

Notes on tokamak equilibrium*

BY YOUJUN HU

Institute of Plasma Physics, Chinese Academy of Sciences
Email: yjhu@ipp.cas.cn

Abstract

A routine in operating a tokamak is to reconstruct axisymmetric poloidal magnetic field under the constraints of MHD force balance and various magnetic measurements. This kind of task can be done by various codes, e.g., EFIT. Another routine in analysing the tokamak discharge is to construct a coordinate system, based on the 2D equilibrium magnetic field, with a desired form of Jacobian by using discrete numerical equilibrium data output by the equilibrium reconstructing codes. These are my notes when learning tokamak equilibrium theory.

These notes are evolving and are written for my own record. I have been keeping revising these notes for more than 10 years. I enjoy seeing the continuous improvement of these notes and my understanding of this simple but important stuff in tokamak physics.

1 Axisymmetric magnetic field

Due to the divergence-free nature of magnetic field, i.e., $\nabla \cdot \mathbf{B} = 0$, magnetic field can be expressed as the curl of a vector field,

$$\mathbf{B} = \nabla \times \mathbf{A}, \quad (1.1)$$

where \mathbf{A} is called the vector potential of \mathbf{B} . (Using a vector potential representation is helpful in that we do not need to worry about the condition $\nabla \cdot \mathbf{B} = 0$ once the magnetic field is in the vector potential form.)

In cylindrical coordinates (R, ϕ, Z) , the above expression is written

$$\mathbf{B} = \left(\frac{1}{R} \frac{\partial A_Z}{\partial \phi} - \frac{\partial A_\phi}{\partial Z} \right) \hat{\mathbf{R}} + \left(\frac{1}{R} \frac{\partial(RA_\phi)}{\partial R} - \frac{1}{R} \frac{\partial A_R}{\partial \phi} \right) \hat{\mathbf{Z}} + \left(\frac{\partial A_R}{\partial Z} - \frac{\partial A_Z}{\partial R} \right) \hat{\phi}. \quad (1.2)$$

We consider axisymmetric magnetic field. The axial symmetry means that, when expressed in the cylindrical coordinate system (R, ϕ, Z) , the components of \mathbf{B} , namely B_R , B_Z , and B_ϕ , are all independent of ϕ . For this case, it can be proved that an axisymmetric vector potential \mathbf{A} suffices for expressing the magnetic field, i.e., all the components of the vector potential \mathbf{A} can also be taken independent of ϕ . Using this, Eq. (1.2) is written

$$\mathbf{B} = -\frac{\partial A_\phi}{\partial Z} \hat{\mathbf{R}} + \frac{1}{R} \frac{\partial(RA_\phi)}{\partial R} \hat{\mathbf{Z}} + \left(\frac{\partial A_R}{\partial Z} - \frac{\partial A_Z}{\partial R} \right) \hat{\phi}. \quad (1.3)$$

*. This article has been written using GNU $\text{\TeX}_{\text{MACS}}$ [14].

In tokamak literature, $\hat{\phi}$ direction is called the toroidal direction, and (R, Z) planes (i.e., $\phi = \text{const}$ planes) are called poloidal planes.

1.1 Poloidal magnetic field

Equation (1.3) indicates that the two poloidal components of \mathbf{B} , namely B_R and B_Z , are determined by a single component of \mathbf{A} , namely A_ϕ . This motivates us to define a function $\Psi(R, Z)$:

$$\Psi(R, Z) \equiv R A_\phi(R, Z). \quad (1.4)$$

Then Eq. (1.3) implies the poloidal components, B_R and B_Z , can be written as

$$B_R = -\frac{1}{R} \frac{\partial \Psi}{\partial Z}, \quad (1.5)$$

$$B_Z = \frac{1}{R} \frac{\partial \Psi}{\partial R}. \quad (1.6)$$

(Note that it is the property of being axisymmetric and divergence-free that enables us to express the two components of \mathbf{B} , namely B_R and B_Z , in terms of a single function $\Psi(R, Z)$.) Furthermore, it is ready to prove that Ψ is constant along a magnetic field line, i.e. $\mathbf{B} \cdot \nabla \Psi = 0$. [Proof:

$$\begin{aligned} \mathbf{B} \cdot \nabla \Psi &= \mathbf{B} \cdot \left(\frac{\partial \Psi}{\partial R} \hat{\mathbf{R}} + \frac{\partial \Psi}{\partial Z} \hat{\mathbf{Z}} \right) \\ &= -\frac{1}{R} \frac{\partial \Psi}{\partial Z} \frac{\partial \Psi}{\partial R} + \frac{1}{R} \frac{\partial \Psi}{\partial R} \frac{\partial \Psi}{\partial Z} \\ &= 0. \end{aligned} \quad (1.7)$$

]

The function Ψ is usually called the “poloidal flux function” in tokamak literature because Ψ can be related to the poloidal magnetic flux, as we will discuss in Sec. 1.7.

Using Eqs. (1.5) and (1.6), the poloidal magnetic field \mathbf{B}_p is written as

$$\begin{aligned} \mathbf{B}_p &= B_R \hat{\mathbf{R}} + B_Z \hat{\mathbf{Z}} \\ &= -\frac{1}{R} \frac{\partial \Psi}{\partial Z} \hat{\mathbf{R}} + \frac{1}{R} \frac{\partial \Psi}{\partial R} \hat{\mathbf{Z}} \\ &= \frac{1}{R} \nabla \Psi \times \hat{\phi} \\ &= \nabla \Psi \times \nabla \phi \end{aligned} \quad (1.8)$$

1.2 Toroidal magnetic field

Next, let's examine the toroidal component B_ϕ . Equation (1.3) indicates that B_ϕ involves both A_R and A_Z , which means that using the potential form does not enable useful simplification for B_ϕ . Therefore we will directly use B_ϕ . Define $g \equiv RB_\phi(R, Z)$, (the reason that we define this quantity will become apparent when we discuss the force balance equation) then the toroidal magnetic field is written

$$\mathbf{B}_\phi = B_\phi \hat{\phi} = \frac{g}{R} \hat{\phi} = g \nabla \phi. \quad (1.9)$$

1.3 General form of axisymmetric magnetic field

Combining Eqs. (1.8) and (1.9), we can write a general axisymmetric magnetic field as

$$\begin{aligned} \mathbf{B} &= \mathbf{B}_p + \mathbf{B}_\phi \\ &= \nabla \Psi \times \nabla \phi + g \nabla \phi. \end{aligned} \quad (1.10)$$

Expression (1.10) is a famous formula in tokamak physics.

1.4 Gauge transformation of Ψ

Next, we discuss the gauge transformation of the vector potential \mathbf{A} in the axisymmetric case. As is well known, magnetic field remains the same under the following gauge transformation:

$$\mathbf{A}^{\text{new}} = \mathbf{A} + \nabla f, \quad (1.11)$$

where f is an arbitrary scalar field. Here we require that ∇f be axisymmetric because, as mentioned above, an axisymmetric vector potential suffices for describing an axisymmetric magnetic field. In cylindrical coordinates, ∇f is given by

$$\nabla f = \frac{\partial f}{\partial R} \hat{\mathbf{R}} + \frac{\partial f}{\partial Z} \hat{\mathbf{Z}} + \frac{1}{R} \frac{\partial f}{\partial \phi} \hat{\phi}. \quad (1.12)$$

Since ∇f is axisymmetric, it follows that all the three components of ∇f are independent of ϕ , i.e., $\partial^2 f / \partial R \partial \phi = 0$, $\partial^2 f / \partial Z \partial \phi = 0$, and $\partial^2 f / \partial \phi^2 = 0$, which implies that $\partial f / \partial \phi$ is independent of R , Z , and ϕ , i.e., $\partial f / \partial \phi$ is actually a spatial constant. Using this, the ϕ component of the gauge transformation (1.11)

is written

$$\begin{aligned} A_\phi^{\text{new}} &= A_\phi + \frac{1}{R} \frac{\partial f}{\partial \phi} \\ &= A_\phi + \frac{C}{R}, \end{aligned} \quad (1.13)$$

where C is a constant. Note that the requirement of being axial symmetry greatly reduces the degree of freedom of the gauge transformation for A_ϕ (and thus for RA_ϕ , i.e., Ψ). Multiplying Eq. (1.13) with R , we obtain the corresponding gauge transformation for Ψ ,

$$\Psi^{\text{new}} = \Psi + C, \quad (1.14)$$

which indicates Ψ has the same gauge transformation as the electric potential, i.e., adding a constant. (Note that the definition $\Psi(R, Z) \equiv RA_\phi$ does not imply $\Psi(R=0, Z)=0$ because A_ϕ can adopt $1/R$ dependence under the gauge transformation (1.13)).

1.5 Contours of Ψ in the poloidal plane

Because Ψ is constant along a magnetic field line and Ψ is independent of ϕ , it follows that the projection of a magnetic field line onto (R, Z) plane is a contour of Ψ . On the other hand, are contours of Ψ on (R, Z) plane also the projections of magnetic field lines onto the plane? Yes, they are. [Proof. A contour of Ψ on (R, Z) plane satisfies

$$d\Psi = 0, \quad (1.15)$$

i.e.,

$$\frac{\partial \Psi}{\partial R} dR + \frac{\partial \Psi}{\partial Z} dZ = 0. \quad (1.16)$$

$$\Rightarrow \frac{1}{R} \frac{\partial \Psi}{\partial R} dR + \frac{1}{R} \frac{\partial \Psi}{\partial Z} dZ = 0. \quad (1.17)$$

Using Eqs. (1.5) and (1.6), the above equation is written

$$B_Z dR - B_R dZ = 0, \quad (1.18)$$

i.e.,

$$\frac{dZ}{dR} = \frac{B_Z}{B_R}, \quad (1.19)$$

which is the equation of the projection of a magnetic field line on (R, Z) plane. Thus, we prove that contours of Ψ are also the projections of magnetic field lines in (R, Z) plane.]

1.6 Magnetic surfaces

For axial symmetry system, magnetic surfaces can be defined in a trivial way. The axial symmetry of tokamak magnetic field allows us to introduce a surface of revolution that is generated by rotating the projection of a magnetic field line on (R, Z) plane around the axis of symmetry, Z axis. The unique property of this revolution surface is that no field line point-intersects it and a field line with one point on it will have the whole field line on it. This revolution surface is called a magnetic surface. For instance, consider an arbitrary magnetic field line, whose projection on the poloidal plane is shown in Fig. 1.1. A magnetic surface is generated by rotating the projection line around the Z axis.

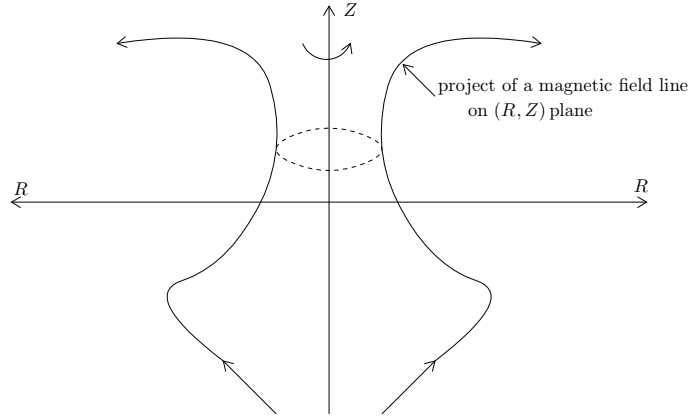


Figure 1.1. A magnetic surface in tokamak is a revolution surface generated by rotating the projection of a magnetic field line in the poloidal plane $[(R, Z)$ plane] around the Z axis. The projection is not necessarily a closed curve in the poloidal plane, as is shown in this figure.

The value of Ψ is constant on a magnetic surface (since Ψ is constant along a magnetic field line and Ψ is independent of ϕ). On the other hand, the value of Ψ is usually different on different magnetic surfaces. The above two facts enable Ψ to be used as labels of magnetic surfaces.

1.7 Relation of Ψ with the poloidal magnetic flux

Note that Ψ is defined by $\Psi = RA_\phi$, which is just a component of the vector potential \mathbf{A} , thereby having no obvious physical meaning. Next, we try to find the physical meaning of Ψ , i.e., try to find some simple algebraic relation of Ψ with some quantity that can be measured in experiments.

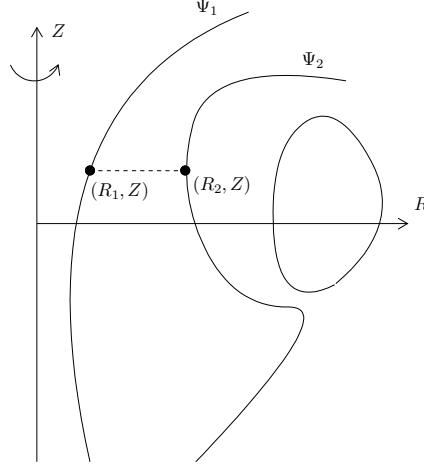


Figure 1.2. The poloidal magnetic flux Ψ_p between the two magnetic surfaces Ψ_1 and Ψ_2 is given by $\Psi_p = 2\pi(\Psi_2 - \Psi_1)$.

In Fig. 1.2, there are two magnetic surfaces labeled, respectively, by $\Psi = \Psi_1$ and $\Psi = \Psi_2$. Using Gauss's theorem in the toroidal volume between the two magnetic surface, we know that the poloidal magnetic flux through any toroidal ribbons between the two magnetic surfaces is equal to each other. Next, we calculate this poloidal magnetic flux. To make the calculation easy, we select a plane perpendicular to the Z axis, as is shown by the dash line in Fig. 1.1. In this case, only B_Z contribute to the poloidal magnetic flux, which is written (the positive direction of the plane is chosen in the direction of $\hat{\mathbf{Z}}$)

$$\begin{aligned}
 \Psi_p &= \int_{R_1}^{R_2} B_z(R, Z) 2\pi R dR \\
 &= \int_{R_1}^{R_2} \frac{1}{R} \frac{\partial \Psi}{\partial R} 2\pi R dR \\
 &= 2\pi \int_{R_1}^{R_2} \frac{\partial \Psi}{\partial R} dR \\
 &= 2\pi[\Psi_2 - \Psi_1].
 \end{aligned} \tag{1.20}$$

Equation (1.20) provides a simple physical meaning for Ψ , i.e., the difference of $2\pi\Psi$ between two magnetic surfaces is equal to the poloidal magnetic flux enclosed by the two magnetic surfaces. Noting that we are considering the axisymmetric case, the physical meaning of Ψ can also be stated as: the difference of Ψ between two magnetic surfaces is equal to the poloidal magnetic flux per radian. Due to this relation, Ψ is usually called the “poloidal magnetic flux function” in tokamak literature. Note that it is the difference of Ψ between two locations that determines the the physical quantities Ψ_p , which is consistent with the fact that gauge transformation (1.14) dose not change the values of physical quantities.

1.8 Measurement of poloidal magnetic flux

By measuring the voltage around a toroidal loop of wire, we can obtain the time derivative of the poloidal magnetic flux through the loop and, after integrating over time, the poloidal magnetic flux itself. Suppose that the loop is located at (R, Z) and denote the magnetic flux through the loop by $\Psi_{\text{pol}}(R, Z, t)$ (only the poloidal magnetic field contribute to this flux since the loop is in the toroidal direction). Then Faraday's law gives

$$\varepsilon = -\frac{d\Psi_{\text{pol}}}{dt}, \quad (1.21)$$

where ε is the emf. If the loop is a coil with N turns, the induced voltage V in the coil is N times the emf ε , i.e., $V = N\varepsilon$. Using this, Eq. (1.21) is written as

$$V = -N\frac{d\Psi_{\text{pol}}}{dt}. \quad (1.22)$$

Integrating the above equation over time, we obtain

$$\Psi_{\text{pol}}(R, Z, t) - \Psi_{\text{pol}}(R, Z, 0) = -\frac{1}{N}\int_0^t V dt. \quad (1.23)$$

The starting time $t=0$ can be chosen at a time when no plasma is present and thus the initial value, $\Psi_{\text{pol}}(R, Z, 0)$, is easy to obtain.

As discussed in Sec. 1.7, the relation between Ψ and Ψ_{pol} is given by

$$\Psi_{\text{pol}}(R, Z, t) = 2\pi[\Psi(R, Z, t) - \Psi(R=0, Z, t)], \quad (1.24)$$

where $\Psi(R=0, Z, t)$ is actually a spatial constant since $R=0$ must be a magnetic field line (required by the axisymmetry). $\Psi(R=0, Z, t)$ is also a time constant if we choose a gauge that is independent of time. We will stick to the time-independent gauge since it does not seem useful to choose a time-dependent gauge for Ψ . Denote the spatial-temporal constant $\Psi(R=0, Z, t)$ by C_g , then Eq. (1.24) is written as

$$\Psi(R, Z, t) = \frac{1}{2\pi}\Psi_{\text{pol}}(R, Z, t) + C_g, \quad (1.25)$$

i.e.,

$$\Psi(R, Z, t) = \frac{1}{2\pi}\left(-\frac{1}{N}\int_0^t V dt + \Psi_{\text{pol}}(R, Z, 0)\right) + C_g, \quad (1.26)$$

which tells us how to obtain Ψ from the measured voltage V .

The above toroidal loop of wire used to measure poloidal magnetic flux is often called “flux loop” by tokamak operators. There are usually many flux loops (e.g. 37 on EAST[30], 41 on DIII-D) at different locations in the poloidal plane (see Fig. 16.1). The measured value of poloidal flux can be used to obtain contours of Ψ and thus obtain a flux surface. With the feedback control (by adjusting the current in the poloidal field coils), these measurement, along with the poloidal field measurement by magnetic probes, can be used to control the shape of the last-closed-flux-surface (LCFS), the distance of the LCFS from the first wall, and the X-point location. This is often called iso-flux control, gap control, or X-point control.

1.9 Closed magnetic surfaces in tokamak

In most part of a tokamak plasma, the contours of Ψ on (R, Z) plane are closed curves. As discussed above, the contours of Ψ are the projection of magnetic lines on the poloidal plane. Closed contours of Ψ implies closed magnetic surfaces, as shown in Fig 1.3.

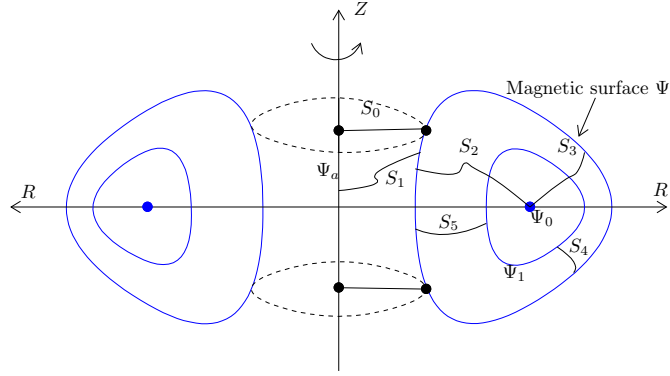


Figure 1.3. Closed magnetic surfaces (blue) and various toroidal surfaces used to define the poloidal magnetic flux. The magnetic flux through the toroidal surface S_2 and S_3 is equal to each other. Also the magnetic flux through S_4 and S_5 is equal to each other; the magnetic flux through S_0 and S_1 is equal to each other.

The innermost magnetic surface reduces to a curve, which is called magnetic axis (in Fig. 1.3, Ψ_0 labels the magnetic axis). Because the magnetic axis is the point of maximum/minimum of $\Psi(R, Z)$, the value of $\nabla\Psi$ is zero at the magnetic axis. As a result, the poloidal component of the equilibrium magnetic field is zero on magnetic axis (refer to Eq. (1.10)), i.e., the magnetic field has only toroidal component there.

As discussed in Sec. 1.7, the poloidal magnetic flux enclosed by a magnetic surface Ψ (the poloidal magnetic flux through the toroidal surfaces S_2) is given by

$$\Psi_p = 2\pi (\Psi_0 - \Psi), \quad (1.27)$$

where Ψ_0 is the value of Ψ at the magnetic axis. Here the positive direction of the surface S_2 is defined to be in the clockwise direction when an observer looks along the direction of $\hat{\phi}$. In practice, we need to pay attention to the positive direction of the toroidal surface used to define the poloidal flux (there can be a sign difference when choosing different positive directions). Also note, in tokamak literature, the poloidal magnetic flux enclosed by a closed magnetic surface can have two different definitions, one of which is the the poloidal magnetic flux through the surface S_2 in Fig. 1.3, the other one is the poloidal magnetic flux through the central hole of the magnetic surface, i.e., the poloidal flux through S_1 in Fig. 1.3. The former definition is adopted in this article, except explicitly specified otherwise. In the latter case, the poloidal magnetic flux is written

$$\Psi_p = 2\pi(\Psi - \Psi_a), \quad (1.28)$$

where the positive direction of the surface S_1 is defined to be in the clockwise direction.

Also note that, since the poloidal magnetic field can be written as $\mathbf{B}_p = \nabla\Psi \times \nabla\phi$, the condition $\Psi_{\text{LCFS}} - \Psi_{\text{axis}} > 0$ means \mathbf{B}_p points in the anticlockwise direction (viewed along ϕ direction), and $\Psi_{\text{LCFS}} - \Psi_{\text{axis}} < 0$ means \mathbf{B}_p points in the clockwise direction.

1.9.1 Shape parameters of a closed magnetic surface

Let us introduce parameters characterizing the shape of a magnetic surface in the poloidal plane. The “midplane” is defined as the plane that passes through the magnetic axis and is perpendicular to the symmetric axis (Z axis). For a up-down symmetric (about the midplane) magnetic surface, its shape can be roughly characterized by four parameters, namely, the R coordinate of the innermost and outermost points in the midplane, R_{in} and R_{out} ; the (R, Z) coordinators of the highest point of the magnetic surface, $(R_{\text{top}}, Z_{\text{top}})$. These four parameters are indicated in Fig. 1.4.

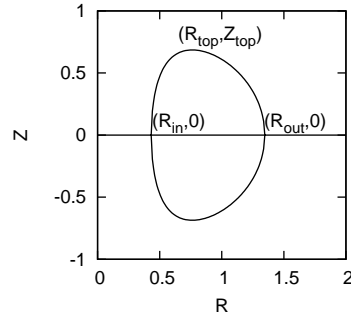


Figure 1.4. Four parameters characterizing the shape of a flux surface: the R coordinate of the innermost and outermost points in the middle-plane, R_{in} and R_{out} ; the (R, Z) coordinators of the highest point of the flux surface, $(R_{\text{top}}, Z_{\text{top}})$.

In terms of these four parameters, we can define the major radius of a magnetic surface

$$R_0 = \frac{R_{\text{in}} + R_{\text{out}}}{2}, \quad (1.29)$$

(which is the R coordinate of the geometric center of the magnetic surface), the minor radius of a magnetic surface

$$a = \frac{R_{\text{out}} - R_{\text{in}}}{2}, \quad (1.30)$$

the triangularity of a magnetic surface

$$\delta_t = \frac{R_0 - R_{\text{top}}}{a}, \quad (1.31)$$

and, the ellipticity (elongation) of a magnetic surface

$$\kappa = \frac{Z_{\text{top}}}{a}. \quad (1.32)$$

Usually, we specify the value of R_0 , a , δ_t , and κ , instead of $(R_{\text{in}}, R_{\text{out}}, R_{\text{top}}, Z_{\text{top}})$, to characterize the shape of a magnetic surface. The value of the triangularity δ_t is usually positive in traditional tokamak operations, but negative triangularity is achievable and potentially useful, which is under active investigation.

Besides, using a and R_0 , we can define another useful parameter $\varepsilon \equiv a / R_0$, which is called the inverse aspect ratio.

The four shape parameters for the typical Last-Closed-Flux-Surface (LCFS) of EAST tokamak are: major radius $R_0 = 1.85m$ (can reach $1.9m$), minor radius $a = 0.45m$, ellipticity $\kappa = 1.8$ (can be in the range from 1.7 to 1.9), triangularity $\delta_t = 0.6$ (can be in the range from 0.5 to 0.7). Note that the major radius R_0 of the LCFS is usually different from R_{axis} (the R coordinate of the magnetic axis). Usually we have $R_{\text{axis}} > R_0$ due to the so-called Shafranov shift.

1.10 Safety factor

Magnetic field lines on closed magnetic surfaces travel closed curves in a poloidal plane. For these field lines, we can define the safety factor q , which is the number of toroidal loops a magnetic field line travels when it makes one poloidal loop, i.e.

$$q \equiv \frac{\Delta\phi}{2\pi}, \quad (1.33)$$

where $\Delta\phi$ as the change of the toroidal angle when a magnetic field line travels a full poloidal loop. The safety factor can also be understood as the average pitch angle of a magnetic field line in (θ, ϕ) plane of a closed magnetic surface.

For open field line region (where a field line touches the wall before its poloidal projection can close itself), the “connection length” is often used to characterize the magnetic field.

1.10.1 Expression of safety factor in terms of magnetic field

The equation of magnetic field lines is given by

$$\frac{Rd\phi}{d\ell_p} = \frac{B_\phi}{B_p}, \quad (1.34)$$

where $d\ell_p$ is the line element along the direction of \mathbf{B}_p on the poloidal plane. Equation (1.34) can be arranged in the form

$$d\phi = \frac{1}{R} \frac{B_\phi}{B_p} d\ell_p, \quad (1.35)$$

which can be integrated over $d\ell_p$ to give

$$\Delta\phi = \oint \frac{1}{R} \frac{B_\phi}{B_p} d\ell_p, \quad (1.36)$$

where the line integration is along the poloidal magnetic field (the contour of Ψ on the poloidal plane). Using this, Eq. (1.33) is written

$$q = \frac{1}{2\pi} \oint \frac{1}{R} \frac{B_\phi}{B_p} d\ell_p. \quad (1.37)$$

1.10.2 Expression of safety factor in terms of magnetic flux

The safety factor given by Eq. (1.37) is expressed in terms of the components of the magnetic field. The safety factor can also be expressed in terms of the magnetic flux. Define $\Delta\Psi_p$ as the poloidal magnetic flux enclosed by two neighboring magnetic surface, then $\Delta\Psi_p$ is given by

$$\Delta\Psi_p = 2\pi R \Delta x B_p \quad (1.38)$$

where Δx is the length of a line segment in the poloidal plane between the two magnetic surfaces, which is perpendicular to the first magnetic surface (so perpendicular to the \mathbf{B}_p). Note that Δx , as well as R and B_p , generally depends on the poloidal location whereas $\Delta\Psi_p$ is independent of the poloidal location.

Using Eq. (1.38), the poloidal magnetic field is written as

$$B_p = \frac{1}{2\pi R} \frac{\Delta\Psi_p}{\Delta x}. \quad (1.39)$$

Substituting Eq. (1.39) into Eq. (1.37), we obtain

$$q = \frac{1}{2\pi} \oint \frac{1}{R} \frac{B_\phi}{B_p} d\ell_p = \frac{1}{2\pi} \oint \frac{1}{R} \frac{2\pi R \Delta x B_\phi}{\Delta\Psi_p} d\ell_p = \oint \frac{\Delta x B_\phi}{\Delta\Psi_p} d\ell_p. \quad (1.40)$$

We know $\Delta\Psi_p$ is a constant independent of the poloidal location, so $\Delta\Psi_p$ can be taken outside the integration to give

$$q = \frac{1}{\Delta\Psi_p} \oint \Delta x B_\phi d\ell_p \quad (1.41)$$

It is ready to realise that the integral appearing in Eq. (1.41) is the toroidal magnetic flux enclosed by the two magnetic surfaces, $\Delta\Psi_t$. Using this, Eq. (1.41) is written as

$$q = \frac{\Delta\Psi_t}{\Delta\Psi_p} \quad (1.42)$$

Equation (1.42) indicates that the safety factor of a magnetic surface is equal to the differential of the toroidal magnetic flux with respect to the poloidal magnetic flux enclosed by the magnetic surface.

1.10.3 Rational surfaces vs. irrational surfaces

If the safety factor of a magnetic surface is a rational number, i.e., $q = m/n$, where m and n are integers, then this magnetic surface is called a rational surface, otherwise an irrational surface. A field line on a rational surface with $q = m/n$ closes itself after it travels n poloidal loops. An example of a field line on a rational surface is shown in Fig. 1.5.

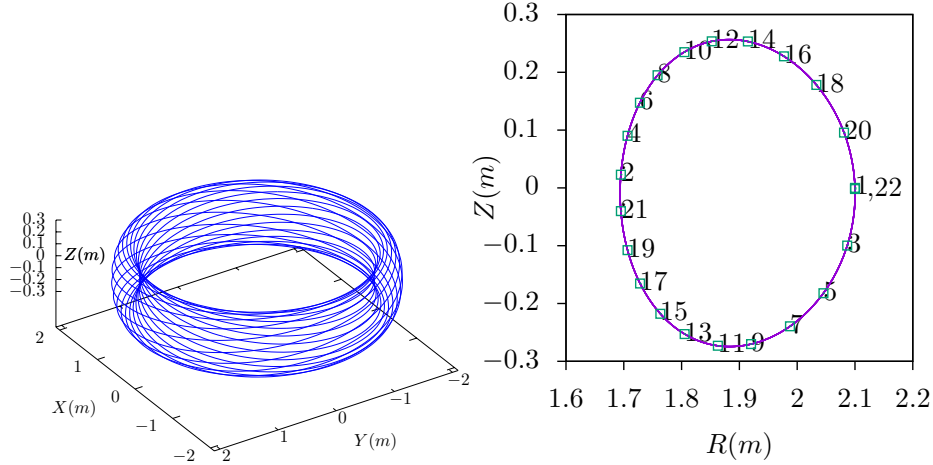


Figure 1.5. Left: A magnetic field line (blue) on a rational surface with $q = 2.1 = 21/10$ (magnetic field is from EAST discharge #59954@3.1s). This field line closes itself after traveling 21 toroidal loops (meanwhile, it travels 10 poloidal loops). Right: The intersecting points of the magnetic field line with the $\phi = 0$ plane when it is traveling toroidally. The sequence of the intersecting points is indicated by the number labels. The 22nd intersecting point coincides with the 1st point and then the intersecting points repeat themselves.

1.11 Non-axisymmetric magnetic perturbations

In the above, the magnetic field is assumed to be axisymmetric. With this assumption, the poloidal magnetic field (having two components) can be expressed in terms of a single component of the vector potential \mathbf{A} , A_ϕ (specifically via $\Psi \equiv A_\phi R$). This kind of simplification can not be achieved if the axisymmetry assumption is dropped, because other components of the vector potential (namely A_R and A_Z) will appear in the expression of the poloidal magnetic field. Let us re-examine Eq. (1.2) for a magnetic perturbation:

$$\begin{aligned} \delta \mathbf{B} = & \left(\frac{1}{R} \frac{\partial \delta A_Z}{\partial \phi} - \frac{\partial \delta A_\phi}{\partial Z} \right) \hat{\mathbf{R}} + \left(\frac{1}{R} \frac{\partial (R \delta A_\phi)}{\partial R} - \frac{1}{R} \frac{\partial \delta A_R}{\partial \phi} \right) \hat{\mathbf{Z}} \\ & + \left(\frac{\partial \delta A_R}{\partial Z} - \frac{\partial \delta A_Z}{\partial R} \right) \hat{\phi}. \end{aligned} \quad (1.43)$$

When studying tearing modes and turbulence, most authors narrow the possible perturbed magnetic field by setting $\delta A_R = \delta A_Z = 0$, i.e.,

$$\delta B_R = -\frac{1}{R} \frac{\partial \delta \Psi}{\partial Z}, \quad (1.44)$$

$$\delta B_Z = \frac{1}{R} \frac{\partial \delta \Psi}{\partial R}, \quad (1.45)$$

$$\delta B_\phi = 0. \quad (1.46)$$

where $\delta \Psi = R \delta A_\phi$. Therefore this kind of magnetic perturbation can be written in the same form of the equilibrium poloidal magnetic field:

$$\delta \mathbf{B} = \nabla \delta \Psi \times \nabla \phi. \quad (1.47)$$

The above approximation is widely used in theory and simulation (e.g., tearing mode theory; turbulence simulation, where δA_ϕ is replaced by δA_\parallel .) Do we miss some magnetic perturbations that is important for plasma transport when using the above specific form?

The total magnetic field is then written as

$$\mathbf{B} = \nabla(\Psi + \delta \Psi) \times \nabla \phi + g \nabla \phi. \quad (1.48)$$

****check**** Can the projection of the total magnetic field line in the poloidal plane can be traced by tracing the contour of $\Psi + \delta \Psi$? No. The contours of $\Psi + \delta \Psi$ will not show island structures in the poloidal plane. To show the expected island structures, we need to subtract non-reconnecting poloidal magnetic field from the total poloidal field. I will discuss this in Sec. (), where I will show that contours of the so-called helical flux will give the expected island structures near the resonant surfaces. ****can be wrong, I have not checked this.**

Next, we return to discuss the 2D case (i.e., assuming axisymmetry).

2 Plasma current density in terms of Ψ and g

When the displacement current term is neglectable (the case we consider in this note), the conductive current is just another representation of the magnetic field. Specifically, the current density can be written in terms of the magnetic field as (Ampère's law):

$$\mu_0 \mathbf{J} = \nabla \times \mathbf{B} = -\frac{\partial B_\phi}{\partial Z} \hat{\mathbf{R}} + \frac{1}{R} \frac{\partial(RB_\phi)}{\partial R} \hat{\mathbf{Z}} + \left(\frac{\partial B_R}{\partial Z} - \frac{\partial B_Z}{\partial R} \right) \hat{\phi}. \quad (2.1)$$

2.1 Poloidal current density

Use Eq. (2.1) and the definition $g \equiv RB_\phi$, the poloidal components of the current density, J_Z and J_R , can be written as

$$\mu_0 J_R = -\frac{1}{R} \frac{\partial g}{\partial Z}, \quad (2.2)$$

and

$$\mu_0 J_Z = \frac{1}{R} \frac{\partial g}{\partial R}, \quad (2.3)$$

respectively.

2.2 Toroidal current density

Ampere's law (2.1) indicates the toroidal current density J_ϕ is given by

$$\begin{aligned} \mu_0 J_\phi &= \frac{\partial B_R}{\partial Z} - \frac{\partial B_Z}{\partial R} \\ &= -\frac{1}{R} \frac{\partial^2 \Psi}{\partial Z^2} - \frac{\partial}{\partial R} \left(\frac{1}{R} \frac{\partial \Psi}{\partial R} \right). \end{aligned} \quad (2.4)$$

Define a differential operator Δ^* by

$$\Delta^* \equiv \frac{\partial^2}{\partial Z^2} + R \frac{\partial}{\partial R} \left(\frac{1}{R} \frac{\partial}{\partial R} \right), \quad (2.5)$$

which is the Laplace operator in cylindrical coordinates for the axisymmetric case, then Eq. (2.4) is written

$$J_\phi = -\frac{1}{\mu_0 R} \Delta^* \Psi. \quad (2.6)$$

3 Constraint of force-balance on magnetic field

Next, we will consider what constraint the force balance imposes on the axisymmetric magnetic field discussed above.

3.1 MHD momentum equation

The MHD momentum equation of plasmas is given by

$$\rho \left(\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u} \right) = \rho_q \mathbf{E} + \mathbf{J} \times \mathbf{B} - \nabla \cdot \mathbb{P} \quad (3.1)$$

where ρ , ρ_q , \mathbb{P} , \mathbf{J} , \mathbf{E} , and \mathbf{B} are mass density, charge density, thermal pressure tensor, current density, electric field, and magnetic field, respectively. The electric field force $\rho_q \mathbf{E}$ is usually ignored due to either $\rho_q = 0$ or $\mathbf{E} = 0$. Further assume that there is no plasma flow ($\mathbf{u} = 0$, the flow effect is discussed in A.2) and the plasma pressure is isotropic, then the steady state momentum equation (force balance equation) is written

$$\mathbf{J} \times \mathbf{B} = \nabla P, \quad (3.2)$$

where P is the scalar plasma pressure.

Is the force balance (3.2) always satisfied in a real tokamak discharge? To answer this question, we need to go back to the original momentum equation (3.1). The imbalance between $\mathbf{J} \times \mathbf{B}$ and ∇P will give rise to the compressional Alfvén waves, the time-scale of which, τ_A , is much shorter than the time-scale τ we are interested in. Therefore, on the time scale τ and for slow flow with $\mathbf{u} < C_s$, where C_s is the sound speed, the leading order of the momentum equation is the force balance (3.2). (to be sure, I need to think about this again). This reasoning is from Youwen Sun[24].

3.2 Force balance in tokamak plasmas

3.2.1 Parallel force balance

Consider the force balance in the direction of \mathbf{B} . Dotting the equilibrium equation (3.2) by \mathbf{B} , we obtain

$$0 = \mathbf{B} \cdot \nabla P, \quad (3.3)$$

which implies that P is constant along a magnetic field line. Since Ψ is also constant along a magnetic field line, P can be expressed in terms of only Ψ on a single magnetic line. Note that this does not necessarily mean P is a single-valued function of Ψ , (i.e. $P = P(\Psi)$). This is because P still has the freedom of taking different value on different magnetic field lines with the same value of Ψ while still satisfying $\mathbf{B} \cdot \nabla P = 0$. This situation can appear when there are saddle points (X points) in Ψ contours (refer to Sec. A.10) and P takes different functions of Ψ in islands of Ψ separated by a X point. For pressure within a single island of Ψ , $P = P(\Psi)$ can be safely assumed.

On the other hand, if $P = P(\Psi)$, then we obtain

$$\mathbf{B} \cdot \nabla P = \frac{dP}{d\Psi} \mathbf{B} \cdot \nabla \Psi = 0,$$

i.e., Eq. (3.3) is satisfied, indicating $P = P(\Psi)$ is a sufficient condition for the force balance in the parallel (to the magnetic field) direction.

3.2.2 Toroidal force balance

Next, consider the ϕ component of Eq. (3.2), which is written

$$J_Z B_R - J_R B_Z = \frac{1}{R} \frac{\partial P}{\partial \phi}. \quad (3.4)$$

Since $P = P(\Psi)$, which implies $\partial P / \partial \phi = 0$, equation (3.4) reduces to

$$J_Z B_R - J_R B_Z = 0 \quad (3.5)$$

Using the expressions of the poloidal current density (2.2) and (2.3) in the force balance equation (3.5) yields

$$\frac{\partial g}{\partial R} B_R + \frac{\partial g}{\partial Z} B_Z = 0, \quad (3.6)$$

which can be further written

$$\mathbf{B} \cdot \nabla g = 0. \quad (3.7)$$

According to the same reasoning for the pressure, we conclude that $g = g(\Psi)$ is a sufficient condition for the toroidal force balance. (The function g defined here is usually called the “poloidal current function” in tokamak literature. The reason for this name is discussed in Sec. A.3.)

3.2.3 Force balance along the major radius: Grad-Shafranov equation

Next, consider the force balance in $\hat{\mathbf{R}}$ direction. The $\hat{\mathbf{R}}$ component of the force balance equation (3.2) is written

$$J_\phi B_Z - J_Z B_\phi = \frac{\partial P}{\partial R} \quad (3.8)$$

Using the expressions of the current density and magnetic field [Eqs. (1.6), (2.3), and (2.6)], equation (3.8) is written

$$-\frac{1}{R} \Delta^* \Psi \frac{1}{R} \frac{\partial \Psi}{\partial R} - \frac{1}{R} \frac{\partial g}{\partial R} \frac{g}{R} = \mu_0 \frac{\partial P}{\partial R}. \quad (3.9)$$

Assuming the sufficient condition discussed above, i.e., P and g are a function of only Ψ , i.e., $P = P(\Psi)$ and $g = g(\Psi)$, Eq. (3.9) is written

$$-\frac{1}{R}\Delta^*\Psi\frac{1}{R}\frac{\partial\Psi}{\partial R}-\frac{1}{R}\frac{dg}{d\Psi}\frac{\partial\Psi}{\partial R}\frac{g}{R}=\mu_0\frac{dP}{d\Psi}\frac{\partial\Psi}{\partial R}, \quad (3.10)$$

which can be simplified to

$$\Delta^*\Psi = -\mu_0 R^2 \frac{dP}{d\Psi} - \frac{dg}{d\Psi} g, \quad (3.11)$$

i.e.,

$$\frac{\partial^2\Psi}{\partial Z^2} + R\frac{\partial}{\partial R}\left(\frac{1}{R}\frac{\partial\Psi}{\partial R}\right) = -\mu_0 R^2 \frac{dP}{d\Psi} - \frac{dg}{d\Psi} g. \quad (3.12)$$

Equation (3.12) is known as Grad-Shafranov (GS) equation.

[Note that the Z component of the force balance equation is written

$$\begin{aligned} J_R B_\phi - J_\phi B_R &= \frac{\partial P}{\partial Z} \\ \Rightarrow -\frac{\partial g}{\partial Z} \frac{1}{R} \frac{g}{R} - \frac{1}{R} \Delta^*\Psi \frac{1}{R} \frac{\partial\Psi}{\partial Z} &= \mu_0 \frac{dP}{d\Psi} \frac{\partial\Psi}{\partial Z} \\ \Rightarrow -\frac{dg}{d\Psi} \frac{\partial\Psi}{\partial Z} \frac{1}{R} \frac{g}{R} - \frac{1}{R} \Delta^*\Psi \frac{1}{R} \frac{\partial\Psi}{\partial Z} &= \mu_0 \frac{dP}{d\Psi} \frac{\partial\Psi}{\partial Z} \\ \Rightarrow -\frac{dg}{d\Psi} \frac{1}{R} \frac{g}{R} - \frac{1}{R} \Delta^*\Psi \frac{1}{R} &= \mu_0 \frac{dP}{d\Psi} \\ \Rightarrow \Delta^*\Psi &= -\mu_0 R^2 \frac{dP}{d\Psi} - \frac{dg}{d\Psi} g \end{aligned}$$

which turns out to be identical with the Grad-Shafranov equation. This is not a coincidence. The reason is that the force balance equation has been satisfied in three different directions (namely, $\hat{\phi}$, $\hat{\mathbf{R}}$, and \mathbf{B} direction) and thus it must be satisfied in all the directions.]

Using the GS equation, J_ϕ can also be expressed as

$$J_\phi = \frac{1}{\mu_0 R} \left(\mu_0 R^2 \frac{dP}{d\Psi} + \frac{dg}{d\Psi} g \right). \quad (3.13)$$

3.3 Axisymmetric equilibrium magnetic field

A general axisymmetric magnetic field (which does not necessarily satisfy the force balance), is given by Eq. (1.10), i.e.,

$$\mathbf{B} = \underbrace{\nabla\Psi \times \nabla\phi}_{\text{poloidal}} + \underbrace{g\nabla\phi}_{\text{toroidal}}, \quad (3.14)$$

For the above axisymmetric magnetic field to be consistent with the force balance equation (3.2), there are additional requirements for Ψ and g . Specifically, Ψ is restricted by the GS equation and g should be a function of only Ψ . Therefore an axisymmetric equilibrium magnetic field is fully determined by two functions, $\Psi = \Psi(R, Z)$ and $g = g(\Psi)$. The Ψ is determined by solving the GS equation with specified RHS source terms and boundary conditions.

The RHS source terms in the GS equation (3.12) are $P(\Psi)$ and $g(\Psi)$, both of which must be specified before the GS equation can be solved. For most cases, the source terms are nonlinear about Ψ and thus the GS equation is a two-dimensional (in R and Z) nonlinear partial differential equation for Ψ .

For most choices of $P(\Psi)$ and $g(\Psi)$, the GS equation (3.12) has to be solved numerically. For some particular choices of P and g profiles, analytical solutions are available, one of which is the Solovév equilibrium and is discussed in Appendix A.1.

Note that we solve the GS equation in order to obtain the poloidal magnetic flux Ψ and thus the poloidal magnetic field. The toroidal magnetic field must be specified in some way before we can solve the GS equation. There are several ways of specifying the toroidal magnetic field: (1) given $g(\Psi)$, (2) given $\langle j_{\parallel} \rangle$, (3) given the safety factor $q(\Psi)$. There are simple relations between g , $\langle j_{\parallel} \rangle$, and q , which allows translation from one to another (discussed later). In transport simulations, $\langle j_{\parallel} \rangle$ is obtained from current drive models and neoclassical bootstrap current models. Note that the specification of the source terms (P , g , q , and $\langle j_{\parallel} \rangle$) usually involve the unknown Ψ (via not only the explicit presence of Ψ , but also the flux-surface averaging which implicit involves Ψ). This indicates that iterations are needed when numerically solving the GS equation.

Many physical processes in tokamak are sensitive to the magnetic configuration. Serious numerical simulations should use magnetic field that satisfies the Grad-Shafranov equation and, if possible, use realistic magnetic fields reconstructed from experimental data.

3.4 Axisymmetric equilibrium current density

Since $\mathbf{J} = \mu_0^{-1} \nabla \times \mathbf{B}$, the current density \mathbf{J} can be inferred from a given magnetic field. The components of \mathbf{J} (expressed in terms of g and Ψ) are given by Eqs. (2.2), (2.3) and (2.6) and these expressions can be further simplified by using the equilibrium constraints, such as $g = g(\Psi)$ and $\Delta^* \Psi = -\mu_0 R^2 \frac{dP}{d\Psi} - \frac{dg}{d\Psi} g$. The simplified \mathbf{J} expressions are given in A.8.

On the other hand, in the equilibrium reconstruction[18], it is the current that is first (partially) specified (e.g., by summing sources of current drive and bootstrap current) and then the current is used as constraints for the GS equation, i.e, constraints for the magnetic field.

3.5 Equilibrium scaling

The GS equation is given by Eq. (3.12), i.e.,

$$\frac{\partial^2 \Psi}{\partial Z^2} + R \frac{\partial}{\partial R} \left(\frac{1}{R} \frac{\partial \Psi}{\partial R} \right) = -\mu_0 R^2 \frac{dP}{d\Psi} - \frac{dg}{d\Psi} g. \quad (3.15)$$

If a solution to the GS equation is obtained, the solution can be scaled to obtain a family of solutions. Given an equilibrium with $\Psi(R, Z)$, $P(\Psi)$, $g(\Psi)$, then it is ready to prove that $\Psi_2 = s\Psi$, $P_2 = s^2 P(\Psi)$, and $g_2 = \pm s g(\Psi)$ is also a solution to the GS equation, where s is a constant. In this case, both the poloidal and toroidal magnetic fields are increased by a factor of s , and thus the safety factor remains unchanged. Also note that the pressure is increased by s^2 factor and thus the value of β (the ratio of the therm pressure to magnetic pressure) remains unchanged. Note that $g_2 = \pm s g(\Psi)$, which indicates that the direction of the toroidal magnetic field can be reversed without breaking the force balance. Also note that $\Psi_2 = s\Psi$ and s can be negative, which indicates that the direction of the toroidal current can also be reversed without breaking the force balance.

The second kind of scaling is to set $\Psi_2 = \Psi$, $P_2 = P(\Psi)$, and $g_2^2 = g^2(\Psi) + c$. It is ready to prove that the scaled expression is still a solution to the GS equation because $g_2 g_2' = g g'$. This scaling keep the pressure and the poloidal field unchanged and thus the poloidal beta β_p remains unchanged. This scaling scales the toroidal field and thus can be used to generate a family of equilibria with different profiles of safety factor.

Another scaling, which is trivial, is to set $\Psi_2 = \Psi$, $P_2 = P(\Psi) + c$, and $g_2 = g(\Psi)$. This scaling can be used to test the effects of the pressure (not the pressure gradient) on various physical processes.

When a numerical equilibrium is obtained, one can use these scaling methods together to generate new equilibria that satisfy particular global conditions. Note that the shape of magnetic surfaces of the scaled equilibrium remains the same as the original one.

The above scaling is made under the constraint that the the GS equation (3.15), is satisfied. In practice, we may scale Ψ by a factor while fixing g and P . This does not satisfy the GS equation, but allows more flexibility in changing the the safety factor profile. Scaling Ψ by a factor corresponds to scaling the plasma current and the poloidal magnetic field.

4 How to solve Grad-Shafranov equation

4.1 Solving by inverting the Laplace operator

If we want to invert the Laplace operator Δ^* of the GS equation to solve for Ψ , we encounter two issues: (1) the RHS of the GS equation involves functions of Ψ , where both the functions and Ψ are unknown; (2) the values of Ψ on a given

boundry should be prescribed, which in turn involves two issues: (a) how to choose the shape of the boundry? (b) how to know the value of Ψ on the boundry? The answer to issues (a) and (b) is that we can choose any boundry we like (e.g., rectangular region) and guess the values of Ψ on it (which will be updated in iterations discussed later). The answer to issue (1) is that the function forms are chosen by users, usually as polynomials in Ψ , e.g. [?]

$$\frac{dP(\Psi)}{d\Psi} = \sum_{n=0}^{n_p} \alpha_n x^n - x^{n_p+1} \sum_{n=0}^{n_p} \alpha_n, \quad (4.1)$$

$$2g \frac{dg}{d\Psi} = \sum_{n=0}^{n_g} \beta_n x^n - x^{n_g+1} \sum_{n=0}^{n_g} \beta_n, \quad (4.2)$$

where $x = (\Psi - \Psi_M) / (\Psi_B - \Psi_M)$, Ψ_M and Ψ_B are values of Ψ at the magnetic axis and LCFS (but the locations of the axis and LCFS are unknown), the free parameters α_n and β_n are assumed known (which will be later adjusted in the least-square-minimizing iteration to make the resulting solution match some measurements and constraints imposed). To make the RHS fully known, we need to guess values of Ψ in the inner region. Then inverting Δ^* to get new Ψ . The values we obtained via inverting Δ^* usually differ from what we used in the RHS, so we need to iterate until convergence. We call this iteration as the first-level iteration.

After the first-level iteration converges, we can calculate J_ϕ through Eq. (3.13) or Eq. (2.6). After this, all the current (current in the plasma, and current in the external coils, which are assumed to be known) perpendicular to the poloidal plane is known, we can calculate the value of Ψ on the boundary, Ψ_b , by using the Biot-Savart Law (see (B.10)):

$$\Psi(R, Z) = \int_P G(R', Z'; R, Z) J_\phi dR' dZ' + \sum_{i=1}^{N_c} G(R_i^c, Z_i^c; R, Z) I_{i,\phi}^c, \quad (4.3)$$

where G is Green's function given by

$$G(R', Z', R, Z) = \frac{\mu_0 R}{4\pi} \int_0^{2\pi} \frac{R' \cos \phi' d\phi'}{\sqrt{(Z - Z')^2 + (R - R' \cos \phi')^2 + (R' \sin \phi')^2}} \quad (4.4)$$

If Ψ_b calculated this way differs from the initial guess of the value of Ψ on the boundary, we need to iterate: use the Ψ_b calculated as a new guess value of Ψ on the boundary and repeat the above procedures until convergence. We call this iteration as the second-level iteration.[?].

The third level iteration is to iterate over the free parameters α_n , β_n , the PF coil currents, to make the solution match some measurements and constraints imposed. This iteration is an optimization problem.

4.2 Solving by using integral form of the GS equation

The Green's function method of calculating Ψ from J_ϕ via Eq. (4.3) can be considered as the integral form of the GS equation. Re-examing the previous subsection, we find, instead of inverting the Laplace operator to solve for Ψ at the interior points, we can just use Eq. (4.3) to calculate them. The plasma toroidal current J_ϕ is calculated through Eq. (3.13). In this case, we only have two levels of iteration: 1. iteration over the rhd of GS equation, 2. iteration over the free parameters (α_n , β_n , PF coil currents). The iteration over the value of Ψ on the boundary is not needed. So, this is a more straightforward method.

5 Fixed boundary equilibrium and choices of coordinates

Plasma physicists love to consider a kind of simplified problem: the fixed boundary equilibrium problem, where the shape of the boundary flux surface is given (the value of Ψ is a constant on this boundary). In dealing with the fixed boundary problem, the curvilinear coordinate system is useful. Specifically, the convenience is that the coordinates can be adjusted to make one of the coordinate surfaces coincide with the given boundary flux surface, so that the boundary condition becomes trivial.

More generally, curvilinear coordinates can be adjusted to make coordinate surfaces coincide with magnetic surfaces (discussed later). This often simplifies analysis of wave and transport problems, especially when we properly choose the poloidal/toroidal coordinate to make magnetic field lines look like straight lines in terms of these coordinates.

Next section discusses the basic theory of curvilinear coordinates system[4].

In many studies of tokamak plasmas, one need construct a curvilinear coordinate system based on a given magnetic configuration in order to make the problem amenable to analytical methods or numerical methods. Specifically, many theories and numerical codes use the curvilinear coordinate systems that are constructed with one coordinate surface coinciding with magnetic surfaces. In these coordinate systems, we need to choose a poloidal coordinate θ and a toroidal coordinate ζ . As mentioned above, a particular choice for θ and ζ is one that makes the magnetic field lines be straight lines in (θ, ζ) plane. These kinds of coordinates are often called magnetic coordinates. That is, "magnetic coordinates" are defined so they conform to the shape of the magnetic surfaces and trivialize the equations for the field lines."

A further tuned magnetic coordinate system is the so-called field aligned (or field-line following) coordinate system, in which changing one of the three coordinates with the other two fixed would correspond to following a magnetic field line. The field aligned coordinates are discussed in Sec. 13.

Next, let us discuss some general properties about coordinates transformation.

6 Curvilinear coordinate system

6.1 Coordinates transformation

In the Cartesian coordinates, a point is described by its coordinates (x, y, z) , which, in the vector form, is written as

$$\mathbf{r} = x\hat{\mathbf{x}} + y\hat{\mathbf{y}} + z\hat{\mathbf{z}}, \quad (6.1)$$

where \mathbf{r} is the location vector of the point; $\hat{\mathbf{x}}$, $\hat{\mathbf{y}}$, and $\hat{\mathbf{z}}$ are the basis vectors of the Cartesian coordinates, which are constant, independent of spacial location. The transformation between the Cartesian coordinates system, (x, y, z) , and a general coordinates system, (x_1, x_2, x_3) , can be expressed as

$$\mathbf{r} = x(x_1, x_2, x_3)\hat{\mathbf{x}} + y(x_1, x_2, x_3)\hat{\mathbf{y}} + z(x_1, x_2, x_3)\hat{\mathbf{z}}. \quad (6.2)$$

For example, cylindrical coordinates (R, ϕ, Z) can be considered as a general coordinate systems, which are defined by $\mathbf{r} = R \cos \phi \hat{\mathbf{x}} + R \sin \phi \hat{\mathbf{y}} + Z \hat{\mathbf{z}}$.

