This note considers solving the adjoint equation for current drive in toroidal geometry using the method of separation of variables. I will also answer the question: what approximation **must** be imposed to make the adjoint equation solvable by the technique of separation of variables.

1 Adjoint equation

In toroidal geometry and in the regime of low collisionality (i.e. the so-called "banana regime"), the adjoint equation is written as

$$\left\langle \frac{1}{f_{em}} C^l(\chi f_{em}) \right\rangle_b = e\frac{L}{\tau_b}, \tag{1}$$

(This equation agrees with Eq. (6) in Karney's paper[1].) where

$$\frac{1}{f_{em}} C^l(\chi f_{em}) = \frac{1}{f_m}[C^{e/e}(f_{em}\chi, f_{em}) + C^{e/e}(f_{em}, f_{em}\chi) + C^{e/i}(f_{em}\chi, f_i)], \tag{2}$$

is the linearized collision operator, $L = \oint dl$ is the length of magnetic field line traveled by a particle in one period of time, $\tau_b = \oint dl/v_\parallel$ is the peroid of the poloidal motion of the particle, the bounce average operator is defined by

$$\langle \ldots \rangle_b = \frac{1}{\tau_b} \oint (\ldots) \frac{dl}{v_\parallel}. \tag{3}$$

Note tha L/τ_b appearing in Eq. (1) is the average parallel velocity of the particle. For current drive problem, we are concerned with only circulating particles, since only circulating particles contribute to the current (trapped particles's contribution to the toroidal current is zero when averaged over thier obits). Thus Eq. (1) will be considered and solved in the phase space of the circulating particles. The boundary condition is set such that χf_{em} is zero on the phase space boundary seperating the circulating and trapped zones.

[We note in passing that, in Lin-Liu's paper[2], the adjoint equation takes the form

$$-\oint \frac{dl}{v_\parallel} C_e^{l+}(\chi_0) = \oint \frac{B}{\langle B^2 \rangle} dl, \tag{4}$$

which can be rewritten as

$$-\frac{1}{\tau_b} \oint \frac{dl}{v_\parallel} \frac{1}{f_{em}} C^l(\chi f_{em}) = \frac{1}{\tau_b} \oint \frac{B}{\langle B^2 \rangle} dl \tag{5}$$

$$\Rightarrow \left\langle \frac{1}{f_{em}} C^l(\chi f_{em}) \right\rangle_b = -\frac{\oint \frac{B}{\langle B^2 \rangle} dl}{\tau_b}$$

$$\Longrightarrow \left\langle \frac{1}{f_{em}} C^l(\chi f_{em}) \right\rangle_b = -\frac{\oint \frac{dl}{B}}{\tau_b} \tag{6}$$

Inspecting the above equation, we find that the left-hand side of the equation is identical to that of Eq. (1) while the right-hand side of the equation (inhomogeneous term) is different from Eq. (1) by a magnetic field factor $-1/Be$ in the integrand of the $\oint dl$ integration.]

2 Bounce averaged collision operator

The first term on the right-hand side of Eq. (2) is given by

$$\frac{1}{f_{em}} C^{e/e}(f_{em}\chi, f_{em}) = \frac{1}{u^2} \frac{\partial}{\partial u}\left(u^2 D_{cuu}^{a/b} \frac{\partial \chi}{\partial u}\right) + F_u^{a/b} \frac{\partial \chi}{\partial u} + \frac{D_{\theta\theta}}{u^2} \frac{1}{\sin\theta} \frac{\partial}{\partial \theta}\left[\sin\theta \left(\frac{\partial \chi}{\partial \theta}\right)\right]. \tag{7}$$

The third term on the right-hand side of Eq. (2) is the electron-ion collision term. This collision term is appproximated by a pitch-angle scattering operator, which can be added to the pitch-angle term of the electron-electron collision, giving a total pitch-angle scattering term. In the following, I will not discuss seperately electron-ion collision. Ion's contribution is understood as being already included in the diffusion coefficient, $D_{\theta\theta}$, of Eq. (7).

It is convenient to use the "midplane" coordinates to present the method of separation of variables. In these coordinates, the bounce averaged pitch-angle scattering operator is written as (refer to another notes for details)

$$\left\langle \frac{D_{\theta\theta}}{u^2} \frac{1}{\sin\theta} \frac{\partial}{\partial\theta}\left[\sin\theta\left(\frac{\partial\chi}{\partial\theta}\right)\right]\right\rangle_b = \frac{D_{\theta\theta}}{u_0^2} \frac{1}{\lambda} \frac{1}{\sin\theta_0} \frac{\partial}{\partial\theta_0}\left[\sin\theta_0\, \lambda \left\langle \frac{\tan^2\theta_0}{\tan^2\theta}\right\rangle_b \left(\frac{\partial\chi_0}{\partial\theta_0}\right)\right], \tag{8}$$

where $\chi_0(u_0, \theta_0) = \chi(u, \theta, l)$ and (u_0, θ_0) is related to (u, θ, l) through the energy and magnetic moment conserving rules, l is the lengh along the magnetic field line which is to label the poloidal position; $\lambda = v_0\cos\theta_0\tau_b/L$, which can be proved to be a function of only θ_0.

