This note discusses the results in Lin-Liu's paper[1], and give some details about the derivation and numerical results.

1 Different choices of velocity space coordinates

We consider axially symetric (about the equilibrium magnetic field) velocity distribution function. The velocity space can be described by w, μ and σ coordinates. Then, the distribution function is written as

$$f = f(w, \mu, \sigma, \boldsymbol{r}). \tag{1}$$

We first consider the nonrelativistic case. In this case the magnetic moment μ is defined as

$$\mu = \frac{mv_\perp^2}{2B}, \tag{2}$$

the kinetic energy is given by $w = mv^2/2$. And $\sigma = \mathrm{sgn}(v_\parallel) = \pm\, 1$. In this choice of velocity space coordinates, $v_\parallel$ is not an independent variable, $v_\parallel$ can be expressed as a function of w, μ and σ,

$$v_\parallel = \sigma\sqrt{\frac{2}{m}(w - \mu B)}. \tag{3}$$

In addition to (w, μ, σ) coordinates used in above, one can also use another velocity space: (v, θ), where θ is the included angle between local magnetic field and velocity(the so-called pitch angle). The transformation relation between this two coordinates is

$$\begin{cases} \mu = \dfrac{mv^2\sin^2\theta}{2B} \\ w = \dfrac{mv^2}{2} \\ \sigma = \mathrm{sgn}(\cos\theta) \end{cases} \text{and} \quad \begin{cases} v = \sqrt{2w/m} \\ \cos\theta = \sigma\sqrt{1 - \dfrac{B\mu}{w}} \end{cases} \tag{4}$$

If the distribution function in (w, μ, σ) and (v, θ) space are given, respectively, by $f(w, \mu, \sigma, \boldsymbol{r})$ and $g(v, \theta, \boldsymbol{r})$, the two distribution function are related to each other by

$$g(v, \theta, \boldsymbol{r}) = f\left(\frac{mv^2}{2}, \frac{mv^2\sin^2\theta}{2B}, \mathrm{sgn}(\cos\theta), \boldsymbol{r}\right), \tag{5}$$

and

$$f(w, \mu, \sigma, \boldsymbol{r}) = g\left(\sqrt{2w/m}, \arccos^{-1}\left(\sigma\sqrt{1 - \frac{B\mu}{w}}\right), \boldsymbol{r}\right).$$

In the banana regime, using the expansion of drift kinetic equation in terms of the small parameter ω_c/ω_b (Refer to another section in this note), $f = f_0 + f_1$, and $f_i \propto (\omega_c/\omega_b)^i$, the leading order of the equation takes the form

$$v_\parallel \hat{\boldsymbol{b}} \cdot \nabla f_0(w, \mu, \sigma, \boldsymbol{r}) = 0, \tag{6}$$

where $\hat{\boldsymbol{b}}$ is the unit vector along the magnetic field and operator ∇ operate on the 4th of the variable list (w, μ, σ, $\boldsymbol{r}$) (i.e. the operator ∇ is taken by holding w and μ constant). In toroidal geometry, flux coordinates (ρ, θ_{pol}, ϕ) is usually used, where ρ is the radial variable (usually is the normalized toroidal flux, $\rho \equiv (\Phi/\pi B_{T0})^{1/2}$, where Φ is toroidal flux), θ_{pol} is the poloidal angle and ϕ is the toroidal angle. We now consider the toroidal symmetric case in which f_0 is independent of toroidal angle ϕ, then $f_0 = f_0(w,\, \mu,\, \sigma,\, \rho,\, \theta_{\mathrm{pol}})$. Then from Eq. (6), one knows that for the phase space point where $v_\parallel \neq 0$, the distribution function f_0 is independent of θ_{pol}, Therefore

$$f = f(w, \mu, \sigma, \rho) \tag{7}$$

Then from Eq. (5), one gets

$$g(v,\theta,\rho,\theta_{\rm pol}) = f\left(\frac{mv^2}{2}, \frac{mv^2\sin^2\theta}{2B(\rho,\theta_{\rm pol})}, {\rm sgn}(\cos\theta), \rho\right), \tag{8}$$

which indicates that g is still a function of $\theta_{\rm pol}$. Define a function which is independent of $\theta_{\rm pol}$,

$$\begin{aligned} G(v,\theta,\rho) &\equiv g(v,\theta,\rho,\theta_{\rm pol}=0) \\ &= f\left(\frac{mv^2}{2}, \frac{mv^2\sin^2\theta}{2B(\rho,0)}, {\rm sgn}(\cos\theta), \rho\right), \end{aligned} \tag{9}$$

which is the distribution function at the $\theta_{\rm pol}=0$ spatial point in terms of the v,θ velocity coordinates. In terms of this function, the distribution at arbitrary $\theta_{\rm pol}$, Eq. (8), can be written as,

$$g(v,\theta,\rho,\theta_{\rm pol}) = G(v_0(v,\theta),\theta_0(v,\theta),\rho), \tag{10}$$

where

$$v_0 = v, \tag{11}$$

$$\sin^2\theta_0 = \sin^2\theta\frac{B(\rho,0)}{B(\rho,\theta_{\rm pol})} \Rightarrow \theta_0 = \arcsin\left[\sin\theta\sqrt{\frac{B(\rho,0)}{B(\rho,\theta_{\rm pol})}}\right]. \tag{12}$$

These agree with the results given in CQL3D manual. Note that the above result is obtained using only the transformation rule between the two systems of velocity coordinates. This transformation has nothing to do with the motion of particles; the above derivation has nothing to do with whether w or μ is constant of the motion.

