P_\alpha} S_1\right\}, \tag{23b}$$

$$G_1^{P_\theta} = [\delta A_\psi - \delta A_\zeta \delta(\psi,\theta)]\partial_\theta \psi + \delta A_\theta + \delta A_\zeta[q(\psi_0) + \tilde{q}(\psi_0) - \tilde{q}(\psi)] + \partial_{P_\theta} S_1, \tag{23c}$$

$$G_1^{P_\alpha} = -\delta A_\zeta + \partial_{\alpha_c} S_1, \tag{23d}$$

$$G_1^{\xi} = -\{\delta \mathbf{A}\cdot \partial_M \boldsymbol{\rho} + \partial_M S_1\}, \tag{23e}$$

$$G_1^{M} = \delta \mathbf{A}\cdot \partial_\xi \boldsymbol{\rho} + \partial_\xi S_1, \tag{23f}$$

$$G_1^{-U} = -\delta\phi + \partial_t S_1, \tag{23g}$$

$$G_1^{t} = 0. \tag{23h}$$

From Eq. (15a), the transformation of the extended phase-space Hamiltonian can be expanded in powers of ϵ_δ up to $O(\epsilon_\delta)$, written as

$$\bar{h}_0 = h_0, \tag{24a}$$

$$\bar{H}_1 = \bar{h}_1 = -G_1^i \partial_i h_0 \tag{24b}$$

$$= -(\partial_j S_1 + \Gamma_{1j})\dot{Z}_0^j \tag{24c}$$

$$\equiv K_1 - \partial_j S_1 \dot{Z}_0^j, \tag{24d}$$

where $\dot{Z}_0^j$ are the components of zero-order extended phase-space velocity.

To decouple the gyrocenter motion from the gyromotion, the gyrocenter Hamiltonian $\bar{h}_n$ is chosen to satisfy the condition $\bar{h}_n = \langle \bar{h}_n \rangle$. Here $\langle\cdots\rangle$ denotes gyroaveraging. Thus, the first-order gauge function S_1 can be chosen as

$$\frac{d_0 S_1}{d\tau} = \left(\partial_t + \dot{\theta}_0 \partial_\theta + (\dot{P}_\theta)_0 \partial_{P_\theta} + (\dot{\alpha}_c)_0 \partial_{\alpha_c} + \dot{\xi}_0 \partial_\xi\right) S_1 = \tilde{K}_1, \tag{25a}$$

$$\tilde{K}_1 \equiv K_1 - \langle K_1 \rangle, \tag{25b}$$

$$K_1 = \delta\phi - \left(\dot{\psi}_0 \delta A_\psi + \dot{\theta}_0 \delta A_\theta + \dot{\zeta}_0 \delta \mathbf{A}_\zeta + \dot{\xi}_0 \delta A\cdot \partial_\xi \boldsymbol{\rho}\right), \tag{25c}$$

$$\left(\dot{\psi}_0, \dot{\theta}_0, \dot{\zeta}_0\right) = \frac{\partial(\psi,\theta,\zeta)}{\partial Z^i}\dot{Z}_0^i. \tag{25d}$$

Here, the subscript 0 denotes the unperturbed motion.

Then the first-order gyrocenter Hamiltonian $\bar{H}_1$ is written as

$$\bar{H}_1 = \langle K_1 \rangle = \langle \delta\phi \rangle - \left(\dot{\psi}_0 \langle \delta A_\psi \rangle + \dot{\theta}_0 \langle \delta A_\theta \rangle + \dot{\zeta}_0 \langle \delta A_\zeta \rangle + \dot{\xi}_0 \langle \delta \mathbf{A}\cdot \partial_\xi \boldsymbol{\rho} \rangle\right), \tag{26}$$

which is independent of the gyroangle. The detailed treatment for the first-order gyrocenter gauge function S_1 and Hamiltonian $\bar{H}_1$ will be given in Appendix A. The gyrocenter equations of motion are written in terms of the Hamiltonian and the Poisson bracket as

$$d\bar{Z}^i/d\tau = \left\{\bar{Z}^i, \bar{h}\right\}_{gy} = J^{ij}\partial_j \bar{h}, \tag{27}$$

where the canonical Poisson bracket is defined as

$$\{f,g\}_{gy} = \frac{\partial f}{\partial\bar{\theta}}\frac{\partial g}{\partial \bar{P}_\theta} - \frac{\partial f}{\partial \bar{P}_\theta}\frac{\partial g}{\partial\bar{\theta}} + \frac{\partial f}{\partial\bar{\alpha}_c}\frac{\partial g}{\partial \bar{P}_\alpha} - \frac{\partial f}{\partial \bar{P}_\alpha}\frac{\partial g}{\partial\bar{\alpha}_c} + \frac{\partial f}{\partial\bar{\xi}}\frac{\partial g}{\partial\bar{\mu}} - \frac{\partial f}{\partial\bar{\mu}}\frac{\partial g}{\partial\bar{\xi}} + \frac{\partial f}{\partial t}\frac{\partial g}{\partial(-\bar{U})} - \frac{\partial f}{\partial(-\bar{U})}\frac{\partial g}{\partial t}. \tag{28}$$

The gyrocenter Hamilton canonical equations can be explicitly given as

$$\frac{d\bar{\xi}}{d\tau} = \frac{\partial\bar{h}}{\partial\bar{M}} = \frac{\partial(\bar{H}_0 + \bar{H}_1)}{\partial\bar{M}}, \tag{29a}$$

$$\frac{d\bar{M}}{d\tau} = -\frac{\partial\bar{h}}{\partial\bar{\xi}} = 0, \tag{29b}$$

$$\frac{d\bar{\alpha}_c}{d\tau} = \frac{\partial\bar{h}}{\partial\bar{P}_\alpha} = \frac{\partial(\bar{H}_0 + \bar{H}_1)}{\partial\bar{P}_\alpha}, \tag{29c}$$

$$\frac{d\bar{P}_\alpha}{d\tau} = -\frac{\partial\bar{h}}{\partial\bar{\alpha}_c} = -\frac{\partial\bar{H}_1}{\partial\bar{\alpha}_c}, \tag{29d}$$

$$\frac{d\bar{\theta}}{d\tau} = \frac{\partial\bar{h}}{\partial\bar{P}_\theta} = \frac{\partial(\bar{H}_0 + \bar{H}_1)}{\partial\bar{P}_\theta}, \tag{29e}$$

$$\frac{d\bar{P}_\theta}{d\tau} = -\frac{\partial\bar{h}}{\partial\bar{\theta}} = -\frac{\partial(\bar{H}_0 + \bar{H}_1)}{\partial\bar{\theta}}, \tag{29f}$$

$$\frac{dt}{d\tau} = \frac{\partial\bar{h}}{\partial(-\bar{U})} = 1, \tag{29g}$$

$$\frac{d(-\bar{U})}{d\tau} = -\frac{\partial\bar{h}}{\partial t} = -\frac{\partial\bar{H}_1}{\partial t}. \tag{29h}$$

Note that Eq. (29b) means that the gyrocenter motion is decoupled from the gyromotion, and that Eqs. (29d) and (29h) have no zero-order term in the right-hand-side. In Ref. 8, it is shown that the canonical gyrokinetic equations is equivalent to the noncannonical one. In this paper we show that in the linear drift approximation the canonical gyrocenter equations of motion are equivalent to the usual guiding-center equations of motion. This will be given in Appendix B.