Assuming separation of variables, we seek solutions of the form

$$\chi_0(u_0, \theta_0) = H(\theta_0)G(u_0) \tag{9}$$

Substituting this into Eq. (8), gives

$$\left\langle \frac{D_{\theta\theta}}{u^2} \frac{1}{\sin\theta} \frac{\partial}{\partial\theta}\left[\sin\theta\left(\frac{\partial\chi}{\partial\theta}\right)\right]\right\rangle_b = \frac{D_{\theta\theta}}{u_0^2} G(u_0) \frac{1}{\lambda} \frac{1}{\sin\theta_0} \frac{\partial}{\partial\theta_0}\left[\sin\theta_0\, \lambda \left\langle \frac{\tan^2\theta_0}{\tan^2\theta}\right\rangle_b \left(\frac{\partial H(\theta_0)}{\partial\theta_0}\right)\right], \tag{10}$$

which takes the form of $g(u_0)h(\theta_0)$ with $g(u_0)$ and $h(\theta_0)$ given respectively by

$$g(u_0) = \frac{2D_{\theta\theta}}{u_0^2} G(u_0), \tag{11}$$

and

$$h(\theta_0) = \frac{1}{2}\frac{1}{\lambda} \frac{1}{\sin\theta_0} \frac{\partial}{\partial\theta_0}\left[\sin\theta_0\, \lambda \left\langle \frac{\tan^2\theta_0}{\tan^2\theta}\right\rangle_b \left(\frac{\partial H(\theta_0)}{\partial\theta_0}\right)\right]. \tag{12}$$

We now consider the rest four terms of the adjoint equation, i.e.,

$$\left\langle \frac{1}{u^2}\frac{\partial}{\partial u}\left(u^2 D_{cuu}^{a/b}\frac{\partial\chi}{\partial u}\right)\right\rangle_b + \left\langle F_u^{a/b}\frac{\partial\chi}{\partial u}\right\rangle_b + \left\langle \frac{1}{f_m} C^{e/e}(f_{em}, f_{em}\chi)\right\rangle_b - e\frac{L}{\tau_b}. \tag{13}$$

If the sum of the above four terms can be written in the form $\alpha(u_0)\beta(\theta_0)$, then, it is obvious that separation of variables will work for the adjoint equation. We now check whether this applies when we keep only the first Legendre harmonic of χ in these terms. In this case, $\chi(u, \theta, l)$ is approximated by its first Legendre harmonic

$$\chi(u, \theta, l) \approx \tilde{\chi}_1(u, l)\cos\theta. \tag{14}$$

Using this in the first term of Eq. (13) gives

$$\begin{aligned}
\left\langle \frac{1}{u^2}\frac{\partial}{\partial u}\left(u^2 D_{cuu}^{a/b}\frac{\partial\chi}{\partial u}\right)\right\rangle_b &= \frac{1}{\tau_b}\int \frac{dl}{v_\parallel}\left[\frac{1}{u^2}\frac{\partial}{\partial u}\left(u^2 D_{cuu}^{a/b}\frac{\partial\chi}{\partial u}\right)\right] \\
&\approx \frac{1}{\tau_b}\oint \frac{dl}{v_\parallel}\left[\frac{1}{u^2}\frac{\partial}{\partial u}\left(u^2 D_{cuu}^{a/b}\frac{\partial}{\partial u}\tilde{\chi}_1(u, l)\cos\theta\right)\right] \\
&= \frac{1}{\tau_b}\oint \frac{dl}{v_\parallel}\cos\theta\left[\frac{1}{u_0^2}\frac{\partial}{\partial u_0}\left(u_0^2 D_{cuu}^{a/b}\frac{\partial}{\partial u_0}\tilde{\chi}_1(u_0, l)\right)\right] \\
&= \frac{1}{\tau_b v_0}\oint dl\left[\frac{1}{u_0^2}\frac{\partial}{\partial u_0}\left(u_0^2 D_{cuu}^{a/b}\frac{\partial}{\partial u_0}\tilde{\chi}_1(u_0, l)\right)\right] \\
&= \frac{1}{\tau_b v_0}\frac{1}{u_0^2}\frac{\partial}{\partial u_0}\left[u_0^2 D_{cuu}^{a/b}\frac{\partial}{\partial u_0}\left(\int \tilde{\chi}_1(u_0, l)dl\right)\right] \\
&= \frac{1}{\tau_b v_0} A(u_0)
\end{aligned} \tag{15}$$

where $A(u_0)$ is defined as

$$A(u_0) = \frac{1}{u_0^2}\frac{\partial}{\partial u_0}\left[u_0^2 D_{cuu}^{a/b}\frac{\partial}{\partial u_0}\left(\oint \tilde{\chi}_1(u_0, l)dl\right)\right] \tag{16}$$

Similarly, the second term of Eq. (13) can be written as

$$\begin{aligned}
\left\langle F_u^{a/b}\frac{\partial\chi}{\partial u}\right\rangle_b &\approx \frac{1}{\tau_b}\oint \frac{dl}{v_\parallel}\left[F_u^{a/b}\frac{\partial\tilde{\chi}_1(u_0, l)}{\partial u_0}\cos\theta\right] \\
&= \frac{1}{\tau_b v_0} Q(u_0),
\end{aligned} \tag{17}$$

with $Q(u_0)$ given by

$$Q(u_0) = F_u^{a/b} \frac{\partial}{\partial u_0} \oint \tilde{\chi}_1(u_0, l) dl. \tag{18}$$

The third term of Eq. (13) can be written as

$$\begin{aligned}
\left\langle \frac{1}{f_m} C^{e/e}(f_{em}, f_{em}\chi) \right\rangle_b &= \frac{1}{\tau_b} \oint \frac{dl}{v_\parallel} \left[\frac{1}{f_m} C^{e/e}(f_{em}, f_{em}\chi) \right] \\
&\approx \frac{1}{\tau_b} \oint \frac{dl}{v_\parallel} \left[\frac{1}{f_m} C^{e/e}(f_{em}, f_{em}\tilde{\chi}_1(u,l)\cos\theta) \right] \\
&= \frac{1}{\tau_b} \oint \frac{dl}{v_\parallel} I_1[\tilde{\chi}_1(u,l)]\cos\theta). \qquad (19)
\end{aligned}$$