2 Current contributed by trapped electrons is zero

In terms of (w,μ) coordinates, the trapped and passing regions of pahse space are shown in Fig. 1.

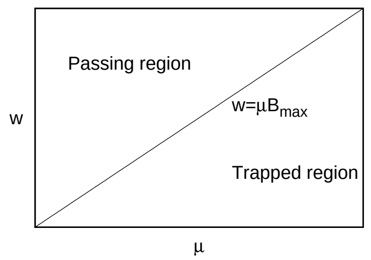

Figure 1. Passing and trapped regions of phase space $(w,\ \mu)$. The boundary between passing and trapped region is given by $w=\mu B_{\rm max}$, where $B_{\rm max}$ is the maximum value of magnetic field on one flux surface.

Because we consider time scale much longer than the bounce period of trapped electrons, the distribution function we are interested should be the time average of the distribution over many peroids of bounce motion. For trapped electrons, σ is peroid in bounce motion. Thus in the trapped region the averaged distribution should a even function about the variable σ, i.e.,

$$f(w,\mu,-\sigma) = f(w,\mu,\sigma). \tag{13}$$

Using this result, we can prove that the parallel current contributed by trapped electrons is zero, i.e.,

$$\int_{\rm Trap} v_\parallel f(w,\mu,\sigma)d^3v = 0. \tag{14}$$

Proof: Using

$$d^3v = \frac{2\pi}{m^2}\sum_{\sigma=\pm1}\frac{B}{|v_\parallel|}d\mu d\varepsilon$$

The left hand side of the above equation is written as

$$
\begin{aligned}
\int_{\mathrm{Trap}} v_\parallel f(w,\mu,\sigma) d^3 v &= \frac{2\pi}{m^2}\sum_{\sigma=\pm 1}\int_{\mathrm{Trap}} v_\parallel f(w,\mu,\sigma)\frac{B}{|v_\parallel|} d\mu d\varepsilon \\
&= \frac{2\pi}{m^2}\sum_{\sigma=\pm 1}\int_{\mathrm{Trap}} \sigma f(w,\mu,\sigma) B d\mu d\varepsilon \\
&= \frac{2\pi}{m^2}\Big[\int_{\mathrm{Trap}} (1) f(w,\mu,1) B d\mu d\varepsilon + \int_{\mathrm{Trap}} (-1) f(w,\mu,-1) B d\mu d\varepsilon\Big] \\
&= \frac{2\pi}{m^2}\Big[\int_{\mathrm{Trap}} f(w,\mu,1) B d\mu d\varepsilon - \int_{\mathrm{Trap}} f(w,\mu,1) B d\mu d\varepsilon\Big] \\
&= 0. \qquad (15)
\end{aligned}
$$

Thus Eq. (13) is proved.

3 Definition of adjoint operator

We call a operator C^{l+} is the adjoint operator of C^l, if C^{l+} has the following relation with C^l,

$$\int d\Gamma\, g C_e^{l+}(f) = \int d\Gamma\, f C_e^l(g) \qquad (16)$$

where f and g are two arbitrary functions.

4 Theory of the adjoint method

The perturbed distribution function satisfies the linearized Fokker-Planck equation,

$$v_\parallel \hat{\boldsymbol{b}}\cdot\nabla f_1(\varepsilon,\mu,\sigma,\boldsymbol{r}) - C_e^l(f_1) = S_w(f_m), \qquad (17)$$

where $\hat{\boldsymbol{b}}$ is the unit vector along the equilibrium magnetic field, $v_\parallel$ is the velocity component parallel to the magnetic field, $C_e^l(f)$ is the linearized collision operator. We want to solve this equation to determine the first moment of f_1

$$j_\parallel = q_e \int f_1 v_\parallel d\boldsymbol{v}, \qquad (18)$$

where $d\boldsymbol{v}$ is the volume element of velocity space. Before introducing the adjoint method, let us prove that $j_\parallel/B$ is a function of magnetic flux. First, it can be proved that the divergence of the parallel current $\boldsymbol{j}_\parallel$ is zero (Refer to another note),

$$\nabla\cdot(j_\parallel \hat{\boldsymbol{b}}) = 0 \qquad (19)$$

Then from this, it's straightforward to obtain

$$
\begin{aligned}
&\Rightarrow \nabla\cdot\Big(j_\parallel \frac{\boldsymbol{B}}{B}\Big) = 0 \\
&\Rightarrow \frac{j_\parallel}{B}\nabla\cdot\boldsymbol{B} + \boldsymbol{B}\cdot\nabla\frac{j_\parallel}{B} = 0 \\
&\Rightarrow \boldsymbol{B}\cdot\nabla\frac{j_\parallel}{B} = 0, \qquad (20)
\end{aligned}
$$

