Electrical conductivity of plasma

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Abstract

This note considers the calculation of the electrical conductivity of non-relativistic plasmas. The results indicate that the electrical conductivity σ is a function of T_e and Z_i, where T_e is the temperature of electrons and Z_i is the charge number of ions. The dependence of σ on T_e is given analytically by $\sigma \propto T_e^{3/2}$ while the dependence on Z_i has to be calculated numerically. The numerical results indicates that, as expected, σ decreases with the increasing of Z_i. The analytic results indicates that σ is independent of the number density of electrons n_e.

1 Equation for electrons distribution function

The steady state of the electrons distribution is determined by the balance of the collision term with the electrical field term:

$$C(f_e, f_e) + C(f_e, f_i) = \frac{q_e \boldsymbol{E}}{m_e} \cdot \nabla_v f_e, \tag{1}$$

where ∇_v denotes the gradient operator in velocity space. We use perturbation theory to solve Eq. (1) for f_e. The electrical field is treated as a perturbation. Expanding f_e as

$$f_e = f_{e0} + f_{e1} + f_{e2} + \ldots \tag{2}$$

and using this in Eq. (1), the zeroth order equation is written as

$$C(f_{e0}, f_{e0}) + C(f_{e0}, f_i) = 0. \tag{3}$$

The first order equation of Eq. (1) is written as

$$C(f_{e1}, f_{e0}) + C(f_{e0}, f_{e1}) + C(f_{e1}, f_i) = \frac{q_e \boldsymbol{E}}{m_e} \cdot \nabla_v f_{e0}. \tag{4}$$

The zeroth order equation gives that $f_{e0} = f_{em}$, where f_{em} is a Maxwellian distribution with temperature T_e and density n_e. Using this in the first order equation (4) gives

$$C(f_{e1}, f_{em}) + C(f_{em}, f_{e1}) + C(f_{e1}, f_i) = \frac{q_e \boldsymbol{E}}{m_e} \cdot \nabla_v f_{em}. \tag{5}$$

The right-hand side of Eq. (5) is written as

$$\begin{aligned} \frac{q_e \boldsymbol{E}}{m_e} \cdot \nabla_v f_{em} &= \frac{q_e \boldsymbol{E}}{m_e} \cdot \left(-\frac{\boldsymbol{v}}{v_{ta}^2} \right) f_{em} \\ &= -\frac{q_e E}{m_e v_{te}} \frac{v_\parallel}{v_{te}} f_{em}, \end{aligned} \tag{6}$$

where $v_\parallel$ is the velocity parallel to the electric field. Using Eq. (6), equation (5) is written as

$$C(f_{e1}, f_{em}) + C(f_{em}, f_{e1}) + C(f_{e1}, f_i) = -\frac{q_e E}{m_e v_{te}} \frac{v_\parallel}{v_{te}} f_{em}. \tag{7}$$

It can be proved that the solution to the above equation, f_{e1}, consists of only the first Legendre harmonics (The proof is given in another note). Thus we write

$$f_{e1} = f_{em}(v) \chi(v) \cos\theta, \tag{8}$$

then Eq. (7) is written as

$$\frac{1}{v^2}\frac{\partial}{\partial v}\left(v^2 D_{cvv}^{e/e}\frac{\partial\chi}{\partial v}\right)-\frac{m_e v}{T_e}D_{cvv}^{e/e}\frac{\partial\chi}{\partial v}-\frac{2D_{c\theta\theta}^{e/e}+\Gamma^{e/e}Z_i/v}{v^2}\chi$$
$$+\frac{C(f_{em},f_{em}\chi\cos\theta)}{f_{em}\cos\theta}+\frac{q_e E}{m_e v_{te}}\frac{v}{v_{te}}=0, \tag{9}$$

which is the equation (98) in Karney's 1986 paper[1]. (The detailed derivation of this equation is given in another note.).

