# 托卡马克几何与平衡

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Aug. 7, 2012

# Contents

# 导言

几何的复杂性是聚变等离子体物理的难题之一。本讲义旨在介绍托卡马克平衡的基本概念及其在稳定性与输运研究中的应用；力求通过简要描述阐明复杂几何的处理方法。
此讲义追求的主要特点概括如下：

**一般性**

适用于一般形变下的偏滤器托卡马克平衡位型

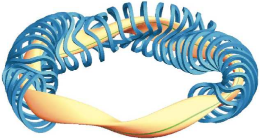
**基础性**

突出最重要最常用的基本概念

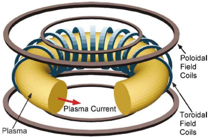
**实用性**

突出平衡位型的处理和应用，而不是平衡理论本身

# 环形磁约束位型



1.1 Non-axisymmetric[1]

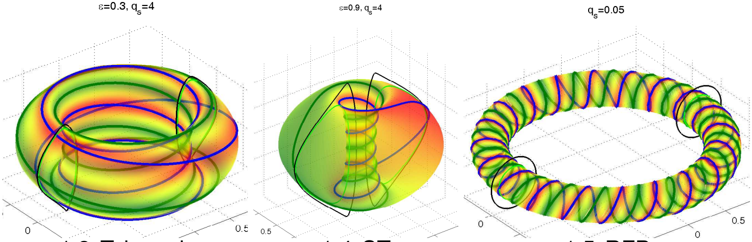



1.2 Axisymmetric

- 为了克服损失锥效应，必须采用闭合环位型。为了克服漂移损失，磁力线必须有旋转变换。
- 螺旋磁力线的**旋转变换**【rotational transform, $\iota \equiv B^\theta/B^\zeta = 1/q$, $q$ 是**安全因子**（safety factor）】使其形成一层层嵌套**磁面**（flux surfaces）结构。不同磁面上的磁力线的夹角可能不一样，即具有**磁剪切**(magnetic shear $s \equiv \rho q'/q$)。
- 非轴对称环位型如仿星器和螺旋器的旋转变换由外部磁场线圈提供。轴对称位型如托卡马克和反场pinch的旋转变换主要是由等离子体电流自身产生。

[1] Allen H. Boozer. In: *Rev. Mod. Phys.* 76 (2004), p. 1071.

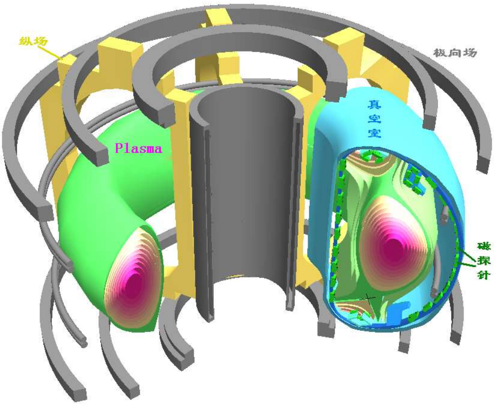
# 轴对称磁约束位型




1.3 Tokamak 1.4 ST 1.5 RFP

- 不同位型下的磁力线的空间**曲率**（Curvature）及其分布差异很大。
- 有些磁面上的磁力线经过若干圈后可以回到原来的出发点，即，$q = m/n$, $m$和$n$分别为绕环向和极向旋转的圈数，该磁面称为**有理面**（rational surface）。由于其周期性存在共振效应，因此又称为**共振面**（resonant surface）。有些磁面上磁力线无论经过多少圈后都回不到原来的出发点，即q 是一个无理数，这些称为**无理面**（irrational surface）或**非共振面**（non-resonant surface）。

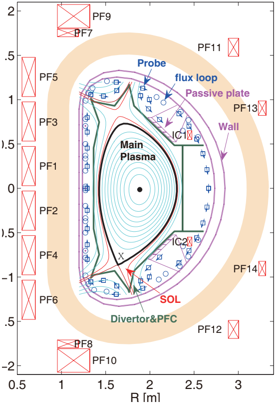
# EAST 电磁结构



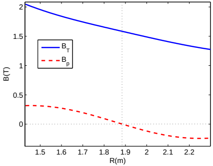

1.6 3D view EAST EM

- $R_0 \sim 1.8 - 1.9$m, $a \sim 0.45$m, $B_T \sim 1.5 - 3.5$T, $I_p \sim 0.3 - 1.5$MA.
- 极向场一体化设计。同时实现位移、形状控制和欧姆等离子体电流驱动。
- 中心螺线管主要承担电流驱动，偏滤器线圈主要承担拉长、成形，外测大线圈主要承担位移控制。内部快控线圈用来快速控制垂直位移不稳定性。
- 纵场、极向场全超导。

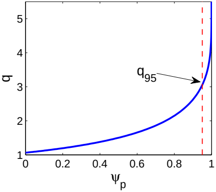
# EAST 偏滤器磁场位型特征




1.7 EAST configuration



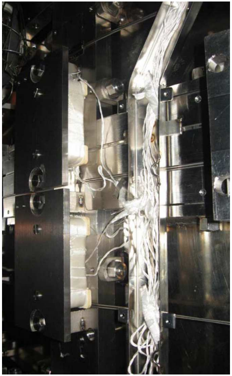

1.8 Magnetic field



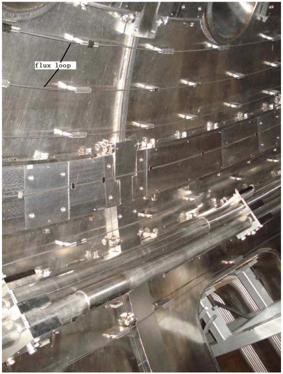

1.9 q profile

- 最外层磁面以内由嵌套的闭合磁面组成，形成**约束区**。
- 最外层磁面以外有一层很薄的（中平面位置径向1-2cm）开放磁场部分，称为**刮削层**（SOL），磁力线的尽头通向**偏滤器**（Divertor），将约束区输运出来的粒子与热导向偏滤器。
- 最外层磁面和SOL区之间**分界线**（speratrix）下方有个$B_p = 0$的鞍点称为**X point**.
- $B_p/B_T \sim \mathcal{O}(\epsilon = a/R_0)$
- $B \approx B_T \propto 1/R$
- 最外层磁面处$q \to \infty$, 定义归一化极向磁通$\hat{\psi}_p = 0.95$处的安全因子为$q_{95}$, 其功能与限制器位型下的边界$q$类似。

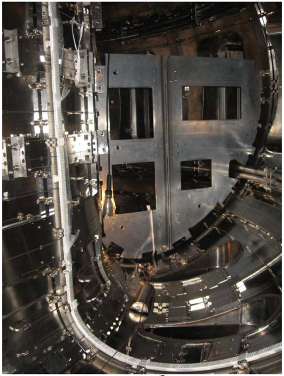
# EAST位型相关电磁测量



1.10 磁探针




1.11 单匝环



1.12 罗珂

电磁测量原理简单，精确有效，是平衡位型反演最重要的基本诊断。磁探针测量局域磁场，单匝环测量局域磁通，罗珂测量总电流。

# 动量方程

Momentum equation for single fluid model:

$$\boxed{\rho_m \frac{d}{dt}\vec{V} = \vec{J}\times\vec{B} - \nabla P - \nabla\cdot\overleftrightarrow{\Pi} + \vec{S}_M + O} \quad (1)$$

where

$$\frac{d}{dt} \equiv \frac{\partial}{\partial t} + \vec{V}\cdot\nabla \quad (2)$$