V. LINEAR GYROKINETIC VLASOV EQUATION IN TERMS OF THE CANONICAL VARIABLES

The linear gyrokinetic Vlasov equation in terms of the canonical variables is presented in this section. The six independent gyrocenter phase-space coordinates $(\bar{M}, \bar{P}_\alpha, \bar{P}_\theta, \bar{\xi}, \bar{\alpha}_c, \bar{\theta})$ are used to derive the gyrokinetic Vlasov equation, which can be written as

$$\frac{\partial\bar{F}}{\partial t} + \frac{d\bar{\theta}}{dt}\frac{\partial\bar{F}}{\partial\bar{\theta}} + \frac{d\bar{\alpha}_c}{dt}\frac{\partial\bar{F}}{\partial\bar{\alpha}_c} + \frac{d\bar{P}_\alpha}{dt}\frac{\partial\bar{F}}{\partial\bar{P}_\alpha} + \frac{d\bar{P}_\theta}{dt}\frac{\partial\bar{F}}{\partial\bar{P}_\theta} = 0, \tag{30}$$

where we have made use of the fact that $d\bar{M}/dt = 0$ and the distribution function $\bar{F}$ is independent of the gyrophase angle $\bar{\xi}$. Expanding in powers of ϵ_δ up to $O(\epsilon_\delta)$, the distribution function can be written as

$$\bar{F} = \bar{F}_0 + \bar{F}_1, \tag{31}$$

where $\bar{F}_0 = \bar{F}_0(\bar{M}, \bar{P}_\alpha, \bar{U})$ is the zero-order distribution function.[8] The linear gyrokinetic Vlasov equation can be written as

$$\left(\frac{d}{dt}\right)_0 \bar{F}_1 = -\left(\frac{d}{dt}\right)_1 \bar{F}_0, \tag{32a}$$

$$\left(\frac{d}{dt}\right)_0 = \frac{\partial}{\partial t} + \left(\frac{d\bar{\theta}}{dt}\right)_0 \frac{\partial}{\partial \bar{\theta}} + \left(\frac{d\bar{\alpha}_c}{dt}\right)_0 \frac{\partial}{\partial \bar{\alpha}_c} + \left(\frac{d\bar{P}_\theta}{dt}\right)_0 \frac{\partial}{\partial \bar{P}_\theta}, \tag{32b}$$

$$\left(\frac{d}{dt}\right)_1 = \left(\frac{d\bar{P}_\theta}{dt}\right)_1 \frac{\partial}{\partial \bar{P}_\theta} + \left(\frac{d\bar{P}_\alpha}{dt}\right)_1 \frac{\partial}{\partial \bar{P}_\alpha} + \left(\frac{d\bar{\theta}}{dt}\right)_1 \frac{\partial}{\partial \bar{\theta}} + \left(\frac{d\bar{\alpha}_c}{dt}\right)_1 \frac{\partial}{\partial \bar{\alpha}_c}. \tag{32c}$$

Due to the fact that the zero-order distribution function $\bar{F}_0(\bar{M}, \bar{P}_\alpha, \bar{U})$ is independent of $\bar{\theta}$ and $\bar{\alpha}_c$, and the energy $\bar{U}$ can be understood as a function of the six independent phase-space coordinates, that is, $\bar{U} = \bar{U}(\bar{M}, \bar{P}_\alpha, \bar{P}_\theta, \bar{\xi}, \bar{\alpha}_c, \bar{\theta}; t)$, we have

$$\left(\frac{d}{dt}\right)_1 \bar{F}_0 = \left[\left(\frac{d\bar{P}_\alpha}{dt}\right)_1 \frac{\partial}{\partial \bar{P}_\alpha} + \left(\frac{d\bar{U}}{dt}\right)_1 \frac{\partial}{\partial \bar{U}}\right] \bar{F}_0. \tag{33}$$

The guiding-center distribution function F can be obtained by the pull-back operator,

$$F = \bar{F}_0 + G_1^M \frac{\partial \bar{F}_0}{\partial M} + G_1^{-U} \frac{\partial \bar{F}_0}{\partial(-U)} + G_1^{P_\alpha} \frac{\partial \bar{F}_0}{\partial P_\alpha} + \bar{F}_1. \tag{34}$$

VI. THE NUMERICAL CODE GYCAVA FOR ADVANCING THE GYROCENTER MOTION

The code GYCAVA for advancing the gyrocenter motion is developed, which is based on the code GCM (Ref. 19) for computing the unperturbed guiding-center motion. GYCAVA is benchmarked against GCM, i.e., the guiding-center motion without perturbation computed by GYCAVA agrees with the result obtained by GCM. The code GYCAVA can compute the guiding-center motion in arbitrary perturbed electromagnetic fields, not in a special form of perturbation as Ref. 22.

A. Algorithm of GYCAVA

The magnetohydrodynamics equilibrium data evaluated on the grid are read by the code for determining the tokamak equilibrium configuration. The values of these functions at the particle locations are evaluated by using the cubic spline interpolation from the grid.

The stepping procedure is carried out by using the variable time step fourth-order Runge–Kutta algorithm. The gyrocenter canonical equations of motion are used to advance the particle in gyrocenter variables. A subroutine GYROTRAN is used to transform from gyrocenter variables to guiding-center variables or backward.

The initial guiding-center canonical variables $Z^i = (P_\theta, \theta, P_\alpha, \alpha_c)$ and the noncanonical variables $X^i = (\psi(\mathbf{Z}), \zeta(\mathbf{Z}), \rho_\parallel g(\mathbf{Z}))$ are transformed to the initial gyrocenter canonical variables $\bar{Z}^i = (\bar{P}_\theta, \bar{\theta}, \bar{P}_\alpha, \bar{\alpha}_c)$ and the noncanonical variables $\bar{X}^i = (\psi(\bar{\mathbf{Z}}), \zeta(\bar{\mathbf{Z}}), \rho_\parallel g(\bar{\mathbf{Z}}))$ by

$$\bar{Z}^i = Z^i + \langle G_1^i \rangle, \tag{35a}$$

$$\bar{X}^i = X^i + \frac{\partial X^i}{\partial Z^j} \langle G_{1j} \rangle. \tag{35b}$$