2 Relaxation method of calculating steady state solution

Equation (9) can be further written

$$D_{cvv}^{e/e}\frac{\partial^2\chi}{\partial v^2}+\left[\frac{1}{v^2}\frac{\partial}{\partial v}(v^2 D_{cvv}^{e/e})-\frac{m_e v}{T_e}D_{cvv}^{e/e}\right]\frac{\partial\chi}{\partial v}-\frac{2D_{c\theta\theta}^{e/e}+\Gamma^{e/e}Z_i/v}{v^2}\chi$$
$$+\frac{C(f_{em},f_{em}\chi(v)\cos\theta)}{f_{em}\cos\theta}+\frac{q_e E}{m_e v_{te}}\frac{v}{v_{te}}=0 \tag{10}$$

A method of finding a solution to this time-independent problem is the relaxation method, in which we consider a time-dependent problem and calculate the steady solution. We consider the following time dependent problem:

$$\frac{\partial\chi}{\partial t}=a\frac{\partial^2\chi}{\partial v^2}+b\frac{\partial\chi}{\partial v}+c\chi+\frac{C(f_{em},f_{em}\chi(v)\cos\theta)}{f_{em}\cos\theta}+\frac{q_e E}{m_e v_{te}}\frac{v}{v_{te}}, \tag{11}$$

where

$$a = D_{cvv}^{e/e}, \tag{12}$$
$$b = \frac{1}{v^2}\frac{\partial}{\partial v}(v^2 D_{cvv}^{e/e})-\frac{m_e v}{T_e}D_{cvv}^{e/e}, \tag{13}$$

and

$$c=-\frac{2D_{c\theta\theta}^{e/e}}{v^2}-\frac{\Gamma^{e/e}Z_i}{v^3}. \tag{14}$$

The Fokker-Planck coefficients are given by

$$D_{cvv}^{e/e}=\frac{4\pi\Gamma^{e/e}}{3n_e}\left(\int_0^v\frac{(v')^4}{v^3}f_{em}(v')dv'+\int_v^\infty v' f_{em}(v')dv'\right), \tag{15}$$

$$D_{c\theta\theta}^{e/e}=\frac{4\pi\Gamma^{e/e}}{3n_e}\left[\int_0^v\frac{v'^2}{2v^3}(3v^2-(v')^2)f_{em}(v')dv'+\int_v^\infty v' f_{em}(v')dv'\right], \tag{16}$$

$$\begin{aligned}\frac{C(f_{em},f_{em}\chi(v)\cos\theta)}{f_{em}\cos\theta} &= \frac{4\pi\Gamma^{e/e}}{n_e}\left[f_{em}\chi(v)+\int_0^v\frac{v'^2}{v_{te}^2}\left(\frac{v'^3}{5v_{te}^2v^2}-\frac{v'}{3v^2}\right)f_{em}(v')\chi(v')dv'\right.\\ &+\left.\int_v^\infty\frac{v'^2}{v_{te}^2}\left(\frac{v^3}{5v_{te}^2v'^2}-\frac{v}{3v'^2}\right)f_{em}(v')\chi(v')dv'\right].\end{aligned} \tag{17}$$

$$\begin{aligned}\frac{1}{v^2}\frac{\partial}{\partial v}(v^2 D_{cvv}^{e/e}) &= \frac{4\pi\Gamma^{e/e}}{3n_e}\frac{1}{v^2}\frac{\partial}{\partial v}\left[\left(\frac{1}{v}\int_0^v(v')^4 f_{em}(v')dv'+v^2\int_v^\infty v' f_{em}(v')dv'\right)\right]\\ &= \frac{4\pi\Gamma^{e/e}}{3n_e}\left[-\frac{1}{v^4}\int_0^v(v')^4 f_{em}(v')dv'+\frac{2}{v}\int_v^\infty v' f_{em}(v')dv'\right]\end{aligned}$$

where

$$\Gamma^{e/e}=\frac{n_e e^4\ln\Lambda^{e/e}}{4\pi\varepsilon_0^2 m_e^2}. \tag{18}$$

3 Normalization

Define normalized quantities $\overline{f} \equiv f/f_0$, $\overline{v} \equiv v/v_0$, and $\tau \equiv t/t_0$ where

$$f_0 = \frac{n_e}{v_{te}^3},\ v_0 = v_{te},\ t_0 = \frac{v_{te}^3}{\Gamma^{e/e}}. \tag{19}$$

