This is my notes when reading Liu Chen's book[1].

1 Vlasov equation

The linearized Vlasov equation is

$$\left[\frac{\partial}{\partial t}+\boldsymbol{v}\cdot\frac{\partial}{\partial \boldsymbol{x}}+\frac{q}{m}\left(\boldsymbol{E}_0+\frac{\boldsymbol{v}\times\boldsymbol{B}_0}{c}\right)\cdot\frac{\partial}{\partial \boldsymbol{v}}\right]\delta f=-\frac{q}{m}\left(\delta\boldsymbol{E}+\frac{\boldsymbol{v}\times\delta\boldsymbol{B}}{c}\right)\cdot\frac{\partial}{\partial \boldsymbol{v}}f_0, \tag{1}$$

where f_0 and δf are the equilibrium and perturbed distribution functions, respcetively, $\boldsymbol{E}_0$, $\boldsymbol{B}_0$, $\delta\boldsymbol{E}$, and $\delta\boldsymbol{B}$ are the equilibrim and perturbed electromagnetic field. We consider the case of $\boldsymbol{E}_0=0$. Define the unperturbed Vlasov propagator

$$\mathcal{F}\equiv\frac{\partial}{\partial t}+\boldsymbol{v}\cdot\frac{\partial}{\partial \boldsymbol{x}}+\frac{q}{mc}(\boldsymbol{v}\times\boldsymbol{B}_0)\cdot\frac{\partial}{\partial \boldsymbol{v}}, \tag{2}$$

then Eq. (1) is written as

$$\mathcal{F}\delta f=-\frac{q}{m}\left(\delta\boldsymbol{E}+\frac{\boldsymbol{v}\times\delta\boldsymbol{B}}{c}\right)\cdot\frac{\partial}{\partial \boldsymbol{v}}f_0. \tag{3}$$

2 Guiding-center transformation

We now consider the guiding-center transformation in uniform magnetic field. The transformation from the particle variables $(\boldsymbol{x},\boldsymbol{v})$ to the guiding-center variables $(\boldsymbol{X},\varepsilon,\mu,\alpha,\sigma)$ is defined as

$$\boldsymbol{X}=\boldsymbol{x}+\boldsymbol{v}\times\frac{\boldsymbol{e}_\parallel}{\Omega}, \tag{4}$$

$$\varepsilon=\frac{v^2}{2}, \tag{5}$$

$$\mu=v_\perp^2/2B_0, \tag{6}$$

$$\sigma=\mathrm{sgn}(v_\parallel), \tag{7}$$

and α is the gyrophase angle which is defiend in the following. Here $\boldsymbol{e}_\parallel=\boldsymbol{B}_0/B_0$, $\Omega=qB_0/(mc)$. In terms of (ε, μ, α, σ), the parallel and perpendicular velocity of a particle are given respectively by

$$v_\parallel=\sigma\sqrt{2(\varepsilon-B_0\mu)}, \tag{8}$$

and

$$\boldsymbol{v}_\perp=\sqrt{2B_0\mu}(\boldsymbol{e}_1\cos\alpha+\boldsymbol{e}_2\sin\alpha), \tag{9}$$

where $\boldsymbol{e}_1$ and $\boldsymbol{e}_2$ are two orthogonal unit vectors perpendicular to $\boldsymbol{B}_0$; $\boldsymbol{e}_1\cdot\boldsymbol{e}_\parallel=0$ and $\boldsymbol{e}_2=\boldsymbol{e}_\parallel\times\boldsymbol{e}_1$, thus Eq. (9) indicates the gyrophase angle is defined as the included angle between $\boldsymbol{v}_\perp$ and $\boldsymbol{e}_1$. We now consider the transformation of the $\partial/\partial\boldsymbol{x}$ and $\partial/\partial\boldsymbol{v}$ operators to the guiding-center variables. For notation convenience, we define $\boldsymbol{V}=(\varepsilon,\mu,\alpha,\sigma)$. Then we have

$$\frac{\partial}{\partial \boldsymbol{x}}=\frac{\partial \boldsymbol{X}}{\partial \boldsymbol{x}}\cdot\frac{\partial}{\partial \boldsymbol{X}}+\frac{\partial \boldsymbol{V}}{\partial \boldsymbol{x}}\cdot\frac{\partial}{\partial \boldsymbol{V}} \tag{10}$$

From Eq. (4), we obtain

$$\frac{\partial \boldsymbol{X}}{\partial \boldsymbol{x}}=\boldsymbol{I}, \tag{11}$$

and, since the equilibrium magnetic field is uniform, the definition of ε, μ, α, and σ do not involve spatial variables, thus $\boldsymbol{V}$ is independent of $\boldsymbol{x}$, i.e.,

$$\frac{\partial \boldsymbol{V}}{\partial \boldsymbol{x}}=0. \tag{12}$$

Using the above results in Eq. (10) gives

$$\begin{aligned} \frac{\partial}{\partial \boldsymbol{x}} &= \boldsymbol{I}\cdot\frac{\partial}{\partial \boldsymbol{X}}+0\cdot\frac{\partial}{\partial \boldsymbol{V}} \\ &= \frac{\partial}{\partial \boldsymbol{X}}. \end{aligned} \tag{13}$$

Now consider the transformation of the gradient in velocity space, $\partial/\partial\boldsymbol{v}$,

$$\frac{\partial}{\partial \boldsymbol{v}}=\frac{\partial \boldsymbol{X}}{\partial \boldsymbol{v}}\cdot\frac{\partial}{\partial \boldsymbol{X}}+\frac{\partial \boldsymbol{V}}{\partial \boldsymbol{v}}\cdot\frac{\partial}{\partial \boldsymbol{V}} \tag{14}$$

From Eq. (4), we get

$$\begin{aligned} \frac{\partial \boldsymbol{X}}{\partial \boldsymbol{v}} &= \frac{\partial}{\partial \boldsymbol{v}}\left(\frac{\boldsymbol{v}\times\boldsymbol{e}_\parallel}{\Omega}\right) \\ &= \frac{1}{\Omega}\frac{\partial}{\partial \boldsymbol{v}}(\boldsymbol{v}\times\boldsymbol{e}_\parallel) \\ &= \frac{1}{\Omega}\boldsymbol{I}\times\boldsymbol{e}_\parallel \end{aligned} \tag{15}$$