- 方程左边为惯性项。其中第二项为对流项，包含**非惯性力和非线性**作用。
- 方程右边的前两项反映压力平衡，特征时间是**压缩阿尔芬波时间**，$\tau_A$。后几项反映的是等离子动量输运过程，特征时间是**动量约束时间**，$\tau_M$。
- 当我们关心的时间尺度满足$\tau_A \ll \tau \ll \tau_M$，并且$V \ll C_s$，方程的零阶项部分为力平衡方程（即Eq. (1)中的黑色部分）。

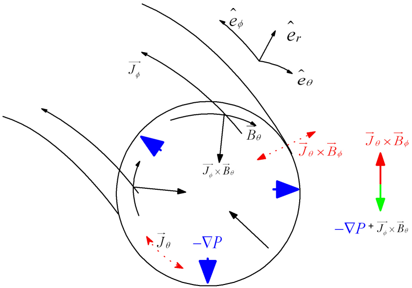
# 力平衡方程

Force balance equation:

$$\boxed{\vec{J}\times\vec{B}-\nabla P=0} \tag{3}$$

- 反映了**热压力**与磁场的**洛伦兹约束力**的平衡。
- 压强是**磁面函数**。因此,压力作用使得等离子体在垂直于磁面方向向外膨胀。
- 电流线躺在磁面上。只有**垂直方向的电流**参与力平衡。
- 平衡方程也可以写成
  $\nabla(P+\frac{B^2}{2\mu_0})-\vec{b}\vec{b}\cdot\nabla\frac{B^2}{2\mu_0}\equiv\nabla_\perp(P+\frac{B^2}{2\mu_0})=\frac{B^2}{\mu_0}\vec{\kappa}$
  这里$\vec{\kappa}\equiv\vec{b}\cdot\nabla\vec{b}$是磁力线的曲率。

# 小半径方向力平衡



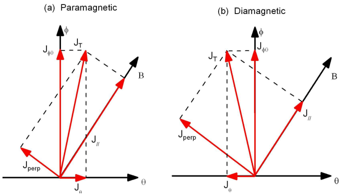

1.13 Force balance

洛伦兹力可以分解成环向电流和极向电流贡献两部分:

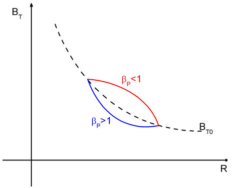
$$\nabla P = \vec{J}_\phi \times \vec{B}_p + \vec{J}_p \times \vec{B}_\phi \qquad (4)$$

- 热压力总是垂直磁面向外。
- 环向电流贡献项总是垂直磁面向内。
- 极向电流贡献项取决于极向电流的方向。

# 顺磁、逆磁和极向比压



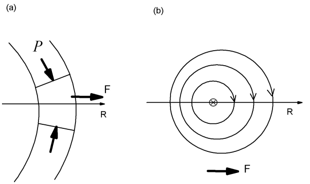

1.14 Effect of $\vec{J}_\theta$ orientation



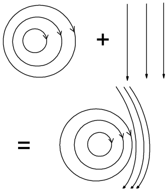

1.15 $B_T$

- 当$\vec{J}_\theta$ 产生的纵场和外部线圈的纵场方向一致时，称为**顺磁**（paramagnetic），该项贡献力沿着径向向外，因此**极向比压** $\beta_p \equiv 2\mu_0 \langle P\rangle_V / \left\langle B_p^2\right\rangle_{\psi lcs} < 1$。
- 当$\vec{J}_\theta$ 产生的纵场和外部线圈的纵场方向相反时，称为**逆磁**（diamagnetic），该项贡献力沿着径向向内，因此极向比压$\beta_p > 1$。
- 通过测量该电流产生的环向磁通即逆磁通量，$\delta\Phi = \frac{(\mu_0 I_p)^2}{8\pi B_t}(\beta_p - 1)$，可以计算出$\beta_p$，因此，得到内能.

# 大半径方向力平衡



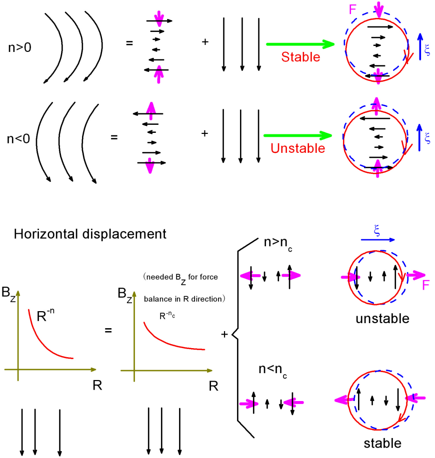

1.16 Expansion in major radial direction




1.17 Necessary vertical field

- 环效应使得等离子体朝大半径方向向外膨胀。
- 需要外加**垂直场**主动控制等离子体水平位移。对于圆截面等离子体，有近似解析解$B_{VZ} = -\frac{\mu_0 I_p}{4\pi R}(\ln\frac{8R}{a} + \Lambda - 0.5)$，这里$\Lambda \equiv \beta_p + l_i/2 - 1$.
- **内感**（internal inductence），$l_i \equiv \left\langle B_p^2 \right\rangle_V / \left\langle B_p^2 \right\rangle_{\psi_{lcs}}$。
- 为抵制向外膨胀的热压力，低场侧磁面变密，即芯部磁面会向外移动，称为**Shafranov shift**。对于圆截面等离子体，有近似解析解$\Delta = \frac{\epsilon}{2}(\Lambda + 1)a$, $\epsilon^{-1} \equiv R/a$ 称为Aspect ratio。

# 轴对称不稳定性



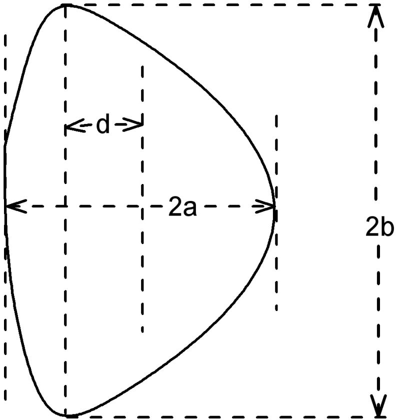

Magnetic decay index:

$$n = -\frac{R}{B_{VZ}}\frac{\partial B_{VZ}}{\partial R} = -\frac{R}{B_{VZ}}\frac{\partial B_{VR}}{\partial Z} \quad (5)$$

and hence $B_{VZ} \propto R^{-n}$.

