Computational electromagnetics

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1 Maxwell's equations as time evolution system

Time evolution of the magnetic field $\mathbf{B}$ is governed by Faraday's law,

$$\frac{\partial \mathbf{B}}{\partial t} = -\nabla \times \mathbf{E} \tag{1}$$

Time evolution of the electric field $\mathbf{E}$ is governed by Ampere's law,

$$\mu_0 \varepsilon_0 \frac{\partial \mathbf{E}}{\partial t} = \nabla \times \mathbf{B} - \mu_0 \mathbf{J}, \tag{2}$$

Time evolution of the current $\mathbf{J}$ is determined by the media in which the electromagnetic field lies. This relation can be symbolically written as

$$\frac{\partial \mathbf{J}}{\partial t} = F(\mathbf{E}, \mathbf{B}, t), \tag{3}$$

where F is a function determined by the properties of the media. The function F can be an integral-differential operator. Equation (3) is a general constitutive equation of media.

The other two Maxwell's equation are considered as the requirement for the initial value of $\mathbf{E}$ and $\mathbf{B}$. The spatial structure of the initial magnetic field must satisfy the divergence-free condition, i.e.,

$$\nabla \cdot \mathbf{B} = 0, \tag{4}$$

The spatial structure of the initial electric field must satisfy Gauss' law

$$\nabla \cdot \mathbf{E} = \frac{\rho_c}{\varepsilon_0}, \tag{5}$$

where the charge density ρ_c, like $\mathbf{J}$, is determined by the media. Specifically, the charge density and current density given by a media satisfy the charge continuity equation, i.e.,

$$\frac{\partial \rho_c}{\partial t} = -\nabla \cdot \mathbf{J}. \tag{6}$$

Once the time evolution of $\mathbf{J}$ is determined, ρ_q is determined by Eq. (6) and thus can be considered as a derived quantity. Therefore, the electromagnetic property of a media is fully determined by the constitutive relation Eq. (3).

1.1 Self-consistency check from the view of initial value problem

It is ready to verify that the Gauss' laws, Eq. (4) and (5), are guaranteed at later time if they are initially satisfied. In fact, the divergence of Faraday's law (1) is written

$$\frac{\partial \nabla \cdot \mathbf{B}}{\partial t} = -\nabla \cdot (\nabla \times \mathbf{E}), \tag{7}$$

i.e.,

$$\frac{\partial \nabla \cdot \mathbf{B}}{\partial t} = 0. \tag{8}$$

which implies that $\nabla \cdot \mathbf{B} = 0$ will hold at later time if it is satisfied at the initial time.

The divergence of Ampere's law is written

$$\mu_0\varepsilon_0\frac{\partial\nabla\cdot\mathbf{E}}{\partial t}=0-\mu_0\nabla\cdot\mathbf{J}. \tag{9}$$

Using the charge continuity equation

$$\nabla\cdot\mathbf{J}=-\frac{\partial\rho_c}{\partial t}, \tag{10}$$

in Eq. (9), we obtain

$$\frac{\partial}{\partial t}(\varepsilon_0\nabla\cdot\mathbf{E}-\rho_q)=0, \tag{11}$$

which implies that $\varepsilon_0\nabla\cdot\mathbf{E}-\rho_c=0$ will hod at later time if it is satisfied at the initial time.

1.2 Keeping the magnetic field divergence-free in numerical methods

When discretizing Maxwell's equations in time and space, there are two kinds of methods to guarantee that the magnetic field is divergence free at later time. One is the divergence-cleaning schemes and another is to use staggered mesh scheme, e.g. Yee cells, which ensure the resulting numerical scheme does not change the divergence of the field.

1.3 One-dimension (in Cartesian coordinates) electromagnetic wave in vacuum : two TEM modes are decoupled

One-dimensional system is a system in which all quantities depends on only one spatial coordinate. Consider the one-dimensional case in Cartesian coordinates and assume the corresponding coordinate is x. Then we have

$$\nabla\times\mathbf{E}=\begin{vmatrix} i & j & k\\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z}\\ E_x(x) & E_y(x) & E_z(x)\end{vmatrix}=-\frac{\partial E_z}{\partial x}e_y+\frac{\partial E_y}{\partial x}e_z,$$

and

$$\nabla\times\mathbf{B}=\begin{vmatrix} i & j & k\\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z}\\ B_x(x) & B_y(x) & B_z(x)\end{vmatrix}=-\frac{\partial B_z}{\partial x}e_y+\frac{\partial B_y}{\partial x}e_z$$

In vacuum, there is no current and charge density, i.e., $\mathbf{J}=0$ and $\rho_q=0$. Then, the equations system (1) and (2) are decoupled into two independent systems. The first system is

$$\frac{\partial B_y}{\partial t}=\frac{\partial E_z}{\partial x}, \tag{12}$$

$$\mu_0\varepsilon_0\frac{\partial E_z}{\partial t}=\frac{\partial B_y}{\partial x}. \tag{13}$$

The second is

$$-\frac{\partial B_z}{\partial t}=\frac{\partial E_y}{\partial x} \tag{14}$$

$$\mu_0\varepsilon_0\frac{\partial E_y}{\partial t}=-\frac{\partial B_z}{\partial x} \tag{15}$$

Note that, for both the systems, the electrical filed and magnetic field are perpendicular (transverse) to the x direction, therefore they are called Transverse Electromagnetic (TEM) modes.

