Analytical classical dynamics

BY YOUJUN HU

Institute of plasma physics, Chinese Academy of Sciences
Email: yjhu@ipp.cas.cn

Abstract

These notes were initially written when I read Fitzpatrick's book[1] and were later revised to add more contents.

1 Generalized coordinates

For a dynamical system with $\mathcal{F}$ degrees of freedom, the Cartesian coordinates can be expressed in terms of generalized coordinates $(q_1, q_2, ..., q_{\mathcal{F}})$,

$$x_j = x_j(q_1, q_2, ..., q_{\mathcal{F}}, t), \text{for } j = 1, ..., \mathcal{F} \tag{1}$$

2 Generalized force

The work on a dynamical system when its Cartesian coordinates changed by δx_j is given by

$$\begin{aligned} \delta W &= \sum_{j=1}^{\mathcal{F}} F_j \delta x_j \\ &= \sum_{j=1}^{\mathcal{F}} F_j \sum_{i=1}^{\mathcal{F}} \frac{\partial x_j}{\partial q_i} \delta q_i \\ &= \sum_{i=1}^{\mathcal{F}} \left(\sum_{j=1}^{\mathcal{F}} F_j \frac{\partial x_j}{\partial q_i} \right) \delta q_i \\ &= \sum_{i=1}^{\mathcal{F}} Q_i \delta q_i, \end{aligned}$$

where Q_i is defined by

$$Q_i = \sum_{j=1}^{\mathcal{F}} F_j \frac{\partial x_j}{\partial q_i}, \tag{2}$$

which is called generalized force. For conservative system, F_j can be written in the form

$$F_j = -\frac{\partial U}{\partial x_j}. \tag{3}$$

Using Eq. (3), the generalized force is written as

$$\begin{aligned} Q_i &= -\sum_{j=1}^{\mathcal{F}} \frac{\partial U}{\partial x_j} \frac{\partial x_j}{\partial q_i} \\ &= -\frac{\partial U}{\partial q_i}. \end{aligned} \tag{4}$$

3 Euler-Lagrange equation

Newton's second law is written as

$$m_j \ddot{x}_j = F_j. \tag{5}$$

Using

$$\dot{x}_j = \frac{\partial x_j}{\partial t} + \sum_{i=1}^{\mathcal{F}} \frac{\partial x_j}{\partial q_i} \dot{q}_i, \tag{6}$$

we obtain

$$\frac{\partial \dot{x}_j}{\partial \dot{q}_I} = \frac{\partial x_j}{\partial q_I}, \tag{7}$$

where the partial derivative on the left-hand side is taken by holding q_1, ..., $q_{\mathcal{F}}$ constant (this convention is important for deriving the Euler-Lagrange equation). Multiplying Eq. (7) by $m\dot{x}_j$ and summing over j, we obtain

$$\sum_{j=1}^{\mathcal{F}} m_j \dot{x}_j \frac{\partial \dot{x}_j}{\partial \dot{q}_I} = \sum_{j=1}^{\mathcal{F}} m_j \frac{\partial x_j}{\partial q_I} \dot{x}_j. \tag{8}$$

Taking time differential, the above equation is written as

$$\frac{d}{dt}\left(\sum_{j=1}^{\mathcal{F}} m_j \dot{x}_j \frac{\partial \dot{x}_j}{\partial \dot{q}_I} \right) = \frac{d}{dt}\left(\sum_{j=1}^{\mathcal{F}} m_j \frac{\partial x_j}{\partial q_I} \dot{x}_j \right) \tag{9}$$

The left-hand side of Eq. (9) is written as

$$\begin{aligned} \frac{d}{dt}\left(\sum_{j=1}^{\mathcal{F}} m_j \dot{x}_j \frac{\partial \dot{x}_j}{\partial \dot{q}_I} \right) &= \frac{d}{dt}\left(\sum_{j=1}^{\mathcal{F}} \frac{1}{2} m_j \frac{\partial \dot{x}_j^2}{\partial \dot{q}_I} \right) \\ &= \frac{d}{dt}\left(\frac{\partial}{\partial \dot{q}_I} \sum_{j=1}^{\mathcal{F}} \frac{1}{2} m_j \dot{x}_j^2 \right) \\ &= \frac{d}{dt}\left(\frac{\partial}{\partial \dot{q}_I} K \right), \end{aligned} \tag{10}$$

where $K \equiv \sum_{j=1}^{\mathcal{F}} \frac{1}{2} m_j \dot{x}_j^2$ is the kinetic energy. The right-hand side of Eq. (9) is written as

$$\sum_{j=1}^{\mathcal{F}} m_j \frac{\partial x_j}{\partial q_I} \ddot{x}_j + \sum_{j=1}^{\mathcal{F}} m_j \frac{d}{dt}\left(\frac{\partial x_j}{\partial q_I} \right) \dot{x}_j. \tag{11}$$