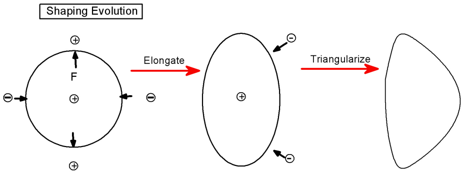
- 垂直位移稳定性条件:
  $n > 0$
- 水平位移位移稳定性条件:
  $n < n_c$
  圆截面等离子下，$n_c = 3/2$。

# Plasma Shape

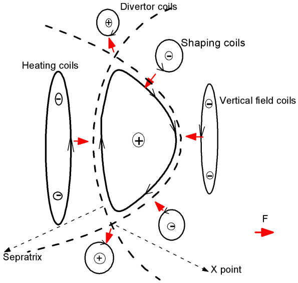



- Elongation: $\kappa = b/a$
- Triangularity: $\delta = d/a$
- 上下非对称时，上下拉长与三角度可以不同。

# Plasma Shaping



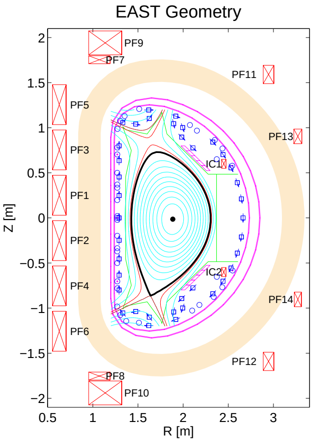

1.21 plasma shaping

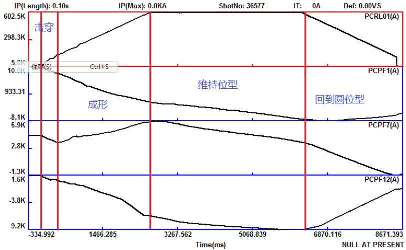



1.22 Forces for shaping

- 四极场形成拉长(elongation)；六极场形成三角形变(triangularity)。
- 四极场显然造成$n < 0$，因此，拉长使得垂直位移不稳定。

# Example: Shaping in EAST




1.24 EAST discharge

- 拉长主要依靠上下偏滤器线圈。
- $B_{pV} = \sum_j G_j^B I_{PF}^j$, 位移与shaping
- $V_\phi = -\frac{d\psi_p}{dt} = \sum_j G_j^V \frac{dI_{PF}^j}{dt}$, 电流驱动

# 重要平衡量小结

- 基本参数:
  纵场$B_T$, 电流$I_p$, 大半径$R_0$,小半径$a$和$\epsilon \equiv a/R_0$，拉长比$\kappa$，三角度$\delta$, 归一化极向磁通0.95处的安全因子$q_{95}$.
- 基本分布:
  等离子体电流分布($q$分布)、压强分布。
- 等离子体比压:
  $\beta_p \equiv 2\mu_0 \langle P\rangle_V / \left\langle B_p^2 \right\rangle_{\psi_{lcs}} = 1 + \frac{8\pi B_t \delta\Phi}{(\mu_0 I_p)^2} \sim O(1)$，
  $\beta \equiv 2\mu_0 \langle P\rangle_V / B_0^2 \sim O(1\%)$，
  $\beta_N \equiv \beta(\%)[a(m)B(T)/I(MA)] \sim O(1)$。
- 等离子体内感:
  $l_i \equiv \left\langle B_p^2 \right\rangle_V / \left\langle B_p^2 \right\rangle_{\psi_{lcs}} \sim O(1)$。

# Curvilinear coordinates

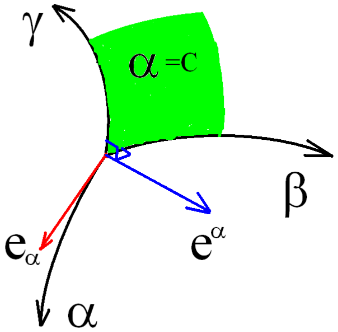
- In Cartesian coordinates $(x, y, z)$, the position of a point $P(x, y, z)$ is determined by the intersection of three mutually perpendicular planes, $x = const$, $y = const$, $z = const$. It can also be determined by **general curvilinear coordinates** $(\alpha, \beta, \gamma)$, if $(\alpha, \beta, \gamma)$ are the smooth functions of $(x, y, z)$ and the Jacobian $\mathcal{J} \neq 0$. The Jacobian is defined as

$$\mathcal{J} \equiv \frac{\partial(x, y, z)}{\partial(\alpha, \beta, \gamma)} = [(\nabla\alpha \times \nabla\beta) \cdot \nabla\gamma]^{-1} \tag{6}$$

  The differential volume can be written as, $dv = dxdydz = \mathcal{J} d\alpha d\beta d\gamma$.

- In order to describe direction of the fields in the general coordinates, we need to define the base vectors. $(e^\alpha, e^\beta, e^\gamma) \equiv (\nabla\alpha, \nabla\beta, \nabla\gamma)$ are **contravariant coordinates**, and $(e_\alpha, e_\beta, e_\gamma) \equiv (\partial_\alpha \vec{X}, \partial_\beta \vec{X}, \partial_\gamma \vec{X}) = \mathcal{J}(\nabla\beta \times \nabla\gamma, \nabla\gamma \times \nabla\alpha, \nabla\alpha \times \nabla\beta)$ are the **covariant coordinates**. They have the relationship, $e^i \cdot e_j = \delta_{ij}$, and $e_\alpha \times e_\beta = \mathcal{J} e^\gamma$.

# Geometric meaning of the Curvilinear coordinates




1.25 Geometric meaning of the Curvilinear coordinates

- $(\alpha = \mathit{const}, \beta = \mathit{const}, \gamma = \mathit{const})$ are the **coordinate surfaces**.
- $(\alpha, \beta, \gamma)$ are the **coordinate lines**.
- The contravariant coordinates $(e^\alpha, e^\beta, e^\gamma) \equiv (\nabla\alpha, \nabla\beta, \nabla\gamma)$ are **normal** to the coordinate surfaces
- The covariant coordinates $(e_\alpha, e_\beta, e_\gamma)$ are **tangential** to the coordinate lines.

# 度规（Metrics）

- The metrics of the coordinates are defined as: $g^{\alpha\beta} = e^{\alpha} \cdot e^{\beta}$ and $g_{\alpha\beta} = e_{\alpha} \cdot e_{\beta}$.
- It has $g_{ij}g^{jk} = \delta_{ik}$ (two $j$ means summary over all the coordinates, this will also be used as default in the following) and $e_i = g_{ij}e^j$.
- The relationship between the metrics and the Jacobian can be written as,

$$det(g^{ij}) = 1/\mathcal{J}^2 \quad and \quad det(g_{ij}) = \mathcal{J}^2 \tag{7}$$

- The covariant of the metrics can be calculated from the contravariant of the metrics with

$$g_{\alpha\alpha} = \mathcal{J}^2[g^{\beta\beta}g^{\gamma\gamma} - (g^{\beta\gamma})^2] \tag{8a}$$

$$g_{\alpha\beta} = \mathcal{J}^2[g^{\alpha\gamma}g^{\beta\gamma} - g^{\alpha\beta}g^{\gamma\gamma}] \tag{8b}$$