Then coefficients used in Eq. (11) can be written

$$\begin{aligned} D_{cvv}^{e/e} &= \frac{4\pi\Gamma^{e/e}}{3n_e} f_0 v_0^2 \left(\int_0^{\overline{v}} \frac{(\overline{v}')^4}{\overline{v}^3} \overline{f}_{em} d\,\overline{v}' + \int_{\overline{v}}^{\infty} \overline{v}' \overline{f}_{em}(\overline{v}') d\,\overline{v}' \right) \\ &= \frac{4\pi}{3} \frac{v_{te}^2}{t_0} \left(\int_0^{\overline{v}} \frac{(\overline{v}')^4}{\overline{v}^3} \overline{f}_{em} d\,\overline{v}' + \int_{\overline{v}}^{\infty} \overline{v}' \overline{f}_{em}(\overline{v}') d\,\overline{v}' \right) \end{aligned} \tag{20}$$

$$\begin{aligned} D_{c\theta\theta}^{e/e} &= \frac{4\pi\Gamma^{e/e}}{3n_e} \left[\int_0^{v} \frac{v'^2}{2v^3} (3v^2 - (v')^2) f_{em}(v') dv' + \int_v^{\infty} v' f_{em}(v') dv' \right] \\ &= \frac{4\pi}{3} \frac{v_{te}^2}{t_0} \left[\int_0^{\overline{v}} \frac{\overline{v}'^2}{2\,\overline{v}^3} (3\overline{v}^2 - (\overline{v}')^2) \overline{f}_{em}(\overline{v}') d\,\overline{v}' + \int_{\overline{v}}^{\infty} \overline{v}' f_{em}(\overline{v}') d\,\overline{v}' \right] \end{aligned} \tag{21}$$

Note the dimension of $D_{cvv}^{e/e}$ and $D_{c\theta\theta}^{e/e}$ is $D_0 = v_{te}^2/t_0$.

$$\begin{aligned} \frac{C(f_{em}, f_{em}\chi(v)\cos\theta)}{f_{em}\cos\theta} &= \frac{4\pi\Gamma^{e/e}}{n_e} \Bigg[f_{em}\chi(v) + \int_0^{v} \frac{v'^2}{v_{te}^2} \left(\frac{v'^3}{5v_{te}^2 v^2} - \frac{v'}{3v^2} \right) f_{em}(v')\chi(v') dv' \\ &+ \int_v^{\infty} \frac{v'^2}{v_{te}^2} \left(\frac{v^3}{5v_{te}^2 v'^2} - \frac{v}{3v'^2} \right) f_{em}(v')\chi(v') dv' \Bigg] \\ &= \frac{4\pi}{t_0} \Bigg[\overline{f}_{em}\chi(v) + \int_0^{\overline{v}} \overline{v}'^2 \left(\frac{\overline{v}'^3}{5\overline{v}^2} - \frac{\overline{v}'}{3\overline{v}^2} \right) \overline{f}_{em}(\overline{v}')\chi(\overline{v}') d\,\overline{v}' \\ &+ \int_v^{\infty} \overline{v}'^2 \left(\frac{\overline{v}^3}{5\overline{v}'^2} - \frac{\overline{v}}{3\overline{v}'^2} \right) \overline{f}_{em}(\overline{v}')\chi(\overline{v}') d\overline{v}' \Bigg] \end{aligned} \tag{22}$$

$$\frac{1}{v^2} \frac{\partial}{\partial v} \left(v^2 D_{cvv}^{e/e} \right) = \frac{4\pi}{3} \frac{v_{te}}{t_0} \left[-\frac{1}{\overline{v}^4} \int_0^{\overline{v}} (\overline{v}')^4 \overline{f}_{em}(\overline{v}') d\overline{v}' + \frac{2}{\overline{v}} \int_{\overline{v}}^{\infty} \overline{v}' \overline{f}_{em}(\overline{v}') d\overline{v}' \right] \tag{23}$$

The normalized form of Eq. (11) is

$$\frac{\partial\chi}{\partial\tau} = \frac{t_0}{v_{te}^2} a \frac{\partial^2\chi}{\partial\overline{v}^2} + \frac{t_0}{v_{te}} b \frac{\partial\chi}{\partial\overline{v}} + t_0 c\chi + t_0 \frac{C(f_{em}, f_{em}\chi(v)\cos\theta)}{f_{em}\cos\theta} + \beta\overline{v} \tag{24}$$

where

$$a = D_{cvv}^{e/e}, \tag{25}$$

$$b = \frac{1}{v^2} \frac{\partial}{\partial v} \left(v^2 D_{cvv}^{e/e} \right) - \frac{v}{v_{te}^2} D_{cvv}^{e/e}, \tag{26}$$

$$c = -\frac{2D_{c\theta\theta}^{e/e}}{v^2} - \frac{\Gamma^{e/e} Z_i}{v^3}, \tag{27}$$