1.4 Two dimension (in Cartesian coordinates) vacuum electromagnetic wave: TM and TE modes are decoupled

If the system is two-dimensional, i.e., all quantities depends on only two variables, e.g., x and y. Then the curl of $\mathbf{E}$ is written as

$$\nabla\times\mathbf{E}=\begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ E_x(x,y) & E_y(x,y) & E_z(x,y)\end{vmatrix}=e_x\frac{\partial E_z}{\partial y}-e_y\frac{\partial E_z}{\partial x}+e_z\left[\frac{\partial E_y}{\partial x}-\frac{\partial E_x}{\partial y}\right],$$

and similar for the curl of $\mathbf{B}$. In this case, the system Eqs. (1) and (2) is also decoupled into two independent systems. The first system is

$$\mu_0\varepsilon_0\frac{\partial E_z}{\partial t}=\frac{\partial B_y}{\partial x}-\frac{\partial B_x}{\partial y}, \tag{16}$$

$$-\frac{\partial B_x}{\partial t}=\frac{\partial E_z}{\partial y}, \tag{17}$$

$$-\frac{\partial B_y}{\partial t}=-\frac{\partial E_z}{\partial x}. \tag{18}$$

The modes described by these three equations are called TM modes because the magnetic field is transverse with respect to the z direction.

The second system is

$$-\frac{\partial B_z}{\partial t}=\frac{\partial E_y}{\partial x}-\frac{\partial E_x}{\partial y}, \tag{19}$$

$$\mu_0\varepsilon_0\frac{\partial E_x}{\partial t}=\frac{\partial B_z}{\partial y}, \tag{20}$$

$$\mu_0\varepsilon_0\frac{\partial E_y}{\partial t}=-\frac{\partial B_z}{\partial x}. \tag{21}$$

The modes described by these three equations are called TE modes because the electrical field is transverse with respect to the z direction.

1.5 Three-Dimension

For three-dimension problems, Maxwell's equations system can not be exactly decoupled to subsystems: all the components are coupled together. Using

$$\nabla\times\mathbf{E}=\begin{pmatrix} \mathbf{e}_x & \mathbf{e}_y & \mathbf{e}_z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ E_x & E_y & E_z\end{pmatrix}=\mathbf{e}_x\left(\frac{\partial E_z}{\partial y}-\frac{\partial E_y}{\partial z}\right)+\mathbf{e}_y\left(\frac{\partial E_x}{\partial z}-\frac{\partial E_z}{\partial x}\right)+\mathbf{e}_z\left(\frac{\partial E_y}{\partial x}-\frac{\partial E_x}{\partial y}\right), \tag{22}$$

and

$$\nabla\times\mathbf{B}=\begin{pmatrix} \mathbf{e}_x & \mathbf{e}_y & \mathbf{e}_z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ B_x & B_y & B_z\end{pmatrix}=\mathbf{e}_x\left(\frac{\partial B_z}{\partial y}-\frac{\partial B_y}{\partial z}\right)+\mathbf{e}_y\left(\frac{\partial B_x}{\partial z}-\frac{\partial B_z}{\partial x}\right)+\mathbf{e}_z\left(\frac{\partial B_y}{\partial x}-\frac{\partial B_x}{\partial y}\right), \tag{23}$$

the three component equations of Faraday's law, $\frac{\partial \mathbf{B}}{\partial t}=-\nabla\times\mathbf{E}$, are written

$$\frac{\partial B_x}{\partial t}=\frac{\partial E_y}{\partial z}-\frac{\partial E_z}{\partial y}, \tag{24}$$

$$\frac{\partial B_y}{\partial t}=\frac{\partial E_z}{\partial x}-\frac{\partial E_x}{\partial z}, \tag{25}$$

$$\frac{\partial B_z}{\partial t}=\frac{\partial E_x}{\partial y}-\frac{\partial E_y}{\partial x}, \tag{26}$$

and the component equations of Ampere's law, $\mu_0\varepsilon_0\frac{\partial \mathbf{E}}{\partial t} = \nabla \times \mathbf{B} - \mu_0 \mathbf{J}$, are written

$$\mu_0\varepsilon_0\frac{\partial E_x}{\partial t} = \frac{\partial B_z}{\partial y} - \frac{\partial B_y}{\partial z} - \mu_0 J_x, \tag{27}$$

$$\mu_0\varepsilon_0\frac{\partial E_y}{\partial t} = \frac{\partial B_x}{\partial z} - \frac{\partial B_z}{\partial x} - \mu_0 J_y, \tag{28}$$

$$\mu_0\varepsilon_0\frac{\partial E_z}{\partial t} = \frac{\partial B_y}{\partial x} - \frac{\partial B_x}{\partial y} - \mu_0 J_z, \tag{29}$$

1.6 Particle-in-cell simulation of waves in uniform magnetized plasmas

Assume the uniform equilibrium magnetic field is along $\mathbf{e}_z$ direction.

1.6.1 Parallel field equation

Taking time derivative on both sides of Eq. (29), we obtain

$$\mu_0\varepsilon_0\frac{\partial^2 E_z}{\partial t^2} = \frac{\partial^2 E_z}{\partial x^2} - \frac{\partial^2 E_x}{\partial z \partial x} - \frac{\partial^2 E_y}{\partial z \partial y} + \frac{\partial^2 E_z}{\partial y^2} - \mu_0 \frac{\partial J_z}{\partial t}, \tag{30}$$

where use has been made of Eqs. () and () to eliminate B_y and B_x.