In obtaining the last equality, I make use of the fact that Legendre harmonics are eigen functions of the collision operator to write $f_{em}^{-1} C^{e/e}(f_{em}, f_{em}\tilde{\chi}_1(u,l)\cos\theta) = I_1[\tilde{\chi}_1(u,l)]\cos\theta$, here I_1 is an integration operator, which will be specified later. Eq. (19) is further written

$$\begin{aligned}
\left\langle \frac{1}{f_m} C^{e/e}(f_{em}, f_{em}\chi) \right\rangle_b &= \frac{1}{\tau_b v_0} \oint dl I_1[\tilde{\chi}_1(u_0, l)] \\
&= \frac{1}{\tau_b v_0} M(u_0), \qquad (20)
\end{aligned}$$

with $M(u_0)$ given by

$$M(u_0) = \oint dl I_1[\tilde{\chi}_1(u_0, l)] \tag{21}$$

Using Eqs. (15), (17), and (20), the sum of the rest four terms of the adjoint equation is written as

$$\frac{1}{\tau_b v_0} A(u_0) + \frac{1}{\tau_b v_0} Q(u_0) + \frac{1}{\tau_b v_0} M(u_0) - \frac{1}{\tau_b v_0} e L v_0, \tag{22}$$

which can be put in the form of $\alpha(u_0)\beta(\theta_0)$, with $\alpha(u_0)$ and $\beta(\theta_0)$ given respectively by

$$\beta(\theta_0) = \frac{1}{\tau_b v_0}, \tag{23}$$

$$\alpha(u_0) = A(u_0) + Q(u_0) + M(u_0) - e L v_0 \tag{24}$$

(It can be proved that $1/\tau_b v_0$ is a function of only θ_0. Refer to Sec. 4.) Thus the adjoint equation now takes the form

$$g(u_0)h(\theta_0) + \alpha(u_0)\beta(\theta_0) = 0, \tag{25}$$

which is obviously separable in variables u_0 and θ_0.

Examining Eq. (22), one can find the θ_0 independence is contained in τ_b, and the four terms have the same θ_0 dependence, so θ_0 dependent part can be factored out to give the form of $\alpha(\theta_0)\beta(u_0)$. It is obvious that additional different θ_0 dependence will appear when other Legendre harmonics are evaluated. So when other Legendre harmonics are included in χ, the sum of these terms can not be reduced to the form of $\alpha(\theta_0)\beta(u_0)$, thus separation of variables does not work.

3 Separation of variables

Eq. (25) can be written as

$$\frac{h(\theta_0)}{\beta(\theta_0)} = -\frac{\alpha(u_0)}{g(u_0)}, \tag{26}$$

which separates into two ordinary differential equations

$$\frac{h(\theta_0)}{\beta(\theta_0)} = -c_0 L \tag{27}$$

and

$$\frac{\alpha(u_0)}{g(u_0)} = c_0 L, \tag{28}$$

where c_0 is the separation constant.

3.1 Angular direction equation

Using Eqs. (12) and (23) in the angular equation (27), gives

$$\frac{1}{2}\frac{1}{\lambda}\frac{1}{\sin\theta_0}\frac{\partial}{\partial\theta_0}\left[\sin\theta_0\,\lambda\left\langle\frac{\tan^2\theta_0}{\tan^2\theta}\right\rangle_b\left(\frac{\partial H(\theta_0)}{\partial\theta_0}\right)\right]=-\frac{c_0 L}{\tau_b v_0} \tag{29}$$

Using $\lambda=\tau_b v_0\cos\theta_0/L$ in the above equation gives

$$\frac{1}{\sin\theta_0}\frac{\partial}{\partial\theta_0}\left[\sin\theta_0\,\lambda\left\langle\frac{\tan^2\theta_0}{\tan^2\theta}\right\rangle_b\left(\frac{\partial H(\theta_0)}{\partial\theta_0}\right)\right]=-2c_0\cos\theta_0 \tag{30}$$

$$\Longrightarrow\frac{\partial}{\partial\xi_0}\left[\sin\theta_0\,\lambda\left\langle\frac{\tan^2\theta_0}{\tan^2\theta}\right\rangle_b\left(\frac{\partial H(\theta_0)}{\partial\theta_0}\right)\right]=2c_0\xi_0, \tag{31}$$

where $\xi_0\equiv\cos\theta_0$. Integrating the above equation from $\xi_0=\xi_0$ to $\xi_0=1$, we obtain

$$\left[\sin\theta_0\,\lambda\left\langle\frac{\tan^2\theta_0}{\tan^2\theta}\right\rangle_b\left(\frac{\partial H(\theta_0)}{\partial\theta_0}\right)\right]_{\xi_0=1}-\left[\sin\theta_0\,\lambda\left\langle\frac{\tan^2\theta_0}{\tan^2\theta}\right\rangle_b\left(\frac{\partial H(\theta_0)}{\partial\theta_0}\right)\right]_{\xi_0=\xi_0}=\int_{\xi_0}^{1}2c_0\xi_0 \tag{32}$$

Noting the first term of the above equation is zero, the above equation is written as

$$\sin\theta_0\,\lambda\left\langle\frac{\tan^2\theta_0}{\tan^2\theta}\right\rangle_b\frac{\partial H(\theta_0)}{\partial\theta_0}=c_0\left(\xi_0^2-1\right) \tag{33}$$

$$\Longrightarrow\sin^2\theta_0\,\lambda\left\langle\frac{\tan^2\theta_0}{\tan^2\theta}\right\rangle_b\frac{\partial H(\theta_0)}{\partial\xi_0}=c_0\left(1-\xi_0^2\right) \tag{34}$$

$$\Longrightarrow\lambda\left\langle\frac{\tan^2\theta_0}{\tan^2\theta}\right\rangle_b\frac{\partial H(\theta_0)}{\partial\xi_0}=c_0 \tag{35}$$