The transformation function in Eq. (6.2) can be written as

$$\begin{aligned} x &= x(x_1, x_2, x_3) \\ y &= y(x_1, x_2, x_3) \\ z &= z(x_1, x_2, x_3) \end{aligned} \quad (6.3)$$

6.2 Jacobian

A useful quality characterizing coordinate transformation is the Jacobian determinant (or simply called Jacobian), which, for the transformation in Eq. (6.3), is defined by

$$\mathcal{J} = \begin{vmatrix} \frac{\partial x}{\partial x_1} & \frac{\partial x}{\partial x_2} & \frac{\partial x}{\partial x_3} \\ \frac{\partial y}{\partial x_1} & \frac{\partial y}{\partial x_2} & \frac{\partial y}{\partial x_3} \\ \frac{\partial z}{\partial x_1} & \frac{\partial z}{\partial x_2} & \frac{\partial z}{\partial x_3} \end{vmatrix}, \quad (6.4)$$

which can also be written as

$$\mathcal{J} = \frac{\partial \mathbf{r}}{\partial x_1} \times \frac{\partial \mathbf{r}}{\partial x_2} \cdot \frac{\partial \mathbf{r}}{\partial x_3}. \quad (6.5)$$

It is easy to prove that the Jacobian \mathcal{J} in Eq. (6.5) can also be written (the derivation is given in my notes on Jacobian)

$$\mathcal{J} = (\nabla x_1 \times \nabla x_2 \cdot \nabla x_3)^{-1}. \quad (6.6)$$

Conventionally, the Jacobian of the transformation from the Cartesian coordinates to a particular coordinate system σ is called the Jacobian of σ , without explicitly mentioning that this transformation is with respect to the Cartesian coordinates.

Using the definition in Eq. (6.4), the Jacobian \mathcal{J} of the Cartesian coordinates can be calculated, yielding 1. Likewise, the Jacobian of the cylindrical coordinates (R, ϕ, Z) can be calculated as follows:

$$\begin{aligned}\mathcal{J} &= \begin{vmatrix} \frac{\partial x}{\partial R} & \frac{\partial x}{\partial \phi} & \frac{\partial x}{\partial Z} \\ \frac{\partial y}{\partial R} & \frac{\partial y}{\partial \phi} & \frac{\partial y}{\partial Z} \\ \frac{\partial z}{\partial R} & \frac{\partial z}{\partial \phi} & \frac{\partial z}{\partial Z} \end{vmatrix} = \begin{vmatrix} \frac{\partial R \cos \phi}{\partial R} & \frac{\partial R \cos \phi}{\partial \phi} & \frac{\partial R \cos \phi}{\partial Z} \\ \frac{\partial R \sin \phi}{\partial R} & \frac{\partial R \sin \phi}{\partial \phi} & \frac{\partial R \sin \phi}{\partial Z} \\ \frac{\partial Z}{\partial R} & \frac{\partial Z}{\partial \phi} & \frac{\partial Z}{\partial Z} \end{vmatrix} \\ &= \begin{vmatrix} \cos \phi & -R \sin \phi & 0 \\ \sin \phi & R \cos \phi & 0 \\ 0 & 0 & 1 \end{vmatrix} = R\end{aligned}$$

If the Jacobian of a coordinate system is greater than zero, it is called a right-handed coordinate system. Otherwise it is called a left-handed system.

6.3 Orthogonality relation between two sets of basis vectors

In a curvilinear coordinate system (x_1, x_2, x_3) , there are two kinds of basis vectors: ∇x_i and $\partial \mathbf{r} / \partial x_i$, with $i = 1, 2, 3$. These two kinds of basis vectors satisfy the following orthogonality relation:

$$\nabla x_i \cdot \frac{\partial \mathbf{r}}{\partial x_j} = \delta_{ij}, \quad (6.7)$$

where δ_{ij} is the Kronecker delta function. [Proof: Working in a Cartesian coordinate system (x, y, z) with the corresponding basis vectors denoted by $(\hat{\mathbf{x}}, \hat{\mathbf{y}}, \hat{\mathbf{z}})$, then the left-hand side of Eq. (6.7) can be written as

$$\begin{aligned}\nabla x_i \cdot \frac{\partial \mathbf{r}}{\partial x_j} &= \left[\left(\frac{\partial x_i}{\partial x} \right) \hat{\mathbf{x}} + \left(\frac{\partial x_i}{\partial y} \right) \hat{\mathbf{y}} + \left(\frac{\partial x_i}{\partial z} \right) \hat{\mathbf{z}} \right] \cdot \left[\frac{\partial x}{\partial x_j} \hat{\mathbf{x}} + x \frac{\partial \hat{\mathbf{x}}}{\partial x_j} + \frac{\partial y}{\partial x_j} \hat{\mathbf{y}} + y \frac{\partial \hat{\mathbf{y}}}{\partial x_j} + \right. \\ &\quad \left. \frac{\partial z}{\partial x_j} \hat{\mathbf{z}} + z \frac{\partial \hat{\mathbf{z}}}{\partial x_j} \right] \\ &= \left[\left(\frac{\partial x_i}{\partial x} \right) \hat{\mathbf{x}} + \left(\frac{\partial x_i}{\partial y} \right) \hat{\mathbf{y}} + \left(\frac{\partial x_i}{\partial z} \right) \hat{\mathbf{z}} \right] \cdot \left[\frac{\partial x}{\partial x_j} \hat{\mathbf{x}} + 0 + \frac{\partial y}{\partial x_j} \hat{\mathbf{y}} + 0 + \frac{\partial z}{\partial x_j} \hat{\mathbf{z}} + \right. \\ &\quad \left. 0 \right] \quad (6.8)\end{aligned}$$

$$\begin{aligned}&= \frac{\partial x_i}{\partial x} \frac{\partial x}{\partial x_j} + \frac{\partial x_i}{\partial y} \frac{\partial y}{\partial x_j} + \frac{\partial x_i}{\partial z} \frac{\partial z}{\partial x_j} \\ &= \frac{\partial x_i}{\partial x_j} \quad (6.9) \\ &= \delta_{ij},\end{aligned}$$

where the second equality is due to $\partial \hat{\mathbf{x}} / \partial x^j = 0$, $\partial \hat{\mathbf{y}} / \partial x^j = 0$, $\partial \hat{\mathbf{z}} / \partial x^j = 0$ since $\hat{\mathbf{x}}, \hat{\mathbf{y}}, \hat{\mathbf{z}}$ are constant vectors independent of spatial location; the chain rule has been used in obtaining Eq. (6.9)]

[The cylindrical coordinate system (R, ϕ, Z) is an example of general coordinates. As an exercise, we can verify that the cylindrical coordinates have the property given in Eq. (6.7). In this case, $x = x_1 \cos x_2$, $y = x_1 \sin x_2$, $z = x_3$, where $x_1 \equiv R$, $x_2 \equiv \phi$, $x_3 \equiv Z$.]

It can be proved that ∇x^i is a contravariant vector while $\partial \mathbf{r} / \partial x^i$ is a covariant vector (I do not prove this and do not bother with the meaning of these names, just using this as a naming scheme for easy reference).

The orthogonality relation in Eq. (6.7) is fundamental to the theory of general coordinates. The orthogonality relation allows one to write the covariant basis vectors in terms of contravariant basis vectors and vice versa. For example, the orthogonality relation tells that $\partial \mathbf{r} / \partial x_1$ is orthogonal to ∇x_2 and ∇x_3 , thus, $\partial \mathbf{r} / \partial x_1$ can be written as

$$\frac{\partial \mathbf{r}}{\partial x_1} = A \nabla x_2 \times \nabla x_3, \quad (6.10)$$

where A is a unknown variable to be determined. To determine A , dotting Eq. (6.10) by ∇x_1 , and using the orthogonality relation again, we obtain

$$1 = A (\nabla x_2 \times \nabla x_3) \cdot \nabla x_1, \quad (6.11)$$

which gives

$$\begin{aligned} A &= \frac{1}{(\nabla x_2 \times \nabla x_3) \cdot \nabla x_1} \\ &= \mathcal{J} \end{aligned} \quad (6.12)$$

Thus $\partial \mathbf{r} / \partial x_1$ is written, in terms of ∇x_1 , ∇x_2 , and ∇x_3 , as

$$\frac{\partial \mathbf{r}}{\partial x_1} = \mathcal{J} \nabla x_2 \times \nabla x_3. \quad (6.13)$$

Similarly, we obtain

$$\frac{\partial \mathbf{r}}{\partial x_2} = \mathcal{J} \nabla x_3 \times \nabla x_1 \quad (6.14)$$

and

$$\frac{\partial \mathbf{r}}{\partial x_3} = \mathcal{J} \nabla x_1 \times \nabla x_2. \quad (6.15)$$

Equations (6.13)-(6.15) can be generally written

$$\frac{\partial \mathbf{r}}{\partial x_i} = \mathcal{J} \nabla x_j \times \nabla x_k, \quad (6.16)$$

where (i, j, k) represents the cyclic order in the variables (x_1, x_2, x_3) . Equation (6.16) expresses the covariant basis vectors in terms of the contravariant basis vectors. On the other hand, from Eq. (6.13)-(6.15), we obtain

$$\nabla x_i = \mathcal{J}^{-1} \frac{\partial \mathbf{r}}{\partial x_j} \times \frac{\partial \mathbf{r}}{\partial x_k}, \quad (6.17)$$

which expresses the contravariant basis vectors in terms of the covariant basis vectors.

6.4 An example: (ψ, θ, ζ) coordinates

Suppose (ψ, θ, ζ) is an arbitrary general coordinate system. Following Einstein's notation, contravariant basis vectors are denoted with upper indices as

$$\mathbf{e}^\psi \equiv \nabla \psi; \quad \mathbf{e}^\theta \equiv \nabla \theta; \quad \mathbf{e}^\zeta \equiv \nabla \zeta. \quad (6.18)$$

and the covariant basis vectors are denoted with low indices as

$$\mathbf{e}_\psi \equiv \frac{\partial \mathbf{r}}{\partial \psi}; \quad \mathbf{e}_\theta \equiv \frac{\partial \mathbf{r}}{\partial \theta}; \quad \mathbf{e}_\zeta \equiv \frac{\partial \mathbf{r}}{\partial \zeta}. \quad (6.19)$$

Then the orthogonality relation, Eq. (6.7), is written as

$$\mathbf{e}^\alpha \cdot \mathbf{e}_\beta = \delta_{\alpha\beta}. \quad (6.20)$$

In term of the contravariant basis vectors, \mathbf{A} is written

$$\mathbf{A} = A_\psi \mathbf{e}^\psi + A_\theta \mathbf{e}^\theta + A_\zeta \mathbf{e}^\zeta, \quad (6.21)$$

where the components are easily obtained by taking scalar product with \mathbf{e}_ψ , \mathbf{e}_θ , and \mathbf{e}_ζ , yielding $A_\psi = \mathbf{A} \cdot \mathbf{e}_\psi$, $A_\theta = \mathbf{A} \cdot \mathbf{e}_\theta$, and $A_\zeta = \mathbf{A} \cdot \mathbf{e}_\zeta$. Similarly, in term of the covariant basis vectors, \mathbf{A} is written

$$\mathbf{A} = A^\psi \mathbf{e}_\psi + A^\theta \mathbf{e}_\theta + A^\zeta \mathbf{e}_\zeta, \quad (6.22)$$

where $A^\psi = \mathbf{A} \cdot \mathbf{e}^\psi$, $A^\theta = \mathbf{A} \cdot \mathbf{e}^\theta$, and $A^\zeta = \mathbf{A} \cdot \mathbf{e}^\zeta$.

Using the above notation, the relation in Eq. (6.16) is written as

$$\mathbf{e}_\psi = \mathcal{J} \mathbf{e}^\theta \times \mathbf{e}^\zeta \quad (6.23)$$

$$\mathbf{e}_\theta = \mathcal{J} \mathbf{e}^\zeta \times \mathbf{e}^\psi \quad (6.24)$$

$$\mathbf{e}_\zeta = \mathcal{J} \mathbf{e}^\psi \times \mathbf{e}^\theta \quad (6.25)$$

where $\mathcal{J} = [(\nabla \psi \times \nabla \theta) \cdot \nabla \zeta]^{-1}$. Similarly, the relation in Eq. (6.17) is written as

$$\mathbf{e}^\psi = \mathcal{J}^{-1} \mathbf{e}_\theta \times \mathbf{e}_\zeta \quad (6.26)$$

$$\mathbf{e}^\theta = \mathcal{J}^{-1} \mathbf{e}_\zeta \times \mathbf{e}_\psi \quad (6.27)$$

$$\mathbf{e}^\zeta = \mathcal{J}^{-1} \mathbf{e}_\psi \times \mathbf{e}_\theta \quad (6.28)$$

6.5 Gradient and directional derivative in general coordinates (ψ, θ, ζ)

The gradient of a scalar function $f(\psi, \theta, \zeta)$ is readily calculated from the chain rule,

$$\nabla f = \frac{\partial f}{\partial \psi} \nabla \psi + \frac{\partial f}{\partial \theta} \nabla \theta + \frac{\partial f}{\partial \zeta} \nabla \zeta. \quad (6.29)$$

Note that the gradient of a scalar function is in the covariant representation. The inverse form of this expression is obtained by dotting the above equation respectively by the three contravariant basis vectors, yielding

$$\frac{\partial f}{\partial \psi} = (\mathcal{J} \nabla \theta \times \nabla \zeta) \cdot \nabla f = \frac{\partial \mathbf{r}}{\partial \psi} \cdot \nabla f \quad (6.30)$$

$$\frac{\partial f}{\partial \theta} = (\mathcal{J} \nabla \zeta \times \nabla \psi) \cdot \nabla f = \frac{\partial \mathbf{r}}{\partial \theta} \cdot \nabla f \quad (6.31)$$

$$\frac{\partial f}{\partial \zeta} = (\mathcal{J} \nabla \psi \times \nabla \theta) \cdot \nabla f = \frac{\partial \mathbf{r}}{\partial \zeta} \cdot \nabla f \quad (6.32)$$

Using Eq. (6.29), the directional derivative in the direction of $\nabla \psi$ is written as

$$\nabla \psi \cdot \nabla f = |\nabla \psi|^2 \frac{\partial f}{\partial \psi} + (\nabla \theta \cdot \nabla \psi) \frac{\partial f}{\partial \theta} + (\nabla \zeta \cdot \nabla \psi) \frac{\partial f}{\partial \zeta}. \quad (6.33)$$

6.6 Divergence operator in general coordinates (ψ, θ, ζ)

To calculate the divergence of a vector, it is desired that the vector should be in the contravariant form because we can make use of the fact:

$$\nabla \cdot (\nabla \alpha \times \nabla \beta) = 0, \quad (6.34)$$

for any scalar quantities α and β . Therefore we write vector \mathbf{A} as

$$\mathbf{A} = A^{(\psi)} \mathcal{J} \nabla \theta \times \nabla \zeta + A^{(\theta)} \mathcal{J} \nabla \zeta \times \nabla \psi + A^{(\zeta)} \mathcal{J} \nabla \psi \times \nabla \theta, \quad (6.35)$$

where $A^{(\psi)} = \mathbf{A} \cdot \nabla \psi$, $A^{(\theta)} = \mathbf{A} \cdot \nabla \theta$, $A^{(\zeta)} = \mathbf{A} \cdot \nabla \zeta$. Then the divergence of \mathbf{A} is readily calculated as

$$\nabla \cdot \mathbf{A} = (\nabla \theta \times \nabla \zeta) \cdot \nabla (A^{(\psi)} \mathcal{J}) + (\nabla \zeta \times \nabla \psi) \cdot \nabla (A^{(\theta)} \mathcal{J}) + (\nabla \psi \times \nabla \theta) \cdot \nabla (A^{(\zeta)} \mathcal{J}) \quad (6.36)$$

$$= \frac{1}{\mathcal{J}} \left(\frac{\partial A^{(\psi)} \mathcal{J}}{\partial \psi} + \frac{\partial A^{(\theta)} \mathcal{J}}{\partial \theta} + \frac{\partial A^{(\zeta)} \mathcal{J}}{\partial \zeta} \right), \quad (6.37)$$

where the second equality is obtained by using Eqs. (6.30), (6.31), and (6.32).

6.7 Laplacian operator in general coordinates (ψ, θ, ζ)

The Laplacian operator is defined by $\nabla^2 \equiv \nabla \cdot \nabla$. Then $\nabla^2 f$ is written as (f is an arbitrary function)

$$\begin{aligned}\nabla^2 f &= \nabla \cdot \nabla f \\ &= \nabla \cdot \left(\frac{\partial f}{\partial \psi} \nabla \psi + \frac{\partial f}{\partial \theta} \nabla \theta + \frac{\partial f}{\partial \zeta} \nabla \zeta \right).\end{aligned}\quad (6.38)$$

To proceed, we can use the divergence formula (6.37) to express the divergence in the above expression. However, the vector in the above (blue term) is not in the covariant form desired by the divergence formula (6.37). If we want to directly use the formula (6.37), we need to transform the vector (blue term in expression (6.38)) to the covariant form. This process seems to be a little complicated. Therefore, I choose not to use this method. Instead, I try to simplify expression (6.38) by using basic vector identities:

$$\begin{aligned}\nabla^2 f &= \nabla \left(\frac{\partial f}{\partial \psi} \right) \cdot \nabla \psi + \nabla \left(\frac{\partial f}{\partial \theta} \right) \cdot \nabla \theta + \nabla \left(\frac{\partial f}{\partial \zeta} \right) \cdot \nabla \zeta \\ &\quad + \frac{\partial f}{\partial \psi} \nabla^2 \psi + \frac{\partial f}{\partial \theta} \nabla^2 \theta + \frac{\partial f}{\partial \zeta} \nabla^2 \zeta.\end{aligned}\quad (6.39)$$

Using the gradient formula, the above expression is further written as

$$\begin{aligned}\nabla^2 f &= \frac{\partial^2 f}{\partial \psi^2} |\nabla \psi|^2 + \frac{\partial^2 f}{\partial \psi \partial \theta} \nabla \psi \cdot \nabla \theta + \frac{\partial^2 f}{\partial \psi \partial \zeta} \nabla \psi \cdot \nabla \zeta \\ &\quad + \frac{\partial^2 f}{\partial \theta^2} |\nabla \theta|^2 + \frac{\partial^2 f}{\partial \theta \partial \zeta} \nabla \theta \cdot \nabla \zeta \\ &\quad + \frac{\partial^2 f}{\partial \zeta^2} |\nabla \zeta|^2 + \frac{\partial^2 f}{\partial \zeta \partial \theta} \nabla \zeta \cdot \nabla \theta + \frac{\partial^2 f}{\partial \zeta \partial \psi} \nabla \zeta \cdot \nabla \psi \\ &\quad + \frac{\partial f}{\partial \psi} \nabla^2 \psi + \frac{\partial f}{\partial \theta} \nabla^2 \theta + \frac{\partial f}{\partial \zeta} \nabla^2 \zeta,\end{aligned}\quad (6.40)$$

and can be simplified as

$$\begin{aligned}\nabla^2 f &= \frac{\partial^2 f}{\partial \psi^2} |\nabla \psi|^2 + 2 \frac{\partial^2 f}{\partial \psi \partial \theta} \nabla \psi \cdot \nabla \theta + 2 \frac{\partial^2 f}{\partial \psi \partial \zeta} \nabla \psi \cdot \nabla \zeta \\ &\quad + \frac{\partial^2 f}{\partial \theta^2} |\nabla \theta|^2 + 2 \frac{\partial^2 f}{\partial \theta \partial \zeta} \nabla \theta \cdot \nabla \zeta \\ &\quad + \frac{\partial^2 f}{\partial \zeta^2} |\nabla \zeta|^2 \\ &\quad + \frac{\partial f}{\partial \psi} \nabla^2 \psi + \frac{\partial f}{\partial \theta} \nabla^2 \theta + \frac{\partial f}{\partial \zeta} \nabla^2 \zeta.\end{aligned}\quad (6.41)$$

Assume (ψ, θ, ζ) are field-line following coordinates with $\partial \mathbf{r} / \partial \theta$ along the field line direction, then neglect all the parallel derivatives, i.e., derivative over θ , then the above expression is reduced to

$$\begin{aligned} \nabla^2 f = & \frac{\partial^2 f}{\partial \psi^2} |\nabla \psi|^2 + 2 \frac{\partial^2 f}{\partial \psi \partial \zeta} \nabla \psi \cdot \nabla \zeta + \frac{\partial^2 f}{\partial \zeta^2} |\nabla \zeta|^2 \\ & + \frac{\partial f}{\partial \psi} \nabla^2 \psi + \frac{\partial f}{\partial \zeta} \nabla^2 \zeta. \end{aligned} \quad (6.42)$$

This approximation reduces the Laplacian operator from being three-dimensional to being two-dimensional. This approximation is often called the high- n approximation, where n is the toroidal mode number (mode number along ζ direction).

6.8 Curl operator in general coordinates (ψ, θ, ζ)

To take the curl of a vector, it should be in the covariant representation since we can make use of the fact that $\nabla \times \nabla \alpha = 0$. Thus the curl of \mathbf{A} is written as

$$\begin{aligned} \nabla \times \mathbf{A} &= \nabla \times (A_1 \nabla \psi + A_2 \nabla \theta + A_3 \nabla \zeta) \\ &= \nabla A_1 \times \nabla \psi + \nabla A_2 \times \nabla \theta + \nabla A_3 \times \nabla \zeta \\ &= \left(\frac{\partial A_1}{\partial \theta} \nabla \theta + \frac{\partial A_1}{\partial \zeta} \nabla \zeta \right) \times \nabla \psi + \left(\frac{\partial A_2}{\partial \psi} \nabla \psi + \frac{\partial A_2}{\partial \zeta} \nabla \zeta \right) \times \nabla \theta + \\ &\quad \left(\frac{\partial A_3}{\partial \psi} \nabla \psi + \frac{\partial A_3}{\partial \theta} \nabla \theta \right) \times \nabla \zeta \\ &= \frac{1}{\mathcal{J}} \left(\frac{\partial A_2}{\partial \psi} - \frac{\partial A_1}{\partial \theta} \right) \mathcal{J} \nabla \psi \times \nabla \theta + \frac{1}{\mathcal{J}} \left(\frac{\partial A_1}{\partial \zeta} - \frac{\partial A_3}{\partial \psi} \right) \mathcal{J} \nabla \zeta \times \nabla \psi + \\ &\quad \frac{1}{\mathcal{J}} \left(\frac{\partial A_3}{\partial \theta} - \frac{\partial A_2}{\partial \zeta} \right) \mathcal{J} \nabla \theta \times \nabla \zeta. \end{aligned} \quad (6.43)$$

Note that taking the curl of a vector in the covariant form leaves the vector in the contravariant form.

6.9 Metric tensor for general coordinate system

Consider a general coordinate system (ψ, θ, ζ) . I define the metric tensor as the transformation matrix between the covariant basis vectors and the contravariant ones. Equations (6.16) and (6.17) express the relation between the two sets of basis vectors using cross product. Next, let us express the relation in matrix form. To obtain the metric matrix, we write the contravariant basis vectors in terms of the covariant ones, such as

$$\nabla \psi = a^1 \mathcal{J} \nabla \theta \times \nabla \zeta + a^2 \mathcal{J} \nabla \zeta \times \nabla \psi + a^3 \mathcal{J} \nabla \psi \times \nabla \theta. \quad (6.44)$$

Taking the scalar product respectively with $\nabla\psi$, $\nabla\theta$, and $\nabla\zeta$, Eq. (6.44) is written as

$$a^1 = |\nabla\psi|^2, \quad (6.45)$$

$$a^2 = \nabla\psi \cdot \nabla\theta, \quad (6.46)$$

$$a^3 = \nabla\psi \cdot \nabla\zeta. \quad (6.47)$$

Similarly, we write

$$\nabla\theta = b^1 \mathcal{J} \nabla\theta \times \nabla\zeta + b^2 \mathcal{J} \nabla\zeta \times \nabla\psi + b^3 \mathcal{J} \nabla\psi \times \nabla\theta, \quad (6.48)$$

Taking the scalar product with $\nabla\psi$, $\nabla\theta$, and $\nabla\zeta$, respectively, the above becomes

$$b^1 = \nabla\theta \cdot \nabla\psi \quad (6.49)$$

$$b^2 = |\nabla\theta|^2, \quad (6.50)$$

$$b^3 = \nabla\theta \cdot \nabla\zeta. \quad (6.51)$$

The same situation applies for the $\nabla\zeta$ basis vector,

$$\nabla\zeta = c^1 \nabla\theta \times \nabla\zeta \mathcal{J} + c^2 \nabla\zeta \times \nabla\psi \mathcal{J} + c^3 \nabla\psi \times \nabla\theta \mathcal{J}, \quad (6.52)$$

Taking the scalar product with $\nabla\psi$, $\nabla\theta$, and $\nabla\zeta$, respectively, the above equation becomes

$$c^1 = \nabla\zeta \cdot \nabla\psi \quad (6.53)$$

$$c^2 = \nabla\zeta \cdot \nabla\theta \quad (6.54)$$

$$c^3 = |\nabla\zeta|^2 \quad (6.55)$$

Summarizing the above results in matrix form, we obtain

$$\begin{pmatrix} \nabla\psi \\ \nabla\theta \\ \nabla\zeta \end{pmatrix} = \begin{pmatrix} |\nabla\psi|^2 & \nabla\psi \cdot \nabla\theta & \nabla\psi \cdot \nabla\zeta \\ \nabla\theta \cdot \nabla\psi & |\nabla\theta|^2 & \nabla\theta \cdot \nabla\zeta \\ \nabla\zeta \cdot \nabla\psi & \nabla\zeta \cdot \nabla\theta & |\nabla\zeta|^2 \end{pmatrix} \begin{pmatrix} \nabla\theta \times \nabla\zeta \mathcal{J} \\ \nabla\zeta \times \nabla\psi \mathcal{J} \\ \nabla\psi \times \nabla\theta \mathcal{J} \end{pmatrix} \quad (6.56)$$

Similarly, to convert contravariant basis vector to covariant one, we write

$$\nabla\theta \times \nabla\zeta \mathcal{J} = d_1 \nabla\psi + d_2 \nabla\theta + d_3 \nabla\zeta \quad (6.57)$$

Taking the scalar product respectively with $\nabla\theta \times \nabla\zeta \mathcal{J}$, $\nabla\zeta \times \nabla\psi \mathcal{J}$, and $\nabla\psi \times \nabla\theta \mathcal{J}$, the above equation becomes

$$d_1 = |\nabla\theta \times \nabla\zeta|^2 \mathcal{J}^2 \quad (6.58)$$

$$d_2 = (\nabla\theta \times \nabla\zeta \mathcal{J}) \cdot (\nabla\zeta \times \nabla\psi \mathcal{J}) \quad (6.59)$$

$$d_3 = (\nabla\theta \times \nabla\zeta \mathcal{J}) \cdot (\nabla\psi \times \nabla\theta \mathcal{J}) \quad (6.60)$$

For the second contravariant basis vector

$$\nabla\phi \times \nabla\psi\mathcal{J} = e_1\nabla\psi + e_2\nabla\theta + e_3\nabla\zeta \quad (6.61)$$

$$e_1 = (\nabla\zeta \times \nabla\psi) \cdot (\nabla\theta \times \nabla\zeta) \mathcal{J}^2 \quad (6.62)$$

$$e_2 = (\nabla\zeta \times \nabla\psi) \cdot (\nabla\zeta \times \nabla\psi) \mathcal{J}^2 \quad (6.63)$$

$$e_3 = (\nabla\zeta \times \nabla\psi) \cdot (\nabla\psi \times \nabla\theta) \mathcal{J}^2 \quad (6.64)$$

For the third contravariant basis vector

$$\nabla\psi \times \nabla\theta\mathcal{J} = f_1\nabla\psi + f_2\nabla\theta + f_3\nabla\zeta \quad (6.65)$$

$$f_1 = (\nabla\psi \times \nabla\theta) \cdot (\nabla\theta \times \nabla\zeta) \mathcal{J}^2 \quad (6.66)$$

$$f_2 = (\nabla\psi \times \nabla\theta) \cdot (\nabla\zeta \times \nabla\psi) \mathcal{J}^2 \quad (6.67)$$

$$f_3 = (\nabla\psi \times \nabla\theta) \cdot (\nabla\psi \times \nabla\theta) \mathcal{J}^2 \quad (6.68)$$

Summarizing these results, we obtain

$$\begin{pmatrix} \nabla\theta \times \nabla\zeta\mathcal{J} \\ \nabla\zeta \times \nabla\psi\mathcal{J} \\ \nabla\psi \times \nabla\theta\mathcal{J} \end{pmatrix} = M \begin{pmatrix} \nabla\psi \\ \nabla\theta \\ \nabla\zeta \end{pmatrix}, \quad (6.69)$$

where

$$M = \begin{pmatrix} |\nabla\theta \times \nabla\zeta|^2 \mathcal{J}^2 & (\nabla\theta \times \nabla\zeta) \cdot (\nabla\zeta \times \nabla\psi) \mathcal{J}^2 & (\nabla\theta \times \nabla\zeta) \cdot (\nabla\psi \times \nabla\theta) \mathcal{J}^2 \\ (\nabla\zeta \times \nabla\psi) \cdot (\nabla\theta \times \nabla\zeta) \mathcal{J}^2 & (\nabla\zeta \times \nabla\psi) \cdot (\nabla\zeta \times \nabla\psi) \mathcal{J}^2 & (\nabla\zeta \times \nabla\psi) \cdot (\nabla\psi \times \nabla\theta) \mathcal{J}^2 \\ (\nabla\psi \times \nabla\theta) \cdot (\nabla\theta \times \nabla\zeta) \mathcal{J}^2 & (\nabla\psi \times \nabla\theta) \cdot (\nabla\zeta \times \nabla\psi) \mathcal{J}^2 & (\nabla\psi \times \nabla\theta) \cdot (\nabla\psi \times \nabla\theta) \mathcal{J}^2 \end{pmatrix},$$

This matrix and the matrix in Eqs. (6.56) should be the inverse of each other. It is ready to prove this by directly calculating the product of the two matrix.

6.9.1 Special case: metric tensor for (ψ, θ, ϕ) coordinate system

Suppose that (ψ, θ, ϕ) are arbitrary general coordinates except that ϕ is the usual toroidal angle in cylindrical coordinates. Then $\nabla\phi = 1/R\hat{\phi}$ is perpendicular to both $\nabla\psi$ and $\nabla\theta$. Using this, Eq. (6.56) is simplified to

$$\begin{pmatrix} \nabla\psi \\ \nabla\theta \\ \nabla\phi \end{pmatrix} = \begin{pmatrix} |\nabla\psi|^2 & \nabla\psi \cdot \nabla\theta & 0 \\ \nabla\psi \cdot \nabla\theta & |\nabla\theta|^2 & 0 \\ 0 & 0 & 1/R^2 \end{pmatrix} \begin{pmatrix} \nabla\theta \times \nabla\phi\mathcal{J} \\ \nabla\phi \times \nabla\psi\mathcal{J} \\ \nabla\psi \times \nabla\theta\mathcal{J} \end{pmatrix} \quad (6.70)$$

Similarly, Eq. (6.69) is simplified to

$$\begin{pmatrix} \nabla\theta \times \nabla\phi\mathcal{J} \\ \nabla\phi \times \nabla\psi\mathcal{J} \\ \nabla\psi \times \nabla\theta\mathcal{J} \end{pmatrix} = \begin{pmatrix} |\nabla\theta|^2 \mathcal{J}^2 / R^2 & -\nabla\theta \cdot \nabla\psi \mathcal{J}^2 / R^2 & 0 \\ -\nabla\psi \cdot \nabla\theta \mathcal{J}^2 / R^2 & |\nabla\psi|^2 \mathcal{J}^2 / R^2 & 0 \\ 0 & 0 & R^2 \end{pmatrix} \begin{pmatrix} \nabla\psi \\ \nabla\theta \\ \nabla\phi \end{pmatrix} \quad (6.71)$$

[Note that the matrix in Eqs. (6.70) and (6.71) should be the inverse of each other. The product of the two matrix,

$$\begin{pmatrix} |\nabla\theta|^2 \mathcal{J}^2 / R^2 & -\nabla\theta \cdot \nabla\psi \mathcal{J}^2 / R^2 & 0 \\ -\nabla\psi \cdot \nabla\theta \mathcal{J}^2 / R^2 & \frac{\mathcal{J}^2}{R^2} |\nabla\psi|^2 & 0 \\ 0 & 0 & R^2 \end{pmatrix} \begin{pmatrix} |\nabla\psi|^2 & \nabla\psi \cdot \nabla\theta & 0 \\ \nabla\psi \cdot \nabla\theta & |\nabla\theta|^2 & 0 \\ 0 & 0 & 1/R^2 \end{pmatrix}, \quad (6.72)$$

can be calculated to give

$$\begin{pmatrix} A & 0 & 0 \\ 0 & A & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

where

$$A = |\nabla\theta|^2 |\nabla\psi|^2 \mathcal{J}^2 / R^2 - (\nabla\theta \cdot \nabla\psi)^2 \mathcal{J}^2.$$

By using the definition of the Jacobian in Eq. (6.6), it is easy to verify that $A = 1$, i.e.,

$$|\nabla\theta|^2 |\nabla\psi|^2 - (\nabla\theta \cdot \nabla\psi)^2 = \frac{R^2}{\mathcal{J}^2}, \quad (6.73)$$

]

7 Covariant/contravariant representation of equilibrium magnetic field

The axisymmetric equilibrium magnetic field is given by Eq. (3.14), i.e.,

$$\mathbf{B} = \nabla\Psi \times \nabla\phi + g\nabla\phi. \quad (7.1)$$

In a general coordinate system (ψ, θ, ϕ) (not necessarily magnetic surface coordinates), the above expression can be written as

$$\mathbf{B} = -\Psi_\psi \nabla\phi \times \nabla\psi - \Psi_\theta \nabla\phi \times \nabla\theta + g\nabla\phi, \quad (7.2)$$

where the subscripts denote the partial derivatives with the corresponding subscripts. Note that Eq. (7.2) is a mixed representation, which involves both covariant and contravariant basis vectors. Equation (7.2) can be converted to the contravariant form by using the metric tensor, giving

$$\mathbf{B} = -\Psi_\psi \nabla\phi \times \nabla\psi - \Psi_\theta \nabla\phi \times \nabla\theta + g \frac{\mathcal{J}}{R^2} \nabla\psi \times \nabla\theta. \quad (7.3)$$

Similarly, Eq. (7.2) can also be transformed to the covariant form, giving

$$\mathbf{B} = \left(\Psi_\psi \frac{\mathcal{J}}{R^2} \nabla\psi \cdot \nabla\theta + \Psi_\theta \frac{\mathcal{J}}{R^2} |\nabla\theta|^2 \right) \nabla\psi + \left(-\Psi_\psi \frac{\mathcal{J}}{R^2} |\nabla\psi|^2 - \Psi_\theta \frac{\mathcal{J}}{R^2} \nabla\theta \cdot \nabla\psi \right) \nabla\theta + g\nabla\phi. \quad (7.4)$$

For the convenience of notation, define

$$h^{\alpha\beta} = \frac{\mathcal{J}}{R^2} \nabla\alpha \cdot \nabla\beta, \quad (7.5)$$

then Eq. (7.4) is written as

$$\mathbf{B} = (\Psi_\psi h^{\psi\theta} + \Psi_\theta h^{\theta\theta}) \nabla\psi + (-\Psi_\psi h^{\psi\psi} - \Psi_\theta h^{\psi\theta}) \nabla\theta + g \nabla\phi. \quad (7.6)$$

8 Magnetic surface coordinates (ψ, θ, ϕ)

A coordinate system (ψ, θ, ϕ) , where ϕ is the usual cylindrical toroidal angle, is called a magnetic surface coordinate system if Ψ is a function of only ψ , i.e., $\partial\Psi/\partial\theta=0$ (we also have $\partial\Psi/\partial\phi=0$ since we are considering axially symmetrical case). In terms of (ψ, θ, ϕ) coordinates, the contravariant form of the magnetic field, Eq. (7.3), is written as

$$\mathbf{B} = -\Psi' \nabla\phi \times \nabla\psi + g \frac{\mathcal{J}}{R^2} \nabla\psi \times \nabla\theta, \quad (8.1)$$

where $\Psi' \equiv d\Psi/d\psi$. The covariant form of the magnetic field, Eq. (7.4), is written as

$$\mathbf{B} = \left(\Psi' \frac{\mathcal{J}}{R^2} \nabla\psi \cdot \nabla\theta \right) \nabla\psi + \left(-\Psi' \frac{\mathcal{J}}{R^2} |\nabla\psi|^2 \right) \nabla\theta + g \nabla\phi. \quad (8.2)$$

8.1 Local safety factor

The local safety factor \hat{q} is defined by

$$\hat{q} = \frac{\mathbf{B} \cdot \nabla\phi}{\mathbf{B} \cdot \nabla\theta}, \quad (8.3)$$

which characterizes the local pitch angle of a magnetic field line in (θ, ϕ) plane of a magnetic surface. Substituting the contravariant representation of the magnetic field, Eq. (8.1), into the above equation, the local safety factor is written

$$\hat{q}(\psi, \theta) = -\frac{g}{R^2} \frac{\mathcal{J}}{\Psi'}. \quad (8.4)$$

Note that the expression \hat{q} in Eq. (8.4) depends on the Jacobian \mathcal{J} . This is because the definition of \hat{q} depends on the definition of θ , which in turn depends on the the Jacobian \mathcal{J} .

In terms of \hat{q} , the contravariant form of the magnetic field, Eq. (8.1), is written

$$\mathbf{B} = -\Psi' (\nabla\phi \times \nabla\psi + \hat{q} \nabla\psi \times \nabla\theta). \quad (8.5)$$

and the parallel differential operator $\mathbf{B}_0 \cdot \nabla$ is written as

$$\mathbf{B}_0 \cdot \nabla = -\Psi'(\nabla \phi \times \nabla \psi + \hat{q} \nabla \psi \times \nabla \theta) \cdot \nabla = -\Psi' \mathcal{J}^{-1} \left(\frac{\partial}{\partial \theta} + \hat{q} \frac{\partial}{\partial \phi} \right). \quad (8.6)$$

If \hat{q} happens to be independent of θ (i.e., field lines are straight in (θ, ϕ) plane), then the above operator becomes a constant coefficient differential operator (after divided by \mathcal{J}^{-1}). This simplification is useful because different poloidal harmonics are decoupled in this case. We will discuss this issue further in Sec. 12.

8.2 Global safety factor

The global safety factor defined in Eq. (1.37) is actually the poloidal average of the local safety factor, i.e.,

$$q(\psi) \equiv \frac{1}{2\pi} \int_0^{2\pi} \hat{q} d\theta \quad (8.7)$$

$$= -\frac{1}{2\pi} \frac{g}{\Psi'} \int_0^{2\pi} \frac{\mathcal{J}}{R^2} d\theta. \quad (8.8)$$

Note that q and \hat{q} defined this way can be negative, which depends on the choice of the positive direction of ϕ and θ coordinates (note that the safety factor given in G-eqds file is always positive, i.e. it is the absolute value of the safety factor defined here).

Next, let us transform the θ integration in expression (8.8) to a curve integral in the poloidal plane. Using the relation $d\ell_p$ and $d\theta$ [Eq. (?)], expression (8.8) is further written

$$\begin{aligned} q(\psi) &= -\frac{1}{2\pi} \frac{g}{\Psi'} \oint \text{sign}(\mathcal{J}) \frac{d\ell_p}{R|\nabla \psi|} \\ &= -\frac{1}{2\pi} g \frac{\text{sign}(\mathcal{J})}{\text{sign}(\Psi')} \oint \frac{d\ell_p}{R|\nabla \Psi|}. \end{aligned} \quad (8.9)$$

Expression (8.9) is used in the GTAW code to numerically calculate the value of q on magnetic surfaces (as a benchmarking of the q profile specified in the G-eqds file). Expression (8.9) can also be considered as a relation between q and g . In the equilibrium problem where q is given (fixed- q equilibrium), we can use expression (8.9) to obtain the corresponding g (which explicitly appears in the GS equation):

$$g = -2\pi q \left(\oint \frac{d\ell_p}{R|\nabla \Psi|} \right)^{-1} \frac{\text{sign}(\Psi')}{\text{sign}(\mathcal{J})}. \quad (8.10)$$

We note that expression (8.10) involves magnetic surface averaging, which is unknown before we know Ψ . Therefore iteration is usually needed in solving the fixed- q equilibrium (i.e., we guess the unknown Ψ , so that the magnetic surface averaging in expression (8.10) can be performed, yielding the values of g .)

Using $B_p = |\nabla\Psi|/R$ and $B_\phi = g/R$, the absolute value of q in expression (8.9) is written

$$|q| = \frac{1}{2\pi} g \oint \frac{d\ell_p}{R|\nabla\Psi|}. \quad (8.11)$$

$$= \frac{1}{2\pi} \oint \frac{1}{R} \frac{|B_\phi|}{B_p} d\ell_p \quad (8.12)$$

which is the familiar formula we see in textbooks.

8.3 Relation between Jacobian and poloidal angle θ

Given the definition of a magnetic surface coordinate system (ψ, θ, ϕ) , the Jacobian of this system is fully determined. On the other hand, given the definition of ψ , ϕ , and the Jacobian, the definition of θ is fully determined (can have some trivial shifting freedoms). Next, let us discuss how to calculate θ in this case. In (ψ, θ, ϕ) coordinates, a line element is written

$$d\mathbf{l} = \frac{\partial \mathbf{r}}{\partial \psi} d\psi + \frac{\partial \mathbf{r}}{\partial \theta} d\theta + \frac{\partial \mathbf{r}}{\partial \phi} d\phi \quad (8.13)$$

The line element that lies on a magnetic surface (i.e., $d\psi = 0$) and in a poloidal plane (i.e., $d\phi = 0$) is then written

$$\begin{aligned} d\ell_p &= \frac{\partial \mathbf{r}}{\partial \theta} d\theta \\ &= \mathcal{J} \nabla \phi \times \nabla \psi d\theta. \end{aligned} \quad (8.14)$$

We use the convention that $d\ell_p$ and $d\theta$ take the same sign, i.e.,

$$d\ell_p = |\mathcal{J} \nabla \phi \times \nabla \psi| d\theta. \quad (8.15)$$

Using the fact that $\nabla \psi$ and $\nabla \phi$ are orthogonal and $\nabla \phi = \hat{\phi}/R$, the above equation is written as

$$d\theta = \frac{R}{|\mathcal{J} \nabla \psi|} dl_p \quad (8.16)$$

Given $|\mathcal{J} \nabla \psi|$, Eq. (8.16) can be integrated to determine the θ coordinate of points on a magnetic surface.

$$\bar{\delta} = \int_0^\theta \hat{q} d\theta = - \int_0^\theta \frac{g}{R^2} \frac{\mathcal{J}}{\Psi'} d\theta \quad (8.17)$$

8.4 Calculating poloidal angle

Once $|\mathcal{J} \nabla \psi|$ is known, the value of θ of a point can be obtained by integrating expression (8.16), i.e.,

$$\theta_{i,j} = \theta_{\text{ref},j} + \int_{\mathbf{x}_{\text{ref},j}}^{\mathbf{x}_{i,j}} \frac{R}{|\mathcal{J} \nabla \psi|} dl_p, \quad (8.18)$$

where the curve integration is along the contour $\Psi = \Psi_j$, $\mathbf{x}_{\text{ref},j}$ is a reference point on the contour, where value of the poloidal angle is chosen as $\theta_{\text{ref},j}$. The choice of the positive direction of θ is up to users. Depending on the positive direction chosen, the sign of the Jacobian of the constructed coordinates can have a sign difference from the \mathcal{J} appearing in Eq. (8.18). Denote the Jacobian of the constructed coordinates by \mathcal{J}' , then

$$\mathcal{J}' = \pm \mathcal{J} \quad (8.19)$$

This sign can be determined after the radial coordinate and the positive direction of the poloidal angle are chosen. In GTAW code, I choose the positive direction of θ to be in anticlockwise direction when observers look along the direction of $\hat{\phi}$. To achieve this, the line integration in Eq. (8.18) should be along the anticlockwise direction. (I use the determination of the direction matrix, a well known method in graphic theory, to determine the direction from a given set of discrete points on a magnetic surface.)

8.4.1 Normalized poloidal angle

The span of θ defined by Eq. (8.18) is usually not 2π in one poloidal loop. This poloidal angle can be scaled by $s(\psi)$ to define a new poloidal coordinate $\bar{\theta}$, whose span is 2π in one poloidal loop, where $s(\psi)$ is a magnetic surface function given by

$$s(\psi) = \frac{2\pi}{\oint \frac{R}{|\mathcal{J}\nabla\psi|} dl_p}. \quad (8.20)$$

Then $\bar{\theta}$ is written as

$$\begin{aligned} \bar{\theta}_{i,j} &= s(\psi_j)\theta_{i,j} \\ &= s(\psi_j) \left(\theta_{\text{ref},j} + \int_{\mathbf{x}_{\text{ref},j}}^{\mathbf{x}_{i,j}} \frac{R}{|\mathcal{J}\nabla\psi|} dl_p \right) \\ &= \bar{\theta}_{\text{ref},j} + s(\psi_j) \int_{\mathbf{x}_{\text{ref},j}}^{\mathbf{x}_{i,j}} \frac{R}{|\mathcal{J}\nabla\psi|} dl_p, \end{aligned} \quad (8.21)$$

where $\bar{\theta}_{\text{ref},j} = s(\psi_j)\theta_{\text{ref},j}$. Since we have modified the definition of the poloidal angle, the Jacobian of the new coordinates $(\psi, \bar{\theta}, \phi)$ is different from that of (ψ, θ, ϕ) . The Jacobian \mathcal{J}_{new} of the new coordinates $(\psi, \bar{\theta}, \phi)$ is written as

$$\begin{aligned} \mathcal{J}_{\text{new}} &\equiv \frac{1}{(\nabla\psi \times \nabla\bar{\theta}) \cdot \nabla\phi} \\ &= \frac{1}{\{\nabla\psi \times \nabla[s(\psi)\theta]\} \cdot \nabla\phi} \\ &= \frac{1}{s(\psi)} \frac{1}{(\nabla\psi \times \nabla\theta) \cdot \nabla\phi} \\ &= \frac{1}{s(\psi)} \mathcal{J}' \\ &= \pm \frac{1}{s(\psi)} \mathcal{J} \end{aligned} \quad (8.22)$$

The Jacobian \mathcal{J} can be set to various forms to achieve various poloidal coordinates, which will be discussed in the next section. After the Jacobian and the radial coordinate ψ is chosen, all the quantities on the right-hand side of Eq. (8.21) are known and the integration can be performed to obtain the value of $\bar{\theta}_{i,j}$ of each point on each flux surface.

[The reference points $\mathbf{x}_{\text{ref},j}$ and the values of poloidal angle at these points can be chosen by users. One choice of the reference points $\mathbf{x}_{\text{ref},j}$ are those points on the horizontal ray in the midplane that starts from the magnetic axis and points to the low field side of the device and $\bar{\theta}_{\text{ref},j}$ at these points is chosen as zero (this is my choice in the GTAW code). In the TEK code, the reference points are chosen at the high-field side of the midplane and $\bar{\theta}_{\text{ref},j} = -\pi$ at the reference points.]

8.4.2 Jacobian for equal-arc-length poloidal angle

If the Jacobian \mathcal{J} is chosen to be of the following form

$$\mathcal{J} = \frac{R}{|\nabla\psi|}. \quad (8.23)$$

Then $\bar{\theta}_{i,j}$ in Eq. (8.21) is written

$$\bar{\theta}_{i,j} = \bar{\theta}_{\text{ref},j} + \frac{2\pi}{\oint dl_p} \int_{\mathbf{x}_{\text{ref},j}}^{\mathbf{x}_{i,j}} dl_p. \quad (8.24)$$

and the Jacobian of new coordinates $(\psi, \bar{\theta}, \phi)$, \mathcal{J}_{new} , which is given by Eq. (8.22), now takes the form

$$\mathcal{J}_{\text{new}} = \pm \frac{\oint dl_p}{2\pi} \frac{R}{|\nabla\psi|} = \pm \frac{\oint dl_p}{2\pi} \frac{R}{|\nabla\Psi|} \frac{d\Psi}{d\psi}. \quad (8.25)$$

Equation (8.24) indicates a set of poloidal points with equal arc intervals corresponds to a set of uniform θ_i points. Therefore this choice of the Jacobian is called the equal-arc-length Jacobian. Note that Eq. (8.24) does not involve the radial coordinate ψ . Therefore the values of θ of points on any magnetic surface can be determined before the radial coordinate is chosen.

8.4.3 Jacobian for equal-volume poloidal angle (Hamada coordinate)

The volume element in (ψ, θ, ϕ) coordinates is given by $dV = |\mathcal{J}| d\theta d\phi d\psi$. If we choose a Jacobian that is independent of θ , then uniform θ grids will correspond to grids with uniform volume interval. In this case, \mathcal{J} is written as

$$\mathcal{J} = h(\psi), \quad (8.26)$$

where $h(\psi)$ is a function independent of θ . Then $\bar{\theta}_{i,j}$ in Eq. (8.21) is written

$$\bar{\theta}_{i,j} = \bar{\theta}_{\text{ref},j} + \frac{2\pi}{\oint \frac{R}{|\nabla\psi|} dl_p} \int_{\mathbf{x}_{\text{ref},j}}^{\mathbf{x}_{i,j}} \frac{R}{|\nabla\psi|} dl_p \quad (8.27)$$

and the Jacobian of the new coordinates $(\psi, \bar{\theta}, \phi)$, \mathcal{J}_{new} , is given by Eq. (8.22), which now takes the following form:

$$\mathcal{J}_{\text{new}} = \pm \frac{1}{2\pi} \oint \frac{R}{|\nabla\psi|} dl_p. \quad (8.28)$$

Note that both $\bar{\theta}_{i,j}$ and \mathcal{J}_{new} are independent of the function $h(\psi)$ introduced in Eq. (8.26). ($h(\psi)$ is eliminated by the normalization procedure specified in Sec. 8.4.1 due to the fact that $h(\psi)$ is constant on a magnetic surface.) The equal-volume poloidal angle is also called Hamada poloidal angle.

The equal-volume poloidal angle is useful in achieving loading balance for parallel particle simulations. Assume that markers are loaded uniform in space and the poloidal angle is domain decomposed and assigned to different MPI process. Then the equal-volume poloidal angle can make marker number in each MPI process be equal to each other and thus work loading to each process be equal. (**check** If the domain decomposition is also applied to the radial direction, to achieve loading balance, then the radial coordinate ψ should be chosen in a way that makes $\oint (R/|\nabla\psi|) dl_p$ be independent of ψ , so that \mathcal{J}_{new} in Eq. (8.28) is constant in space.**)

8.4.4 Jacobian of Boozer's poloidal angle

If the Jacobian \mathcal{J} is chosen to be of the Boozer form:

$$\mathcal{J}(\psi, \theta) = \frac{h(\psi)}{B^2}. \quad (8.29)$$

then the poloidal angle in Eq. (8.21) is written as

$$\theta_{i,j} = \theta_{\text{ref},j} + \frac{2\pi}{\oint \frac{B^2 R}{|\nabla\psi|} dl_p} \int_{\mathbf{x}_{\text{ref},j}}^{\mathbf{x}_{i,j}} \frac{B^2 R}{|\nabla\psi|} dl_p. \quad (8.30)$$

The final Jacobian is given by

$$\mathcal{J}_{\text{new}} = \pm \frac{\oint \frac{B^2 R}{|\nabla\psi|} dl_p}{2\pi} \frac{1}{B^2}. \quad (8.31)$$

The usefulness of Boozer poloidal angle will be further discussed in Sec. 12.8 after we introduce a generalized toroidal angle.

8.4.5 Jacobian for PEST poloidal angle (straight-field-line poloidal angle)

If the Jacobian \mathcal{J} is chosen to be of the following form

$$\mathcal{J}(\psi, \theta) = R^2,$$

then Eq. (8.4) implies that the local safety factor, $\hat{q}(\psi, \theta) = -g/\Psi'$, is a magnetic surface function, i.e., the magnetic field lines are straight in (ψ, θ) plane. Then the poloidal angle in Eq. (8.21) is written

$$\bar{\theta}_{i,j} = \bar{\theta}_{\text{ref},j} + \frac{2\pi}{\oint \frac{1}{R|\nabla\psi|} dl_p} \int_{\mathbf{x}_{\text{ref},j}}^{\mathbf{x}_{i,j}} \frac{1}{R|\nabla\psi|} dl_p, \quad (8.32)$$

The Jacobian \mathcal{J}_{new} given by Eq. (8.22) now takes the form

$$\mathcal{J}_{\text{new}} = \pm R^2 \frac{\oint \frac{1}{R|\nabla\psi|} dl_p}{2\pi}. \quad (8.33)$$

Let us denote an arbitrary poloidal angle by θ and the above straight-field-line poloidal angle by ϑ , then it is ready to find the following relation between θ and ϑ :

$$\vartheta = \vartheta_{\text{ref},j} + \frac{1}{q} \int_{\theta_{\text{ref},j}}^{\theta} \hat{q} d\theta, \quad (8.34)$$

where \hat{q} is the local safety factor corresponding to the arbitrary poloidal angle θ , i.e., $\hat{q} = \mathbf{B} \cdot \nabla \phi / (\mathbf{B} \cdot \nabla \theta)$. [Proof: Using $d\theta = \frac{R}{|\mathcal{J}\nabla\psi|} dl_p$, the poloidal angle ϑ given in Eq. (8.32) is written as

$$\begin{aligned} \vartheta &= \vartheta_{\text{ref},j} + \frac{2\pi}{\oint \frac{1}{R|\nabla\psi|} dl_p} \int_{\mathbf{x}_{\text{ref},j}}^{\mathbf{x}_{i,j}} \frac{1}{R|\nabla\psi|} dl_p \\ &= \vartheta_{\text{ref},j} + \frac{2\pi}{\oint \frac{1}{R|\nabla\psi|} \frac{|\mathcal{J}\nabla\psi|}{R} d\theta} \int_{\mathbf{x}_{\text{ref},j}}^{\mathbf{x}_{i,j}} \frac{1}{R|\nabla\psi|} \frac{|\mathcal{J}\nabla\psi|}{R} d\theta \\ &= \vartheta_{\text{ref},j} + \frac{2\pi}{\oint \frac{|\mathcal{J}|}{R^2} d\theta} \int_{\mathbf{x}_{\text{ref},j}}^{\mathbf{x}_{i,j}} \frac{|\mathcal{J}|}{R^2} d\theta. \end{aligned} \quad (8.35)$$

Using $\hat{q} = -\frac{g}{R^2} \frac{\mathcal{J}}{\Psi'}$, where \mathcal{J} is the Jacobian of (ψ, θ, ϕ) coordinates, then the above ϑ is reduced to expression (8.34).]