Eq. (20) indicates that $j_\parallel/B$ is a flux surface function. Now we discuss the adjoint method. Instead of solving Eq. (17) directly to get the f_1 and then calculate the first moment, it turns out that we can get $j_\parallel$ though the following way. First solve the following adjoint equation:

$$-v_\parallel \hat{\boldsymbol{b}}\cdot\nabla\chi - C_e^{l+}(\chi) = \frac{v_\parallel B}{\langle B^2\rangle}, \qquad (21)$$

where $\langle ...\rangle$ is the flux surface average. Then multiplying Eq. (21) by f_1, we obtain

$$- f_1 v_\parallel \hat{\boldsymbol{b}} \cdot \nabla \chi - f_1 C_e^{l+}(\chi) = \frac{f_1 v_\parallel B}{\langle B^2\rangle}. \tag{22}$$

Integrate the two sides of the above equation in velocity space, we obtain

$$\int d\Gamma\Big[- f_1 v_\parallel \hat{b} \cdot \nabla \chi - f_1 C_e^{l+}(\chi)\Big] = \frac{1}{q_e}\frac{j_\parallel B}{\langle B^2\rangle}. \tag{23}$$

Flux averaging the two sides of the above equation yields

$$\left\langle \int d\Gamma\Big[- f_1 v_\parallel \hat{b} \cdot \nabla \chi - f_1 C_e^{l+}(\chi)\Big]\right\rangle = \frac{1}{q_e}\left\langle\frac{j_\parallel B}{\langle B^2\rangle}\right\rangle \tag{24}$$

Now we make use of the most important properties of the operators $v_\parallel \hat{b} \cdot \nabla$ and C_e^{l+}, i.e., adjoint properties,

$$\left\langle \int d\Gamma f v_\parallel \hat{b} \cdot \nabla g\right\rangle = -\left\langle \int d\Gamma g v_\parallel \hat{b} \cdot \nabla f\right\rangle \tag{25}$$

and

$$\int d\Gamma f_1 C_e^{l+}(\chi) = \int d\Gamma \chi C_e^l(f_1). \tag{26}$$

[Refer to another note for the proof of Eq. (25).] Using the above two properties, Eq. (24) is written as

$$\left\langle \int d\Gamma \chi\Big[v_\parallel \hat{b} \cdot \nabla f_1 - C_e^l(f_1)\Big]\right\rangle = \frac{1}{q_e}\left\langle\frac{j_\parallel B}{\langle B^2\rangle}\right\rangle \tag{27}$$

Using Eq. (17) to rewrite the term in the bracket of Eq. (27), we obtain

$$\left\langle \int d\Gamma \chi S_w\right\rangle = \frac{1}{q_e}\left\langle\frac{j_\parallel B}{\langle B^2\rangle}\right\rangle. \tag{28}$$

Making use of the fact that $j_\parallel / B$ is a flux surface function, the above equation is reduced to

$$\Longrightarrow \frac{j_\parallel}{B} = q_e\left\langle \int d\Gamma \chi S_w\right\rangle. \tag{29}$$

5 Efficiencies of current drive

Multiplying the drift kinetic equation

$$\frac{\partial f_1}{\partial t} + v_\parallel \hat{b} \cdot \nabla f_1 - C_e^l(f_1) = S_w, \tag{30}$$

by w and then integrating in velocity space, one gets

$$\frac{\partial}{\partial t}\Big(\int d\Gamma w f_1\Big) + \int d\Gamma\Big(v_\parallel \hat{b} \cdot w \nabla f_1\Big) - 0 = \int d\Gamma w S_w, \tag{31}$$

where use has been made of the fact that linearized collision operator C_e^l conserve energy, $\int d\Gamma w C_e^l(f_1) = 0$. Then flux averaging Eq. (31) gives

$$\frac{\partial}{\partial t}\Big(\Big\langle \int d\Gamma w f_1\Big\rangle\Big) + \Big\langle \int d\Gamma\Big(v_\parallel \hat{b} \cdot w \nabla f_1\Big)\Big\rangle = \Big\langle \int d\Gamma w S_w\Big\rangle \tag{32}$$

Using the fact that (Question here***!!! please check carefully that the following equation is correct)

$$\Big\langle \int d\Gamma\Big(v_\parallel \hat{b} \cdot w \nabla f_1\Big)\Big\rangle = -\Big\langle \int d\Gamma\Big[v_\parallel \hat{b} \cdot f_1 \nabla(w)\Big]\Big\rangle = 0 \tag{33}$$

Eq. (32) is reduced to

$$\frac{\partial}{\partial t}\left(\left\langle \int d\Gamma w f_1 \right\rangle\right) = \left\langle \int d\Gamma w S_w \right\rangle, \tag{34}$$

which indicates that the averaged energy absorbed per unit volume per unit time by electrons from the wave is given by

$$P = \left\langle \int d\Gamma w S_w \right\rangle \tag{35}$$

Define averaged efficiency of current drive at a flux surface as

$$\zeta^\star \equiv \frac{e^3}{\varepsilon_0^2}\frac{n_e}{T_e}\frac{\langle j_\parallel \rangle}{2\pi P} \tag{36}$$