1.6.2 Perpendicular field equations

Using the following implicit scheme for Eqs. ()-():

$$\frac{B_x^{(n+1)} - B_x^{(n)}}{\Delta t} = \left(\frac{\partial E_y}{\partial z}\right)^{(n+1)} - \left(\frac{\partial E_z}{\partial y}\right)^{(n+1)}, \tag{31}$$

$$\frac{B_y^{(n+1)} - B_y^{(n)}}{\Delta t} = \left(\frac{\partial E_z}{\partial x}\right)^{(n+1)} - \left(\frac{\partial E_x}{\partial z}\right)^{(n+1)}, \tag{32}$$

$$\frac{B_z^{(n+1)} - B_z^{(n)}}{\Delta t} = \left(\frac{\partial E_x}{\partial y}\right)^{(n+1)} - \left(\frac{\partial E_y}{\partial x}\right)^{(n+1)}, \tag{33}$$

where Δt is the length of time-step, the superscripts n indicate the current time step and $n+1$ future time step. Applying the same implicit scheme as above to Eqs. ()-(), we obtain

$$\mu_0\varepsilon_0\frac{E_x^{(n+1)} - E_x^{(n)}}{\Delta t} = \left(\frac{\partial B_z}{\partial y}\right)^{(n+1)} - \left(\frac{\partial B_y}{\partial z}\right)^{(n+1)} - \mu_0 J_x^{(n+1)}, \tag{34}$$

$$\mu_0\varepsilon_0\frac{E_y^{(n+1)} - E_y^{(n)}}{\Delta t} = \left(\frac{\partial B_x}{\partial z}\right)^{(n+1)} - \left(\frac{\partial B_z}{\partial x}\right)^{(n+1)} - \mu_0 J_y^{(n+1)}, \tag{35}$$

Using Eqs. ()-() to eliminate $B_x^{(n+1)}$, $B_y^{(n+1)}$, and $B_z^{(n+1)}$, we obtain

$$\mu_0\varepsilon_0\frac{E_x^{(n+1)} - E_x^{(n)}}{\Delta t} = \frac{\partial}{\partial y}\left[B_z^{(n)} + \Delta t\left(\frac{\partial E_x}{\partial y}\right)^{(n+1)} - \Delta t\left(\frac{\partial E_y}{\partial x}\right)^{(n+1)}\right] - \frac{\partial}{\partial z}\left[B_y^{(n)} + \Delta t\left(\frac{\partial E_z}{\partial x}\right)^{(n+1)} - \Delta t\left(\frac{\partial E_x}{\partial z}\right)^{(n+1)}\right] - \mu_0 J_x^{(n+1)}, \tag{36}$$

$$\mu_0\varepsilon_0\frac{E_y^{(n+1)} - E_y^{(n)}}{\Delta t} = \frac{\partial}{\partial z}\left[B_x^{(n)} + \Delta t\left(\frac{\partial E_y}{\partial z}\right)^{(n+1)} - \Delta t\left(\frac{\partial E_z}{\partial y}\right)^{(n+1)}\right] - \frac{\partial}{\partial x}\left[B_z^{(n)} + \Delta t\left(\frac{\partial E_x}{\partial y}\right)^{(n+1)} - \Delta t\left(\frac{\partial E_y}{\partial x}\right)^{(n+1)}\right] - \mu_0 J_y^{(n+1)}, \tag{37}$$

which can be organized as

$$\Delta t\left(\frac{\partial^2 E_z}{\partial x\partial z}\right)^{(n+1)} - \Delta t\left(\frac{\partial^2 E_x}{\partial z^2}\right)^{(n+1)} - \left[\Delta t\left(\frac{\partial^2 E_x}{\partial y^2}\right)^{(n+1)} - \Delta t\left(\frac{\partial^2 E_y}{\partial x\partial y}\right)^{(n+1)}\right] + \frac{\mu_0\varepsilon_0}{\Delta t}E_x^{(n+1)} = \frac{\mu_0\varepsilon_0}{\Delta t}E_x^{(n)} + \frac{\partial B_z^{(n)}}{\partial y} - \frac{\partial B_y^{(n)}}{\partial z} - \mu_0 J_x^{(n+1)}, \tag{38}$$

$$\Delta t\left(\frac{\partial^2 E_x}{\partial x\partial y}\right)^{(n+1)} - \Delta t\left(\frac{\partial^2 E_y}{\partial x^2}\right)^{(n+1)} - \left[\Delta t\left(\frac{\partial^2 E_y}{\partial z^2}\right)^{(n+1)} - \Delta t\left(\frac{\partial^2 E_z}{\partial y\partial z}\right)^{(n+1)}\right] + \frac{\mu_0\varepsilon_0}{\Delta t}E_y^{(n+1)} = \frac{\mu_0\varepsilon_0}{\Delta t}E_y^{(n)} + \frac{\partial B_x^{(n)}}{\partial z} - \frac{\partial B_z^{(n)}}{\partial x} - \mu_0 J_y^{(n+1)}, \tag{39}$$

2 Potential form of Maxwell's equations

In the above, **E** and **B** are used as field variables in presenting the formulas. For simulations of low-frequency phenomena in plasmas, it is often convenient to use the potential form of Maxwell's equation. Next, let us first review the basic of the potential form. Gauss's law for magnetism, Eq. (4), is automatically satisfied if we write **B** as

$$\mathbf{B} = \nabla \times \mathbf{A}, \tag{40}$$

where **A** is called the vector potential. Similarly, Faraday's law (1) is automatically satisfied if we write **E** as

$$\mathbf{E} = -\nabla\varphi - \frac{\partial \mathbf{A}}{\partial t}, \tag{41}$$

where φ is called the scalar potential. Using equations (40) and (41), Ampere's law (2) is written

$$\mu_0\varepsilon_0\left(\frac{\partial \nabla\varphi}{\partial t} + \frac{\partial^2 \mathbf{A}}{\partial t^2}\right) + \nabla \times \nabla \times \mathbf{A} = \mu_0 \mathbf{J}. \tag{42}$$