The second term of r.h.s. of Eq. (14) can be written as

$$\frac{\partial \boldsymbol{V}}{\partial \boldsymbol{v}}\frac{\partial}{\partial \boldsymbol{V}}=\frac{\partial\varepsilon}{\partial \boldsymbol{v}}\frac{\partial}{\partial\varepsilon}+\frac{\partial\mu}{\partial \boldsymbol{v}}\frac{\partial}{\partial\mu}+\frac{\partial\alpha}{\partial \boldsymbol{v}}\frac{\partial}{\partial\alpha} \tag{16}$$

Using

$$\frac{\partial\varepsilon}{\partial \boldsymbol{v}}=\boldsymbol{v}, \tag{17}$$

$$\frac{\partial\mu}{\partial \boldsymbol{v}}=\frac{\boldsymbol{v}_\perp}{B_0}, \tag{18}$$

and

$$\frac{\partial\alpha}{\partial \boldsymbol{v}}=\frac{1}{v_\perp}\left(\boldsymbol{e}_\parallel\times\frac{\boldsymbol{v}_\perp}{v_\perp}\right) \tag{19}$$

Eq. (16) is written as

$$\frac{\partial \boldsymbol{V}}{\partial \boldsymbol{v}}\frac{\partial}{\partial \boldsymbol{V}}=\boldsymbol{v}\frac{\partial}{\partial\varepsilon}+\boldsymbol{v}_\perp\frac{\partial}{B_0\partial\mu}+\frac{\boldsymbol{e}_\alpha}{v_\perp}\frac{\partial}{\partial\alpha}, \tag{20}$$

where

$$\boldsymbol{e}_\alpha=\boldsymbol{e}_\parallel\times\frac{\boldsymbol{v}_\perp}{v_\perp}. \tag{21}$$

Using Eqs. (15) and (20) in Eq. (14) yields

$$\frac{\partial}{\partial \boldsymbol{v}}=\frac{\boldsymbol{I}\times\boldsymbol{e}_\parallel}{\Omega}\cdot\frac{\partial}{\partial \boldsymbol{X}}+\boldsymbol{v}\frac{\partial}{\partial\varepsilon}+\boldsymbol{v}_\perp\frac{\partial}{B_0\partial\mu}+\frac{\boldsymbol{e}_\alpha}{v_\perp}\frac{\partial}{\partial\alpha} \tag{22}$$

Using Eqs. (13) and (22), the unperturbed Vlasov propagator

$$\mathcal{F}\equiv\frac{\partial}{\partial t}+\boldsymbol{v}\cdot\frac{\partial}{\partial \boldsymbol{x}}+\frac{q}{mc}(\boldsymbol{v}\times\boldsymbol{B}_0)\cdot\frac{\partial}{\partial \boldsymbol{v}}, \tag{23}$$

can be written, term by term, as

$$\frac{\partial}{\partial t}=\frac{\partial}{\partial t} \tag{24}$$

$$\begin{aligned} \boldsymbol{v}\cdot\frac{\partial}{\partial \boldsymbol{x}} &= \boldsymbol{v}\cdot\frac{\partial}{\partial \boldsymbol{X}} \\ &= \boldsymbol{v}_\perp\cdot\frac{\partial}{\partial \boldsymbol{X}_\perp}+v_\parallel\frac{\partial}{\partial X_\parallel} \end{aligned} \tag{25}$$

$$\begin{aligned} \frac{q}{mc}(\boldsymbol{v}\times\boldsymbol{B}_0)\cdot\frac{\partial}{\partial \boldsymbol{v}} &= (\boldsymbol{v}\times\boldsymbol{\Omega})\cdot\left(\frac{\boldsymbol{I}\times\boldsymbol{e}_\parallel}{\Omega}\cdot\frac{\partial}{\partial \boldsymbol{X}}+\boldsymbol{v}\frac{\partial}{\partial\varepsilon}+\boldsymbol{v}_\perp\frac{\partial}{B_0\partial\mu}+\frac{\boldsymbol{e}_\alpha}{v_\perp}\frac{\partial}{\partial\alpha}\right) \\ &= (\boldsymbol{v}\times\boldsymbol{\Omega})\cdot\left(\frac{\boldsymbol{I}\times\boldsymbol{e}_\parallel}{\Omega}\cdot\frac{\partial}{\partial \boldsymbol{X}}+\frac{\boldsymbol{e}_\alpha}{v_\perp}\frac{\partial}{\partial\alpha}\right) \\ &= (\boldsymbol{v}\times\boldsymbol{e}_\parallel)\cdot\left[(\boldsymbol{I}\times\boldsymbol{e}_\parallel)\cdot\frac{\partial}{\partial \boldsymbol{X}}\right]+(\boldsymbol{v}\times\boldsymbol{\Omega})\cdot\frac{\boldsymbol{e}_\alpha}{v_\perp}\frac{\partial}{\partial\alpha} \\ &= -\boldsymbol{v}_\perp\cdot\frac{\partial}{\partial \boldsymbol{X}_\perp}-\Omega\frac{\partial}{\partial\alpha} \end{aligned} \tag{26}$$

Using Eqs. (24), (25) and (26), the unperturbed Vlasov propagator, $\mathcal{F}$, is written as

$$\mathcal{F} = \frac{\partial}{\partial t} + v_{\|}\frac{\partial}{\partial X_{\|}} - \Omega\frac{\partial}{\partial \alpha} \equiv \mathcal{F}_g. \tag{27}$$