# Vector representation and differential operators

- Any vector field can be expressed as $\vec{A} \equiv A^i e_i \equiv A_i e^i$ . $A_i = \vec{A} \cdot e_i$ and $A^i = \vec{A} \cdot e^i$ are the covariant and contravariant parts of the vector, respectively. It has $A_i = g_{ij} A^j$, $A^i = g^{ij} A_j$.
- Usually, either $e^i$ or $e_i$ is not a unit vector. Therefore, both the covariant and contravariant parts of the vector are different from the physical values in the corresponding direction.
- The gradient, divergence, curl and Laplacian can be written as,

$$\nabla f = \nabla\alpha \frac{\partial}{\partial\alpha} f \quad (9a)$$

$$\nabla \cdot \vec{V} = \mathcal{J}^{-1} \frac{\partial}{\partial\alpha} (\mathcal{J} V^\alpha) \quad (9b)$$

$$\nabla\gamma \cdot (\nabla \times \vec{V}) = \mathcal{J}^{-1} \left( \frac{\partial}{\partial\alpha} V_\beta - \frac{\partial}{\partial\beta} V_\alpha \right) \quad (9c)$$

$$\nabla^2 f = \nabla \cdot \nabla f = \mathcal{J}^{-1} \frac{\partial}{\partial\alpha} \left( \mathcal{J} g^{\alpha\beta} \frac{\partial}{\partial\beta} f \right) \quad (9d)$$

# Other useful formulas

- Some other useful identities about operators:

$$\nabla\times\nabla f = 0 \tag{10a}$$
$$\nabla\cdot(\nabla f\times\nabla g) = 0 \tag{10b}$$
$$\nabla\cdot(\vec{A}\times\vec{B}) = \vec{B}\cdot\nabla\times\vec{A}-\vec{A}\cdot\nabla\times\vec{B} \tag{10c}$$
$$\nabla\times(\vec{A}\times\vec{B}) = [\nabla\cdot\vec{B}+\vec{B}\cdot\nabla]\vec{A}-[\nabla\cdot\vec{A}+\vec{A}\cdot\nabla]\vec{B} \tag{10d}$$
$$\vec{A}\times(\nabla\times\vec{B}) = (\nabla\vec{B})\cdot\vec{A}-(\vec{A}\cdot\nabla)\vec{B} \tag{10e}$$
$$\nabla(\vec{A}\cdot\vec{B}) = \vec{A}\times(\nabla\times\vec{B})+\vec{B}\times(\nabla\times\vec{A}) \tag{10f}$$
$$+(\vec{A}\cdot\nabla)\vec{B}+(\vec{B}\cdot\nabla)\vec{A} \tag{10g}$$
$$\nabla^2\vec{A} = \nabla(\nabla\cdot\vec{A})-\nabla\times(\nabla\times\vec{A}) \tag{10h}$$
$$\nabla\cdot(\vec{A}\vec{B}) = (\nabla\cdot\vec{A})\vec{B}+(\vec{A}\cdot\nabla)\vec{B} \tag{10i}$$
$$\oint_v\nabla\cdot\vec{A}dv = \oint_s\vec{A}\cdot d\vec{s} \tag{10j}$$
$$\oint_s\nabla\times\vec{A}\cdot d\vec{s} = \oint_l\vec{A}\cdot d\vec{l} \tag{10k}$$

# 任意磁场位型表征

From $\nabla \cdot \vec{B} = 0$, the contravariant representation of the magnetic field can be written as,

$$\vec{B} = \nabla\psi_t \times \nabla\theta - \nabla\psi_p(\psi_t, \theta, \zeta) \times \nabla\zeta \tag{11}$$

From Eq. (11), the magnetic field line trajectory can be determined by the equations,

$$\frac{d\psi_t}{d\zeta} = -\frac{\partial\psi_p}{\partial\theta} \tag{12a}$$

$$\frac{d\theta}{d\zeta} = \frac{\partial\psi_p}{\partial\psi_t} \tag{12b}$$

It is equivalent to a Hamiltonian system with a Hamiltonian $\psi_p(\psi_t, \theta, \zeta)$. $\psi_t$, $\theta$, $\zeta$ are the canonical momentum, coordinate and time, respectively.

# 闭合环位型磁场表征

如果满足$\psi_p = \psi_p(\psi_t)$, 磁力线形成闭合磁面。
It has $\frac{\partial \psi_t}{\partial \psi_p} = \frac{d\psi_t}{d\psi_p} \equiv q(\psi) = 1/\iota(\psi)$.
For the magnetic field with a closed flux surface, it can be written as so called Clebsch representation,

$$\vec{B} = q\nabla\psi_p \times \nabla\theta - \nabla\psi_p \times \nabla\zeta \quad (13)$$

$$= \nabla\psi_p \times \nabla\alpha \quad (14)$$

where $\alpha \equiv q\theta - \zeta$ .
From $\vec{B} = \nabla \times \vec{A}$, one obtains $\vec{A} = \psi_p \nabla\alpha$.
Note: Here $\psi_p = -\Psi_p/2\pi$ , $\psi_t = \Psi_t/2\pi$, and $\Psi_p$ and $\Psi_t$ are the physical poloidal and toroidal magnetic flux, respectively.

# 轴对称磁场表征

For toroidal symmetric tokamak case, the magnetic field can be written as,

$$\vec{B} = \nabla\phi \times \nabla\psi_p + g(\psi_p)\nabla\phi \tag{15}$$

where $\phi$ is the geometric toroidal angle.
From $|\nabla\phi| = 1/R$, it is obtained that $|\nabla\psi_p| = RB_p$ and $g = RB_t$, and $B_p$ and $B_t$ are the poloidal and toroidal magnetic field, respectively.

# Force free equilibrium

If the thermal force can be neglected, i.e. $\nabla P = 0$, the force free solution of the force balance equation can be written as,

$$\nabla \times \vec{B} = \lambda \vec{B} \tag{16}$$

where $\lambda \equiv \mu_0 J_{||}/B$.
From the Taylor relaxation[2], one obtains that the stable equilibrium, i.e. minimum energy state, requires

$$\lambda \equiv constant \tag{17}$$

The finite radial gradient of $\lambda$ is the driving force for the tearing instability.

[2] J. B. Taylor. In: *Rev. Mod. Phys.* 58 (1986), p. 741.

# Grad-Shafralov平衡方程

- By using the magnetic field with axisymmetry,

$$\begin{aligned}\vec{B} &= \nabla\phi\times\nabla\psi_p + g(\psi_p)\nabla\phi \\ &= B_p\hat{e}_p + ge^{\phi}\end{aligned} \tag{18}$$

where $\hat{e}_p = \frac{\nabla\phi\times\nabla\psi_p}{|\nabla\phi\times\nabla\psi_p|}$, and then the plasma current can be written as,

$$\mu_0\vec{J} = \mu_0 J_p\hat{e}_p + \mu_0 J_\phi e^{\phi} \tag{19}$$

where $J_\phi = RJ_t = \frac{1}{\mu_0}\triangle^*\psi_p$, $\triangle^* \equiv R^2\nabla\cdot(\frac{1}{R^2}\nabla) = R\frac{\partial}{\partial R}(\frac{1}{R}\frac{\partial}{\partial R}) + \frac{\partial^2}{\partial Z^2}$, $J_p = -\frac{1}{\mu_0}B_p g'$, and the prime denotes the derivative with respect to $\psi_p$. $J_p$ and $J_t$ are the poloidal and toroidal current density respectively.