Now I try to write Eq. (35) in the form adopted in Lin-Liu's paper. The main task is to express the bounce-average using the flux average and using a new angular variable, Λ, instead of θ_0. The factor on the left-hand side of Eq. (35) can be written

$$\begin{aligned}\lambda\left\langle\frac{\tan^2\theta_0}{\tan^2\theta}\right\rangle_b &= \lambda\frac{1}{\tau_b}\oint\frac{\tan^2\theta_0}{\tan^2\theta}\frac{dl}{v_\parallel}\\ &= \lambda\frac{1}{\tau_b}\oint\frac{\tan^2\theta_0}{\tan^2\theta}\frac{B}{v_\parallel}\frac{dl}{B}\end{aligned} \tag{36}$$

Using $\sin^2\theta=b\sin^2\theta_0$ and

$$\tan^2\theta=\frac{b\sin^2\theta_0}{1-b\sin^2\theta_0}, \tag{37}$$

Eq. (36) is written

$$\begin{aligned}\lambda\left\langle\frac{\tan^2\theta_0}{\tan^2\theta}\right\rangle_b &= \lambda\frac{1}{\tau_b}\oint\frac{1-b\sin^2\theta_0}{b\cos^2\theta_0}\frac{B}{v_\parallel}\frac{dl}{B}\\ &= \lambda\frac{1}{\tau_b v_0}\oint\frac{1-b\sin^2\theta_0}{b\cos^2\theta_0}\frac{B}{\cos\theta}\frac{dl}{B}\end{aligned}$$

Using $\cos\theta=\mathrm{sgn}(\xi_0)\sqrt{1-b\sin^2\theta_0}$ (note that for passing particle $\mathrm{sgn}(\cos\theta)=\mathrm{sgn}(\cos\theta_0)$, i.e. the direction of parallel velocity does not change sign), the above equation is written

$$\begin{aligned}\lambda\left\langle\frac{\tan^2\theta_0}{\tan^2\theta}\right\rangle_b &= \mathrm{sgn}(\xi_0)\lambda\frac{1}{\tau_b v_0}\oint\sqrt{1-b\sin^2\theta_0}\frac{B}{b\cos^2\theta_0}\frac{dl}{B}\\ &= \mathrm{sgn}(\xi_0)\lambda\frac{1}{\tau_b v_0}\frac{B_0}{\cos^2\theta_0}\oint\sqrt{1-b\sin^2\theta_0}\frac{dl}{B}\\ &= \mathrm{sgn}(\xi_0)\left(\frac{B_0}{L}\oint\frac{dl}{B}\right)\frac{1}{\cos\theta_0}\left\langle\sqrt{1-b\sin^2\theta_0}\right\rangle_f,\end{aligned} \tag{38}$$

where $\langle\ldots\rangle_f$ is the flux average defined by

$$\langle\ldots\rangle_f=\frac{\oint(\ldots)\frac{dl}{B}}{\oint\frac{dl}{B}}. \tag{39}$$

Substituting Eq. (38) into Eq. (35), gives

$$\frac{\partial H(\theta_0)}{\partial \xi_0} = \mathrm{sgn}(\xi_0) \frac{c_0}{\frac{B_0}{L} \oint \frac{dl}{B}} \frac{\xi_0}{\left\langle \sqrt{1 - b\sin^2\theta_0} \right\rangle_f}. \tag{40}$$

Eq. (40) determines $H(\theta_0)$, i.e., the angular part of χ. The boundary condition for Eq. (40) is $H(\theta_0 = \theta_{\mathrm{tr}}) = 0$, where θ_{tr} is the critical pitch angle between circulating and trapped region, $\sin^2\theta_{\mathrm{tr}} = B_0/B_{\max}$. Here it is convenient to use a new angular variable, Λ, instead of θ_0,

$$\Lambda \equiv \frac{\mu B_{\max}}{m u^2/2} \tag{41}$$

which is related to θ_0 by

$$\Lambda = \frac{B_{\max}}{B_0} \big(1 - \cos^2\theta_0\big). \tag{42}$$

Using this, Eq. (40) is written as

$$\frac{\partial H}{\partial \Lambda} \frac{\partial \Lambda}{\partial \xi_0} = \mathrm{sgn}(\xi_0) \frac{c_0}{\frac{B_0}{L} \oint \frac{dl}{B}} \frac{\xi_0}{\left\langle \sqrt{1 - b\sin^2\theta_0} \right\rangle_f},$$

which simplifies to

$$\frac{\partial H}{\partial \Lambda} \frac{B_{\max}}{B_0} (-2\xi_0) = \mathrm{sgn}(\xi_0) \frac{c_0}{\frac{B_0}{L} \oint \frac{dl}{B}} \frac{\xi_0}{\left\langle \sqrt{1 - b\sin^2\theta_0} \right\rangle_f} \tag{43}$$

$$\Rightarrow \frac{\partial H}{\partial \Lambda} = \frac{1}{2}\ \mathrm{sgn}(\xi_0) \frac{-c_0}{\frac{1}{L} \oint \frac{B_{\max}}{B} dl} \frac{1}{\left\langle \sqrt{1 - \Lambda \frac{B}{B_{\max}}} \right\rangle_f}. \tag{44}$$