The equilibrium equation

$$\mathcal{F}_g f_{0g} = 0, \tag{28}$$

then reduces to (since equilibrium distribution function is independent of time and spatial location)

$$-\Omega\frac{\partial}{\partial \alpha} f_{0g} = 0, \tag{29}$$

Here, the subscript g (standing for guiding-center) denotes a quantity of guiding-cener variables, $(\boldsymbol{X}, \boldsymbol{V})$. The solution to Eq. (29) is obvious, i.e.,

$$f_{0g} = f_{0g}(\varepsilon, \mu, \sigma), \tag{30}$$

or equivalently, in term of the usual coordinators, the equilibrium distribution function is written as

$$f_0 = f_0(v_\perp, v_\|). \tag{31}$$

The linearized Vlasov equation in guiding-center coordinates is written as

$$\begin{aligned}\left(\frac{\partial}{\partial t} + v_{\|}\frac{\partial}{\partial X_{\|}} - \Omega\frac{\partial}{\partial \alpha}\right)\delta f_g &= -\frac{q}{m}\left(\delta\boldsymbol{E} + \frac{\boldsymbol{v}\times\delta\boldsymbol{B}}{c}\right)\cdot\frac{\partial}{\partial \boldsymbol{v}} f_0 \\ &= -\frac{q}{m}\left(\delta\boldsymbol{E}_g + \frac{\boldsymbol{v}\times\delta\boldsymbol{B}_g}{c}\right)\cdot\left(\boldsymbol{v}\frac{\partial}{\partial\varepsilon} + \boldsymbol{v}_\perp\frac{\partial}{B_0\partial\mu}\right) f_{0g}, \end{aligned} \tag{32}$$

where use has been made of that f_0 is independent of $\boldsymbol{X}$ and α.

3 Solution to the linearized equation in the electrostatic limit

In the electrostatic limit, we have

$$\delta\boldsymbol{E} = -\frac{\partial\delta\phi}{\partial\boldsymbol{x}} \tag{33}$$

$$= -\frac{\partial\delta\phi_g}{\partial\boldsymbol{X}} \tag{34}$$

then the linearized Vlasov equation in guiding-center coordinators is written as

$$\mathcal{F}_g\delta f_g = \frac{q}{m}\frac{\partial\delta\phi_g}{\partial\boldsymbol{X}}\cdot\left(\boldsymbol{v}\frac{\partial}{\partial\varepsilon} + \boldsymbol{v}_\perp\frac{\partial}{B_0\partial\mu}\right) f_{0g}(\varepsilon, \mu, \sigma), \tag{35}$$

which can be arranged into the form

$$\mathcal{F}_g\delta f_g = \frac{q}{m}\frac{\partial\delta\phi_g}{\partial\boldsymbol{X}}\cdot\left[\boldsymbol{v}_\|\frac{\partial}{\partial\varepsilon} + \boldsymbol{v}_\perp\left(\frac{\partial}{\partial\varepsilon} + \frac{\partial}{B_0\partial\mu}\right)\right] f_{0g} \tag{36}$$

Noting that

$$\frac{\partial\delta\phi}{\partial\boldsymbol{v}} = 0, \tag{37}$$

Transforming to guiding-center coordinates, Eq. (37) is written as

$$\left(\frac{\boldsymbol{I}\times\boldsymbol{e}_\|}{\Omega}\cdot\frac{\partial}{\partial\boldsymbol{X}} + \boldsymbol{v}\frac{\partial}{\partial\varepsilon} + \boldsymbol{v}_\perp\frac{\partial}{B_0\partial\mu} + \frac{\boldsymbol{e}_\alpha}{v_\perp}\frac{\partial}{\partial\alpha}\right)\delta\phi_g = 0. \tag{38}$$

Dotting the above equation by $\boldsymbol{e}_\alpha$, and noting that $\boldsymbol{e}_\alpha\cdot\boldsymbol{v}_\perp = 0$ and $\boldsymbol{e}_\alpha\cdot\boldsymbol{v} = 0$, we obtain

$$\boldsymbol{v}_\perp\cdot\frac{\partial\phi_g}{\partial\boldsymbol{X}} = -\Omega\frac{\partial\delta\phi_g}{\partial\alpha}. \tag{39}$$

Using this in the r.h.s of Eq. (36) gives

$$\mathcal{F}_g \delta f_g = \frac{q}{m}\left[\frac{\partial \delta\phi_g}{\partial \boldsymbol{X}} \cdot \boldsymbol{v}_{\|} \frac{\partial}{\partial \varepsilon} - \Omega \frac{\partial \delta\phi_g}{\partial \alpha}\left(\frac{\partial}{\partial \varepsilon} + \frac{\partial}{B_0 \partial \mu}\right)\right] f_{0g} \tag{40}$$

Following Chen's book, to make contact with the low-frequency limit, we would like to remove the $\partial/\partial\alpha$ terms in r.h.s of the above equation. Thus we let

$$\delta f_g = \frac{q}{m}\delta\phi_g\left(\frac{\partial}{\partial \varepsilon} + \frac{\partial}{B_0 \partial \mu}\right) f_{0g} + \delta G_g \tag{41}$$

then substitute this into Eq. (40) gives a equation for δG_g

$$\mathcal{F}_g\left[\frac{q}{m}\delta\phi_g\left(\frac{\partial}{\partial \varepsilon} + \frac{\partial}{B_0 \partial \mu}\right) f_{0g}\right] + \mathcal{F}_g \delta G_g = \frac{q}{m}\left[\frac{\partial \delta\phi_g}{\partial \boldsymbol{X}} \cdot \boldsymbol{v}_{\|} \frac{\partial}{\partial \varepsilon} - \Omega \frac{\partial \delta\phi_g}{\partial \alpha}\left(\frac{\partial}{\partial \varepsilon} + \frac{\partial}{B_0 \partial \mu}\right)\right] f_{0g}. \tag{42}$$

Using $\mathcal{F}_g\left(\frac{\partial}{\partial \varepsilon} + \frac{\partial}{B_0 \partial \mu}\right) f_{0g} = 0$, the above equation is reduced to

$$\left(\frac{\partial}{\partial \varepsilon} + \frac{\partial}{B_0 \partial \mu}\right) f_{0g} \mathcal{F}_g\left(\frac{q}{m}\delta\phi_g\right) + \mathcal{F}_g \delta G_g = \frac{q}{m}\left[\frac{\partial \delta\phi_g}{\partial \boldsymbol{X}} \cdot \boldsymbol{v}_{\|} \frac{\partial}{\partial \varepsilon} - \Omega \frac{\partial \delta\phi_g}{\partial \alpha}\left(\frac{\partial}{\partial \varepsilon} + \frac{\partial}{B_0 \partial \mu}\right)\right] f_{0g}, \tag{43}$$