- Substituting the above expressions into the force balance equation, one obtains the **Grad-Shafralov** equation,

$$\boxed{\triangle^*\psi_p(R,Z) = -\mu_0 R^2 P' - gg'} \tag{20}$$

# Pfirsch-Schlüter current

- It is easy to obtain that
$\lambda \equiv \mu_0 J_\| / B = -[(\mu_0 g P')/B^2 + g'],$
which is very useful.
- let's write $\lambda = \langle\lambda\rangle_\psi + \lambda_{ps}$,it has

$$\langle\lambda\rangle_\psi \equiv \langle\mu_0 J_\| / B\rangle_\psi = -[(\mu_0 g P')\left\langle 1/B^2\right\rangle_\psi + g'] \tag{21}$$

is the net parallel current and

$$\lambda_{ps} \equiv \mu_0 J_\|^{ps} / B = -\mu_0 g P'\left[1/B^2 - \left\langle 1/B^2\right\rangle_\psi\right] \tag{22}$$

is the so-called **Pfirsch-Schlüter** current. This is an important toroidal effect. It can be evaluated, after the equilibrium solution is obtained.

# G-S 方程分析

- **Grad-Shafralov** equation,

$$\boxed{\triangle^*\psi_p(R,Z) = -\mu_0 R^2 P' - gg'}$$

- 方程存在三个未知分布。至少限定其中两个分布，方程才有唯一解。
  - 左边为**环向电流**$\mu_0 J_\phi$。
  - 右边第一项是压强梯度项或者说是**垂直电流**项，
  - 第二项是**极向电流**项。
- $P$和$g$是磁面函数，$J_\phi$ 不是磁面函数.

# 平衡反演计算所需的诊断

- **外部电磁测量**。
  分布在真空室上的**极向磁通环**和**磁探针**等电磁测量结果，可以描述上面方程的边界条件。这一测量约束左边环向电流一项。
- **内部磁场测量**
  结合内部磁场测量，如MSE和偏振干涉仪,可以给出更为精确的芯部磁通和环向电流分布。
- **环向逆磁通测量**
  给出极向电流g的一个总体约束条件。
- **动理学分布测量**。
  等离子体温度密度测量,可以给出**压强分布**约束。H模下新经典理论给出的平行的**Bootstraped current** 也可以进一步约束pedestal附近的分布。
- **其他**
  如Soft-X-Ray和ECE阵列给定的锯齿反转面的位置，以及其他高m模式扰动位置信息等等。

# G-S 方程解法

- 在简单几何位型或者动理学分布假设下，G-S方程可以被解析求解。
- 复杂的几何位型如偏滤器位型，同时具有一般的动理学分布，解析求解非常困难，通常需要数值解。
- 平衡反演数值求解方法举例：
  - 丝电流算法。这种方法适合于求解边界处磁通分布，给出最外层磁面分布信息。优点：算法简单，计算量小，适合于等离子体位型控制。缺点：无法给出其他平衡分布。
  - EFIT[3]算法。如果约束条件（各种等离子体分布参数测量）足够，这种算法可以给出很精确的平衡解。做位型实时控制时，为了加快运算速度，通常也需要做简化运算。

[3] L. L. Lao *et al.* In: *Nucl. Fusion* 25 (1985), p. 1611.

# 丝电流平衡反演

- 如果我们只是关心最外层磁面附近的极向磁通分布，可以忽略G-S方程的右边的分布形式，把内部的环向等离子体电流等效成$M$ 个电流丝$I_{t,k}|_{k\in[1,M]}$.
- By using the least square fitting,

$$\chi^2 = \frac{1}{N}\sum_{j=1}^{N}\frac{[\psi^*_{ps,j} - \psi^c_{ps,j}(I_t)]^2}{\sigma_j^2} \tag{23}$$

where

$$\psi^c_{ps,j} = \sum_{k=1}^{k=M} G_{ps,j} I_{t,k} + \sum_i G_{exs,ij} I_i^{ex} \tag{24}$$

and $\psi^*_{ps,j}$ is the measured value of the $j^{th}$ sensor, $G_*$ are the mutual inductances,and $\sigma_j$ is the error bar of the $j^{th}$ sensor.
- The poloidal flux distribution can be found by using the Green function method,

$$\psi_{p,j} = \sum_{k=1}^{k=M} G_{pp,j} I_{t,k} + \sum_i G_{exp,ij} I_i^{ex} \tag{25}$$

# EFIT算法– Fitting toroidal current profile

- Iteration method with an initial guess of $\psi_p = \psi_{p,0}$ is used to solve this non-linear second order differential equation.
- The terms $P'$ and $gg'$ are represented by some truncated base functions of $\psi_p$,

$$P' \equiv \sum_{m=0}^{M} \alpha_m \Phi_m(\psi_p) \quad and \quad gg' \equiv \sum_{n=0}^{N} \beta_n \Phi_n(\psi_p) \tag{26}$$

At the step $l$, the coefficients $\alpha_{ml}$ and $\beta_{nl}$ , and hence the toroidal current profile, are determined by minimization

$$\chi^2 = \frac{1}{k} \sum_{j=1}^{k} \frac{[\psi^*_{ps,j} - \psi^c_{ps,j}(\alpha_m, \beta_n)]^2}{\sigma_j^2} \tag{27}$$

where

$$\psi^c_{ps,j} = \int G_{ps,j} J_t(\alpha_{ml}, \beta_{nl}, \psi_{p,l-1}) ds_\phi + \sum_i G_{exs,ij} I_i^{ex} \tag{28}$$

# EFIT算法– G-S solution

The poloidal flux distribution can be found by using one of the following two methods:

- Green function method.
  the poloidal flux $\psi_p$ at the $j^{th}$ grid point of the calculation area $(R, Z)$ at this step can be upgraded from,

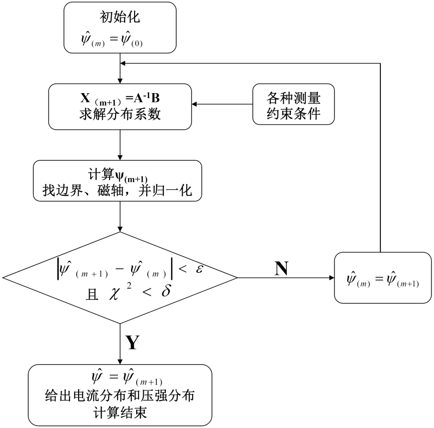
$$\psi_{p,l,j} = \int G_{pp,j} J_t(\alpha_{ml}, \beta_{nl}, \psi_{p,l-1}) ds_\phi + \sum_i G_{exp,ij} I_i^{ex} \tag{29}$$

- Solving the second order differential equation,

$$\triangle^* \psi_p(R, Z) = \mu_0 R J_t \tag{30}$$

The newly obtained poloidal flux replaces the initial one and the whole process is repeated until it converges.