Using $\langle j_\parallel \rangle = (j_\parallel / B)\langle B \rangle$ and Eqs.(29) and (35), $\zeta^\star$ can be written as

$$\begin{aligned} \zeta^\star &= \frac{e^3}{\varepsilon_0^2}\frac{n_e}{T_e}\frac{\langle B \rangle}{2\pi P}\frac{j_\parallel}{B} \\ &= -\frac{e^4}{\varepsilon_0^2}\frac{n_e}{T_e}\frac{\langle B \rangle}{2\pi P}\left\langle \int d\Gamma \chi S_w \right\rangle \\ &= -\frac{e^4}{\varepsilon_0^2}\frac{n_e}{T_e}\frac{\langle B \rangle}{2\pi}\frac{\langle \int \ d\Gamma \chi S_w \rangle}{\langle \int \ d\Gamma \varepsilon S_w \rangle}. \end{aligned} \tag{37}$$

Define

$$v_e = \sqrt{\frac{2T_e}{m_e}},\ \nu_{e0} = \frac{e^4 n_e \ln\Lambda}{4\pi\varepsilon_0^2 m_e^2 v_e^3},\ \overline{\chi} = \chi \nu_{e0}\frac{B_{\max}}{v_e}$$

then $\zeta^\star$ in Eq. (37) can be written as

$$\zeta^\star \ = \ -\frac{4}{\ln\Lambda}\left\langle \frac{B}{B_{\max}} \right\rangle \frac{\langle \int \ d\Gamma \overline{\chi} S_w \rangle}{\langle \int \ d\Gamma \varepsilon / m_e v_e^2 S_w \rangle}. \tag{38}$$

Eq. (38) agrees with Eq. (11) in Lin-Liu's paper[1].

6 Bounce-averaged adjoint equation

The adjoint equation [Eq. (21)] is

$$-v_\parallel \hat{\boldsymbol{b}} \cdot \nabla \chi - C_e^{l+}(\chi) = \frac{v_\parallel B}{\langle B^2 \rangle}, \tag{39}$$

where $\chi = \chi(u, \lambda, \sigma, \rho, \theta_p)$. Expanding χ as $\chi = \chi_0 + \chi_1 + \ldots$, the leading order, χ_0, satisfies the equation (Refer to another note),

$$v_\parallel \hat{b} \cdot \nabla \chi_0 = 0 \tag{40}$$

From this, one gets, for $v_\parallel \neq 0$, χ_0 is independent of space coordinate θ_p, $\chi_0 = \chi_0(u, \lambda, \sigma, \rho)$.

The next order gives the equation

$$-v_\parallel \hat{b} \cdot \nabla \chi_1 - C_e^{l+}(\chi_0) = \frac{v_\parallel B}{\langle B^2 \rangle}, \tag{41}$$

where $\chi_1 = \chi_1(u,\ \lambda,\ \sigma,\ \rho,\ \theta_{\mathrm{pol}})$. Dividing Eq. (41) by $v_\parallel$, then integrating it along the field line for a full circle around the poloidal direction (a full circle integration can be taken only for $\lambda < 1$, i.e., for passing particles), one gets

$$\oint dl \hat{b} \cdot \nabla \chi_1 - \oint \frac{dl}{v_\parallel} C_e^{l+}(\chi_0) = \oint \frac{B}{\langle B^2 \rangle} dl \tag{42}$$

It is obvious that $\oint dl\hat{b}\cdot\nabla\chi_1 = 0$. Then Eq. (42) reduces to

$$-\oint \frac{dl}{v_{\|}} C_e^{l+}(\chi_0) = \oint \frac{B}{\langle B^2\rangle} dl \tag{43}$$

This equation [Eq. (43)] agrees with Eq.(12) in Lin-Liu's paper[1]. The left hand of Eq.(43) (bounce averaged adjoint collision operator) can be rewritten

$$-\oint \frac{dl}{v_{\|}} C_e^{l+}(\chi_0) = -\frac{1}{v}\oint \frac{dl}{B}\frac{B}{\xi} C_e^{l+}(\chi_0) = -\frac{1}{v}\left\langle \frac{B}{\xi} C_e^{l+}(\chi_0)\right\rangle \oint \frac{dl}{B}$$

The right hand of Eq.(43) can be further written as,

$$\oint \frac{B}{\langle B^2\rangle} dl = \oint \frac{B^2}{\langle B^2\rangle}\frac{dl}{B} = \frac{\langle B^2\rangle}{\langle B^2\rangle}\oint \frac{dl}{B} = \oint \frac{dl}{B}$$

Then Eq. (43) reduces to

$$\frac{1}{v}\left\langle \frac{B}{\xi} C_e^{l+}(\chi_0)\right\rangle = -1 \tag{44}$$

which can also be written as,

$$\frac{1}{v}\left\langle \frac{B}{|\xi|} C_e^{l+}(\chi_0)\right\rangle = -\,\mathrm{sgn}(v_{\|}) \tag{45}$$

This equation [Eq.(45)] agrees with Eq.(13) in Lin-Liu's paper[1]. Several remarks on the results obtained so far are in order. We note that only space integration is needed in the above derivation. This integration seems to have nothing to do with the motion of the particles.

checkHowever for passing particle, its orbit is the same as the above space integration path. For trapped particle, its orbit is not a full circle around the poloidal direction, so is different from the above integration path. [According to the well-known view, the integration should be along the orbits of the particles. I for the present accept this view. (I notice that I do not need this view to derive the following.). [I can now answer the question "why the phase space integration should be along particle's orbit." Refer to another note: karney1989_aip_conf.tm]**check**