Using the form of $\mathcal{F}_g$ given by Eq. (27) in the above equation gives

$$\left(\frac{\partial}{\partial \varepsilon} + \frac{\partial}{B_0 \partial \mu}\right) f_{0g}\left(\frac{\partial}{\partial t} + v_{\|}\frac{\partial}{\partial X_{\|}} - \Omega \frac{\partial}{\partial \alpha}\right)\left(\frac{q}{m}\delta\phi_g\right) + \mathcal{F}_g \delta G_g = \frac{q}{m}\left[\frac{\partial \delta\phi_g}{\partial \boldsymbol{X}} \cdot \boldsymbol{v}_{\|} \frac{\partial}{\partial \varepsilon} - \Omega \frac{\partial \delta\phi_g}{\partial \alpha}\left(\frac{\partial}{\partial \varepsilon} + \frac{\partial}{B_0 \partial \mu}\right)\right] f_{0g},$$

which can be simplified to

$$\left(\frac{\partial}{\partial \varepsilon} + \frac{\partial}{B_0 \partial \mu}\right) f_{0g}\left[\frac{q}{m}\left(\frac{\partial}{\partial t} + v_{\|}\frac{\partial}{\partial X_{\|}}\right)\delta\phi_g\right] + \mathcal{F}_g \delta G_g = \frac{q}{m}\left(\frac{\partial \delta\phi_g}{\partial \boldsymbol{X}} \cdot \boldsymbol{v}_{\|} \frac{\partial}{\partial \varepsilon}\right) f_{0g},$$

$$\Longrightarrow \mathcal{F}_g \delta G_g = \frac{q}{m}\left\{\left(\frac{\partial \delta\phi_g}{\partial \boldsymbol{X}} \cdot \boldsymbol{v}_{\|} \frac{\partial}{\partial \varepsilon}\right) f_{0g} - \left[\left(\frac{\partial}{\partial t} + v_{\|}\frac{\partial}{\partial X_{\|}}\right)\delta\phi_g\right]\left(\frac{\partial}{\partial \varepsilon} + \frac{\partial}{B_0 \partial \mu}\right) f_{0g}\right\}$$

$$\Longrightarrow \mathcal{F}_g \delta G_g = -\frac{q}{m}\left[\frac{\partial f_{0g}}{\partial \varepsilon}\frac{\partial \delta\phi_g}{\partial t} + \frac{\partial f_{0g}}{B_0 \partial \mu}\left(\frac{\partial}{\partial t} + v_{\|}\frac{\partial}{\partial X_{\|}}\right)\delta\phi_g\right], \tag{44}$$

which agrees with Eq. (III.2.7) in Chen's book[1].

Noting that δG_g must be a periodic function in the gyrophase angle α, i.e.,

$$\delta G_g(\boldsymbol{X}, \mu, \varepsilon, \alpha + 2\pi, \sigma, t) = \delta G_g(\boldsymbol{X}, \mu, \varepsilon, \alpha, \sigma, t), \tag{45}$$

thus, δG_g can be expressed as

$$\delta G_g = \sum_{n=-\infty}^{n=+\infty} \langle \delta G_g \rangle_n \exp(-in\alpha), \tag{46}$$

where $\langle \delta G_g \rangle_n$ is independent of α. Similarly, $\delta\phi_g(\boldsymbol{X}, \varepsilon, \mu, \sigma, \alpha, t)$ must be a periodic function in the gyrophase angle α, thus, $\delta\phi_g$ can also be expressed as

$$\delta\phi_g = \sum_{n=-\infty}^{n=+\infty} \langle \delta\phi_g \rangle_n \exp(-in\alpha), \tag{47}$$

where $\langle \delta\phi_g \rangle_n$ is independent of α. Substituting the above expressions for δG_g and $\delta\phi_g$ into Eq. (44), yields the following equation for $\langle \delta G_g \rangle_n$

$$\mathcal{F}_{\mathrm{gn}}\langle \delta G_g \rangle_n \equiv \left(\frac{\partial}{\partial t} + v_{\|}\frac{\partial}{\partial X_{\|}} + i\Omega n\right)\langle \delta G_g \rangle_n = -\frac{q}{m}\left\{\frac{\partial f_{0g}}{\partial \varepsilon}\frac{\partial}{\partial t} + \frac{\partial f_{0g}}{B_0 \partial \mu}\left(\frac{\partial}{\partial t} + v_{\|}\frac{\partial}{\partial X_{\|}}\right)\right\}\langle \delta\phi_g \rangle_n. \tag{48}$$

(In obtaining the above equation, we have made use of the fact that different n harmonics are not coupled.) We note in passing that

$$\frac{1}{2\pi}\int_0^{2\pi}\delta G_g=\langle\delta G_g\rangle_0. \quad (49)$$

Eq. (48) is similar to the unmagnetic case, hence can be readily solved by Laplace transformation in time and Fourier in space. Using the following notation

$$\delta\hat{A}_g(\omega,\boldsymbol{k})\equiv L_p(t)F_r(\boldsymbol{X})[\delta A_g(\boldsymbol{X},t)] \quad (50)$$

Eq. (48) is solved to give

$$\left\langle\delta\hat{G}_g\right\rangle_n=\frac{1}{\omega-k_\parallel v_\parallel-\Omega n}\left(-\frac{q}{m}\right)\left\{\frac{\partial f_{0g}}{\partial\varepsilon}\omega+\frac{\partial f_{0g}}{B_0\partial\mu}(\omega-k_\parallel v_\parallel)\right\}\left\langle\delta\hat{\phi}_g\right\rangle_n. \quad (51)$$

Now, in order to make contact with later discussions on nonuniform plasmas where $v_\parallel$ is not a constant of the motion due to varying $\boldsymbol{B}_0$, we want to remove the parallel propagator, $\partial/\partial X_\parallel$, from the r.h.s of Eq. (44). Thus, we furhter write

$$\delta G_g=-\frac{q}{m}\delta\phi_g\frac{\partial f_{0g}}{B_0\mathrm{d}\mu}+\delta h_g. \quad (52)$$