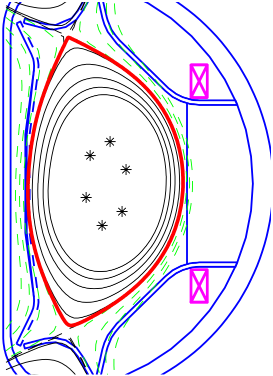
# EFIT反演算法– 小结



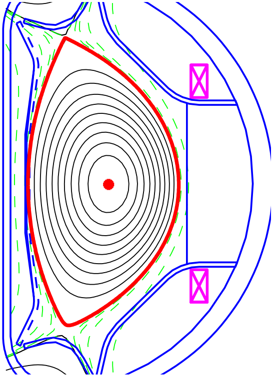

1.26 EFIT平衡反演流程

- EFIT输出包括:
  $\psi_p(R,Z)$;
  $P(\psi)$, $P'$;
  $g(\psi)$, $g'$;
  $q(\psi)$, $l_i$, $\beta$ *etc.*
- EFIT平衡算法可以通过将反演算法中的测量值改为设定值来实现。

# EAST上平衡反演磁通分布举例



1.27 丝电流平衡反演



1.28 EFIT平衡反演

# 磁面坐标系

- 由于磁约束聚变中的强磁场效应，使得等离子体具有很强的**各向异性**，平行于磁力线和垂直于磁力线方向的等离子体动力学行为差异很大，因此，建立在磁力线分布基础上的坐标系能够使很多分析变得简单方便，**磁面坐标系**（flux coordinates）在聚变等离子体物理各项研究中经常用到。引入这种坐标系，使得几何的复杂性都归结到微分算符的度规系数中。
- 磁面坐标系$(\rho,\theta,\zeta)$下，磁场可以写成

$$\vec{B} = \psi'_p[q\nabla\rho\times\nabla\theta - \nabla\rho\times\nabla\zeta] \tag{31}$$

where prime denotes the derivative over $\rho$.

# 磁面坐标选取

- 径向坐标通常取磁面函数，$\rho=\rho(\psi_p)$，如$\psi_p,\psi_t$，$\sqrt{\psi_p}$，$\sqrt{\psi_t}$，$V(\psi_p),q$等等。另外两个类角坐标有不同的取法。
- 从$\vec{B}\cdot\nabla(q\theta-\zeta)=0$可知，磁力线在磁面坐标系中是直线，因此，磁面坐标又称为**直场线坐标系**（straight field line coordinates）。
- 极向磁面坐标通常取为$\theta=\theta(R,Z)$, 因此$\nabla\theta\cdot\nabla\phi=0$。环向磁通坐标写成

$$\zeta\equiv\phi-q(\psi_p)\delta(\psi_p,\theta) \tag{32}$$

这里$\phi$是装置几何环向角，$(R,\phi,Z)$为柱坐标系，$\delta$是$\theta$的周期函数。

# 磁面坐标计算

- 从磁面坐标的$\mathcal{J}=\mathcal{J}(\rho,\theta)$的定义，

$$\mathcal{J}^{-1}\equiv(\nabla\rho\times\nabla\theta)\cdot\nabla\zeta=\frac{B_p}{|\psi_p'|}\left.\frac{\partial\theta}{\partial l}\right|_{\psi_p,\phi} \tag{33}$$

可知

$$\left.\frac{\partial\theta}{\partial l}\right|_{\psi_p,\phi}=\frac{|\psi_p'|}{\mathcal{J}B_p} \tag{34}$$

- 从$B^{\phi}=q(1+\partial_\theta\delta)\psi_p'/\mathcal{J}=\frac{g}{R^2}$可知，

$$\partial_\theta\delta=\frac{g\mathcal{J}}{qR^2}\frac{1}{\psi_p'}-1 \tag{35}$$

因此

$$\left.\frac{\partial\delta}{\partial l}\right|_{\psi_p,\phi}=\partial_l\theta\partial_\theta\delta=\frac{|\psi_p'|}{\mathcal{J}B_p}[\frac{g\mathcal{J}}{qR^2}\frac{1}{\psi_p'}-1] \tag{36}$$

# $\mathcal{J}$形式

- 定义$\mathcal{J} \equiv \frac{|\psi_p'|}{c(\psi_p)}\mathcal{J}_0$，从$\theta \in [0, 2\pi]$，可知$c = \frac{2\pi}{\oint (1/\mathcal{J}_0)\frac{dl}{B_p}} = \frac{|\psi_p'|}{\hat{V}'\langle 1/\mathcal{J}_0\rangle_\psi}$。这里$\langle ... \rangle_\psi$表示磁面平均，$\hat{V} = V/(4\pi^2)$，$V(\psi_p)$是磁面包围的体积。因此$\mathcal{J}$可以写成，

$$\mathcal{J} = \hat{V}'\mathcal{J}_0 \langle 1/\mathcal{J}_0\rangle_\psi \tag{37}$$

- 给定一个$\mathcal{J}_0$的形式，利用前面的公式，通过极向截面上作磁面的几个积分即可计算出磁面坐标的分布。
- 如果我们定义

$$\mathcal{J}_0 \equiv \frac{R^i}{|\nabla\psi_p|^j B^k} \tag{38}$$

给定一组$(i, j, k)$的值，即可以得到一个磁面坐标系。这种形式的$\mathcal{J}$可以产生包括Hamada, Boozer, PEST, Equal-Arc等等各种常用的磁面坐标系。
- 此外，安全因子可以通过以下公式计算

$$q(\psi_p) \equiv \frac{1}{2\pi}\oint \frac{B^\zeta}{B^\theta}d\theta = \frac{\hat{V}'}{\psi_p'}g\left\langle 1/R^2\right\rangle_\psi \tag{39}$$

# 常用磁面坐标系

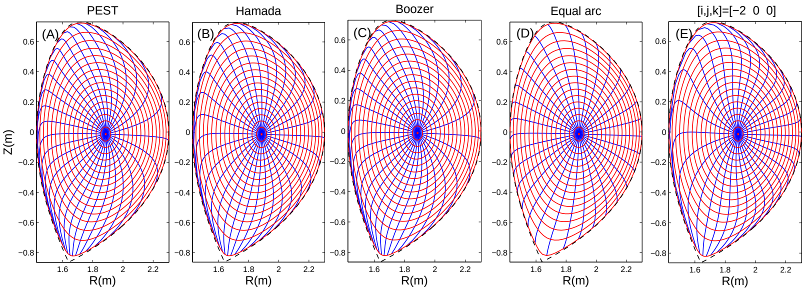
The most often used straight field line coordinates are listed in the following.

- **PEST coordinates:**
  If $i = 2,\ j = k = 0$, the coordinates will be PEST coordinates. It has $\mathcal{J}_0 = R^2, \zeta = \phi$ and $\mathcal{J} = \hat{V}' < 1/R^2 >_\psi R^2 = \hat{V}' < B_t^2 >_\psi /B_t^2$. It is also named as the basic straight field line coordinates.
- **Hamada coordinates:**
  If $i = j = k = 0$, the coordinates will be Hamada coordinates. It has $\mathcal{J}_0 = 1/B_0^2$, and $\mathcal{J} = \hat{V}' < B_0^2 >_\psi /B_0^2 = \hat{V}'$.
- **Equal-arc coordinates:**
  If $i = j = 1,\ k = 0$, the coordinates will be Equal-arc coordinates. It has $\mathcal{J}_0 = 1/B_p$, $\partial_l\theta|_{\psi_p,\phi} = 2\pi/l$, and $\mathcal{J} = \hat{V}' < B_p >_\psi /B_p$.
- **Boozer coordinates:**
  If $i = 0,\ j = 0,\ k = 2$, the coordinates will be Boozer coordinates. It has $\mathcal{J}_0 = 1/B^2$, and $\mathcal{J} = \hat{V}' < B^2 >_\psi /B^2$.