Using the boundary condition $H(\Lambda = 1) = 0$, Eq. (44) is integrated to give

$$H = \frac{1}{2} \frac{c_0}{\frac{1}{L} \oint \frac{B_{\max}}{B} dl} \mathrm{sgn}(\xi_0) \int_\Lambda^1 \frac{d\Lambda'}{\left\langle \sqrt{1 - \Lambda' \frac{B}{B_{\max}}} \right\rangle_f} \tag{45}$$

Except for the prefactor, Eq. (44) is idential with Eq. (29) in Lin-Liu's paper[2]. For later use, define

$$\hat{H}(\Lambda) = \frac{1}{2} \int_\Lambda^1 \frac{d\Lambda'}{\left\langle \sqrt{1 - \Lambda' \frac{B}{B_{\max}}} \right\rangle_f}, \tag{46}$$

then

$$H = \frac{c_0}{\frac{1}{L} \oint \frac{B_{\max}}{B} dl} \mathrm{sgn}(\xi_0) \hat{H}(\Lambda). \tag{47}$$

3.2 Velocity direction equation

Now we consider the equation for the velocity direction, Eq. (28), i.e.,

$$\frac{\alpha(u_0)}{g(u_0)} = c_0 L. \tag{48}$$

Using Eqs. (24) and (11) in Eq. (48) gives

$$A(u_0) + Q(u_0) + M(u_0) - eLv_0 = c_0 L \frac{2D_{\theta\theta}}{u^2} G(u_0), \tag{49}$$

where $A(u_0)$, $Q(u_0)$, and $M(u_0)$ are given respectively by Eqs. (16), (18), and (21).

The expansion coefficient of the first Legendre harmonic of χ is given by

$$\tilde{\chi}_1(u, l) = \frac{3}{2} \int_0^\pi \chi(u, \theta, l) \cos\theta \sin\theta d\theta \tag{50}$$

Note that $\chi(u,\theta,l)$ is an odd function about $\cos\theta$, and $P_k(-\cos\theta)=(-1)^k P_k(\cos\theta)$, thus, for odd value of k, $\chi(v,\theta,l)P_k(\cos\theta)$ is an even function about $\cos\theta$. Therefore Eq. (50) reduces to

$$\tilde{\chi}_k(u,l)=3\int_0^{\pi/2}\chi(u,\theta,l)\cos\theta\sin\theta d\theta \tag{51}$$

Now let us transform the above integration to (u_0,θ_0) coordinates. Using $\chi(u,\theta,l)=\chi_0(u_0,\theta_0)$, then Eq. (51) is written

$$\tilde{\chi}_1(u,l)=3\int_0^{\pi/2}\chi_0(u_0,\theta_0)\cos\theta\sin\theta d\theta \tag{52}$$

Using $\sin^2\theta=b(l)\sin^2\theta_0$, we obtain $\sin\theta\cos\theta d\theta=b(l)\sin\theta_0\cos\theta_0 d\theta_0$. Using this in Eq. (52) gives

$$\tilde{\chi}_1(u,l)=3b(l)\int_0^{\theta_l}\chi_0(u_0,\theta_0)\cos\theta_0\sin\theta_0 d\theta_0$$

Note that the new upper limit of the integration, $\theta_l=\arcsin\left(\sqrt{1/b(l)}\right)$, varies as l changes. Note that, for any value of l, $\theta_l\geqslant\theta_{\rm tr}=\arcsin(\sqrt{1/b_{\max}})$. Futher note that $\chi_0(u_0,\theta_0)$ is zero in the trapped region (i.e., $\theta_0>\theta_{\rm tr}$). Thus, the upper limit of the integration, θ_l, can be chosen to be $\theta_{\rm tr}$ for any value of l,

$$\tilde{\chi}_1(u,l)=3b(l)\int_0^{\theta_{\rm tr}}\chi_0(u_0,\theta_0)\cos\theta_0\sin\theta_0 d\theta_0. \tag{53}$$

Using $\chi_0(u_0,\theta_0)=H(\theta_0)G(u_0)$ in the above equation gives

$$\tilde{\chi}_1(u,l)=b(l)\left(3\int_0^{\theta_{\rm tr}}H(\theta_0)\cos\theta_0\sin\theta_0 d\theta_0\right)G_0(u_0) \tag{54}$$

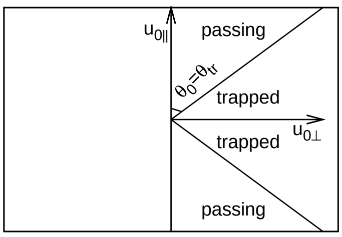

Figure 1. Passing and trapped region of phase space. Here u_0, θ_0 is the middle-plane coordinates. The boundary between passing and trapped region is given by $\theta_0=\theta_{\rm tr}$, where $\theta_{\rm tr}$ is determined by $\sin^2\theta_{\rm tr}=B_{\min}/B_{\max}$; $B_{\min}$ and $B_{\max}$ are respectively the minmum and maximum value of magnetic field on one flux surface.