Note that Boozer poloidal angle is very close to the poloidal angle discussed here because the two Jacobians are very similar:

$$\mathcal{J} = \frac{1}{B^2} \sim R^2. \quad (8.36)$$

8.4.6 Comparison between different types of poloidal angle

All Jacobians introduced above can be written in a general form:

$$\mathcal{J} = \frac{R^i}{|\nabla\psi|^j B^k}, \quad (8.37)$$

The choice of $(i = 2, j = k = 0)$ gives the PEST coordinate, $(i = j = 0, k = 2)$ give the Boozer coordinate, $(i = j = 1, k = 0)$ gives the equal-arc coordinate, $(i = j = k = 0)$ gives the Hammada coordinate.

Figure 8.1 compares the equal-arc-poloidal angle and the straight-line poloidal angle, which shows that the resolution of the straight-line poloidal angle is not good near the low-field-side midplane. Since ballooning modes take larger amplitude near the low-field-side midplane, better resolution is desired there. This is one reason that I often avoid using the straight-line poloidal angle in my numerical codes.

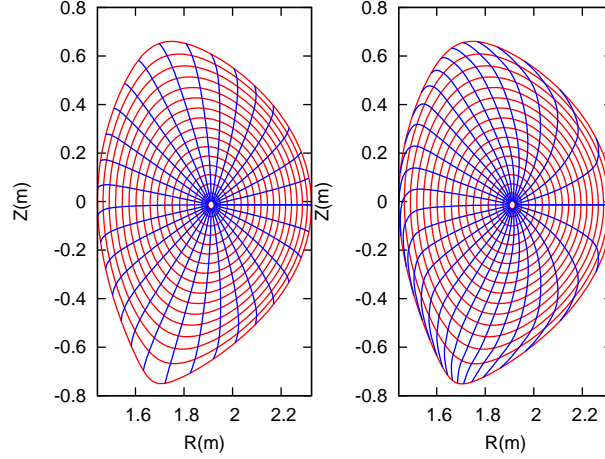


Figure 8.1. The equal-arc poloidal angle (left) and the straight-line poloidal angle (right) for EAST equilibrium #38300@3.9s. The blue lines correspond to the equal- θ lines, with uniform θ interval between neighbour lines. The red lines correspond to the magnetic surfaces, which start from $\sqrt{\Psi_t} = 0.01714$ (the innermost magnetic surface) and end at $\sqrt{\Psi_t} = 0.9851$ (the boundary magnetic surface), and are equally spaced in $\sqrt{\Psi_t}$.

8.4.7 Verification of Jacobian

After the magnetic coordinates are constructed, we can evaluate the Jacobian \mathcal{J}_{new} by using directly the definition of the Jacobian, i.e.,

$$\mathcal{J}_{\text{new}} = \frac{1}{(\nabla\psi \times \nabla\bar{\theta}) \cdot \nabla\phi}, \quad (8.38)$$

which can be further written as

$$\mathcal{J}_{\text{new}} = R(R_{\bar{\theta}}Z_{\psi} - R_{\psi}Z_{\bar{\theta}}), \quad (8.39)$$

where the partial differential can be evaluated by using numerical differential schemes. The results obtained by this way should agree with results obtained from the analytical form of the Jacobian. This consistency check provide a verification for the correctness of the theory derivation and numerical implementation. In evaluating the Jacobian by using the analytical form, we may need to evaluate $\nabla\psi$, which finally reduces to evaluating $\nabla\Psi$. The value of $|\nabla\Psi|$ is obtained numerically based on the numerical data of Ψ given in cylindrical coordinate grids. Then the cubic spline interpolating formula is used to obtain the value

of $|\nabla\Psi|$ at desired points. (\mathcal{J}_{new} calculated by the second method (i.e. using analytic form) is used in the GTAW code; the first methods are also implemented in the code for the benchmark purpose.) In the following sections, for notation ease, the Jacobian of the constructed coordinate system will be denoted by \mathcal{J} , rather than \mathcal{J}_{new} .

8.4.8 Radial coordinate

The radial coordinate ψ can be chosen to be various surface function, e.g., volume, poloidal or toroidal magnetic flux within a magnetic surface. The frequently used radial coordinates include $\bar{\Psi}$, and $\sqrt{\bar{\Psi}}$, where $\bar{\Psi}$ is defined by

$$\bar{\Psi} = \frac{\Psi - \Psi_0}{\Psi_a - \Psi_0}, \quad (8.40)$$

where Ψ_0 and Ψ_a are the values of Ψ at the magnetic axis and LCFS, respectively. Other choices of the radial coordinates: the toroidal magnetic flux and its square root, $\bar{\Psi}_t$, and $\sqrt{\bar{\Psi}_t}$, where Ψ_t and $\bar{\Psi}_t$ are defined by

$$\frac{d\Psi_t}{d\Psi} = 2\pi q, \Psi_t(0) = 0 \quad (8.41)$$

and

$$\bar{\Psi}_t = \frac{\Psi_t}{\Psi_t(1)}, \quad (8.42)$$

respectively, where $\Psi_t(0)$ and $\Psi_t(1)$ are the values of Ψ_t at the magnetic axis and LCFS, respectively.

If $\psi = \sqrt{\bar{\Psi}_t}$, then

$$\frac{d\Psi}{d\psi} = \frac{d\Psi}{d\Psi_t} \frac{d\Psi_t}{d\psi} = \frac{1}{2\pi q} \frac{d\Psi_t}{d\psi} = \frac{1}{2\pi q} \frac{d\Psi_t}{d\bar{\Psi}_t} \frac{d\bar{\Psi}_t}{d\psi} = \frac{1}{2\pi q} \Psi_t(1) 2\psi \quad (8.43)$$

The cylindrical coordinates (R, ϕ, Z) is a right-hand system, with the positive direction of Z pointing vertically up. In GTAW code, the positive direction of θ is chosen in the anticlockwise direction when observers look along the direction of $\hat{\phi}$. Then the definition $\mathcal{J}^{-1} = \nabla\psi \times \nabla\theta \cdot \nabla\phi$ indicates that (1) \mathcal{J} is negative if $\nabla\psi$ points from the magnetic axis to LCFS; (2) \mathcal{J} is positive if $\nabla\psi$ points from the LCFS to the magnetic axis. This can be used to determine the sign of Jacobian after using the analytical formula to obtain the absolute value of Jacobian.

If $\psi = \bar{\Psi}$ or $\psi = \sqrt{\bar{\Psi}}$, $\nabla\psi$ points from the magnetic axis to LCFS.

The volume between magnetic surfaces can also be used as a radial coordinate. The differential volume element is written as

$$d^3V = \mathcal{J} d\psi d\theta d\phi. \quad (8.44)$$

Integrating over the toroidal angle, we obtain

$$d^2V = 2\pi \mathcal{J} d\psi d\theta \quad (8.45)$$

Further integrating over the poloidal angle, we obtain

$$d^1V = 2\pi d\psi \oint \mathcal{J} d\theta, \quad (8.46)$$

i.e.,

$$\frac{d^1V}{d\psi} = 2\pi \oint \mathcal{J} d\theta. \quad (8.47)$$

In codes I wrote, I stick to using $\bar{\Psi}$ as the radial coordinate when doing computation, and transform to other radial coordinates when presenting the results if needed.

9 Constructing magnetic surface coordinate system from discrete $\Psi(R, Z)$ data

Given an axisymmetric tokamak equilibrium in (R, ϕ, Z) coordinates (e.g., 2D data $\Psi(R, Z)$ on a rectangular grids (R, Z) in G-file), we can construct a magnetic surface coordinates (ψ, θ, ϕ) by the following two steps. (1) Find out a series of magnetic surfaces on (R, Z) plane and select radial coordinates for each magnetic surface (e.g. the poloidal flux within each magnetic surface). (2) Specify the Jacobian or some property that we want the poloidal angle to have. Then calculate the poloidal angle of each point on each flux surface (on the $\phi = \text{const}$ plane) by using Eq. (8.16) (if the Jacobian is specified) or some method specified by us to achieve some property we prefer for the poloidal angle (if a Jacobian is not directly specified). Then we obtain the magnetic surface coordinates system (ψ, θ, ϕ) .

9.1 Finding magnetic surfaces

Two-dimensional data $\Psi(R, Z)$ on a rectangular grids (R, Z) is read from the G_EQDSK file (G-file) of EFIT code. Based on the 2D array data, I use 2D cubic spline interpolation to construct an interpolating function $\Psi = \Psi(R, Z)$. To construct a magnetic surface coordinate system, I need to find the contours of Ψ , i.e., magnetic surfaces. The values of Ψ on the magnetic axis, Ψ_0 , and the value of Ψ on the last closed flux surface (LCFS), Ψ_b , are given in G-file. Using these two values, I construct a 1D array “psival” with value of elements changing uniform from Ψ_0 to Ψ_b . Then I try to find the contours of Ψ with contour level value ranging from Ψ_0 to Ψ_b . This is done in the following way: construct a series of straight line (in the poloidal plane) that starts from the location of the magnetic axis and ends at one of the points on the LCFS. Combine the straight line equation, $Z = Z(R)$, with the interpolating function $\Psi(R, Z)$, we obtain a one variable function $h = \Psi(R, Z(R))$. Then finding the location where Ψ is equal to a specified value Ψ_i , is reduced to finding the root of the equation $\Psi(R, Z(R)) - \Psi_i = 0$. Since this is a one variable equation, the root can be easily found by using simple root finding scheme, such as bisection method (bisection method is used in GTAW code). After finding the roots for each value in the array “psival” on each straight lines, the process of finding the contours of Ψ is finished. The contours of Ψ found this way are plotted in Fig. 9.1.

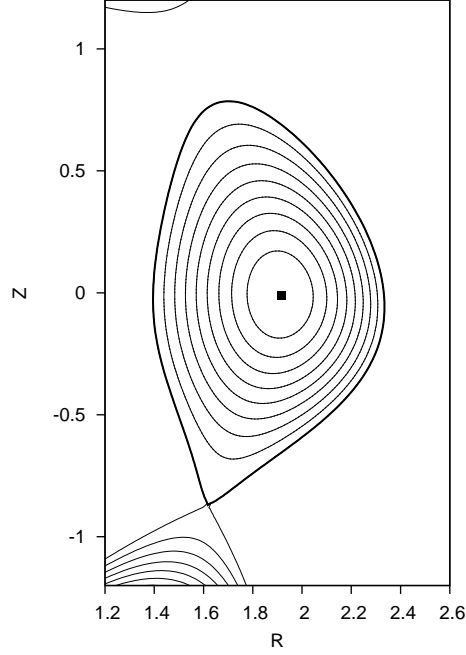


Figure 9.1. Verification of the numerical code that calculates the contours of the poloidal flux Ψ . The bold line in the figure indicates the LCFS. The contour lines (solid lines) given by the gnuplot program agrees well with the results I calculate by using interpolation and root-finding method (the two sets of contours are indistinguishable in this scale). My code only calculate the contour lines within the LCFS, while those given by gnuplot contains additional contour lines below the X point and on the left top in the figure. Eqdisk file of the equilibrium was provided by Dr. Guoqiang Li (filename: g013606.07104).

In the above, we mentioned that the point of magnetic axis and points on the LCFS are needed to construct the straight lines. In G-file, points on LCFS are given explicitly in an array. The location of magnetic axis is also explicitly given in G-file. It is obvious that some of the straight lines $Z = Z(R)$ that pass through the location of magnetic axis and points on the LCFS will have very large or even infinite slope. On these lines, finding the accurate root of the equation $\Psi(R, Z(R)) - \Psi_i = 0$ is difficult or even impossible. The way to avoid this situation is obvious: switch to use function $R = R(Z)$ instead of $Z = Z(R)$ when the slope of $Z = Z(R)$ is large (the switch condition I used is $|dZ/dR| > 1$).

In constructing the flux surface coordinate with desired Jacobian, we will need the absolute value of the gradient of Ψ , $|\nabla\Psi|$, on some specified spatial points. To achieve this, we need to construct a interpolating function for $|\nabla\Psi|$.

The $|\nabla\Psi|$ can be written as

$$|\nabla\Psi| = \sqrt{\left(\frac{\partial\Psi}{\partial R}\right)^2 + \left(\frac{\partial\Psi}{\partial Z}\right)^2}, \quad (9.1)$$

By using the center difference scheme to evaluate the partial derivatives with respect to R and Z in the above equation (using one side difference scheme for the points on the rectangular boundary), we can obtain an 2D array for the value of $|\nabla\Psi|$ on the rectangular (R, Z) grids. Using this 2D array, we can construct an interpolating function for $\nabla\Psi$ by using the cubic spline interpolation scheme.

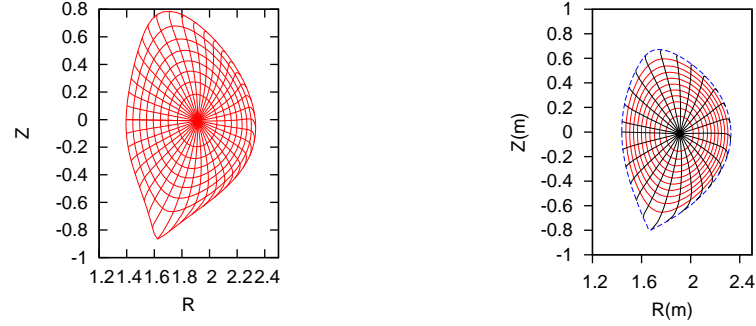


Figure 9.2. Grid points (the intersecting points of two curves in the figure) corresponding to uniform poloidal flux and uniform poloidal arc length for EAST equilibrium shot 13606 at 7.1s (left) (G-file name: g013606.07104) and shot 38300 at 3.9s (right) (G-file name: g038300.03900).

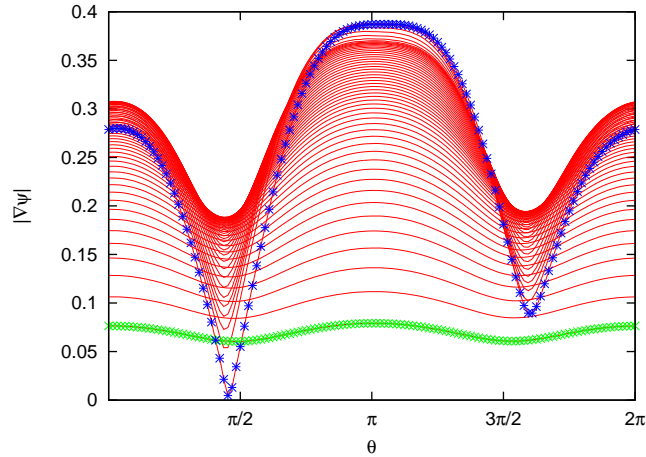


Figure 9.3. $|\nabla\Psi|$ as a function of the poloidal angle. The different lines corresponds to the values of $|\nabla\Psi|$ on different magnetic surfaces. The stars correspond to the values on the boundary magnetic surface while the plus signs correspond to the value on the innermost magnetic surface (the magnetic surface adjacent to the magnetic axis). The equilibrium is for EAST shot 38300 at 3.9s.

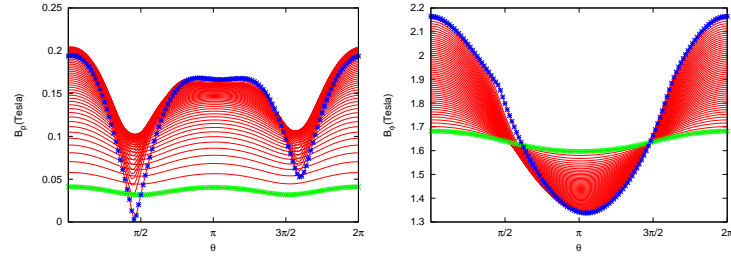


Figure 9.4. The Poloidal magnetic field $B_p = |\nabla\Psi|/R$ (left) and toroidal magnetic field $B_\phi = g/R$ (right) as a function of the poloidal angle. The different lines corresponds to the values on different magnetic surfaces. The stars correspond to the values on the boundary magnetic surface while the plus signs correspond to the value on the innermost magnetic surface (the magnetic surface adjacent to the magnetic axis). The equilibrium is for EAST shot 38300 at 3.9s.

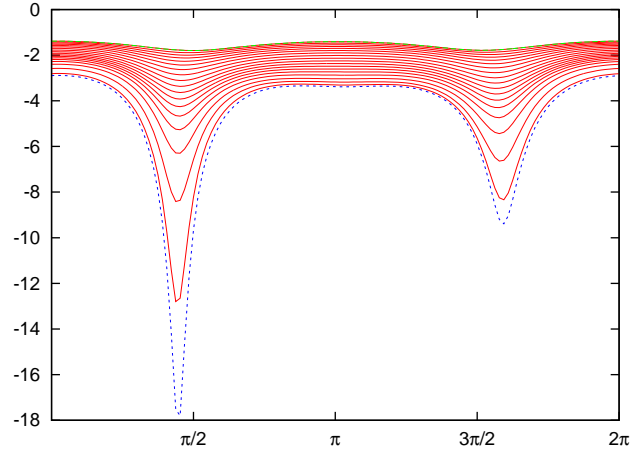


Figure 9.5. Equal-arc Jacobian as a function of the poloidal angle on different magnetic surfaces. The dotted line corresponds to the values of Jacobian on the boundary magnetic surface. The equilibrium is for EAST shot 38300 at 3.9s.

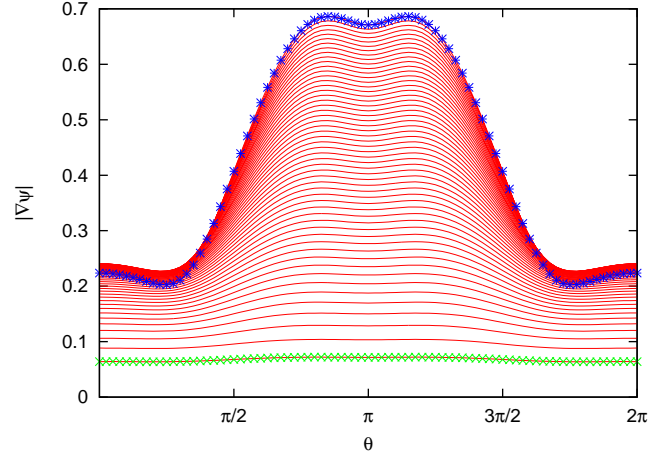


Figure 9.6. $|\nabla\Psi|$ as a function of the poloidal angle. The different lines corresponds to values of $|\nabla\Psi|$ on different magnetic surfaces. The stars correspond to the values of $|\nabla\Psi|$ on the boundary magnetic surface while the plus signs correspond to the value on the innermost magnetic surface (the magnetic surface adjacent to the magnetic axis). The equilibrium is a Solovév equilibrium.

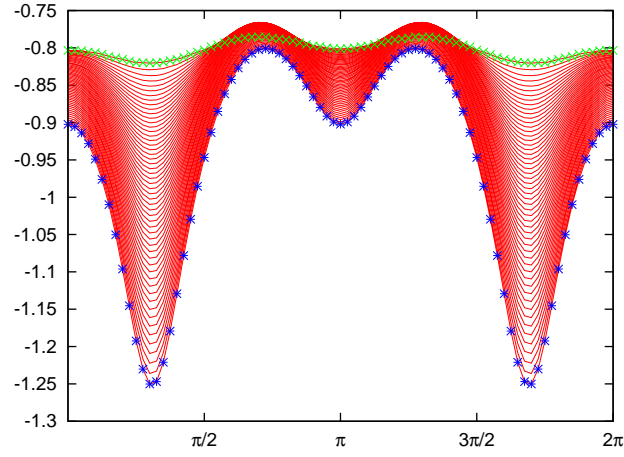


Figure 9.7. Jacobian on different magnetic surfaces as a function of the poloidal angle. The equilibrium is a Solovév equilibrium and the Jacobian is an equal-arc Jacobian. The stars correspond to the values of Jacobian on the boundary magnetic surface while the plus signs correspond to the value on the innermost magnetic surface (the magnetic surface adjacent to the magnetic axis).

9.2 Expression of metric elements of magnetic coordinates (ψ, θ, ϕ)

Metric elements of the (ψ, θ, ϕ) coordinates, e.g., $\nabla\psi \cdot \nabla\theta$, are often needed in practical calculations. Next, we express these metric elements in terms of the cylindrical coordinates (R, Z) and their partial derivatives with respect to ψ and θ . Note that, in this case, the coordinate system is (ψ, θ, ϕ) while R and Z are functions of ψ and θ , i.e.,

$$R = R(\psi, \theta), \quad (9.2)$$

$$Z = Z(\psi, \theta). \quad (9.3)$$

Then ∇R and ∇Z are written as

$$\nabla R = \hat{\mathbf{R}} = R_\psi \nabla\psi + R_\theta \nabla\theta, \quad (9.4)$$

$$\nabla Z = \hat{\mathbf{Z}} = Z_\psi \nabla\psi + Z_\theta \nabla\theta, \quad (9.5)$$

where $R_\psi \equiv \partial R / \partial \psi$, etc. Equations (9.4) and (9.5) can be solved to give

$$\nabla\psi = \frac{1}{R_\psi Z_\theta - Z_\psi R_\theta} (Z_\theta \hat{\mathbf{R}} - R_\theta \hat{\mathbf{Z}}), \quad (9.6)$$

$$\nabla\theta = \frac{1}{Z_\psi R_\theta - R_\psi Z_\theta} (Z_\psi \hat{\mathbf{R}} - R_\psi \hat{\mathbf{Z}}). \quad (9.7)$$

Using the above expressions, the Jacobian of (ψ, θ, ϕ) coordinates, \mathcal{J} , is written as

$$\begin{aligned} \mathcal{J}^{-1} &= \nabla\psi \times \nabla\theta \cdot \nabla\phi \\ &= -\frac{1}{(R_\psi Z_\theta - Z_\psi R_\theta)^2} (Z_\theta R_\psi \hat{\phi} - R_\theta Z_\psi \hat{\phi}) \times \frac{\hat{\phi}}{R} \\ &= -\frac{1}{(Z_\theta R_\psi - R_\theta Z_\psi)R}, \end{aligned} \quad (9.8)$$

i.e.,

$$\mathcal{J} = R(R_\theta Z_\psi - R_\psi Z_\theta). \quad (9.9)$$

Using this, Expressions (9.6) and (9.7) are written as

$$\nabla\psi = -\frac{R}{\mathcal{J}} (Z_\theta \hat{\mathbf{R}} - R_\theta \hat{\mathbf{Z}}) \quad (9.10)$$

and

$$\nabla\theta = \frac{R}{\mathcal{J}} (Z_\psi \hat{\mathbf{R}} - R_\psi \hat{\mathbf{Z}}). \quad (9.11)$$

Then the elements of the metric matrix are written as

$$|\nabla\psi|^2 = \frac{R^2}{\mathcal{J}^2}(Z_\theta^2 + R_\theta^2), \quad (9.12)$$

$$|\nabla\theta|^2 = \frac{R^2}{\mathcal{J}^2}(Z_\psi^2 + R_\psi^2), \quad (9.13)$$

and

$$\nabla\psi \cdot \nabla\theta = -\frac{R^2}{\mathcal{J}^2}(Z_\theta Z_\psi + R_\theta R_\psi). \quad (9.14)$$

Equations (9.12), (9.13), and (9.14) are the expressions of the metric elements in terms of R , R_ψ , R_θ , Z_ψ , and Z_θ . [Combining the above results, we obtain

$$\frac{\nabla\psi \cdot \nabla\theta}{|\nabla\psi|^2} = -\frac{Z_\theta Z_\psi + R_\theta R_\psi}{Z_\theta^2 + R_\theta^2}. \quad (9.15)$$

Equation (9.14) is used in GTAW code. Using the above results, $h^{\alpha\beta} = \frac{\mathcal{J}}{R^2} \nabla\alpha \cdot \nabla\beta$ are written as

$$h^{\psi\psi} = \frac{\mathcal{J}}{R^2} |\nabla\psi|^2 = \frac{1}{\mathcal{J}}(Z_\theta^2 + R_\theta^2) \quad (9.16)$$

$$h^{\theta\theta} = \frac{\mathcal{J}}{R^2} |\nabla\theta|^2 = \frac{1}{\mathcal{J}}(Z_\psi^2 + R_\psi^2), \quad (9.17)$$

$$h^{\psi\theta} = \frac{\mathcal{J}}{R^2} \nabla\psi \cdot \nabla\theta = -\frac{1}{\mathcal{J}}(Z_\theta Z_\psi + R_\theta R_\psi) \quad (9.18)$$

As a side product of the above results, we can calculate the arc length in the poloidal plane along a constant ψ surface, $d\ell_p$, which is expressed as

$$\begin{aligned} d\ell_p &= \sqrt{(dR)^2 + (dZ)^2} \\ &= \sqrt{(R_\psi d\psi + R_\theta d\theta)^2 + (Z_\psi d\psi + Z_\theta d\theta)^2}. \end{aligned}$$

Note that $d\psi = 0$ since we are considering the arc length along a constant ψ surface in (R, Z) plane. Then the above expression is reduced to

$$\begin{aligned} d\ell_p &= \sqrt{(R_\theta d\theta)^2 + (Z_\theta d\theta)^2} \\ &= \sqrt{R_\theta^2 + Z_\theta^2} d\theta \\ &= \frac{|\mathcal{J} \nabla\psi|}{R} d\theta, \end{aligned} \quad (9.19)$$

which agrees with Eq. (8.16).]

10 Constructing model tokamak magnetic field

In some cases (e.g., turbulence simulation), model tokamak magnetic field, which is not an exact solution to the GS equation, is often used. In the model, the safety factor profile $q(\psi)$, toroidal field function $g(\psi)$, and the magnetic surface shape $(R(\psi, \theta), Z(\psi, \theta))$ are given. To use this model in a simulation, we need to calculate its poloidal magnetic field, which is determined by the poloidal magnetic flux function Ψ . To determine Ψ , we need to use Eq. (8.8), i.e.,

$$q(\psi) = -\frac{1}{2\pi} \frac{g}{\Psi'} \int_0^{2\pi} \frac{\mathcal{J}}{R^2} d\theta, \quad (10.1)$$

and re-organize the formula as

$$\frac{d\Psi}{d\psi} = -\frac{1}{2\pi} \frac{g(\psi)}{q(\psi)} \int_0^{2\pi} \frac{\mathcal{J}}{R^2} d\theta, \quad (10.2)$$

which can be integrated to obtain Ψ and thus the poloidal magnetic field $\mathbf{B}_p = \nabla\Psi \times \nabla\phi$, where $\mathcal{J} = [(\nabla\psi \times \nabla\theta) \cdot \nabla\phi]^{-1}$. The toroidal magnetic field can be obtained from $g(\psi)$ by $B_\phi = g/R$. In most papers, $g(\psi)$ is chosen to be a constant.

A typical magnetic surface shape used in simulations is the Miller shape, which is given by

$$\begin{aligned} R(r, \theta) &= R_0(r) + r \cos[\theta + (\sin^{-1}\delta(r)) \sin\theta] \\ Z(r, \theta) &= \kappa(r) r \sin\theta \end{aligned} \quad (10.3)$$

where $r = \psi$ is the radial coordinate, $R_0(r)$, $\delta(r)$, and $\kappa_r(r)$ are the Shafranov shift, triangularity, and elongation profiles, which can be arbitrarily specified. For the special case of R_0 being a constant, $\delta(r) = 0$, and $\kappa(r) = 1$, this shape reduces to a concentric-circular magnetic field. An example of calculating the poloidal field of the model field is given in Sec. 14. This kind of model field can be called theoretical physicists' tokamak. Computational physicists often use this model for code benchmarking purpose. The famous DIII-D cyclone base case is an example, which was extensively used for benchmarking gyrokinetic simulation of ion temperature driven turbulence.

11 Magnetic surface averaging

11.1 Definition

The magnetic surface average of a physical quantity $G(\psi, \theta, \phi)$ is defined by

$$\langle G \rangle \equiv \lim_{\Delta\Psi \rightarrow 0} \left(\frac{\int \int \int_{\Delta\Psi} G d^3V}{\int \int \int_{\Delta\Psi} d^3V} \right), \quad (11.1)$$

where the volume integration is over a small volume between two adjacent flux surfaces with Ψ differing by $\Delta\Psi$. [This formula (with finite $\Delta\Psi$) is used in TEK code to calculate the radial heat flux.]

The above 3D volume integration can also be written as a 2D surface integration. The differential volume element is given by $d^3V = |\mathcal{J}|d\psi d\theta d\phi$, where \mathcal{J} is the Jacobian of (ψ, θ, ϕ) coordinates. Using this, equation (11.1) is written as

$$\begin{aligned}\langle G \rangle &= \lim_{\Delta\Psi \rightarrow 0} \left(\frac{\iint \iint_{\Delta\Psi} G |\mathcal{J}| d\psi d\theta d\phi}{\iint \iint_{\Delta\Psi} |\mathcal{J}| d\psi d\theta d\phi} \right) \\ &= \frac{\iint G |\mathcal{J}| d\theta d\phi}{\iint |\mathcal{J}| d\theta d\phi},\end{aligned}\tag{11.2}$$

which is a 2D averaging over a magnetic surface and thus is called magnetic surface average. Note that the surface averaging of any $n \neq 0$ harmonic is zero (n is the toroidal mode number). Therefore the magnetic surface average contains only the contribution from the $n=0$ component, i.e., axisymmetric component. (On the other hand, $m \neq 0$ poloidal harmonics of G can contribute to the surface average since the Jacobian has a poloidal angle dependence.) Using this and noting that \mathcal{J} is axisymmetric, then expression (11.2) is written as

$$\langle G \rangle = \frac{\int G_0(\psi, \theta) |\mathcal{J}| d\theta}{\int |\mathcal{J}| d\theta},\tag{11.3}$$

where $G_0(\theta)$ is defined by the following Fourier expansion:

$$G = \sum_{n=-\infty}^{+\infty} G_n(\psi, \theta) \exp(-in\phi).\tag{11.4}$$

“Zonal” and “mean” components

$\langle G \rangle$ is sometimes called the “zonal” component of G if the radial wavelength of $\langle G \rangle$ is much smaller than the equilibrium scale length. If the radial wavelength of $\langle G \rangle$ is comparable to the equilibrium scale length, $\langle G \rangle$ is usually called “mean” component in tokamak literature. For example, mean flows are of system space scale and thus are easy to be observed in experiments. On the other hand, the “zonal” flow, which usually refers to the turbulence generated secondary flow, is of much smaller radial scale (the radial wavelength of zonal flow is of several Larmor radius) and thus is difficult to observe in experiments.

Sometimes, we do not want the Jacobian to explicitly appear in the formula. This can be achieved by writing the differential volume element as

$$d^3V = R d\phi \frac{d\Psi}{|\nabla\Psi|} dl_p.\tag{11.5}$$

Using $B_p = |\nabla\Psi|/R$, the volume element is further written as

$$d^3V = d\phi \frac{d\Psi}{B_p} dl_p \quad (11.6)$$

Using this, the averaging defined in Eq. (11.1) is written as

$$\begin{aligned} \langle G \rangle &= \lim_{\Delta\Psi \rightarrow 0} \frac{\iint \int_{\Delta\Psi} G d\phi \frac{d\Psi}{B_p} dl_p}{\iint \int_{\Delta\Psi} d\phi \frac{d\Psi}{B_p} dl_p} \\ &= \frac{\iint G \frac{1}{B_p} d\phi dl_p}{\iint \frac{1}{B_p} d\phi dl_p}. \end{aligned} \quad (11.7)$$

If G is axisymmetric, then the above equation is written as

$$\langle G \rangle = \frac{\oint G \frac{1}{B_p} dl_p}{\oint \frac{1}{B_p} dl_p}. \quad (11.8)$$

(Equation (11.8) is used in the GTAW code to calculate the magnetic surface averaging.) Using Eq. (8.16) and $B_p = |\nabla\Psi|/R$, equation (11.8) can also be written as

$$\langle G \rangle = \frac{\int_0^{2\pi} G |\mathcal{J}| d\theta}{\int_0^{2\pi} |\mathcal{J}| d\theta}. \quad (11.9)$$

Using the expression of the volume element $d\tau = |\mathcal{J}| d\theta d\phi d\psi$, the volume within a magnetic surface is written

$$V(\psi) = \int d\tau = \int |\mathcal{J}| d\theta d\phi d\psi = 2\pi \int_{\psi_0}^{\psi} \int_0^{2\pi} |\mathcal{J}| d\theta d\psi. \quad (11.10)$$

Using this, the differential of V with respect to ψ is written as

$$V' \equiv \frac{dV}{d\psi} = 2\pi \int_0^{2\pi} |\mathcal{J}| d\theta. \quad (11.11)$$

Using this, Eq. (11.9) is written as

$$\langle G \rangle = \frac{2\pi}{V'} \int_0^{2\pi} G |\mathcal{J}| d\theta$$

11.2 Flux Surface Functions—to be deleted

Next, examine the meaning of the following volume integral

$$D(\psi) \equiv \int_V \mathbf{B} \cdot \nabla \theta d\tau, \quad (11.12)$$

where the volume $V = V(\psi)$ is the volume within the magnetic surface labeled by ψ . Using $\nabla \cdot \mathbf{B} = 0$, the quantity D can be further written as

$$D = \int_V \nabla \cdot (\theta \mathbf{B}) d\tau. \quad (11.13)$$

Note that θ is not a single-value function of the spacial points. In order to evaluate the integration in Eq. (11.13), we need to select one branch of θ , which can be chosen to be $0 \leq \theta < 2\pi$. Note that function $\theta = \theta(R, Z)$ is not continuous in the vicinity of the contour of $\theta = 0$. Next, we want to use the Gauss's theorem to convert the above volume integration to surface integration. Noting the discontinuity of the integrand $\theta \mathbf{B}$ in the vicinity of the contour of $\theta = 0$, the volume should be cut along the contour, thus, generating two surfaces. Denote these two surfaces by S_1 and S_2 , then equation (11.13) is written as

$$D = \int_{S_1} \theta \mathbf{B} \cdot d\mathbf{S} + \int_{S_2} \theta \mathbf{B} \cdot d\mathbf{S} + \int_{S_3} \theta \mathbf{B} \cdot d\mathbf{S},$$

where the direction of surface S_1 is in the negative direction of θ , the direction of S_2 is in the positive direction of θ , and the surface S_3 is the toroidal magnetic surface $\psi = \psi_0$. The surface integration through S_3 is obviously zero since \mathbf{B} lies in this surface. Therefore, we have

$$\begin{aligned} D &= \int_{S_1} \theta \mathbf{B} \cdot d\mathbf{S} + \int_{S_2} \theta \mathbf{B} \cdot d\mathbf{S} + 0 \\ &= \int_{S_1} 0 \mathbf{B} \cdot d\mathbf{S} + \int_{S_2} 2\pi \mathbf{B} \cdot d\mathbf{S} \\ &= 2\pi \int_{S_2} \mathbf{B} \cdot d\mathbf{S}. \end{aligned} \quad (11.14)$$

Eq. (11.14) indicates that D is 2π times the magnetic flux through the S_2 surface. Thus, the poloidal flux through S_2 is written as

$$\Psi_p = \frac{1}{2\pi} D = \frac{1}{2\pi} \int_V \mathbf{B} \cdot \nabla \theta d\tau. \quad (11.15)$$

Using the expression of the volume element $d\tau = |\mathcal{J}| d\theta d\phi d\psi$, Ψ_p can be further written in terms of flux surface averaged quantities.

$$\begin{aligned} \Psi_p &= \frac{1}{2\pi} \int_V \mathbf{B} \cdot \nabla \theta |\mathcal{J}| d\theta d\phi d\psi \\ &= \int_0^\psi d\psi \int_0^{2\pi} \mathbf{B} \cdot \nabla \theta |\mathcal{J}| d\theta \\ &= \int_0^\psi d\psi \int_0^{2\pi} \Psi' \nabla \psi \times \nabla \phi \cdot \nabla \theta |\mathcal{J}| d\theta \end{aligned}$$

$$\begin{aligned}
&= -\text{sign}(\mathcal{J}) \int_0^\psi d\psi \int_0^{2\pi} \Psi'(\psi) d\theta \\
&= -2\pi \text{sign}(\mathcal{J}) \int_0^\psi \Psi'(\psi) d\psi \\
&= -2\pi \text{sign}(\mathcal{J}) [\Psi(\psi) - \Psi(0)].
\end{aligned} \tag{11.16}$$

Note that the sign of the Jacobian appears in Eq. (11.16), which is due to the positive direction of surface S_2 is determined by the positive direction of θ , which in turn is determined by the sign of the Jacobian (In my code, however, the positive direction of θ is chosen by me and the sign of the Jacobian is determined by the positive direction of θ). We can verify the sign of Eq. (11.16) is exactly consistent with that in Eq. (1.27).

Similarly, the toroidal flux within a flux surface is written as

$$\Psi_t = \frac{1}{2\pi} \int_V \mathbf{B} \cdot \nabla \phi d\tau, \tag{11.17}$$

the poloidal current within a flux surface is written as

$$K(\psi) = \frac{1}{2\pi} \int_V \mathbf{J} \cdot \nabla \theta d\tau, \tag{11.18}$$

and toroidal current within a flux surface is written as

$$I(\psi) = \frac{1}{2\pi} \int_V \mathbf{J} \cdot \nabla \phi d\tau. \tag{11.19}$$

(**check**)The toroidal magnetic flux is written as

$$\begin{aligned}
\Psi_t &= \frac{1}{2\pi} \int \mathbf{B} \cdot \nabla \phi |\mathcal{J}| d\theta d\phi d\psi \\
&= \int_0^\psi d\psi \int_0^{2\pi} g \frac{1}{R^2} |\mathcal{J}| d\theta \\
&= \int_0^\psi \left[g \frac{V'}{2\pi} \left\langle \frac{1}{R^2} \right\rangle \right] d\psi.
\end{aligned} \tag{11.20}$$

$$\Rightarrow \Psi'_t = g \frac{V'}{2\pi} \left\langle \frac{1}{R^2} \right\rangle$$

$$\Rightarrow \frac{d\Psi_t}{dV} = g \frac{1}{2\pi} \left\langle \frac{1}{R^2} \right\rangle$$

$$\Rightarrow \frac{d\Psi}{dV} = \frac{g}{2\pi q} \frac{1}{2\pi} \left\langle \frac{1}{R^2} \right\rangle. \tag{11.21}$$

Next, calculate the derivative of the toroidal flux with respect to the poloidal flux.

$$\begin{aligned}\frac{d\Psi_t}{d\Psi_p} &= \frac{\Psi'_t}{\Psi'_p} \\ &= -\frac{gV'}{(2\pi)^2\Psi'} \left\langle \frac{1}{R^2} \right\rangle,\end{aligned}\tag{11.22}$$

Comparing this result with Eq. (15.53) indicates that it is equal to the safety factor, i.e.,

$$\frac{d\Psi_t}{d\Psi_p} = q(\psi).\tag{11.23}$$

By using the contravariant representation of current density (16.1), the poloidal current within a magnetic surface is written as

$$\begin{aligned}K(\psi) &= \frac{1}{2\pi} \int \mathbf{J} \cdot \nabla \theta \mathcal{J} d\theta d\phi d\psi \\ &= \frac{1}{\mu_0} \int (-g') \nabla \phi \times \nabla \psi \cdot \nabla \theta \mathcal{J} d\theta d\psi \\ &= -\frac{1}{\mu_0} \int g' d\theta d\psi \\ &= -\frac{2\pi}{\mu_0} \int_0^\psi g' d\psi \\ &= -\frac{2\pi}{\mu_0} [g(\psi) - g(0)].\end{aligned}\tag{11.24}$$

Note that the poloidal current is proportional to g , which explains why g is sometimes called poloidal current function in tokamak literature.

$$-\left[\left(\Psi' \frac{\mathcal{J}}{R^2} |\nabla \psi|^2 \right)_\psi + \left(\Psi' \frac{\mathcal{J}}{R^2} \nabla \psi \cdot \nabla \theta \right)_\theta \right] \nabla \psi \times \nabla \theta - g' \nabla \phi \times \nabla \psi,$$

The toroidal current is written as

$$\begin{aligned}I_\phi(\psi) &= \frac{1}{2\pi} \int \mathbf{J} \cdot \nabla \phi \mathcal{J} d\theta d\phi d\psi \\ &= -\frac{1}{2\pi\mu_0} \int \left[\left(\Psi' \frac{\mathcal{J}}{R^2} |\nabla \psi|^2 \right)_\psi + \left(\Psi' \frac{\mathcal{J}}{R^2} \nabla \psi \cdot \nabla \theta \right)_\theta \right] \nabla \psi \times \nabla \theta \cdot \\ &\quad \nabla \phi \mathcal{J} d\theta d\phi d\psi \\ &= -\frac{1}{\mu_0} \int \left[\left(\Psi' \frac{\mathcal{J}}{R^2} |\nabla \psi|^2 \right)_\psi + \left(\Psi' \frac{\mathcal{J}}{R^2} \nabla \psi \cdot \nabla \theta \right)_\theta \right] d\theta d\psi\end{aligned}$$

$$\begin{aligned}
&= -\frac{1}{\mu_0} \int \left[\left(\Psi' \frac{\mathcal{J}}{R^2} |\nabla \psi|^2 \right) \right]_{\psi} d\theta d\psi \\
&= -\frac{1}{\mu_0} \int_0^{2\pi} d\theta \int_0^{\psi} d\psi \left(\Psi' \frac{\mathcal{J}}{R^2} |\nabla \psi|^2 \right)_{\psi} \\
&= -\frac{1}{\mu_0} \int_0^{2\pi} d\theta \left(\Psi' \frac{\mathcal{J}}{R^2} |\nabla \psi|^2 - 0 \right). \tag{11.25}
\end{aligned}$$

The last equality is due to $\nabla \psi = 0$ at $\psi = 0$. By using the flux surface average operator, Eq. (11.25) is written

$$I_{\phi}(\psi) = -\frac{V' \Psi'}{2\pi \mu_0} \left\langle \frac{|\nabla \psi|^2}{R^2} \right\rangle. \tag{11.26}$$

Next, calculate another useful surface-averaged quantity,

$$\begin{aligned}
\frac{\langle \mathbf{J} \cdot \mathbf{B} \rangle}{\langle \mathbf{B} \cdot \nabla \phi \rangle} &= \frac{\left\langle \frac{g^2}{\mathcal{J}} \left[\left(\frac{1}{g} \Psi' \frac{\mathcal{J}}{R^2} |\nabla \psi|^2 \right)_{\psi} + \left(\frac{1}{g} \Psi' \nabla \psi \cdot \nabla \theta \frac{\mathcal{J}}{R^2} \right)_{\theta} \right] \right\rangle}{\mu_0 \langle g/R^2 \rangle} \\
&= \frac{\frac{2\pi}{V'} \int_0^{2\pi} d\theta g^2 \left[\left(\frac{1}{g} \Psi' \frac{\mathcal{J}}{R^2} |\nabla \psi|^2 \right)_{\psi} + \left(\frac{1}{g} \Psi' \nabla \psi \cdot \nabla \theta \frac{\mathcal{J}}{R^2} \right)_{\theta} \right]}{\mu_0 g \langle R^{-2} \rangle} \\
&= \frac{\frac{2\pi}{V'} g^2 \int_0^{2\pi} d\theta \left[\left(\frac{1}{g} \Psi' \frac{\mathcal{J}}{R^2} |\nabla \psi|^2 \right)_{\psi} + \left(\frac{1}{g} \Psi' \nabla \psi \cdot \nabla \theta \frac{\mathcal{J}}{R^2} \right)_{\theta} \right]}{\mu_0 g \langle R^{-2} \rangle} \\
&= \frac{\frac{2\pi}{V'} g \int_0^{2\pi} d\theta \left[\left(\frac{1}{g} \Psi' \frac{\mathcal{J}}{R^2} |\nabla \psi|^2 \right)_{\psi} \right]}{\mu_0 \langle R^{-2} \rangle} \\
&= \frac{\frac{2\pi}{V'} g \int_0^{2\pi} d\theta \left[\left(\frac{1}{g} \Psi' \frac{\mathcal{J}}{R^2} |\nabla \psi|^2 \right)_{\psi} \right]}{\mu_0 \langle R^{-2} \rangle} \tag{11.27}
\end{aligned}$$

The differential with respect to ψ and the integration with respect to θ can be interchanged, yielding

$$\begin{aligned}
\frac{\langle \mathbf{J} \cdot \mathbf{B} \rangle}{\langle \mathbf{B} \cdot \nabla \phi \rangle} &= \frac{\frac{2\pi}{V'} g \left[\frac{1}{g} \Psi' \left(\int_0^{2\pi} d\theta \frac{\mathcal{J}}{R^2} |\nabla \psi|^2 \right) \right]_{\psi}}{\mu_0 \langle R^{-2} \rangle} \\
&= \frac{\frac{1}{V'} g \left[\frac{1}{g} \Psi' V' \left\langle \frac{|\nabla \psi|^2}{R^2} \right\rangle \right]_{\psi}}{\mu_0 \langle R^{-2} \rangle} \\
&= \frac{g}{\mu_0 V' \langle R^{-2} \rangle} \left[\frac{\Psi' V'}{g} \left\langle \frac{|\nabla \psi|^2}{R^2} \right\rangle \right]_{\psi} \tag{11.28}
\end{aligned}$$

12 Magnetic coordinates (ψ, θ, ζ) with general toroidal angle ζ

12.1 General toroidal angle ζ

In Sec. 8.1, we introduced the local safety factor $\hat{q}(\psi, \theta)$. Equation (8.4) indicates that if the Jacobian is chosen to be of the particular form $\mathcal{J} = h(\psi)R^2$, then the local safety factor is independent of θ , i.e., magnetic line is straight in (θ, ϕ) plane. On the other hand, if we want to make field line straight in (θ, ϕ) plane, the Jacobian must be chosen to be of the specific form $\mathcal{J} = h(\psi)R^2$. We note that, as mentioned in Sec. 8.3, the poloidal angle is fully determined by the choice of the Jacobian. The specific choice of $\mathcal{J} = \alpha(\psi)R^2$ is usually too restrictive for achieve a desired poloidal resolution (for example, the equal-arc poloidal angle can not be achieved by this choice of Jacobian). Is there any way that we can make the field line straight in a coordinate system at the same time ensure that the Jacobian can be freely adjusted to obtain desired poloidal angle? The answer is yes. The obvious way to achieve this is to define a new toroidal angle ζ that generalizes the usual toroidal angle ϕ . Define a new toroidal angle ζ by [10]

$$\zeta = \phi - q(\psi)\delta(\psi, \theta), \quad (12.1)$$

where $\delta = \delta(\psi, \theta)$ is a unknown function to be determined by the constraint of field line being straight in (θ, ζ) plane. Using Eq. (8.5), the new local safety factor in (ψ, θ, ζ) coordinates is written as

$$\begin{aligned} \hat{q}_{\text{new}} &\equiv \frac{\mathbf{B} \cdot \nabla \zeta}{\mathbf{B} \cdot \nabla \theta} \\ &= \frac{(\nabla \phi \times \nabla \psi + \hat{q} \nabla \psi \times \nabla \theta) \cdot \nabla \zeta}{(\nabla \phi \times \nabla \psi + \hat{q} \nabla \psi \times \nabla \theta) \cdot \nabla \theta} \\ &= \frac{\nabla \phi \times \nabla \psi \cdot \nabla \zeta + \hat{q} \nabla \psi \times \nabla \theta \cdot \nabla \zeta}{(\nabla \phi \times \nabla \psi) \cdot \nabla \theta} \\ &= \frac{\nabla \phi \times \nabla \psi \cdot \nabla(-q\delta) + \hat{q} \nabla \psi \times \nabla \theta \cdot \nabla \phi}{(\nabla \phi \times \nabla \psi) \cdot \nabla \theta} \\ &= -q \frac{\partial \delta}{\partial \theta} + \hat{q}. \end{aligned} \quad (12.2)$$

To make the new local safety factor be independent of θ , the right-hand side of Eq. (12.2) should be independent of θ , i.e.,

$$-q \frac{\partial \delta}{\partial \theta} + \hat{q} = c(\psi), \quad (12.3)$$

where $c(\psi)$ can be an arbitrary function of ψ . A convenient choice for $c(\psi)$ is $c(\psi) = q$, i.e., making the new local safety factor be equal to the original global safety factor, i.e., $\hat{q}_{\text{new}} = q$. In this case, equation (12.3) is written as

$$\frac{\partial \delta}{\partial \theta} = \frac{\hat{q}}{q} - 1, \quad (12.4)$$

which, on a magnetic surface labeled by ψ , can be integrated over θ to give

$$\delta(\psi, \theta) = \delta(\psi, \theta_{\text{ref}}) - (\theta - \theta_{\text{ref}}) + \frac{1}{q} \int_{\theta_{\text{ref}}}^{\theta} \hat{q} d\theta, \quad (12.5)$$

where θ_{ref} is an starting poloidal angle arbitrarily chosen for the integration, and $\delta(\psi, \theta_{\text{ref}})$ is the constant of integration. In the following, both θ_{ref} and $\delta(\psi, \theta_{\text{ref}})$ will be chosen to be zero. Then the above equations is written

$$\delta = -\theta + \frac{1}{q} \int_0^{\theta} \hat{q} d\theta. \quad (12.6)$$

Substituting the above expression into the definition of ζ (Eq. 12.1), we obtain

$$\zeta = \phi + q\theta - \int_0^{\theta} \hat{q} d\theta, \quad (12.7)$$

which is the formula for calculating the general toroidal angle. If θ is a straight-field line poloidal angle, then ζ in Eq. (12.7) reduces to the usual toroidal angle ϕ .

In summary, magnetic field line is straight in (θ, ζ) plane with slope being q if ζ is defined by Eq. (12.7). In this method, we make the field line straight by defining a new toroidal angle, instead of requiring the Jacobian to take particular forms. Thus, the freedom of choosing the form of the Jacobian is still available to be used later to choose a good poloidal angle coordinate. Note that the Jacobian of the new coordinates (ψ, θ, ζ) is equal to that of (ψ, θ, ϕ) . [Proof:

$$\begin{aligned} \mathcal{J}_{\text{new}}^{-1} &= \nabla \psi \times \nabla \theta \cdot \nabla \zeta \\ &= \nabla \psi \times \nabla \theta \cdot \nabla (\phi - q\delta) \\ &= \nabla \psi \times \nabla \theta \cdot \nabla \phi - \nabla \psi \times \nabla \theta \cdot \nabla (q\delta) \\ &= \nabla \psi \times \nabla \theta \cdot \nabla \phi - 0 \\ &= \mathcal{J}^{-1}. \end{aligned}$$

] Also note that $\delta(\psi, \theta)$ defined by Eq. (12.6) is a periodic function of θ . [Proof: Equation (12.6) implies that

$$\begin{aligned} \delta(\psi, \theta + 2\pi) &= \frac{1}{q} \int_0^{\theta+2\pi} \hat{q} d\theta - (\theta - \theta_{\text{ref}}) - 2\pi \\ &= \frac{1}{q} \int_0^{\theta} \hat{q} d\theta - (\theta - \theta_{\text{ref}}) - 2\pi + \frac{1}{q} \int_{\theta}^{\theta+2\pi} \hat{q} d\theta \\ &= \delta(\psi, \theta) - 2\pi + \frac{1}{q} \int_{\theta}^{\theta+2\pi} \hat{q} d\theta \\ &= \delta(\psi, \theta). \end{aligned} \quad (12.8)$$

]

[In numerical implementation, the term $\int_0^\theta \frac{\mathbf{B} \cdot \nabla \phi}{\mathbf{B} \cdot \nabla \theta} d\theta$ appearing in δ is computed by using

$$\begin{aligned} \int_0^\theta \frac{\mathbf{B} \cdot \nabla \phi}{\mathbf{B} \cdot \nabla \theta} d\theta &= \int_0^\theta \frac{g}{R^2} \frac{\mathcal{J}}{\Psi'} d\theta = \int_0^\theta \frac{g}{R^2} \frac{1}{\Psi'} \frac{R}{|\nabla \psi|} dl_p \\ &= \int_0^\theta \frac{g}{R} \frac{1}{|\nabla \Psi|} dl_p \\ &= \int_0^\theta \frac{1}{R} \frac{B_\phi}{B_p} d\ell_p. \end{aligned} \quad (12.9)$$

For later use, from Eq. (12.6), we obtain

$$\frac{\partial(\delta q)}{\partial \psi} = -\frac{d}{d\psi} \left(\frac{g}{\Psi'} \right) \int_0^\theta \frac{\mathcal{J}}{R^2} d\theta - \frac{g}{\Psi'} \frac{\partial}{\partial \psi} \int_0^\theta \frac{\mathcal{J}}{R^2} d\theta - \frac{dq}{d\psi} \theta. \quad (12.10)$$

This formula is used in GTAW code, where the derivative $\partial(g/\Psi')/\partial\psi$ is calculated numerically by using the central difference scheme.]