Substituting this into Eq. (44) yields an equation for δh_g

$$\left(\frac{\partial}{\partial t}+v_\parallel\frac{\partial}{\partial X_\parallel}-\Omega\frac{\partial}{\partial\alpha}\right)\left(-\frac{q}{m}\delta\phi_g\frac{\partial f_{0g}}{B_0\mathrm{d}\mu}\right)+\mathcal{F}_g\delta h_g=-\frac{q}{m}\left\{\frac{\partial\delta\phi_g}{\partial t}\frac{\partial f_{0g}}{\partial\varepsilon}+\left[\left(\frac{\partial}{\partial t}+v_\parallel\frac{\partial}{\partial X_\parallel}\right)\delta\phi_g\right]\frac{\partial f_{0g}}{B_0\partial\mu}\right\},$$

$$\Rightarrow\left(-\frac{q}{m}\frac{\partial f_{0g}}{B_0\mathrm{d}\mu}\right)\left(\frac{\partial}{\partial t}+v_\parallel\frac{\partial}{\partial X_\parallel}-\Omega\frac{\partial}{\partial\alpha}\right)\delta\phi_g+\mathcal{F}_g\delta h_g=-\frac{q}{m}\left\{\frac{\partial f_{0g}}{\partial\varepsilon}\frac{\partial\delta\phi_g}{\partial t}+\frac{\partial f_{0g}}{B_0\partial\mu}\left(\frac{\partial}{\partial t}+v_\parallel\frac{\partial}{\partial X_\parallel}\right)\delta\phi_g\right\},$$

$$\Rightarrow\left(-\frac{q}{m}\frac{\partial f_{0g}}{B_0\mathrm{d}\mu}\right)\left(-\Omega\frac{\partial}{\partial\alpha}\right)\delta\phi_g+\mathcal{F}_g\delta h_g=-\frac{q}{m}\left\{\frac{\partial f_{0g}}{\partial\varepsilon}\frac{\partial\delta\phi_g}{\partial t}\right\},$$

$$\Rightarrow\mathcal{F}_g\delta h_g=-\frac{q}{m}\left\{\frac{\partial\delta\phi_g}{\partial t}\frac{\partial f_{0g}}{\partial\varepsilon}+\Omega\frac{\partial\delta\phi_g}{\partial\alpha}\frac{\partial f_{0g}}{B_0\partial\mu}\right\}, \quad (53)$$

Then, for the nth harmonic in α, we obtain

$$\left(\frac{\partial}{\partial t}+v_\parallel\frac{\partial}{\partial X_\parallel}+in\Omega\right)\langle\delta h_g\rangle_n=-\frac{q}{m}\left\{\frac{\partial f_{0g}}{\partial\varepsilon}\frac{\partial}{\partial t}-in\Omega\frac{\partial f_{0g}}{B_0\mathrm{d}\mu}\right\}\langle\delta\phi_g\rangle_n, \quad (54)$$

which further gives (Laplace in time and Fourier in space)

$$\left\langle\delta\hat{h}_g\right\rangle_n=-\frac{q}{m}\frac{1}{\omega-k_\parallel v_\parallel-n\Omega}\left(\frac{\partial f_{0g}}{\partial\varepsilon}\omega+n\Omega\frac{\partial f_{0g}}{B_0\mathrm{d}\mu}\right)\left\langle\delta\hat{\phi}_g\right\rangle_n. \quad (55)$$

Using Eqs. (41) and (52), δf_g can be expressed in terms of δh_g as

$$\delta f_g=\frac{q}{m}\delta\phi_g\frac{\partial}{\partial\varepsilon}f_{0g}+\delta h_g, \quad (56)$$

which can be further written as

$$\delta f_g=\frac{q}{m}\delta\phi_g\frac{\partial f_{0g}}{\partial\varepsilon}+\sum_{n=-\infty}^{\infty}\langle\delta h_g\rangle_n\exp(-in\alpha). \quad (57)$$

Laplace transforming in time and Fourier transforming in space, the above equation is written as

$$\delta\hat{f}_g=\frac{q}{m}\delta\hat{\phi}_g\frac{\partial f_{0g}}{\partial\varepsilon}+\sum_{n=-\infty}^{\infty}\left\langle\delta\hat{h}_g\right\rangle_n\exp(-in\alpha). \quad (58)$$

Substituting $\left\langle \delta\hat{h}_g \right\rangle_n$ given by Eq. (55) into the obve equation, gives

$$\delta\hat{f}_g = \frac{q}{m}\delta\hat{\phi}_g\frac{\partial f_{0g}}{\partial\varepsilon} - \frac{q}{m}\sum_{n=-\infty}^{\infty}\frac{1}{\omega - k_\parallel v_\parallel - n\Omega}\left(\frac{\partial f_{0g}}{\partial\varepsilon}\omega + n\Omega\frac{\partial f_{0g}}{B_0 \mathrm{d}\mu}\right)\left\langle\delta\hat{\phi}_g\right\rangle_n \exp(-in\alpha), \tag{59}$$

For the electromagnetic field, we have $\delta A = \delta A(\boldsymbol{x},\, t)$. Transforming to guiding-center coordinates, δA_g

$$\delta A(\boldsymbol{x}, t) = \delta A_g(\boldsymbol{X}, \boldsymbol{V}, t). \tag{60}$$

Note that δA_g depends on $\boldsymbol{V}$. We now derive the relation between $\delta\hat{A}$ and $\delta\hat{A}_g$,

$$\begin{aligned} \delta A \equiv L_p^{-1}(\omega)F_r^{-1}(\boldsymbol{k})\delta\hat{A} &= \int\frac{d\omega}{2\pi}\int\frac{d^3\boldsymbol{k}}{(2\pi)^3}\delta\hat{A}(\omega,\boldsymbol{k})\exp(-i\omega t + i\boldsymbol{k}\cdot\boldsymbol{x}) \\ &= \delta A_g = L_p^{-1}(\omega)F_r^{-1}(\boldsymbol{k})\delta\hat{A}_g \\ &= \int\frac{d\omega}{2\pi}\int\frac{d^3\boldsymbol{k}}{(2\pi)^3}\delta\hat{A}_g(\omega,\boldsymbol{k},\boldsymbol{V})\exp(-i\omega t + i\boldsymbol{k}\cdot\boldsymbol{X}) \end{aligned} \tag{61}$$