Substituting Eq. (54) into the equations (16), (18), and (21), gives

$$\begin{aligned} A(u_0) &= \frac{1}{u_0^2}\frac{\partial}{\partial u_0}\left[u_0^2 D_{cuu}^{a/b}\frac{\partial}{\partial u_0}\left(\oint\tilde{\chi}_1(u_0,l)dl\right)\right] \\ &= \left(\oint b(l)dl\right)\left(3\int_0^{\theta_{\rm tr}}H(\theta_0)\cos\theta_0\sin\theta_0 d\theta_0\right)\frac{1}{u_0^2}\frac{\partial}{\partial u_0}\left[u_0^2 D_{cuu}^{a/b}\frac{\partial}{\partial u_0}G(u_0)\right], \end{aligned} \tag{55}$$

$$\begin{aligned} Q(u_0) &= F_u^{a/b}\frac{\partial}{\partial u_0}\oint\tilde{\chi}_1(u_0,l)dl \\ &= \left(\oint b(l)dl\right)\left(3\int_0^{\theta_{\rm tr}}H(\theta_0)\cos\theta_0\sin\theta_0 d\theta_0\right)F_u^{a/b}\frac{\partial}{\partial u_0}G(u_0), \end{aligned} \tag{56}$$

$$\begin{aligned} M(u_0) &= \oint dl I_1[\tilde{\chi}_1(u_0,l)] \\ &= \left(\oint b(l)dl\right)\left(3\int_0^{\theta_{\rm tr}}H(\theta_0)\cos\theta_0\sin\theta_0 d\theta_0\right)I_1[G(u_0)] \end{aligned} \tag{57}$$

Define

$$f_c^\star = \left(\frac{1}{L}\oint b(l)dl\right)\left(3\int_0^{\theta_{\rm tr}} H(\theta_0)\cos\theta_0 \sin\theta_0 d\theta_0\right) \tag{58}$$

(Refer to Sec. 5 for the relation of $f_c^\star$ defined here and the f_c defined in Lin-Liu's paper[2, 3].) then

$$A(u_0) = \frac{1}{u_0^2}\frac{\partial}{\partial u_0}\left[u_0^2 D_{cuu}^{a/b}\frac{\partial G(u_0)}{\partial u_0}\right] L f_c^\star, \tag{59}$$

$$Q(u_0) = F_u^{a/b}\frac{\partial G(u_0)}{\partial u_0} L f_c^\star. \tag{60}$$

$$M(u_0) = I_1[G(u_0)] L f_c^\star \tag{61}$$

Substituting Eqs. (59), (60), and (61) into Eq. (49) gives

$$\frac{1}{u_0^2}\frac{\partial}{\partial u_0}\left[u_0^2 D_{cuu}^{e/e}\frac{\partial G(u_0)}{\partial u_0}\right] L f_c^\star + F_u^{e/e}\frac{\partial G(u_0)}{\partial u_0} L f_c^\star + I_1[G(u_0)] L f_c^\star - c_0 L\frac{2D_{\theta\theta}}{u_0^2} G(u_0) = e L v_0 \tag{62}$$

$$\Rightarrow \frac{1}{u_0^2}\frac{\partial}{\partial u_0}\left[u_0^2 D_{cuu}^{e/e}\frac{\partial G(u_0)}{\partial u_0}\right] + F_u^{e/e}\frac{\partial G(u_0)}{\partial u_0} + I_1[G(u_0)] - \frac{1}{f_c^\star} c_0 \frac{2D_{\theta\theta}}{u_0^2} G(u_0) = \frac{e\, v_0}{f_c^\star} \tag{63}$$

Define

$$\overline{G}\,(u_0) = \frac{G(u_0)}{ec/\nu_c} \tag{64}$$

then Eq. (63) is written as

$$\frac{1}{u_0^2}\frac{\partial}{\partial u_0}\left[u_0^2 D_{cuu}^{e/e}\frac{\partial \overline{G}\,(u_0)}{\partial u_0}\right] + F_u^{e/e}\frac{\partial \overline{G}\,(u_0)}{\partial u_0} + I_1[\overline{G}\,(u_0)] - \frac{c_0}{f_c^\star}\frac{2D_{\theta\theta}}{u_0^2}\overline{G}\,(u_0) = \frac{v_0}{f_c^\star}\frac{\nu_c}{c} \tag{65}$$

$$\Rightarrow \frac{1}{\overline{u}_0^2}\frac{\partial}{\partial \overline{u}_0}\left[\overline{u}_0^2\, \overline{D}_{cuu}^{e/e}\frac{\partial \overline{G}\,(u_0)}{\partial \overline{u}_0}\right]\frac{1}{t_c} + \frac{1}{t_c}\overline{F}_u^{e/e}\frac{\partial \overline{G}\,(u_0)}{\partial \overline{u}_0} + I_1[\overline{G}\,(u_0)] - \frac{c_0}{f_c^\star}\frac{1}{t_c}\frac{2\overline{D}_{\theta\theta}}{\overline{u}_0^2}\overline{G}\,(u_0) = \frac{v_0}{f_c^\star}\frac{\nu_c}{c} \tag{66}$$

$$\Rightarrow \frac{1}{\overline{u}_0^2}\frac{\partial}{\partial \overline{u}_0}\left[\overline{u}_0^2\, \overline{D}_{cuu}^{e/e}\frac{\partial \overline{G}\,(u_0)}{\partial \overline{u}_0}\right] + \overline{F}_u^{e/e}\frac{\partial \overline{G}\,(u_0)}{\partial \overline{u}_0} + I_1[\overline{G}\,(u_0)] - \frac{c_0}{f_c^\star}\frac{2\overline{D}_{\theta\theta}}{\overline{u}_0^2}\overline{G}\,(u_0) = \frac{1}{f_c^\star}\,\overline{v}_0 \tag{67}$$