12.2 Contravariant form of magnetic field in (ψ, θ, ζ) coordinates

Recall that the contravariant form of the magnetic field in (ψ, θ, ϕ) coordinates is given by Eq. (8.5), i.e.,

$$\mathbf{B} = -\Psi'(\nabla \phi \times \nabla \psi + \hat{q} \nabla \psi \times \nabla \theta). \quad (12.11)$$

Next, let us derive the corresponding form in (ψ, θ, ζ) coordinates. Using the definition of ζ , equation (12.11) is written as

$$\begin{aligned} \mathbf{B} &= -\Psi' \nabla(\zeta + q\delta) \times \nabla \psi - \Psi' \hat{q} \nabla \psi \times \nabla \theta \\ &= -\Psi' \nabla \zeta \times \nabla \psi - \Psi' \nabla(q\delta) \times \nabla \psi - \Psi' \hat{q} \nabla \psi \times \nabla \theta \\ &= -\Psi' \nabla \zeta \times \nabla \psi - \Psi' q \frac{\partial \delta}{\partial \theta} \nabla \theta \times \nabla \psi - \Psi' \hat{q} \nabla \psi \times \nabla \theta \end{aligned} \quad (12.12)$$

Using Eq. (12.4), the above equation is simplified as

$$\mathbf{B} = -\Psi'(\nabla \zeta \times \nabla \psi + q \nabla \psi \times \nabla \theta). \quad (12.13)$$

Equation (12.13) is the contravariant form of the magnetic field in (ψ, θ, ζ) coordinates.

The expression of the magnetic field in Eq. (12.13) can be rewritten in terms of the flux function Ψ_p and Ψ_t discussed in Sec. 11.2. Equation (12.13) is

$$\mathbf{B} = \nabla \Psi \times \nabla \zeta + q \nabla \theta \times \nabla \Psi, \quad (12.14)$$

which, by using Eq. (11.16), i.e., $\nabla \Psi = \nabla \Psi_p / (2\pi)$, is rewritten as

$$\mathbf{B} = \frac{1}{2\pi} (\nabla \zeta \times \nabla \Psi_p + q \nabla \Psi_p \times \nabla \theta), \quad (12.15)$$

which, by using Eq. (11.23), i.e., $q = d\Psi_t/d\Psi_p$, is further written as

$$\mathbf{B} = \frac{1}{2\pi}(\nabla\zeta \times \nabla\Psi_p + \nabla\Psi_t \times \nabla\theta). \quad (12.16)$$

12.3 Relation between the partial derivatives in (ψ, θ, ϕ) and (ψ, θ, ζ) coordinates

Noting the simple fact that

$$\frac{d}{dx} = \frac{d}{d(x+c)}, \quad (12.17)$$

where c is a constant, we conclude that

$$\left(\frac{\partial f}{\partial \zeta}\right)_{\psi, \theta} = \left(\frac{\partial f}{\partial \phi}\right)_{\psi, \theta}, \quad (12.18)$$

(since $\phi = \zeta + q(\psi)\delta(\psi, \theta)$, where the part $q(\psi)\delta(\psi, \theta)$ acts as a constant when we hold ψ and θ constant), i.e., the symmetry property with respect to the new toroidal angle ζ is identical with the one with respect to the old toroidal angle ϕ . On the other hand, generally,

$$\left(\frac{\partial f}{\partial \psi}\right)_{\theta, \zeta} \neq \left(\frac{\partial f}{\partial \psi}\right)_{\theta, \phi} \quad (12.19)$$

and

$$\left(\frac{\partial f}{\partial \theta}\right)_{\psi, \zeta} \neq \left(\frac{\partial f}{\partial \theta}\right)_{\psi, \phi}. \quad (12.20)$$

In the special case that f is axisymmetric (i.e., f is independent of ϕ in (ψ, θ, ϕ) coordinates), then two sides of Eqs. (12.19) and (12.20) are equal to each other. Note that the partial derivatives $\partial/\partial\psi$ and $\partial/\partial\theta$ in Sec. 12.1 and 12.2 are taken in (ψ, θ, ϕ) coordinates. Because the quantities involved in Sec. 12.1 and 12.2 are axisymmetric, these partial derivatives are equal to their counterparts in (ψ, θ, ζ) coordinates.

12.4 Steps to construct a straight-line magnetic coordinate system

In Sec. 9, we have provided the steps to construct the magnetic surface coordinate system (ψ, θ, ϕ) . Only one additional step is needed to construct the straight-line flux coordinate system (ψ, θ, ζ) . The additional step is to calculate the generalized toroidal angle ζ according to Eq. (12.1), where δ is obtained from Eq. (12.6). Also note that the Jacobian of (ψ, θ, ζ) coordinates happens to be equal to that of (ψ, θ, ϕ) coordinates.

12.5 Form of operator $\mathbf{B} \cdot \nabla$ in (ψ, θ, ζ) coordinates

The usefulness of the contravariant form [Eq. (12.13)] of the magnetic field lies in that it allows a simple form of $\mathbf{B} \cdot \nabla$ operator in a coordinate system. (The operator $\mathbf{B}_0 \cdot \nabla$ is usually called magnetic differential operator.) In (ψ, θ, ζ) coordinate system, by using the contravariant form Eq. (12.13), the operator is written as

$$\begin{aligned} \mathbf{B} \cdot \nabla f &= -\Psi'(\nabla \zeta \times \nabla \psi) \cdot \nabla f(\psi, \theta, \zeta) - \Psi'q(\nabla \psi \times \nabla \theta) \cdot \nabla f(\psi, \theta, \zeta) \\ &= -\Psi' \mathcal{J}^{-1} \left(\frac{\partial}{\partial \theta} + q \frac{\partial}{\partial \zeta} \right) f. \end{aligned} \quad (12.21)$$

Next, consider the solution of the following magnetic differential equation:

$$\mathbf{B} \cdot \nabla f = h. \quad (12.22)$$

where $h = h(\psi, \theta, \zeta)$ is some known function. Using Eq. (12.21), the magnetic differential equation is written as

$$\left(\frac{\partial}{\partial \theta} + q(\psi) \frac{\partial}{\partial \zeta} \right) f = -\frac{1}{\Psi'} \mathcal{J}h(\psi, \theta, \zeta). \quad (12.23)$$

Note that the coefficients before the two partial derivatives of the above equation are all independent of θ and ζ . This indicates that different Fourier harmonics in θ and ζ are decoupled. As a result of this fact, if f is Fourier expanded as

$$f(\psi, \theta, \zeta) = \sum_{m,n} f_{mn}(\psi) e^{i(m\theta - n\zeta)}, \quad (12.24)$$

(note that, following the convention adopted in tokamak literature[10], the Fourier harmonics are chosen to be $e^{i(m\theta - n\zeta)}$, instead of $e^{i(m\theta + n\zeta)}$), and the right-hand side is expanded as

$$-\frac{1}{\Psi'} \mathcal{J}h(\psi, \theta, \zeta) = \sum_{m,n} \gamma_{mn}(\psi) e^{i(m\theta - n\zeta)}, \quad (12.25)$$

then Eq. (12.23) can be readily solved to give

$$f_{mn} = \frac{\gamma_{mn}}{i[m - nq]}. \quad (12.26)$$

The usefulness of the straight line magnetic coordinates (ψ, θ, ζ) lies in that, as mentioned previously, it makes the coefficients before the two partial derivatives both independent of θ and ζ , thus, allowing a simple solution to the magnetic differential equation.

12.6 Resonant surface of a perturbation

Equation (12.26) indicates that, for the differential equation (12.22), there is a resonant response to a perturbation $e^{i(m\theta-n\zeta)}$ on a magnetic surface with $m-nq=0$. Therefore, the magnetic surface with $q=m/n$ is called the “resonant surface” for the perturbation $e^{i(m\theta-n\zeta)}$.

The phase change of the perturbation $e^{i(m\theta-n\zeta)}$ along a magnetic field is given by $m\Delta\theta-n\Delta\zeta$, which can be written as $\Delta\theta(m-nq)$. Since $m-nq=0$ on a resonant surface, this indicates that there is no phase change along a magnetic field line on a resonant surface, i.e., the parallel wavenumber k_{\parallel} is zero on a resonant surface.

12.7 Helical angle used in tearing mode theory

Next, we discuss a special poloidal angle, which is useful in describing a perturbation of single harmonic (m, n) . This poloidal angle is defined by

$$\chi = \theta - \frac{n}{m}\zeta, \quad (12.27)$$

where (m, n) are the mode numbers of the perturbation. The poloidal angle χ is often called helical angle and is special in that its definition is associated with a perturbation (the mode numbers of the perturbation appear in the definition) while the definition of the poloidal angles discussed previously only involve the equilibrium quantities.

The poloidal angle χ is designed to make 3D perturbations of the form $\sim f(\psi, m\theta - n\zeta)$ reduce to 2D perturbations, i.e.,

$$\left. \frac{\partial f}{\partial \zeta} \right|_{\psi, \chi} = 0. \quad (12.28)$$

It is ready to verify that the Jacobian of coordinates (ψ, χ, ζ) is equal to that of coordinates (ψ, θ, ζ) [proof: $(\mathcal{J}')^{-1} = \nabla\psi \times \nabla\chi \cdot \nabla\zeta = \nabla\psi \times \nabla(\theta - n\zeta/m) \cdot \nabla\zeta = \nabla\psi \times \nabla\theta \cdot \nabla\zeta = \mathcal{J}^{-1}$].

The component of \mathbf{B} along $\nabla\chi$ direction (i.e., the covariant component) is written

$$\begin{aligned} B^{(\chi)} &\equiv \mathbf{B} \cdot \nabla\chi \\ &= -\Psi'(\nabla\zeta \times \nabla\psi + q\nabla\psi \times \nabla\theta) \cdot \nabla(\theta - n\zeta/m) \\ &= -\Psi'(\nabla\zeta \times \nabla\psi \cdot \nabla\theta) + \Psi' \frac{n}{m} (q\nabla\psi \times \nabla\theta \cdot \nabla\zeta) \\ &= -\Psi' \mathcal{J}^{-1} + \Psi' \frac{n}{m} q \mathcal{J}^{-1} \\ &= \Psi' \mathcal{J}^{-1} \left(\frac{nq}{m} - 1 \right). \end{aligned} \quad (12.29)$$

At the resonant surface $q = m/n$, equation (12.29) implies $B^{(\chi)} = 0$. The direction $\nabla\chi$ defines the reconnecting component of the magnetic field?

On the other hand, the component of \mathbf{B} along $\nabla\theta$ direction is written

$$\begin{aligned}
 B^{(\theta)} &\equiv \mathbf{B} \cdot \nabla\theta \\
 &= -\Psi'(\nabla\zeta \times \nabla\psi + q\nabla\psi \times \nabla\theta) \cdot \nabla\theta \\
 &= -\Psi'\nabla\zeta \times \nabla\psi \cdot \nabla\theta \\
 &= -\Psi'\mathcal{J}^{-1}.
 \end{aligned} \tag{12.30}$$

Using (12.30) and (12.29), the relation between $B^{(\theta)}$ and $B^{(\chi)}$ is written as

$$B^{(\chi)} = B^{(\theta)} \left(1 - \frac{nq}{m} \right). \tag{12.31}$$

12.8 Covariant form of magnetic field in (ψ, θ, ζ) coordinate system

In the above, we have obtained the covariant form of the magnetic field in (ψ, θ, ϕ) coordinates (i.e., Eq. (8.2)). Next, we derive the corresponding form in (ψ, θ, ζ) coordinate. In order to do this, we need to express the $\nabla\phi$ basis vector in terms of $\nabla\psi$, $\nabla\theta$, and $\nabla\zeta$ basis vectors. Using the definition of the generalized toroidal angle, we obtain

$$\begin{aligned}
 g\nabla\phi &= g\nabla(\zeta + q\delta) \\
 &= g\nabla\zeta + gq\nabla\delta + g\delta\nabla q \\
 &= g\nabla\zeta + gq\left(\frac{\partial\delta}{\partial\psi}\nabla\psi + \frac{\partial\delta}{\partial\theta}\nabla\theta\right) + g\delta q'\nabla\psi \\
 &= \left(gq\frac{\partial\delta}{\partial\psi} + g\delta q'\right)\nabla\psi + gq\frac{\partial\delta}{\partial\theta}\nabla\theta + g\nabla\zeta \\
 &= g\frac{\partial(q\delta)}{\partial\psi}\nabla\psi + gq\frac{\partial\delta}{\partial\theta}\nabla\theta + g\nabla\zeta.
 \end{aligned} \tag{12.32}$$

Using Eq. (12.32), the covariant form of the magnetic field, Eq. (8.2), is written as

$$\mathbf{B} = \left(\Psi' \frac{\mathcal{J}}{R^2} \nabla\psi \cdot \nabla\theta + g \frac{\partial(q\delta)}{\partial\psi} \right) \nabla\psi + \left(gq \frac{\partial\delta}{\partial\theta} - \Psi' \frac{\mathcal{J}}{R^2} |\nabla\psi|^2 \right) \nabla\theta + g\nabla\zeta. \tag{12.33}$$

This expression can be further simplified by using equation (12.4) to eliminate $\partial\delta/\partial\theta$, which gives

$$\begin{aligned}
 \mathbf{B} &= \left(\Psi' \frac{\mathcal{J}}{R^2} \nabla\psi \cdot \nabla\theta + g \frac{\partial(q\delta)}{\partial\psi} \right) \nabla\psi + \left(-\frac{g^2}{\Psi'} - gq \frac{R^2}{\mathcal{J}} - \Psi' |\nabla\psi|^2 \right) \frac{\mathcal{J}}{R^2} \nabla\theta + g\nabla\zeta \\
 &= \left(\Psi' \frac{\mathcal{J}}{R^2} \nabla\psi \cdot \nabla\theta + g \frac{\partial(q\delta)}{\partial\psi} \right) \nabla\psi + \left(-\frac{g^2 + |\nabla\Psi|^2}{\Psi'} - gq \frac{R^2}{\mathcal{J}} \right) \frac{\mathcal{J}}{R^2} \nabla\theta + g\nabla\zeta.
 \end{aligned} \tag{12.34}$$

Using $B^2 = (|\nabla\Psi|^2 + g^2)/R^2$, the above equation is written as

$$\begin{aligned}\mathbf{B} &= \left(\Psi' \frac{\mathcal{J}}{R^2} \nabla\psi \cdot \nabla\theta + g \frac{\partial(q\delta)}{\partial\psi} \right) \nabla\psi + \left(-\frac{B^2 R^2}{\Psi'} - gq \frac{R^2}{\mathcal{J}} \right) \frac{\mathcal{J}}{R^2} \nabla\theta + g \nabla\zeta \\ &= \left(\Psi' \frac{\mathcal{J}}{R^2} \nabla\psi \cdot \nabla\theta + g \frac{\partial(q\delta)}{\partial\psi} \right) \nabla\psi + \left(-\frac{B^2}{\Psi'} \mathcal{J} - gq \right) \nabla\theta + g \nabla\zeta.\end{aligned}\quad (12.35)$$

Equation (12.35) is the covariant form of the magnetic field in (ψ, θ, ζ) coordinate system. For the particular choice of the radial coordinate $\psi = -\Psi$ and the Jacobian $\mathcal{J} = h(\psi)/B^2$ (i.e., Boozer's Jacobian, discussed in Sec. 12.9), equation (12.35) reduces to

$$\mathbf{B} = \left(-\frac{\mathcal{J}}{R^2} \nabla\psi \cdot \nabla\theta + g \frac{\partial(q\delta)}{\partial\psi} \right) \nabla\psi + I(\psi) \nabla\theta + g(\psi) \nabla\zeta, \quad (12.36)$$

with $I(\psi) = h(\psi) - gq$. The magnetic field expression in Eq. (12.36) frequently appears in tokamak literature[28]. In this form, the coefficients before both $\nabla\theta$ and $\nabla\zeta$ depends on only the radial coordinate. In terms of $I(\psi)$, the Jacobian can also be written as

$$\mathcal{J} = \frac{gq + I}{B^2}. \quad (12.37)$$

12.9 Form of operator $(\mathbf{B} \times \nabla\psi/B^2) \cdot \nabla$ in (ψ, θ, ζ) coordinates

In solving the MHD eigenmode equations in toroidal geometries, besides the $\mathbf{B} \cdot \nabla$ operator, we will also encounter another surface operator $(\mathbf{B} \times \nabla\psi/B^2) \cdot \nabla$. Next, we derive the form of this operator in (ψ, θ, ζ) coordinate system. Using the covariant form of the equilibrium magnetic field [Eq. (12.35)], we obtain

$$\frac{\mathbf{B} \times \nabla\psi}{B^2} = \frac{1}{B^2} \left(-\frac{B^2}{\Psi'} \mathcal{J} - gq \right) \nabla\theta \times \nabla\psi + \frac{g}{B^2} \nabla\zeta \times \nabla\psi. \quad (12.38)$$

Using this, the $(\mathbf{B} \times \nabla\psi/B^2) \cdot \nabla$ operator is written as

$$\frac{\mathbf{B} \times \nabla\psi}{B^2} \cdot \nabla = \frac{1}{B^2} \left(\frac{B^2}{\Psi'} \mathcal{J} + gq \right) \mathcal{J}^{-1} \frac{\partial}{\partial\zeta} + \frac{g}{B^2} \mathcal{J}^{-1} \frac{\partial}{\partial\theta} \quad (12.39)$$

$$= \left(\frac{1}{\Psi'} + g \frac{\mathcal{J}^{-1}}{B^2} q \right) \frac{\partial}{\partial\zeta} + g \frac{\mathcal{J}^{-1}}{B^2} \frac{\partial}{\partial\theta}, \quad (12.40)$$

which is the form of the operator in (ψ, θ, ζ) coordinate system.

Examining Eq. (12.40), we find that the coefficients before the two partial derivatives will be independent of θ and ζ if the Jacobian \mathcal{J} is chosen to be of the form $\mathcal{J} = h(\psi)/B^2$, where h is some magnetic surface function. It is obvious that the independence of the coefficients on θ and ζ will be advantageous to some applications. The coordinate system (ψ, θ, ζ) with the particular choice of $\mathcal{J} = h(\psi)/B^2$ is called the Boozer coordinates. The usefulness of the new toroidal angle ζ is highlighted in Boozer's choice of the Jacobian, which makes

both $\mathbf{B} \cdot \nabla$ and $(\mathbf{B} \times \nabla \psi / B^2) \cdot \nabla$ be a constant-coefficient differential operator. For other choices of the Jacobian, only the $\mathbf{B} \cdot \nabla$ operator is a constant-coefficient differential operator.

12.10 Radial differential operator

In solving the MHD eigenmode equations in toroidal geometry, we also need the radial differential operator $\nabla \psi \cdot \nabla$. Next, we derive the form of the operator in (ψ, θ, ζ) coordinates. Using

$$\nabla f = \frac{\partial f}{\partial \psi} \nabla \psi + \frac{\partial f}{\partial \theta} \nabla \theta + \frac{\partial f}{\partial \zeta} \nabla \zeta,$$

the radial differential operator is written as

$$\begin{aligned} \nabla \psi \cdot \nabla f &= |\nabla \psi|^2 \frac{\partial f}{\partial \psi} + (\nabla \theta \cdot \nabla \psi) \frac{\partial f}{\partial \theta} + (\nabla \zeta \cdot \nabla \psi) \frac{\partial f}{\partial \zeta} \\ &= |\nabla \psi|^2 \frac{\partial f}{\partial \psi} + (\nabla \theta \cdot \nabla \psi) \frac{\partial f}{\partial \theta} + \{\nabla[\phi - q\delta(\psi, \theta)] \cdot \nabla \psi\} \frac{\partial f}{\partial \zeta} \\ &= |\nabla \psi|^2 \frac{\partial f}{\partial \psi} + (\nabla \theta \cdot \nabla \psi) \frac{\partial f}{\partial \theta} - \nabla[q\delta] \cdot \nabla \psi \frac{\partial f}{\partial \zeta} \\ &= |\nabla \psi|^2 \frac{\partial f}{\partial \psi} + (\nabla \theta \cdot \nabla \psi) \frac{\partial f}{\partial \theta} - [q\nabla\delta + \delta\nabla q] \cdot \nabla \psi \frac{\partial f}{\partial \zeta} \\ &= |\nabla \psi|^2 \frac{\partial f}{\partial \psi} + (\nabla \theta \cdot \nabla \psi) \frac{\partial f}{\partial \theta} - \left[q \left(\frac{\partial \delta}{\partial \psi} \nabla \psi + \frac{\partial \delta}{\partial \theta} \nabla \theta \right) + \delta q' \nabla \psi \right] \cdot \nabla \psi \frac{\partial f}{\partial \zeta} \\ &= |\nabla \psi|^2 \frac{\partial f}{\partial \psi} + (\nabla \theta \cdot \nabla \psi) \frac{\partial f}{\partial \theta} - \left[\frac{\partial(q\delta)}{\partial \psi} |\nabla \psi|^2 + q \frac{\partial \delta}{\partial \theta} \nabla \theta \cdot \nabla \psi \right] \frac{\partial f}{\partial \zeta}, \end{aligned} \tag{12.41}$$

where $\partial(q\delta)/\partial\psi$ and $q\partial\delta/\partial\theta$ are given respectively by Eqs. (12.10) and (12.4). Using the above formula, $\nabla \psi \cdot \nabla \zeta$ is written as

$$\nabla \psi \cdot \nabla \zeta = - \left[\frac{\partial(q\delta)}{\partial \psi} |\nabla \psi|^2 + q \frac{\partial \delta}{\partial \theta} \nabla \theta \cdot \nabla \psi \right]. \tag{12.42}$$

This formula is used in GTAW code.

13 Field-line-following coordinates

13.1 Definition of the field-line-following coordinates (ψ, θ, α)

In (ψ, θ, ζ) coordinates, a magnetic field line is straight in (θ, ζ) plane with slope

being q . Then the equation for a magnetic field line is written as

$$\zeta = q\theta + \alpha, \quad (13.1)$$

where α is a constant in (θ, ζ) plane and can be used to label magnetic field lines on a magnetic surface. This motivates us to use α , i.e.,

$$\alpha \equiv \zeta - q\theta, \quad (13.2)$$

to replace ζ . Then the magnetic field in Eq. (12.13) is written as

$$\mathbf{B} = \Psi' \nabla \psi \times \nabla \alpha, \quad (13.3)$$

which is called the Clebsch form. The direction

$$\frac{\partial \mathbf{r}}{\partial \theta} \Big|_{\psi, \alpha} = \mathcal{J} \nabla \alpha \times \nabla \psi, \quad (13.4)$$

is parallel (or anti-parallel) to the magnetic field direction. Due to this fact, (ψ, θ, α) coordinates are usually called “field-line-following coordinates” or “field-aligned coordinates” [2, 6].

Equation (13.3) implies that

$$\mathbf{B} \cdot \nabla \alpha = 0, \quad (13.5)$$

and

$$\mathbf{B} \cdot \nabla \psi = 0, \quad (13.6)$$

i.e., both α and ψ are constant along a magnetic field line. Taking scalar product of Eq. (13.3) with $\nabla \theta$, we obtain

$$\mathbf{B} \cdot \nabla \theta = -\frac{\Psi'}{\mathcal{J}}, \quad (13.7)$$

which is nonzero, i.e., only θ among (ψ, θ, α) is changing along a magnetic field line. (Here $\mathcal{J} = (\nabla \psi \times \nabla \theta \cdot \nabla \alpha)^{-1}$ is the Jacobian of the coordinate system (ψ, θ, α) , which happens to be equal to the Jacobian of (ψ, θ, ζ) coordinates.)

Using Eq. (13.3), the magnetic differential operator $\mathbf{B} \cdot \nabla$ in the new coordinate system (ψ, θ, α) is written

$$\mathbf{B} \cdot \nabla f = -\frac{\Psi'}{\mathcal{J}} \frac{\partial}{\partial \theta} f, \quad (13.8)$$

which is just a partial derivative over θ , as is expected, since only θ is changing along a magnetic field line.

By the way, note that $(\mathbf{B} \cdot \nabla \alpha) / (\mathbf{B} \cdot \nabla \theta) = 0$, i.e., the magnetic field lines are straight with zero slope on (θ, α) plane.

Using Eqs. (13.2) and (12.7), α can be written as

$$\alpha = \phi - \int_0^\theta \hat{q} d\theta, \quad (13.9)$$

where $\hat{q} = \mathbf{B} \cdot \nabla \phi / \mathbf{B} \cdot \nabla \theta$ is the local safety factor. (If we choose the straight-field-line θ , then α is written as $\alpha = \phi - q\theta$.) Define $\bar{\delta} = \int_0^\theta \hat{q} d\theta$, which is called `tor_shift` in TEK code, then $\alpha = \phi - \bar{\delta}$. In TEK, I choose $\theta \in [-\pi, \pi)$

with $\theta = -\pi$ corresponding to the high-field-side midplane, and θ is increasing along the counter-clockwise direction viewed along $\nabla\phi$. The θ cut, i.e., $\theta = \pm\pi$ is far away from the low-field-side where ballooning modes often have larger amplitude. The (x, y) grid near the θ cut is highly twisted in real space and interpolation is needed in mapping physical quantity from the grid at $\theta = -\pi$ plane to that at $\theta = +\pi$. Numerical errors more likely appear there. So we prefer that the θ cut is located in less important area (area where mode amplitude is small).

It is widely believed that turbulence responsible for energy transport in tokamak plasmas usually has $k_{\parallel} \ll k_{\perp}$, where k_{\parallel} and k_{\perp} are the parallel and perpendicular wavenumbers, respectively. Due to this elongated structure along the parallel direction, less grids can be used in the parallel direction than that in the perpendicular direction in turbulence simulation. In this case, the field-aligned coordinates (ψ, θ, α) provide suitable coordinates to be used, where less gridpoints can be used for θ coordinate in simulations and even some $\partial/\partial\theta$ derivatives can be neglected (high- n approximation), which simplifies the equations that need to be solved. There is another reason why almost all gyrokinetic codes use field-aligned coordinates: the stability of numerical algorithms is improved when we use coarse grids in the parallel direction because the parallel Courant condition (for explicit schemes) $\Delta t \leq \Delta L_{\parallel}/v_{\parallel}$ can be more easily satisfied (especially for the cases with kinetic electrons), where ΔL_{\parallel} is the parallel grid spacing, which is larger when coarse grids are used in the parallel direction. This is also mentioned in Ref. [21] and it seems to be right from my experiences of testing several algorithm but a strict test is needed to verify this. This can also be understood in the following way: the coarse parallel grid automatically filters out physically irrelevant but numerically problematic high- k_{\parallel} modes, permitting much longer time steps for explicit time stepping, in both particle and fluid codes[12].

13.2 Some discussions

The fact $\mathbf{B} \cdot \nabla\alpha = 0$ implies that α is constant along a magnetic field line. At first glance, a magnetic line on an irrational surface seems to sample all the points on the surface. This seems to indicate that α is a flux surface label for irrational surface. However, α must be a non-flux-surface-function so that it can provide a suitable toroidal coordinate. I had once been confused by this conflict for a long time. The key point to resolve this confusion is to realize that it is wrong to say there is only one magnetic line on an irrational surface, i.e. it is wrong to say a magnetic line on an irrational surface samples all the points on the surface. There are still infinite number of magnetic field lines that can not be connected with each other on an irrational surface. Then the fact $\mathbf{B} \cdot \nabla\alpha = 0$ does not imply that α must be the same on these different magnetic field lines. In fact, although $\mathbf{B} \cdot \nabla\alpha = 0$, the gradient of α on a flux-surface along the perpendicular (to \mathbf{B}) direction is nonzero, i.e., $\mathbf{B} \times \nabla\Psi \cdot \nabla\alpha \neq 0$. [Proof:

$$\begin{aligned} \mathbf{B} \times \nabla\Psi \cdot \nabla\alpha &= \nabla\Psi \times \nabla\alpha \cdot \mathbf{B} \\ &= B^2 \end{aligned} \tag{13.10}$$

which is obviously nonzero.] This indicates that α is not constant on a flux-surface.

In (ψ, θ, ϕ) coordinates, $\nabla\phi$ is perpendicular to $\nabla\psi$. However, in field-line-following coordinates (ψ, θ, α) , $\nabla\alpha$ is not perpendicular to $\nabla\psi$. Therefore $\nabla\alpha$ is not along the binormal direction $\mathbf{B} \times \nabla\psi$.

13.3 Expression of $\nabla\alpha$, $\nabla\psi \cdot \nabla\alpha$, and $\nabla\alpha \cdot \nabla\theta$

The generalized toroidal angle α is defined by Eq. (13.9), i.e., $\alpha = \phi - \bar{\delta}$, where $\bar{\delta} = \int_0^\theta \hat{q} d\theta$. The gradient of α is then written as

$$\begin{aligned} \nabla\alpha &= \nabla\phi - \nabla\bar{\delta} \\ &= \frac{\hat{\phi}}{R} - \frac{\partial\bar{\delta}}{\partial\psi}\nabla\psi - \frac{\partial\bar{\delta}}{\partial\theta}\nabla\theta. \end{aligned} \quad (13.11)$$

Then

$$\nabla\psi \cdot \nabla\alpha = -\frac{\partial\bar{\delta}}{\partial\psi}|\nabla\psi|^2 - \frac{\partial\bar{\delta}}{\partial\theta}\nabla\psi \cdot \nabla\theta \quad (13.12)$$

$$\nabla\alpha \cdot \nabla\theta = -\frac{\partial\bar{\delta}}{\partial\psi}\nabla\psi \cdot \nabla\theta - \frac{\partial\bar{\delta}}{\partial\theta}|\nabla\theta|^2 \quad (13.13)$$

Another method of computing the above quantities: Using Eqs. (9.10) and (9.11), expression (13.11) is written as

$$\begin{aligned} \nabla\alpha &= \frac{\hat{\phi}}{R} + \frac{\partial\bar{\delta}}{\partial\psi} \frac{R}{\mathcal{J}} (Z_\theta \hat{\mathbf{R}} - R_\theta \hat{\mathbf{Z}}) - \frac{\partial\bar{\delta}}{\partial\theta} \frac{R}{\mathcal{J}} (Z_\psi \hat{\mathbf{R}} - R_\psi \hat{\mathbf{Z}}) \\ &= \frac{\hat{\phi}}{R} + \left(\frac{\partial\bar{\delta}}{\partial\psi} \frac{R}{\mathcal{J}} Z_\theta - \frac{\partial\bar{\delta}}{\partial\theta} \frac{R}{\mathcal{J}} Z_\psi \right) \hat{\mathbf{R}} + \left(\frac{\partial\bar{\delta}}{\partial\theta} \frac{R}{\mathcal{J}} R_\psi - \frac{\partial\bar{\delta}}{\partial\psi} \frac{R}{\mathcal{J}} R_\theta \right) \hat{\mathbf{Z}}. \end{aligned} \quad (13.14)$$

(Note that $\partial\bar{\delta}/\partial\psi$ is discontinuous across the θ cut.) Then

$$\begin{aligned} \nabla\psi \cdot \nabla\alpha &= \left(-\frac{R}{\mathcal{J}} Z_\theta \hat{\mathbf{R}} + \frac{R}{\mathcal{J}} R_\theta \hat{\mathbf{Z}} \right) \cdot \left[\frac{\hat{\phi}}{R} + \left(\frac{\partial\bar{\delta}}{\partial\psi} \frac{R}{\mathcal{J}} Z_\theta - \frac{\partial\bar{\delta}}{\partial\theta} \frac{R}{\mathcal{J}} Z_\psi \right) \hat{\mathbf{R}} + \left(\frac{\partial\bar{\delta}}{\partial\theta} \frac{R}{\mathcal{J}} R_\psi - \frac{\partial\bar{\delta}}{\partial\psi} \frac{R}{\mathcal{J}} R_\theta \right) \hat{\mathbf{Z}} \right] \\ &= -\frac{R}{\mathcal{J}} Z_\theta \left(\frac{\partial\bar{\delta}}{\partial\psi} \frac{R}{\mathcal{J}} Z_\theta - \frac{\partial\bar{\delta}}{\partial\theta} \frac{R}{\mathcal{J}} Z_\psi \right) + \frac{R}{\mathcal{J}} R_\theta \left(\frac{\partial\bar{\delta}}{\partial\theta} \frac{R}{\mathcal{J}} R_\psi - \frac{\partial\bar{\delta}}{\partial\psi} \frac{R}{\mathcal{J}} R_\theta \right). \end{aligned} \quad (13.15)$$

$$\begin{aligned} \nabla\alpha \cdot \nabla\theta &= \left[\frac{\hat{\phi}}{R} + \left(\frac{\partial\bar{\delta}}{\partial\psi} \frac{R}{\mathcal{J}} Z_\theta - \frac{\partial\bar{\delta}}{\partial\theta} \frac{R}{\mathcal{J}} Z_\psi \right) \hat{\mathbf{R}} + \left(\frac{\partial\bar{\delta}}{\partial\theta} \frac{R}{\mathcal{J}} R_\psi - \frac{\partial\bar{\delta}}{\partial\psi} \frac{R}{\mathcal{J}} R_\theta \right) \hat{\mathbf{Z}} \right] \cdot \\ &\quad \left(\frac{R}{\mathcal{J}} Z_\psi \hat{\mathbf{R}} - \frac{R}{\mathcal{J}} R_\psi \hat{\mathbf{Z}} \right) \end{aligned}$$

$$= \left(\frac{\partial \bar{\delta}}{\partial \psi} \frac{R}{\mathcal{J}} Z_\theta - \frac{\partial \bar{\delta}}{\partial \theta} \frac{R}{\mathcal{J}} Z_\psi \right) \frac{R}{\mathcal{J}} Z_\psi - \left(\frac{\partial \bar{\delta}}{\partial \theta} \frac{R}{\mathcal{J}} R_\psi - \frac{\partial \bar{\delta}}{\partial \psi} \frac{R}{\mathcal{J}} R_\theta \right) \frac{R}{\mathcal{J}} R_\psi. \quad (13.16)$$

The figures below plot the spatial variation of some quantities.

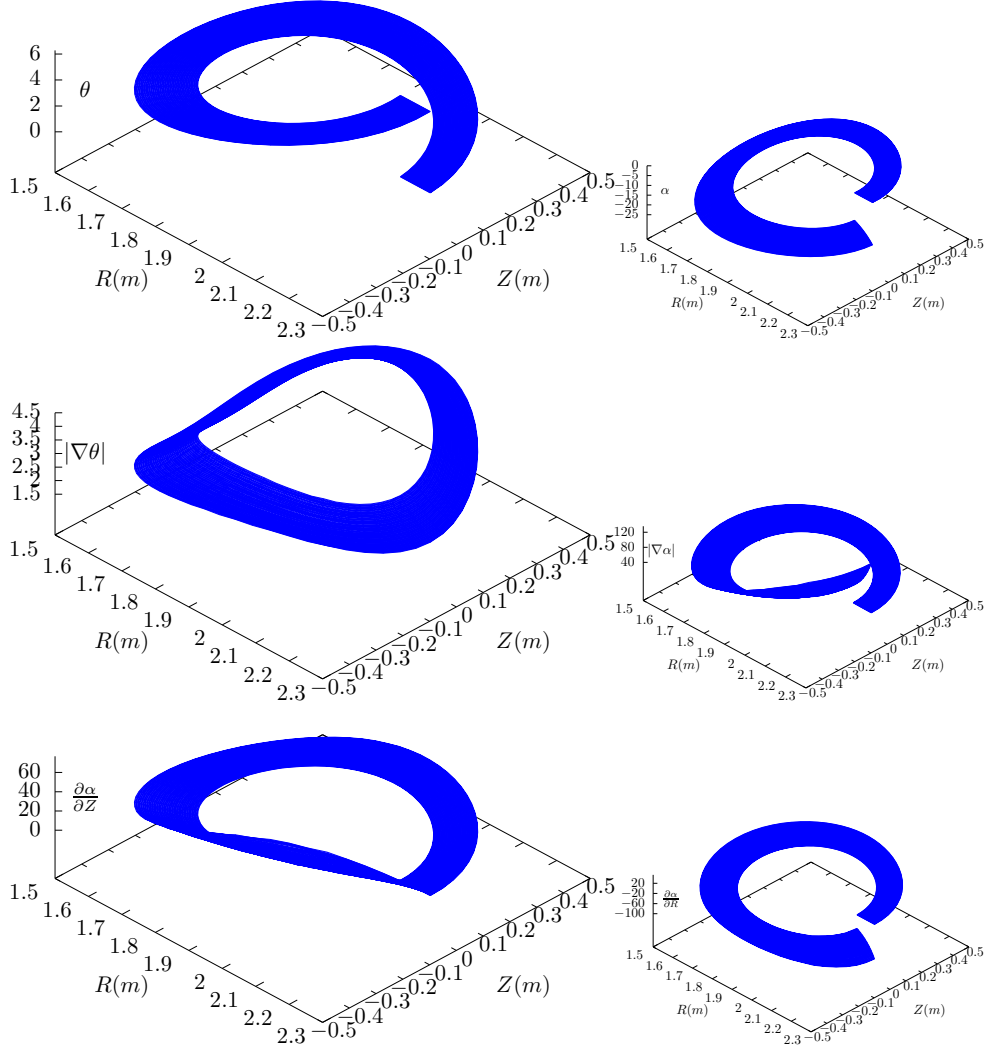


Figure 13.1. The value of θ , α and their gradients on an annulus on the poloidal plane (R, Z) . Note that $\nabla\theta$ is single-valued while $\partial\alpha/\partial R$ and $\nabla\alpha$ are multi-valued and thus there is a jump near the branch cut when a single branch is chosen.

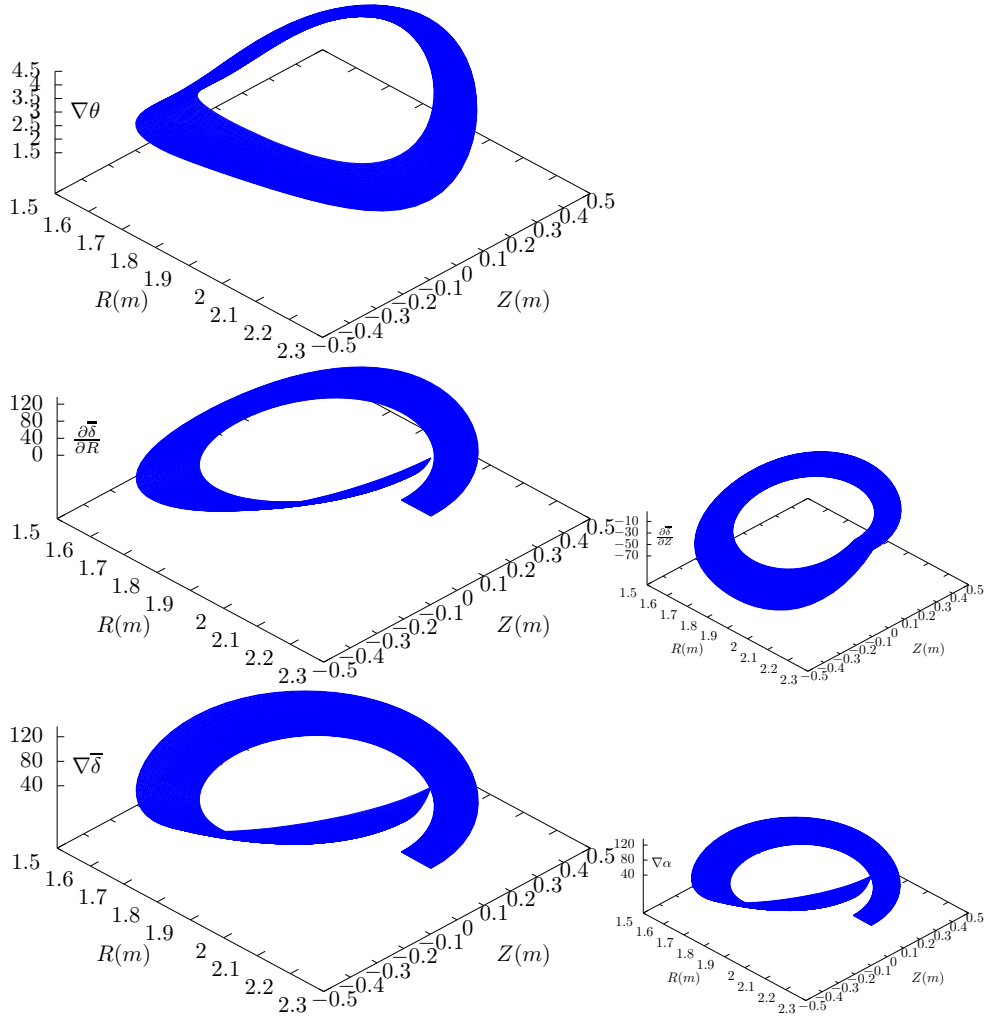


Figure 13.2. The same as Fig. 13.1, but gradients are computed in cylindrical coordinates. The results agrees with those of Fig. 13.1, which provides the confidence in the correctness of the numerical implementation.

13.4 Field-aligned coordinates in GEM[7] and GENE[13] codes

In GEM[7] and GENE[13] codes, the field-aligned coordinates (x, y, z) are defined by

$$x = r - r_0, \quad (13.17)$$

$$y = \alpha \frac{r_0}{q_0}, \quad (13.18)$$

$$z = \theta q_0 R_0, \quad (13.19)$$

where r is an arbitrary flux surface label with length dimension, which is often chosen in GEM to be the minor radius of a magnetic surface in the midplane. Here r_0 and R_0 are constant quantities of length dimension, r_0 is the minor radius of a reference magnetic surface (usually corresponding to the center of the radial simulation box), R_0 is the major radius of the magnetic axis, q_0 is the safety factor value on the $r = r_0$ surface. The constant length $q_0 R_0$ introduced in the definition of z is to make z approximately correspond to the length along the field line in the large-aspect ratio limit. The constant length r_0 / q_0 introduced in the definition of y is to make y corresponds to the arc-length in the poloidal plane traced by a field line when its usual toroidal angle increment $\Delta\phi$ is α . This explanation makes y look like a poloidal coordinate whereas y is actually a toroidal coordinate.

Next, let us calculate the wave number along the y direction, k_y , for a mode with toroidal wave number n . The wavelength along the y direction, λ_y , is given

by

$$\lambda_y = \frac{2\pi}{n} \frac{r_0}{q_0}. \quad (13.20)$$

Then the wavenumber k_y is written as

$$k_y = \frac{2\pi}{\lambda_y} = \frac{n q_0}{r_0}. \quad (13.21)$$

On the other hand, the poloidal wavenumber k_θ is given by

$$k_\theta = \frac{2\pi}{\lambda_\theta} \approx \frac{2\pi}{2\pi r_0/m} = \frac{m}{r_0}. \quad (13.22)$$

where m is the poloidal mode number of the mode in (ψ, θ, ϕ) coordinates. If the mode has the property $k_\parallel \approx 0$, i.e., $m \approx n q_0$, then k_θ is equal to the k_y . The motivation of introducing the constant length r_0/q_0 in the definition of y is to make $k_y \approx k_\theta$ for a mode with $k_\parallel \approx 0$.

Some authors call k_y or k_θ by the name “binormal wavenumber”, which is not an appropriate name in my opinion. Some authors call y the binormal direction, which is also an inappropriate name since neither ∇y nor $\partial \mathbf{r} / \partial y$ is along the binormal direction $\mathbf{B}_0 \times \nabla \psi$.

13.5 Visualization of gridpoints in field aligned coordinate system

In this section, I try to visualize gridpoints in the field aligned coordinates. The directions of the covariant basis vectors of (ψ, θ, α) coordinates are as follows:

$$\frac{\partial \mathbf{r}}{\partial \alpha} \Big|_{\psi, \theta} \longrightarrow \text{usual toroidal direction, } \hat{\phi}, \quad (13.23)$$

$$\frac{\partial \mathbf{r}}{\partial \theta} \Big|_{\psi, \alpha} \longrightarrow (\text{parallel or antiparallel to}) \text{ field line direction} \quad (13.24)$$

$$\frac{\partial \mathbf{r}}{\partial \psi} \Big|_{\theta, \alpha} \longrightarrow \text{combination of the usual radial and toroidal direction} \quad (13.25)$$

Here $\partial \mathbf{r} / \partial \psi \Big|_{\theta, \alpha}$ is a combination of the usual radial and toroidal direction, which needs some clarification. Note that, ϕ is related to α by Eq. (13.9), i.e.,

$$\phi = \alpha + \int_0^\theta \frac{\mathbf{B} \cdot \nabla \phi}{\mathbf{B} \cdot \nabla \theta} d\theta \approx \alpha + q(\psi)\theta, \quad (13.26)$$

(where the second equality becomes exact if θ is the straight-field-line poloidal angle defined in Sec. 8.4.5.), which indicates that, for $q'(\psi) \neq 0$ and $\theta \neq 0$, the usual toroidal angle ϕ is changing when changing ψ and holding θ and α fixed. Figure 13.3b shows how the usual toroidal angle ϕ changes when we change ψ and hold θ and α fixed.

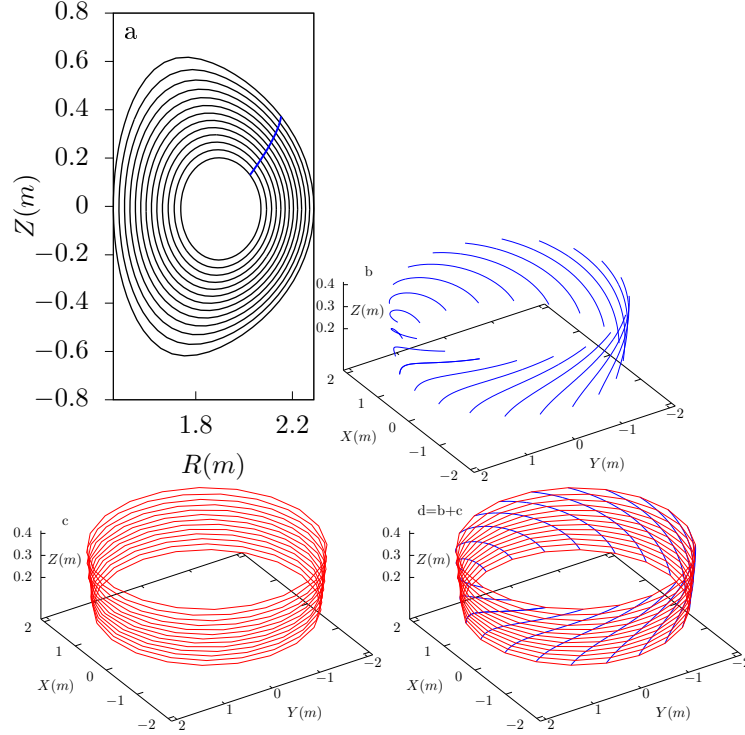


Figure 13.3. (a): A θ contour on $\phi = 0$ plane. Here $\theta = 9 \times 2\pi / 63$. This is the direction of $\partial \mathbf{r} / \partial \psi|_{\theta, \phi}$. (b): ψ coordinate lines in (ψ, θ, α) coordinates ($\partial \mathbf{r} / \partial \psi|_{\theta, \alpha}$ are the tangent lines to these curves) on the isosurface of $\theta = 9 \times 2\pi / 63$. Here different lines correspond to different values of α . (c): α coordinate lines ($\partial \mathbf{r} / \partial \alpha|_{\psi, \theta}$ are tangent lines to these curves), which are along the usual toroidal direction $\hat{\phi}$. (d): Grid on the isosurface of $\theta = 9 \times 2\pi / 63$, where the red lines are α coordinate lines while the blue lines are ψ coordinate lines. Magnetic field from EAST discharge #59954@3.03s.

The relation $\phi \approx \alpha + q(\psi)\theta$ given by Eq. (13.26) indicates that the toroidal shift along $\partial \mathbf{r} / \partial \psi|_{\alpha, \theta}$ for a radial change from ψ_1 to ψ_2 is given by $(q(\psi_2) - q(\psi_1))\theta$, which is larger on θ isosurface with larger value of θ . An example for this is shown in Fig. 13.4 for $\theta = 19 \times 2\pi / 63$, where $\partial \mathbf{r} / \partial \psi|_{\alpha, \theta}$ has larger toroidal shift than that in Fig. 13.3 for $\theta = 9 \times 2\pi / 63$.

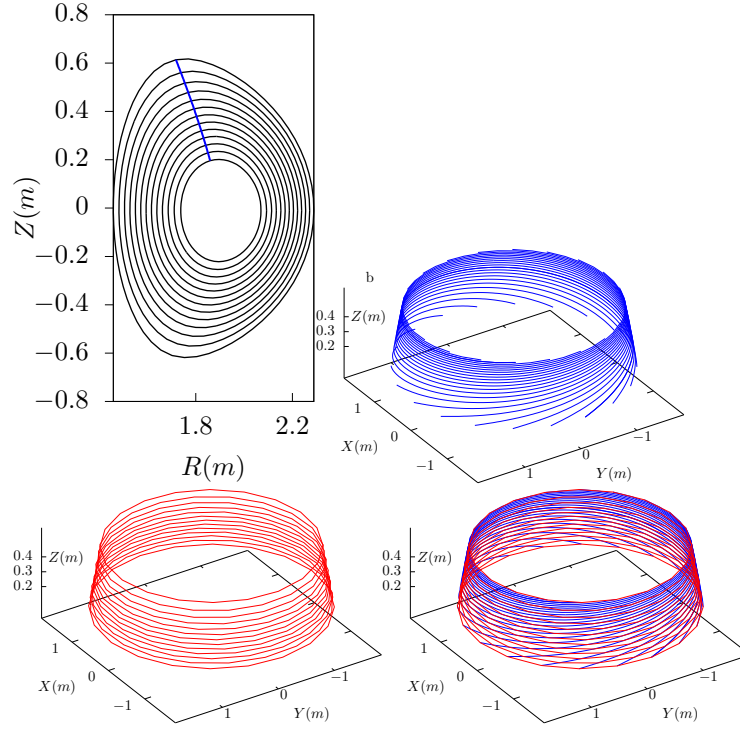


Figure 13.4. (a) $\theta = 19 \times 2\pi/63$ contour on $\phi=0$ plane, (b) $\partial\mathbf{r}/\partial\psi$ curves, (c) $\partial\mathbf{r}/\partial\alpha$ curves on isosurface of $\theta = 19 \times 2\pi/63$. $\partial\mathbf{r}/\partial\alpha$ lines are identical with $\partial\mathbf{r}/\partial\phi$ lines. (d): Grid on the isosurface of $\theta = 19 \times 2\pi/63$, which are the combinations of $\partial\mathbf{r}/\partial\psi$ curves and $\partial\mathbf{r}/\partial\alpha$ curves.

The $\partial\mathbf{r}/\partial\psi|_{\theta,\alpha}$ curves can be understood from another perspective. Examine a family of magnetic field lines that start from $\theta = 0$ and $\phi = \phi_1$ but different radial coordinates. These starting points all have the same value of α , which is equal to ϕ_1 . When following these field lines to another isosurface of θ , the intersecting points of these field lines with the θ isosurface will trace out a $\partial\mathbf{r}/\partial\psi|_{\theta,\alpha}$ line with $\alpha = \phi_1$. Examine another family of magnetic field lines similar to the above but with the starting toroidal angle $\phi = \phi_2$. They will trace out another $\partial\mathbf{r}/\partial\psi|_{\theta,\alpha}$ line (with $\alpha = \phi_2$) on the θ isosurface. Continue the process, we finally get those curves in Fig. 13.3b and Fig. 13.4b.

13.5.1 Radial wavenumber in field-aligned coordinates (ψ, θ, α)

For a harmonic in (ψ, θ, ϕ) coordinates given by $A(\psi, \theta, \phi) = \exp(i k_\psi \psi + i m \theta + i n \phi)$, the radial wave number is k_ψ . Let us calculate the corresponding radial wavenumber k_ψ^* in the new coordinates (ψ, θ, α) , which is defined by

$$k_\psi^* = \left. \frac{\partial \text{phase}}{\partial \psi} \right|_{\theta, \alpha}, \quad (13.27)$$

where the phase is given by $\text{phase} = k_\psi \psi + m\theta + n\phi$. Then the above expression is written as

$$\begin{aligned} k_\psi^* &= \left. \frac{\partial(k_\psi \psi + m\theta + n\phi)}{\partial \psi} \right|_{\theta, \alpha} \\ &= k_\psi + n \left. \frac{\partial \phi}{\partial \psi} \right|_{\theta, \alpha} \end{aligned} \quad (13.28)$$

Using $\phi \approx \alpha + q(\psi)\theta$, the above expression is written as

$$k_\psi^* = k_\psi + nq'\theta, \quad (13.29)$$

where $q' = dq/d\psi$. This result indicates that, compared with the radial wavenumber in coordinate system (ψ, θ, ϕ) , the radial wavenumber in the new coordinate system (ψ, θ, α) has an increment $n\theta q'$. For nonzero magnetic shear ($q' \neq 0$) and nonzero poloidal location ($\theta \neq 0$), the increment $n\theta q'$ can be large for modes with toroidal mode number $n \gg 1$. Then we need to use more radial grid number (compared what is needed in (ψ, θ, ϕ) coordinates) to resolve the radial variation. This is one of disadvantage of using field-aligned coordinates. If the saving associated with using less parallel grid number out-weights the cost associated with using more radial grid number, we obtain a net saving in using the field-aligned coordinates.

Let us examine how many ψ grid points are needed to resolve the ψ dependence in (ψ, θ, α) coordinates on the high-field side ($\theta = \pi$). Assume $k_\psi \approx 0$, then k_ψ^* at $\theta = \pi$ is given by $k_\psi^* = n\pi q'$. The corresponding wave-length is given by $\lambda_\psi^* = 2\pi/k_\psi^*$. The grid spacing $\Delta\psi$ should be less than half of this wave-length (sampling theorem). Then the grid number should satisfy that

$$N_\psi^* = \frac{L_\psi}{\Delta\psi} \geq \frac{L_\psi}{\lambda_\psi^*/2} = nq'L_\psi, \quad (13.30)$$

where L_ψ is the radial width of the computational domain.