Using Newton's second law, the above expression is written as

$$\sum_{j=1}^{\mathcal{F}} \frac{\partial x_j}{\partial q_I} F_j + \sum_{j=1}^{\mathcal{F}} m_j \frac{d}{dt}\left(\frac{\partial x_j}{\partial q_I} \right) \dot{x}_j, \tag{12}$$

which can be further written as, by using the definition of the generalized force,

$$Q_I + \sum_{j=1}^{\mathcal{F}} m_j \frac{d}{dt}\left(\frac{\partial x_j}{\partial q_I} \right) \dot{x}_j. \tag{13}$$

In order to simplify the second term in expression (13), we try to prove that

$$\frac{d}{dt}\left(\frac{\partial x_j}{\partial q_I} \right) = \frac{\partial \dot{x}_j}{\partial q_I}, \tag{14}$$

where the partial derivative with respect to q_I on the right-hand side is taken by holding $\dot{q}_1$, $\dot{q}_2, \ldots,$ $\dot{q}_{\mathcal{F}}$ constant. [Proof: The right-hand side of the above equation is written as

$$\begin{aligned} \frac{\partial \dot{x}_j}{\partial q_I} &= \frac{\partial}{\partial q_I}\left(\frac{\partial x_j}{\partial t} + \sum_{i=1}^{\mathcal{F}} \frac{\partial x_j}{\partial q_i} \dot{q}_i \right) \\ &= \frac{\partial x_j}{\partial q_I \partial t} + \sum_{i=1}^{\mathcal{F}} \frac{\partial x_j}{\partial q_I \partial q_i} \dot{q}_i, \end{aligned} \tag{15}$$

In obtaining the last equality, we have used the fact that the order of the partial derivative of x_j with respect to q_i, q_I, and t is interchangeable. The right-hand side of Eq. (15) can be further written as

$$\frac{\partial}{\partial t}\left(\frac{\partial x_j}{\partial q_I}\right)+\sum_{i=1}^{\mathcal{F}}\frac{\partial}{\partial q_i}\left(\frac{\partial x_j}{\partial q_I}\right)\dot{q}_i, \tag{16}$$

which is obviously the total time derivative of $\partial x_j/\partial q_I$, i..e.,

$$\frac{d}{dt}\left(\frac{\partial x_j}{\partial q_I}\right), \tag{17}$$

which is exactly the left-hand side of Eq. (14). Thus Eq. (14) is proved.] Using Eq. (14), the expression (13) is written as

$$Q_I+\sum_{j=1}^{\mathcal{F}} m_j\frac{\partial \dot{x}_j}{\partial q_I}\dot{x}_j,$$

which can be further written as

$$Q_I+\sum_{j=1}^{\mathcal{F}}\frac{1}{2}m_j\frac{\partial \dot{x}_j^2}{\partial q_I},$$

i.e.,

$$Q_I+\frac{\partial K}{\partial q_I} \tag{18}$$

Thus we obtain

$$\frac{d}{dt}\left(\frac{\partial}{\partial \dot{q}_I}K\right)=Q_I+\frac{\partial K}{\partial q_I}. \tag{19}$$

If the generalized force is given by Eq. (4), equation (19) is written as

$$\frac{d}{dt}\left(\frac{\partial}{\partial \dot{q}_I}K\right)=-\frac{\partial U}{\partial q_I}+\frac{\partial K}{\partial q_I}. \tag{20}$$

Define

$$\mathcal{L}=K-U \tag{21}$$

and noting that U is independent of $\dot{q}_1, \dot{q}_1, \ldots \dot{q}_{\mathcal{F}}$, Eq. (20) is written as