# 径向坐标选取

- If the radial coordinates is chosen as $\rho = \hat{V}$, then it has $\hat{V}' = 1$.
- If we choose the radial coordinate as $\rho = \rho_t \equiv \sqrt{2\psi_t/B_0}$, then $\psi_p' = \rho B_0/q$ and $\hat{V}' = \rho B_0/(g\left\langle 1/R^2\right\rangle_\psi)$. It reduces to the minor radius in the cylindrical coordinates.
- If we choose the radial coordinate as $\rho = \rho_p \equiv \sqrt{(\psi_p - \psi_{axis})/(\psi_{bdy} - \psi_{axis})}$, then $\psi_p' = 2\rho(\psi_{bdy} - \psi_{axis})$ and $\hat{V}' = 2\rho q(\psi_{bdy} - \psi_{axis})/(g\left\langle 1/R^2\right\rangle_\psi)$. Here $\psi_{bdy}$ and $\psi_{axis}$ are $\psi_p$ at the plasma boundary and magnetic axis, respectively.
- If we choose $\rho = q$ as the radial coordinate, it has $\psi_p' = 1/d_{\psi_p}q$, and $\hat{V}' = d_{\psi_p}\hat{V}/d_{\psi_p}q = \frac{q}{g\langle 1/R^2\rangle_\psi d_{\psi_p}q}$.

# 磁面坐标举例




1.29 Examples for EAST equilibrium

# 度规（Metrics）计算

All the metrics $g^{\alpha\beta}$ (contra-variants), $g_{\alpha\beta}$ (co-variants) can be written as a combination of $g^{\rho\rho}, g^{\rho\theta}, g^{\theta\theta}$.
The basic metrics $g^{\rho\rho}$, $g^{\rho\theta}$ and $g^{\theta\theta}$ can be calculated from the following equations,

$$g^{\rho\rho} = \left(\frac{R}{\mathcal{J}}\right)^2 [(\partial_\theta R)^2 + (\partial_\theta Z)^2] \quad (40)$$

$$g^{\rho\theta} = -\left(\frac{R}{\mathcal{J}}\right)^2 [\partial_\rho R \partial_\theta R + \partial_\rho Z \partial_\theta Z] \quad (41)$$

$$g^{\theta\theta} = [(g^{\rho\theta})^2 + R^2/\mathcal{J}^2]/g^{\rho\rho} \quad (42)$$

The covariants of the metric can be transformed form the contravariant ones by using their definitions.

# 沿磁力线坐标系（Field aligned coordinates）

几何效应都体现在微分算符中。通过引入磁面坐标，使得微分算符运算更简便。
Using the field aligned coordinates $(\rho, \alpha = q\theta - \zeta, \theta)$, it is obvious that $\vec{B}\cdot\nabla = B^{\theta}\partial_{\theta}$, and the Jacobian of $(\rho, \alpha, \theta)$ can be written as $\mathcal{J}^{-1} \equiv (\nabla\rho \times \nabla\alpha)\cdot\nabla\theta = B^{\theta}/\psi_p'$.
It has

$$
\begin{aligned}
B_{\rho} &= [q'\theta + (q\delta)']g - \frac{\mathcal{J}\psi_p'}{R^2}g^{\rho\theta} &\quad (43a)\\
B_{\alpha} &= -g &\quad (43b)\\
B_{\theta} &= B^2/B^{\theta} = \mathcal{J}B^2/\psi_p' &\quad (43c)
\end{aligned}
$$

If we choose the radial coordinate $\rho = \rho_t \equiv \sqrt{2\psi_t/B_0}$, it has $\mathcal{J} = \hat{V}'G_J = \frac{\rho B_0}{G}G_J$ with $G_J \equiv \mathcal{J}_0\langle 1/\mathcal{J}_0\rangle_{\psi}$, and $G \equiv g\left\langle 1/R^2\right\rangle_{\psi}$. It also has $\frac{\psi_p'}{\mathcal{J}B} = \frac{R\langle 1/R^2\rangle_{\psi}}{qG_J} \equiv G_{\parallel}\frac{1}{qR_0}$, and $G_{\parallel} \equiv \frac{RR_0\langle 1/R^2\rangle_{\psi}}{G_J} \approx 1$.

# 重要微分几何算子

$$\nabla\cdot\vec{F} \xrightarrow{\vec{F}_\perp\equiv\frac{\vec{B}\times\nabla f}{B^2}} B\nabla_\parallel(F_\parallel/B)+\mathcal{K}(f) \tag{44a}$$

$$\nabla_\parallel f \approx \frac{G_\parallel}{qR_0}\partial_\theta f-\frac{qB}{\rho B_0}\{A/B,f\}_{\rho\alpha} \tag{44b}$$

$$\nabla_\perp^2 f \approx \frac{G}{\rho}\partial_\rho[\frac{\rho}{G}(g^{\rho\rho}\partial_\rho+g^{\rho\alpha}\partial_\alpha)]f+\partial_\alpha[(g^{\rho\alpha}\partial_\rho+g^{\alpha\alpha}\partial_\alpha)]f \tag{44c}$$

$$\vec{V}_E\cdot\nabla f \equiv \frac{\vec{B}\times\nabla\Phi}{B^2}\cdot\nabla f\approx\frac{q}{\rho B_0}\{\Phi,f\}_{\rho\alpha} \tag{44d}$$

$$\mathcal{K}(f) \equiv \nabla\cdot\left(\frac{\vec{B}\times\nabla f}{B^2}\right)\approx\frac{q}{\rho B_0 B^2}\left[\partial_\rho B^2\partial_\alpha f-\frac{RG_\parallel}{qR_0}\partial_\theta B^2\partial_\rho f\right] \tag{44e}$$

where the Poison bracket $\{A,B\}_{\rho\alpha}=\partial_\rho A\partial_\alpha B-\partial_\alpha A\partial_\rho B$. Here the perturbation field $\vec{B}_1=\nabla\times(A\vec{b})\approx-\vec{B}\times\nabla(A/B)$ is used. $k_\parallel\ll k_\perp$ is also used in the above formula.