Using

$$\boldsymbol{X} = \boldsymbol{x} + \frac{\boldsymbol{v}\times\boldsymbol{e}_\parallel}{\Omega}, \tag{62}$$

we obtain

$$\delta\hat{A}_g \exp(iL_k) = \delta\hat{A}, \tag{63}$$

where

$$L_k = \boldsymbol{k}\cdot\frac{\boldsymbol{v}\times\boldsymbol{e}_\parallel}{\Omega} \tag{64}$$

Without any loss of generality, we define $\boldsymbol{k} = k_\perp\boldsymbol{e}_1 + k_\parallel\boldsymbol{e}_\parallel$, then we have $L_k = \lambda\sin\alpha$, where $\lambda = k_\perp v_\perp/\Omega$, α is the gyrophase angle, which is defined as the included angle between $\boldsymbol{v}_\perp$ and $\boldsymbol{e}_1$. Using the identity

$$\exp(\pm i\lambda\sin\alpha) = \sum_{n=-\infty}^{\infty} J_n(\lambda)\exp(\pm in\alpha) \tag{65}$$

in Eq. (63) gives

$$\begin{aligned} \delta\hat{A}_g &= \delta\hat{A}\exp(-iL_k) \\ &= \delta\hat{A}\exp(-i\lambda\sin\alpha) \\ &= \sum_{n=-\infty}^{\infty}\delta\hat{A}J_n(\lambda)\exp(-in\alpha) \end{aligned} \tag{66}$$

For the electrical potential, the above equation is written as

$$\delta\hat{\phi}_g = \sum_{n=-\infty}^{\infty}\delta\hat{\phi}J_n(\lambda)\exp(-in\alpha) \tag{67}$$

Comparing the above equation with Eq. (47), we obtain

$$\left\langle\delta\hat{\phi}_g\right\rangle_n = \delta\hat{\phi}J_n(\lambda). \tag{68}$$

Using this in Eq. (59) gives

$$\delta\hat{f}_g = \frac{q}{m}\delta\hat{\phi}_g\frac{\partial f_{0g}}{\partial\varepsilon} - \frac{q}{m}\sum_{n=-\infty}^{\infty}\frac{1}{\omega - k_\parallel v_\parallel - n\Omega}\left(\frac{\partial f_{0g}}{\partial\varepsilon}\omega + n\Omega\frac{\partial f_{0g}}{B_0 \mathrm{d}\mu}\right)\delta\hat{\phi}J_n(\lambda)\exp(-in\alpha), \tag{69}$$

Using Eq. (67), the above equation is written as

$$\delta\hat{f}_g = \frac{q}{m}\delta\hat{\phi}\frac{\partial f_{0g}}{\partial\varepsilon}\sum_{n=-\infty}^{\infty}J_n(\lambda)\exp(-in\alpha) - \frac{q}{m}\delta\hat{\phi}\sum_{n=-\infty}^{\infty}\frac{1}{\omega - k_\parallel v_\parallel - n\Omega}\left(\frac{\partial f_{0g}}{\partial\varepsilon}\omega + n\Omega\frac{\partial f_{0g}}{B_0 \mathrm{d}\mu}\right)J_n(\lambda)\exp(-in\alpha), \tag{70}$$

Using that $\delta\hat{f}_g\exp(i\lambda\sin\alpha) = \delta\hat{f}$, Eq. (70) is written as

$$\begin{aligned}\delta\hat{f} &= \frac{q}{m}\delta\hat{\phi}\frac{\partial f_{0g}}{\partial\varepsilon}\sum_{n=-\infty}^{\infty} J_n(\lambda)\exp(-in\alpha+i\lambda\sin\alpha)\\ &- \frac{q}{m}\delta\hat{\phi}\sum_{n=-\infty}^{\infty}\frac{1}{\omega-k_\parallel v_\parallel-n\Omega}\left(\frac{\partial f_{0g}}{\partial\varepsilon}\omega+n\Omega\frac{\partial f_{0g}}{B_0\mathrm{d}\mu}\right)J_n(\lambda)\exp(-in\alpha+i\lambda\sin\alpha)\end{aligned} \tag{71}$$

Define the gyrophase average

$$\left\langle\delta\hat{f}\right\rangle_O = \frac{1}{2\pi}\int_0^{2\pi} d\alpha\delta\hat{f}_g, \tag{72}$$

then Eq. (71) can be integrated to give

$$\begin{aligned}\left\langle\delta\hat{f}\right\rangle_O &= \frac{q}{m}\delta\hat{\phi}\frac{\partial f_{0g}}{\partial\varepsilon}\sum_{n=-\infty}^{\infty} J_n(\lambda)\frac{1}{2\pi}\int_0^{2\pi}\exp(-in\alpha+i\lambda\sin\alpha)d\alpha\\ &- \frac{q}{m}\delta\hat{\phi}\sum_{n=-\infty}^{\infty}\frac{1}{\omega-k_\parallel v_\parallel-n\Omega}\left(\frac{\partial f_{0g}}{\partial\varepsilon}\omega+n\Omega\frac{\partial f_{0g}}{B_0\mathrm{d}\mu}\right)J_n(\lambda)\frac{1}{2\pi}\int_0^{2\pi}\exp(-in\alpha+\\ &\quad i\lambda\sin\alpha)d\alpha\end{aligned} \tag{73}$$

We note that (the integral representation of the Bessel function)

$$J_n(\lambda) = \frac{1}{2\pi}\int_0^{2\pi} e^{-i(n\alpha-\lambda\sin\alpha)}d\alpha, \tag{74}$$

then, Eq. (73) is written as

$$\left\langle\delta\hat{f}\right\rangle_O = \frac{q}{m}\delta\hat{\phi}\frac{\partial f_{0g}}{\partial\varepsilon}\sum_{n=-\infty}^{\infty}J_n^2(\lambda) - \frac{q}{m}\delta\hat{\phi}\sum_{n=-\infty}^{\infty}\frac{1}{\omega-k_\parallel v_\parallel-n\Omega}\left(\frac{\partial f_{0g}}{\partial\varepsilon}\omega+n\Omega\frac{\partial f_{0g}}{B_0\mathrm{d}\mu}\right)J_n^2(\lambda). \tag{75}$$