After a particle is advanced every a time step the gyrocenter variables is transformed to the guiding-center variables by

$$Z^i = \bar{Z}^i - \langle G_1^i \rangle, \tag{36a}$$

$$X^i = \bar{X}^i - \frac{\partial X^i}{\partial Z^j} \langle G_{1j} \rangle. \tag{36b}$$

The gyrocenter equations of motion used in the code GYCAVA are

$$\frac{d\bar{\theta}}{dt} = \frac{\partial \bar{h}}{\partial \bar{P}_\theta}, \tag{37a}$$

$$\frac{d\bar{P}_\theta}{dt} = -\frac{\partial \bar{h}}{\partial \bar{\theta}}, \tag{37b}$$

$$\frac{d\bar{\alpha}_c}{dt} = \frac{\partial \bar{h}}{\partial \bar{P}_\alpha}, \tag{37c}$$

$$\frac{d\bar{P}_\alpha}{dt} = -\frac{\partial \bar{h}}{\partial \bar{\alpha}_c}, \tag{37d}$$

$$\frac{d(-\bar{U})}{dt} = -\frac{\partial \bar{h}}{\partial t}, \tag{37e}$$

$$\frac{d\bar{\psi}}{dt} = \frac{\partial \psi}{\partial \theta}\frac{d\bar{\theta}}{dt} + \frac{\partial \psi}{\partial P_\theta}\frac{d\bar{P}_\theta}{dt} + \frac{\partial \psi}{\partial \alpha_c}\frac{d\bar{\alpha}_c}{dt} + \frac{\partial \psi}{\partial P_\alpha}\frac{d\bar{P}_\alpha}{dt}, \tag{37f}$$

$$\bar{h} = \bar{H} - \bar{U} = \bar{H}_0 + \bar{H}_1 - \bar{U}, \tag{37g}$$

$$\bar{H}_0 = H_0, \tag{37h}$$

$$\bar{H}_1 = \langle \delta\phi \rangle - \left(\dot{\psi}_0 \langle \delta A_\psi \rangle + \dot{\theta}_0 \langle \delta A_\theta \rangle + \dot{\zeta}_0 \langle \delta A_\zeta \rangle + \dot{\xi}_0 \langle \delta \mathbf{A} \cdot \partial_\xi \boldsymbol{\rho} \rangle \right), \tag{37i}$$

where the variables $\overline{\rho_\parallel g}, \bar{\zeta}$ are obtained by

$$\overline{\rho_\parallel g} = \bar{\psi} - \bar{P}_\alpha, \tag{38a}$$

$$\bar{\zeta} = -\bar{\alpha}_c + q(\bar{\psi})\bar{\theta} - \lambda(\bar{\psi}, \bar{\theta}, \psi_0). \tag{38b}$$

The detailed treatment for $S_1, \bar{H}_1$ is given in Appendix A.

B. Gyrocenter motion in a tokamak with a static magnetic island

As a numerical example of GYCAVA, we discuss in this subsection the gyrocenter motion in a tokamak with a static magnetic island.

The equilibrium configuration used here is the Shafranov equilibrium, in large-aspect-ratio, with the magnetic flux coordinates constructed. The main parameters of the tokamak equilibrium are as follows. The major radius is 1.67 m, and the minor radius is 0.62 m. The magnetic field is 2.4 T. The values of the safety factor at the magnetic axis and the boundary are 1.2 and 3.7, respectively.

To model a static magnetic island, we choose the perturbation of the poloidal magnetic flux as

$$\psi_1 = \psi_{1(m,n)}(\psi)\cos(m\theta - n\zeta - \omega t), \tag{39a}$$

$$\psi_{1(m,n)}(\psi) = \epsilon_\psi(\psi - \psi_{\text{axis}})(\psi_b - \psi). \tag{39b}$$

Here, ϵ_ψ is a small constant. $(\psi_{\text{axis}} = 0, \psi_b = 0098\text{Wb})$ are the values of the equilibrium poloidal magnetic flux at the magnetic axis and the boundary of the tokamak, respectively. The parameters of the perturbation are chosen as $m = 2, n = 1$, $\omega = 0, \psi_{1(m,n)}(\psi_s)/\psi_s = 10^{-5}$, with ψ_s the value of the equilibrium poloidal magnetic flux on the rational surface where $q = 2$. The scale length of the imposed perturbation of the poloidal magnetic flux has the same order as the minor radius of the tokamak; note that $|\nabla \ln \psi_1|^{-1} \sim r$. Thus the finite-Larmor-radius effect is not important in this numerical example.

The Poincare section plots of the magnetic flux surfaces, and the particle orbits are set at $\zeta = 0$. The initial particle energy is set as $E_0 = 1$ keV. Particles are launched from two different locations $(\psi = 5.734 \times 10^{-2}\text{Wb}, \theta = 0, \zeta = 0)$ and $(\psi = 5.756 \times 10^{-2}\text{Wb}, \theta = 0, \zeta = 0)$ for the electrons, and from two different locations $(\psi = 5.837 \times 10^{-2}\text{Wb}, \theta = 0, \zeta = 0)$ and $(\psi = 5.864 \times 10^{-2}\text{Wb}, \theta = 0, \zeta = 0)$ for the ions. The magnetic flux surfaces where the initial positions locate are plotted for comparing with particle orbits.

The Poincare section plots for passing electron orbits with and without perturbation are shown in Figs. 1(a) and 1(b), respectively. The electron orbits slightly deviate from the magnetic flux surfaces. The passing ion orbits with and without perturbation are shown in Figs. 1(c) and 1(d), respectively. The ion orbits significantly deviate from the magnetic flux surfaces. It is clearly seen that a static magnetic island induces the island structure of the passing particle orbits near the rational surface.

The trapped electron orbits with and without perturbation are shown in Figs. 2(a) and 2(b), respectively. The trapped ion orbits with and without perturbation are shown in Figs. 2(c) and 2(d), respectively. It is seen from Fig. 2 that a static magnetic island has little effect on the Poincare section plots of the trapped particles.