The number of Fourier harmonics that need to be included is given by

$$N_{k_\psi^*} = \frac{k_\psi^*}{2\pi/L_\psi} = \frac{nq'L_\psi}{2}. \quad (13.31)$$

For DIII-D cyclone base case, choose the radial coordinate ψ as r . At the radial location $\psi = r_0 = 0.24m$, $q_0 = 1.4$, $\hat{s}_0 = 0.78$, then $q'_0 = s_0 q_0 / r_0 = 4.5m^{-1}$. Then $N_r^* = n \times 0.45$ for the radial width $L_r = 0.10m$.

Figure 13.5 plots $\partial \mathbf{r} / \partial \psi|_{\theta, \alpha}$ lines on the $\theta = 0, 2\pi$ isosurfaces, which are chosen to be on the low-field-side midplane. On $\theta = 0$ surface, $\partial \mathbf{r} / \partial \psi|_{\theta, \alpha}$ lines are identical to $\partial \mathbf{r} / \partial \psi|_{\theta, \phi}$ lines. On $\theta = 2\pi$ surface, each $\partial \mathbf{r} / \partial \psi|_{\theta, \alpha}$ line has large ϕ shift. In old version of my code, $\theta = 0, 2\pi$ surfaces are chosen as the θ cuts (in the

new version $\theta = [-\pi, \pi]$). A connection condition for the perturbations is needed between these two surfaces. This connection condition is discussed in Sec. 13.5.2.

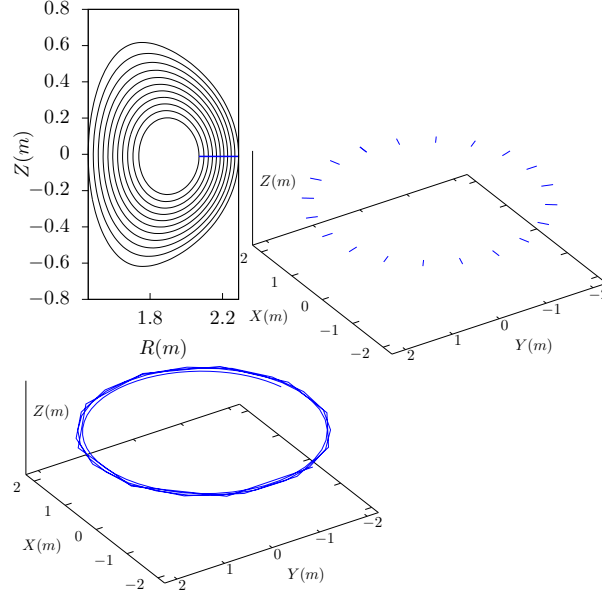


Figure 13.5. (a) $\theta = 0$ contour (blue line) in $\phi = 0$ plane; (b) a series of $\partial \mathbf{r} / \partial \psi|_{\theta, \alpha}$ curves (with $\alpha_j = j2\pi/20$, $j = 0, 1, 2, \dots, 20$) on $\theta = 0$ isosurface, (c) a single $\partial \mathbf{r} / \partial \psi|_{\theta, \alpha}$ curve (with $\alpha = 0$) on $\theta = 2\pi$ isosurface. This curve finish about 4 torodial loops because $(q_{\max} - q_{\min})2\pi = (5.56 - 1.79) \times 2\pi \approx 4 \times 2\pi$. The radial range is $\psi = 0.2 \rightarrow 0.9$, where ψ is the normalized poloidal magnetic flux. Magnetic field from EAST discharge #59954@3.03s (gfile g059954.003030 provided by Hao BaoLong).

13.5.2 Periodic conditions of physical quantity along θ and α in field-line-following coordinates (ψ, θ, α)

Since (ψ, θ, ϕ) and $(\psi, \theta + 2\pi, \phi)$ correspond to the same spatial point, a real space continuous quantity f expressed in terms of coordinates (ψ, θ, ϕ) , i.e., $f = f(\psi, \theta, \phi)$, must satisfy the following periodic conditions along θ :

$$f(\psi, \theta + 2\pi, \phi) = f(\psi, \theta, \phi). \quad (13.32)$$

Since (ψ, θ, ϕ) and $(\psi, \theta, \phi + 2\pi)$ correspond to the same spatial point, f must satisfy the following periodic conditions along ϕ :

$$f(\psi, \theta, \phi + 2\pi) = f(\psi, \theta, \phi). \quad (13.33)$$

Since (ψ, θ, α) and $(\psi, \theta, \alpha + 2\pi)$ correspond to the same spatial point, a real space continuous quantity g expressed in terms of field-line-following coordinates

(ψ, θ, α) , i.e., $g = g(\psi, \theta, \alpha)$, must satisfy the following periodic condition along α :

$$g(\psi, \theta, \alpha + 2\pi) = g(\psi, \theta, \alpha). \quad (13.34)$$

However, generally there is no periodic condition along θ ,

$$g(\psi, \theta + 2\pi, \alpha) \neq g(\psi, \theta, \alpha), \quad (13.35)$$

because $P_1 = (\psi, \theta, \alpha)$ and $P_2 = (\psi, \theta + 2\pi, \alpha)$ are generally not the same spatial point. In fact, equation (13.9) implies, for point P_1 , its toroidal angle ϕ_1 is given by

$$\phi_1 = \alpha + \int_0^\theta \frac{\mathbf{B} \cdot \nabla \phi}{\mathbf{B} \cdot \nabla \theta} d\theta, \quad (13.36)$$

while for point P_2 , its toroidal angle ϕ_2 is given by

$$\phi_2 = \alpha + \int_0^{\theta+2\pi} \frac{\mathbf{B} \cdot \nabla \phi}{\mathbf{B} \cdot \nabla \theta} d\theta = \phi_1 + 2\pi q, \quad (13.37)$$

i.e., ϕ_1 and ϕ_2 are different by $2\pi q$. From this, we know that (ψ, θ, α) and $(\psi, \theta + 2\pi, \alpha - 2\pi q)$ correspond to the same spatial point. Therefore we have the following periodic condition:

$$g(\psi, \theta, \alpha) = g(\psi, \theta + 2\pi, \alpha - 2\pi q), \quad (13.38)$$

or equivalently

$$g(\psi, \theta + 2\pi, \alpha) = g(\psi, \theta, \alpha + 2\pi q). \quad (13.39)$$

13.5.3 Numerical implementation of periodic condition (13.38)

For the fully kinetic ion module of GEM code that I am developing, θ is chosen in the range $[0:2\pi]$. The condition (13.39) imposes the following boundary condition:

$$g(\psi, 2\pi, \alpha) = g(\psi, 0, \alpha + 2\pi q). \quad (13.40)$$

If α is on a grid, $\alpha + 2\pi q$ is usually not on a grid. Therefore, to get the value of $g(\psi, 0, \alpha + 2\pi q)$, an interpolation of the discrete data over the generalized toroidal angle α (or equivalently ϕ) is needed, as is shown in Fig. 13.6.

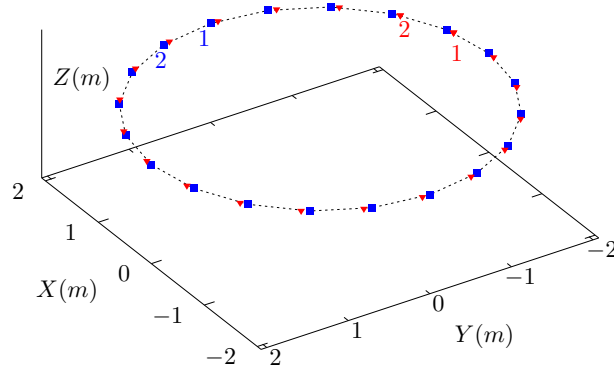


Figure 13.6. Twenty magnetic field lines (on $\psi = 0.2$ magnetic surface) starting at different toroidal angle (blue points) on the midplane ($\theta = 0$) go a full poloidal loop (i.e., $\Delta\theta = 2\pi$), arriving at a toroidal angle (red points) which are different from their respective starting toroidal angle. The field values on the red points can be obtained by interpolating the field values on the blue points. The safety factor of the magnetic surface $q = 1.79$. Magnetic field from EAST discharge #59954@3.03s.

13.5.4 α contours on a magnetic surface

Figure 13.7 compares a small number of ϕ contours and α contours on a magnetic surface.

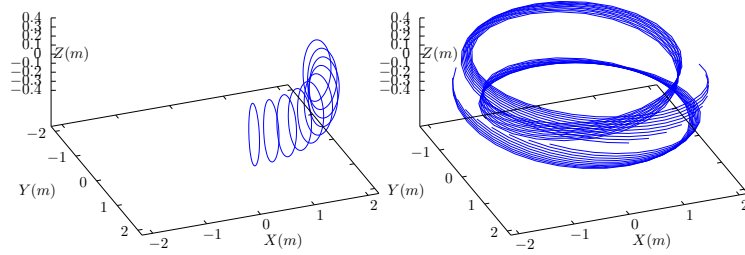


Figure 13.7. Comparison between a series of ϕ contours (left) and a series of α contours (right) on a magnetic surface. Here contour levels are $\phi_j = \alpha_j = (j - 1)2\pi/4/(10 - 1)$ with $j = 1, 2, \dots, 10$, i.e., only $1/4$ of the full torus. Every ϕ and α contour start from the lower-field-side midplane and go one poloidal loop. Magnetic field from EAST discharge #59954@3.03s.

As is shown in the left panel of Fig. 13.7, with ϕ fixed, an θ curve reaches its starting point when θ changes from zero to 2π . However, as shown in the right panel of Fig. 13.7, with α fixed, an θ curve (i.e. a magnetic field line) does not necessarily reach its starting point when θ changes from zero to 2π . There is a toroidal shift, $2\pi q$, between the starting point and ending point. Therefore there is generally no periodic condition along θ since q is not always an integer. A mixed periodic condition involves both θ and α is given in (13.38).

In field-line-following coordinates (ψ, θ, α) , a toroidal harmonic of a physical perturbation can be written as

$$\delta A(\psi, \theta, \alpha) = \delta A_0(\psi) \cos(m'\theta + n\alpha + \alpha_0) \quad (13.41)$$

where n is the toroidal mode number, m' , which is not necessary an integer, is introduced to describe the variation along a field line. The periodic condition given by Eq. (13.38) requires that

$$\cos(m'\theta + n\alpha + \alpha_0) = \cos[m'(\theta + 2\pi) + n(\alpha - 2\pi q) + \alpha_0], \quad (13.42)$$

To satisfy the above condition, we can choose

$$m'2\pi - n2\pi q = N2\pi, \quad (13.43)$$

where N is an arbitrary integer, i.e.,

$$m' = N + nq. \quad (13.44)$$

We are interested in perturbation with a slow variation along the field line direction (i.e., along $\partial \mathbf{r} / \partial \theta|_{\psi, \alpha}$) and thus we want the value of m' to be small. One of the possible small values given by expression (13.44) is to choose $N = -n \times \text{NINT}((q_{\max} + q_{\min})/2)$, so that m' is given by

$$m'(\psi) = nq - n \times \text{NINT}\left(\frac{q_{\max} + q_{\min}}{2}\right), \quad (13.45)$$

where NINT is a function that return the nearest integer of its argument, q_{\max} and q_{\min} is the maximal and minimal value of the safety factor in the radial region in which we are interested. [In the past, I choose $m'(\psi) = nq - \text{NINT}(nq)$. However, $m'(\psi)$ in this case is not a continuous function of ψ and thus is not physical.] This form is used to set the initial density perturbation in the fully kinetic code I am developing. Note that m' in Eq. (13.45) depends on the radial coordinate ψ through $q(\psi)$. Also note that m' here is different from the poloidal mode number m in (ψ, θ, ϕ) coordinate system. It is ready to show that the perturbation given by Eq. (13.41) with $m' \sim 1$ and $n \gg 1$ has large poloidal mode number m when expressed in (ψ, θ, ϕ) coordinates. [Proof: Expression (13.41) can be written as

$$\delta A = \delta A_0(\psi) \cos[m'\theta + n(\phi - \bar{\delta}(\psi, \theta)) + \alpha_0] \quad (13.46)$$

Assume that θ is the straight-field-line poloidal angle in (ψ, θ, ϕ) coordinate system, then $\bar{\delta}(\psi, \theta) = q\theta$ and the above equation is written as

$$\delta A = \delta A_0(\psi) \cos[(m' - nq)\theta + n\phi + \alpha_0], \quad (13.47)$$

which indicates the poloidal mode number m in (ψ, θ, ϕ) coordinates is given by $m = m' - nq$. For the case with $m' \sim 1$ and $n \gg 1$, m is much larger than one.]

Since α contours on a magnetic surface are magnetic field lines, they span out the 3D shape of magnetic surface when there are many α contours on a magnetic surface, as is shown by the right-panel of Fig. 13.8.

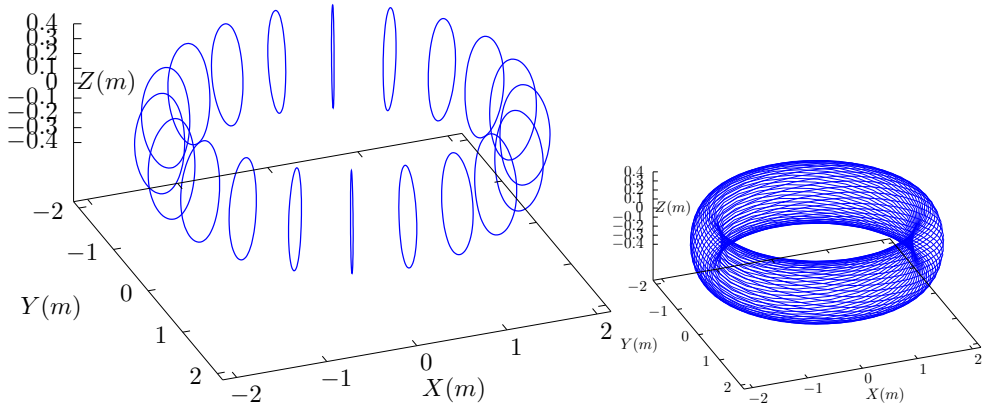


Figure 13.8. Comparison between a series of ϕ contours(left) and a series of α contours (left) on a magnetic surface. The α contours correspond to magnetic field lines. Here the α values of adjacent α contours differ by $d\alpha = 2\pi/20$ and each α contour goes one full poloidal loop. Magnetic field from EAST discharge #59954@3.03s.

13.5.5 α contours in a toroidal annulus

Figure 13.9 compares the ϕ coordinate surface of (ψ, θ, ϕ) coordinates with the α coordinate surface of (ψ, θ, α) coordinates.

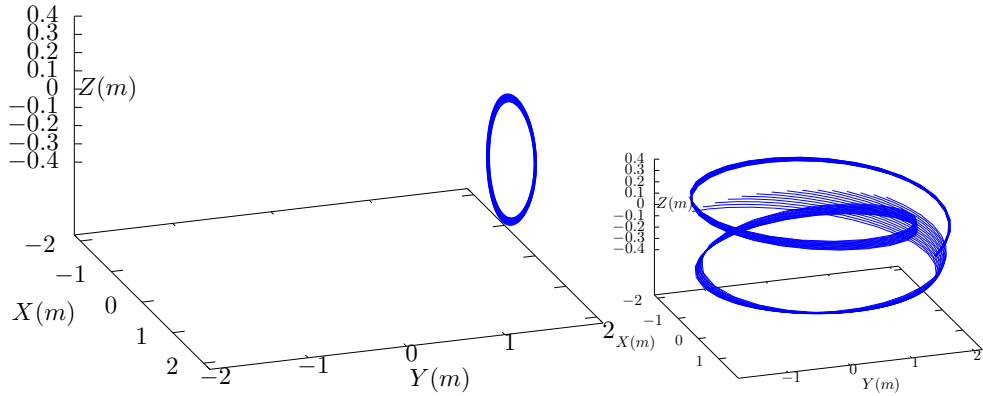


Figure 13.9. Comparison between isosurface of $\phi = 2\pi/8$ (projection of magnetic field lines onto $\phi = 2\pi/8$ plane) and isosurface of $\alpha = 2\pi/8$. The α isosurface is made of a family of contours of $\alpha = 2\pi/8$, which are all magnetic field lines. These field lines are traced by starting from a series of points on the low-field-side midplane ($\theta = 0$) at different radial locations and the field lines are followed by one poloidal loop. The radial range is given by $\psi_N \in [0.4; 0.5]$, where ψ_N is the normalized poloidal magnetic flux. Magnetic field from EAST discharge #59954@3.03s.

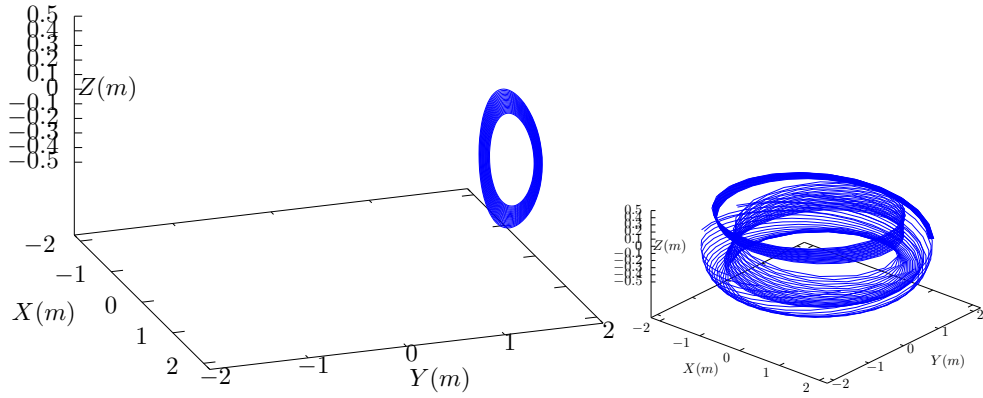


Figure 13.10. The same plot as in Fig. 13.9 but with a larger radial range. $\psi_N \in [0.4:0.7]$, where ψ_N is the normalized poloidal magnetic flux.

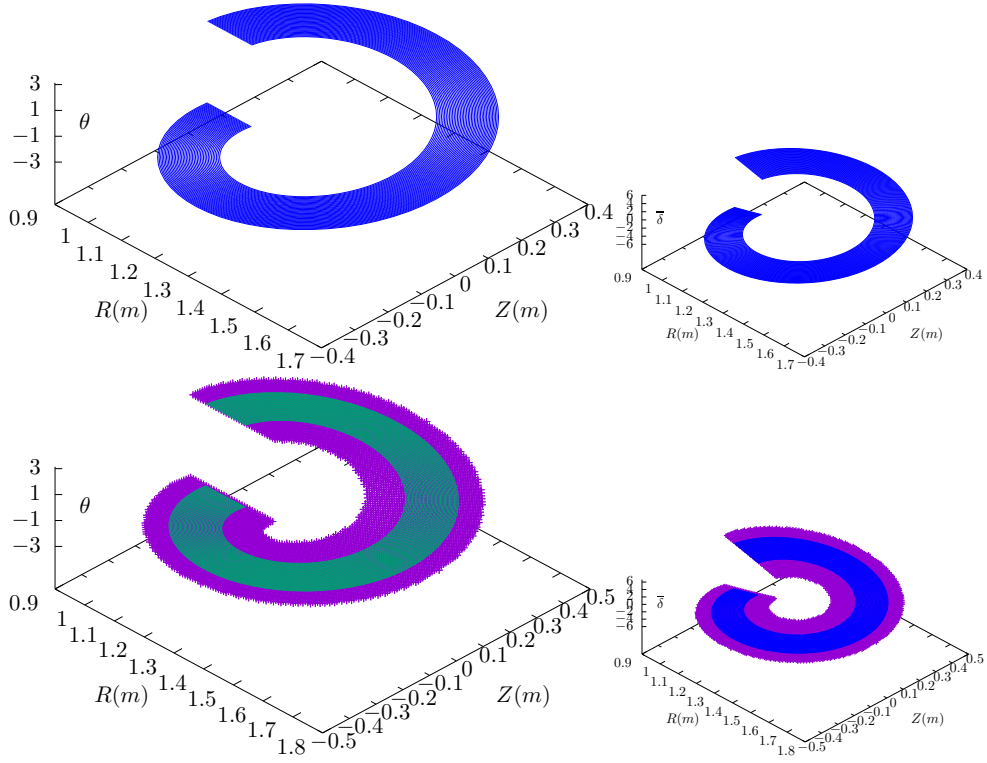


Figure 13.11. Values of θ and $\bar{\delta} \equiv \int_0^\theta \hat{q} d\theta$ on an annulus in the poloidal plane (R, Z) . Upper panel: using magnetic coordinates. Lower panel: comparison with the results computed in cylindrical coordinates. Both θ and $\bar{\delta}$ are discontinuous at the θ cut, $\theta = \pm\pi$, which is chosen on the high-field-side.

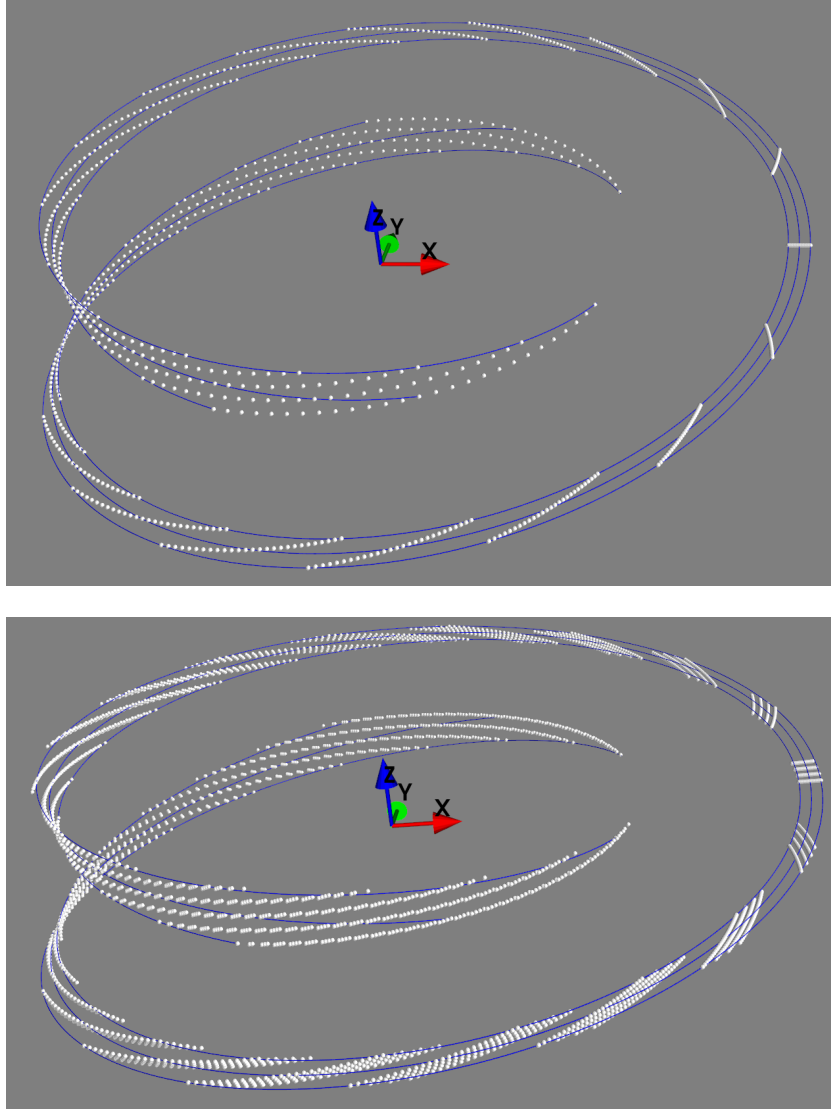


Figure 13.12. Upper pannel: 2D grid (ψ, θ) of resolution $(27, 33)$ for a fixed value of α . Lower pannel: 3D grid (ψ, α, θ) of resolution $(27, 4, 33)$ for a toroidal wedge of $2\pi/40$. The blue lines are field lines. The 3D grid is generated by rotating the 2D grid toroidally. Only 3 field-lines are shown for both cases.

The above grid is often called “flux-tube”, since it is created by following field lines, and it looks like a tube if the field lines are very near to each other (e.g.,

when we choose a very narrow radial and toroidal region to start from). Since no magnetic field line passes through the sides of the tube, the flux through any cross section of the tube is equal. The term “flux tube” is often used in astrophysics.

13.6 Field-line-following mesh in gyrokinetic turbulence simulation codes

Most gyrokinetic simulation codes use field-line-following coordinates in constructing spatial mesh. The mesh is conceptually generated by the following three steps: (1) selecting some initial points; (2) tracing out magnetic-field-lines passing through these points; (3) choose the intersecting points of these field-lines with a series of chosen surfaces as the final grid-points. The initial points and the chosen surfaces differ among various codes and thus the resulting grid differs. Next, let us discuss some examples.

13.6.1 Mesh in GENE, GYRO, and GEM codes

Given the definition of (ψ, θ) coordinates, choose toroidally symmetrical points (can be a toroidal wedge) in $\theta = 0$ plane ($\theta = 0$ plane is usually chosen to be the low-field-side midplane). Then trace out the magnetic-field-lines passing through these points for one poloidal circuit (usually chosen in the range $\theta \in [-\pi, \pi]$) and record the intersection points of these field-lines with various $\theta = \text{const}$ planes. Note that the gridpoints in $\theta = -\pi$ plane usually do not coincide those in $\theta = +\pi$ plane due to the toroidal shift arising when the safety factor is irrational. Interpolation can be used to map physical variables defined on gridpoints in $\theta = -\pi$ plane to those in $\theta = +\pi$ plane.

13.6.2 Mesh in GTC and GTS codes[26]

Given the definition of (ψ, θ) coordinates, 2D grid-points can be chosen on $\phi = 0$ plane based on (ψ, θ) coordinates. Then trace out the magnetic-field-lines starting from these points for one toroidal circuit and record the intersection points of these field-lines with $\phi_j = j\Delta\phi$ planes, where $j = 1, 2, \dots, N_t - 1$, $\Delta\phi = 2\pi / (N_t - 1)$. It is obvious that the resulting mesh are not toroidally symmetrical. And also the grids on $\phi = 0$ plane differ from those on $\phi = 2\pi$ plane. Interpolation can be used to mapping physical variables defined on grid-points of $\phi = 0$ to those of $\phi = 2\pi$ plane. In this case, the number of “toroidal grid-points” N_t (i.e., the number of poloidal planes) is actually the number of grid-points in the parallel direction within one toroidal circuit.

13.6.3 Mesh in XGC1 code ***check**

XGC1 can handle the region outside of the LCFS. Here we only discuss the region inside the LCFS. At each radial gridpoint on $\phi = 0$ plane, follow the magnetic field line starting from this point for one poloidal loop and record the intersection points of this field-line with $\phi_j = j\Delta\phi$ planes, where $j = 1, 2, \dots, N_t$, $\Delta\phi = 2\pi / N_t$. For the case $q > 1$, one magnetic-field-line will have more than one intersection points on some poloidal planes. Repeat tracing the field line for each radial location. Then project (toroidally) all the intersection points on different poloidal planes to a single poloidal plane and rotate (toroidally) these 2D grids to define a 3D grid that are toroidally symmetrical.

13.7 Numerical verification of the field-aligned coordinates

The generalized toroidal angle α is numerically calculated in my code. To verify $\mathbf{B} \cdot \nabla \alpha = 0$ along a magnetic field-line, figure 13.13 plots the values of α along a magnetic field line, which indicates that α is constant. This indicates the numerical implementation of the field-aligned coordinates is correct.

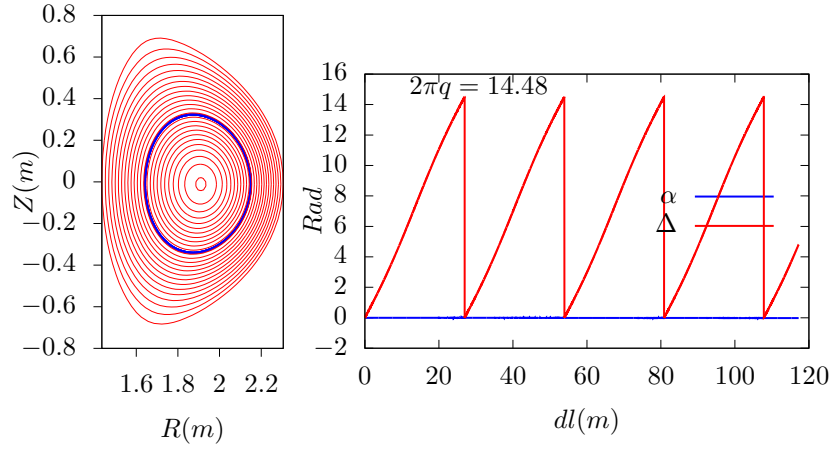


Figure 13.13. Left: Projection of a field line on the poloidal plane. Right: the value of Δ and α along the magnetic field line. Here α is defined by $\alpha = \phi - \Delta$, where ϕ is the usual cylindrical toroidal angle and $\Delta = \int_0^\theta \frac{\mathbf{B} \cdot \nabla \phi}{\mathbf{B} \cdot \nabla \theta} d\theta$.

13.7.1 Binormal wavenumber

Let us introduce the binormal wavenumber, which is frequently used in presenting turbulence simulation results. Define the binormal direction \mathbf{s} by

$$\mathbf{s} = \frac{\mathbf{B} \times \nabla \Psi}{|\mathbf{B} \times \nabla \Psi|},$$

which is a unit vector lying on a magnetic surface and perpendicular to \mathbf{B} . The binormal wavenumber of a mode is defined by

$$k_{bn} = \mathbf{s} \cdot \nabla p, \quad (13.48)$$

where p is the phase of the mode. Consider a mode given by $\exp(ik_\psi\psi + im\theta - in\zeta)$, then the phase $p = k_\psi\psi + m\theta - n\zeta$. Then k_{bn} is written as

$$\begin{aligned} k_{bn} &= \frac{\mathbf{B} \times \nabla \Psi}{|\mathbf{B} \times \nabla \Psi|} \cdot \nabla (k_\psi\psi + m\theta - n\zeta) \\ &= \frac{\mathbf{B} \times \nabla \Psi}{|\mathbf{B} \times \nabla \Psi|} \cdot \nabla (m\theta - n\zeta), \end{aligned} \quad (13.49)$$

where the radial phase $k_\psi\psi$ does not appear since $\mathbf{B} \times \nabla \Psi \cdot \nabla \psi = 0$. The above expression can be further written as

$$\begin{aligned} k_{bn} &= \frac{1}{|\mathbf{B} \times \nabla \Psi|} [m\mathbf{B} \times \nabla \Psi \cdot \nabla \theta - n\mathbf{B} \times \nabla \Psi \cdot \nabla \zeta] \\ &= \frac{1}{|\mathbf{B} \times \nabla \Psi|} [m\mathbf{B} \cdot \nabla \Psi \times \nabla \theta - n\mathbf{B} \cdot \nabla \Psi \times \nabla \zeta] \\ &= \frac{n\mathbf{B} \cdot}{|\mathbf{B} \times \nabla \Psi|} \left[\frac{m}{n} \nabla \Psi \times \nabla \theta - \nabla \Psi \times \nabla \zeta \right]. \end{aligned} \quad (13.50)$$

Equation (13.50) is the general expression of the binormal wavenumber. On the resonant surface of the mode, i.e., $q(\psi) = m/n$, then the above expression is written as

$$k_{bn} = \frac{n\mathbf{B} \cdot}{|\mathbf{B} \times \nabla \Psi|} [q\nabla \Psi \times \nabla \theta - \nabla \Psi \times \nabla \zeta]. \quad (13.51)$$

Using Eq. (12.12), i.e., $\mathbf{B} = -(\nabla \zeta \times \nabla \Psi + q\nabla \Psi \times \nabla \theta)$, the above expression is written as

$$\begin{aligned} k_{bn} &= \frac{-n\mathbf{B} \cdot}{|\mathbf{B} \times \nabla \Psi|} \mathbf{B} \\ &= -n \frac{B}{|\nabla \Psi|} \end{aligned}$$

Using $B_p = |\nabla \Psi|/R$, the above equation is written

$$k_{bn} = -n \frac{B}{RB_p} \quad (13.52)$$

which indicates the binormal wavenumber generally depends on the poloidal angle. For large aspect-ratio tokamak, we have $B_\phi \approx B$, $q \approx B_\phi r / (B_p R)$. Then Eq. (13.52) is written

$$k_{bn} \approx -\frac{nq}{r}, \quad (13.53)$$

which indicates the binormal wavenumber are approximately independent of the poloidal angle. Since $m = nq$ on a resonant surface, the above equation is written $|k_{bn}| \approx m/r$, which is the usual poloidal wave number. Due to this relation, the binormal wavenumber k_{bn} is often called the poloidal wavenumber and denoted by k_θ in papers on tokamak turbulence. In the GENE code, y coordinate is defined by $y = \alpha r_0 / q_0$. Then the k_y of a mode of toroidal mode number n is given by $k_y = 2\pi / \lambda_y$ where $\lambda_y = \lambda_\alpha r_0 / q_0$ and $\lambda_\alpha = 2\pi / n$. Then k_y is written as $k_y = n q_0 / r_0$, which is similar to the binormal defined above. For this reason, k_y of GENE code is also called binormal wave-vector, which is in fact not reasonable because neither $\partial \mathbf{r} / \partial y$ or ∇y is along the binormal direction.

14 Concentric-circular magnetic configuration with a given safety factor profile

Assume magnetic surfaces of a magnetic configuration are known and given by

$$R(r, \theta) = R_0 + r \cos \theta, \quad (14.1)$$

$$Z(r, \theta) = r \sin \theta, \quad (14.2)$$

where (r, θ) are two parameters and r is magnetic surface label (i.e., $\partial \Psi / \partial \theta|_{r=0}$). The above parametric equations specify a series of concentric-circular magnetic surfaces.

Assume the toroidal field function $g(r) = R B_\phi$ is given. Then the toroidal magnetic field is determined by $B_\phi = g / R$. Further assume the safety factor profile $q(r)$ is given, then the magnetic field is fully determined. Next, let us derive the explicit form of the poloidal magnetic field \mathbf{B}_p , which is given by

$$\mathbf{B}_p = \nabla \Psi \times \nabla \phi = \frac{1}{2\pi} \nabla \Psi_p \times \nabla \phi, \quad (14.3)$$

which involves the poloidal magnetic flux Ψ_p . Therefore our task is to express Ψ_p in terms of q and g . Using $q(r) = d\Psi_t / d\Psi_p$, we obtain

$$d\Psi_p = \frac{1}{q} d\Psi_t,$$

Integrate the above equation over r , we obtain

$$\int_0^r d\Psi_p = \int_0^r \frac{1}{q} d\Psi_t, \quad (14.4)$$

which can be written as

$$\Psi_p(r) - \Psi_p(0) = \int_0^r \frac{1}{q(r)} d \left(\int_0^r \int_{-\pi}^{\pi} B_\phi r dr d\theta \right), \quad (14.5)$$

where use has been made of $\Psi_t = \int_0^r \int_{-\pi}^{\pi} B_\phi r dr d\theta$. Using $B_\phi = g / R$ and $R = R_0 + r \cos\theta$, the above equation is written

$$\Psi_p(r) - \Psi_p(0) = \int_0^r \frac{1}{q(r)} d\left(\int_0^r \int_{-\pi}^{\pi} \frac{g}{R_0 + r \cos\theta} r dr d\theta \right) \quad (14.6)$$

Using `maxima` (an open-source computer algebra system), the above integration over θ can be performed analytically, giving

$$\int_{-\pi}^{\pi} \frac{1}{R_0 + r \cos\theta} d\theta = \frac{2\pi}{\sqrt{R_0^2 - r^2}}. \quad (14.7)$$

Using this, equation (14.6) is written as

$$\Psi_p(r) - \Psi_p(0) = \int_0^r \frac{1}{q(r)} d\left(\int_0^r \frac{2\pi g r}{\sqrt{R_0^2 - r^2}} dr \right), \quad (14.8)$$

which can be simplified as

$$\Psi_p(r) - \Psi_p(0) = \int_0^r \frac{1}{q(r)} \frac{2\pi g r}{\sqrt{R_0^2 - r^2}} dr. \quad (14.9)$$

This is what we want—the expression of the poloidal magnetic flux in terms of q and g . [Another way of obtaining Eq. (14.9) is to use Eq. (8.8), i.e.,

$$\frac{d\Psi_p}{dr} = -\frac{g}{q} \int_0^{2\pi} \frac{\mathcal{J}}{R^2} d\theta, \quad (14.10)$$

where \mathcal{J} is the Jacobian of the (r, θ, ϕ) coordinates and is given by $\mathcal{J} = R(R_\theta Z_r - R_r Z_\theta) = -Rr$. Then Eq. (14.10) is simplified as

$$\frac{d\Psi_p}{dr} = \frac{g}{q} \int_0^{2\pi} \frac{r}{R} d\theta = \frac{g}{q} \int_0^{2\pi} \frac{r}{R_0 + r \cos\theta} d\theta = \frac{g}{q} r \frac{2\pi}{\sqrt{R_0^2 - r^2}}, \quad (14.11)$$

which, after being integrated over r , gives Eq. (14.9).]

Using Eq. (14.9), the poloidal magnetic field in Eq. (14.3) is written as

$$\begin{aligned} \mathbf{B}_p &= \frac{\nabla r \times \nabla \phi}{2\pi} \frac{\partial}{\partial r} \left(\int_0^r \frac{1}{q(r)} \frac{2\pi g r}{\sqrt{R_0^2 - r^2}} dr \right) \\ &= \nabla r \times \nabla \phi \frac{1}{q(r)} \frac{gr}{\sqrt{R_0^2 - r^2}}. \end{aligned} \quad (14.12)$$

[Using the formulas $\nabla r = -\frac{R}{\mathcal{J}}(Z_\theta \hat{\mathbf{R}} - R_\theta \hat{\mathbf{Z}})$ and $\mathcal{J} = R(R_\theta Z_r - R_r Z_\theta)$, where \mathcal{J} is the Jacobian of the (r, θ, ϕ) coordinates, we obtain $\mathcal{J} = -Rr$ and $\nabla r = \cos\theta \hat{\mathbf{R}} + \sin\theta \hat{\mathbf{Z}}$, $\nabla \phi = \hat{\phi} / R$. Then Eq. (14.12) is written as

$$\mathbf{B}_p = \frac{\cos\theta \hat{\mathbf{Z}} - \sin\theta \hat{\mathbf{R}}}{R} \frac{1}{q(r)} \frac{gr}{\sqrt{R_0^2 - r^2}} \quad (14.13)$$

This is the explicit form of the poloidal magnetic field in terms of g and q . The magnitude of \mathbf{B}_p is written as

$$B_p = \frac{1}{(R_0 + r \cos \theta)} \frac{1}{q(r)} \frac{gr}{\sqrt{R_0^2 - r^2}} \quad (14.14)$$

Note that both B_p and B_ϕ depend on the poloidal angle θ .]

I use Eq. (14.9) to compute the 2D data of Ψ ($\Psi = \Psi_p / 2\pi$) on the poloidal plane when creating a numerical G-eqsk file for the above magnetic configuration (Fortran code is at `/home/yj/project_new/circular_configuration_with_q_given`).

Assume that the poloidal plasma current is zero, then $g(r) = RB_\phi$ is a constant independent of r . This is always assumed by the authors who use concentric-circular configuration but is seldom explicitly mentioned.

In analytical work, $1/R$ dependence on (r, θ) is often approximated as

$$\frac{1}{R} = \frac{1}{R_0 + r \cos \theta} = \frac{1}{R_0} \frac{1}{1 + \varepsilon \cos \theta} \approx \frac{1}{R_0} (1 - \varepsilon \cos \theta),$$

where $\varepsilon = r/R_0$ is the local inverse aspect ratio.

14.1 Is the above \mathbf{B} divergence-free?

Since the above field is derived from the general form given by Eq. (1.10), it is guaranteed that the field is divergence-free. In case of any doubt, let us directly verify this. Write \mathbf{B} as

$$\mathbf{B} = B^{(1)} \mathcal{J} \nabla \theta \times \nabla \phi + B^{(2)} \mathcal{J} \nabla \phi \times \nabla r + B^{(3)} \mathcal{J} \nabla r \times \nabla \theta, \quad (14.15)$$

where \mathcal{J} is the Jacobian of (r, θ, ϕ) coordinates; $B^{(1)}$, $B^{(2)}$, and $B^{(3)}$ are given by

$$B^{(1)} = \mathbf{B} \cdot \nabla r \quad (14.16)$$

$$B^{(2)} = \mathbf{B} \cdot \nabla \theta \quad (14.17)$$

$$B^{(3)} = \mathbf{B} \cdot \nabla \phi \quad (14.18)$$

Use \mathbf{B}_p given by (14.12), then $B^{(1)}$, $B^{(2)}$, and $B^{(3)}$ are written as

$$B^{(1)} = 0, \quad (14.19)$$

$$B^{(2)} = -\frac{1}{\mathcal{J}} \frac{1}{q(r)} \frac{g_0 r}{\sqrt{R_0^2 - r^2}}, \quad (14.20)$$

and

$$B^{(3)} = \frac{B_\phi}{R}, \quad (14.21)$$

respectively. Then, by using the divergence formula in (r, θ, ϕ) coordinates, $\nabla \cdot \mathbf{B}$ is written as

$$\begin{aligned}
 \nabla \cdot \mathbf{B} &= \frac{1}{\mathcal{J}} \left(\frac{\partial B^{(1)} \mathcal{J}}{\partial r} + \frac{\partial B^{(2)} \mathcal{J}}{\partial \theta} + \frac{\partial B^{(3)} \mathcal{J}}{\partial \phi} \right) \\
 &= \frac{1}{\mathcal{J}} \left(0 + \frac{\partial B^{(2)} \mathcal{J}}{\partial \theta} + 0 \right) \\
 &= \frac{1}{\mathcal{J}} \frac{\partial}{\partial \theta} \left(-\frac{1}{q(r)} \frac{g_0 r}{\sqrt{R_0^2 - r^2}} \right) \\
 &= 0
 \end{aligned} \tag{14.22}$$

i.e., \mathbf{B} in this case is indeed divergence-free.

14.2 Is the above \mathbf{B} a solution to the GS equation?

The answer is no. It is ready to verify that the poloidal magnetic flux function $\Psi = \Psi_p/2\pi$ given by Eq. (14.9) is not a solution to the GS equation (?) even if the plasma pressure in the GS equation is set to zero. The above expression is a solution to the GS equation in the limit of infinite aspect ratio and zero plasma pressure. The finite aspect ratio and plasma pressure requires the magnetic surfaces to have the Shafranov shift in order to satisfy the GS equation (see Sec. A.14).

14.3 Explicit expression for the generalized toroidal angle

For the above magnetic field, the toroidal shift involved in the definition of the generalized toroidal angle can be expressed in simple analytical form. The toroidal shift is given by

$$\bar{\delta} = \int_0^\theta \hat{q} d\theta, \tag{14.23}$$

where the local safety factor \hat{q} can be written as

$$\hat{q} = -\frac{g_0}{R^2} \frac{\mathcal{J}}{\Psi'}. \tag{14.24}$$

Using $\mathcal{J} = -Rr$ and

$$\Psi' = \frac{1}{q(r)} \frac{g_0 r}{\sqrt{R_0^2 - r^2}} \tag{14.25}$$

The local safety factor \hat{q} in Eq. (14.24) is written as

$$\hat{q} = q \sqrt{R_0^2 - r^2} \frac{1}{R}. \tag{14.26}$$

Using this, expression (14.23) is written

$$\bar{\delta} = q \sqrt{R_0^2 - r^2} \int_0^\theta \frac{1}{R} d\theta \quad (14.27)$$

Assume $\theta \in (-\pi, \pi)$, then the integration $\int_0^\theta 1/R d\theta$ can be analytically performed (using maxima), yielding

$$\int_0^\theta \frac{1}{R} d\theta = \frac{2 \arctan\left(\frac{\sin\theta(R_0 - r)}{(\cos\theta + 1)\sqrt{R_0^2 - r^2}}\right)}{\sqrt{R_0^2 - r^2}}. \quad (14.28)$$

Then expression (14.27) is written

$$\bar{\delta} = 2q \arctan\left(\frac{(R_0 - r)}{\sqrt{R_0^2 - r^2}} \tan\left(\frac{\theta}{2}\right)\right), \quad (14.29)$$

where use has been made of $\sin\theta / (\cos\theta + 1) = \tan(\theta/2)$. Using this, the generalized toroidal angle can be written as

$$\begin{aligned} \alpha &= \phi - \bar{\delta} \\ &= \phi - 2q \arctan\left(\frac{(R_0 - r)}{\sqrt{R_0^2 - r^2}} \tan\left(\frac{\theta}{2}\right)\right). \end{aligned} \quad (14.30)$$

The results given by the formula (14.29) are compared with the results from my code that assumes a general numerical configuration. The results from the two methods agree with each other, as is shown in Fig. 14.1, which provides confidence in both the analytical formula and the numerical code.

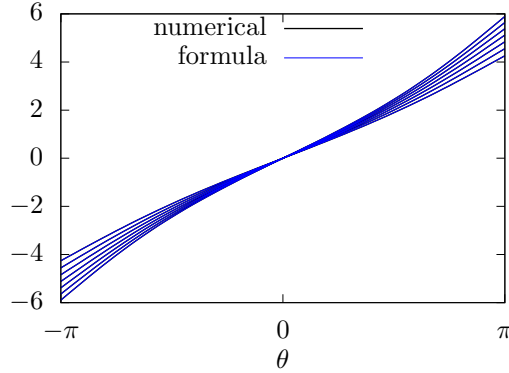


Figure 14.1. The results of $\bar{\delta} = \int_0^\theta \hat{q} d\theta$ computed by using formula (14.29) and the numerical code agree with each other. The different lines correspond to values of $\bar{\delta}$ on different magnetic surfaces. In the numerical code, two kinds of poloidal angles can be selected: one is the equal-volume poloidal angle, and another is the equal-arc-length angle. Make sure that the latter is selected when doing the comparison because the the poloidal angle θ appearing in the analytical formula is the equal-arc-length poloidal angle.

In passing, we note that the straight-field-line poloidal angle θ_f can also be considered to be defined by

$$\alpha = \phi - q\theta_f, \quad (14.31)$$

i.e.,

$$\theta_f = \frac{1}{q}(\phi - \alpha) = \frac{\bar{\delta}}{q}. \quad (14.32)$$

Then using Eq. (14.30), θ_f is written as

$$\theta_f = 2 \arctan \left(\frac{(R_0 - r)}{\sqrt{R_0^2 - r^2}} \tan \left(\frac{\theta}{2} \right) \right), \quad (14.33)$$

which agrees with Eq. (A2) in Gorler's paper[13].

14.4 Metric elements

Let $\psi = r$ and define $h^{\psi R} = \nabla \psi \cdot \hat{\mathbf{R}}$, $h^{\alpha R} = \nabla \alpha \cdot \hat{\mathbf{R}}$, etc. Explicit expressions for these elements can be written as

$$\begin{aligned} h^{\psi R} &= \nabla \psi \cdot \hat{\mathbf{R}} \\ &= \nabla r \cdot \hat{\mathbf{R}} \\ &= -\frac{R}{\mathcal{J}}(Z_\theta \hat{\mathbf{R}} - R_\theta \hat{\mathbf{Z}}) \cdot \hat{\mathbf{R}} \\ &= -\frac{R}{\mathcal{J}}Z_\theta \\ &= \cos \theta \\ h^{\psi Z} &= \nabla \psi \cdot \hat{\mathbf{Z}} \\ &= \nabla r \cdot \hat{\mathbf{Z}} \\ &= -\frac{R}{\mathcal{J}}(Z_\theta \hat{\mathbf{R}} - R_\theta \hat{\mathbf{Z}}) \cdot \hat{\mathbf{Z}} \\ &= \frac{R}{\mathcal{J}}R_\theta \\ &= \sin \theta \\ h^{\psi \phi} &= \nabla \psi \cdot \hat{\boldsymbol{\phi}} \\ &= 0 \end{aligned} \quad (14.34)$$

$$\begin{aligned} h^{\theta R} &= \nabla \theta \cdot \hat{\mathbf{R}} \\ &= \frac{R}{\mathcal{J}}(Z_\psi \hat{\mathbf{R}} - R_\psi \hat{\mathbf{Z}}) \cdot \hat{\mathbf{R}} \\ &= \frac{R}{\mathcal{J}}Z_\psi \end{aligned}$$

$$\begin{aligned}
&= -\frac{1}{r}\sin\theta \\
h^{\theta Z} &= \nabla\theta \cdot \hat{\mathbf{Z}} \\
&= \frac{R}{\mathcal{J}}(Z_\psi \hat{\mathbf{R}} - R_\psi \hat{\mathbf{Z}}) \cdot \hat{\mathbf{Z}} \\
&= -\frac{R}{\mathcal{J}}R_\psi \\
&= \frac{1}{r}\cos\theta \\
h^{\theta\phi} &= \nabla\theta \cdot \hat{\phi} \\
&= 0 \\
h^{\alpha R} &= \nabla\alpha \cdot \hat{\mathbf{R}} \\
&= \left[\frac{\hat{\phi}}{R} + \left(\frac{\partial\bar{\delta}}{\partial\psi} \frac{R}{\mathcal{J}} Z_\theta - \frac{\partial\bar{\delta}}{\partial\theta} \frac{R}{\mathcal{J}} Z_\psi \right) \hat{\mathbf{R}} + \left(\frac{\partial\bar{\delta}}{\partial\theta} \frac{R}{\mathcal{J}} R_\psi - \frac{\partial\bar{\delta}}{\partial\psi} \frac{R}{\mathcal{J}} R_\theta \right) \hat{\mathbf{Z}} \right] \cdot \hat{\mathbf{R}} \\
&= \frac{\partial\bar{\delta}}{\partial\psi} \frac{R}{\mathcal{J}} Z_\theta - \frac{\partial\bar{\delta}}{\partial\theta} \frac{R}{\mathcal{J}} Z_\psi \\
&= \frac{\partial\bar{\delta}}{\partial\psi} \frac{R}{\mathcal{J}} r \cos\theta - \frac{\partial\bar{\delta}}{\partial\theta} \frac{R}{\mathcal{J}} \sin\theta \\
&= -\frac{\partial\bar{\delta}}{\partial\psi} \cos\theta + \frac{\partial\bar{\delta}}{\partial\theta} \frac{1}{r} \sin\theta \\
&= -\frac{\partial\bar{\delta}}{\partial r} \cos\theta + \hat{q} \frac{1}{r} \sin\theta \\
h^{\alpha Z} &= \nabla\alpha \cdot \hat{\mathbf{Z}} \\
&= \left[\frac{\hat{\phi}}{R} + \left(\frac{\partial\bar{\delta}}{\partial\psi} \frac{R}{\mathcal{J}} Z_\theta - \frac{\partial\bar{\delta}}{\partial\theta} \frac{R}{\mathcal{J}} Z_\psi \right) \hat{\mathbf{R}} + \left(\frac{\partial\bar{\delta}}{\partial\theta} \frac{R}{\mathcal{J}} R_\psi - \frac{\partial\bar{\delta}}{\partial\psi} \frac{R}{\mathcal{J}} R_\theta \right) \hat{\mathbf{Z}} \right] \cdot \hat{\mathbf{Z}} \\
&= \frac{\partial\bar{\delta}}{\partial\theta} \frac{R}{\mathcal{J}} R_\psi - \frac{\partial\bar{\delta}}{\partial\psi} \frac{R}{\mathcal{J}} R_\theta \\
&= -\hat{q} \frac{1}{r} \cos\theta - \frac{\partial\bar{\delta}}{\partial r} \sin\theta \\
h^{\alpha\phi} &= \frac{1}{R}. \tag{14.35}
\end{aligned}$$

Using expression (14.29), $d\bar{\delta}/dr$ can be evaluated analytically, yielding

$$\begin{aligned}
\frac{d\bar{\delta}}{dr} &= 2\frac{dq}{dr} \arctan\left(\frac{(R_0-r)}{\sqrt{R_0^2-r^2}} \tan\left(\frac{\theta}{2}\right)\right) + 2q \frac{1}{1 + \left(\frac{(R_0-r)}{\sqrt{R_0^2-r^2}} \tan\left(\frac{\theta}{2}\right)\right)^2} \tan\left(\frac{\theta}{2}\right) \frac{d}{dr} \left[\frac{(R_0-r)}{\sqrt{R_0^2-r^2}} \right] \\
&= 2\frac{dq}{dr} \arctan\left(\frac{(R_0-r)}{\sqrt{R_0^2-r^2}} \tan\left(\frac{\theta}{2}\right)\right)
\end{aligned}$$

$$+ 2q \frac{1}{1 + \left(\frac{(R_0 - r)}{\sqrt{R_0^2 - r^2}} \tan\left(\frac{\theta}{2}\right) \right)^2} \tan\left(\frac{\theta}{2}\right) \frac{-R_0}{(R_0 + r)\sqrt{R_0^2 - r^2}}$$

where use has been made of

$$\frac{d}{dx} \arctan(x) = \frac{1}{1 + x^2}$$

(I did not remember this formula and I use SymPy to obtain this.) These expressions are used to benchmark the numerical code that assume general flux surface shapes. The results show that the code gives correct result when concentric circular flux surfaces are used.

Taking the θ derivative of $\bar{\delta}$, equation (14.29) is written as (using SymPy)

$$\frac{\partial \bar{\delta}}{\partial \theta} = 2qA \left(\frac{\tan^2\left(\frac{\theta}{2}\right)}{2} + \frac{1}{2} \right) \frac{1}{A^2 \tan^2\left(\frac{\theta}{2}\right) + 1} \quad (14.36)$$

where

$$A = \frac{(R_0 - r)}{\sqrt{R_0^2 - r^2}} \quad (14.37)$$

Equation (14.26) should be equal to \hat{q} given by Eq. (14.26). This was verified numerically.