# Shift metric

The magnetic shear can leads to strong deformation of the coordinate cells in the plane perpendicular to the field. One idea of the shift metric procedure has been proposed to solve this difficulty[4].
The shifted field aligned coordinates can be written as,

$$\alpha_k = q(\theta - \theta_k) - \zeta + \delta_k(\rho) = \alpha - q\theta_k + \delta_k \tag{45}$$

Where $\delta_k$ is a peoridic function of $\theta_k$. It has $d\alpha|_{\theta=\theta_k} = d\alpha_k|_{\theta=\theta_k}$.
From $g^{\rho\alpha_k}|_{\theta=\theta_k} = 0$, one obtains,

$$\partial_\rho \delta_k = [g^{\rho\zeta} - q g^{\rho\theta}]/g^{\rho\rho} \tag{46}$$

After the shift metric procedure $\alpha \to \alpha_k$,

$$\nabla_\perp^2 f \quad \approx \quad \frac{G}{\rho}\partial_\rho[\frac{\rho}{G}(g^{\rho\rho}\partial_\rho)]f + \frac{q^2}{\rho^2}\partial_{\alpha_k}^2 f \tag{47}$$

[4] B. Scott. In: *Phys. Plasmas* 8 (2001), p. 447.

# 磁面函数与磁面平均

- 只依赖于磁通坐标的称为**磁面函数**，如$q(\psi)$, $g(\psi)$, 平衡的$P(\psi)$, $T(\psi)$, $N(\psi)$ 等等.
- 而有些物理量依赖于几何位型，如$R$,$B$,$J_\phi$等等都不是磁面函数。
- 很多分析中需要在磁面上计算，如输运分析中研究粒子、动量和热在垂直于磁面方向的一维的扩散与对流的过程，这时需要对非磁面函数做**磁面平均**。
- 磁面平均定义为:

$$\langle F\rangle_\psi = \frac{\oint_s F\mathcal{J}d\theta d\phi}{\oint_s \mathcal{J}d\theta d\phi} = \frac{\oint_l F\mathcal{J}d\theta}{\oint_l \mathcal{J}d\theta} = \frac{1}{2\pi\widehat{V}'}\oint_l F\mathcal{J}d\theta = \frac{\oint_l F(dl/B_p)}{\oint_l (dl/B_p)} \qquad (48)$$

where $\widehat{V}' \equiv d\widehat{V}/d\rho = \frac{1}{2\pi}\oint_l \mathcal{J}d\theta$. If $\rho=\rho_t$, then $\hat{V}' = \frac{\rho B_0}{g\langle 1/R^2\rangle_\psi}$.

- 输运方程中重要的磁面平均项:

$$\left\langle \nabla\cdot\vec{F}\right\rangle_\psi = \left\langle \mathcal{J}^{-1}\partial_\rho(\mathcal{J}F^\rho)\right\rangle_\psi = \frac{\partial_\rho[\langle F^\rho\rangle_\psi \oint \mathcal{J}d\theta]}{\oint \mathcal{J}d\theta} = \frac{1}{\hat{V}'}\partial_\rho[\hat{V}'\langle F^\rho\rangle_\psi] \qquad (49)$$

# 平衡与不稳定性及输运约束的关系

- 平衡分布梯度是宏观和微观不稳定性的驱动源。
- 不稳定性的边界决定了稳定平衡的存在区间。
- 输运决定了等离子体平衡的演化。
- 平衡决定的几何位型对不稳定性和输运有着重要影响。
- 平衡的磁场位型影响粒子的运动轨道，进而影响等离子体输运与约束。

# 曲率效应

By using the Clebsch representation of the magnetic field $\vec{B} = \psi_p' \nabla\rho \times \nabla\alpha$, the curvature of the magnetic field line can be written as

$$\begin{aligned}
\vec{\kappa} &\equiv \vec{b}\cdot\nabla\vec{b} = -\vec{b}\times(\nabla\times\vec{b}) = \frac{\mu_0}{B^2}\vec{J}\times\vec{B} - \frac{1}{B^2}\vec{b}\times[\vec{b}\times\nabla(B^2/2)] \\
&= \frac{\mu_0}{B^2}\left[\nabla(P+\frac{B^2}{2\mu_0}) - \vec{b}\vec{b}\cdot\nabla(\frac{B^2}{2\mu_0})\right] \\
&= \kappa_\rho\nabla\rho + \kappa_\alpha\nabla\alpha
\end{aligned} \tag{50}$$

where

$$\kappa_\rho \equiv \vec{\kappa}\cdot e_\rho = \frac{\partial_\rho(\mu_0 P + B^2/2)}{B^2} - \frac{B_\rho}{B^4}[\vec{B}\cdot\nabla(B^2/2)] \tag{51a}$$

$$\kappa_\alpha \equiv \vec{\kappa}\cdot e_\alpha = -\frac{B_\alpha}{B^4}[\vec{B}\cdot\nabla(B^2/2)] + \frac{\partial_\alpha(B^2/2)}{B^2} \tag{51b}$$

气球模（Ballooning mode）的驱动项，$-2(\vec{\xi}_\perp\cdot\nabla P)(\vec{\xi}_\perp^*\cdot\vec{\kappa})$。显然，当$\vec{\kappa}\cdot\nabla P > 0$时起解稳作用。
测地（Geodesic）曲率可以将离子声波和阿尔芬波耦合起来形成Geodesic Acoustic Mode (GAM)和Beta induced Alfven Eigenmode (BAE) 等等。

# ∇B与环效应

$\nabla B = \vec{b}\frac{B^\theta}{B}\partial_\theta B + B\vec{\kappa} - \frac{\mu_0}{B}\nabla P$，环效应和数学上的$\nabla B$在环内外侧的非对称性是相对应的.

- 第一项平行梯度形成磁镜场，可能导致粒子被捕获（trapped）形成香蕉轨道（banana orbit）。
  沿着磁力线存在$B_m \le B \le B_M$,如果粒子的最大平行速度满足$v_{\parallel,M}^2 \le 2\mu(B_M - B_m)$,就会被捕获形成香蕉粒子.
- 垂直磁场梯度会引起粒子导向中心（guiding center）的漂移(drift),

$$\vec{v}_{DB} = -\frac{M(\mu\nabla B + v_\parallel^2\vec{\kappa}) \times \vec{B}}{eB^2} \approx -\frac{(W_\perp + 2W_\parallel)\nabla B \times \vec{B}}{eB^3} \qquad (52)$$

- 环效应引起的离子平行粘滞（parallel viscosity）会导致极向流的衰减（poloidal flow damping）。
- 电子平行粘滞会引起靴带电流（Bootstrapped current）。

# Shaping效应

Shaping对于等离子体稳定性和约束都具有重要的影响。

- 拉长可以增加等离子体体积。
- 拉长引起$n = 0$垂直位移不稳定性。
- 一定的拉长与三角形变对宏观的MHD和微观湍流具有致稳作用。
- Squareness对于等离子体稳定性和约束都有重要影响。

# 安全因子与磁剪切

安全因子（或螺旋度）与系统的稳定性密切相关。

- $q_{95} = \frac{aB}{I}S$ 对于Global MHD极限具有重要意义。这里$S$是个形状因子，在大纵横比柱几何下$S \sim a/R$.
- $\beta_N^{crit} = 4l_i$ 说明宏观稳定性与电流密度分布密切相关。
- 电流密度梯度或磁剪切是撕裂模（tearing mode）和内部电阻扭曲模(resistive internal kink mode)的主要自由能源。
- 磁剪切对于交换模（interchange mode）具有致稳效应。
- 气球模的第二稳定区与局域磁剪切的增加相关。

# 结束

到此结束，希望有点用！