The deviations of Hamiltonian and the longitudinal invariant are shown in Fig. 3 for the passing particles and in Fig. 4 for the trapped particles, which are defined as

$$\Delta\bar{H}/[\bar{H}]_0 \equiv \frac{\bar{H}(\bar{Z}) - [\bar{H}(\bar{Z})]_{t=0}}{[\bar{H}(\bar{Z})]_{t=0}}, \tag{40a}$$

$$\Delta\mathcal{J}/\mathcal{J}_0 \equiv \frac{\mathcal{J} - \mathcal{J}_0}{\mathcal{J}_0}. \tag{40b}$$

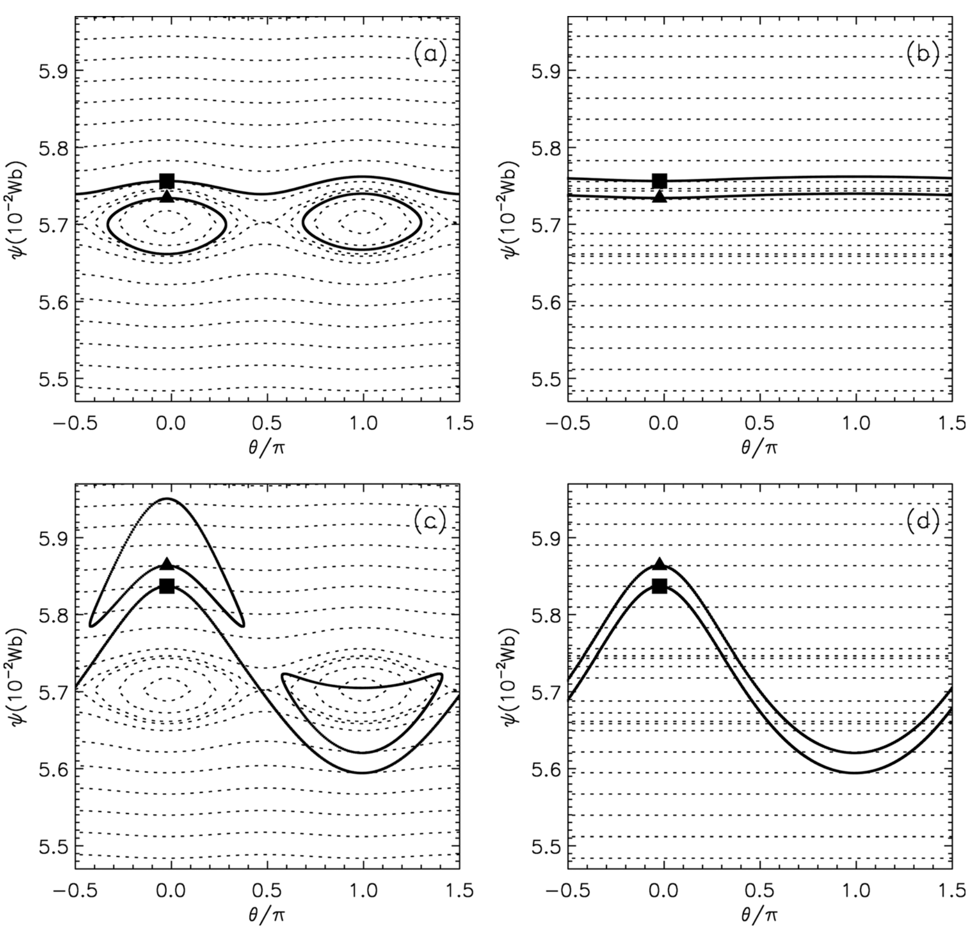

FIG. 1. Poincare section plots of the magnetic flux surfaces (dotted line) and the passing particle orbits (dotted symbol) with the initial energy $E_0 = 1$ keV and the pitch $v_\parallel/v = 1.0$. Two different launch points are labeled by the square and the triangle symbols, respectively. (a), (b) The passing electrons with and without perturbation, respectively. (c), (d) The passing ions with and without perturbation, respectively.

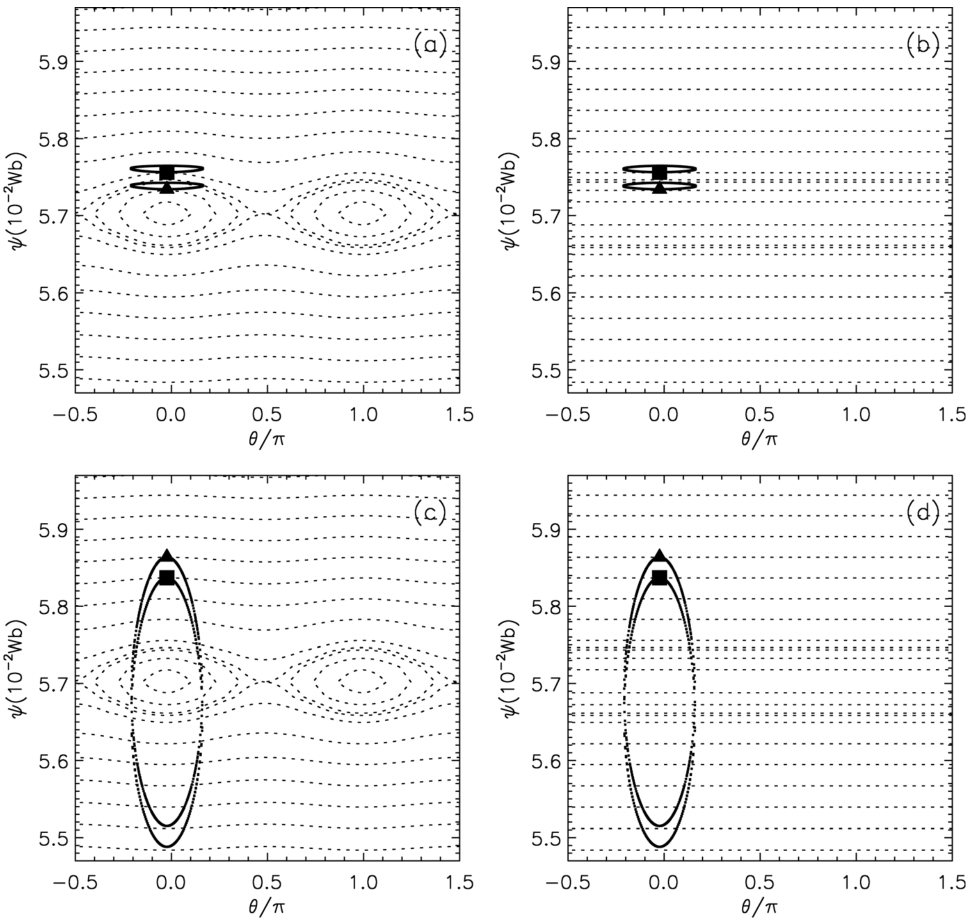

FIG. 2. Poincare section plots of the magnetic flux surfaces (dotted line) and the trapped particle orbits (dotted symbol) with the initial energy $E_0 = 1$ keV and the pitch $v_\parallel/v = 0.2$. Two different launch points are labeled by the square and the triangle symbols, respectively. (a), (b) The trapped electrons with and without perturbation, respectively. (c), (d) The trapped ions with and without perturbation, respectively.