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}_I}\right)=\frac{\partial \mathcal{L}}{\partial q_I}, \tag{22}$$

which is the Euler-Lagrange equation, where, recalling the remarks below Eqs. (7) and (14), we know that the partial derivative with respect to q_I and $\dot{q}_I$ are taken by treating $\dot{q}_i$ and q_i, respectively, as independent variables. Define the canonical momentum

$$p_I=\frac{\partial \mathcal{L}}{\partial \dot{q}_I}, \tag{23}$$

then the Euler-Lagrange equation is written as

$$\frac{d}{dt}(p_I)=\frac{\partial \mathcal{L}}{\partial q_I}. \tag{24}$$

4 Prove that Euler-Lagrange equation is coordinates independent

Try to use

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{x}_j}\right)=\frac{\partial \mathcal{L}}{\partial x_j}, \tag{25}$$

where x_j are rectangular coordinates, to prove that

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}_i}\right)=\frac{\partial \mathcal{L}}{\partial q_i}. \tag{26}$$

where q_i are arbitrary generalized coordinates.

Proof: The left-hand side of Eq. (26) is written as

$$\begin{aligned}\frac{d}{dt}\left(\frac{\partial\mathcal{L}}{\partial\dot{q}_i}\right) &= \frac{d}{dt}\left(\sum_j \frac{\partial\mathcal{L}}{\partial x_j}\frac{\partial x_j}{\partial\dot{q}_i}+\sum_j \frac{\partial\mathcal{L}}{\partial\dot{x}_j}\frac{\partial\dot{x}_j}{\partial\dot{q}_i}\right) \quad (27)\\ &= \frac{d}{dt}\left(0+\sum_j \frac{\partial\mathcal{L}}{\partial\dot{x}_j}\frac{\partial\dot{x}_j}{\partial\dot{q}_i}\right)\\ &= \sum_j \frac{d}{dt}\left(\frac{\partial\mathcal{L}}{\partial\dot{x}_j}\frac{\partial\dot{x}_j}{\partial\dot{q}_i}\right)\\ &= \sum_j\left[\frac{d}{dt}\left(\frac{\partial\mathcal{L}}{\partial\dot{x}_j}\right)\frac{\partial\dot{x}_j}{\partial\dot{q}_i}+\frac{\partial\mathcal{L}}{\partial\dot{x}_j}\frac{d}{dt}\left(\frac{\partial\dot{x}_j}{\partial\dot{q}_i}\right)\right] \quad (28)\end{aligned}$$

Using Eq. (25), the above equation is written as

$$\frac{d}{dt}\left(\frac{\partial\mathcal{L}}{\partial\dot{q}_i}\right)=\sum_j\left[\frac{\partial\mathcal{L}}{\partial x_j}\frac{\partial\dot{x}_j}{\partial\dot{q}_i}+\frac{\partial\mathcal{L}}{\partial\dot{x}_j}\frac{d}{dt}\left(\frac{\partial\dot{x}_j}{\partial\dot{q}_i}\right)\right] \quad (29)$$

Using the fact that

$$\frac{d}{dt}\left(\frac{\partial\dot{x}_j}{\partial\dot{q}_i}\right)=\frac{\partial\dot{x}_j}{\partial q_i}. \quad (30)$$

Eq. (29) is written as

$$\frac{d}{dt}\left(\frac{\partial\mathcal{L}}{\partial\dot{q}_i}\right)=\sum_j\left[\frac{\partial\mathcal{L}}{\partial x_j}\frac{\partial\dot{x}_j}{\partial\dot{q}_i}+\frac{\partial\mathcal{L}}{\partial\dot{x}_j}\frac{\partial\dot{x}_j}{\partial q_i}\right] \quad (31)$$

Using the fact that

$$\frac{\partial\dot{x}_j}{\partial\dot{q}_i}=\frac{\partial x_j}{\partial q_i} \quad (32)$$

Eq. (31) is further written as

$$\begin{aligned}\frac{d}{dt}\left(\frac{\partial\mathcal{L}}{\partial\dot{q}_i}\right) &= \sum_j\left[\frac{\partial\mathcal{L}}{\partial x_j}\frac{\partial x_j}{\partial q_i}+\frac{\partial\mathcal{L}}{\partial\dot{x}_j}\frac{\partial\dot{x}_j}{\partial q_i}\right]\\ &= \frac{\partial\mathcal{L}}{\partial q_i}, \quad (33)\end{aligned}$$

i.e.,

$$\frac{d}{dt}\left(\frac{\partial\mathcal{L}}{\partial\dot{q}_i}\right)=\frac{\partial\mathcal{L}}{\partial q_i}, \quad (34)$$

which is what we want.