Using $\sum_{n=-\infty}^{\infty}J_n^2(\lambda)=1$, the above equation is written

$$\left\langle\delta\hat{f}\right\rangle_O = \frac{q}{m}\delta\hat{\phi}\left[\frac{\partial f_{0g}}{\partial\varepsilon} - \sum_{n=-\infty}^{\infty}\frac{J_n^2(\lambda)}{\omega-k_\parallel v_\parallel-n\Omega}\left(\frac{\partial f_{0g}}{\partial\varepsilon}\omega+n\Omega\frac{\partial f_{0g}}{B_0\mathrm{d}\mu}\right)\right], \tag{76}$$

which agrees with Eq. (III.2.23) in Chen's book[1].

3.1 Dispersion relation

Poisson's equation is

$$\nabla^2\delta\phi = -\frac{1}{\varepsilon_0}\sum_j q_j\delta n_j. \tag{77}$$

Laplace in time and Fourier in space, we obtain

$$-k^2\delta\hat{\phi} = -\frac{1}{\varepsilon_0}\sum_j q_j\delta\hat{n}_j, \tag{78}$$

where the perturbed density is given by

$$\begin{aligned}\hat{n}_j &= \int\hat{f}_j d^3\boldsymbol{v}\\ &= \sum_\sigma\int\hat{f}_j\frac{B_0 d\varepsilon d\mu}{|v_\parallel|}d\alpha\\ &= \sum_\sigma 2\pi\int\frac{B_0 d\varepsilon d\mu}{|v_\parallel|}\left\langle\delta\hat{f}_j\right\rangle_O\end{aligned} \tag{79}$$

Using Eqs. (76) and (79), Eq. (78) is written as

$$-k^2\delta\hat{\phi} = -2\pi\sum_j\frac{1}{\varepsilon_0}\frac{q_j^2}{m_j}\delta\hat{\phi}\sum_\sigma\int\frac{B_0 d\varepsilon d\mu}{|v_\parallel|}\left[\frac{\partial f_{0g}}{\partial\varepsilon} - \sum_{n=-\infty}^{\infty}\frac{J_n^2(\lambda)}{\omega-k_\parallel v_\parallel-n\Omega}\left(\frac{\partial f_{0g}}{\partial\varepsilon}\omega+n\Omega\frac{\partial f_{0g}}{B_0\mathrm{d}\mu}\right)\right], \tag{80}$$

from which we obtain the dispersion relation

$$D_{e.s} = 1 + \sum_j \chi_j = 0 \tag{81}$$

where χ_j, the jth-species susceptibility, is given by

$$\begin{aligned}
\chi_j &= -2\pi \frac{1}{\varepsilon_0} \frac{q_j^2}{m_j} \frac{1}{k^2} \sum_\sigma \int \frac{B_0 d\varepsilon d\mu}{|v_\parallel|} \left[\frac{\partial f_{0g}}{\partial \varepsilon} - \sum_{n=-\infty}^{\infty} \frac{J_n^2(\lambda)}{\omega - k_\parallel v_\parallel - n\Omega} \left(\frac{\partial f_{0g}}{\partial \varepsilon} \omega + n\Omega \frac{\partial f_{0g}}{B_0 \mathrm{d}\mu} \right) \right] \\
&= -2\pi \frac{\omega_{pj}^2}{k^2} \frac{1}{n_j} \sum_\sigma \int \frac{B_0 d\varepsilon d\mu}{|v_\parallel|} \left[\frac{\partial f_{0g}}{\partial \varepsilon} - \sum_{n=-\infty}^{\infty} \frac{J_n^2(\lambda)}{\omega - k_\parallel v_\parallel - n\Omega} \left(\frac{\partial f_{0g}}{\partial \varepsilon} \omega + n\Omega \frac{\partial f_{0g}}{B_0 \mathrm{d}\mu} \right) \right],
\end{aligned}$$

where $\omega_{pj}^2 = n_j e^2 / m_j \varepsilon_0$, n_j is the equilibrium density of the jth-species. Using $\sum_{n=-\infty}^{\infty} J_n^2(\lambda) = 1$, the above equation can also be written as

$$\begin{aligned}
\chi_j &= 2\pi \frac{\omega_{pj}^2}{k^2} \frac{1}{n_j} \sum_\sigma \int \frac{B_0 d\varepsilon d\mu}{|v_\parallel|} \left[-\sum_n J_n^2 \frac{\partial f_{0g}}{\partial \varepsilon} + \sum_n \frac{J_n^2(\lambda)}{\omega - k_\parallel v_\parallel - n\Omega} \left(\frac{\partial f_{0g}}{\partial \varepsilon} \omega + n\Omega \frac{\partial f_{0g}}{B_0 \mathrm{d}\mu} \right) \right] \\
&= 2\pi \frac{\omega_{pj}^2}{k^2} \frac{1}{n_j} \sum_\sigma \int \frac{B_0 d\varepsilon d\mu}{|v_\parallel|} \left[\sum_n \frac{-\omega + k_\parallel v_\parallel + n\Omega}{\omega - k_\parallel v_\parallel - n\Omega} J_n^2 \frac{\partial f_{0g}}{\partial \varepsilon} + \sum_n \frac{J_n^2(\lambda)}{\omega - k_\parallel v_\parallel - n\Omega} \left(\frac{\partial f_{0g}}{\partial \varepsilon} \omega + \right.\right. \\
&\quad \left.\left. n\Omega \frac{\partial f_{0g}}{B_0 \mathrm{d}\mu} \right) \right] \\
&= 2\pi \frac{\omega_{pj}^2}{k^2} \frac{1}{n_j} \sum_\sigma \int \frac{B_0 d\varepsilon d\mu}{|v_\parallel|} \sum_n \frac{J_n^2(\lambda)}{\omega - k_\parallel v_\parallel - n\Omega} \left[(k_\parallel v_\parallel + n\Omega) \frac{\partial f_{0g}}{\partial \varepsilon} + n\Omega \frac{\partial f_{0g}}{B_0 \mathrm{d}\mu} \right],
\end{aligned} \tag{82}$$

which agrees with Eq. (III.2.26) in Chen's book[1].