Using the vector identity $\nabla \times \nabla \times \mathbf{A} = \nabla(\nabla \cdot \mathbf{A}) - \nabla^2 \mathbf{A}$, equation (42) is written

$$\mu_0\varepsilon_0\frac{\partial^2 \mathbf{A}}{\partial t^2} + \nabla\left(\mu_0\varepsilon_0\frac{\partial\varphi}{\partial t} + \nabla\cdot\mathbf{A}\right) - \nabla^2\mathbf{A} = \mu_0\mathbf{J}. \tag{43}$$

Similarly, using the potential form of the electric field, Gauss's law (5) is written as

$$-\nabla^2\varphi - \frac{\partial \nabla\cdot\mathbf{A}}{\partial t} = \frac{\rho_c}{\varepsilon_0}. \tag{44}$$

2.1 Lorenz gauge

Choose the Lorenz gauge, i.e.,

$$\nabla\cdot\mathbf{A} = -\mu_0\varepsilon_0\frac{\partial\varphi}{\partial t} \tag{45}$$

then Ampere's law (43) is reduced to

$$\mu_0\varepsilon_0\frac{\partial^2\mathbf{A}}{\partial t^2} - \nabla^2\mathbf{A} = \mu_0\mathbf{J}, \tag{46}$$

and Gauss's law (44) is reduced to

$$\mu_0\varepsilon_0\frac{\partial^2\varphi}{\partial t^2} - \nabla^2\varphi = \frac{\rho_c}{\varepsilon_0}. \tag{47}$$

2.2 Coulomb gauge

Choose the Coulomb gauge, i.e., $\nabla \cdot \mathbf{A} = 0$, then Ampere's law (43) is reduced to

$$\mu_0 \varepsilon_0 \frac{\partial^2 \mathbf{A}}{\partial t^2} - \nabla^2 \mathbf{A} = \mu_0 \mathbf{J} - \mu_0 \varepsilon_0 \nabla \left(\frac{\partial \varphi}{\partial t} \right), \tag{48}$$

and Gauss's law (44) is reduced to

$$-\nabla^2 \varphi = \frac{\rho_c}{\varepsilon_0}. \tag{49}$$

2.2.1 Darwin approximation (Magnetoinductive approximation)

For phenomena of low frequency with $\omega \ll kc$, where ω and k are typical frequency and wavenumber of the phenomena, it is obvious that the first term in Eq. (48) is much smaller than the second term. Dropping the first term, equation (48) is written as

$$-\nabla^2 \mathbf{A} = \mu_0 \mathbf{J} - \mu_0 \varepsilon_0 \nabla \left(\frac{\partial \varphi}{\partial t} \right). \tag{50}$$

This form is called Darwin approximation or magnetoinductive approximation. Another way of explaining this approximation is that it drops the divergence-free part of the displacement current [Proof: The displacement current is $\mu_0 \varepsilon_0 \partial \mathbf{E} / \partial t$, which, in terms of the potentials, is written as $-\mu_0 \varepsilon_0 \nabla(\partial \varphi / \partial t) - \mu_0 \varepsilon_0 \partial^2 \mathbf{A} / \partial t^2$, where the last term is divergence-free in Coulomb gauge.) This approximation eliminates vacuum light waves from the system. [Proof: In vacuum, the original system has the dispersion relation $-\omega^2 / c^2 + k^2 = 0$, which is the dispersion relation of vacuum light waves, whereas the dispersion relation of the modified system in vacuum is $k^2 = 0$, i.e., no vacuum light waves are permitted.) Ref. [3] provided a clear explanation for the Darwin approximation.

If the displacement current is totally neglected, then Ampere's law (48) is reduced to

$$-\nabla^2 \mathbf{A} = \mu_0 \mathbf{J}, \tag{51}$$

which is a more rough approximation compared with Darwin approximation. Equation (51) is usually adopted in gyrokinetic simulations of tokamak micro-instabilities. The advantage of using potential forms with Coulomb gauge for low-frequency modes is that the time derivative terms in Maxwellian equations disappear, yielding field equations in only space.

2.2.2 On electromagnetic PIC simulation of plasmas

As is discussed above, working in the Coulomb gauge and dropping the displacement current, Maxwell's equations reduce to Eqs. (49) and (51) with the $\mathbf{E}$ and $\mathbf{B}$ given by Eqs. (40) and (41), respectively. That is

$$-\nabla^2 \delta \varphi = \frac{\delta \rho_c}{\varepsilon_0}, \tag{52}$$

$$-\nabla^2 \delta \mathbf{A} = \mu_0 \delta \mathbf{J}, \tag{53}$$

with $\delta \mathbf{E}$ and $\delta \mathbf{B}$ given by

$$\delta \mathbf{B} = \nabla \times \delta \mathbf{A}, \tag{54}$$

$$\delta \mathbf{E} = -\nabla \delta \varphi - \frac{\partial \delta \mathbf{A}}{\partial t}, \tag{55}$$

The above electromagnetic fields are assumed to be perturbations relative to background fields in a plasma.

In the so-called electrostatic simulation, the effect of the vector potential $\delta \mathbf{A}$ on the plasma is neglected, which means that the contribution of $\delta \mathbf{A}$ in both Eqs. (54) and (55) are neglected, i.e., particles in plasmas respond to only an electric field given by $\delta \mathbf{E} = -\nabla \delta \varphi$. In this case, we do not need to solve Eq. (53) since we do not need $\delta \mathbf{A}$ in computing the perturbed field.