7 Another choice of velocity coordinators

For axial symmetric velocity distribution, velocity space coordinators can be chosen to $(u,\ \xi)$, where $\xi = u_{\|}/u$. Another choice is (u, λ, σ), where $\sigma = \mathrm{sgn}(u_{\|}) = \pm 1$ and

$$\lambda = \frac{\mu B_{\max}}{mu^2/2} = \frac{B_{\max}}{B}\frac{u_{\perp}^2}{u^2} = \frac{B_{\max}}{B}\sin^2\theta = \frac{B_{\max}}{B}(1-\xi^2) \tag{46}$$

whose value is in the range of $0 \leqslant \lambda \leqslant B_{\max}/B$. And λ is a proper velocity coordinate of the drift kinetic equation (by saying "a proper velocity coordinate of DKE", I mean the space operator ∇ does not operate on it.) (Or it may be said λ is a constant of motion, though this is very misleading, because this actually has nothing to do with the motion of particles.)

In relativistic case, magnetic moment is defined by

$$\mu = \frac{mu_{\perp}^2}{2B}. \tag{47}$$

Then parallel momentum per unit mass can be written

$$u_{\|} = \mathrm{sgn}(\xi)\sqrt{u^2 - u_{\perp}^2} = \mathrm{sgn}(\xi)\sqrt{u^2 - \frac{2B\mu}{m_e}} \tag{48}$$

From this we know if a particle satisfies $u^2 > 2\mu B_{\max}/m_e$ the particle are passing, otherwise the particle is trapped. The passing condition corresponds to $\lambda < 1$ while the trapped condition corresponds to $\lambda > 1$.

8 High-velocity limit collision operator

Fisch's relativistic high-velocity limit collision operator is given by

$$C^l(f) = [\nu_{ei}(u) + \nu_D(u)]L(f) + \frac{1}{u^2}\frac{\partial}{\partial u}\left(u^2\lambda_s(u)f\right) \tag{49}$$

with

$$L(f) = \frac{1}{2}\frac{\partial}{\partial \xi}(1-\xi^2)\frac{\partial}{\partial \xi}f, \tag{50}$$

$$\nu_{ei}(u) = Z_{\rm eff}\nu_{e0}\gamma\left(\frac{u_e}{u}\right)^3,\ \nu_D = \nu_{e0}\gamma\left(\frac{u_e}{u}\right)^3, \tag{51}$$

and

$$\lambda_s(u) = \nu_{e0}u_e\gamma^2\left(\frac{u_e}{u}\right)^2, \tag{52}$$

where $u_e \equiv \sqrt{2T_e/m_e}$, $\nu_{e0} = \Gamma^{e/e}/u_e^3$, $\Gamma^{e/e} = (n_e e^4 \ln\Lambda^{e/e})/(4\pi\varepsilon_0^2 m_e^2)$. By comparing the high-velocity limit operator with the genearl Fokker-Planck operator, which takes the form

$$C^{a/b}(f_a, f_m) = \frac{1}{u^2}\frac{\partial}{\partial u}u^2\left(D_{cuu}^{a/b}\frac{\partial f_a}{\partial u} - F_{cu}^{a/b}f_a\right) + \frac{1}{u\sin\theta}\frac{\partial}{\partial\theta}\left[\sin\theta\left(D_{c\theta\theta}^{a/b}\frac{1}{u}\frac{\partial f_a}{\partial\theta}\right)\right], \tag{53}$$

we can identify the Fokker-Planck coefficients of the high-velocity limit operator, specificly,

$$D_{\theta\theta} = \frac{\nu_{ei}(u) + \nu_D(u)}{2} = \frac{Z_{\rm eff}+1}{2}\nu_{e0}\gamma\left(\frac{u_e}{u}\right)^3, \tag{54}$$

$$F_u = -\lambda_s(u) = -\nu_{e0}u_e\gamma^2\left(\frac{u_e}{u}\right)^2, \tag{55}$$

and

$$D_{uu} = 0. \tag{56}$$

Note that the Fokker-Planck coefficients of the high-velocity limit collision operator, $D_{\theta\theta}$ and F_u., become infinite when $u \to 0$. This is because the high-velocity limit operator is derived in the limit of $u \to \infty$, thus, it is not valid for small value of u.

The high-velocity limitcollision operator given by Eq. (49) does not have the self-adjointness property. Therefore, to use the adjoint method, we need to construct its adjoint operator, which satisfies the Eq. (16). It can be proved that the following operator is the adjoint operator of the high-velocity operator: (Refer to another note for the proof.)

$$C^{l+}(f) = [\nu_{ei}(u) + \nu_D(u)]L(f) - \lambda_s(u)\frac{\partial}{\partial u}f. \tag{57}$$

9 Solution to the bounce-averaged adjoint equation

The adjoint operator for the high velocity limit collision operator is given by

$$C^{l+}(\chi_0) = [\nu_{ei}(u) + \nu_D(u)]L(\chi_0) - \lambda_s(u)\frac{\partial}{\partial u}\chi_0 \tag{58}$$