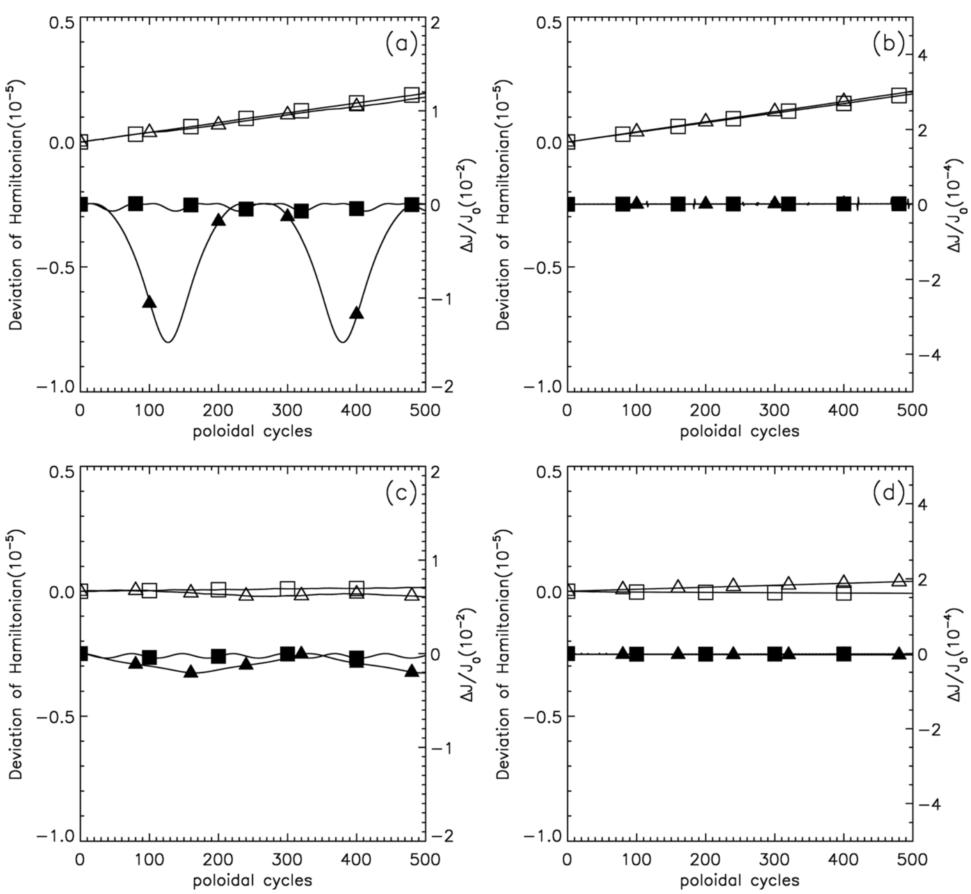

FIG. 3. The deviations of Hamiltonian $\Delta\bar{H}/[\bar{H}]_0$ (labeled by "Δ" and "□") and the longitudinal invariant $\Delta\mathcal{J}/\mathcal{J}_0$ (labeled by "▲" and "■") for passing particles launched from two different positions (same as Fig. 1) with the initial energy $E_0 = 1$ keV and the pitch $v_\parallel/v = 1.0$. (a), (b) The passing electrons with and without perturbation, respectively. (c), (d) The passing ions with and without perturbation, respectively.

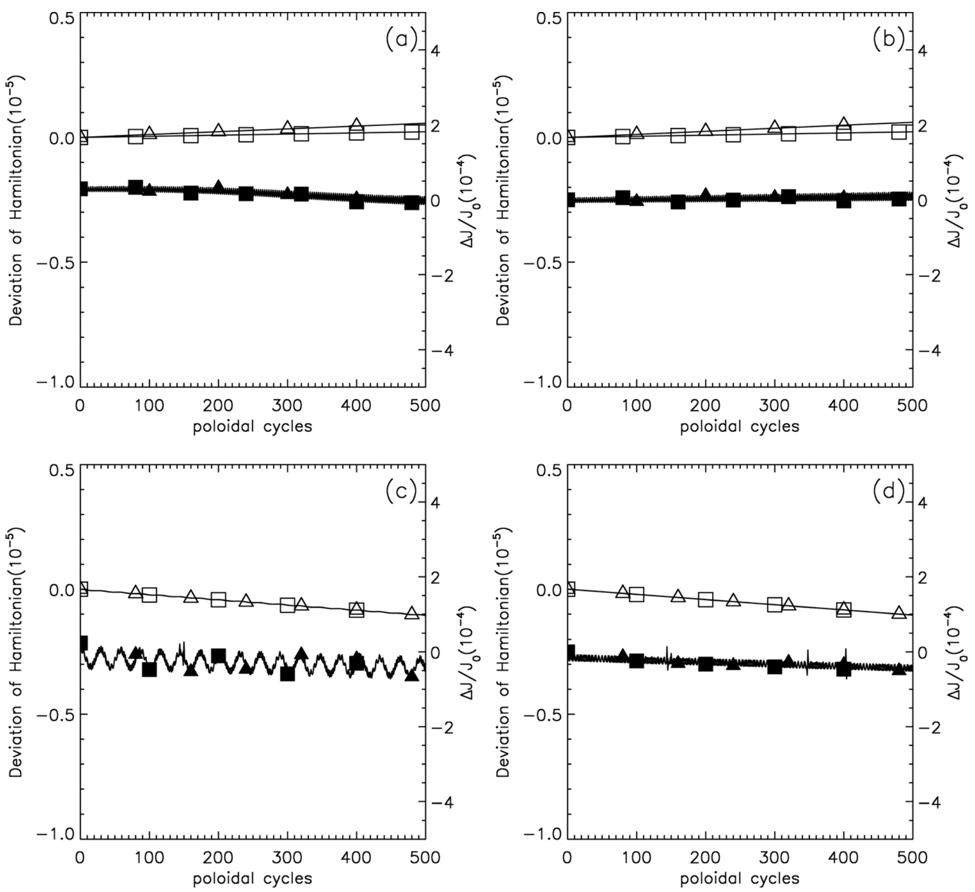

FIG. 4. The deviations of Hamiltonian $\Delta\bar{H}/[\bar{H}]_0$ (labeled by "Δ" and "□") and the longitudinal invariant $\Delta\mathcal{J}/\mathcal{J}_0$ (labeled by "▲" and "■") for trapped particles launched from two different positions (same as Fig. 2) with the initial energy $E_0 = 1$ keV and the pitch $v_{\parallel}/v = 0.2$. (a), (b) The trapped electrons with and without perturbation, respectively. (c), (d) Trapped ions with and without perturbation, respectively.

Here, $\bar{H}(\bar{Z})$ and J are the gyrocenter Hamiltonian function evaluated at the gyrocenter coordinates and the longitudinal invariant, respectively; $[\bar{H}(\bar{Z})]_{t=0}$ and J_0 are the initial values of $\bar{H}(\bar{Z})$ and J, respectively.

The longitudinal invariant is defined as[17]

$$\mathcal{J} = \oint P_\theta d\theta. \tag{41}$$

The definition of the longitudinal invariant here is different from Ref. 22 due to the different definitions of P_θ. The definition of the longitudinal invariant here is valid not only for the trapped particles but also for the passing particles, as has been pointed out in Ref. 17, while the conventional definition[26] is only valid for the trapped particles.