5 Using Euler-Lagrange equation to derive Hamilton's equation

The Euler-Lagrange equation is given by

$$\frac{d}{dt}(p_I)=\frac{\partial\mathcal{L}}{\partial q_I}, \quad (35)$$

with p_I defined by

$$p_I=\frac{\partial\mathcal{L}}{\partial\dot{q}_I}, \quad (36)$$

The Lagrangian $\mathcal{L}$ is defined as the difference of the kinetic energy and the potential energy of the system, i.e.,

$$\mathcal{L}=K-U. \quad (37)$$

In the Euler-Lagrange equation, the independent variables are chosen to be q_i and $\dot{q}_i$ with $i=1$, 2...$\mathcal{F}$, where $\mathcal{F}$ is the freedom of the system, i.e.,

$$\mathcal{L}=\mathcal{L}(q_1,q_2,\ldots q_{\mathcal{F}},\dot{q}_1,\ldots\dot{q_{\mathcal{F}}}). \quad (38)$$

Now we use the above results to derive Hamilton's equation. In Hamilton's equation the independent variables are chosen to be q_i and p_i with $i=1,2,\ldots.\mathcal{F}$. The Hamiltonian is equal to the total energy of the system, i.e.

$$\mathcal{H}=K+U. \quad (39)$$

And the Hamiltonian must be expressed as a function of q_i and p_i with $i = 1, 2 \ldots \mathcal{F}$ (i.e., it is the total energy expressed in terms of q_i and p_i that can be called Hamiltonian). Noting that

$$2K = \sum_i p_i \dot{q_i}. \tag{40}$$

(The proof of this result is given in Sec. 6), we can obtain the relation between the Lagrangian and Hamiltonian

$$\mathcal{L} + \mathcal{H} = \sum_i p_i \dot{q_i}. \tag{41}$$

Since the independent variables in Hamilton's equation are q_i and p_i, thus the $\dot{q_i}$ should be view as a function of general coordinates and general momentum, i.e.,

$$\dot{q_i} = \dot{q_i}(q_1, q_2, \ldots q_{\mathcal{F}}, p_1, p_2, \ldots p_{\mathcal{F}}). \tag{42}$$

Understanding this dependence, we can take the differential of Eq. (41) with respect to p_I, which gives (using the chain rule)

$$\sum_i \frac{\partial \mathcal{L}}{\partial q_i}\frac{\partial q_i}{\partial p_I} + \sum_i \frac{\partial \mathcal{L}}{\partial \dot{q_i}}\frac{\partial \dot{q_i}}{\partial p_I} + \frac{\partial \mathcal{H}}{\partial p_I} = \sum_i p_i \frac{\partial \dot{q_i}}{\partial p_I} + \dot{q_I}. \tag{43}$$

Noting that q and p are independent variables, thus $\partial q_i / \partial p_I = 0$, the above equation is written as

$$0 + \sum_i \frac{\partial \mathcal{L}}{\partial \dot{q_i}}\frac{\partial \dot{q_i}}{\partial p_I} + \frac{\partial \mathcal{H}}{\partial p_I} = \sum_i p_i \frac{\partial \dot{q_i}}{\partial p_I} + \dot{q_I}. \tag{44}$$

Noting the definition of the general momentum, the two summation terms cancel each other, we are left with

$$\dot{q_I} = \frac{\partial \mathcal{H}}{\partial p_I}, \tag{45}$$

which is the first Hamilton's equation. Similarly, we take the differential of Eq. (41) with respect to q_I, which gives

$$\frac{\partial \mathcal{L}}{\partial q_I} + \sum_i \frac{\partial \mathcal{L}}{\partial \dot{q_i}}\frac{\partial \dot{q_i}}{\partial q_I} + \frac{\partial \mathcal{H}}{\partial q_I} = \sum_i p_i \frac{\partial \dot{q_i}}{\partial q_I}. \tag{46}$$

Noting the definition of the general momentum, the two summation terms cancel each other, we are left with

$$\frac{\partial \mathcal{L}}{\partial q_I} = -\frac{\partial \mathcal{H}}{\partial q_I}. \tag{47}$$

Using the Euler-Lagrange equation (35), the above equation is written

$$\dot{p_I} = -\frac{\partial \mathcal{H}}{\partial q_I},$$

which is the second Hamilton's equation.