and a dimensionless parameter β

$$\beta \equiv \frac{t_0 q_e E}{m_e v_{te}}. \tag{28}$$

Define normalized coefficients

$$\overline{a} \equiv \frac{t_0}{v_{te}^2} a = \frac{t_0}{v_{te}^2} D_{cvv}^{e/e} \tag{29}$$

$$\overline{b} \equiv \frac{t_0}{v_{te}} b = \frac{t_0}{v_{te}} \frac{1}{v^2} \frac{\partial}{\partial v} \left(v^2 D_{cvv}^{e/e} \right) - \overline{v}\ \overline{a} \tag{30}$$

$$\overline{c} \equiv t_0 c = -\frac{2}{\overline{v}^2} \left(\frac{t_0}{v_{te}^2} D_{c\theta\theta}^{e/e} \right) - \frac{Z_i}{\overline{v}^3} \tag{31}$$

then Eq. (24) is written as

$$\frac{\partial\chi}{\partial\tau} = \overline{a}\frac{\partial^2\chi}{\partial\overline{v}^2} + \overline{b}\frac{\partial\chi}{\partial\overline{v}} + \overline{c}\chi + t_0\frac{C(f_{em}, f_{em}\chi(v)\cos\theta)}{f_{em}\cos\theta} + \beta\overline{v}. \tag{32}$$

4 Numerical scheme

The first order derivative is written as,

$$\left[\frac{\partial\chi}{\partial v}\right]_i^n = \frac{\chi_{i+1}^n - \chi_{i-1}^n}{2\Delta v} \tag{33}$$

The second order derivative is written as

$$\left[\frac{\partial^2\chi}{\partial v^2}\right]_i^n = \frac{\chi_{i+1}^n - 2\chi_i^n + \chi_{i-1}^n}{\Delta v^2} \tag{34}$$

Time derivative:

$$\left[\frac{\partial\chi}{\partial t}\right]_i^n = \frac{\chi_i^{n+1} - \chi_i^n}{\Delta t} \tag{35}$$

Using implicit scheme for the first three terms (differential terms) on the right-hand of Eq. (32), we obtain

$$\begin{aligned}\frac{\chi_i^{n+1} - \chi_i^n}{\Delta\overline{t}} &= \overline{a}_i\frac{\chi_{i+1}^{n+1} - 2\chi_i^{n+1} + \chi_{i-1}^{n+1}}{\Delta\overline{v}^2} + \overline{b}_i\frac{\chi_{i+1}^{n+1} - \chi_{i-1}^{n+1}}{2\Delta\overline{v}} + \overline{c}_i\chi_i^{n+1} \\ &\quad + t_0\left[\frac{C(f_{em}, f_{em}\chi(v)\cos\theta)}{f_{em}\cos\theta}\right]_i^n + \beta\overline{v}_i,\end{aligned}$$

which can be arranged to the form

$$\begin{aligned}&\left(-\frac{\overline{a}_i\Delta t}{\Delta\overline{v}^2} + \frac{\overline{b}_i\Delta t}{2\Delta\overline{v}}\right)\chi_{i-1}^{n+1} + \left(1 + \frac{2\overline{a}_i\Delta\overline{t}}{\Delta\overline{v}^2} - \overline{c}_i\Delta\overline{t}\right)\chi_i^{n+1} + \left(-\frac{\overline{a}_i\Delta t}{\Delta\overline{v}^2} - \frac{\overline{b}_i\Delta t}{2\Delta\overline{v}}\right)\chi_{i+1}^{n+1} \\ &= \chi_i + \Delta\overline{t}\left\{t_0\left[\frac{C(f_{em}, f_{em}\chi(v)\cos\theta)}{f_{em}\cos\theta}\right]_i^n + \beta\overline{v}_i\right\},\end{aligned} \tag{36}$$