4 Kinetic theory of low-frequency Magnetohydrodynamic waves

According to Eqs. (41) and (44), we have

$$\delta f_g = \frac{q}{m} \delta \phi_g \left(\frac{\partial}{\partial \varepsilon} + \frac{\partial}{B_0 \partial \mu} \right) f_{0g} + \delta G_g, \tag{83}$$

and

$$\left(\frac{\partial}{\partial t} + v_\parallel \frac{\partial}{\partial X_\parallel} - \Omega \frac{\partial}{\partial \alpha} \right) \delta G_g = -\frac{q}{m} \left\{ \frac{\partial \delta \phi_g}{\partial t} \frac{\partial f_{0g}}{\partial \varepsilon} + \left[\left(\frac{\partial}{\partial t} + v_\parallel \frac{\partial}{\partial X_\parallel} \right) \delta \phi_g \right] \frac{\partial f_{0g}}{B_0 \partial \mu} \right\}. \tag{84}$$

We consider $\Omega \partial / \partial \alpha$ term to be at the fastest times scale, at the order of $O(1)$, all other terms, namely, $\partial / \partial t$ and $v_\parallel \partial / \partial X_\parallel$, are at the order of $O(\eta)$, where η is a small parameter. Expanding δG_g as

$$\delta G_g = \delta G_{g0} + \delta G_{g1} + \ldots, \tag{85}$$

where $\delta G_{gn} \sim O(\eta^n)$. Similarly for $\delta \phi_g$, i.e.,

$$\delta \phi_g = \delta \phi_{g0} + \delta \phi_{g1} + \ldots. \tag{86}$$

Substituting the expansions into Eq. (84), we have, to the order of $O(1)$,

$$\frac{\partial}{\partial \alpha} \delta G_{g0} = 0, \tag{87}$$

which indicates δG_{g0} is independent of gyrophase α. To the next order, $O(\eta)$, we have

$$\left(\frac{\partial}{\partial t} + v_\parallel \frac{\partial}{\partial X_\parallel} \right) \delta G_{g0} - \Omega \frac{\partial}{\partial \alpha} \delta G_{g1} = -\frac{q}{m} \left\{ \frac{\partial \delta \phi_{g0}}{\partial t} \frac{\partial f_{0g}}{\partial \varepsilon} + \left[\left(\frac{\partial}{\partial t} + v_\parallel \frac{\partial}{\partial X_\parallel} \right) \delta \phi_{g0} \right] \frac{\partial f_{0g}}{B_0 \partial \mu} \right\}. \tag{88}$$

Making use of the periodicity of δG_{g1} in gyrophase α, we can average the above equation over gyrophase to eliminate the last term on the l.h.s, which gives (noting that δG_{g0} is independent of α)

$$\left(\frac{\partial}{\partial t}+v_{\parallel}\frac{\partial}{\partial X_{\parallel}}\right)\delta G_{g0}=\langle r.h.s\rangle_O, \tag{89}$$

where

$$\begin{aligned}\langle r.h.s\rangle_O &\equiv \frac{1}{2\pi}\int_0^{2\pi} d\alpha(r.h.s)\\ &= -\frac{q}{m}\frac{1}{2\pi}\int_0^{2\pi} d\alpha\left\{\frac{\partial\delta\phi_{g0}}{\partial t}\frac{\partial f_{0g}}{\partial\varepsilon}+\left[\left(\frac{\partial}{\partial t}+v_{\parallel}\frac{\partial}{\partial X_{\parallel}}\right)\delta\phi_{g0}\right]\frac{\partial f_{0g}}{B_0\partial\mu}\right\}\\ &= -\frac{q}{m}\left\{\frac{\partial\langle\delta\phi_{g0}\rangle_0}{\partial t}\frac{\partial f_{0g}}{\partial\varepsilon}+\left[\left(\frac{\partial}{\partial t}+v_{\parallel}\frac{\partial}{\partial X_{\parallel}}\right)\langle\delta\phi_{g0}\rangle_0\right]\frac{\partial f_{0g}}{B_0\partial\mu}\right\},\end{aligned} \tag{90}$$

where, in obtaining the last equality, we have used the fact that f_{0g} is independent of α. Also we have used that

$$\frac{1}{2\pi}\int_0^{2\pi}\delta\phi_{g0}d\alpha=\langle\delta\phi_{g0}\rangle_0, \tag{91}$$

where $\langle\delta\phi_{g0}\rangle_0$ is the expansion coefficient when expanding $\delta\phi_{g0}$ as series of α harmonics. Equation (89) along with Eq. (90) gives the electrostatic low-frequency linear grokinetic equation for uniform magnetized plasmas. (The equations agree with Eq .(III.7.7) and Eq. (III.7.8) in Chen's book[1].)

Noting that

$$\left\langle\delta\hat{\phi}_{g0}\right\rangle_0=J_0(\lambda)\hat{\phi}_0 \tag{92}$$

and making Laplace transformation in time and Fourier in space to both sides of Eq. (89), we obtain

$$(-i\omega+\mathrm{i}k_{\parallel}v_{\parallel})\delta\hat{G}_{g0}=-\frac{q}{m}\left\{-i\omega J_0(\lambda)\hat{\phi}_0\frac{\partial f_{0g}}{\partial\varepsilon}+\left[(-i\omega+k_{\parallel}v_{\parallel})J_0(\lambda)\hat{\phi}_0\right]\frac{\partial f_{0g}}{B_0\partial\mu}\right\} \tag{93}$$

$$\Rightarrow\delta\hat{G}_{g0}=-\frac{q}{m}J_0(\lambda)\hat{\phi}_0\left\{\frac{\omega}{\omega-k_{\parallel}v_{\parallel}}\frac{\partial f_{0g}}{\partial\varepsilon}+\frac{\partial f_{0g}}{B_0\partial\mu}\right\} \tag{94}$$

Bibliography

[1] Liu chen. *Wave and Instabilities in Plasmas*. World Scientific Pub Co Inc, 1987.