Define

$$C_L^{l+}(\chi_0) = [\nu_{ei}(u) + \nu_D(u)]L(\chi_0) \tag{59}$$

and

$$C_s^{l+}(\chi_0) = -\lambda_s(u)\frac{\partial}{\partial u}\chi_0, \tag{60}$$

then the adjoint operator is written as

$$C^{l+}(\chi_0) = C_L^{l+}(\chi_0) + C_s^{l+}(\chi_0). \tag{61}$$

Now Let us calculate the left-hand side of the adjoint equation (44), which is now written

$$\frac{1}{v}\left\langle\frac{B}{\xi}C_e^{l+}(\chi_0)\right\rangle=\frac{1}{v}\left\langle\frac{B}{\xi}C_L^{l+}(\chi_0)\right\rangle+\frac{1}{v}\left\langle\frac{B}{\xi}C_s^{l+}(\chi_0)\right\rangle \tag{62}$$

The first term (pitch angle scattering part) on the rigth-hand side of the above equation is written

$$\begin{aligned}\frac{1}{v}\left\langle\frac{B}{\xi}C_L^{l+}(\chi_0)\right\rangle &= \frac{1}{v}\left\langle\frac{B}{\xi}[\nu_{ei}(u)+\nu_D(u)]L(\chi_0)\right\rangle\\ &= \frac{\nu_{ei}(u)+\nu_D(u)}{v}\left\langle\frac{B}{\xi}\frac{1}{2}\frac{\partial}{\partial\xi}(1-\xi^2)\frac{\partial}{\partial\xi}\chi_0\right\rangle\end{aligned} \tag{63}$$

Since χ_0 is considered as a function of u, λ, and σ, and the above equation involves derivative with respect to ξ, we need to use the chain rule to transform the derivative about ξ to one about λ. From Eq. (46), we obtain

$$\frac{\partial\lambda}{\partial\xi}=-2\xi\frac{B_{\max}}{B}. \tag{64}$$

Using this, Eq. (63) is written

$$\begin{aligned}\frac{1}{v}\left\langle\frac{B}{\xi}C_L^{l+}(\chi_0)\right\rangle &= \frac{\nu_{ei}(u)+\nu_D(u)}{v}\left\langle\frac{B}{\xi}\frac{1}{2}\left[-2\xi\frac{B_{\max}}{B}\right]\frac{\partial}{\partial\lambda}(1-\xi^2)\left(-2\xi\frac{B_{\max}}{B}\right)\frac{\partial}{\partial\lambda}\chi_0\right\rangle\\ &= 2B_{\max}\frac{\nu_{ei}(u)+\nu_D(u)}{v}\left\langle\frac{\partial}{\partial\lambda}\left(\lambda\xi\frac{\partial\chi_0}{\partial\lambda}\right)\right\rangle\end{aligned} \tag{65}$$

Noting the flux surface average can be interchanged with the differential with respect to λ, Eq. (65) is written

$$\frac{1}{v}\left\langle\frac{B}{\xi}C_L^{l+}(\chi_0)\right\rangle=2B_{\max}\frac{\nu_{ei}(u)+\nu_D(u)}{v}\frac{\partial}{\partial\lambda}\left(\lambda\langle\xi\rangle\frac{\partial\chi_0}{\partial\lambda}\right). \tag{66}$$

The second term (slowing-down term) on the right hand of Eq. (62) is written

$$\frac{1}{v}\left\langle\frac{B}{\xi}C_s^{l+}(\chi_0)\right\rangle=-\frac{1}{v}\left\langle\frac{B}{\xi}\lambda_s(u)\frac{\partial}{\partial u}\chi_0\right\rangle, \tag{67}$$

To make the adjoint equation take the form desired for the method of separation of variables, we will approximate the χ_0 in the slowing-down term by its first Legendre component. Before doing this, we introduce a distribution function, $g(u,\,\xi,\,\theta_p)$, which is the representation of $\chi_0(u,\,\lambda,\,\sigma)$ in $(u,\,\xi)$ coordinates. Note that the poloidal angle variable θ_p appears because the transformating rules involves θ_p. It is obvious that the two distribution functions, g and χ, are related to each other by

$$g(u,\xi,\theta_p)=\chi_0\left(u,\frac{B_{\max}}{B(\theta_p)}(1-\xi^2),\mathrm{sgn}(\xi)\right), \tag{68}$$

or equivalently

$$\chi_0(u,\lambda,\sigma)=g\left(u,\sigma\sqrt{1-\lambda\frac{B(\theta_p)}{B_{\max}}},\theta_p\right). \tag{69}$$

Now Let us approximate $g(u,\xi,\theta_p)$ by its first Legendre harmonic, i.e.,

$$g(u,\xi,\theta_p)\approx g^{(1)}(u,\theta_p)\xi, \tag{70}$$

where the coefficient, $g^{(1)}(u,\theta_p)$, is given by

$$g^{(1)}(u,\theta_p)=\frac{3}{2}\int_{-1}^{1}d\xi g(u,\xi,\theta_p)\xi. \tag{71}$$