Note that the particle energy conserves for a static magnetic island. It is seen from Figs. 3 and 4 that the deviations of Hamiltonian have the order of 10^{-6}, which numerically demonstrates the conservation of the particle energy.

From Figs. 3(b), 3(d), 4(b), and 4(d), it is clearly seen that without perturbation, $\Delta\mathcal{J}/\mathcal{J}_0 \sim 10^{-4}$ for both passing and trapped particles in equilibrium fields, which numerically demonstrates the conservation of the longitudinal invariant without perturbation.[17] From Figs. 3(a), 3(c), 4(a), and 4(c), it is seen that with perturbation, $\Delta\mathcal{J}/\mathcal{J}_0 \sim 10^{-2}$ for passing particles, and $\Delta\mathcal{J}/\mathcal{J}_0 \sim 10^{-4}$ for trapped particles. Clearly, even with the perturbation, $\mathcal{J}$ is still a good invariant for the trapped particles, as was pointed out previously.[22,26,27] However, $\mathcal{J}$ is not a good invariant for the passing particles in perturbed fields; this is not hard to understand. For a trapped particle, it returns to nearly the same location after a bounce period; therefore, a trapped particle's motion is quasiperiodic in the perturbed field, and the longitudinal invariant hence conserved as an adiabatic invariant. For a passing particle, after a poloidal (transit) period, it does not return to the same location (in toroidal angle); therefore, its periodic motion only exists in equilibrium with toroidal symmetry (where the toroidal angle is an ignorable variable), while in a perturbed field with toroidal symmetry broken, its motion is not quasiperiodic, and $\mathcal{J}$ is hence not a good invariant anymore.

Following the above discussion, it is not hard to understand the results presented in Figs. 1 and 2. The magnetic island has little effects on the trapped particles due to the conservation of the longitudinal invariant. The magnetic island induces the island structure of the orbits of passing particles due to the nonconservation of $\mathcal{J}$ induced by the toroidal symmetry breaking. The passing electron almost follows the magnetic field line due to its small Larmor radius. The passing ion orbits (including the island structure) deviate significantly from the magnetic flux surfaces due to the large ion Larmor radius; this behavior of ion orbits in a magnetic island was noted previously.[28]

VII. SUMMARY

In conclusion, the linear gyrokinetic theory in the exact canonical variables is presented, which is based on the Lie-transform perturbation method. The equivalence of the gyrocenter and the guiding-center equations of motion is shown in the linear drift approximation.

A numerical code GYCAVA in terms of the canonical variables is developed for advancing the gyrocenter motion in a tokamak with arbitrary perturbation fields. The numerical results of GYCAVA for the gyrocenter orbits in a static magnetic island are discussed. It is found that a static magnetic island in a tokamak has little effect on the Poincare section plots of the trapped particle orbits due to the conservation of the longitudinal invariant; and it induces the island structure of passing particle orbits due to the fact that the longitudinal invariant for the passing particles is broken by the asymmetric perturbation.

The code GYCAVA for advancing the gyrocenter canonical variables can be used to simulate the motion of charged particles in a tokamak with arbitrary perturbation fields; it will be useful in the future numerical application of the canonical gyrokinetic theory in simulation of tokamak fusion plasmas.

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APPENDIX A: THE DETAILED TREATMENT FOR S_1, $\bar{H}_1$

The gyroradius and perpendicular velocity are written as

$$\boldsymbol{\rho} = \rho_\perp \hat{a}, \tag{A1a}$$

$$\mathbf{v} = v_\parallel \hat{b} + v_\perp \hat{c}. \tag{A1b}$$

Here, $\hat{a}, \hat{c}$ are the directions of the gyroradius and perpendicular velocity, respectively. $\hat{b}$ is the parallel unit vector. Rotating unit vectors $(\hat{a}, \hat{b}, \hat{c})$ are written in terms of fixed unit vectors $(\hat{1}, \hat{2}, \hat{b})$ as

$$\hat{a} = \cos\xi \hat{1} - \sin\xi \hat{2}, \tag{A2a}$$

$$\hat{c} = -\sin\xi \hat{1} - \cos\xi \hat{2}, \tag{A2b}$$

where fixed unit vectors $(\hat{1}, \hat{2}, \hat{b})$ are independent of the gyroangle. $\hat{1}$ and $\hat{2}$ are chosen as $\frac{\nabla\psi}{|\nabla\psi|}$ and $\frac{\hat{b}\times\nabla\psi}{|\nabla\psi|}$, respectively. Using Eqs. (A1) and (A2) and

$$\begin{aligned} \boldsymbol{\rho}\cdot\nabla &= \boldsymbol{\rho}\cdot\nabla X^j \partial_j = \rho_\perp \hat{1}\cdot\nabla X^j \cos\xi \partial_j - \rho_\perp \hat{2}\cdot\nabla X^j \sin\xi \partial_j \\ &\equiv \rho_c^j \cos\xi \partial_j + \rho_s^j \sin\xi \partial_j, \end{aligned} \tag{A3}$$

we have

$$e^{\boldsymbol{\rho}\cdot\nabla} = e^{\rho_s^j \sin\xi \partial_j} e^{\rho_c^j \cos\xi \partial_j} = \sum_{m,n} i^{-m} I_m(\rho_s^j \partial_j) I_{n-m}(\rho_c^j \partial_j) e^{in\xi}, \tag{A4a}$$

$$e^{\boldsymbol{\rho}\cdot\nabla} \cos\xi = \sum_{m,n} i^{-m} I_m(\rho_s^j \partial_j) I'_{n-m}(\rho_c^j \partial_j) e^{in\xi}, \tag{A4b}$$

$$e^{\boldsymbol{\rho}\cdot\nabla} \sin\xi = \sum_{m,n} i^{-m} I'_m(\rho_s^j \partial_j) I_{n-m}(\rho_c^j \partial_j) e^{in\xi}, \tag{A4c}$$

where $X^j = (\psi, \theta, \zeta)$.

From Eq. (25c), K_1 can be rewritten as

$$\begin{aligned} K_1 = e^{\boldsymbol{\rho}\cdot\nabla}\Big(\delta\phi_c - \big(\dot{\psi}_0 \delta A_{\psi c} + \dot{\theta}_0 \delta A_{\theta c} + \dot{\zeta}_0 \delta A_{\zeta c}\big)\Big) \\ + e^{\boldsymbol{\rho}\cdot\nabla} \cos\xi v_\perp \hat{1}\cdot\delta\boldsymbol{A}_c + e^{\boldsymbol{\rho}\cdot\nabla} \sin\xi v_\perp \hat{2}\cdot\delta\boldsymbol{A}_c. \end{aligned} \tag{A5}$$

Here, ∇ only acts on $(\delta\phi_c, \delta\boldsymbol{A}_c)$. The subscript c denotes that the corresponding function is evaluated at the guiding-center position.