In the electromagnetic simulation, we retain the contribution from both $\delta\varphi$ and $\delta\mathbf{A}$. Therefore we need to solve both Eqs. (52) and (53). Before solving these two equations, we need to compute the source terms (right-hand side of the equations). In PIC simulations, we can compute both the density $\delta\rho_c$ and $\delta\mathbf{J}$ by depositing markers onto grids. The resulting $\delta\rho_c$ and $\delta\mathbf{J}$ should satisfy the continuity equation if we computed the them from an accurate distribution function. However, due to noise or error in PIC simulations, $\delta\rho_c$ and $\delta\mathbf{J}$ computed directly from the marker distribution will significantly deviate from the continuity equation. This inconsistency between $\delta\rho_c$ and $\delta\mathbf{J}$ will result in inconsistency in $\delta\varphi$ and $\delta\mathbf{A}$ when we solve them from Eqs. (52) and (53), respectively. This inconsistency will finally give significant error in $\delta\mathbf{E}$ when we calculate $\delta\mathbf{E}$ via Eq. (55), which involves the difference between $\delta\varphi$ and $\delta\mathbf{A}$. This is one of the "cancellation problems" in electromagnetic gyrokinetic PIC simulation of tokamak plasmas[2, 1]. One obvious method of solving this cancellation problem is to first compute $\delta\mathbf{J}$ via depositing markers and then use continuity equation to obtain $\delta\rho_c$. This guarantees that $\delta\mathbf{J}$ and $\delta\rho_c$ satisfy the continuity equation.

3 Responce of dielectric media to electric field

Any media that has a response to an applied electric field can be called a dielectric media (similarly, any media that has a response to an applied magnetic field is called magnetic media). Dielectric media is usually considered as an insulator, in which there is no free charge, i.e., all charges are bound to their initial equilibrium position by a restoring force. A conductor can be considered as a special kind of dielectric media in which the restoring force is zero. When a dielectric insulator is placed in an electric field, electric charges are displaced from their average equilibrium positions, which produces electric dipoles at molecule length scale. This generation of electric dipoles in dielectric media by electric field is called polarization. The degree of polarization can be characterized by the density of electric-dipoles, $\mathbf{P}$, which is defined by

$$\mathbf{P} = \sum_j n_j q_j \mathbf{s}_j, \tag{56}$$

where j is the species label, $\mathbf{s}_j$ is the average displace vector of the charge q_j from their original equilibrium location, n_j is the number density. Next let us derive the relation between the applied electric field and the resulting $\mathbf{P}$.

Since the mass of electrons are much lighter than that of atomic nuclei, the response of a dielectric to an applied electric field is dominated by electrons. Thus, to a first approximation, the polarization $\mathbf{P}$ is determined by the electron response, i.e., $\mathbf{P}$ can be written as

$$\mathbf{P} = -e n_e \mathbf{s} \tag{57}$$

Let us calculate the displacement $\mathbf{s}$ due to an electric field $\mathbf{E}$. Let us assume that the electrons are bound "quasi-elastically" to their rest positions, i.e., when the electron is displaced by $\mathbf{s}$ from the rest position, the restoring force is given by $\mathbf{F}_{\text{restore}} = -f\mathbf{s}$. Further assume there is a friction force proportional to the velocity of electrons, $\mathbf{F}_{\text{frict}} = m\gamma_0\dot{\mathbf{s}}$. Then the equation of motion for electrons is written

$$\ddot{\mathbf{s}} = -\omega_0^2\mathbf{s} - \frac{e\mathbf{E}}{m} - \gamma_0\dot{\mathbf{s}}, \tag{58}$$

where $\omega_0 = \sqrt{f/m}$ is the characteristic oscillation frequency of the electrons. Fourier transform both sides of Eq. (58), yielding

$$-\omega^2\hat{\mathbf{s}} = -\omega_0^2\hat{\mathbf{s}} - \frac{e\hat{\mathbf{E}}}{m} + i\gamma_0\omega\hat{\mathbf{s}}, \tag{59}$$

from which we obtain

$$\hat{\mathbf{s}} = \frac{-e\hat{\mathbf{E}}/m}{\omega_0^2 - \omega^2 - i\gamma_0\omega} \tag{60}$$

Then the Fourier transform of $\mathbf{P}$ is written as

$$\hat{\mathbf{P}} = \frac{n_e e^2\hat{\mathbf{E}}/m}{\omega_0^2 - \omega^2 - i\gamma_0\omega} \tag{61}$$

In general cases

$$\hat{\mathbf{P}} = \varepsilon_0 \boldsymbol{\chi}_e \cdot \hat{\mathbf{E}}, \tag{62}$$

where $\boldsymbol{\chi}_e$ is called electric susceptibility, which is a tensor for an-isotropic media. In our case, comparing Eqs. (61) with (62), we obtain

$$\chi_e = \frac{n_e e^2 / (m\varepsilon_0)}{\omega_0^2 - \omega^2 - i\gamma_0\omega}. \tag{63}$$

3.1 Polarization charge and polarization current

It can be proved that the charge density associated with the electric-dipole density $\mathbf{P}$ is the divergence of $\mathbf{P}$, i.e., (refer to Richard Fitzpatrick's book)

$$\rho_{qb} = -\nabla \cdot \mathbf{P}. \tag{64}$$

It can be proved that the time-variation of $\mathbf{P}$ gives rise to a current given by (I do not prove this)

$$\mathbf{j}_b = \frac{\partial \mathbf{P}}{\partial t}, \tag{65}$$

which is called the polarization current. Equations (64) and (65) imply that

$$\frac{\partial \rho_{qb}}{\partial t} + \nabla \cdot \mathbf{j}_b = 0, \tag{66}$$

i.e., ρ_{qb} and $\mathbf{j}_b$ satisfy the charge conservation.