6 Prove that $2K = \sum_i p_i \dot{q_i}$

To prove that

$$2K = \sum_i p_i \dot{q_i}. \tag{48}$$

Proof: Using the definition of the generalized momentum p_i, the right-hand side of the above equation is written as

$$\sum_i p_i \dot{q_i} = \sum_i \frac{\partial \mathcal{L}}{\partial \dot{q_i}} \dot{q_i}. \tag{49}$$

Noting that $\mathcal{L} = K - U$, where the potential U is independent of $\dot{q}_i$, Eq. (49) is written as

$$\sum_i p_i \dot{q_i} = \sum_i \frac{\partial K}{\partial \dot{q_i}} \dot{q_i}. \tag{50}$$

Using the definition of the kinetic energy, Eq. (50) is written as

$$\begin{aligned} \sum_i p_i \dot{q_i} &= \sum_i \frac{\partial \sum_j \frac{1}{2} m_j \dot{x}_j^2}{\partial \dot{q_i}} \dot{q_i} \\ &= \sum_i \sum_j \frac{1}{2} m_j \frac{\partial \dot{x}_j^2}{\partial \dot{q_i}} \dot{q_i} \\ &= \sum_i \sum_j m_j \dot{x}_j \frac{\partial \dot{x}_j}{\partial \dot{q_i}} \dot{q_i}. \end{aligned} \tag{51}$$

Using the fact that

$$\frac{\partial \dot{x}_j}{\partial \dot{q_i}} = \frac{\partial x_j}{\partial q_i}, \tag{52}$$

Eq. (51) is written as

$$\begin{aligned} \sum_i p_i \dot{q_i} &= \sum_i \sum_j m_j \dot{x}_j \frac{\partial x_j}{\partial q_i} \dot{q_i} \\ &= \sum_j m_j \dot{x}_j \sum_i \frac{\partial x_j}{\partial q_i} \dot{q_i} \end{aligned} \tag{53}$$

If the coordinates transformation $x_j = x_j(q_1, ..., q_{\mathcal{F}}, t)$ does not explicitly depends on t, i.e., $x_j = x_j(q_1, ..., q_{\mathcal{F}})$, then the above equation is written as

$$\begin{aligned} \sum_i p_i \dot{q_i} &= \sum_j m_j \dot{x}_j \dot{x}_j \\ &= 2K \end{aligned} \tag{54}$$

Thus Eq. (48) is proved.

7 Variational principle

The derivation of Lagrange's equation given in the last section starts with Newton's law. The derivation can also starts with a variational principle. Define the action integral

$$J = \int_{t_1}^{t_2} \mathcal{L} dt, \tag{55}$$

Then the action principle says that the equation of motion is given by

$$\delta J = 0. \tag{56}$$

From Eq. (56), we can derive the Euler-Lagrange equation

$$\Rightarrow \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}_I}\right) = \frac{\partial \mathcal{L}}{\partial q_I} \tag{57}$$

8 Field theory—not finished

Generally speaking, the Lagrangian of a dynamical system is a function that summarizes the dynamics of the system. Modern formulations of classical field theories tend to be expressed by using Lagrangian. The Lagrangian is a function that, when subjected to an action principle, gives rise to the field equations and a conservation law for the theory.

The Lagrangian in many classical systems is a function of generalized coordinates q_i and their velocities $\dot{q}_i$. These coordinates (and velocities) are, in turn, parametric functions of time. In the classical view, time is an independent variable and $(q_i,\ \dot{q}_i)$ are dependent variables. This formalism was generalized further to handle field theory. In field theory, the independent variable is replaced by an event in space-time (x, y, z, t), or more generally by a point s on a manifold. And the dependent variables q are replaced by φ, the value of a field at that point in space-time, so that field equations are obtained by means of an action principle, written as:

$$\frac{\delta S}{\delta \varphi} = 0. \tag{58}$$

Lagrangian densities in field theory

The time integral of the Lagrangian is called the action denoted by S. In field theory, a distinction is occasionally made between the Lagrangian L, of which the action is the time integral:

$$S = \int L\, dt \tag{59}$$

and the Lagrangian density $\mathcal{L}$, which one integrates over all space-time to get the action:

$$S[\varphi] = \int \mathcal{L}[\varphi(x)] d^4 x. \tag{60}$$

The Lagrangian is then the spatial integral of the Lagrangian density.

$$\mathcal{L} = \mathcal{L}(\varphi, \partial\varphi, \partial\partial\varphi, \ldots, x) \tag{61}$$