5 Boundary conditions

The computational domain is $0 < v < v_{\max}$ and the boundary conditions are set to be

$$\chi(v=0) = 0 \tag{37}$$

$$\frac{\partial^2\chi}{\partial v^2}|_{v=v_{\max}} = 0 \tag{38}$$

6 Electrical Conductivity

Electrical conductivity is defined as the ratio of the steady state electrical current density to the applied electrical field:

$$\begin{aligned}\sigma &\equiv \frac{J}{E} \\ &= \frac{q_e\int v\cos\theta f_e(\boldsymbol{v})d\boldsymbol{v}}{E}.\end{aligned} \tag{39}$$

where f_e is the electron distribution function determined by Eq. (1). As discussed above, in the perturbation method of calculating f_e, f_e is expanded as $f_e = f_{e0} + f_{e1} + f_{e2} + \ldots$. We consider only the linear approximation, in which only terms up to the first order are included, i.e., $f_e \approx f_{e0} + f_{e1}$. As discussed above, f_{e0} is a Maxwellian distribution, which is isotropic, thus, does not contribute to the current. Using this, Eq. (39) is written as

$$\begin{aligned} \sigma &\approx \frac{q_e \int v \cos\theta f_{e1}(\boldsymbol{v}) d\boldsymbol{v}}{E} \\ &= \frac{q_e 2\pi \int_0^\pi \sin\theta d\theta \int_0^\infty v^2 [v \cos\theta f_{e1}] dv}{E}. \end{aligned} \tag{40}$$

The first order distribution function is $f_{e1}(\boldsymbol{v}) = f_{em}(v)\chi(v)\cos\theta$. Using this, Eq. (40) is written as

$$\begin{aligned} \sigma &= \frac{q_e 2\pi \int_0^\pi \sin\theta d\theta \int_0^\infty v^3 f_{em}(v)\chi(v)\cos^2\theta dv}{E} \\ &= \frac{4\pi q_e}{3E} \int_0^\infty v^3 f_{em}(v)\chi(v) dv \\ &= \frac{4\pi q_e}{3E} \frac{n_e}{v_{te}^3} v_{te}^4 \int_0^\infty \bar{v}^3 \bar{f}_{em}(\bar{v})\chi(\bar{v}) d\bar{v} \\ &= \frac{4\pi q_e}{3E} n_e v_{te} \int_0^\infty \bar{v}^3 \bar{f}_{em}(\bar{v})\chi(\bar{v}) d\bar{v} \end{aligned}$$

Define

$$\sigma_0 \equiv \frac{n_e q_e^2 t_0}{m_e} = \frac{n_e q_e^2}{m_e} \frac{v_{te}^3}{\frac{n_e e^4 \ln\Lambda^{e/e}}{4\pi\varepsilon_0^2 m_e^2}} = \frac{4\pi\varepsilon_0^2}{e^2 \ln\Lambda^{e/e}\sqrt{m_e}} T_e^{3/2} \tag{41}$$

Then the normalized electrical conductivity can be written

$$\bar{\sigma} \equiv \frac{\sigma}{\sigma_0} = \frac{4\pi}{3} \int_0^\infty \bar{v}^3 \bar{f}_{em}(\bar{v}) \frac{\chi(\bar{v})}{\beta} d\bar{v} \tag{42}$$

Note that both $\chi(\bar{v}) / \beta$ and $\bar{f}_{em}(\bar{v})$ are independent of T_e. All the dependence of σ on T_e is contained in σ_0, which is proportional to $T_e^{3/2}$ if the weak dependence of the Coulomb logarithm on T_e is ignored. Thus the electric conductivity scales with T_e approximately as $\sigma \propto T_e^{3/2}$. Also note that all the terms in Eqs. (41) and (42) are independent of the electron number density n_e (the weak dependence of the Coulomb logarithm on n_e is ignored), which indicates that σ is independent of n_e. Also not that $\chi(\bar{v})/\beta$ only depends on Z_i. The dependence of Z_i has to be solved numerically. The results are given in the next section.

7 Numerical results of dependence of σ on Z_i

Table (1) gives the electrical conductivity for various values of ion Z_i. These results agree with the results in Karney1986 paper. Here Z_i is defined as

$$Z_i = -\frac{q_i}{q_e} \frac{\ln\Lambda^{e/i}}{\ln\Lambda^{e/e}} \tag{43}$$

Z_i	1	2	5	10
σ/σ_0	7.421	4.375	2.077	1.132

Table 1. The electrical conductivity for various values of Z_i. Numerical parameters: grid number $N = 5000$, $v_{\max} = 15 v_{te}$, $\Delta t = 1000 t_0$. The iteration process usually converges after about 50 steps (The absolute change in χ_1 per step was less than 10^{-10}). The iteration process converges faster for bigger values of Z_i.