3.2 Electric conductivity of dielectric media

In dielectric media, all the charge can be considered as polarization charge and all the current can be considered as polarization current. Let us derive the relation between the electric conductivity and the electric susceptibility of a dielectric media. Fourier transform both sides of Eq. (65), yielding

$$\hat{\mathbf{j}}_b = -i\omega \hat{\mathbf{P}}. \tag{67}$$

Using Eq. (62), the above equation is written as

$$\hat{\mathbf{J}}_b = -i\omega\varepsilon_0 \boldsymbol{\chi}_e \cdot \hat{\mathbf{E}}, \tag{68}$$

from which we see the conductivity is given by

$$\boldsymbol{\sigma} = -i\omega\varepsilon_0 \boldsymbol{\chi}_e \tag{69}$$

In our case, using Eq. (63), Eq. (69) is written

$$\sigma = -i\omega\varepsilon_0 \frac{n_e e^2 / (m\varepsilon_0)}{\omega_0^2 - \omega^2 - i\gamma_0\omega} \tag{70}$$

In the limit $\omega_0 \to 0$, i.e., zero restoring force, i.e., conductor, the above equation is written

$$\sigma = -i\varepsilon_0 \frac{n_e e^2 / (m\varepsilon_0)}{{}^2 - \omega - i\gamma_0}. \tag{71}$$

Plasma can be considered as a dielectric media, i.e., all the charge in plasma is considered as polarization charge and all the current in plasma is considered as polarization current. There are many characteristic frequencies ω_0 in plasmas. For example, in magnetized plasmas, the cyclotron frequency $\omega_c = B_0 e / B$ is a characteristic frequency for the response to an electric field perpendicular to the equilibrium magnetic field.

4 Maxwell's equations in dielectric media

Using this, Gauss's law in dielectric media is written as

$$\varepsilon_0 \nabla \cdot \mathbf{E} = \rho_{qf} - \nabla \cdot \mathbf{P}, \tag{72}$$

which can be written as

$$\nabla \cdot (\varepsilon_0 \mathbf{E} + \mathbf{P}) = \rho_{qf} \tag{73}$$

Define

$$\mathbf{D} = \varepsilon_0 \mathbf{E} + \mathbf{P}, \tag{74}$$

then Eq. (73) is written

$$\nabla \cdot \mathbf{D} = \rho_{qf} \tag{75}$$

Using Eq. (), $\mathbf{D}$ is written as

$$\mathbf{D} = \varepsilon_0 (1 + \chi_e) \mathbf{E} \tag{76}$$

Define

$$\varepsilon = 1 + \chi_e, \tag{77}$$

which is known as the relative permittivity of the medium (or dielectric constant), then $\mathbf{D}$ is written as

$$\mathbf{D} = \varepsilon_0 \varepsilon \mathbf{E}. \tag{78}$$

5 Antenna

electrical impedance

Impedance is defined as the frequency domain ratio of the voltage to the current

6 Approximations

For some cases, the displacement current term $\mu_0 \varepsilon_0 \partial \mathbf{E} / \partial t$ is negligibly small compared with the conduction current term $\mu_0 \mathbf{J}$. For these cases, the displacement current term can be dropped and the Ampere's law is written as

$$\nabla \times \mathbf{B} = \mu_0 \mathbf{J}. \tag{79}$$

Integrating Eq. (79) over a surface S, we obtain

$$\int \int_S \nabla \times \mathbf{B} \cdot d\mathbf{S} = \mu_0 \int \int_S \mathbf{J} \cdot d\mathbf{S}, \tag{80}$$

which can be written as

$$\oint \mathbf{B} \cdot d\mathbf{l} = \mu_0 \int \int_S \mathbf{J} \cdot d\mathbf{S}, \tag{81}$$

where the line integration is along the boundary loop C of the surface S. Since Eq. (79) implies $\nabla \cdot \mathbf{J} = 0$, we know that the surface integration of $\mathbf{J}$ over any surface attached to loop C is equal to each other. Use I to denote this electric current, then Eq. (81) is written

$$\oint \mathbf{B} \cdot d\mathbf{l} = \mu_0 I \tag{82}$$

This equation is called Ampere's circuital law.

7 Connection condition

$$\nabla \cdot \mathbf{B} = 0, \tag{83}$$

then we have

$$\int_V \nabla \cdot \mathbf{B} dV = 0, \tag{84}$$

which can be further written as

$$\int_S \mathbf{B} \cdot d\mathbf{S} = 0. \tag{85}$$

Using this on a interface, we obtain

$$(\mathbf{n} \cdot \mathbf{B})_1 = (\mathbf{n} \cdot \mathbf{B})_2 \tag{86}$$

That is, the normal component of magnetic field is continuous on any interface.

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \tag{87}$$

$$\Rightarrow \int (\nabla \times \mathbf{E}) \cdot d\mathbf{S} = -\int \frac{\partial \mathbf{B}}{\partial t} \cdot d\mathbf{S} \tag{88}$$

$$\Rightarrow \int (\nabla \times \mathbf{E}) \cdot d\mathbf{S} = 0$$

$$\Rightarrow \int \mathbf{E} \cdot d\mathbf{l} = 0 \tag{89}$$

That is, the tangential component of electric field is continuous on any interface.