Taking the r derivative of Eq. (14.9), we obtain

$$\frac{d\Psi_p}{dr} = \frac{1}{q(r)} \frac{2\pi g_0 r}{\sqrt{R_0^2 - r^2}} \quad (14.38)$$

i.e.,

$$\frac{d\Psi}{dr} = \frac{1}{q(r)} \frac{g_0 r}{\sqrt{R_0^2 - r^2}}. \quad (14.39)$$

$$\frac{d\psi}{dr} = \frac{1}{\Psi(a) - \Psi(0)} \frac{1}{q(r)} \frac{g_0 r}{\sqrt{R_0^2 - r^2}}. \quad (14.40)$$

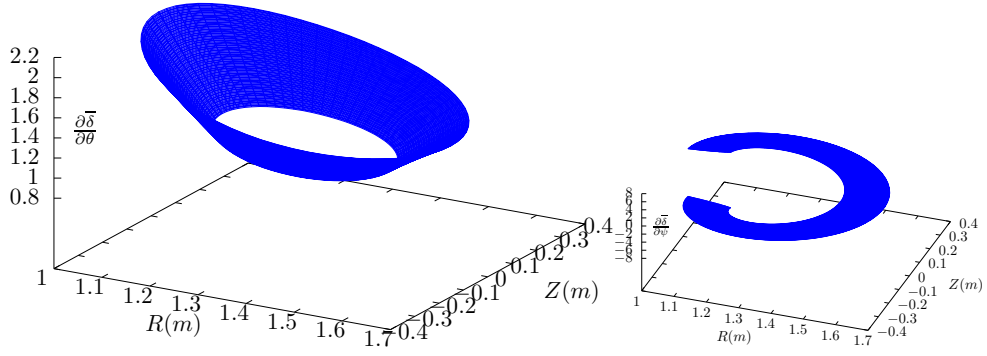


Figure 14.2. $\partial\bar{\delta}/\partial\theta$ and $\partial\bar{\delta}/\partial\psi$. Numerical and analytical results are plotted, which are so close to each other that they can not be distinguished by eyes. $\partial\bar{\delta}/\partial\theta$ is equal to the local safety factor \hat{q} . Note that $\partial\bar{\delta}/\partial\psi$ is discontinuous at the θ cut, which in this case is on the high-field-side.

14.5 Safety factor profile

The magnetic shear for a concentric-circular configuration is defined by

$$\hat{s} = \frac{dq}{dr} \frac{r}{q}, \quad (14.41)$$

where r is the minor radius of a magnetic surface. The above expression can be re-arranged as

$$\hat{s} \frac{dr}{r} = \frac{dq}{q} \quad (14.42)$$

Integrating the above equation over r and assuming \hat{s} is a constant, we obtain

$$\hat{s} \int_{r_0}^r \frac{dr}{r} = \int_{r_0}^r \frac{dq}{q}, \quad (14.43)$$

Performing the integration, the above equation is written as

$$\hat{s}(\ln r - \ln r_0) = \ln q - \ln q_0, \quad (14.44)$$

where $q_0 = q(r_0)$. Equation (14.44) can be finally written as

$$q = q_0 \left(\frac{r}{r_0} \right)^{\hat{s}}. \quad (14.45)$$

This is a profile with a constant magnetic shear s . In Ben's toroidal ITG simulation, the following q profile is used:

$$q = q_0 + (r - r_0)q'(r_0), \quad (14.46)$$

with $q'(r_0) = \hat{s}q_0/r_0$. This is a linear profile over r , with the values of q and the shear at $r = r_0$ being q_0 and \hat{s} , respectively.

15 Fixed boundary tokamak equilibrium problem

The fixed boundary equilibrium problem (also called the “inverse equilibrium problem” by some authors) refers to the case where the shape of a boundary magnetic surface is given and one is asked to solve the equilibrium within this magnetic surface. To make it convenient to deal with the shape of the boundary, one usually uses a general coordinates system which has one coordinate surface coinciding with the given magnetic surface. This makes it trivial to deal with the irregular boundary. To obtain the equilibrium, one needs to solve the GS equation in the general coordinate system.

15.1 Toroidal elliptic operator in general coordinates

Next we derive the form of the GS equation in a general coordinate system. The main task is to derive the form of the toroidal elliptic operator in the general coordinate system. The toroidal elliptic operator takes the form

$$\Delta^* \Psi \equiv R^2 \nabla \cdot \left(\frac{1}{R^2} \nabla \Psi \right) \quad (15.1)$$

For an arbitrary general coordinate system (ψ, θ, ϕ) (the (ψ, θ, ϕ) coordinate system here is an arbitrary general coordinate system except that $\nabla \phi$ is perpendicular to both $\nabla \psi$ and $\nabla \theta$), the toroidal elliptic operator is written

$$\Delta^* \Psi = \frac{R^2}{\mathcal{J}} \left\{ \left[\Psi_\psi \frac{\mathcal{J}}{R^2} |\nabla \psi|^2 \right]_\psi + \left[\Psi_\theta \frac{\mathcal{J}}{R^2} |\nabla \theta|^2 \right]_\theta + \left[\Psi_\psi \frac{\mathcal{J}}{R^2} (\nabla \psi \cdot \nabla \theta) \right]_\theta + \left[\Psi_\theta \frac{\mathcal{J}}{R^2} \nabla \theta \cdot \nabla \psi \right]_\psi \right\}, \quad (15.2)$$

where the subscripts denotes partial derivatives, \mathcal{J} is the Jacobian of the coordinate system (ψ, θ, ϕ) . [Next, we provide the proof of Eq. (15.2). The gradient of Ψ is written as (note that Ψ is independent of ϕ)

$$\nabla \Psi = \frac{\partial \Psi}{\partial \psi} \nabla \psi + \frac{\partial \Psi}{\partial \theta} \nabla \theta \quad (15.3)$$

Using this expression and the divergence formula (6.37), the elliptic operator in Eq. (15.1) is written

$$\begin{aligned} \Delta^* \Psi &= R^2 \nabla \cdot \left(\frac{1}{R^2} \frac{\partial \Psi}{\partial \psi} \nabla \psi + \frac{1}{R^2} \frac{\partial \Psi}{\partial \theta} \nabla \theta \right) \\ &= R^2 \frac{1}{\mathcal{J}} \frac{\partial}{\partial \psi} \left(\mathcal{J} \frac{1}{R^2} \frac{\partial \Psi}{\partial \psi} \nabla \psi \cdot \nabla \psi + \mathcal{J} \frac{1}{R^2} \frac{\partial \Psi}{\partial \theta} \nabla \theta \cdot \nabla \psi \right) \\ &\quad + R^2 \frac{1}{\mathcal{J}} \frac{\partial}{\partial \theta} \left(\mathcal{J} \frac{1}{R^2} \frac{\partial \Psi}{\partial \psi} \nabla \psi \cdot \nabla \theta + \mathcal{J} \frac{1}{R^2} \frac{\partial \Psi}{\partial \theta} \nabla \theta \cdot \nabla \theta \right) \end{aligned}$$

$$= \frac{R^2}{\mathcal{J}} \left[\left(\Psi_\psi \frac{\mathcal{J}}{R^2} |\nabla \psi|^2 \right)_\psi + \left(\Psi_\theta \frac{\mathcal{J}}{R^2} \nabla \psi \cdot \nabla \theta \right)_\psi + \left(\Psi_\psi \frac{\mathcal{J}}{R^2} \nabla \psi \cdot \nabla \theta \right)_\theta + \left(\Psi_\theta \frac{\mathcal{J}}{R^2} |\nabla \theta|^2 \right)_\theta \right], \quad (15.4)$$

which is Eq. (15.2).]

Using Eq. (15.2), the GS equation (3.12) is written

$$\frac{R^2}{\mathcal{J}} \left[\left(\Psi_\psi \frac{\mathcal{J}}{R^2} |\nabla \psi|^2 \right)_\psi + \left(\Psi_\theta \frac{\mathcal{J}}{R^2} |\nabla \theta|^2 \right)_\theta + \left(\Psi_\theta \frac{\mathcal{J}}{R^2} \nabla \psi \cdot \nabla \theta \right)_\psi + \left(\Psi_\psi \frac{\mathcal{J}}{R^2} \nabla \psi \cdot \nabla \theta \right)_\theta \right] = -\mu_0 R^2 \frac{dP}{d\Psi} - \frac{dg}{d\Psi} g, \quad (15.5)$$

which is the form of the GS equation in (ψ, θ, ϕ) coordinate system.

15.2 Finite difference form of toroidal elliptic operator in general coordinate system

The toroidal elliptic operator in Eq. (15.2) can be written

$$\Delta^* \Psi = \frac{R^2}{\mathcal{J}} [(h^{\psi\psi} \Psi_\psi)_\psi + (h^{\theta\theta} \Psi_\theta)_\theta + (h^{\psi\theta} \Psi_\theta)_\psi + (h^{\psi\theta} \Psi_\psi)_\theta], \quad (15.6)$$

where $h^{a\beta}$ is defined by Eq. (7.5), i.e.,

$$h^{\alpha\beta} = \frac{\mathcal{J}}{R^2} \nabla \alpha \cdot \nabla \beta. \quad (15.7)$$

Next, we derive the finite difference form of the toroidal elliptic operator. The finite difference form of the term $(h^{\psi\psi} \Psi_\psi)_\psi$ is written

$$\begin{aligned} (h^{\psi\psi} \Psi_\psi)_\psi|_{i,j} &= \frac{1}{\delta\psi} \left[h_{i,j+1/2}^{\psi\psi} \left(\frac{\Psi_{i,j+1} - \Psi_{i,j}}{\delta\psi} \right) - h_{i,j-1/2}^{\psi\psi} \left(\frac{\Psi_{i,j} - \Psi_{i,j-1}}{\delta\psi} \right) \right] \\ &= H_{i,j+1/2}^{\psi\psi} (\Psi_{i,j+1} - \Psi_{i,j}) - H_{i,j-1/2}^{\psi\psi} (\Psi_{i,j} - \Psi_{i,j-1}), \end{aligned} \quad (15.8)$$

where

$$H^{\psi\psi} = \frac{h^{\psi\psi}}{(\delta\psi)^2}. \quad (15.9)$$

The finite difference form of $(h^{\theta\theta} \Psi_\theta)_\theta$ is written

$$\begin{aligned} (h^{\theta\theta} \Psi_\theta)_\theta|_{i,j} &= \frac{1}{\delta\theta} \left[h_{i+1/2,j}^{\theta\theta} \left(\frac{\Psi_{i+1,j} - \Psi_{i,j}}{\delta\theta} \right) - h_{i-1/2,j}^{\theta\theta} \left(\frac{\Psi_{i,j} - \Psi_{i-1,j}}{\delta\theta} \right) \right] \\ &= H_{i+1/2,j}^{\theta\theta} (\Psi_{i+1,j} - \Psi_{i,j}) - H_{i-1/2,j}^{\theta\theta} (\Psi_{i,j} - \Psi_{i-1,j}), \end{aligned} \quad (15.10)$$

where

$$H^{\theta\theta} = \frac{h^{\theta\theta}}{(\delta\theta)^2}. \quad (15.11)$$

The finite difference form of $(h^{\psi\theta}\Psi_\theta)_\psi$ is written as

$$\begin{aligned} (\Psi_\theta h^{\psi\theta})_\psi|_{i,j} &= \frac{1}{\delta\psi} \left[h_{i,j+1/2}^{\psi\theta} \left(\frac{\Psi_{i+1,j+1/2} - \Psi_{i-1,j+1/2}}{2\delta\theta} \right) - \right. \\ &\quad \left. h_{i,j-1/2}^{\psi\theta} \left(\frac{\Psi_{i+1,j-1/2} - \Psi_{i-1,j-1/2}}{2\delta\theta} \right) \right]. \end{aligned} \quad (15.12)$$

Approximating the value of Ψ at the grid centers by the average of the value of Ψ at the neighbor grid points, Eq. (15.12) is written as

$$\begin{aligned} (\Psi_\theta h^{\psi\theta})_\psi|_{i,j} &= H_{i,j+1/2}^{\psi\theta} (\Psi_{i+1,j} + \Psi_{i+1,j+1} - \Psi_{i-1,j} - \Psi_{i-1,j+1}) - \\ &\quad H_{i,j-1/2}^{\psi\theta} (\Psi_{i+1,j-1} + \Psi_{i+1,j} - \Psi_{i-1,j-1} - \Psi_{i-1,j}). \end{aligned} \quad (15.13)$$

where

$$H^{\psi\theta} = \frac{h^{\psi\theta}}{4\delta\psi d\theta}. \quad (15.14)$$

Similarly, the finite difference form of $(h^{\psi\theta}\Psi_\psi)_\theta$ is written as

$$\begin{aligned} (\Psi_\psi h^{\psi\theta})_\theta|_{i,j} &= \frac{1}{\delta\theta} \left[h_{i+1/2,j}^{\psi\theta} \left(\frac{\Psi_{i+1/2,j+1} - \Psi_{i+1/2,j-1}}{2\delta\psi} \right) - \right. \\ &\quad \left. h_{i-1/2,j}^{\psi\theta} \left(\frac{\Psi_{i-1/2,j+1} - \Psi_{i-1/2,j-1}}{2\delta\psi} \right) \right] \\ &= H_{i+1/2,j}^{\psi\theta} (\Psi_{i+1,j+1} + \Psi_{i,j+1} - \Psi_{i+1,j-1} - \Psi_{i,j-1}) - \\ &\quad H_{i-1/2,j}^{\psi\theta} (\Psi_{i,j+1} + \Psi_{i-1,j+1} - \Psi_{i,j-1} - \Psi_{i-1,j-1}). \end{aligned} \quad (15.15)$$

Using the above results, the finite difference form of the operator $\mathcal{J}\Delta^*\Psi/R^2$ is written as

$$\begin{aligned} \frac{\mathcal{J}}{R^2}\Delta^*\Psi|_{i,j} &= (h^{\psi\theta}\Psi_\theta)_\psi + (h^{\psi\psi}\Psi_\psi)_\psi + (h^{\theta\theta}\Psi_\theta)_\theta + (h^{\psi\theta}\Psi_\psi)_\theta \\ &= H_{i,j+1/2}^{\psi\psi} (\Psi_{i,j+1} - \Psi_{i,j}) - H_{i,j-1/2}^{\psi\psi} (\Psi_{i,j} - \Psi_{i,j-1}) \\ &\quad + H_{i+1/2,j}^{\theta\theta} (\Psi_{i+1,j} - \Psi_{i,j}) - H_{i-1/2,j}^{\theta\theta} (\Psi_{i,j} - \Psi_{i-1,j}) \\ &\quad + H_{i,j+1/2}^{\psi\theta} (\Psi_{i+1,j} + \Psi_{i+1,j+1} - \Psi_{i-1,j} - \Psi_{i-1,j+1}) - \\ &\quad H_{i,j-1/2}^{\psi\theta} (\Psi_{i+1,j-1} + \Psi_{i+1,j} - \Psi_{i-1,j-1} - \Psi_{i-1,j}) \\ &\quad + H_{i+1/2,j}^{\psi\theta} (\Psi_{i+1,j+1} + \Psi_{i,j+1} - \Psi_{i+1,j-1} - \Psi_{i,j-1}) - \\ &\quad H_{i-1/2,j}^{\psi\theta} (\Psi_{i,j+1} + \Psi_{i-1,j+1} - \Psi_{i-1,j-1} - \Psi_{i,j-1}) \\ &= \Psi_{i,j} (-H_{i,j+1/2}^{\psi\psi} - H_{i,j-1/2}^{\psi\psi} - H_{i+1/2,j}^{\theta\theta} - H_{i-1/2,j}^{\theta\theta}) \\ &\quad + \Psi_{i-1,j-1} (H_{i,j-1/2}^{\psi\theta} + H_{i-1/2,j}^{\psi\theta}) + \Psi_{i,j-1} (H_{i,j-1/2}^{\psi\psi} - H_{i+1/2,j}^{\psi\theta} + \\ &\quad H_{i-1/2,j}^{\psi\theta}) \\ &\quad + \Psi_{i+1,j-1} (-H_{i,j-1/2}^{\psi\theta} - H_{i+1/2,j}^{\psi\theta}) + \Psi_{i-1,j} (H_{i-1/2,j}^{\theta\theta} - H_{i,j+1/2}^{\psi\theta} + \\ &\quad H_{i,j-1/2}^{\psi\theta}) \end{aligned}$$

$$\begin{aligned}
& + \Psi_{i+1,j}(H_{i+1/2,j}^{\theta\theta} + H_{i,j+1/2}^{\psi\theta} - H_{i,j-1/2}^{\psi\theta}) + \Psi_{i-1,j+1}(-H_{i,j+1/2}^{\psi\theta} - H_{i-1/2,j}^{\psi\theta}) \\
& + \Psi_{i,j+1}(H_{i,j+1/2}^{\psi\psi} + H_{i+1/2,j}^{\psi\theta} - H_{i-1/2,j}^{\psi\theta}) + \Psi_{i+1,j+1}(H_{i,j+1/2}^{\psi\theta} + H_{i+1/2,j}^{\psi\theta})
\end{aligned}$$

The coefficients are given by

$$h^{\psi\psi} = \frac{\mathcal{J}}{R^2} |\nabla\psi|^2 = \frac{1}{\mathcal{J}} (R_\theta^2 + Z_\theta^2), \quad (15.16)$$

$$h^{\theta\theta} = \frac{\mathcal{J}}{R^2} |\nabla\theta|^2 = \frac{1}{\mathcal{J}} (R_\psi^2 + Z_\psi^2), \quad (15.17)$$

and

$$h^{\psi\theta} = \frac{\mathcal{J}}{R^2} \nabla\psi \cdot \nabla\theta = -\frac{1}{\mathcal{J}} (R_\theta R_\psi + Z_\theta Z_\psi), \quad (15.18)$$

where the Jacobian

$$\mathcal{J} = R(R_\theta Z_\psi - R_\psi Z_\theta). \quad (15.19)$$

The partial derivatives, R_θ , R_ψ , Z_θ , and Z_ψ , appearing in Eqs. (15.16)-(15.19) are calculated by using the central difference scheme. The values of $h^{\psi\psi}$, $h^{\theta\theta}$, $h^{\psi\theta}$ and \mathcal{J} at the middle points are approximated by the linear average of their values at the neighbor grid points.

15.3 Special treatment at coordinate origin, wrong! to be deleted

$$\Psi \approx a^{00} + a^{10}R + a^{01}Z + a^{11}RZ + a^{20}R^2 + a^{02}Z^2. \quad (15.20)$$

$$R = R_0 + b^{11}\sqrt{\psi}\cos\theta \quad (15.21)$$

$$Z = Z_0 + c^{11}\sqrt{\psi}\sin\theta \quad (15.22)$$

$$\begin{aligned}
\Psi & \approx a^{00} + a^{10}(R_0 + b^{11}\sqrt{\psi}\cos\theta) + a^{01}c^{11}\sqrt{\psi}\sin\theta + a^{11}(R_0 + b^{11}\sqrt{\psi}\cos\theta)c^{11}\sqrt{\psi}\sin\theta + a^{20}(R_0 + b^{11}\sqrt{\psi}\cos\theta)^2 + a^{02}(c^{11}\sqrt{\psi}\sin\theta)^2 \\
& = a^{00} + a^{10}R_0 + a^{10}b^{11}\sqrt{\psi}\cos\theta + a^{01}c^{11}\sqrt{\psi}\sin\theta + a^{11}R_0c^{11}\sqrt{\psi}\sin\theta + \frac{1}{2}a^{11}b^{11}\sqrt{\psi}c^{11}\sqrt{\psi}\sin 2\theta + 2a^{20}R_0^2 + a^{20}(b^{11})^2\psi\cos^2\theta + 2a^{20}R_0b^{11}\sqrt{\psi}\cos\theta + a^{02}(c^{11})^2\psi\sin^2\theta \\
& = d_0 + d_1\sqrt{\psi}\cos\theta + d_2\sqrt{\psi}\sin\theta + d_3\psi\sin 2\theta + d_4\psi\cos^2\theta + d_5\psi\sin^2\theta
\end{aligned}$$

$$\Psi = d_0 + d_1\sqrt{\psi}\cos\theta + d_2\sqrt{\psi}\sin\theta$$

$$R = R_0 + b^{11}\sqrt{\psi}\cos\theta + b^{12}\sqrt{\psi}\cos 2\theta + b^{21}(\sqrt{\psi})^2\cos\theta + b^{22}(\sqrt{\psi})^2\cos 2\theta \quad (15.23)$$

$$Z = Z_0 + c^{11}\sqrt{\psi}\sin\theta + c^{12}\sqrt{\psi}\sin 2\theta + c^{21}(\sqrt{\psi})^2\sin\theta + c^{22}(\sqrt{\psi})^2\sin 2\theta \quad (15.24)$$

15.4 Pressure and toroidal field function profile

The function $p(\Psi)$ and $g(\Psi)$ in the GS equation are free functions which must be specified by users before solving the GS equation. Next, we discuss one way to specify the free functions. Following Ref. [16], we take $P(\Psi)$ and $g(\Psi)$ to be of the forms

$$P(\Psi) = P_0 - (P_0 - P_b)\hat{p}(\bar{\Psi}), \quad (15.25)$$

$$\frac{1}{2}g^2(\Psi) = \frac{1}{2}g_0^2(1 - \gamma\hat{g}(\bar{\Psi})), \quad (15.26)$$

with \hat{p} and \hat{g} chosen to be of polynomial form:

$$\hat{p}(\bar{\Psi}) = \bar{\Psi}^\alpha, \quad (15.27)$$

$$\hat{g}(\bar{\Psi}) = \bar{\Psi}^\beta, \quad (15.28)$$

where

$$\bar{\Psi} \equiv \frac{\Psi - \Psi_a}{\Psi_b - \Psi_a}, \quad (15.29)$$

with Ψ_b the value of Ψ on the boundary, Ψ_a the value of Ψ on the magnetic axis, α , β , γ , P_0 , P_b , and g_0 are free parameters. Using the profiles of P and g given by Eqs. (15.25) and (15.26), we obtain

$$\frac{dP(\Psi)}{d\Psi} = -\frac{1}{\Delta}(P_0 - P_b)\alpha\bar{\Psi}^{\alpha-1} \quad (15.30)$$

where $\Delta = \Psi_b - \Psi_a$, and

$$g\frac{dg}{d\Psi} = -\frac{1}{2}g_0^2\gamma\frac{1}{\Delta}\beta\bar{\Psi}^{\beta-1}. \quad (15.31)$$

Then the term on the r.h.s (nonlinear source term) of the GS equation is written

$$-\mu_0 R^2 \frac{dP}{d\Psi} - \frac{dg}{d\Psi}g = \mu_0 R^2 \left[\frac{1}{\Delta}(P_0 - P_b)\alpha\bar{\Psi}^{\alpha-1} \right] + \frac{1}{2}g_0^2\gamma\frac{1}{\Delta}\beta\bar{\Psi}^{\beta-1}. \quad (15.32)$$

The value of parameters P_0 , P_b , and g_0 in Eqs. (15.25) and (15.26), and the value of α and β in Eqs. (15.27) and (15.28) can be chosen arbitrarily. The parameter γ is used to set the value of the total toroidal current. The toroidal current density is given by Eq. (A.44), i.e.,

$$J_\phi = R \frac{dP}{d\Psi} + \frac{1}{\mu_0} \frac{dg}{d\Psi} \frac{g}{R}, \quad (15.33)$$

which can be integrated over the poloidal cross section within the boundary magnetic surface to give the total toroidal current,

$$\begin{aligned}
I_\phi &= \int J_\phi dS \\
&= \int \left(R \frac{dP}{d\Psi} + \frac{1}{\mu_0} \frac{dg}{d\Psi} \frac{g}{R} \right) dR dZ \\
&= \int \left[R \left(-\frac{1}{\Delta} \right) (P_0 - P_b) \hat{p}'(\bar{\Psi}) - \frac{1}{\mu_0 R} \frac{1}{2} g_0^2 \gamma \frac{1}{\Delta} \hat{g}'(\bar{\Psi}) \right] dR dZ \quad (15.34)
\end{aligned}$$

Using

$$dR dZ = \frac{1}{R} \mathcal{J} d\psi d\theta, \quad (15.35)$$

Eq. (15.34) is written as

$$I_\phi = \int \left[\left(-\frac{1}{\Delta} \right) (P_0 - P_b) \hat{p}'(\bar{\Psi}) - \frac{1}{\mu_0 R^2} \frac{1}{2} g_0^2 \gamma \frac{1}{\Delta} \hat{g}'(\bar{\Psi}) \right] \mathcal{J} d\psi d\theta, \quad (15.36)$$

from which we solve for γ , giving

$$\gamma = \frac{-\Delta I_\phi - (P_0 - P_b) \int [\hat{p}'(\bar{\Psi})] \mathcal{J} d\psi d\theta}{\frac{1}{2\mu_0} g_0^2 \int \left[\frac{1}{R^2} \hat{g}'(\bar{\Psi}) \right] \mathcal{J} d\psi d\theta}. \quad (15.37)$$

If the total toroidal current I_ϕ is given, Eq. (15.37) can be used to determine the value of γ .

15.5 Boundary magnetic surface and initial coordinates

In the fixed boundary equilibrium problem, the shape of the boundary magnetic surface (it is also the boundary of the computational region) is given while the shape of the inner flux surface is to be solved. A simple analytical expression for a D-shaped magnetic surface takes the form

$$R = R_0 + a \cos(\theta + \Delta \sin \theta), \quad (15.38)$$

$$Z = \kappa a \sin \theta, \quad (15.39)$$

with θ changing from 0 to 2π . According to the definition in Eqs. (1.29), (1.30), and (1.32) we can readily verify that the parameters a , R_0 , κ appearing in Eqs. (15.38) and (15.39) are indeed the minor radius, major radius, and ellipticity, respectively. According to the definition of triangularity Eq. (1.31), the triangularity δ for the magnetic surface defined by Eqs. (15.38) and (15.39) is written as

$$\delta = \sin \Delta. \quad (15.40)$$

Another common expression for the shape of a magnetic surface was given by Miller[8, 20], which is written as

$$R = R_0 + a \cos[\theta + \arcsin(\Delta \sin \theta)], \quad (15.41)$$

$$Z = \kappa a \sin \theta. \quad (15.42)$$

Note that Miller's formula is only slightly different from the formula (15.38). For Miller's formula, it is easy to prove that the triangularity δ is equal to Δ (instead of $\delta = \sin\Delta$ as given in Eq. (15.40)).

In the iterative metric method[11] for solving the fixed boundary equilibrium problem, we need to provide an initial guess of the shape of the inner flux surface (this initial guess is used to construct a initial generalized coordinates system). A common guess of the inner flux surfaces is given by

$$R = \psi^\alpha (R_{\text{LCFS}} - R_0) + R_0, \quad (15.43)$$

$$Z = \psi^\alpha Z_{\text{LCFS}}, \quad (15.44)$$

where α is a parameter, ψ is a label parameter of flux surface. If the shape of the LCFS is given by Eqs. (15.38) and (15.39), then Eqs. (15.43) and (15.44) are written as

$$R = R_0 + \psi^\alpha a \cos(\theta + \Delta \sin\theta), \quad (15.45)$$

$$Z = \psi^\alpha \kappa a \sin\theta. \quad (15.46)$$

Fig. 15.1 plots the shape given by Eqs. (15.45) and (15.46) for $a = 0.4$, $R_0 = 1.7$, $\kappa = 1.7$, $\Delta = \arcsin(0.6)$, and $\alpha = 1$ with ψ varying from zero to one.

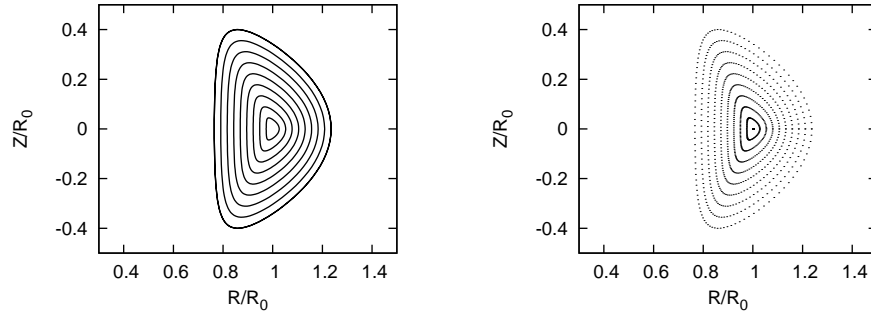


Figure 15.1. Shape of flux surface given by Eqs. (15.45) and (15.46). Left figure: points with the same value of ψ are connected to show ψ coordinate surface; Right figure: dot plot. Parameters are $a = 0.4m$, $R_0 = 1.7m$, $\kappa = 1.7$, $\Delta = \arcsin(0.6)$, $\alpha = 1$ with ψ varying from zero (center) to one (boundary). The shape parameters of LCFS are chosen according to the parameters of EAST tokamak.

15.6 Fixed boundary equilibrium numerical code

The tokamak equilibrium problem where the shape of the LCFS is given is called fixed boundary equilibrium problem. I wrote a numerical code that uses the iterative metric method[11] to solve this kind of equilibrium problem. Figure 15.2 describes the steps involved in the iterative metric method.

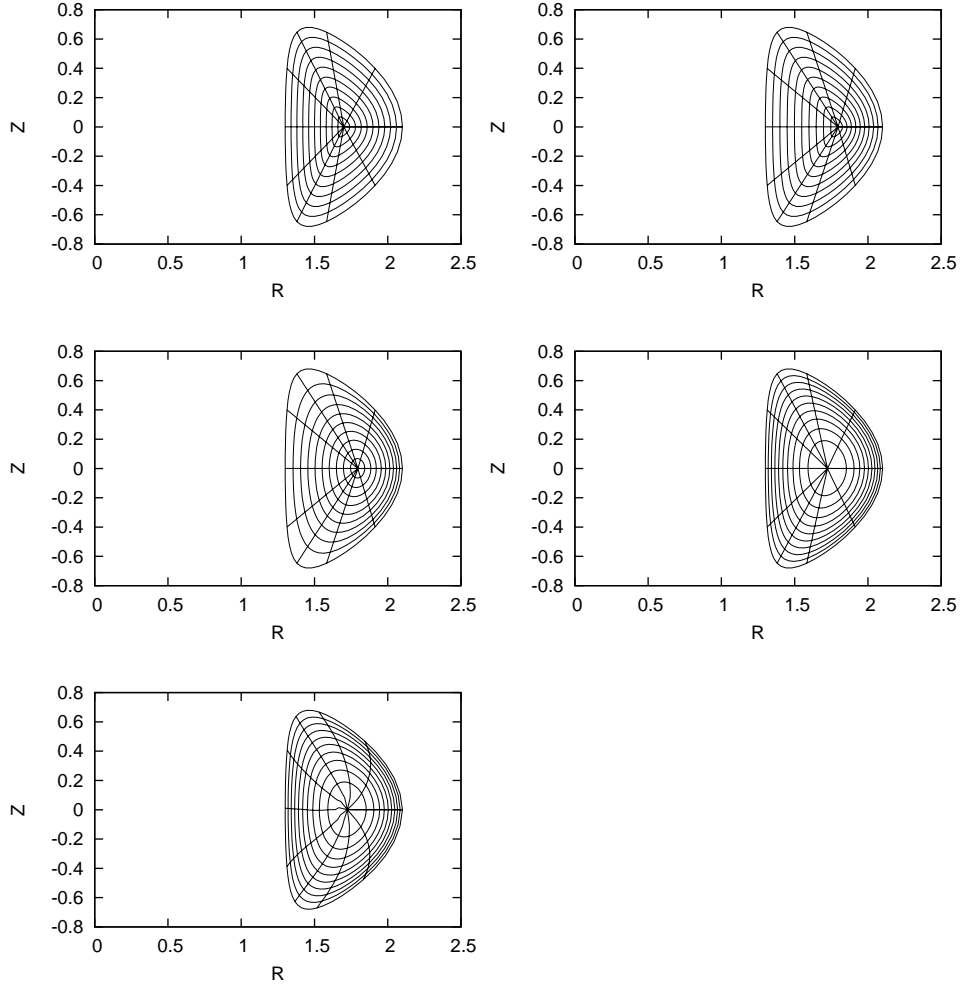


Figure 15.2. Upper left figure plots the initial coordinate surfaces. After solving the GS equation to obtain the location of the magnetic axis, I shift the origin point of the initial coordinate system to the location of the magnetic axis (upper right figure). Then, reshape the coordinate surface so that the coordinate surfaces $\psi = \text{const}$ lies on magnetic surfaces (middle left figure). Recalculate the radial coordinate ψ that is consistent with the Jacobian constraint and interpolate flux surface to uniform ψ coordinates (middle right figure). Recalculate the poloidal coordinate θ that is consistent with the Jacobian constraint and interpolate poloidal points on every flux surface to uniform θ coordinates (bottom left figure).

15.7 Benchmark of the code

To benchmark the numerical code, I set the profile of P and g according to Eqs. (?) and (?) with the parameters $c_2 = 0$, $c_1 = B_0(\kappa_0^2 + 1)/(R_0^2 \kappa_0 q_0)$, $\kappa_0 = 1.5$, and

$q_0 = 1.5$. The comparison of the analytic and numerical results are shown in Fig. 15.3.

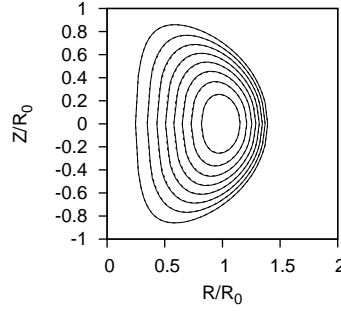


Figure 15.3. Agreement between the magnetic surfaces (contours of Ψ) given by the Solovév analytic formula and those calculated by the numerical code. The two sets of magnetic surfaces are indistinguishable at this scale. The boundary magnetic surface is selected by the requirement that $\bar{\Psi} = 0.11022$ in the Solovév formula (?). Parameters: $\kappa_0 = 1.5$, $q_0 = 1.5$.

Note that the parameter c_0 in the Solovév equilibrium seems to be not needed in the numerical calculation. In fact this impression is wrong: the c_0 parameter is actually needed in determining the boundary magnetic surface of the numerical equilibrium (in the case considered here c_0 is chosen as $c_0 = B_0 / (R_0^2 \kappa_0 q_0)$).

15.8 Low-beta equilibrium vs. high-beta equilibrium

With the pressure increasing, the magnetic axis usually shifts to the low-field-side of the device, as is shown in Fig. 15.4.

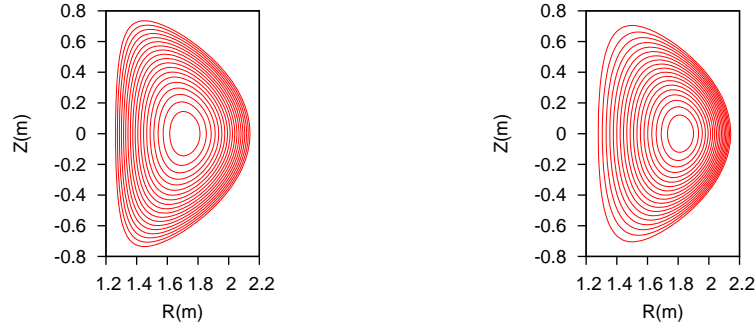


Figure 15.4. Comparison of two equilibria obtained with $P_0 = 10^4$ Pasca (left) and $P_0 = 10^5$ Pasca (right), respectively, where P_0 is the pressure at the magnetic axis. All the other parameters are the same for the two equilibria, $\alpha = 1.0$, $\beta = 1.0$, $P_b = 10^{-1}$ Pasca, $g_0 = 1.0 Tm$, $I_\phi = 500 kA$, and the LCFS is given by miller's formulas (15.41) and (15.42) with $R_0 = 1.7$, $a = 0.45$, $\kappa = 1.7$, and $\delta = 0.6$.

15.9 Analytical form of Jacobian (need cleaning up)

Consider the Jacobian of the form

$$\mathcal{J} = \mu \left(\frac{R}{R_0} \right)^m \psi^n, \quad (15.47)$$

where m and n are arbitrary integers which can be appropriately chosen by users, μ_0 and R_0 are constants included for normalization. In the iterative metric method of solving fixed boundary equilibrium problem, we first construct a coordinates transformation $(\theta, \psi) \rightarrow (R(\theta, \psi), Z(\theta, \psi))$ (this transformation is arbitrary except for that surface $\psi = 1$ coincides with the last closed flux surface), then solve the GS equation in (θ, ψ) coordinate system to get the value of Ψ at grid points, and finally adjust the value of $(R(\theta, \psi), Z(\theta, \psi))$ to make surface $\psi = \text{const}$ lies on a magnetic surface. It is obvious the Jacobian of the final transformation we obtained usually does not satisfy the constraint given by Eq. (15.47) since we do not use any information of Eq. (15.47) in the above steps. Now comes the question: how to make the transformation obtained above satisfy the constraint Eq. (15.47) through adjusting the values of θ ? To make the constraint Eq. (15.47) satisfied, θ and ψ should satisfy the relation

$$\mathcal{J}(\theta, \psi) = \mu \left[\frac{R(\theta, \psi)}{R_0} \right]^m \psi^n, \quad (15.48)$$

which

Using Eq. (), we obtain

$$\begin{aligned} \int_0^\theta d\theta &= \int_0^\theta d\theta \frac{\mathcal{J}}{\psi^n \mu} \left(\frac{R_0}{R} \right)^m \\ \theta &= \int_0^\theta d\theta \frac{\mathcal{J}}{\psi^n \mu} \left(\frac{R_0}{R} \right)^m \end{aligned} \quad (15.49)$$

normalized to 2π , the normalized θ is written as

$$\theta = 2\pi \frac{\int_0^\theta d\theta \frac{\mathcal{J}}{\psi^n \mu} \left(\frac{R_0}{R} \right)^m}{\int_0^{2\pi} d\theta \frac{\mathcal{J}}{\psi^n \mu} \left(\frac{R_0}{R} \right)^m} = 2\pi \frac{\int_0^\theta d\theta \mathcal{J} \left(\frac{1}{R} \right)^m}{\int_0^{2\pi} d\theta \mathcal{J} \left(\frac{1}{R} \right)^m} \quad (15.50)$$

15.10 Grad-Shafranov equation with prescribed safety factor profile (to be finished)

In the GS equation, $g(\Psi)$ is one of the two free functions which can be prescribed by users. In some cases, we want to specify the safety factor profile $q(\Psi)$, instead of $g(\Psi)$, in solving the GS equation. Next, we derive the form of the GS equation

that contains $q(\Psi)$, rather than $g(\Psi)$, as a free function. The safety factor defined in Eq. (8.8) can be written

$$q(\psi) = -\frac{1}{2\pi} \frac{g}{\Psi'} \int_0^{2\pi} \frac{\mathcal{J}}{R^2} d\theta \quad (15.51)$$

$$\begin{aligned} &= -\frac{1}{2\pi} \frac{g}{\Psi'} \frac{\int_0^{2\pi} R^{-2} \mathcal{J} d\theta}{\int_0^{2\pi} \mathcal{J} d\theta} \int_0^{2\pi} \mathcal{J} d\theta \\ &= -\frac{1}{2\pi} \frac{g}{\Psi'} \langle R^{-2} \rangle \int_0^{2\pi} \mathcal{J} d\theta \end{aligned} \quad (15.52)$$

$$= -\text{sgn}(\mathcal{J}) \frac{V'}{(2\pi)^2} \frac{\langle R^{-2} \rangle}{\Psi'} g. \quad (15.53)$$

Equation (15.53) gives the relation between the safety factor q and the toroidal field function g . This relation can be used in the GS equation to eliminate g in favor of q , which gives

$$g = -\text{sgn}(\mathcal{J}) \frac{(2\pi)^2}{V' \langle R^{-2} \rangle} \Psi' q. \quad (15.54)$$

$$\Rightarrow \frac{dg}{d\Psi} g = \frac{(2\pi)^2}{V' \langle R^{-2} \rangle} q \left[\frac{(2\pi)^2}{V' \langle R^{-2} \rangle} q \Psi' \right]_{\psi}. \quad (15.55)$$

Note that this expression involves the flux surface average, which depends on the flux surface shape and the shape is unknown before Ψ is determined.

Multiplying Eq. (?) by R^{-2} gives

$$\frac{1}{\mathcal{J}} \left[\left(\Psi' \frac{\mathcal{J}}{R^2} |\nabla \psi|^2 \right)_{\psi} + \left(\Psi' \frac{\mathcal{J}}{R^2} (\nabla \psi \cdot \nabla \theta) \right)_{\theta} \right] + \mu_0 \frac{dp}{d\Psi} + R^{-2} \frac{dg}{d\Psi} g = 0. \quad (15.56)$$

Surface-averaging the above equation, we obtain

$$\frac{2\pi}{V'} \int d\theta \left[\left(\Psi' \frac{\mathcal{J}}{R^2} |\nabla \psi|^2 \right)_{\psi} + \left(\Psi' \frac{\mathcal{J}}{R^2} (\nabla \psi \cdot \nabla \theta) \right)_{\theta} \right] + \mu_0 \frac{dp}{d\Psi} + \langle R^{-2} \rangle \frac{dg}{d\Psi} g = 0, \quad (15.57)$$

$$\Rightarrow \frac{2\pi}{V'} \int d\theta \left(\Psi' \frac{\mathcal{J}}{R^2} |\nabla \psi|^2 \right)_{\psi} + \mu_0 \frac{dp}{d\Psi} + \langle R^{-2} \rangle \frac{dg}{d\Psi} g = 0, \quad (15.58)$$

$$\Rightarrow \frac{2\pi}{V'} \left[\Psi' \int d\theta \left(\mathcal{J} \frac{|\nabla \psi|^2}{R^2} \right)_{\psi} \right] + \mu_0 \frac{dp}{d\Psi} + \langle R^{-2} \rangle \frac{dg}{d\Psi} g = 0, \quad (15.59)$$

$$\Rightarrow \frac{1}{V'} \left[V' \Psi' \left\langle \frac{|\nabla \psi|^2}{R^2} \right\rangle_{\psi} \right] + \mu_0 \frac{dp}{d\Psi} + \langle R^{-2} \rangle \frac{dg}{d\Psi} g = 0. \quad (15.60)$$

$$\Rightarrow \left[V' \Psi' \left\langle \frac{|\nabla \psi|^2}{R^2} \right\rangle_{\psi} \right] + \mu_0 V' \frac{dp}{d\Psi} + \langle R^{-2} \rangle V' \frac{dg}{d\Psi} g = 0, \quad (15.61)$$

Substitute Eq. (15.55) into the above equation to eliminate $gdg/d\Psi$, we obtain

$$\Rightarrow \left[V' \Psi' \left\langle \frac{|\nabla \psi|^2}{R^2} \right\rangle \right]_{\psi} + \mu_0 V' \frac{dp}{d\Psi} + q(2\pi)^4 \left[\frac{q\Psi'}{V' \langle R^{-2} \rangle} \right]_{\psi} = 0, \quad (15.62)$$

Eq. (15.62) agrees with Eq. (5.55) in Ref. [16].

$$\begin{aligned} \Rightarrow & V' \left\langle \frac{|\nabla \psi|^2}{R^2} \right\rangle \Psi'' + \left[V' \left\langle \frac{|\nabla \psi|^2}{R^2} \right\rangle \right]_{\psi} \Psi' + \mu_0 V' \frac{dp}{d\Psi} + q(2\pi)^4 \left[\frac{q}{V' \langle R^{-2} \rangle} \right] \Psi'' + \\ & q(2\pi)^4 \left[\frac{q}{V' \langle R^{-2} \rangle} \right]_{\psi} \Psi' = 0 \end{aligned} \quad (15.63)$$

$$\begin{aligned} \Rightarrow \Psi'' = & - \frac{1}{V'D} \left\{ \left[V' \left\langle \frac{|\nabla \psi|^2}{R^2} \right\rangle \right]_{\psi} \Psi' + q(2\pi)^4 \left[\frac{q}{V' \langle R^{-2} \rangle} \right]_{\psi} \Psi' + \right. \\ & \left. \mu_0 V' \frac{dp}{d\Psi} \right\} \end{aligned} \quad (15.64)$$

where

$$D = \left\langle \frac{|\nabla \psi|^2}{R^2} \right\rangle + \frac{(2\pi)^4}{\langle R^{-2} \rangle} \left(\frac{q}{V'} \right)^2 \quad (15.65)$$

The GS equation is

$$\Delta^* \Psi = -\mu_0 R^2 \frac{dp}{d\Psi} - g \frac{dg}{d\Psi} \quad (15.66)$$

$$\Rightarrow \Delta^* \Psi = -\mu_0 R^2 \frac{dp}{d\Psi} - \frac{q(2\pi)^4}{V' \langle R^{-2} \rangle} \left[\frac{q\Psi'}{V' \langle R^{-2} \rangle} \right]_{\psi} \quad (15.67)$$

$$\Rightarrow \Delta^* \Psi = -\mu_0 R^2 \frac{dp}{d\Psi} - \frac{q(2\pi)^4}{V' \langle R^{-2} \rangle} \left\{ \left[\frac{q}{V' \langle R^{-2} \rangle} \right]_{\psi} \Psi' + \left[\frac{q}{V' \langle R^{-2} \rangle} \right] \Psi'' \right\} \quad (15.68)$$

Using Eq. (15.64) to eliminate Ψ'' in the above equation, the coefficients before $(-\mu_0 dp/d\Psi)$ is written as

$$\begin{aligned} B &= R^2 - \frac{q(2\pi)^4}{V' \langle R^{-2} \rangle} \left\{ \left[\frac{q}{V' \langle R^{-2} \rangle} \right] \frac{1}{V'D} V' \right\} \\ &= \frac{1}{D} \left\{ R^2 D - \left(\frac{q}{V'} \right)^2 \frac{(2\pi)^4}{\langle R^{-2} \rangle^2} \right\}. \end{aligned} \quad (15.69)$$

Substituting the expression of D into the above equation, we obtain

$$\begin{aligned} B &= \frac{1}{D} \left\{ R^2 \left\langle \frac{|\nabla \psi|^2}{R^2} \right\rangle + R^2 \frac{(2\pi)^4}{\langle R^{-2} \rangle} \left(\frac{q}{V'} \right)^2 - \left(\frac{q}{V'} \right)^2 \frac{(2\pi)^4}{\langle R^{-2} \rangle^2} \right\} \\ &= \frac{1}{D} \left\{ R^2 \left\langle \frac{|\nabla \psi|^2}{R^2} \right\rangle + \left(\frac{q}{V'} \right)^2 \frac{(2\pi)^4}{\langle R^{-2} \rangle} \left(R^2 - \frac{1}{\langle R^{-2} \rangle} \right) \right\}, \end{aligned} \quad (15.70)$$

which is equal to the expression (5.58) in Ref. [16]. The coefficient before Ψ' is written as

$$A = -\frac{q(2\pi)^4}{V'\langle R^{-2}\rangle} \left\{ \left[\frac{q}{V'\langle R^{-2}\rangle} \right]_{\psi} - \left[\frac{q}{V'\langle R^{-2}\rangle} \right] \frac{1}{V'D} \left[\left(V' \left\langle \frac{|\nabla\psi|^2}{R^2} \right\rangle \right)_{\psi} + q(2\pi)^4 \left[\frac{q}{V'\langle R^{-2}\rangle} \right]_{\psi} \right] \right\}$$

Define

$$\alpha = \left\langle \frac{|\nabla\psi|^2}{R^2} \right\rangle,$$

$$\beta = \frac{q}{V'\langle R^{-2}\rangle}.$$

$$A = -(2\pi)^4 \beta \left\{ \beta_{\psi} - \beta \frac{1}{V'D} [(V'\alpha)_{\psi} + q(2\pi)^4 \beta_{\psi}] \right\}$$

$$\implies \frac{D}{-(2\pi)^4 \beta} A = D\beta_{\psi} - \beta \frac{1}{V'} [(V'\alpha)_{\psi} + q(2\pi)^4 \beta_{\psi}] \quad (15.71)$$

$$\implies \frac{D}{-\beta(2\pi)^4} A = D\beta_{\psi} - \beta \frac{1}{V'} [V''\alpha + V'\alpha_{\psi} + q(2\pi)^4 \beta_{\psi}] \quad (15.72)$$

Using

$$D = \alpha + (2\pi)^4 \langle R^{-2} \rangle \beta^2 \quad (15.73)$$

Eq. () is written as

$$\implies \frac{D}{-\beta(2\pi)^4} A = \alpha\beta_{\psi} + (2\pi)^4 \langle R^{-2} \rangle \beta^2 \beta_{\psi} - \beta \frac{1}{V'} V''\alpha - \beta \frac{1}{V'} V'\alpha_{\psi} - \beta \frac{1}{V'} q(2\pi)^4 \beta_{\psi} \quad (15.74)$$

$$\implies \frac{D}{-\beta(2\pi)^4} A = \alpha\beta_{\psi} + (2\pi)^4 \langle R^{-2} \rangle \beta^2 \beta_{\psi} - \beta \frac{1}{V'} V''\alpha - \beta\alpha_{\psi} - \beta \frac{1}{V'} q(2\pi)^4 \beta_{\psi} \quad (15.75)$$

$$\frac{A}{-\beta(2\pi)^4} D = \alpha\beta_{\psi} + (2\pi)^4 \langle R^{-2} \rangle \beta^2 \beta_{\psi} - \beta \frac{1}{V'} V''\alpha - \beta\alpha_{\psi} - \beta^2 \langle R^{-2} \rangle (2\pi)^4 \beta_{\psi} \quad (15.76)$$

$$\implies \frac{A}{-\beta(2\pi)^4} D = \alpha\beta_{\psi} - \beta \frac{1}{V'} V''\alpha - \beta\alpha_{\psi} \quad (15.77)$$

$$\implies \frac{A}{-\beta(2\pi)^4} D = -\beta^2 \left(\frac{\alpha}{\beta} \right)_{\psi} - \beta \frac{1}{V'} V''\alpha \quad (15.78)$$

$$\Rightarrow \frac{A}{-\beta(2\pi)^4} D = -\beta^2 \left(\frac{\alpha}{\beta} \right)_\psi - \beta^2 \frac{1}{V'} V'' \frac{\alpha}{\beta} \quad (15.79)$$

$$\Rightarrow \frac{A}{-\beta(2\pi)^4} D = -\beta^2 \left[\left(\frac{\alpha}{\beta} \right)_\psi + \frac{1}{V'} V'' \frac{\alpha}{\beta} \right] \quad (15.80)$$

$$\Rightarrow \frac{A}{-\beta(2\pi)^4} D = -\beta^2 \frac{1}{V'} \left[\left(\frac{\alpha}{\beta} \right)_\psi V' + V'' \frac{\alpha}{\beta} \right] \quad (15.81)$$

$$\Rightarrow \frac{A}{-\beta(2\pi)^4} D = -\beta^2 \frac{1}{V'} \left(\frac{\alpha}{\beta} V' \right)_\psi \quad (15.82)$$

$$\Rightarrow A = \frac{(2\pi)^4}{D} \left(\frac{q}{V' \langle R^{-2} \rangle} \right)^3 \frac{1}{V'} \left[\left\langle \frac{|\nabla \psi|^2}{R^2} \right\rangle \frac{\langle R^{-2} \rangle}{q} V'^2 \right]_\psi \quad (15.83)$$

But the expression of A is slightly different from that given in Ref. [16] [Eq. (5.57)]. Using the above coefficients, the GS equation with the q -profile held fixed is written as

$$\left(\Delta^* - A \frac{\partial}{\partial \psi} \right) \Psi = -B \mu_0 \frac{dp}{d\Psi}. \quad (15.84)$$

16 Misc contents

16.1 Expression of current density

Let us derive the contravariant expression for the current density. Using

$$\mu_0 \mathbf{J} = \nabla \times \mathbf{B},$$

along with magnetic field expression (8.2) and the curl formula (6.43), we obtain

$$\mu_0 \mathbf{J} = - \left[\left(\Psi' \frac{\mathcal{J}}{R^2} |\nabla \psi|^2 \right)_\psi + \left(\Psi' \frac{\mathcal{J}}{R^2} \nabla \psi \cdot \nabla \theta \right)_\theta \right] \nabla \psi \times \nabla \theta - g' \nabla \phi \times \nabla \psi, \quad (16.1)$$

which is the contravariant form of the current density vector. Next, for later use, calculate the parallel current. By using Eqs. (8.2) and (16.1), the parallel current density is written as

$$\begin{aligned} \mu_0 \mathbf{J} \cdot \mathbf{B} &= - \left[\left(\Psi' \frac{\mathcal{J}}{R^2} |\nabla \psi|^2 \right)_\psi + \left(\Psi' \frac{\mathcal{J}}{R^2} \nabla \psi \cdot \nabla \theta \right)_\theta \right] g \nabla \psi \times \nabla \theta \cdot \nabla \phi - \\ &\quad g' \left(-\Psi' \frac{\mathcal{J}}{R^2} |\nabla \psi|^2 \right) \nabla \phi \times \nabla \psi \cdot \nabla \theta \\ &= - \left[\left(\Psi' \frac{\mathcal{J}}{R^2} |\nabla \psi|^2 \right)_\psi + \left(\Psi' \frac{\mathcal{J}}{R^2} \nabla \psi \cdot \nabla \theta \right)_\theta \right] g \mathcal{J}^{-1} + g' \mathcal{J}^{-1} \Psi' \frac{\mathcal{J}}{R^2} |\nabla \psi|^2 \end{aligned}$$

$$\begin{aligned}
&= -g^2 \mathcal{J}^{-1} \left\{ \frac{1}{g} \left(\Psi' \frac{\mathcal{J}}{R^2} |\nabla \psi|^2 \right)_{\psi} + \frac{1}{g} \left(\Psi' \frac{\mathcal{J}}{R^2} \nabla \psi \cdot \nabla \theta \right)_{\theta} - \frac{g'}{g^2} \Psi' \frac{\mathcal{J}}{R^2} |\nabla \psi|^2 \right\} \\
&= -g^2 \mathcal{J}^{-1} \left[\left(\frac{\Psi'}{g} \frac{\mathcal{J}}{R^2} |\nabla \psi|^2 \right)_{\psi} + \left(\frac{\Psi'}{g} \frac{\mathcal{J}}{R^2} \nabla \psi \cdot \nabla \theta \right)_{\theta} \right]. \tag{16.2}
\end{aligned}$$

16.2 Normalized internal inductance

The self-inductance of a current loop is defined as the ratio of the magnetic flux Φ traversing the loop and its current I :

$$L = \frac{\Phi}{I}, \tag{16.3}$$

where

$$\Phi = \int_S \mathbf{B} \cdot d\mathbf{S}. \tag{16.4}$$

It can be proved that L is independent of the current I in the loop, i.e., L is fully determined by the shape of the loop.