By using $\frac{d_0 S_1}{dt} \simeq \dot{\xi}_0 \partial_\xi S_1$ and Eq. (25a), the first-order gauge function S_1 is written as

$$S_1 = \int \frac{\tilde{K}_1}{\Omega_c} d\xi, \tag{A6}$$

where Ω_c is the gyrofrequency.

Let

$$S_1 = \sum_{n=-\infty}^{n=\infty} S_{1n} e^{in\xi}, \tag{A7a}$$

$$K_1 = \sum_{n=-\infty}^{n=\infty} K_{1n} e^{in\xi}. \tag{A7b}$$

Combining Eqs. (A4)–(A7), we have

$$S_1 = \sum_{n\neq 0} \frac{1}{in\Omega_c} K_{1n}, \tag{A8}$$

$$\begin{aligned} K_{1n} = \sum_m \{ & i^{-m} I_m(\rho_s^j \partial_j) I_{n-m}(\rho_c^j \partial_j) \\ & \Big[\delta\phi_c - \big(\dot{\psi}_0 \delta A_{\psi c} + \dot{\theta}_0 \delta A_{\theta c} + \dot{\zeta}_0 \delta A_{\zeta c}\big)\Big] \\ & + i^{-m} I_m(\rho_s^j \partial_j) I'_{n-m}(\rho_c^j \partial_j) v_\perp \hat{1}\cdot\delta\boldsymbol{A}_c \\ & + i^{-m} I'_m(\rho_s^j \partial_j) I_{n-m}(\rho_c^j \partial_j) v_\perp \hat{2}\cdot\delta\boldsymbol{A}_c \}. \end{aligned} \tag{A9}$$

Note that $\bar{H}_1 = \langle K_1 \rangle = K_{10}$. In the code GYCAVA, the Bessel functions $I_n(x), I'_n(x)$ are expanded to $O(x^2)$; this means that the finite-Larmor-radius effects are kept to the second order.

APPENDIX B: THE EQUIVALENCE OF THE GYROCENTER AND THE USUAL GUIDING-CENTER EQUATIONS OF MOTION

In the linear drift approximation, we have four small parameters $\epsilon_B, \epsilon_\delta, \epsilon_\perp, \epsilon_\omega$,

$$\epsilon_B \equiv \rho_s / L_B \ll 1, \tag{B1a}$$

$$\left|\frac{\boldsymbol{v}_s \cdot \boldsymbol{A}_1}{T_s}\right| \sim \frac{\phi_1}{T_s} \sim \epsilon_\delta \ll 1, \tag{B1b}$$

$$|\mathbf{k}_\perp| \rho_s \equiv \epsilon_\perp \ll 1, \tag{B1c}$$

$$\frac{\omega}{\Omega_s} \sim \epsilon_\omega \ll 1. \tag{B1d}$$

Using Eqs. (B1) and (25a), we found the first-order gauge function,

$$S_1 \approx -\delta \boldsymbol{A}_c \cdot \boldsymbol{\rho}. \tag{B2}$$

Substituting Eq. (B2) into Eq. (22), we found the first-order generating function,

$$G_1^i = \delta A_{kc}\partial_j X^k J^{ji} \tag{B3a}$$

$$= (\delta A_{\psi c}\partial_j \psi + \delta A_{\theta c}\partial_j \theta + \delta A_{\zeta c}\partial_j \zeta + \delta \phi_c \partial_j t)J^{ji} \tag{B3b}$$

The components of the first-order generating function can be explicitly written as

$$G_1^\theta = -[\delta A_{\psi c} - \delta A_{\zeta c}\delta(\psi,\theta)]\partial_{P_\theta}\psi, \tag{B4a}$$

$$G_1^{\alpha_c} = -[\delta A_{\psi c} - \delta A_{\zeta c}\delta(\psi,\theta)]\partial_{P_\alpha}\psi, \tag{B4b}$$

$$G_1^{P_\theta} = -[\delta A_{\psi c} - \delta A_{\zeta c}\delta(\psi,\theta)]\partial_\theta \psi + \delta A_{\theta c} + \delta A_{\zeta c}[q(\psi_0) + \tilde{q}(\psi_0) - \tilde{q}(\psi)], \tag{B4c}$$

$$G_1^{P_\alpha} = -\delta A_{\zeta c}, \tag{B4d}$$

$$G_1^\xi = 0, \tag{B4e}$$

$$G_1^M = 0, \tag{B4f}$$

$$G_1^{-U} = -\delta \phi_c, \tag{B4g}$$

$$G_1^t = 0. \tag{B4h}$$

Substituting Eq. (B3a) into Eq. (24b), we found the first-order gyrocenter Hamiltonian

$$\bar{H}_1 = -G_1^i \partial_i h_0 = -\delta A_{kc}\partial_j X^k J^{ji}\partial_i h_0 \tag{B5a}$$

$$= -(\delta A_{\psi c}\partial_j \psi + \delta A_{\theta c}\partial_j \theta + \delta A_{\zeta c}\partial_j \zeta + \delta \phi_c \partial_j t)J^{ji}\partial_i h_0. \tag{B5b}$$

Using the relations of gyrocenter and guiding-center variables, we can obtain the guiding-center equations of motion from the linear drift approximation of the gyrocenter equations of motion. Substituting Eqs. (20) and (B5a) into Eq. (27), we found the guiding-center equations of motion in terms of the canonical variables,

$$\frac{dZ^i}{dt} = \frac{d}{dt}\left(\bar{Z}^i - G_1^i(\bar{\boldsymbol{Z}})\right) \tag{B6a}$$

$$= \left\{\bar{Z}^i, \bar{h}_0\right\}_{gy} + \left\{\bar{Z}^i, \bar{h}_1\right\}_{gy} - \left\{G_1^i, \bar{h}_0\right\}_{gy} \tag{B6b}$$

$$= \left\{\bar{Z}^i, h_0\right\}_{gc} + G_1^j \partial_j \left\{\bar{Z}^i, h_0\right\}_{gc} + G_1^{-U}\partial_{-U}\left\{\bar{Z}^i, h_0\right\}_{gc} + \left\{\bar{Z}^i, \phi_1 - G_1^j \partial_j h_0\right\}_{gc} - \partial_j G_1^i \left\{\bar{Z}^j, h_0\right\}_{gc} - \partial_t G_1^i \{t, h_0\}_{gc} \tag{B6c}$$

$$= \dot{Z}_0^i - J^{ij}[\omega_{1jk}\dot{Z}_0^k + E_{1j}], \tag{B6d}$$

where $Z^i = (P_\theta, \theta, P_\alpha, \alpha_c)$. Then, the guiding-center equations of motion in terms of the noncanonical variables are thus obtained from the canonical gyrokinetic theory in the linear drift approximation,

$$\frac{dX^\alpha}{dt} = \frac{\partial X^\alpha}{\partial Z^i}\frac{dZ^i}{dt}, \tag{B7a}$$