7.1 Taking the electrostatic limit–seems wrong!

Define the electrostatic limit as the case where the magnetic perturbation can be neglected. Let us examine how Maxwell's equations for general electromagnetic field reduce to the electrostatic limit and which equations are needed to be solved to determined the electric field in the case. Write $\mathbf{E}$, $\mathbf{B}$, $\mathbf{J}$, and ρ_c as

$$\mathbf{E} = \mathbf{E}_0 + \delta \mathbf{E}, \tag{90}$$

$$\mathbf{B} = \mathbf{B}_0 + \delta \mathbf{B}, \tag{91}$$

$$\rho_c = \rho_{c0} + \delta \rho_c, \tag{92}$$

where $\mathbf{E}_0$, $\mathbf{B}_0$, $\mathbf{J}_0$, and ρ_{q0} is a solution to the above system of equations. All the above equations are linear, except for the constitutive relation (3) where nonlinear terms can be present. Then the equations for the perturbations $\delta\mathbf{E}$, $\delta\mathbf{B}$, $\delta\mathbf{J}$, and $\delta\rho_q$ are written as

$$\frac{\partial \delta \mathbf{B}}{\partial t} = -\nabla \times \delta \mathbf{E}, \tag{93}$$

$$\mu_0 \varepsilon_0 \frac{\partial \delta \mathbf{E}}{\partial t} = \nabla \times \delta \mathbf{B} - \mu_0 \delta \mathbf{J}, \tag{94}$$

$$\frac{\partial \delta \rho_c}{\partial t} = -\nabla \cdot \delta \mathbf{J}, \tag{95}$$

which take the same form as the original equations because of the linearity of these equations. Now, take the electrostatic limit, i.e., setting $\delta\mathbf{B}$ to zero, then the above equations are written as

$$0 = -\nabla \times \delta \mathbf{E}, \tag{96}$$

$$\mu_0\varepsilon_0\frac{\partial\delta\mathbf{E}}{\partial t} = -\mu_0\delta\mathbf{J}, \tag{97}$$

$$\frac{\partial\delta\rho_c}{\partial t} = -\nabla\cdot\delta\mathbf{J}, \tag{98}$$

Equation (96) is equivalent to that $\delta\mathbf{E} = -\nabla\delta\phi$, where $\delta\phi$ is an arbitrary function. Since $\delta\phi$ is a scalar, we only need one equation in order to determine it. This equation can be obtained by taking the divergence of Eq. (97) and using Eq. (98), yielding

$$\frac{\partial}{\partial t}(\varepsilon_0\nabla\cdot\delta\mathbf{E} - \delta\rho_c) = 0,$$

Poisson's equation requires that $\varepsilon_0\nabla\cdot\delta\mathbf{E} - \delta\rho_q = 0$ at initial time. Therefore Eq. (98) is equivalent to

$$\varepsilon_0\nabla\cdot\delta\mathbf{E} - \delta\rho_q = 0$$

Using $\delta\mathbf{E} = -\nabla\delta\phi$, the above equation is written as

$$-\varepsilon_0\nabla^2\delta\phi - \delta\rho_c = 0. \tag{99}$$

This is the equation for $\delta\phi$.

Since only one component of Eq. (97), the divergence, is required to be satisfied, do we need to worry about whether the other components of Eq. (97) are satisfied?

8 Lorentz transformation

The Lorentz transformation for frames in standard configuration can be shown to be:

$$\begin{aligned} t' &= \gamma\left(t - \frac{vx}{c^2}\right) \\ x' &= \gamma(x - vt) \\ y' &= y \\ z' &= z \end{aligned}$$

where

$$\gamma = \frac{1}{\sqrt{1 - v^2/c^2}} \tag{100}$$

is called the Lorentz factor. Lorentz factor may take another form,

$$\gamma = \sqrt{1 + \frac{u^2}{c^2}} \tag{101}$$

where u is the momentum per unit rest mass, $u = p/m_0$. Using the relation $v = u/\gamma$, one can easily prove that these two form of Lorentz factor is identical.

This Lorentz transformation is often expressed in matrix form as

$$\begin{pmatrix} ct' \\ x' \\ y' \\ z' \end{pmatrix} = \begin{pmatrix} \gamma & -\beta\gamma & 0 & 0 \\ -\beta\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} ct \\ x \\ y \\ z \end{pmatrix}$$

where $\beta = v/c$.

9 Problem

Consider two particles, particle 1 and particle 2. They are observed have velocity $\mathbf{v}_1$ and $\mathbf{v}_2$ in the lab frame. What is the velocity $\mathbf{v}_1'$ of particle 1 observed in the rest frame of particle 2?

Lorentz factor γ_1' of particle 1 in the rest frame of particle 2 can be expressed as

$$\gamma_1' = \frac{1}{\sqrt{1 - v_1'^2/c^2}}$$

From this we get

$$v_1' = \frac{c\sqrt{\gamma_1'^2 - 1}}{\gamma_1'}$$

Now the problem is how to express γ_1' as the function of $\mathbf{v}_1$ and $\mathbf{v}_2$.

10 Four-vector

Four-vector differs from a vector in that it can be transformed by Lorentz transformations.