On the other hand, the energy contained in the magnetic field produced by the loop current is given by

$$W = \int_{V_0} \frac{B^2}{2\mu_0} dV, \tag{16.5}$$

where the volume includes all space where B is not negligible. It can be proved that (to be proved) W , L , and I are related to each other by:

$$W = \frac{1}{2} L I^2, \tag{16.6}$$

i.e.,

$$L = \frac{2W}{I^2}, \tag{16.7}$$

which can be considered an equivalent definition of the self-inductance.

The internal inductance L_i of tokamak plasma is defined in such a way that W only includes the magnetic energy within the plasma. Specifically, L_i is defined by

$$L_i \equiv \frac{2W_i}{I^2} = \frac{2}{I^2} \int_P \frac{B_{\theta}^2}{2\mu_0} dV \tag{16.8}$$

where the integration over the plasma volume P and only the poloidal field B_{θ} appears in the integration since plasma current produces only the poloidal magnetic field.

The normalized internal inductance l_i is defined as

$$l_i \equiv \frac{L_i}{\mu_0 R_0 / 2}, \tag{16.9}$$

where R_0 is the major radius of the device (equal to R of the magnetic axis). Using Eq. (16.8), expression (16.9) is written as

$$l_i = \frac{1}{\mu_0 R_0 / 2} \frac{2}{I^2} \int_P \frac{B_\theta^2}{2\mu_0} dV = \frac{2V}{I^2 \mu_0^2 R_0} \frac{\int_P B_\theta^2 dV}{V}, \quad (16.10)$$

which is the definition of l_i used in the ITER design.

Another way of defining the normalized internal inductance is

$$l_i = \frac{\langle B_\theta^2 \rangle_P}{\langle B_\theta^2 \rangle_S}, \quad (16.11)$$

where $\langle \dots \rangle_S$ is the surface average over the plasma boundary. For circular cross section with minor radius a and assuming B_θ is independent of the poloidal angle, then, Ampere's law gives $B_\theta(a) = \mu_0 I / (2\pi a)$. Then $\langle B_\theta^2 \rangle_S$ is approximated as

$$\langle B_\theta^2 \rangle_S \approx B_0^2(a) = \frac{\mu_0^2 I^2}{4\pi^2 a^2} \quad (16.12)$$

Using this and $V \approx \pi a^2 2\pi R_0$, Eq. (16.11) is written as

$$l_i = \frac{4\pi^2 a^2}{\mu_0^2 I^2} \langle B_\theta^2 \rangle_P = \frac{4\pi^2 a^2 R_0}{\mu_0^2 I^2 R_0} \langle B_\theta^2 \rangle_P = \frac{2V}{\mu_0^2 I^2 R_0} \langle B_\theta^2 \rangle_P, \quad (16.13)$$

which agrees with the definition in Eq. (16.10).

The normalized internal inductance reflects the peakness of the current density profile in the toroidal plasma: a small value of l_i corresponds to a broad current profile.

16.3 Cylindrical tokamak

For a large aspect-ratio, circular cross section tokamak, the R on a magnetic surface is nearly constant, $R \approx R_0$. The poloidal angle dependence of the magnetic field can be neglected, i.e., $B_\phi(r, \theta) \approx B_\phi(r)$, and $B_p(r, \theta) \approx B_p(r)$, where r is the minor radius of the relevant magnetic surface. Using these, the safety factor in Eq. (1.37) is approximated to

$$q = \frac{1}{2\pi} \oint \frac{1}{R} \frac{B_\phi}{B_p} d\ell_p \approx \frac{1}{2\pi} \frac{B_\phi(r)}{B_p(r)} \frac{1}{R_0} \oint d\ell_p = \frac{1}{2\pi} \frac{B_\phi(r)}{B_p(r)} \frac{1}{R_0} 2\pi r = \frac{r}{R_0} \frac{B_\phi(r)}{B_p(r)}, \quad (16.14)$$

16.4 Coils system of EAST tokamak

16.4.1 Poloidal field (PF) coils

EAST has 12 independently powered superconducting poloidal field (PF)

coils, as is shown in Fig. 16.1.

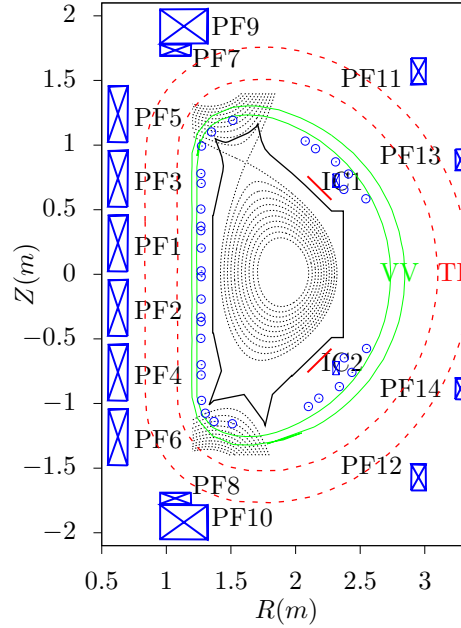


Figure 16.1. Coils system of EAST tokamak. Coils PF1 to PF6 are Central Solenoid (CS) coils with 120 turns/coil. Both PF11 and PF12 have 64 turns, Both PF13 and PF14 have 32 turns. PF7 and PF9 are adjacent to each other and connected in series and thus are considered as one independent coil, with total turns being 248. Similarly PF8 and PF10 are connected in series with total 248 turns. All the PF coils can be considered as shaping coils since they all have effects in shaping the plasmas. In practice, they are further classified according to their main roles. PF1 to PF6 form a solenoid in the center of the torus and thus called Central Solenoid (CS) coils. Their main role is to induce electric field to drive current in the plasma and heat the plasma. As a result, they are often called “Ohm heating coils”. PF13 and PF14 are mainly used to control (slow) vertical plasma displacement and thus are often called “vertical field coils” or “position control coils”. PF11 and PF12 are used to triangularize the plasma and thus is called “shaping coils”. PF7+PF9 and PF8+PF10 are often called (by EAST operators) as “big coils” or “divertor coils” since they have the largest number of turns and current and are used to elongate/shape the plasma to divertor configurations. IC1 and IC2 are copper coils (2turn/coil), which are connected in anti-series and thus have opposite currents. They are close to the plasma (just behind the first wall and within the vacuum vessel) and are used to control fast plasma displacements, specifically VDEs (vertical displacement events). IC1 and IC2 are often called “fast control coils”. Maximum current per turn in PF coils is 14.5kA. Locations of PF coils are from Refs. [24][9]. The inner green D-shaped structure corresponds to the vacuum vessel wall and the outer red dashed D-shape corresponds to the TF-coils. Note that PF1-14 coils are all located outside of the TF-coils. The small blue circles are “flux loops” measuring the poloidal magnetic flux. Red coils are RMP copper coils.

16.4.2 Toroidal magnetic Field (TF) coils and toroidal magnetic field of EAST tokamak

Using Ampere's circuital law

$$\oint \mathbf{B} \cdot d\mathbf{l} = \mu_0 I, \quad (16.15)$$

along the toroidal direction and assuming perfect toroidal symmetry, we obtain

$$2\pi R B_\phi = \mu_0 I, \quad (16.16)$$

which gives

$$B_\phi = \frac{\mu_0 I}{2\pi R} = 2 \times 10^{-7} \frac{I}{R}. \quad (16.17)$$

Neglecting the poloidal current contributed by the plasma, the poloidal current is determined solely by the current in the TF coils. The EAST tokamak has 16 groups of TF coils with 132 turns/coil (I got to know the number of turns from ZhaoLiang Wang: $\varphi \times R = 12 \times 11 = 132$). Denote the current in a single turn by I_s , then Eq. (16.17) is written

$$B_\phi = \frac{\mu_0 \times 16 \times 132 \times I_s}{2\pi R} = 4.224 \times 10^{-4} \frac{I_s}{R} \quad (16.18)$$

Using this formula, the strength of the toroidal magnetic field at $R = 1.8m$ for $I_s = 10^4 A$ is calculated to be $B_\phi = 2.34T$. This was one of the two scenarios often used in EAST experiments (another scenario is $I_s = 8 \times 10^3 A$). (The limit of the current in a single turn of the TF coils is 14.5kA (from B. J. Xiao's paper [29]).

Note that the exact equilibrium toroidal magnetic field B_ϕ is given by $B_\phi = g(\Psi) / R$. Compare this with Eq. (16.17), we know that the approximation made to obtain Eq. (16.18) is equivalent to $g(\Psi) \approx \mu_0 I_{TF} / 2\pi$, i.e. assuming g is a constant. The poloidal plasma current density j_{pol} is related to g by $j_{pol} = g'(\Psi) B_p / \mu_0$. The constant g corresponds to zero plasma poloidal current, which is consistent to the assumption used to obtain Eq. (16.18).

Let us estimate the safety factor value near the plasma edge using the total plasma current and the current in a single turn of TF coils I_s . For divertor magnetic configuration, the plasma edge is at the separatrix, where $q \rightarrow \infty$. To get a characteristic safety factor value that is finite, one often chooses the edge to be the magnetic surface that encloses 95% of the poloidal magnetic flux. Denote this surface by S_{95} and the value of q at this surface by q_{95} , which is given by

$$q_{95} \approx \frac{B_{\phi, \text{axis}}}{\bar{B}_p} \frac{a}{R_{\text{axis}}}, \quad (16.19)$$

where a is the minor radius of the surface S_{95} , and R_{axis} the major radius of the magnetic axis, $B_{\phi, \text{axis}}$ is the the magnitude of toroidal magnetic field at the magnetic axis, and \bar{B}_p is the average poloidal magnetic field on the surface,

$\bar{B}_p \approx \mu_0 I_p / (2\pi a)$. Using Eq. (16.18), Eq. (16.19) is written as

$$q_{95} \approx \frac{4.224 \times 10^{-4} I_s}{\mu_0 I_p} \frac{2\pi a^2}{R_{\text{axis}}^2}, \quad (16.20)$$

For EAST, typically $R_{\text{axis}} = 1.85m$ and $a = 0.45m$. Using this, we obtain

$$q_{95} \approx 125 \frac{I_s}{I_p}. \quad (16.21)$$

16.4.3 RMP coils of EAST

The so-called resonant magnetic perturbation (RMP) coils are 3D coils that are used to suppress or mitigate edge localized modes. The shape and location of RMP coils of EAST tokamak are plotted in Fig. 16.2.

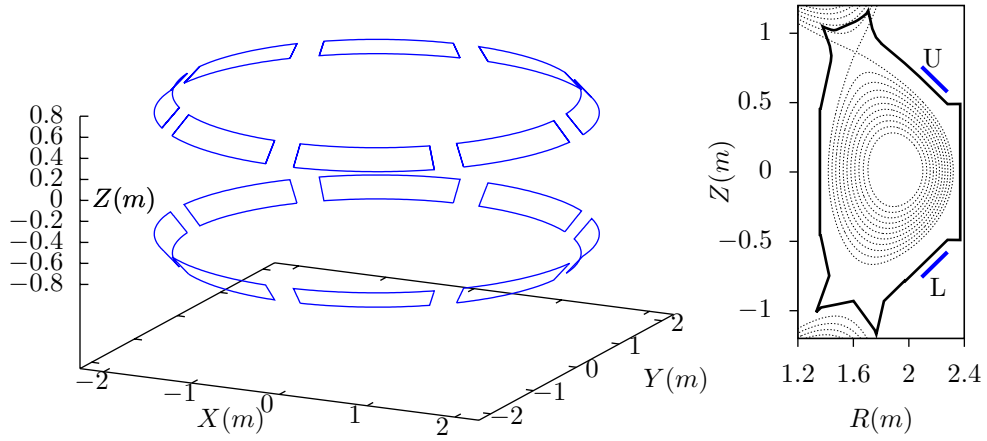


Figure 16.2. Location of the RMP coils on EAST tokamak in 3D view (left) and poloidal view (right).

16.5 Comparison of major tokamaks in the world

The size of EAST is similar to that of DIII-D tokamak. The main parameters are summarized in Table. 16.1. A significant difference between EAST and DIII-D is that DIII-D has a larger minor radius, which makes DIII-D able to operate with a larger toroidal current than that EAST can do for the same current density. Another significant difference between EAST and DIII-D is that the coils of EAST are super-conducting while the coils of DIII-D are not. The super-conducting coils enable EAST to operate at longer pulse.

	EAST	DIII-D[19]	KSTAR[17]	SPARC	WEST	JET
Major radius R_0	1.85m	1.67m	1.8m	1.85m	2.5m	2.96m
minor radius a	0.45m	0.67m	0.5m	0.57m	0.5m	0.9m
elongation						
Plasma volume				20m ³		
No. of TF coils, turns, current	16, 130, 14.5kA	24, 6, 126kA	16, 56, 35.2kA			
B_t at R_0	3.26T	2.17T	3.5T	12.2T	3.7T	3.45T
CS coil module \times turn \times current	6 \times 120 \times 14.5kA					
No. of independent PF coils	6+6	?+18				
Available solenoid magnetic flux	12Vs	10.5Vs	17Vs			
Maximum plasma current	1.0MA	3.0MA	2MA	8.7MA	1MA	5MA
Pulse length	400s	10s	300s			
superconducting?	Yes	No	Yes		Yes	No

Table 16.1. Comparison of main parameters of EAST, DIII-D, JET, and ITER tokamaks. The B_t of EAST is calculated at $R = 1.85m$ by using Eq. (16.18) with $I_s = 14.5kA$ (the maximal current allowed). The currents listed in the table are maximum plasma current whereas typical plasma currents for EAST are 0.5MA and for DIII-D are 2MA or less.

	ITER[1]	CFETR(old version)	CFETR (new)	BEST
Major radius R_0	6.2m	5.7m	7.2m	3.6m
minor radius a	2.0m	1.6m	2.2m	
elongation				
No. of TF coils, turns, current	18, 134, 68kA	16, 132, 67.5kA	16, ?, ?	16, 152, 45.6kA
B_t at R_0	5.29T	5.00T	6.5T	
CS coil module \times turn \times current				
No. of independent PF coils				
Available solenoid magnetic flux				
Maximum plasma current	15MA	10MA	14MA	
Pulse length	400s			
superconducting?	Yes	Yes	Yes	Yes

Table 16.2. Continued from Table 16.1.

	ASDEX-U	HL-2M	NSTX	MAST
Major radius R_0	1.65m	1.78m	0.85m	0.9m
minor radius a	0.7m	0.65m	0.68m	0.6m
elongation				
No. of TF coils, turns, current	16,?,?	20,7,190kA		
B_t at R_0		2.99T	0.3T	0.55T
CS coil module \times turn \times current		1,?,?		
No. of independent PF coils		1+16		
Available solenoid magnetic flux		14Vs		
typical plasma current	1.6MA	2.5MA		
Pulse length		5s		
superconducting?	No	No	No	

Table 16.3. Continued from Table 16.2.

DIII-D has 24 groups of TF coils with 6turns/coil, i.e., total turns are $24 \times 6 = 144$, with a maximum current of $I_s = 126\text{kA}$ in a single turn[19]. Using formula (16.17), the toroidal field at $R = 1.67\text{m}$ can be calculated, giving $2.17T$.

DIII-D is special in that its poloidal field (PF) coils are located inside of the TF-coils, which makes the PF-coils more close to the plasma and thus more efficient in shaping the plasma. However, this nested structure is difficult to assemble. In superconducting tokamaks (e.g., EAST, KSTAR, ITER), PF coils are all placed outside of the TF-coils.

I noticed that HL-2M also has the PF coils located within the TF-coils, similar to DIII-D. This remind me that this layout may apply to all non-superconducting tokamaks (to be confirmed, No, non-superconducting tokamak ASDEX-U has PF coils outside of TF coils).

KSTAR has 16 TF coils and 14 PF coils. Both of the TF and PF coil system use internally cooled superconductors. The nominal current in TF coils is 35.2kA / turn with 56turns / coil and all coils connected in series[23]. Using these information and formula (16.17), the toroidal field at $R = 1.8\text{m}$ can be calculated, giving $3.5T$. The PF coil system consists of 8 Central Solenoid (CS) coils and 6 outer PF coils and can provide 17 V-sec.

ITER has 18 TF coils with number of turns in one coil being 134 and current per turn 68kA [5]. Using these information and formula (16.17), the toroidal field at $R = 6.2\text{m}$ can be calculated, giving $5.29T$.

16.6 Miller's formula for shaped flux surfaces

According to Refs. [8, 20], Miller's formula for a series of shaped flux surfaces is given by

$$R = R_0(r) + r \cos\{\theta + \arcsin[\delta(r)\sin\theta]\}, \quad (16.22)$$

$$Z = \kappa(r)r \sin\theta, \quad (16.23)$$

where $\kappa(r)$ and $\delta(r)$ are elongation and triangularity profile, $R_0(r)$ is the Shafranov shift profile, which is given by

$$R_0(r) = R_0(a) - \frac{aR'_0}{2} \left[1 - \left(\frac{r}{a} \right)^2 \right], \quad (16.24)$$

where R'_0 is a constant, $R_0(a)$ is the major radius of the center of the boundary flux surface. The triangularity profile is

$$\delta(r) = \delta_0 \left(\frac{r}{a} \right)^2, \quad (16.25)$$

and the elongation profile is

$$\kappa(r) = \kappa_0 - 0.3 + 0.3 \left(\frac{r}{a} \right)^4. \quad (16.26)$$

The nominal ITER parameters are $\kappa_0 = 1.8$, $\delta_0 = 0.5$ and $R'_0 = -0.16$. I wrote a code to plot the shapes of the flux surface (/home/yj/project/miller_flux_surface). An example of the results is given in Fig. 16.3.

[⟨image|/home/yj/project/miller_flux_surface/plt.eps||||⟩](#)

Figure 16.3. Flux-surfaces given by Eqs. (16.22) and (16.23) with r/a varying from 0.1 to 1.0 (corresponding boundary surface). Other parameters are $R_0(a)/a = 3$, $\kappa_0 = 1.8$, $\delta_0 = 0.5$, $R'_0 = -0.16$.

16.7 Double transport barriers pressure profile

An analytic expression for the pressure profile of double (inner and external) transport barriers is given by

$$P(\psi) = a_i \left(1 + \tanh \left(\frac{-(\psi - \psi_i)}{w_i} \right) \right) + a_e \left(1 + \tanh \left(\frac{-(\psi - \psi_e)}{w_e} \right) \right) - c \quad (16.27)$$

where ψ is the normalized poloidal flux, w_i and w_e are the width of the inner and external barriers, ψ_i and ψ_e are the locations of the barriers, a_i and a_e is the height of the barriers, c is a constant to ensure $P(\psi) = 0$ at $\psi = 1$.

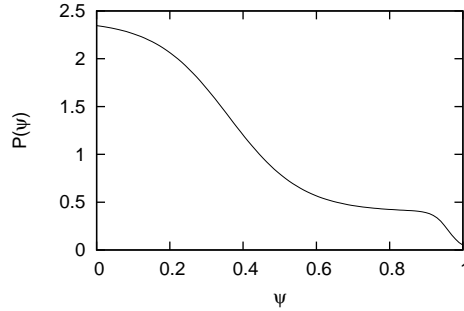


Figure 16.4. Pressure profile of double (inner and external) transport barriers given by Eq. (16.27) with $a_i = 1$, $b_i = 0.2$, $\psi_i = 0.36$, $\psi_e = 0.96$, $w_i = 0.2$, $w_e = 0.04$, $c = 0$.

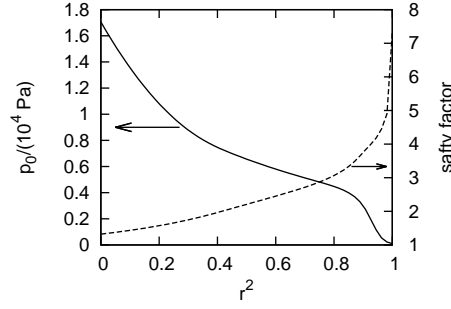


Figure 16.5. Equilibrium pressure profile for EAST discharge #38300 at 3.9s (reconstructed by EFIT code, gfile name: g038300.03900), which shows a boundary transport barrier.

16.8 Ballooning transformation

To construct a periodic function about θ , we introduce a function $\bar{z}(\theta)$ which is defined over $-\infty < \theta < \infty$ and vanishes sufficiently fast as $|\theta| \rightarrow \infty$ so that the following infinite summation converge:

$$\sum_{l=-\infty}^{\infty} \bar{z}(\theta + 2\pi l). \quad (16.28)$$

If we use the above sum to define a function

$$z(\theta) = \sum_{l=-\infty}^{\infty} \bar{z}(\theta + 2\pi l), \quad (16.29)$$

then it is obvious that

$$z(\theta + 2\pi) = z(\theta), \quad (16.30)$$

i.e., $z(\theta)$ is a periodic function about θ with period of 2π .

If we use the right-hand-side of Eq. (16.29) to represent $z(\theta)$, then we do not need to worry about the periodic property of $z(\theta)$ (the periodic property is guaranteed by the representation)

A

[In passing, we note that $\Psi \equiv A_\phi R$ is the covariant toroidal component of \mathbf{A} in cylindrical coordinates (R, ϕ, Z) . The proof is as follows. Note that the covariant form of \mathbf{A} should be expressed in terms of the contravariant basis vector $(\nabla R, \nabla \phi, \text{ and } \nabla Z)$, i.e.,

$$\mathbf{A} = A_1 \nabla R + A_2 \nabla \phi + A_3 \nabla Z. \quad (\text{A.1})$$

where A_2 is the covariant toroidal component of \mathbf{A} . To obtain A_2 , we take scalar product of Eq. (A.1) with $\partial \mathbf{r} / \partial \phi$ and use the orthogonality relation (6.7), which gives

$$\mathbf{A} \cdot \frac{\partial \mathbf{r}}{\partial \phi} = A_2. \quad (\text{A.2})$$

In cylindrical coordinates (R, ϕ, Z) , the location vector is written as

$$\mathbf{r}(R, Z, \phi) = R\hat{\mathbf{R}}(\phi) + Z\hat{\mathbf{Z}} + 0\hat{\phi} \quad (\text{A.3})$$

where $\hat{\mathbf{R}}$, $\hat{\mathbf{Z}}$, and $\hat{\phi}$ are unit vectors along $\partial \mathbf{r} / \partial R$, $\partial \mathbf{r} / \partial Z$, and $\partial \mathbf{r} / \partial \phi$, respectively, i.e.

$$\hat{\mathbf{R}} = \frac{\partial \mathbf{r}}{\partial R} \left| \frac{\partial \mathbf{r}}{\partial R} \right|^{-1}, \hat{\mathbf{Z}} = \frac{\partial \mathbf{r}}{\partial Z} \left| \frac{\partial \mathbf{r}}{\partial Z} \right|^{-1}, \hat{\phi} = \frac{\partial \mathbf{r}}{\partial \phi} \left| \frac{\partial \mathbf{r}}{\partial \phi} \right|^{-1} \quad (\text{A.4})$$

Using this, we obtain

$$\frac{\partial \mathbf{r}}{\partial \phi} = R\hat{\phi}, \quad (\text{A.5})$$

Use Eq. (A.5) in Eq. (A.2) giving

$$A_2 = A_\phi R, \quad (\text{A.6})$$

with A_ϕ defined by $A_\phi = \mathbf{A} \cdot \hat{\phi}$. Equation (A.6) indicates that $\Psi = A_\phi R$ is the covariant toroidal component of the vector potential.]

A.1 Solovév equilibrium

For most choices of $P(\Psi)$ and $g(\Psi)$, the GS equation (3.12) has to be solved numerically. For the particular choice of P and g profiles given by

$$\frac{dP}{d\Psi} = -\frac{c_1}{\mu_0}, \quad (\text{A.7})$$

$$g \frac{dg}{d\Psi} = -c_2 R_0^2, \quad (\text{A.8})$$

analytical solution to the GS equation can be found, which is given by[16]

$$\Psi = \frac{1}{2}(c_2 R_0^2 + c_0 R^2)Z^2 + \frac{1}{8}(c_1 - c_0)(R^2 - R_0^2)^2, \quad (\text{A.9})$$

where c_0 , c_1 , c_2 , and R_0 are arbitrary constants. [Proof: By direct substitution, we can verify Ψ of this form is indeed a solution to the GS equation (3.12).] A useful choice for tokamak application is to set $c_0 = B_0 / (R_0^2 \kappa_0 q_0)$, $c_1 = B_0(\kappa_0^2 + 1) / (R_0^2 \kappa_0 q_0)$, and $c_2 = 0$. Then Eq. (A.9) is written

$$\Psi = \frac{B_0}{2R_0^2 \kappa_0 q_0} \left[R^2 Z^2 + \frac{\kappa_0^2}{4} (R^2 - R_0^2)^2 \right], \quad (\text{A.10})$$

which can be solved analytically to give the explicit form of the contour of Ψ on (R, Z) plane:

$$Z = \pm \frac{1}{R} \sqrt{\frac{2R_0^2 \kappa_0 q_0}{B_0} \Psi - \frac{\kappa_0^2}{4} (R^2 - R_0^2)^2}, \quad (\text{A.11})$$

which indicates the magnetic surfaces are up-down symmetrical. Using Eq. (A.7), i.e.,

$$\frac{dP}{d\Psi} = -\frac{c_1}{\mu_0} = -\frac{B_0(\kappa_0^2 + 1)}{\mu_0 R_0^2 \kappa_0 q_0}, \quad (\text{A.12})$$

the pressure is written

$$P = P_0 - \frac{B_0(\kappa_0^2 + 1)}{\mu_0 R_0^2 \kappa_0 q_0} \Psi, \quad (\text{A.13})$$

where P_0 is a constant of integration. Note Eq. (A.10) indicates that that $\Psi = 0$ at the magnetic axis ($R = R_0, Z = 0$). Therefore, Eq. (A.13) indicates that P_0 is the pressure at the magnetic axis. The toroidal field function g is a constant in this case, which implies there is no poloidal current in this equilibrium. (For the Solovév equilibrium (A.10), I found numerically that the value of the safety factor at the magnetic axis ($R = R_0, Z = 0$) is equal to $q_0 g / (R_0 B_0)$. This result should be able to be proved analytically. I will do this later. In calculating the safety factor, we also need the expression of $|\nabla \Psi|$, which is given analytically by

$$\begin{aligned} |\nabla \Psi| &= \sqrt{\left(\frac{\partial \Psi}{\partial R}\right)^2 + \left(\frac{\partial \Psi}{\partial Z}\right)^2} \\ &= \frac{B_0}{2R_0^2 \kappa_0 q_0} \sqrt{[2RZ^2 + \kappa_0^2(R^2 - R_0^2)R]^2 + (2R^2 Z)^2}. \end{aligned} \quad (\text{A.14})$$

)

Define $\Psi_0 = B_0 R_0^2$, and $\bar{\Psi} = \Psi / \Psi_0$, then Eq. (A.10) is written as

$$\bar{\Psi} = \frac{1}{2\kappa_0 q_0} \left[\bar{R}^2 \bar{Z}^2 + \frac{\kappa_0^2}{4} (\bar{R}^2 - 1)^2 \right], \quad (\text{A.15})$$

where $\bar{R} = R / R_0$, $\bar{Z} = Z / R_0$. From Eq. (A.15), we obtain

$$\bar{Z} = \pm \frac{1}{\bar{R}} \sqrt{2\kappa_0 q_0 \bar{\Psi} - \frac{\kappa_0^2}{4} (\bar{R}^2 - 1)^2}. \quad (\text{A.16})$$

Given the value of κ_0, q_0 , for each value of $\bar{\Psi}$, we can plot a magnetic surface on (\bar{R}, \bar{Z}) plane. An example of the nested magnetic surfaces is shown in Fig. A.1.

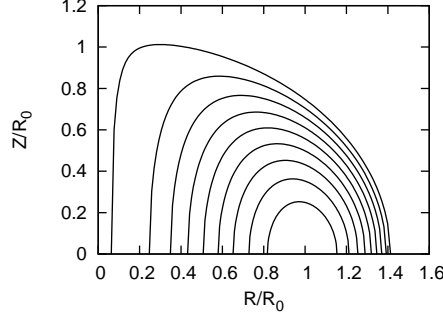


Figure A.1. Flux surfaces of Solovév equilibrium for $\kappa_0 = 1.5$ and $q_0 = 1.5$, with $\bar{\Psi}$ varying from zero (center) to 0.123 (edge). The value of $\bar{\Psi}$ on the edge is determined by the requirement that the minimum of \bar{R} is equal to zero. (To prevent “divided by zero” that appears in Eq. (A.16) when $R=0$, the value of $\bar{\Psi}$ on the edge is shifted to $0.123 - \varepsilon$ when plotting the above figure, where ε is a small number, $\varepsilon = 10^{-3}$ in this case.)

The minor radius of a magnetic surface of the Solovév equilibrium can be calculated by using Eq. (A.11), which gives

$$R_{\text{in}} = \sqrt{R_0^2 - \sqrt{A\Psi}}, \quad (\text{A.17})$$

$$R_{\text{out}} = \sqrt{R_0^2 + \sqrt{A\Psi}}, \quad (\text{A.18})$$

and thus

$$a = \frac{R_{\text{out}} - R_{\text{in}}}{2} = \frac{\sqrt{R_0^2 + \sqrt{A\Psi}} - \sqrt{R_0^2 - \sqrt{A\Psi}}}{2}. \quad (\text{A.19})$$

where $A = 8R_0^2 q_0 / (B_0 \kappa_0)$. In my code of constructing Solovév magnetic surface, the value of a is specified by users, and then Eq. (A.19) is solved numerically to obtain the value of Ψ of the flux surface. Note that the case $\Psi = 0$ corresponds to $R_{\text{in}} = R_{\text{out}} = R_0$, i.e., the magnetic axis, while the case $\Psi = R_0^2 B_0 \kappa_0 / (8q_0)$ corresponds to $R_{\text{in}} = 0$. Therefore, the reasonable value of Ψ of a magnetic surface should be in the range $0 \leq \Psi < R_0^2 B_0 \kappa_0 / (8q_0)$. This range is used as the interval bracketing a root in the bisection root finder.

Using Eq. (A.19), the inverse aspect ratio of a magnetic surface labeled by Ψ can be approximated as[16]

$$\varepsilon \approx \sqrt{\frac{2q_0\Psi}{\kappa_0 R_0^2 B_0}}. \quad (\text{A.20})$$

Therefore, the value of Ψ of a magnetic surface with the inverse aspect ratio ε is approximately given by

$$\Psi = \frac{\varepsilon^2 \kappa_0 R_0^2 B_0}{2q_0}. \quad (\text{A.21})$$

A.2 Plasma rotation

In deriving the Grad-Shafranov equation, we have assumed that there is no plasma flow. Next, let us examine whether this assumption is justified for plasmas in EAST tokamak.

The complete momentum equation is given by

$$\rho_m \left[\frac{\partial \mathbf{U}}{\partial t} + \mathbf{U} \cdot \nabla \mathbf{U} \right] = \rho_q \mathbf{E} + \mathbf{J} \times \mathbf{B} - \nabla \cdot \mathbb{P}, \quad (\text{A.22})$$

where $\rho_q \mathbf{E}$ term can be usually neglected due to either $\rho_q \approx 0$ or $\mathbf{E} \approx 0$, \mathbb{P} is a pressure tensor, which is different from the scalar pressure considered in this note. The equilibrium with pressure tensor can be important for neutral beam heating plasma, where pressure contributed by NBI fast ions can be a tensor. With plasma flow and scalar pressure and neglecting electric force term, the steady state momentum equation is written

$$\rho_m \mathbf{U} \cdot \nabla \mathbf{U} = \mathbf{J} \times \mathbf{B} - \nabla P, \quad (\text{A.23})$$

where the term on the left-hand side is the contribution of plasma flow to the force balance. Let us estimate the magnitude of this term. Macroscopic flows in tokamak are usually along the toroidal direction (the poloidal flow is usually heavily damped). The toroidal flow usually has the same toroidal angular frequency on a magnetic surface, i.e., the flow can be written as

$$\mathbf{U} = R\omega_T(\psi)\mathbf{e}_\phi, \quad (\text{A.24})$$

where $\omega_T = \omega_T(\psi)$ is the toroidal angular frequency of the flow, which can have radial variation. Using this expression, the left-hand side of Eq. (A.23) is written as

$$\begin{aligned} \rho_m \mathbf{U} \cdot \nabla \mathbf{U} &= \rho_m R\omega_T \mathbf{e}_\phi \cdot \nabla (R\omega_T \mathbf{e}_\phi) \\ &= \rho_m R\omega_T \frac{1}{R} \frac{\partial}{\partial \phi} (R\omega_T \mathbf{e}_\phi) \\ &= -\rho_m R\omega_T^2 \mathbf{e}_R, \end{aligned} \quad (\text{A.25})$$

which is just the centripetal force. For typical EAST plasma, the rotation frequency $f_T = \omega_T / 2\pi$ is less than 10kHz. For $f_T = 10\text{kHz}$, $R = 1.8\text{m}$ and $\rho_m = 1.3 \times 10^{-7} \text{kg} / \text{m}^3$ [Mass density of EAST#38300@3.9s, $\rho_m = n_D m_D = 4 \times 10^{19} \text{m}^{-3} \times 2 \times 1.6726 \times 10^{-27} \text{kg} = 1.3 \times 10^{-7} \text{kg} / \text{m}^3$], the above fore is about 9.23×10^2 newton. On the other hand, the pressure gradient force in Eq. (A.23) is about 4×10^4 newton, which is one order larger than the force contributed by the rotation. [Pressure gradient force is estimated by using $\nabla P \approx P_0 / a$, where P_0 is the thermal pressure at the magnetic axis and a is the minor radius of plasma. For typical EAST plasmas (EAST#38300@3.9s), $P_0 \approx 2 \times 10^4 \text{Pa}$, $a = 0.45\text{m}$. Then $\nabla P \approx P_0 / a = 4 \times 10^4 \text{N} / \text{m}^3$.] This indicates the rotation has little influence on the force balance. In other words, the EAST tokamak equilibrium can be well described by the static equilibrium without rotation.

A.3 Poloidal plasma current

As is discussed in Sec. 3.2, to satisfy the force balance, $g \equiv RB_\phi$ must be a magnetic surface function, i.e., $g = g(\Psi)$. Using this, expression (2.2) and (2.3) are written

$$\mu_0 J_R = -\frac{1}{R} \frac{dg}{d\Psi} \frac{\partial \Psi}{\partial Z} = \frac{dg}{d\Psi} B_R, \quad (\text{A.26})$$

and

$$\mu_0 J_Z = \frac{1}{R} \frac{dg}{d\Psi} \frac{\partial \Psi}{\partial R} = \frac{dg}{d\Psi} B_Z, \quad (\text{A.27})$$

respectively. The above two equations imply that

$$\frac{J_R}{J_Z} = \frac{B_R}{B_Z}, \quad (\text{A.28})$$

which implies that the projections of \mathbf{B} lines and \mathbf{J} lines in the poloidal plane are identical to each other. This indicates that the \mathbf{J} surfaces coincide with the magnetic surfaces.

The poloidal plasma current density is usually small (compared with the toroidal plasma current density) but is important for some cases of interest and thus could not be safely neglected. Many model equilibria (e.g., Solovév equilibria, DIII-D cyclone base cases) frequently used in simulations assume that g is a spatial constant, i.e., neglecting the poloidal plasmas current. The conclusions drawn from these simulations could be misleading.

Using this and $\nabla \cdot \mathbf{J} = 0$, and following the same steps in Sec. 1.7, we obtain

$$I_{\text{pol}} = \frac{1}{\mu_0} 2\pi [g(\Psi_2) - g(\Psi_1)], \quad (\text{A.29})$$

where I_{pol} is the poloidal current enclosed by the two magnetic surfaces, the positive direction of I_{pol} is chosen to be in the clockwise direction when observers look along $\hat{\phi}$. Equation (A.29) indicates that the difference of g between two magnetic surface is proportional to the poloidal current. For this reason, g is usually call the “poloidal current function”.

In the above, we see that the relation of g with the poloidal electric current is similar to that of Ψ with the poloidal magnetic flux. This similarity is due to the following differential relations:

$$\begin{cases} \mathbf{B} = \nabla \times \mathbf{A} \\ \Psi = RA_\phi \end{cases} \quad \begin{cases} \mu_0 \mathbf{J} = \nabla \times \mathbf{B} \\ g = RB_\phi \end{cases}$$

The poloidal plasma current density \mathbf{J}_p can be further written as

$$\begin{aligned} \mathbf{J}_p &\equiv J_R \hat{\mathbf{R}} + J_Z \hat{\mathbf{Z}} \\ &= \frac{1}{\mu_0} \left(-\frac{1}{R} \frac{\partial g}{\partial Z} \hat{\mathbf{R}} + \frac{1}{R} \frac{\partial g}{\partial R} \hat{\mathbf{Z}} \right) \\ &= \frac{1}{\mu_0} \nabla g \times \nabla \phi \end{aligned} \quad (\text{A.30})$$

Using $g = g(\Psi)$, Eq. (A.30) can also be written as

$$\begin{aligned}\mathbf{J}_p &= \frac{1}{\mu_0} \frac{dg}{d\Psi} \nabla\Psi \times \nabla\phi \\ &= \frac{1}{\mu_0} \frac{dg}{d\Psi} \mathbf{B}_p.\end{aligned}\tag{A.31}$$

A.4 Efficiency of tokamak magnetic field in confining plasma: Plasma beta

To characterize the efficiency of the magnetic field of tokamaks in confining plasmas, define the plasma β , which is the ratio of the thermal pressure to the magnetic pressure, i.e.,

$$\beta = \frac{p}{B^2/2\mu_0}.\tag{A.32}$$

Since the pressure in tokamak plasmas is not uniform, the volume averaged pressure is usually used to define the beta. In tokamak plasmas, the toroidal beta β_t and the poloidal beta β_p are defined, respectively, by

$$\beta_t = \frac{\langle p \rangle}{B_{t0}^2/2\mu_0},\tag{A.33}$$

$$\beta_p = \frac{\langle p \rangle}{\langle B_\theta^2 \rangle_s/2\mu_0},\tag{A.34}$$

where $\langle \dots \rangle$ is the volume averaging, $\langle \dots \rangle_s$ is the surface averaging over the plasma boundary, B_{t0} is the vacuum toroidal magnetic field at the magnetic axis (or geometrical center of the plasma). In tokamaks, the toroidal magnetic field is dominant and thus the toroidal beta β_t (not β_p) is the usual way to characterize the efficiency of the magnetic field in confining plasmas. Why do we need β_p ? The short answer is that β_p is proportional to an important current, the so-called bootstrap current, which is important for tokamak steady state operation. Alternatively, β_p can be understood as characterizing the efficiency of the plasma current in confining the plasma. This can be seen by using Ampere's law to approximately write the average poloidal magnetic field $B_{pa} \approx \mu_0 I_p / (2\pi a)$. Then β_p is written

$$\beta_p \approx \frac{\langle p \rangle}{\mu_0 I_p^2 / (8\pi^2 a^2)},\tag{A.35}$$

which is the ratio of the pressure to the plasma current, and thus characterizes the efficiency of the plasma current in confining the plasma.

Tokamak experiments have found that it is easier to achieve higher β_t in low B_t plasmas than in higher B_t plasmas, which indicates that the efficiency of the magnetic field in confining plasma is a decreasing function of the magnitude of the magnetic field.

A.5 Beta limit

Beta limit means there is a limit for the value of beta beyond which the plasma will encounter a serious disruption. Early calculation of the beta limit on JET shows that the maximal β_t obtained is proportional to $I_p/(10^6 a B_{t0})$, where all quantities are in SI units. This scaling relation $\beta_{t\max} = C_T I_p/(10^6 a B_{t0})$ is often called Troyon scaling, where the coefficient C_T was determined numerically by Troyon to 0.028. Often C_T is expressed in percent, in which case $C_T = 2.8$. This motivates us to define

$$\beta_N = 10^8 \frac{a B_{t0}}{I_p} \beta_t. \quad (\text{A.36})$$

which is called “normalized beta”. The normalized beta β_N is an operational parameter indicating how close the plasma is to reach destabilizing major MHD activities. Its typical value is of order unit.

As mentioned above, β_N calculated by Troyon is 2.8. Empirical evaluation from the data of different tokamaks raises this value slightly to 3.5, although significantly higher values, e.g., $\beta_N = 7.2$, have been achieved in the low aspect ratio tokamak NSTX[22].

The value of β_N indicates how close one is to the onset of deleterious instability. The ability to increase the value of β_N can be considered to be the ability of controlling the major MHD instabilities, and thus can be used to characterize how well a tokamak device is operated. One goal of EAST tokamak during 2015-2016 is to sustain a plasma with $\beta_N \geq 2$ for at least 10 seconds.

(check** The tearing mode, specifically the neoclassical tearing mode (NTM) is expected to set the beta limit in a reactor.)

(**check: Tokamak experiments have found that it is easier to achieve high β_N in large I_p plasmas than in small I_p plasmas. However, experiments found it is easier to achieve high β_p in small I_p plasmas than in large I_p plasmas. Examining the expression of β_N and β_p given by Eqs. (A.35) and Eq. (A.36), respectively, we recognize that pressure limit should have a scaling of $\langle p \rangle \propto I_p^\alpha$ with $1 < \alpha < 2$.)

A.6 Why bigger tokamaks with larger plasma current are better at fusion?

As is mentioned in Sec. A.4, the beta limit study on JET tokamak shows that the maximal β_t obtained is proportional to $I_p/a B_{t0}$. This means the maximal plasmas pressure obtained is proportional to I_p/a , i.e.,

$$\langle p \rangle \propto \frac{I_p}{a} \quad (\text{A.37})$$

The total plasma energy is given by $E \approx \langle p \rangle 2\pi R \pi a^2$, where R is the major radius of the device. Using Eq. (A.37), we obtain

$$E \propto I_p a R. \quad (\text{A.38})$$

Since fusion power is proportional to the plasma energy, the above relation indicates, to obtain larger fusion power, we need bigger tokamaks with larger plasma current.

The reason why larger plasma current is desired can also be appreciated by examining an empirical scaling of the the energy confinement time τ_E given by

$$\tau_E \propto \frac{I^2}{T}, \quad (\text{A.39})$$

which is proportional to I^2 .

Another fundamental reason for building larger tokamaks is that the energy confinement time $\tau_E \sim a^2 / \chi$ increases with the machine size, where χ is the heat diffusivity. In addition, the heat diffusivity χ decreases with increasing machine size if the diffusivity satisfies the gyro-Bohm scaling, which is given by

$$\chi_{GB} = \frac{\sqrt{m_i} T_i^{3/2}}{a B^2 q_i^2}, \quad (\text{A.40})$$

which is inverse proportional to the machine size a . However, if the diffusivity satisfies the Bohm scaling, which is given by

$$\chi_B = \frac{T}{|q| B}, \quad (\text{A.41})$$

then, the diffusivity is independent of the machine size. The Bohm diffusivity χ_B is a / ρ_s times larger than the gyro-Bohm diffusivity χ_{GB} , where ρ_s is the gyro-radius. Heat diffusivity scaling in the low confinement operation (L mode) in present-day tokamaks is observed to be Bohm or worse than Bohm.

A.7 Density limit

The maximum density that can be obtained in stable plasma operations (without disruption) is empirically given by

$$n_e \approx 1.5 n_G, \quad (\text{A.42})$$

where n_G is the Greenwalt density, which is given by

$$n_G = 10^{14} \frac{I_p}{\pi a^2} \quad (\text{A.43})$$

where I_p is the plasma current, a is the minor radius, all physical quantities are in SI units. The $1.5 n_G$ gives the density limit that can be achieved for a tokamak operation scenario with plasma current I_p and plasma minor radius a . The Greenwalt density limit is an empirical one, which, like other empirical limits, can be exceeded in practice. Equation (A.43) indicates that the Greenwalt density is proportional to the current density. Therefore the ability to operate in large plasma current density means the ability to operate with high plasma density.

Note that neither the pressure limit nor the density limit is determined by the force-balance constraints. They are determined by the stability of the equilibrium. On the other hand, since the stability of the equilibrium is determined by the equilibrium itself, the pressure and density limit is determined by the equilibrium.

A.8 Relation of plasma current density to pressure gradient

Due to the force balance condition, the plasma current is related to the plasma pressure. Using the equilibrium constraint in the \mathbf{R} direction, the toroidal current density J_ϕ given by Eq. (2.6) can be written as

$$J_\phi = R \frac{dP}{d\Psi} + \frac{1}{\mu_0 R} \frac{dg}{d\Psi} g. \quad (\text{A.44})$$

The parallel (to the magnetic field) current density is written as

$$\begin{aligned} J_\parallel &\equiv \frac{\mathbf{J} \cdot \mathbf{B}}{B} \\ &= \frac{J_\phi B_\phi + \mathbf{J}_p \cdot \mathbf{B}_p}{B} \\ &= \frac{\left(R \frac{dP}{d\Psi} + \frac{1}{\mu_0 R} \frac{dg}{d\Psi} g \right) \frac{g}{R} + \frac{1}{\mu_0} \frac{dg}{d\Psi} \left(\frac{\nabla \Psi}{R} \right)^2}{B} \\ &= \frac{g \frac{dP}{d\Psi} + \frac{1}{\mu_0} \frac{dg}{d\Psi} \left[\left(\frac{g}{R} \right)^2 + \left(\frac{\nabla \Psi}{R} \right)^2 \right]}{B} \\ &= \frac{g \frac{dP}{d\Psi} + \frac{1}{\mu_0} \frac{dg}{d\Psi} B^2}{B}. \\ &= g \frac{dP}{d\Psi} \frac{1}{B} + \frac{1}{\mu_0} \frac{dg}{d\Psi} B. \end{aligned} \quad (\text{A.45})$$

For later use, define

$$\begin{aligned} \sigma &\equiv \frac{J_\parallel}{B} \\ &= g \frac{dP}{d\Psi} \frac{1}{B^2} + \frac{1}{\mu_0} \frac{dg}{d\Psi}. \end{aligned} \quad (\text{A.46})$$

Equation (A.46) is used in GTAW code to calculate J_\parallel / B (actually calculated is $\mu_0 J_\parallel / B$) [15]. Note that the expression for J_\parallel / B in Eq. (A.46) is not a magnetic surface function. Define σ_{ps} as

$$\begin{aligned} \sigma_{\text{ps}} &\equiv \sigma - \langle \sigma \rangle \\ &= g \frac{dP}{d\Psi} \left[\frac{1}{B^2} - \left\langle \frac{1}{B^2} \right\rangle \right] \end{aligned} \quad (\text{A.47})$$

$$\equiv \frac{J_\parallel^{\text{ps}}}{B}, \quad (\text{A.48})$$

where J_\parallel^{ps} is called Pfirsch-Schluter (PS) current. In cylindrical geometry, due to the poloidal symmetry, the Pfirsch-Schluter current is zero. In toroidal geometry, due to the poloidal asymmetry, the PS current is generally nonzero. Thus, this quantity characterizes a toroidal effect.

Another useful quantity is $\mu_0 \langle \mathbf{J} \cdot \mathbf{B} \rangle$, which is written as

$$\mu_0 \langle \mathbf{J} \cdot \mathbf{B} \rangle = \mu_0 g \frac{dP}{d\Psi} + \frac{dg}{d\Psi} \langle B^2 \rangle, \quad (\text{A.49})$$

where $\langle \dots \rangle$ is flux surface averaging operator.

A.9 Vacuum magnetic field (not finished)

In the vacuum region that is between the plasma and the first wall, there is no current, i.e.,

$$\nabla \times \mathbf{B} = 0. \quad (\text{A.50})$$

Next, let us examine what constraint this condition imposes on the magnetic field. Using expression (1.10), the above equation is written as

$$\nabla \times (\nabla \Psi \times \nabla \phi) + \nabla g \times \nabla \phi = 0, \quad (\text{A.51})$$

It is not obvious how to draw useful information from the above equation.

In the above, we use the vector potential approach. Next, let us try the scalar potential approach:

$$\mathbf{B} = \nabla \Phi, \quad (\text{A.52})$$

then the constraint $\nabla \cdot \mathbf{B} = 0$ is written as

$$\nabla^2 \Phi = 0. \quad (\text{A.53})$$

A.10 Discussion about the poloidal current function, check!

Note that, on both an irrational surface and a rational surface, there are infinite number of magnetic field lines that are not connected with each other (it is wrong to say there is only one magnetic field line on a irrational surface). Consider a field f that satisfies $\mathbf{B} \cdot \nabla f = 0$, the value f is a constant along any one of the magnetic field lines. Now comes the question: whether the values of f on different field lines are equal to each other? To answer this question, we can choose a direction different from \mathbf{B} on the magnetic surface and examine whether Ψ is constant or not along this direction, i.e., whether $\mathbf{k} \cdot \nabla f$ equals zero or not, where \mathbf{k} is the chosen direction. For axisymmetric magnetic surfaces, it is ready to see that $\mathbf{k} = \hat{\phi}$ is a direction in the magnetic surface and it is usually not identical with \mathbf{B}/B . Then we obtain

$$\mathbf{k} \cdot \nabla f = \hat{\phi} \cdot \nabla f = \frac{1}{R} \frac{\partial f}{\partial \phi}. \quad (\text{A.54})$$

If f is independent of ϕ , then $\mathbf{k} \cdot \nabla f = 0$. Combining with the fact that $\mathbf{B} \cdot \nabla f = 0$ and the fact that \mathbf{B} and \mathbf{k} are two different directions on the magnetic surfaces, we know that f is constant on the surface, i.e., the values of f on different field lines are equal to each other. If f is non-axisymmetric, then we know the values of f on different magnetic field lines on the same magnetic surface are not equal to each other.

This reasoning is for the case of axisymmetric magnetic surfaces. It is ready to do the same reasoning for non-axisymmetric magnetic surface after we find a convenient direction \mathbf{k} on the magnetic surface.

[check***As discussed in Sec. 3.2, the force balance equation of axisymmetric plasma requires that $\mathbf{B} \cdot \nabla g = 0$. From this and the fact $\mathbf{B} \cdot \nabla \Psi = 0$, we conclude that g is a function of Ψ , i.e., $g = g(\Psi)$. However, this reasoning is not rigorous. Note the concept of a function requires that a function can not be a one-to-more map. This means that $g = g(\Psi)$ indicates that the values of g must be equal on two different magnetic field lines that have the same value of Ψ . However, the two equations $\mathbf{B} \cdot \nabla g = 0$ and $\mathbf{B} \cdot \nabla \Psi = 0$ do not require this constraint. To examine whether this constraint removes some equilibria from all the possible ones, we consider a system with an X point. Inside one of the magnetic islands, we use

$$g = \Psi^2 / (B_0 R_0^3), \quad (\text{A.55})$$

and inside the another, we use

$$g = \Psi^3 / (B_0^2 R_0^5), \quad (\text{A.56})$$

Then solve the two GS equations respectively within the boundary of the two islands. It is easy to obtain two magnetic surfaces that have the same value of Ψ respectively inside the two islands. Equations (A.55) and (A.56) indicate that the values of g on the two magnetic surfaces are different from each other. It is obvious the resulting equilibrium that contain the two islands can not be recovered by directly solving a single GS equation with a given function $g(\Psi)$.**check]

A.11 tmp check!

(In practice, I choose the positive direction of θ and ϕ along the direction of toroidal and poloidal magnetic field (i.e., $\mathbf{B} \cdot \nabla \phi$ and $\mathbf{B} \cdot \nabla \theta$ are always positive in (ψ, θ, ϕ) coordinates system). Then, the q_{local} defined by Eq. (8.3) is always positive. It follows that q_{global} should be also positive. Next, let us examine whether this property is correctly preserved by Eq. (8.9). Case 1: The radial coordinate ψ is chosen as $\Psi' \equiv d\Psi / d\psi > 0$. Then the factor before the integration in Eq. (8.9) is negative. We can further verify that \mathcal{J} is always negative for either the case that $\nabla \Psi$ is pointing inward or outward. Therefore the r.h.s. of Eq. (8.9) is always positive for this case. Case 2: The radial coordinate ψ is chosen as $\Psi' \equiv d\Psi / d\psi < 0$. Then the factor before the integration in Eq. (8.9) is positive. We can further verify that \mathcal{J} is always positive for either the case that $\nabla \Psi$ is pointing inward or outward. The above two cases include all possibilities. Therefore, the positivity of q_{global} is always guaranteed)

a magnetic surface forms a central hole around Z axis. Using Gauss's theorem in the volume within the central hole, and noting that no magnetic field line point-intersects a magnetic surface, we know that the magnetic flux through any cross section of the hole is equal to each other. Next we calculate this magnetic flux. To make the calculation easy, we select a plane cross section perpendicular to the Z axis, as is shown by the dash line in Fig. 1.1. In this case, only B_z contribute to the magnetic flux, which is written (the positive direction of the cross section is chosen in the direction of $\hat{\mathbf{e}}_z$)

$$\begin{aligned}\Psi_{p1} &= \int_0^{R_c} B_z(R, Z_c) 2\pi R dR \\ &= \int_0^{R_c} \frac{1}{R} \frac{\partial \Psi}{\partial R} 2\pi R dR \\ &= 2\pi \int_0^{R_c} \frac{\partial \Psi}{\partial R} dR \\ &= 2\pi [\Psi(R_c, Z_c) - \Psi(0, Z_c)].\end{aligned}$$

Note that the axisymmetry requires that the axis of symmetry, Z axis, must be a magnetic field line. Since $\mathbf{B} \cdot \nabla \Psi = 0$, it follows that the value of Ψ on the Z axis is a constant (denoted by Ψ_a below). Further note that the value of Ψ is constant on a magnetic surface. Thus, Ψ_{p1} in Eq. (1.20) is written

$$\Psi_{p1} = 2\pi(\Psi - \Psi_a). \quad (\text{A.57})$$

be generalized to any revolution surface that is generated by rotating a curve segment on the poloidal plane around Z axis. For instance, a curve on the poloidal plane that connects the magnetic axis and a point on a flux surface can form a toroidal surface (e.g., surface S_2 in Fig. 3.12). The magnetic flux through the toroidal surface S_2 is given by

i.e.

$$\Psi - \Psi_a = \frac{\Psi_{p1}}{2\pi}, \quad (\text{A.58})$$

which indicates that the difference of Ψ between the Z axis and a magnetic surface is the poloidal magnetic flux per radian through the central hole, $\Psi_{p1}/2\pi$.