$$= \dot{X}_0^\alpha - J^{\alpha\beta}\left(\omega_{1\beta\gamma}\dot{X}_0^\gamma + E_{1\beta}\right) \tag{B7b}$$

where $\omega_{1\beta\gamma} = \partial_\beta A_{1\gamma} - \partial_\gamma A_{1\beta}$ and $X^\alpha = (\psi, \theta, \zeta, u)$, with u the guiding-center parallel velocity. The usual guiding-center motion can be expressed as Morozov–Solovev–Boozer's formula[29,30] or Littlejohn's formulae.[23] Littlejohn's guiding-center equations of motion are written as

$$\dot{\boldsymbol{R}} = u\frac{\boldsymbol{B}^*}{B_\parallel^*} + \boldsymbol{E}^* \times \frac{\hat{b}}{B_\parallel^*}, \tag{B8a}$$

$$\dot{u} = \frac{\boldsymbol{B}^*}{B_\parallel^*}\cdot \boldsymbol{E}^*, \tag{B8b}$$

where $\boldsymbol{B}^*, B_\parallel^*, \boldsymbol{E}^*$ are defined as

$$\boldsymbol{B}^* \equiv \nabla \times (\boldsymbol{A} + u\hat{b}), \tag{B9a}$$

$$B_\parallel^* \equiv \boldsymbol{B}^* \cdot \hat{b}, \tag{B9b}$$

$$\boldsymbol{E}^* \equiv \boldsymbol{E} - M\nabla B - u\frac{\partial \hat{b}}{\partial t}. \tag{B9c}$$

The equilibrium scalar potential Φ_0 is set to zero. Let $\epsilon_B \to 0, \epsilon_\delta \to 0$, then we have the usual guiding-center equations of motion in the linear drift approximation,

$$\dot{\boldsymbol{R}} = u\hat{b} + \boldsymbol{E}_1 \times \frac{\hat{b}_0}{B_0} \tag{B10a}$$

$$\dot{u} = \hat{b}_0 \cdot \boldsymbol{E}_1. \tag{B10b}$$

The zero-order usual guiding-center equations of motion in ϵ_δ are written as

$$(\dot{X}^\alpha)_0 = (\dot{\boldsymbol{R}})_0 \cdot \nabla X^\alpha = u\hat{b}_0 \cdot \nabla X^\alpha. \tag{B11}$$

In Eq. (B11) and the following equations, X^α is understood as (ψ, θ, ζ). The first-order usual guiding-center equations of motion are written as

$$(\dot{X}^\alpha)_1 = \frac{u}{B_0^3}\left(B_0^2 \boldsymbol{B}_1 \cdot \nabla X^\alpha - (\boldsymbol{B}_0 \cdot \boldsymbol{B}_1)\boldsymbol{B}_0 \cdot \nabla X^\alpha\right) + \boldsymbol{E}_1 \times \frac{\hat{b}_0}{B_0} \cdot \nabla X^\alpha, \tag{B12a}$$

$$(\dot{u})_1 = \hat{b}_0 \cdot \boldsymbol{E}_1. \tag{B12b}$$

The components of the noncanonical Poisson matrix[23] in the drift approximation are written as

$$J^{\alpha\beta} \equiv \{X^\alpha, X^\beta\} = -\frac{1}{B_0}\hat{b}_0 \cdot \nabla X^\alpha \times \nabla X^\beta = -\frac{1}{J_{(\psi,\theta,\zeta)}B_0^2} B_{0\gamma}\epsilon^{\alpha\beta\gamma}, \tag{B13a}$$

$$J^{u\alpha} \equiv \{u, X^\alpha\} = \{u, \boldsymbol{R}\} \cdot \nabla X^\alpha = -\hat{b}_0 \cdot \nabla X^\alpha = -\frac{(\dot{X}^\alpha)_0}{u}, \tag{B13b}$$

where $\epsilon^{\alpha\beta\gamma}$ is the permutation tensor.

Using Eq. (B13), we have

$$\boldsymbol{B}_0 \cdot \boldsymbol{B}_1 = B_{0\gamma}\nabla X^\gamma \cdot \nabla \times \left(A_{1\beta}\nabla X^\beta\right) = J^{-1}_{(\psi,\theta,\zeta)} B_{0\gamma}\partial_\alpha A_{1\beta}\epsilon^{\alpha\beta\gamma} = -J^{\alpha\beta}\partial_\alpha A_{1\beta}B_0^2, \tag{B14a}$$

$$\boldsymbol{E}_1 \times \frac{\hat{b}_0}{B_0} \cdot \nabla X^\alpha = \frac{1}{B_0^2}\nabla X^\alpha \cdot \left(E_{1\beta}\nabla X^\beta \times B_{0\gamma}\nabla X^\gamma\right) = \frac{1}{J_{(\psi,\theta,\zeta)}B_0^2} B_{0\gamma}\epsilon^{\alpha\beta\gamma}E_{1\beta} = -J^{\alpha\beta}E_{1\beta}, \tag{B14b}$$

$$\boldsymbol{B}_1 \cdot \nabla X^\alpha = \nabla X^\alpha \cdot \nabla \times \left(A_{1\gamma}\nabla X^\gamma\right) = J^{-1}_{(\psi,\theta,\zeta)}\partial_\beta A_{1\gamma}\epsilon^{\alpha\beta\gamma}, \tag{B14c}$$

$$= \frac{1}{J_{(\psi,\theta,\zeta)}B_0^2}\left(B_0^\alpha B_{0\alpha} + B_0^\beta B_{0\beta} + B_0^\gamma B_{0\gamma}\right)\partial_\beta A_{1\gamma}\epsilon^{\alpha\beta\gamma}, \tag{B14d}$$

$$= -\frac{1}{u}\left[(\dot{X}^\alpha)_0 J^{\beta\gamma} + (\dot{X}^\beta)_0 J^{\gamma\alpha} + (\dot{X}^\gamma)_0 J^{\alpha\beta}\right]\partial_\beta A_{1\gamma}. \tag{B14e}$$

Substituting Eqs. (B11) and (B14) into Eq. (B12), we found the guiding-center equations of motion from the usual noncanonical drift kinetic theory,

$$(\dot{X}^\alpha)_1 = -J^{\alpha\beta}\left(\omega_{1\beta\gamma}\dot{X}_0^\gamma + E_{1\beta}\right), \tag{B15a}$$

$$(\dot{u})_1 = -J^{u\beta}\left(\omega_{1\beta\gamma}\dot{X}_0^\gamma + E_{1\beta}\right). \tag{B15b}$$

Note that $J^{u\beta}\omega_{1\beta\gamma}\dot{X}_0^\gamma = 0$ is used to derive Eq. (B15b).

By comparing Eqs. (B7) and (B15), we demonstrated that the gyrocenter equations of motion based on the canonical gyrokinetic theory is equivalent to the usual guiding-center equations of motion.

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