Position 4-vector for an event:

$$\mathbf{X} := (ct, x, y, z) = (X^0, X^1, X^2, X^3)$$

The 4-velocity is the rate of change of both time and space coordinates with respect to the proper time of the object, i.e. 4-velocity of an $\mathbf{X}(\tau)$ world line is defined by (Note that τ is the local time on the moving object)

$$\mathbf{U} \equiv \frac{d\mathbf{X}}{d\tau} = \frac{d\mathbf{X}}{dt}\frac{dt}{d\tau} = (c, \mathbf{v})\gamma$$

where

$$\mathbf{v} \equiv \frac{d}{dt}(X^1, X^2, X^3) = \frac{d}{dt}(x, y, z)$$

and use has been made of $dt/d\tau = \gamma$. So we have the following transformation for the 4-velocity:

$$\begin{pmatrix} c\gamma' \\ v_x'\gamma' \\ v_y'\gamma' \\ v_z'\gamma' \end{pmatrix} = \begin{pmatrix} \gamma_t & -\beta_t\gamma_t & 0 & 0 \\ -\beta_t\gamma_t & \gamma_t & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} c\gamma \\ v_x\gamma \\ v_y\gamma \\ v_z\gamma \end{pmatrix}$$

Set $\gamma \longrightarrow \gamma_1$, $\gamma' \longrightarrow \gamma_1'$, $\gamma_t \longrightarrow \gamma_2$, $\beta_t \longrightarrow \beta_2$. That is we transform the quantity of particle 1 from the lab frame to the rest frame of particle 2. The velocity of particle 2 in the lab frame is in the x direction. It follows that

$$c\gamma_1' = \gamma_2 c\gamma_1 - \beta_2\gamma_2 v_{1x}\gamma_1 \tag{102}$$

$$v_{1x}'\gamma_1' = -\beta_2\gamma_2 c\gamma_1 + \gamma_2 v_{1x}\gamma_1 \tag{103}$$

$$v_{1y}'\gamma_1' = v_{1y}\gamma_1 \tag{104}$$

$$v_{1z}'\gamma_1' = v_{1z}\gamma_1 \tag{105}$$

From Eq.(102), we get

$$c\gamma_1' = \gamma_2 c\gamma_1 - \frac{v_2}{c}\gamma_2 u_{1\parallel}$$

$$\Longrightarrow \gamma_1' = \gamma_2\gamma_1 - \frac{u_2 u_{1\parallel}}{c^2}$$

$$\Longrightarrow \gamma_1' = \gamma_2\gamma_1 - \frac{\mathbf{u}_1 \cdot \mathbf{u}_2}{c^2}, \tag{106}$$

where $\mathbf{u}_1 = \gamma_1 \boldsymbol{v}_1$, $\mathbf{u}_2 = \gamma_2 \mathbf{v}_2$. Eq.(106) is the Eq.(4a) in Karney's 1989 paper.

$$\gamma_1' = \frac{1}{\sqrt{1 - \frac{v_r^2}{c^2}}} \Rightarrow 1 - \frac{1}{\gamma_1^2} = \frac{v_r^2}{c^2}$$

11 Relativistic Kinetic Energy

The energy of a particle is given by

$$E = mc^2 = \gamma m_0 c^2 = m_0 c^2 \left(1 - \frac{v^2}{c^2}\right)^{-1/2} \tag{107}$$

Now if v is not incredibly close to c, we can use the expansion

$$(1-x)^{-1/2} = 1 + \frac{1}{2}x + \frac{3}{8}x^2 \tag{108}$$

to get

$$E \approx m_0 c^2 + \frac{1}{2} m_0 v^2 + \frac{3}{8} m_0 \frac{v^4}{c^2} + \ldots$$

We see that the second term is just the usual kinetic energy term valid for low speeds. The firs term represents "rest mass energy". Examining the above, one can represent the relativistic kinetic energy by the following

$$\text{kinetic energy} = E - m_0 c^2 \tag{109}$$

Thus

$$\text{kinetic energy} = \gamma m_0 c^2 - m_0 c^2 = m_0 c^2 (\gamma - 1) \tag{110}$$

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13 Inner product of two four-vectors

The inner product of two four-vectors $\mathbf{U}$ and $\mathbf{V}$ is defined as

$$\mathbf{U} \cdot \mathbf{V} = \eta_{\mu\gamma} U^\mu V^\gamma = (\mathbf{U} \cdot \boldsymbol{\eta}) \cdot \mathbf{V}$$

where $\boldsymbol{\eta}$ is the Minkowski metric. I prefer the following definition for $\boldsymbol{\eta}$

$$\boldsymbol{\eta} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

in which case

$$\mathbf{U} \cdot \mathbf{V} = U^0 V^0 - U^1 V^1 - U^2 V^2 - U^3 V^3.$$

An important property of the inner product is that it is invariant (that is, a scalar): a change of coordinates does not result in a change in value of the inner product.

For 4-velocity this turns out

$$\|\mathbf{U}\| \equiv \mathbf{U} \cdot \mathbf{U} = \gamma^2 c^2 - \gamma^2 v^2 = c^2$$

14 Velocity-addition formula

$$s = \frac{u + v}{1 + uv/c^2}$$

$$u_{1\parallel}'^2 + u_{1\perp}'^2 = u_{1\parallel}'^2 + u_{1\perp}^2 = (-u_2\gamma_1 + \gamma_2 u_{1\parallel})^2 + u_{1\perp}^2$$

$$u_2^2\gamma_1^2 + \gamma_2^2 u_{1\parallel}^2 - 2u_2\gamma_1\gamma_2 u_{1\parallel} + u_{1\perp}^2$$

$$u_2^2\gamma_1^2 + \gamma_2^2 u_{1\parallel}^2 - 2\mathbf{u}_1 \cdot \mathbf{u}_2\gamma_1\gamma_2 + u_{1\perp}^2$$

From Eq.(103), we get

$$u_{1\parallel}' = -\frac{v_2}{c}\gamma_2 c\gamma_1 + \gamma_2 u_{1\parallel}$$

$$\Longrightarrow u_{1\parallel}' = -u_2\gamma_1 + \gamma_2 u_{1\parallel}$$

Eqs.(104)(105) give

$$u_{1\perp}' = u_{1\perp}$$

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