For typical plasmas in the EAST tokamak, we have $T_e = 1$keV and $Z_i = 2$. Using these parameters, Eqs. (41) and (42), and the data in Table 1, we obtain the conductivity of the plasmas $\sigma = 2.38 \times 10^7 S/m$, which is comparable to the conductivity of iron ($1.00 \times 10^7 S/m$), Aluminium ($3.5 \times 10^7 S/m$), and copper ($5.96 \times 10^7 S/m$) (data from en.wikipedia.org).

Figure (1) gives the steady profile of $\chi(v)$ and the perturbed distribution function $f_{e1}(v)$. These results are obtained using $\Delta t = 10 t_0$, and advancing in time for $300\Delta t$. The perturbed distribution function had arrived at steady state.

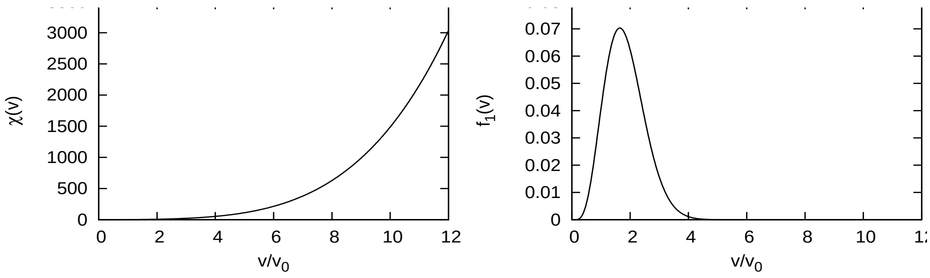

Figure 1. Steady profile of $\chi(v)$ and the perturbed distribution function $f_1(v)$ for parameter $Z_i = 2$. Here is for the case of negative electrical field.

8 Analytic expression of Spitzer Function

The analytic expression of Spitzer function was given in Ref.[2].

$$f_{em} = \frac{n_e}{v_e^3 \pi^{3/2}} \exp(-\frac{v^2}{v_e^2})$$

Note here $v_e \equiv \sqrt{2T_e/m_e}$, differs from v_{te} defined above by a factor $\sqrt{2}$.

$$C_e(f_1) = \nu_{e0} \frac{v}{v_e} \cos\theta f_{em}$$

where $C_e(f_1) \equiv C(f_1, f_{em}) + C(f_{em}, f_1) + C(f_1, f_i)$.

$$\nu_{e0} = \frac{4\pi n_e e^4 \ln\Lambda}{m_e^2 v_e^3}$$

The solution is written as

$$f_1 = -D\left(\frac{v}{v_e}\right) \cos\theta f_{em}$$

where

$$D(x) = x(d_1 x + d_2 x^2 + d_3 x^3 + d_4 x^4)$$

$$\begin{aligned}
d_1 &= (4.397 - 2.32\overline{Z} - 0.283\overline{Z}^2)/G(\overline{Z}) \\
d_2 &= (0.793\overline{Z}^2 + 8.053\overline{Z} - 4.627)/G(\overline{Z}) \\
d_3 &= (0.0467\overline{Z}^3 + 0.108\overline{Z}^2 - 4.136\overline{Z} + 2.006)/G(\overline{Z}) \\
d_4 &= (-0.011\overline{Z}^2 + 0.716\overline{Z} - 0.304)/G(\overline{Z})
\end{aligned}$$

where $G(\overline{Z}) = \overline{Z}(1 + 0.292\overline{Z})(1 + 1.16\overline{Z} + 0.16\overline{Z}^2)$. It follows that

$$\overline{Z}D(x) = \begin{cases} x^2(0.60 + 1.41x - 0.66x^2 + 0.134x^3) & \text{For } \overline{Z} = 1 \\ 2x^2(-0.11 + 1.17x - 0.44x^2 + 0.086x^3) & \text{For } \overline{Z} = 2 \end{cases}$$

Fig. 2 compares the numerical solution with the above analytic solution.

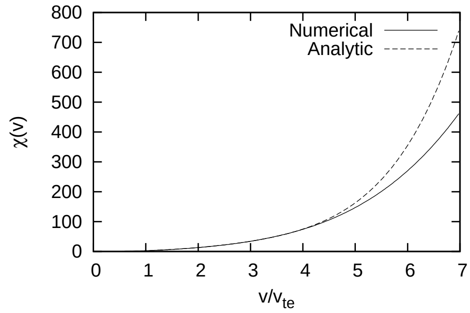

Figure 2. Comparison of the numerical solution with the analytic solution for $Z_i = 1$.