The magnetic surface forms a central hole around Z axis. The magnetic flux through any cross section of the central hole is equal to each other and is given by $\Psi_p = 2\pi(\Psi_b - \Psi_a)$, where Ψ_a and Ψ_b are the value of Ψ at the Z axis and the magnetic surface, respectively.

The conclusion in Eq. (A.57) can be generalized to any revolution surface that is generated by rotating a curve on the poloidal plane about Z axis. For instance, a curve on the poloidal plane that connects the magnetic axis and a point on a flux surface can form a toroidal surface (e.g., surface S_2 in Fig. 3.12).

Also note the difference between Ψ_p and Ψ_{p1} defined in Sec. 1.7: Ψ_{p1} is the magnetic flux through the central hole of a torus and thus includes the flux in the center transformer, and Ψ_p is the magnetic flux through the ribbon between the magnetic axis and the magnetic surface.

A.12 Radial coordinate to be deleted

We know that the toroidal flux ψ_t , safety factor q , and the Ψ in the GS equation are related by the following equations:

$$d\psi_t = 2\pi q d\Psi \quad (\text{A.59})$$

$$\Rightarrow \psi_t = 2\pi \int_0^\Psi q d\Psi \quad (\text{A.60})$$

Define:

$$\rho \equiv \sqrt{\frac{\psi_t}{\pi}} \quad (\text{A.61})$$

(In the Toray_ga code, the radial coordinate ρ is defined as

$$\rho \equiv \sqrt{\frac{\psi_t}{\pi B_{t0}}}, \quad (\text{A.62})$$

where B_{t0} is a constant factor. ρ defined this way is of length dimension, which is an effective geometry radius obtained by approximating the flux surface as circular.)

I use Eq. (A.61) to define ρ . Then we have

$$\psi_t = \pi \rho^2 \quad (\text{A.63})$$

$$\Rightarrow \frac{d\psi_t}{d\rho} = 2\pi \rho \quad (\text{A.64})$$

$$\Rightarrow \frac{d\psi_t}{d\psi} \frac{d\psi}{d\rho} = 2\pi \rho \quad (\text{A.65})$$

$$\Rightarrow 2\pi q \frac{d\psi}{d\rho} = 2\pi \rho \quad (\text{A.66})$$

$$\Rightarrow \frac{d\psi}{d\rho} = \frac{\rho}{q} \quad (\text{A.67})$$

Eq. (A.67) is used to transform between ψ and ρ .

$$\begin{aligned} d\rho &= \frac{1}{\sqrt{\Phi/\pi R_0^2 B_0}} \frac{1}{2} \frac{1}{\pi B_0 R_0^2} d\Phi = \frac{1}{\rho} \frac{1}{2} \frac{1}{\pi B_0 R_0^2} 2\pi q d\psi = \frac{1}{\rho} \frac{1}{B_0 R_0^2} q d\psi \\ &\Rightarrow d\psi = \frac{\rho B_0 R_0^2}{q} d\rho (\pi a^2) \end{aligned}$$

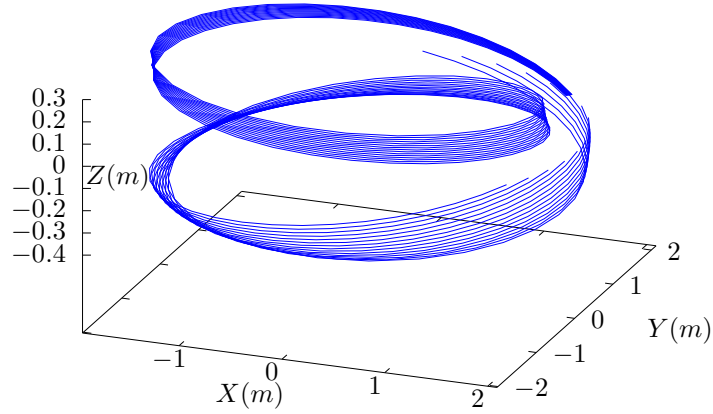


Figure A.2. to be deleted, Isosurface of $\alpha = 2\pi/8$. The surface is made of a family of contours of $\alpha = 2\pi/8$, which are all magnetic field lines. These field lines are traced by starting from a series of points on the low-field-side midplane ($\theta = 0$) at different radial locations and the field lines are followed by a complete poloidal loop. Magnetic field from EAST discharge #59954@3.03s.

A.13 Toroidal elliptic operator in magnetic surface coordinate system

If (ψ, θ, ϕ) are magnetic surface coordinates, i.e., $\partial\Psi/\partial\theta = 0$, then the toroidal elliptic operator in Eq. (15.2) is reduced to

$$\Delta^*\Psi = \frac{R^2}{\mathcal{J}} \left[\left(\Psi_\psi \frac{\mathcal{J}}{R^2} |\nabla\psi|^2 \right)_\psi + \left(\Psi_\psi \frac{\mathcal{J}}{R^2} \nabla\psi \cdot \nabla\theta \right)_\theta \right], \quad (\text{A.68})$$

and the GS equation, Eq. (15.5), is reduced to

$$\frac{R^2}{\mathcal{J}} \left[\left(\Psi_\psi \frac{\mathcal{J}}{R^2} |\nabla\psi|^2 \right)_\psi + \left(\Psi_\psi \frac{\mathcal{J}}{R^2} \nabla\psi \cdot \nabla\theta \right)_\theta \right] = -\mu_0 R^2 \frac{dP}{d\Psi} - \frac{dg}{d\Psi} g(\Psi). \quad (\text{A.69})$$

A.14 Grad-Shafranov equation in (r, θ, ϕ) coordinates

A.14.1 Definition of (r, θ, ϕ) coordinates

Define (r, θ, ϕ) coordinates by

$$R = R_0 + r \cos\theta, \quad (\text{A.70})$$

$$Z = r \sin\theta, \quad (\text{A.71})$$

where (R, ϕ, Z) are the cylindrical coordinates and R_0 is a constant. The above transformation is shown graphically in Fig. A.3.

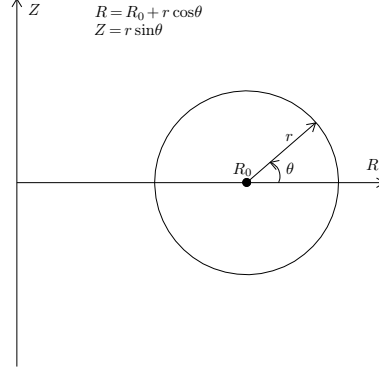


Figure A.3. The relation between coordinates (r, θ) and (R, Z) .

The Jacobian of (r, θ, ϕ) coordinates can be calculated using the definition. Using $x = R \cos \phi$, $y = R \sin \phi$, and $z = Z$, the Jacobian (with respect to the Cartesian coordinates (x, y, z)) is written as

$$\begin{aligned}
 \mathcal{J} &= \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} & \frac{\partial x}{\partial \phi} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} & \frac{\partial y}{\partial \phi} \\ \frac{\partial z}{\partial r} & \frac{\partial z}{\partial \theta} & \frac{\partial z}{\partial \phi} \end{vmatrix} = \begin{vmatrix} \frac{\partial R \cos \phi}{\partial r} & \frac{\partial R \cos \phi}{\partial \theta} & \frac{\partial R \cos \phi}{\partial \phi} \\ \frac{\partial R \sin \phi}{\partial r} & \frac{\partial R \sin \phi}{\partial \theta} & \frac{\partial R \sin \phi}{\partial \phi} \\ \frac{\partial Z}{\partial r} & \frac{\partial Z}{\partial \theta} & \frac{\partial Z}{\partial \phi} \end{vmatrix} \\
 &= \begin{vmatrix} \cos \theta \cos \phi & -r \sin \theta \cos \phi & -R \sin \phi \\ \cos \theta \sin \phi & -r \sin \theta \sin \phi & R \cos \phi \\ \sin \theta & r \cos \theta & 0 \end{vmatrix} \\
 &= \sin \theta (-Rr \sin \theta \cos^2 \phi - Rr \sin \theta \sin^2 \phi) \\
 &\quad + r \cos \theta (-R \sin^2 \phi \cos \theta - R \cos^2 \phi \cos \theta) \\
 &= -Rr \sin^2 \theta - Rr \cos^2 \theta \\
 &= -Rr.
 \end{aligned} \tag{A.72}$$

A.14.2 Toroidal elliptic operator $\Delta^* \Psi$ in (r, θ, ϕ) coordinate system

Next, we transform the GS equation from (R, Z) coordinates to (r, θ) coordinates. Using the relations $R = R_0 + r \cos \theta$ and $Z = r \sin \theta$, we have

$$r = \sqrt{(R - R_0)^2 + Z^2} \Rightarrow \frac{\partial r}{\partial Z} = \frac{Z}{r} = \sin \theta \tag{A.73}$$

$$\sin \theta = \frac{Z}{\sqrt{(R - R_0)^2 + Z^2}} \Rightarrow \cos \theta \frac{\partial \theta}{\partial Z} = \frac{r - Z \frac{Z}{r}}{r^2}. \tag{A.74}$$

$$\Rightarrow \frac{\partial \theta}{\partial Z} = \frac{r - Z \frac{Z}{r}}{r(R - R_0)} = \frac{1 - \sin^2 \theta}{R - R_0} = \frac{\cos^2 \theta}{R - R_0} \quad (\text{A.75})$$

$$\frac{\partial \sin \theta}{\partial Z} = \frac{r - Z \frac{Z}{r}}{r^2} = \frac{1 - \sin^2 \theta}{r} = \frac{\cos^2 \theta}{r} \quad (\text{A.76})$$

The GS equation in (R, Z) coordinates is given by

$$\frac{\partial^2 \Psi}{\partial Z^2} + R \frac{\partial}{\partial R} \left(\frac{1}{R} \frac{\partial \Psi}{\partial R} \right) = -\mu_0 R^2 \frac{dP}{d\Psi} - \frac{dg}{d\Psi} g(\Psi). \quad (\text{A.77})$$

The term $\partial \Psi / \partial Z$ is written as

$$\begin{aligned} \frac{\partial \Psi}{\partial Z} &= \frac{\partial \Psi}{\partial r} \frac{\partial r}{\partial Z} + \frac{\partial \Psi}{\partial \theta} \frac{\partial \theta}{\partial Z} \\ &= \frac{\partial \Psi}{\partial r} \sin \theta + \frac{\partial \Psi}{\partial \theta} \frac{\cos^2 \theta}{R - R_0}. \end{aligned} \quad (\text{A.78})$$

Using Eq. (A.78), $\partial^2 \Psi / \partial Z^2$ is written as

$$\begin{aligned} \frac{\partial^2 \Psi}{\partial Z^2} &= \frac{\partial}{\partial Z} \left(\frac{\partial \Psi}{\partial r} \sin \theta \right) + \frac{\partial}{\partial Z} \left(\frac{\partial \Psi}{\partial \theta} \frac{\cos^2 \theta}{R - R_0} \right) \\ &= \sin \theta \frac{\partial}{\partial Z} \left(\frac{\partial \Psi}{\partial r} \right) + \frac{\partial \Psi}{\partial r} \frac{\partial}{\partial Z} (\sin \theta) + \frac{\cos^2 \theta}{R - R_0} \frac{\partial}{\partial Z} \left(\frac{\partial \Psi}{\partial \theta} \right) + \frac{\partial \Psi}{\partial \theta} \frac{\partial}{\partial Z} \left(\frac{\cos^2 \theta}{R - R_0} \right) \\ &= \sin \theta \left(\frac{\partial^2 \Psi}{\partial r^2} \sin \theta + \frac{\partial^2 \Psi}{\partial r \partial \theta} \frac{\cos^2 \theta}{R - R_0} \right) + \frac{\partial \Psi}{\partial r} \frac{\cos^2 \theta}{r} + \frac{\cos^2 \theta}{R - R_0} \left(\frac{\partial^2 \Psi}{\partial \theta \partial r} \sin \theta + \right. \\ &\quad \left. \frac{\partial^2 \Psi}{\partial \theta^2} \frac{\cos^2 \theta}{R - R_0} \right) \\ &\quad - \frac{\partial \Psi}{\partial \theta} \frac{1}{R - R_0} 2 \cos \theta \sin \theta \frac{\cos^2 \theta}{R - R_0} \end{aligned} \quad (\text{A.79})$$

$$\frac{\partial r}{\partial R} = \frac{\partial}{\partial R} \sqrt{(R - R_0)^2 + Z^2} = \frac{R - R_0}{r} = \cos \theta \quad (\text{A.80})$$

$$\sin \theta = \frac{Z}{\sqrt{(R - R_0)^2 + Z^2}}.$$

$$\cos \theta \frac{\partial \theta}{\partial R} = -Z \frac{R - R_0}{r^3}$$

$$\frac{\partial \theta}{\partial R} = -\frac{Z}{r^2}. \quad (\text{A.81})$$

$$\cos \theta = \frac{R - R_0}{\sqrt{(R - R_0)^2 + Z^2}}$$

$$\frac{\partial}{\partial R} \cos \theta = \frac{r - \frac{R - R_0}{r} (R - R_0)}{r^2} = \frac{1 - \cos^2 \theta}{r} = \frac{\sin^2 \theta}{r} \quad (\text{A.82})$$

$$\begin{aligned}
R \frac{\partial}{\partial R} \left(\frac{1}{R} \frac{\partial \Psi}{\partial R} \right) &= R \frac{\partial}{\partial R} \left[\frac{1}{R} \left(\frac{\partial \Psi}{\partial r} \frac{\partial r}{\partial R} + \frac{\partial \Psi}{\partial \theta} \frac{\partial \theta}{\partial R} \right) \right] \\
&= R \frac{\partial}{\partial R} \left[\frac{1}{R} \left(\frac{\partial \Psi}{\partial r} \cos \theta - \frac{\partial \Psi}{\partial \theta} \frac{Z}{r^2} \right) \right] \\
&= \frac{\partial}{\partial R} \left(\frac{\partial \Psi}{\partial r} \cos \theta - \frac{\partial \Psi}{\partial \theta} \frac{Z}{r^2} \right) + R \left(\frac{\partial \Psi}{\partial r} \cos \theta - \frac{\partial \Psi}{\partial \theta} \frac{Z}{r^2} \right) \left(-\frac{1}{R^2} \right) \\
&= \left(\frac{\partial^2 \Psi}{\partial r^2} \frac{\partial r}{\partial R} + \frac{\partial^2 \Psi}{\partial r \partial \theta} \frac{\partial \theta}{\partial R} \right) \cos \theta + \frac{\partial \Psi}{\partial r} \frac{\partial}{\partial R} \cos \theta - \left(\frac{\partial^2 \Psi}{\partial \theta \partial r} \frac{\partial r}{\partial R} + \right. \\
&\quad \left. \frac{\partial^2 \Psi}{\partial \theta^2} \frac{\partial \theta}{\partial R} \right) \frac{Z}{r^2} - \frac{\partial \Psi}{\partial \theta} \frac{\partial}{\partial R} \left(\frac{Z}{r^2} \right) - \frac{1}{R} \left(\frac{\partial \Psi}{\partial r} \cos \theta - \frac{\partial \Psi}{\partial \theta} \frac{Z}{r^2} \right) \\
&= \left(\frac{\partial^2 \Psi}{\partial r^2} \cos \theta - \frac{\partial^2 \Psi}{\partial r \partial \theta} \frac{Z}{r^2} \right) \cos \theta + \frac{\partial \Psi}{\partial r} \frac{\sin^2 \theta}{r} - \left(\frac{\partial^2 \Psi}{\partial \theta \partial r} \cos \theta - \right. \\
&\quad \left. \frac{\partial^2 \Psi}{\partial \theta^2} \frac{Z}{r^2} \right) \frac{Z}{r^2} + \frac{\partial \Psi}{\partial \theta} Z \frac{1}{r^3} 2 \cos \theta - \frac{1}{R} \left(\frac{\partial \Psi}{\partial r} \cos \theta - \frac{\partial \Psi}{\partial \theta} \frac{Z}{r^2} \right) \\
&= \frac{\partial^2 \Psi}{\partial r^2} \cos^2 \theta + \frac{\partial^2 \Psi}{\partial \theta^2} \frac{\sin^2 \theta}{r^2} - 2 \frac{\partial^2 \Psi}{\partial r \partial \theta} \frac{Z}{r^2} \cos \theta + \frac{\partial \Psi}{\partial r} \frac{\sin^2 \theta}{r} + \\
&\quad \frac{\partial \Psi}{\partial \theta} Z \frac{1}{r^3} 2 \cos \theta \tag{A.83} \\
&- \frac{1}{R} \left(\frac{\partial \Psi}{\partial r} \cos \theta - \frac{\partial \Psi}{\partial \theta} \frac{1}{r} \sin \theta \right) \tag{A.84}
\end{aligned}$$

Summing the the right-hand-side of Eq. (A.79) and the expression on line (A.83) yields

$$\frac{\partial^2 \Psi}{\partial r^2} + \frac{\partial \Psi}{\partial r} \frac{1}{r} + \frac{1}{r^2} \frac{\partial^2 \Psi}{\partial \theta^2}. \tag{A.85}$$

Using these, the GS equation is written as

$$\begin{aligned}
&\frac{\partial^2 \Psi}{\partial r^2} + \frac{\partial \Psi}{\partial r} \frac{1}{r} + \frac{1}{r^2} \frac{\partial^2 \Psi}{\partial \theta^2} - \frac{1}{R_0 + r \cos \theta} \left(\frac{\partial \Psi}{\partial r} \cos \theta - \frac{\partial \Psi}{\partial \theta} \frac{1}{r} \sin \theta \right) = -\mu_0 (R_0 + \\
&r \cos \theta)^2 \frac{dP}{d\Psi} - \frac{dg}{d\Psi} g(\Psi),
\end{aligned}$$

which can be arranged in the form

$$\begin{aligned}
&\left(\frac{1}{r} \frac{\partial}{\partial r} r \frac{\partial}{\partial r} + \frac{1}{r^2} \frac{\partial^2}{\partial \theta^2} \right) \Psi - \frac{1}{R_0 + r \cos \theta} \left(\frac{\partial \Psi}{\partial r} \cos \theta - \frac{\partial \Psi}{\partial \theta} \frac{1}{r} \sin \theta \right) = -\mu_0 (R_0 + \\
&r \cos \theta)^2 \frac{dP}{d\Psi} - \frac{dg}{d\Psi} g(\Psi), \tag{A.86}
\end{aligned}$$

which agrees with Eq. (3.6.2) in Wesson's book[27], where f is defined by $f = RB_\phi / \mu_0$, which is different from $g \equiv RB_\phi$ by a $1/\mu_0$ factor.

A.15 Large aspect ratio expansion

Consider the case that the boundary flux surface is circular with radius $r = a$ and the center of the circle at $(R = R_0, Z = 0)$. Consider the case $\varepsilon = r/R_0 \rightarrow 0$. Expanding Ψ in the small parameter ε ,

$$\Psi = \Psi_0 + \Psi_1 \quad (\text{A.87})$$

where $\Psi_0 \sim O(\varepsilon^0)$, $\Psi_1 \sim O(\varepsilon^1)$. Substituting Eq. (A.87) into Eq. (A.86), we obtain

$$\begin{aligned} & \frac{1}{r} \frac{\partial}{\partial r} r \frac{\partial \Psi_0}{\partial r} + \frac{1}{r} \frac{\partial}{\partial r} r \frac{\partial \Psi_1}{\partial r} + \frac{1}{r^2} \frac{\partial^2 \Psi_0}{\partial \theta^2} + \frac{1}{r^2} \frac{\partial^2 \Psi_1}{\partial \theta^2} - \frac{1}{R_0 + r \cos \theta} \left(\frac{\partial \Psi_0}{\partial r} \cos \theta - \right. \\ & \left. \frac{\partial \Psi_0}{\partial \theta} \frac{1}{r} \sin \theta \right) - \frac{1}{R_0 + r \cos \theta} \left(\frac{\partial \Psi_1}{\partial r} \cos \theta - \frac{\partial \Psi_1}{\partial \theta} \frac{1}{r} \sin \theta \right) = -\mu_0 (R_0 + r \cos \theta)^2 P'(\Psi_0 + \Psi_1) \\ & \Psi_1) - g'(\Psi_0 + \Psi_1) g(\Psi_0 + \Psi_1) \end{aligned}$$

Multiplying the above equation by R_0^2 , we obtain

$$\begin{aligned} & R_0^2 \frac{1}{r} \frac{\partial}{\partial r} r \frac{\partial \Psi_0}{\partial r} + R_0^2 \frac{1}{r} \frac{\partial}{\partial r} r \frac{\partial \Psi_1}{\partial r} + R_0^2 \frac{1}{r^2} \frac{\partial^2 \Psi_0}{\partial \theta^2} + R_0^2 \frac{1}{r^2} \frac{\partial^2 \Psi_1}{\partial \theta^2} - \frac{R_0^2}{R_0 + r \cos \theta} \left(\frac{\partial \Psi_0}{\partial r} \cos \theta - \right. \\ & \left. \frac{\partial \Psi_0}{\partial \theta} \frac{1}{r} \sin \theta \right) - \frac{R_0^2}{R_0 + r \cos \theta} \left(\frac{\partial \Psi_1}{\partial r} \cos \theta - \frac{\partial \Psi_1}{\partial \theta} \frac{1}{r} \sin \theta \right) = -\mu_0 R_0^2 (R_0 + r \cos \theta)^2 P'(\Psi_0 + \Psi_1) \\ & - R_0^2 g'(\Psi_0 + \Psi_1) g(\Psi_0 + \Psi_1) \end{aligned} \quad (\text{A.88})$$

Further assume the following orderings (why?)

$$R_0 \frac{\partial \Psi}{\partial r} \sim O(\varepsilon^{-1}) \Psi, \quad (\text{A.89})$$

and

$$\frac{\partial \Psi}{\partial \theta} \sim O(\varepsilon^0) \Psi. \quad (\text{A.90})$$

Using these orderings, the order of the terms in Eq. (A.88) can be estimated as

$$R_0^2 \frac{1}{r} \frac{\partial}{\partial r} r \frac{\partial \Psi_0}{\partial r} \sim \frac{\Psi_0}{\varepsilon^2} \sim O(\varepsilon^{-2}) \quad (\text{A.91})$$

$$R_0^2 \frac{1}{r} \frac{\partial}{\partial r} r \frac{\partial \Psi_1}{\partial r} \sim \frac{\Psi_1}{\varepsilon^2} \sim O(\varepsilon^{-1}) \quad (\text{A.92})$$

$$R_0^2 \frac{1}{r^2} \frac{\partial^2 \Psi_0}{\partial \theta^2} \sim \frac{1}{\varepsilon^2} \Psi_0 \sim O(\varepsilon^{-2}) \quad (\text{A.93})$$

$$R_0^2 \frac{1}{r^2} \frac{\partial^2 \Psi_1}{\partial \theta^2} \sim \frac{\Psi_1}{\varepsilon^2} \sim O(\varepsilon^{-1}) \quad (\text{A.94})$$

$$\frac{R_0}{R_0 + r \cos \theta} \approx 1 - \frac{r}{R_0} \cos \theta \quad (\text{A.95})$$

$$R_0 \frac{\partial \Psi_0}{\partial r} \cos \theta \sim \frac{\Psi_0}{\varepsilon} \sim O(\varepsilon^{-1}) \quad (\text{A.96})$$

$$R_0 \frac{\partial \Psi_0}{\partial \theta} \frac{1}{r} \sin \theta \sim \frac{\Psi_0}{\varepsilon} \sim O(\varepsilon^{-1}) \quad (\text{A.97})$$

$$R_0 \frac{\partial \Psi_1}{\partial r} \cos \theta \sim \frac{\Psi_1}{\varepsilon} = O(\varepsilon^0) \quad (\text{A.98})$$

$$R_0 \frac{\partial \Psi_1}{\partial \theta} \frac{1}{r} \sin \theta \sim \frac{\Psi_1}{\varepsilon} = O(\varepsilon^0) \quad (\text{A.99})$$

The leading order (ε^{-2} order) balance is given by the following equation:

$$R_0^2 \frac{1}{r} \frac{\partial}{\partial r} r \frac{\partial \Psi_0}{\partial r} + R_0^2 \frac{1}{r^2} \frac{\partial^2 \Psi_0}{\partial \theta^2} = -\mu_0 R_0^4 P'(\Psi_0) - R_0^2 g'(\Psi_0) g(\Psi_0), \quad (\text{A.100})$$

It is reasonable to assume that Ψ_0 is independent of θ since Ψ_0 corresponds to the limit $a/R \rightarrow 0$. (The limit $a/R \rightarrow 0$ can have two cases, one is $r \rightarrow 0$, another is $R \rightarrow \infty$. In the former case, Ψ must be independent of θ since Ψ should be single-valued. The latter case corresponds to a cylinder, for which it is reasonable (really?) to assume that Ψ_0 is independent of θ .) Then Eq. (A.100) is written

$$\frac{1}{r} \frac{\partial}{\partial r} r \frac{\partial \Psi_0}{\partial r} = -\mu_0 R_0^2 P'(\Psi_0) - g'(\Psi_0) g(\Psi_0). \quad (\text{A.101})$$

(My remarks: The leading order equation (A.101) does not corresponds strictly to a cylinder equilibrium because the magnetic field $\mathbf{B} = \nabla \Psi_0 \times \nabla \phi + g \nabla \phi$ depends on θ .) The next order (ε^{-1} order) equation is

$$R_0^2 \frac{1}{r} \frac{\partial}{\partial r} r \frac{\partial \Psi_1}{\partial r} + R_0^2 \frac{1}{r^2} \frac{\partial^2 \Psi_1}{\partial \theta^2} - R_0 \frac{\partial \Psi_0}{\partial r} \cos \theta = -\mu_0 R_0^2 2R_0 r \cos \theta P'(\Psi_0) - \mu_0 R_0^4 P''(\Psi_0) \Psi_1 - R_0^2 [g'(\Psi_0) g(\Psi_0)]' \Psi_1$$

$$R_0^2 \frac{1}{r} \frac{\partial}{\partial r} r \frac{\partial \Psi_1}{\partial r} + R_0^2 \frac{1}{r^2} \frac{\partial^2 \Psi_1}{\partial \theta^2} + \{\mu_0 R_0^4 P''(\Psi_0) + R_0^2 [g'(\Psi_0) g(\Psi_0)]'\} \Psi_1 = -\mu_0 R_0^2 2R_0 r \cos \theta P'(\Psi_0) + R_0 \frac{\partial \Psi_0}{\partial r} \cos \theta \quad (\text{A.102})$$

$$R_0^2 \frac{1}{r} \frac{\partial}{\partial r} r \frac{\partial \Psi_1}{\partial r} + R_0^2 \frac{1}{r^2} \frac{\partial^2 \Psi_1}{\partial \theta^2} + \frac{d}{dr} \{\mu_0 R_0^4 P'(\Psi_0) + R_0^2 g'(\Psi_0) g(\Psi_0)\} \frac{dr}{d\Psi_0} \Psi_1 = -\mu_0 R_0^2 2R_0 r \cos \theta P'(\Psi_0) + R_0 \frac{\partial \Psi_0}{\partial r} \cos \theta \quad (\text{A.103})$$

It is obvious that the simple poloidal dependence of $\cos \theta$ will satisfy the above equation. Therefore, we consider Ψ_1 of the form

$$\Psi_1 = \Delta(r) \frac{d\Psi_0(r)}{dr} \cos \theta, \quad (\text{A.104})$$

where $\Delta(r)$ is a new function to be determined. Substitute this into the Eq. (A.103), we obtain an equation for $\Delta(r)$,

$$R_0^2 \frac{1}{r} \frac{d}{dr} \left(r \frac{d\Psi_0}{dr} \frac{d\Delta}{dr} + r \Delta \frac{d^2 \Psi_0}{dr^2} \right) - R_0^2 \frac{1}{r^2} \Delta \frac{d\Psi_0(r)}{dr} + \frac{d}{dr} \{\mu_0 R_0^4 P'(\Psi_0) + R_0^2 g'(\Psi_0) g(\Psi_0)\} \frac{dr}{d\Psi_0} \Delta \frac{d\Psi_0(r)}{dr} = -\mu_0 R_0^2 2R_0 r P'(\Psi_0) + R_0 \frac{d\Psi_0}{dr} \quad (\text{A.105})$$

$$R_0^2 \frac{1}{r} \frac{d}{dr} \left(r \frac{d\Psi_0}{dr} \frac{d\Delta}{dr} \right) + R_0^2 \frac{1}{r} \frac{d}{dr} \left(r \Delta \frac{d^2\Psi_0}{dr^2} \right) - R_0^2 \frac{1}{r^2} \Delta \frac{d\Psi_0(r)}{dr} + \frac{d}{dr} \{ \mu_0 R_0^4 P'(\Psi_0) + R_0^2 g'(\Psi_0) g(\Psi_0) \} \Delta = -\mu_0 R_0^2 2R_0 r P'(\Psi_0) + R_0 \frac{d\Psi_0}{dr} \quad (\text{A.106})$$

$$R_0^2 \frac{1}{r} \frac{d}{dr} \left(r \frac{d\Psi_0}{dr} \frac{d\Delta}{dr} \right) + \Delta R_0^2 \frac{1}{r} \frac{d}{dr} \left(r \frac{d^2\Psi_0}{dr^2} \right) + R_0^2 \frac{d^2\Psi_0}{dr^2} \frac{d\Delta}{dr} - R_0^2 \frac{1}{r^2} \Delta \frac{d\Psi_0(r)}{dr} + \frac{d}{dr} \{ \mu_0 R_0^4 P'(\Psi_0) + R_0^2 g'(\Psi_0) g(\Psi_0) \} \Delta = -\mu_0 R_0^2 2R_0 r P'(\Psi_0) + R_0 \frac{d\Psi_0}{dr} \quad (\text{A.107})$$

$$R_0^2 \frac{1}{r} \frac{d}{dr} \left(r \frac{d\Psi_0}{dr} \frac{d\Delta}{dr} \right) + \Delta R_0^2 \left[\frac{1}{r} \frac{d}{dr} \left(r \frac{d^2\Psi_0}{dr^2} \right) - \frac{1}{r^2} \frac{d\Psi_0(r)}{dr} \right] + R_0^2 \frac{d^2\Psi_0}{dr^2} \frac{d\Delta}{dr} + \frac{d}{dr} \{ \mu_0 R_0^4 P'(\Psi_0) + R_0^2 g'(\Psi_0) g(\Psi_0) \} \Delta = -\mu_0 R_0^2 2R_0 r P'(\Psi_0) + R_0 \frac{d\Psi_0}{dr} \quad (\text{A.108})$$

Using the identity

$$\frac{d}{dr} \left[\frac{1}{r} \frac{d}{dr} \left(r \frac{d\Psi_0}{dr} \right) \right] = \frac{1}{r} \frac{d}{dr} \left(r \frac{d^2\Psi_0}{dr^2} \right) - \frac{1}{r^2} \frac{d\Psi_0(r)}{dr},$$

equation () is written as

$$R_0^2 \frac{1}{r} \frac{d}{dr} \left(r \frac{d\Psi_0}{dr} \frac{d\Delta}{dr} \right) + \Delta R_0^2 \frac{d}{dr} \left[\frac{1}{r} \frac{d}{dr} \left(r \frac{d\Psi_0}{dr} \right) \right] + R_0^2 \frac{d^2\Psi_0}{dr^2} \frac{d\Delta}{dr} + \frac{d}{dr} \{ \mu_0 R_0^4 P'(\Psi_0) + R_0^2 g'(\Psi_0) g(\Psi_0) \} \Delta = -\mu_0 R_0^2 2R_0 r P'(\Psi_0) + R_0 \frac{d\Psi_0}{dr} \quad (\text{A.109})$$

Using the leading order equation (), we know that the second and fourth term on the l.h.s of the above equation cancel each other, giving

$$R_0^2 \frac{1}{r} \frac{d}{dr} \left(r \frac{d\Psi_0}{dr} \frac{d\Delta}{dr} \right) + R_0^2 \frac{d^2\Psi_0}{dr^2} \frac{d\Delta}{dr} = -\mu_0 R_0^2 2R_0 r P'(\Psi_0) + R_0 \frac{d\Psi_0}{dr} \quad (\text{A.110})$$

$$\Rightarrow \frac{1}{r} \frac{d}{dr} \left(r \frac{d\Psi_0}{dr} \frac{d\Delta}{dr} \right) + \frac{d^2\Psi_0}{dr^2} \frac{d\Delta}{dr} = -\mu_0 2R_0 r P'(\Psi_0) + \frac{1}{R_0} \frac{d\Psi_0}{dr} \quad (\text{A.111})$$

Using the identity

$$\frac{1}{r} \frac{dr}{d\Psi_0} \frac{d}{dr} \left[r \left(\frac{d\Psi_0}{dr} \right)^2 \frac{d\Delta}{dr} \right] = \frac{1}{r} \frac{d}{dr} \left(r \frac{d\Psi_0}{dr} \frac{d\Delta}{dr} \right) + \frac{d^2\Psi_0}{dr^2} \frac{d\Delta}{dr}, \quad (\text{A.112})$$

equation (A.111) is written

$$\frac{1}{r} \frac{dr}{d\Psi_0} \frac{d}{dr} \left[r \left(\frac{d\Psi_0}{dr} \right)^2 \frac{d\Delta}{dr} \right] = -\mu_0 2R_0 r P'(\Psi_0) + \frac{1}{R_0} \frac{d\Psi_0}{dr}, \quad (\text{A.113})$$

$$\Rightarrow \frac{1}{r} \frac{d}{dr} \left[r \left(\frac{d\Psi_0}{dr} \right)^2 \frac{d\Delta}{dr} \right] = -\mu_0 2R_0 r \frac{dP}{dr} + \frac{1}{R_0} \left(\frac{d\Psi_0}{dr} \right)^2, \quad (\text{A.114})$$

Using

$$B_{\theta 0} = \frac{1}{R_0} \frac{d\Psi_0}{dr} \quad (\text{A.115})$$

equation (A.114) is written

$$\frac{1}{r} \frac{d}{dr} \left[r B_{\theta 0}^2 \frac{d\Delta}{dr} \right] = -\mu_0 2 \frac{1}{R_0} r \frac{dP}{dr} + \frac{B_{\theta 0}^2}{R_0}. \quad (\text{A.116})$$

$$\Rightarrow \frac{d}{dr} \left[r B_{\theta 0}^2 \frac{d\Delta}{dr} \right] = \frac{r}{R_0} \left(-2\mu_0 r \frac{dP(\Psi_0)}{dr} + B_{\theta 0}^2 \right), \quad (\text{A.117})$$

which agrees with equation (3.6.7) in Wessson's book[27].

A.16 (s, α) parameters

The normalized pressure gradient, α , which appears frequently in tokamak literature, is defined by[3]

$$\alpha = -R_0 q^2 \frac{1}{B_0^2 / 2\mu_0} \frac{dp}{dr}, \quad (\text{A.118})$$

which can be further written

$$\alpha = -R_0 q^2 \frac{d\bar{p}}{dr}, \quad (\text{A.119})$$

where $\bar{p} = p / (B_0^2 / 2\mu_0)$. Equation (A.119) can be further written as

$$\alpha = -\frac{1}{\varepsilon_a} q^2 \frac{d\bar{p}}{d\bar{r}}, \quad (\text{A.120})$$

where $\varepsilon_a = a / R_0$, $\bar{r} = r / a$, and a is the minor radius of the boundary flux surface. (Why is there a q^2 factor in the definition of α ?)

The global magnetic shear s is defined by

$$s = \frac{r}{q} \frac{dq}{dr}, \quad (\text{A.121})$$

which can be written

$$s = \frac{\bar{r}}{q} \frac{dq}{d\bar{r}}. \quad (\text{A.122})$$

In the case of large aspect ratio and circular flux surface, the leading order equation of the Grad-Shafranov equation in (r, θ) coordinates is written

$$\frac{1}{r} \frac{d}{dr} r \frac{d\Psi}{dr} = -\mu_0 R_0 J_\phi(r), \quad (\text{A.123})$$

which gives concentric circular flux surfaces centered at $(R = R_0, Z = 0)$. Assume that J_ϕ is uniform distributed, i.e., $|J_\phi| = I / (\pi a^2)$, where I is the total current within the flux surface $r = a$. Further assume the current is in the opposite direction of $\nabla \phi$, then $J_\phi = -I / (\pi a^2)$. Using this, Eq. (A.123) can be solved to give

$$\Psi = \frac{\mu_0 I}{4\pi a^2} R_0 r^2. \quad (\text{A.124})$$

Then it follows that the normalized radial coordinate $\rho \equiv (\Psi - \Psi_0) / (\Psi_b - \Psi_0)$ relates to \bar{r} by $\bar{r} = \sqrt{\rho}$ (I check this numerically for the case of EAST discharge #38300). Since in my code, the radial coordinate is Ψ , I need to transform the derivative with respect to \bar{r} to one with respect to Ψ , which gives

$$\alpha = -\frac{1}{\varepsilon} q^2 \frac{d\bar{p}}{d\bar{r}} = -\frac{1}{\varepsilon} q^2 \frac{d\bar{p}}{d\sqrt{\rho}} = -\frac{1}{\varepsilon} q^2 \frac{d\bar{p}}{d\rho} \frac{1}{2\sqrt{\rho}} = -\frac{1}{\varepsilon} q^2 \frac{d\bar{p}}{d\Psi} \frac{1}{2\sqrt{\rho}} (\Psi_b - \Psi_0). \quad (\text{A.125})$$

$$s = \frac{\bar{r}}{q} \frac{dq}{d\bar{r}} = \frac{\sqrt{\rho}}{q} \frac{dq}{d\Psi} \frac{1}{2\sqrt{\rho}} (\Psi_b - \Psi_0) = \frac{1}{2q} \frac{dq}{d\Psi} (\Psi_b - \Psi_0). \quad (\text{A.126})$$

The necessary condition for the existence of TAEs with frequency near the upper tip of the gap is given by[3]

$$\alpha < -s^2 + \varepsilon, \quad (\text{A.127})$$

which is used in my paper on Alfvén eigenmodes on EAST tokamak[15]. Equations (A.125) and (A.126) are used in the GTAW code to calculate s and α .

B Computing magnetic field generated by coils

The vacuum RMP field and ripple field generated by coils can be calculated in the following way.

B.1 Magnetic field

The Biot-Savart law for a zero-thickness wire is given by

$$\mathbf{B}(\mathbf{r}) = \frac{\mu_0}{4\pi} \int I(\mathbf{r}') \frac{d\mathbf{l}(\mathbf{r}') \times (\mathbf{r} - \mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|^3}, \quad (\text{B.1})$$

where $d\mathbf{l}(\mathbf{r}')$ is a line element along the wire (the freedom of choosing which one of the two possible directions along the wire is up to users), $I(\mathbf{r}')$ is the current in the wire (I is positive if the current in the same direction of $d\mathbf{l}$, otherwise, negative).

In Cartesian coordinates, the vector $\mathbf{r} - \mathbf{r}'$ is written as

$$\mathbf{r} - \mathbf{r}' = (x - x')\hat{\mathbf{e}}_x + (y - y')\hat{\mathbf{e}}_y + (z - z')\hat{\mathbf{e}}_z. \quad (\text{B.2})$$

B.2 Magnetic vector potential

Given a current source $\mathbf{J}(\mathbf{r}, t)$, the vector potential can be calculated using

$$\mathbf{A}(\mathbf{r}, t) = \frac{\mu_0}{4\pi} \int_{\Omega} \frac{\mathbf{J}(\mathbf{r}', t')}{|\mathbf{r} - \mathbf{r}'|} d^3\mathbf{r}', \quad (\text{B.3})$$

where

$$t' = t - \frac{|\mathbf{r} - \mathbf{r}'|}{c}, \quad (\text{B.4})$$

is called the retarded time. For a steady-state source, $\mathbf{J}(\mathbf{r}', t') = \mathbf{J}(\mathbf{r})$. Then Eq. (B.3) is simplified as

$$\mathbf{A}(\mathbf{r}) = \frac{\mu_0}{4\pi} \int_{\Omega} \frac{\mathbf{J}(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|} d^3\mathbf{r}'. \quad (\text{B.5})$$

For a current flowing in a thin wire, the above equation is written as

$$\mathbf{A}(\mathbf{r}) = \frac{\mu_0}{4\pi} \int \frac{\mathbf{J}(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|} dS(\mathbf{r}') dl(\mathbf{r}'), \quad (\text{B.6})$$

where dS is a surface element perpendicular to the wire and dl is a line element along the wire. Using $\mathbf{J}(\mathbf{r}')dS(\mathbf{r}') = \mathbf{I}(\mathbf{r}')$, the above equation is written as

$$\begin{aligned} \mathbf{A}(\mathbf{r}) &= \frac{\mu_0}{4\pi} \int \frac{\mathbf{I}(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|} dl(\mathbf{r}') \\ &= \frac{\mu_0}{4\pi} \int \frac{I(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|} d\mathbf{l}(\mathbf{r}'). \end{aligned} \quad (\text{B.7})$$

In cylindrical coordinates, the toroidal component of \mathbf{A} , $A_\phi \equiv \mathbf{A} \cdot \mathbf{e}_\phi(\mathbf{r})$, is written as

$$A_\phi(\mathbf{r}) = \frac{\mu_0}{4\pi} \int \frac{I(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|} d\mathbf{l}(\mathbf{r}') \cdot \mathbf{e}_\phi(\mathbf{r}). \quad (\text{B.8})$$

(A_ϕ is needed in several applications. In solving the free-boundary equilibrium problem, we need to calculate A_ϕ generated by the PF coils. In studying the effects of RMP on particles, we need to calculate P_ϕ , the canonical toroidal angular momentum, whose definition involves A_ϕ . Then we need to calculate A_ϕ generated by the RMP coils.)

A PF coil at (R', Z') with current I (a filamentary current loop) contributes to A_ϕ at $\phi = 0$ via

$$A_\phi(R, Z, \phi = 0) = \frac{\mu_0 I}{4\pi} \int_0^{2\pi} \frac{R' \cos \phi' d\phi'}{\sqrt{(Z - Z')^2 + (R - R' \cos \phi')^2 + (R' \sin \phi')^2}}. \quad (\text{B.9})$$

Since A_ϕ is axisymmetric, the above formula applies to all values of ϕ . Set $I = 1$, we get Green's function:

$$\begin{aligned} &G(R', Z', R, Z) \\ &= R A_\phi \\ &= \frac{\mu_0 R}{4\pi} \int_0^{2\pi} \frac{R' \cos \phi' d\phi'}{\sqrt{(Z - Z')^2 + (R - R' \cos \phi')^2 + (R' \sin \phi')^2}} \end{aligned} \quad (\text{B.10})$$

B.3 For wires in poloidal plane

For a curve in the poloidal plane, the line element $d\mathbf{l}$ in terms of the cylindrical basis is written as $d\mathbf{l} = dR \hat{\mathbf{e}}_R(\phi) + dZ \hat{\mathbf{e}}_z$. Using Cartesian coordinate basis vectors, $d\mathbf{l}$ is written as

$$d\mathbf{l} = dR \cos \phi \hat{\mathbf{e}}_x + dR \sin \phi \hat{\mathbf{e}}_y + dZ \hat{\mathbf{e}}_z. \quad (\text{B.11})$$

Using Eq. (B.11) and (B.2), we obtain

$$d\mathbf{l} \times (\mathbf{r} - \mathbf{r}') = \begin{vmatrix} \hat{\mathbf{e}}_x & \hat{\mathbf{e}}_y & \hat{\mathbf{e}}_z \\ dR \cos \phi & dR \sin \phi & dZ \\ (x - x') & (y - y') & (z - z') \end{vmatrix}. \quad (\text{B.12})$$

B.4 For wires along toroidal direction

For a curve along the toroidal direction, the line element $d\mathbf{l}$ in terms of the cylindrical basis is written as $d\mathbf{l} = R d\phi \hat{\mathbf{e}}_\phi$. Using Cartesian coordinate basis vectors, $d\mathbf{l}$ is written as

$$d\mathbf{l} = R d\phi (-\sin \phi \hat{\mathbf{e}}_x + \cos \phi \hat{\mathbf{e}}_y). \quad (\text{B.13})$$

Using Eq. (B.13) and (B.2), we obtain

$$d\mathbf{l} \times (\mathbf{r} - \mathbf{r}') = \begin{vmatrix} \hat{\mathbf{e}}_x & \hat{\mathbf{e}}_y & \hat{\mathbf{e}}_z \\ -R d\phi \sin \phi & R d\phi \cos \phi & 0 \\ (x - x') & (y - y') & (z - z') \end{vmatrix} \quad (\text{B.14})$$

17 About this document

This document was written by using TeXmacs[25], a structured wysiwyw (what you see is what you want) editor for scientists. The HTML version of this document is generated by first converting the TeXmacs file to TeX format and then using `htlax` to convert the TeX file to HTML format.

Bibliography

- [1] R. Aymar, P. Barabaschi, and Y. Shimomura. The ITER design. *Plasma Physics and Controlled Fusion*, 44(5):519–565, apr 2002.
- [2] M. A. Beer, S. C. Cowley, and G. W. Hammett. Field-aligned coordinates for nonlinear simulations of tokamak turbulence. *Phys. Plasmas (1994-present)*, 2(7):2687–2700, 1995.
- [3] H. L. Berk, J. W. Van Dam, D. Borba, J. Candy, G. T. A. Huysmans, and S. Sharapov. More on core localized toroidal alfvén eigenmodes. *Phys. Plasmas*, 2(9):3401–3406, 1995.
- [4] Allen H. Boozer. Physics of magnetically confined plasmas. *Rev. Mod. Phys.*, 76:1071–1141, Jan 2005.
- [5] R. Gallix, C. Jong, J. Knaster, N. Mitchell C. Sborchia, Y. Fu. Design and specifications of the iter tf coils. *IEEE TRANSACTIONS ON APPLIED SUPERCONDUCTIVITY*, 18(2):463, 2008.
- [6] Yang Chen and Scott E. Parker. A delta-f particle method for gyrokinetic simulations with kinetic electrons and electromagnetic perturbations. *Journal of Computational Physics*, 189(2):463–475, 2003.

- [7] Yang Chen and Scott E. Parker. Electromagnetic gyrokinetic δf particle-in-cell turbulence simulation with realistic equilibrium profiles and geometry. *Journal of Computational Physics*, 220(2):839–855, 2007.
- [8] Yang Chen, Scott E. Parker, J. Lang, and G.-Y. Fu. Linear gyrokinetic simulation of high- n toroidal alfvén eigenmodes in a burning plasma. *Phys. Plasmas*, 17(10):102504, 2010.
- [9] Fu Peng, Gao Ge Chen Yuan-Yang, Bao Xiao-Hua. Plasma shape optimization for east tokamak using orthogonal method. *Chinese Physics B*, 28(1):15201, 2019.
- [10] C.Z Cheng and M.S Chance. Nova: a nonvariational code for solving the mhd stability of axisymmetric toroidal plasmas. *J. of Comput. Phys.*, 71(1):124–146, 1987.
- [11] J DeLucia, S.C Jardin, and A.M.M Todd. An iterative metric method for solving the inverse tokamak equilibrium problem. *Journal of Computational Physics*, 37(2):183–204, 1980.
- [12] A. M. Dimits. Fluid simulations of tokamak turbulence in quasiballooning coordinates. *Phys. Rev. E*, 48:4070–4079, Nov 1993.
- [13] T. Görler, N. Tronko, W. A. Hornsby, A. Bottino, R. Kleiber, C. Nordsieck, V. Grandgirard, F. Jenko, and E. Sonnendrücker. Intercode comparison of gyrokinetic global electromagnetic modes. *Physics of Plasmas*, 23(7):72503, 2016.
- [14] J. van der Hoeven et al. GNU TeXmacs. <https://www.texmacs.org>, 1998.
- [15] Youjun Hu, Guoqiang Li, N. N. Gorelenkov, Huishan Cai, Wenjun Yang, Deng Zhou, and Qilong Ren. Numerical study of alfvén eigenmodes in the experimental advanced superconducting tokamak. *Phys. Plasmas*, 21(5):0, 2014.
- [16] Stephen C. Jardin. *Computational methods in plasma physics*. CRC Press, 2010.
- [17] G.S. Lee, M. Kwon, C.J. Doh, B.G. Hong, K. Kim, M.H. Cho, W. Namkung, C.S. Chang, Y.C. Kim, J.Y. Kim, H.G. Jhang, D.K. Lee, K.I. You, J.H. Han, M.C. Kyum, J.W. Choi, J. Hong, W.C. Kim, B.C. Kim, J.H. Choi, S.H. Seo, H.K. Na, H.G. Lee, S.G. Lee, S.J. Yoo, B.J. Lee, Y.S. Jung, J.G. Bak, H.L. Yang, S.Y. Cho, K.H. Im, N.I. Hur, I.K. Yoo, J.W. Sa, K.H. Hong, G.H. Kim, B.J. Yoo, H.C. Ri, Y.K. Oh, Y.S. Kim, C.H. Choi, D.L. Kim, Y.M. Park, K.W. Cho, T.H. Ha, S.M. Hwang, Y.J. Kim, S. Baang, S.I. Lee, H.Y. Chang, W. Choe, S.G. Jeong, S.S. Oh, H.J. Lee, B.H. Oh, B.H. Choi, C.K. Hwang, S.R. In, S.H. Jeong, I.S. Ko, Y.S. Bae, H.S. Kang, J.B. Kim, H.J. Ahn, D.S. Kim, C.H. Choi, J.H. Lee, Y.W. Lee, Y.S. Hwang, S.H. Hong, K.-H. Chung, D.-I. Choi, and KSTAR Team. Design and construction of the KSTAR tokamak. *Nuclear Fusion*, 41(10):1515–1523, oct 2001.
- [18] G Q Li, Q L Ren, J P Qian, L L Lao, S Y Ding, Y J Chen, Z X Liu, B Lu, and Q Zang. Kinetic equilibrium reconstruction on east tokamak. *Plasma Phys. Controlled Fusion*, 55(12):125008, 2013.
- [19] J.L. Luxon. A design retrospective of the DIII-D tokamak. *Nuclear Fusion*, 42(5):614–633, may 2002.
- [20] R. L. Miller, M. S. Chu, J. M. Greene, Y. R. Lin-Liu, and R. E. Waltz. Noncircular, finite aspect ratio, local equilibrium model. *Phys. Plasmas*, 5(4):973–978, 1998.
- [21] M. Ottaviani. An alternative approach to field-aligned coordinates for plasma turbulence simulations. *Physics Letters A*, 375(15):1677–1685, 2011.
- [22] S.A. Sabbagh, A.C. Sontag, J.M. Bialek, D.A. Gates, A.H. Glasser, J.E. Menard, W. Zhu, M.G. Bell, R.E. Bell, A. Bondeson, C.E. Bush, J.D. Callen, M.S. Chu, C.C. Hegna, S.M. Kaye, L.L. Lao, B.P. LeBlanc, Y.Q. Liu, R. Maingi, D. Mueller, K.C. Shaing, D. Stutman, K. Tritz, and C. Zhang. Resistive wall stabilized operation in rotating high beta nstx plasmas. *Nucl. Fusion*, 46(5):635, 2006.
- [23] I. Song, C. Choi, and M. Cho. Quench protection system for the superconducting coil of the kstar tokamak. *IEEE Transactions on Applied Superconductivity*, 17(1):1–6, March 2007.
- [24] Youwen Sun. Tokamak geometry and equilibrium. http://theory.ipp.ac.cn/~yj/other/TOKAMAK_geometry_and_equilibrium-Youwen_Sun_201208.pdf, 2012. [Online; Institute of Plasma Physics, Chinese Academy of Sciences].

- [25] Joris van der Hoeven. Gnu texmacs, a structured wysiwyw (what you see is what you want) editor for scientists. <http://www.texmacs.org/>, 2007. [Online].
- [26] W. X. Wang, Z. Lin, W. M. Tang, W. W. Lee, S. Ethier, J. L. V. Lewandowski, G. Rewoldt, T. S. Hahm, and J. Manickam. Gyro-kinetic simulation of global turbulent transport properties in tokamak experiments. *Physics of Plasmas*, 13(9):92505, 2006.
- [27] John Wesson. *Tokamaks*. Oxford University Press, 2004.
- [28] R. B. White and M. S. Chance. Hamiltonian guiding center drift orbit calculation for plasmas of arbitrary cross section. *The Physics of Fluids*, 27(10):2455–2467, 1984.
- [29] B J Xiao, D A Humphreys, and M L Walker. East plasma control system. *Fusion Engineering and Design*, 83:181, 2008.
- [30] B. J. Xiao. The first diverted plasma on east tokamak. *34th EPS Conference on Plasma Phys*, 2007.