where $\overline{D}_{cuu}^{e/e} = D_{cuu}^{e/e}/(c^2/t_c)$, $\overline{D}_{c\theta\theta} = D_{c\theta\theta}/(c^2/t_c)$, $\overline{F}_u^{e/e} = F_u^{e/e}/(c/t_c)$. Eq. (67) is the toroidal generalization of equation (98) in Karney's paper[4]. Note that there is a undetermined constant c_0 in Eq. (67). The constant c_0 is a constant of separation, whose value may be determinde by some boundary conditions. In our case, however, the constant c_0 can be chosen arbitrarily, the value of which will not influence χ. The reason is as follows. We note that $H(\theta_0)$ is linearly proportional to c_0, thus, from the definition of $f_c^\star$, Eq. (58), we know $f_c^\star$ is also linearly proportional to c_0. Examining Eq. (67), we find the left-hand side of the equation is actually independent of c_0, while the right-hand side of the equation is linearly proportional to the $1/c_0$. Then it is obvious that the solution to Eq. (67), $\overline{G}\,(u_0)$ is linearly proportional to $1/c_0$. Recalling $H(\theta_0)$ is linearly proportional to c_0, thus the product of $H(\theta_0)$ and $\overline{G}\,(u_0)$ will be independent of c_0, i.e. χ is independent of c_0. I choose $c_0 = 1$ to evalue $f_c^\star$ and $H(\theta_0)$. Then Eq. (67) is solved numerically to determine $\overline{G}\,(u_0)$. (I made minor revision to the homogeneous magnetic field code, specificly, I divide the pitch angle term and the non-homogeneous term by $f_c^\star$.)

Note that the solution to Eq. (67), $\overline{G}$, is negative. Examining carefully the normalization, we find $\overline{G}$ defined in this note corresponds to the minus of F appeaing in Eq. (31) of Lin-Liu's paper[2]. Since Toray code uses the normalization consistent with F, it is obvious that $(-\overline{G})$ can be used to replace F in the code to calculate the results of more general cases where full collision operator is used as the collision model.

4 About $\tau_b v_0$

It is easy to prove that $\tau_b v_0$ is a function of only θ_0.

[Proof:

$$\begin{aligned}\tau_b v_0 &= v_0 \oint \frac{dl}{v_\|}\\ &= \oint \frac{dl}{\cos\theta}\\ &= \oint \frac{dl}{\mathrm{sgn}(\cos\theta)\sqrt{1-\sin^2\theta}}\\ &= \oint \frac{dl}{\mathrm{sgn}(\cos\theta)\sqrt{1-b(l)\sin^2\theta_0}}\end{aligned} \tag{68}$$

For passing particles, $\mathrm{sgn}(\cos\theta)=\mathrm{sgn}(\cos\theta_0)$. Using this in the above equation gives

$$\tau_b v_0 = \oint \frac{dl}{\mathrm{sgn}(\cos\theta_0)\sqrt{1-b(l)\sin^2\theta_0}}, \tag{69}$$

which indicates $\tau_b v_0$ is a function of only θ_0.]

5 Relation of $f_c^\star$ and f_c

Substituting Eq. (47), i.e.,

$$H(\theta_0) = \frac{c_0}{\frac{1}{L}\oint \frac{B_{\max}}{B} dl} \mathrm{sgn}(\xi_0)\hat{H}\left(\frac{B_{\max}}{B_0}\sin^2\theta_0\right), \tag{70}$$

into the definition of the $f_c^\star$ of Eq. (58), gives

$$\begin{aligned}f_c^\star &= 3\left(\frac{1}{L}\oint b(l)dl\right)\int_0^{\theta_{\mathrm{tr}}} H(\theta_0)\cos\theta_0 \sin\theta_0 d\theta_0\\ &= 3\frac{1}{L}\left(\oint b(l)dl\right)\frac{c_0}{\frac{1}{L}\int \frac{B_{\max}}{B}dl}\int_0^{\theta_{\mathrm{tr}}} \hat{H}\left(\frac{B_{\max}}{B_0}\sin^2\theta_0\right)\cos\theta_0 \sin\theta_0 d\theta_0\\ &= \frac{3}{2}c_0 \frac{\oint b(l)dl}{\oint (B_{\max}/B)dl}\int_0^{\theta_{\mathrm{tr}}} \hat{H}\left(\frac{B_{\max}}{B_0}\sin^2\theta_0\right) d\sin^2\theta_0\\ &= \frac{3}{2}c_0 \frac{B_0}{B_{\max}} \frac{\oint b(l)dl}{\oint (B_{\max}/(B)dl}\int_0^{\theta_{\mathrm{tr}}} \hat{H}\left(\frac{B_{\max}}{B_0}\sin^2\theta_0\right) d\frac{B_{\max}}{B_0}\sin^2\theta_0\\ &= \frac{3}{2}c_0 \frac{B_0}{B_{\max}} \frac{\oint b(l)dl}{\oint (B_{\max}/B)dl}\int_0^{1} \hat{H}(\Lambda')d\Lambda'.\end{aligned} \tag{71}$$

The geometric factor in Eq. (71) can be written as

$$\begin{aligned}\frac{B_0}{B_{\max}}\oint b(l)dl &= \frac{B_0}{B_{\max}}\oint \frac{B}{B_0}dl\\ &= B_{\max}\oint \frac{B^2}{B_{\max}^2}\frac{dl}{B}\\ &= B_{\max}\frac{\oint \frac{B^2}{B_{\max}^2}\frac{dl}{B}}{\oint \frac{dl}{B}}\oint \frac{dl}{B}\\ &= \left\langle \frac{B^2}{B_{\max}^2}\right\rangle \oint \frac{B_{\max}}{B}dl\end{aligned} \tag{72}$$

Thus Eq. (71) is written as

$$f_c^\star = \frac{3}{2}c_0\left\langle \frac{B^2}{B_{\max}^2}\right\rangle \int_0^1 \hat{H}(\Lambda')d\Lambda'. \tag{73}$$

We note that f_c in the literature[2] is defined by

$$f_c = \frac{3}{2}\left\langle \frac{B^2}{B_{\max}^2}\right\rangle \int_0^1 \hat{H}(\Lambda')d\Lambda'. \tag{74}$$

If we choose c_0 to be equal one, $f_c^\star$ and f_c are equal.