Bibliography

[1] Charles F. F. Karney. Fokker-planck and quasilinear codes. *Comp. Phys. Rep.*, 4:183–244, 1986.
[2] S. P. Hirshman. Classical collisional theory of beam-driven plasma currents. *Physics of Fluids*, 23(6):1238–1243, 1980.

9 Manuscript: expression for $H(\chi)$ and $I(\chi)$

$$H(\chi) \equiv \frac{C(f_{em}\chi\cos\theta, f_{em})}{f_{em}\cos\theta} = \frac{1}{v^2}\frac{\partial}{\partial v}\left(v^2 D_{cvv}^{a/b}\frac{\partial\chi}{\partial v}\right) - \frac{m_e v}{T_e}D_{cvv}^{a/b}\frac{\partial\chi}{\partial v} - 2D_{c\theta\theta}^{a/b}\frac{1}{v^2}\chi$$

$$\begin{aligned} I(\chi) &\equiv \frac{C(f_{em}, f_{em}\chi(v)\cos\theta)}{f_{em}\cos\theta} \\ &= \frac{4\pi\Gamma^{e/e}}{n_e}\left[f_{em}\chi(v) + \int_0^v \frac{v'^2}{v_{te}^2}\left(\frac{v'^3}{5v_{te}^2 v^2} - \frac{v'}{3v^2}\right) f_{em}(v')\chi(v')dv'\right. \\ &+ \left.\int_v^\infty \frac{v'^2}{v_{te}^2}\left(\frac{v^3}{5v_{te}^2 v'^2} - \frac{v}{3v'^2}\right) f_{em}(v')\chi(v')dv'\right] \end{aligned}$$

Normalized $\overline{H}(\chi) \equiv t_0 H(\chi)$ and $\bar{I}(\chi) = t_0 I(\chi)$,

$$\overline{H}(\chi) = t_0\frac{1}{v^2}\frac{\partial}{\partial v}\left(v^2 D_{cvv}^{a/b}\frac{\partial\chi}{\partial v}\right) - t_0\frac{m_e v}{T_e}D_{cvv}^{a/b}\frac{\partial\chi}{\partial v} - 2t_0 D_{c\theta\theta}^{a/b}\frac{1}{v^2}\chi$$

$$\begin{aligned} \bar{I}(\chi) &= \frac{4\pi\Gamma^{e/e}}{n_e}t_0\left[f_{em}\chi(v) + \int_0^v \frac{v'^2}{v_{te}^2}\left(\frac{v'^3}{5v_{te}^2 v^2} - \frac{v'}{3v^2}\right) f_{em}(v')\chi(v')dv'\right. \\ &+ \left.\int_v^\infty \frac{v'^2}{v_{te}^2}\left(\frac{v^3}{5v_{te}^2 v'^2} - \frac{v}{3v'^2}\right) f_{em}(v')\chi(v')dv'\right] \end{aligned}$$

$\bar{I}(\chi)$ is the quantity actually calculated in the code.

We know collision operator $C(f_a, f_b)$ conserves momentum, i.e.,

$$\int [m_a C(f_a, f_b) + m_b C(f_b, f_a)]\boldsymbol{v} d^3\boldsymbol{v}$$

We want to directly prove numerically that

$$\int [C(\chi f_{em}\cos\theta, f_{em}) + C(f_{em}, \chi f_{em}\cos\theta)]\boldsymbol{v} d^3\boldsymbol{v} = 0$$

The left hand of the above equation

$$\int [H(\chi) + I(\chi)]\boldsymbol{v} f_{em}\cos\theta d^3\boldsymbol{v}$$

$$= \int [H(\chi) + I(\chi)]\boldsymbol{v} f_{em}\cos\theta v^2 d\mu dv d\varphi$$

$$\boldsymbol{v} = v\hat{v}(\theta, \varphi) \tag{44}$$

$$\int_0^{2\pi} \hat{v}(\theta, \phi) d\phi = \hat{z}\cos\theta 2\pi \tag{45}$$

$$\int_0^\pi \cos\theta\sin\theta\cos\theta 2\pi d\theta = 2\pi\int_{-1}^1 \mu^2 d\mu = \frac{4\pi}{3}$$

$$\int_0^\infty [H(\chi) + I(\chi)] f_{em} v^3 dv = 0 \tag{46}$$