Notes on tokamak equilibrium*

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Abstract. A routine in operating a tokamake is to reconstruct axisymmetirc poloidal magnetic field under the constraints of MHD force balance and various magnetic measurements. This kind of task can be done by various codes, e.g., EFIT. Another routine in analsing the tokamak discharge is to constructs a coordinate system, based on the 2D equilibrium magnetic field, with a desired form of Jacobian by using discrete numerical equilibrium data output by the equilibrium reconstructing codes. These are my notes when learning tokamak equilibrium theory. These notes are evolving and are written for my own record. I have been keeping revising these notes for more than 10 years. I enjoy seeing the continuous improvement of these notes and my understanding of this simple but important stuff in tokamak physics.

1 Axisymmetric magnetic field

Due to the divergence-free nature of magnetic field, i.e., $\nabla\cdot\mathbf{B}=0$, magnetic field can be expressed as the curl of a vector field,

$$\mathbf{B}=\nabla\times\mathbf{A}, \tag{1}$$

where $\mathbf{A}$ is called the vector potential of $\mathbf{B}$. (Using a vector potential representation is helpful in that we do not need to worry about the condition $\nabla\cdot\mathbf{B}=0$ once the magnetic field is in the vector potential form.)

In cylindrical coordinates (R,ϕ,Z), the above expression is written

$$\mathbf{B}=\left(\frac{1}{R}\frac{\partial A_Z}{\partial\phi}-\frac{\partial A_\phi}{\partial Z}\right)\hat{\mathbf{R}}+\left(\frac{1}{R}\frac{\partial(RA_\phi)}{\partial R}-\frac{1}{R}\frac{\partial A_R}{\partial\phi}\right)\hat{\mathbf{Z}}+\left(\frac{\partial A_R}{\partial Z}-\frac{\partial A_Z}{\partial R}\right)\hat{\boldsymbol{\phi}}. \tag{2}$$

We consider axisymmetric magnetic field. The axial symmetry means that, when expressed in the cylindrical coordinate system (R,ϕ,Z), the components of $\mathbf{B}$, namely B_R, B_Z, and B_ϕ, are all independent of ϕ. For this case, it can be proved that an axisymmetric vector potential $\mathbf{A}$ suffices for expressing the magnetic field, i.e., all the components of the vector potential $\mathbf{A}$ can also be taken independent of ϕ. Using this, Eq. (2) is written

$$\mathbf{B}=-\frac{\partial A_\phi}{\partial Z}\hat{\mathbf{R}}+\frac{1}{R}\frac{\partial(RA_\phi)}{\partial R}\hat{\mathbf{Z}}+\left(\frac{\partial A_R}{\partial Z}-\frac{\partial A_Z}{\partial R}\right)\hat{\boldsymbol{\phi}}. \tag{3}$$

In tokamak literature, $\hat{\boldsymbol{\phi}}$ direction is called the toroidal direction, and (R,Z) planes (i.e., $\phi=$ const planes) are called poloidal planes.

* *Note:* This document has been written using GNU $\mathrm{T_EX_{MACS}}$ [**?**].

1.1 Poloidal magnetic field

Equation (3) indicates that the two poloidal components of $\mathbf{B}$, namely B_R and B_Z, are determined by a single component of $\mathbf{A}$, namely A_ϕ. This motivates us to define a function $\Psi(R,Z)$:

$$\Psi(R,Z) \equiv RA_\phi(R,Z). \tag{4}$$

Then Eq. (3) implies the poloidal components, B_R and B_Z, can be written as

$$B_R = -\frac{1}{R}\frac{\partial \Psi}{\partial Z}, \tag{5}$$

$$B_Z = \frac{1}{R}\frac{\partial \Psi}{\partial R}. \tag{6}$$

(Note that it is the property of being axisymmetric and divergence-free that enables us to express the two components of $\mathbf{B}$, namely B_R and B_Z, in terms of a single function $\Psi(R,Z)$.) Furthermore, it is ready to prove that Ψ is constant along a magnetic field line, i.e. $\mathbf{B}\cdot\nabla\Psi = 0$. [Proof:

$$\begin{aligned}\mathbf{B}\cdot\nabla\Psi &= \mathbf{B}\cdot\left(\frac{\partial\Psi}{\partial R}\hat{\mathbf{R}} + \frac{\partial\Psi}{\partial Z}\hat{\mathbf{Z}}\right)\\ &= -\frac{1}{R}\frac{\partial\Psi}{\partial Z}\frac{\partial\Psi}{\partial R} + \frac{1}{R}\frac{\partial\Psi}{\partial R}\frac{\partial\Psi}{\partial Z}\\ &= 0.\end{aligned} \tag{7}$$

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The function Ψ is usually called the "poloidal flux function" in tokamak literature because Ψ can be related to the poloidal magnetic flux, as we will discuss in Sec. 1.7.