Using the relation

$$\lambda=\frac{B_{\max}}{B}(1-\xi^2), \tag{72}$$

we obtain

$$d\xi=-\frac{1}{2\xi}\frac{B}{B_{\max}}d\lambda. \tag{73}$$

Changing the integration varible from ξ to λ, Eq. (71) is written

$$\begin{aligned} g^{(1)}(u,\theta_p) &= -\frac{3}{4}\frac{B(\theta_p)}{B_{\max}}\left[\int_{\lambda'=0}^{\lambda'=1} d\lambda' g\left(u, -\sqrt{1-\lambda'\frac{B}{B_{\max}}}, \theta_p\right) + \int_{\lambda'=1}^{\lambda'=0} d\lambda' g\left(u, \sqrt{1-\lambda'\frac{B}{B_{\max}}}, \right.\right. \\ &\left.\left.\theta_p\right)\right] \\ &= \frac{3}{4}\frac{B(\theta_p)}{B_{\max}}\left[\sum_{\mathrm{sgn}(u'_\parallel)=-1,1}\int_0^1 d\lambda' \mathrm{sgn}(u'_\parallel) g\left(u, \mathrm{sgn}(u'_\parallel)\sqrt{1-\lambda'\frac{B}{B_{\max}}}, \theta_p\right)\right], \end{aligned} \tag{74}$$

where I use the fact that $\xi = \mathrm{sgn}(\xi)\sqrt{1-\lambda B/B_{\max}}$ and separate the integration into two regions. Using Eq. (69) to rewrite Eq. (74) gives

$$g^{(1)}(u,\theta_p) = \frac{3}{4}\frac{B(\theta_p)}{B_{\max}}\left[\sum_{\mathrm{sgn}(u'_\parallel)=-1,1}\int_0^1 d\lambda' \mathrm{sgn}(u'_\parallel)\chi_0(u,\lambda',\mathrm{sgn}(u'_\parallel))\right]. \tag{75}$$

Note that the quantity in the bracket is obviously independent of λ. Using $g^{(1)}(u,\ \theta_p)\xi$ to replace χ_0 on the right-hand side of Eq. (67), we obtain

$$\begin{aligned} \frac{1}{v}\left\langle \frac{B}{\xi} C_s^{l+}(\chi_0)\right\rangle &= -\frac{1}{v}\left\langle \frac{B}{\xi}\lambda_s(u)\frac{\partial}{\partial u}\left[g^{(1)}(u,\theta_p)\xi\right]\right\rangle \\ &= -\frac{1}{v}\left\langle B\lambda_s(u)\frac{\partial}{\partial u} g^{(1)}(u,\theta_p)\right\rangle \\ &= -\frac{1}{v}\left\langle B\lambda_s(u)\frac{\partial}{\partial u}\frac{3}{4}\frac{B(\theta_p)}{B_{\max}}\left[\sum_{\mathrm{sgn}(u'_\parallel)}^{\pm 1}\int_0^1 d\lambda' \mathrm{sgn}(u'_\parallel)\chi_0(u,\lambda',\mathrm{sgn}(u'_\parallel))\right]\right\rangle \\ &= -\frac{1}{v}\left\langle \frac{B^2}{B_{\max}}\right\rangle \lambda_s(u)\frac{\partial}{\partial u}\frac{3}{4}\left[\sum_{\mathrm{sgn}(u'_\parallel)}^{\pm 1}\int_0^1 d\lambda' \mathrm{sgn}(u'_\parallel)\chi_0(u,\lambda',\mathrm{sgn}(u'_\parallel))\right] \end{aligned} \tag{76}$$

Define

$$K(u) = \left\langle \frac{B^2}{B_{\max}^2}\right\rangle \frac{3}{4}\left[\sum_{\mathrm{sgn}(u'_\parallel)}^{\pm 1}\int_0^1 d\lambda' \mathrm{sgn}(u'_\parallel)\chi_0(u,\lambda',\mathrm{sgn}(u'_\parallel))\right], \tag{77}$$

which is a function of only u, then Eq. (76) is written as

$$-\frac{1}{v}\left\langle \frac{B}{\xi}\lambda_s(u)\frac{\partial}{\partial u}\chi_0\right\rangle = -\frac{B_{\max}}{v}\lambda_s(u)\frac{\partial}{\partial u}K(u) \tag{78}$$

Using Eqs. (66) and (78), the adjoint equation (44) is written

$$[\nu_{ei}(u)+\nu_D(u)]B_{\max}2\frac{\partial}{\partial\lambda}\left[\lambda\langle\xi\rangle\frac{\partial\chi_0}{\partial\lambda}\right] - B_{\max}\lambda_s(u)\frac{\partial}{\partial u}K(u) = -v. \tag{79}$$

Define

$$\overline{\chi}_0 \equiv \frac{\chi_0}{u_e/(\nu_{e0}B_{\max})}, \tag{80}$$

then Eq. (79) is reduced to,

$$[\nu_{ei}(u)+\nu_D(u)]2\frac{\partial}{\partial\lambda}\left[\lambda\langle\xi\rangle\frac{\partial\overline{\chi}_0}{\partial\lambda}\right] - \lambda_s(u)\frac{\partial}{\partial u}\overline{K}(u) = -\frac{u}{\gamma}\frac{\nu_{e0}}{u_e} \tag{81}$$

where

$$\overline{K}(u) \equiv \nu_{e0}\frac{B_{\max}}{u_e}K(u). \tag{82}$$

[Eq. (81) can also be written as

$$[\nu_{ei}(u)+\nu_D(u)]2\frac{\partial}{\partial\lambda}\left[\lambda\langle|\xi|\rangle\frac{\partial\overline{\chi}_0}{\partial\lambda}\right] - \mathrm{sgn}(u_\parallel)\lambda_s(u)\frac{\partial}{\partial u}\overline{K}(u) = -\mathrm{sgn}(u_\parallel)\frac{u}{\gamma}\frac{\nu_{e0}}{u_e}. \tag{83}$$