9 Noether's theorem—not finished

We know that if the Lagrangian is independent of a coordinate q_i (i.e., q_k is a ignorable coordinate), the corresponding canonical momentum p_k will be conserved. The absence of the ignorable coordinate q_k from the Lagrangian implies that the Lagrangian is unaffected by a change or transformation of q_k. This means the Lagrangian exhibit a symmetry under the transformation. This is the seed idea generalized in Noether's theorem.

$$\mathcal{L} = \mathcal{L}(q_1, q_2, \ldots, q_k, \ldots q_{\mathcal{F}}, \dot{q}_1, \ldots, \dot{q}_k, \ldots., \dot{q}_{\mathcal{F}}, t) \tag{62}$$

If there exists a transformation $q_i = q_i(s)$ $\dot{q}_i = \dot{q}_i(s)$ with $i = 1, \ldots., \mathcal{F}$ and the Lagrangian $\mathcal{L}$ is invariant under this transformation, that is

$$\frac{d\mathcal{L}}{ds} = 0. \tag{63}$$

Then it is easy to prove that C defined in the following is conserved,

$$C = \sum_i p_i \frac{dq_i(s)}{ds}. \tag{64}$$

Proof

$$\begin{aligned}
\frac{dC}{dt} &= \frac{d}{dt}\sum_i p_i \frac{dq_i(s)}{ds} \\
&= \sum_i \left[p_i \frac{d}{dt}\left(\frac{dq_i(s)}{ds}\right) + \frac{dq_i(s)}{ds}\frac{d}{dt}p_i \right] \\
&= \sum_i \left[p_i \frac{d}{dt}\left(\frac{dq_i(s)}{ds}\right) + \frac{dq_i(s)}{ds}\frac{\partial\mathcal{L}}{\partial q_i} \right] \\
&= \sum_i \left[p_i \frac{d}{ds}\left(\frac{dq_i(s)}{dt}\right) + \frac{dq_i(s)}{ds}\frac{\partial\mathcal{L}}{\partial q_i} \right] \\
&= \sum_i \left[\frac{\partial\mathcal{L}}{\partial \dot{q}_i}\frac{d\dot{q}_i}{ds} + \frac{dq_i(s)}{ds}\frac{\partial\mathcal{L}}{\partial q_i} \right] \\
&= \frac{d\mathcal{L}}{ds} \\
&= 0
\end{aligned} \tag{65}$$

QED.

10 Poisson bracket—not finished

Hamilton's equations are

$$\dot{p_I} = -\frac{\partial \mathcal{H}}{\partial q_I} \tag{66}$$

and

$$\dot{q_I} = \frac{\partial \mathcal{H}}{\partial p_I}. \tag{67}$$

Define the Poisson brackets of two functions $f(\boldsymbol{q}, \boldsymbol{p})$ and $g(\boldsymbol{q}, \boldsymbol{p})$,

$$\{f, g\} \equiv \sum_i \left(\frac{\partial f}{\partial q_i} \frac{\partial g}{\partial p_i} - \frac{\partial f}{\partial p_i} \frac{\partial g}{\partial q_i} \right), \tag{68}$$

then Hamilton's equations are written in symmetrical form

$$\dot{q_I} = \{q_I, \mathcal{H}\} \tag{69}$$

and

$$\dot{p_I} = \{p_I, \mathcal{H}\}. \tag{70}$$

More generally, for any function $f = f(\boldsymbol{q}, \boldsymbol{p})$, we obtain

$$\dot{f} = \{f, \mathcal{H}\}. \tag{71}$$

If f explicitly depends on time, i.e., $f = f(\boldsymbol{q}, \boldsymbol{p}, t)$, then

$$\dot{f} = \{f, \mathcal{H}\} + \frac{\partial f}{\partial t}. \tag{72}$$

10.1 Properties of Poisson bracket

From the definition of the Poisson bracket, it is easy to prove that

$$\{f, g\} = -\{g, f\}, \tag{73}$$

$$\{f, g + h\} = \{f, g\} + \{f, h\}, \tag{74}$$

$$\{f + h, g\} = \{f, g\} + \{h, g\}, \tag{75}$$

and

$$\{f, gh\} = \{f, g\}h + \{f, h\}g. \tag{76}$$

Jacobi identity for the Poisson bracket is (I do not check this ind entity)

$$\{f, \{g, h\}\} + \{g, \{h, f\}\} + \{h, \{f, g\}\} = 0. \tag{77}$$

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