Using Eqs. (5) and (6), the poloidal magnetic field $\mathbf{B}_p$ is written as

$$\begin{aligned}\mathbf{B}_p &= B_R\hat{\mathbf{R}} + B_Z\hat{\mathbf{Z}}\\ &= -\frac{1}{R}\frac{\partial\Psi}{\partial Z}\hat{\mathbf{R}} + \frac{1}{R}\frac{\partial\Psi}{\partial R}\hat{\mathbf{Z}}\\ &= \frac{1}{R}\nabla\Psi\times\hat{\boldsymbol{\phi}}\\ &= \nabla\Psi\times\nabla\phi\end{aligned} \tag{8}$$

1.2 Toroidal magnetic field

Next, let's examine the toroidal component B_ϕ. Equation (3) indicates that B_ϕ involves both A_R and A_Z, which means that using the potential form does not enable useful simplification for B_ϕ. Therfore we will directly use B_ϕ. Define $g \equiv RB_\phi(R,Z)$, (the reason that we define this quantity will become apparent when we discuss the forece balance equation) then the toroidal magnetic field is written

$$\mathbf{B}_\phi = B_\phi\hat{\boldsymbol{\phi}} = \frac{g}{R}\hat{\boldsymbol{\phi}} = g\nabla\phi. \tag{9}$$

1.3 General form of axisymmetric magnetic field

Combining Eqs. (8) and (9), we can write a general axisymmetric magnetic field as

$$\begin{aligned}\mathbf{B} &= \mathbf{B}_p + \mathbf{B}_\phi\\ &= \nabla\Psi\times\nabla\phi + g\nabla\phi.\end{aligned} \tag{10}$$

Expression (10) is a famous formula in tokamak physics.

1.4 Gauge transformation of Ψ

Next, we discuss the gauge transformation of the vector potential $\mathbf{A}$ in the axisymmetric case. As is well known, magnetic field remains the same under the following gauge transformation:

$$\mathbf{A}^{\rm new} = \mathbf{A} + \nabla f, \tag{11}$$

where f is an arbitrary scalar field. Here we require that ∇f be axisymmetric because, as mentioned above, an axisymmetric vector potential suffices for describing an axisymmetric magnetic field. In cylindrical coordinates, ∇f is given by

$$\nabla f = \frac{\partial f}{\partial R}\hat{\mathbf{R}} + \frac{\partial f}{\partial Z}\hat{\mathbf{Z}} + \frac{1}{R}\frac{\partial f}{\partial \phi}\hat{\boldsymbol{\phi}}. \tag{12}$$

Since ∇f is axisymmetric, it follows that all the three components of ∇f are independent of ϕ, i.e., $\partial^2 f/\partial R\partial\phi = 0$, $\partial^2 f/\partial Z\partial\phi = 0$, and $\partial^2 f/\partial\phi^2 = 0$, which implies that $\partial f/\partial\phi$ is independent of R, Z, and ϕ, i.e., $\partial f/\partial\phi$ is actually a spatial constant. Using this, the ϕ component of the gauge transformation (11) is written

$$\begin{aligned} A_\phi^{\rm new} &= A_\phi + \frac{1}{R}\frac{\partial f}{\partial\phi} \\ &= A_\phi + \frac{C}{R}, \end{aligned} \tag{13}$$

where C is a constant. Note that the requirement of being axial symmetry greatly reduces the degree of freedom of the gauge transformation for A_ϕ (and thus for RA_ϕ, i..e, Ψ). Multiplying Eq. (13) with R, we obtain the corresponding gauge transformation for Ψ,

$$\Psi^{\rm new} = \Psi + C, \tag{14}$$

which indicates Ψ has the same gauge transformation as the electric potential, i.e., adding a constant. (Note that the definition $\Psi(R, Z) \equiv RA_\phi$ does not imply $\Psi(R = 0, Z) = 0$ because A_ϕ can adopt $1/R$ dependence under the gauge transformation (13)).

1.5 Contours of Ψ in the poloidal plane

Because Ψ is constant along a magnetic field line and Ψ is independent of ϕ, it follows that the projection of a magnetic field line onto (R, Z) plane is a contour of Ψ. On the other hand, are contours of Ψ on (R, Z) plane also the projections of magnetic field lines onto the plane? Yes, they are. [Proof. A contour of Ψ on (R, Z) plane satisfies

$$d\Psi = 0, \tag{15}$$

i.e.,

$$\frac{\partial\Psi}{\partial R}dR + \frac{\partial\Psi}{\partial Z}dZ = 0. \tag{16}$$

$$\Rightarrow \frac{1}{R}\frac{\partial\Psi}{\partial R}dR + \frac{1}{R}\frac{\partial\Psi}{\partial Z}dZ = 0. \tag{17}$$

Using Eqs. (5) and (6), the above equation is written

$$B_Z dR - B_R dZ = 0, \tag{18}$$

i.e.,

$$\frac{dZ}{dR} = \frac{B_Z}{B_R}, \tag{19}$$

which is the equation of the projection of a magnetic field line on (R, Z) plane. Thus, we prove that contours of Ψ are also the projections of magnetic field lines in (R, Z) plane.]