6 Summary

When only $l = 0, 1, 2$ Legendre harmonics are included in the energy diffusion, slowing down, and field particle part of the collision operator (all Legendre harmonics are included in the pitch angle part of the collision operator), the adjoint equation can be solved by separation of variables. (Actually only the $l = 1$ Legendre harmonic is considered in the calculation since the solution to the adjoint equation contains no even order Legendre harmonics.) Separation of variables does not work for the adjoint equation if additional Legendre harmonics with $l \geqslant 3$ are included in the terms mentioned above.

7 Numerical results of efficiencies of electron cyclotron current drive

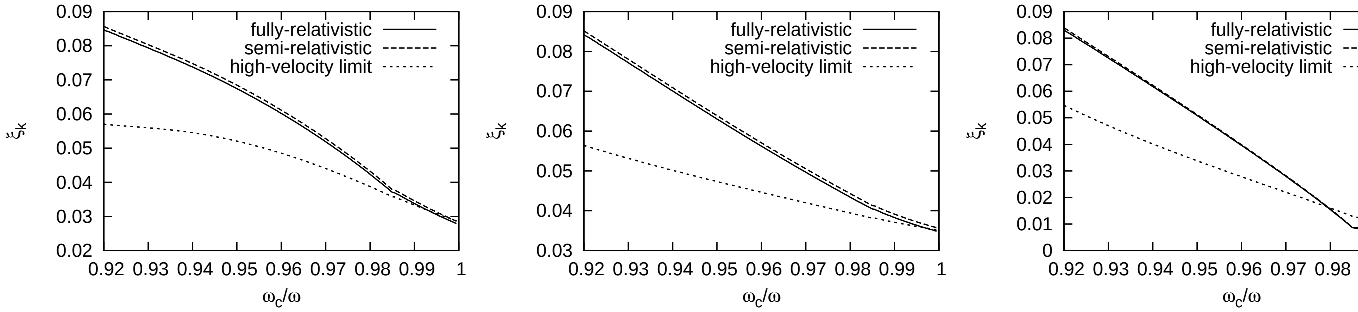

Figure 2. Warm dispersion (left figure), cold dispersion (middle figure), simplified diffusion operator (right figure). Other paramet
$n_{\|} = 0.4$, $l = 1$, O-mode , $Z_i = 1.67$, $T_e = 25$keV, $\varepsilon = 0.1$, $\theta_p = 0°$.

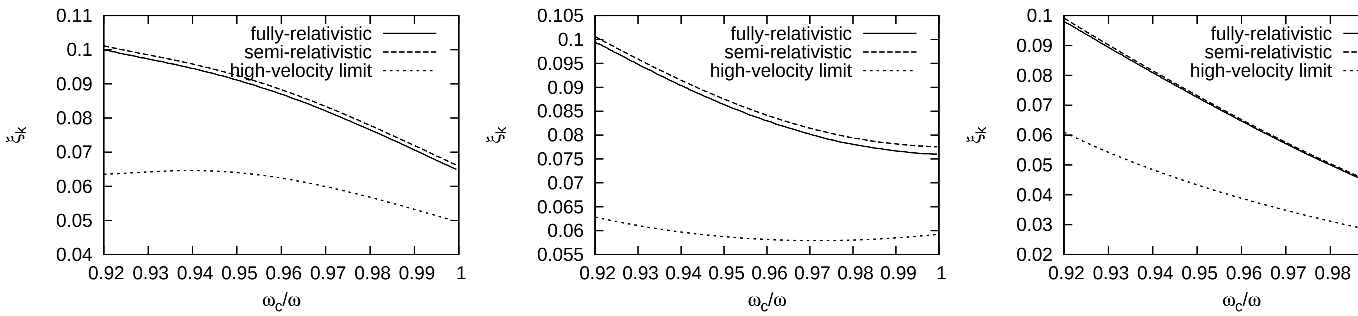

Figure 3. Warm dispersion (left figure), cold dispersion (middle figure), simplified diffusion operator (right figure). Other paramet
$n_{\|} = 0.4$, $l = 1$, O-mode , $Z_i = 1.67$, $T_e = 25$keV, $\varepsilon = 0.1$, $\theta_p = 175°$.

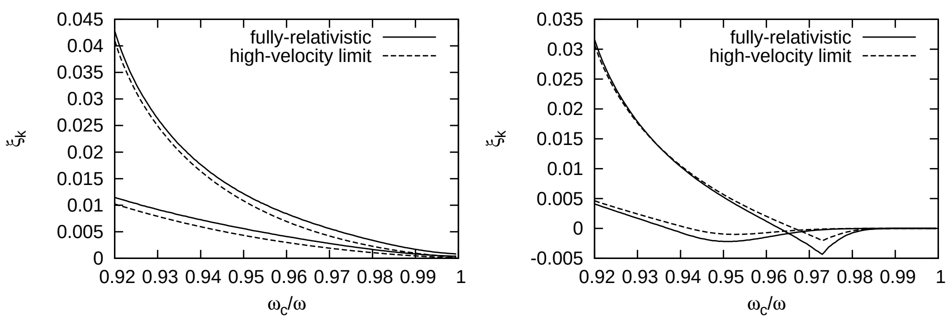

Figure 4. Left figure ($\theta_p = 165°$) for $n_{\|} = 0.4$ and 0.6; right figure ($\theta_p = 15°$) for $n_{\|} = 0.4$ and 0.6. Other parameters: $Z_i = 1.6$, $\varepsilon = 0.2$, $T_e = 2$keV, $l = 1$, O-mode. These figures agree with Fig. 1 of Lin-Liu's paper[2], plasma and wave parameters are the same, however the the normalization of the current drive efficiencies is different.

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