This equation agrees with Eq. (25) in Lin-Liu's paper[1] (however, the expression for $\overline{K}(u)$ is a little different from Eq. (26) in Lin-Liu's paper. My result is correct while Liu-Liu's Eq. (26) is not rigorous in the context. However this does not alter the final result (Lin-Liu knows $\chi_0(\sigma) = -\chi_0(-\sigma)$ and use this to remove the sum over σ in Eq. (82)).]

Now we assume $\overline{\chi_0}$ can be expressed as

$$\overline{\chi_0} = \mathrm{sgn}(u_\parallel) F(u) H(\lambda). \tag{84}$$

Substitute this expression into Eq. (82) gives

$$\begin{aligned}\overline{K}(u) &= \left\langle \frac{B^2}{B_{\max}^2} \right\rangle \frac{3}{4} \left[\sum_{\mathrm{sgn}(u'_\parallel)}^{\pm 1} \int_0^1 d\lambda' \mathrm{sgn}(u'_\parallel) \mathrm{sgn}(u'_\parallel) F(u) H(\lambda') \right] \\ &= \left\langle \frac{B^2}{B_{\max}^2} \right\rangle \frac{3}{4} \left[\sum_{\mathrm{sgn}(u'_\parallel)}^{\pm 1} \int_0^1 d\lambda' F(u) H(\lambda') \right] \\ &= \left\langle \frac{B^2}{B_{\max}^2} \right\rangle \frac{3}{2} \left[\int_0^1 d\lambda' F(u) H(\lambda') \right] \\ &= \left\langle \frac{B^2}{B_{\max}^2} \right\rangle \frac{3}{2} F(u) \left[\int_0^1 d\lambda' H(\lambda') \right] \end{aligned} \tag{85}$$

Define

$$f_c = \left\langle \frac{B^2}{B_{\max}^2} \right\rangle \frac{3}{2} \int_0^1 d\lambda' H(\lambda'), \tag{86}$$

which is a constant independent of u and λ, then $\overline{K}(u)$ is written

$$\overline{K}(u) = f_c F(u), \tag{87}$$

which agrees with Eq. (30) in Lin-Liu's paper[1].

Substituting Eqs. (84) and (87) into Eq. (83), we obtain

$$[\nu_{ei}(u) + \nu_D(u)] \mathrm{sgn}(u_\parallel) F(u) 2 \frac{\partial}{\partial \lambda} \left[\lambda \langle |\xi| \rangle \frac{\partial H(\lambda)}{\partial \lambda} \right] - \mathrm{sgn}(u_\parallel) \lambda_s(u) f_c \frac{\partial F(u)}{\partial u} = -\mathrm{sgn}(u_\parallel) \frac{u}{\gamma} \frac{\nu_{e0}}{u_e}, \tag{88}$$

which is in the form desired for the method of separation of variables. Separating variables to two sides of the equation, we obtain

$$2 \frac{\partial}{\partial \lambda} \left[\lambda \langle |\xi| \rangle \frac{\partial H(\lambda)}{\partial \lambda} \right] = \{[\nu_{ei}(u) + \nu_D(u)] F(u)\}^{-1} \left[\lambda_s(u) f_c \frac{\partial F(u)}{\partial u} - \frac{u}{u_e} \frac{\nu_{e0}}{\gamma} \right], \tag{89}$$

which reduces to two ordinary differential equations:

$$2 \frac{\partial}{\partial \lambda} \left[\lambda \langle |\xi| \rangle \frac{\partial H(\lambda)}{\partial \lambda} \right] = c_0 \tag{90}$$

and

$$\lambda_s(u) f_c \frac{\partial F(u)}{\partial u} - \frac{u}{u_e} \frac{\nu_{e0}}{\gamma} = c_0 [\nu_{ei}(u) + \nu_D(u)] F(u). \tag{91}$$

where c_0 is a constant of separation, which is usually determined by some boundary conditions. In our case, it turns out that c_0 can be chosen arbitrarily (refer to another notes). Here I choose $c_0 = -1$. Then Eq. (90) is written

$$2 \frac{\partial}{\partial \lambda} \left[\lambda \langle |\xi| \rangle \frac{\partial H(\lambda)}{\partial \lambda} \right] = -1, \tag{92}$$

i.e.,

$$2 \frac{\partial}{\partial \lambda} \left[\lambda \left\langle \sqrt{1 - \lambda \frac{B}{B_{\max}}} \right\rangle \frac{\partial H(\lambda)}{\partial \lambda} \right] = -1 \tag{93}$$

Using boundary condition $H(\lambda=1)=0$, the above equation is integrated to give

$$H(\lambda)=\frac{1}{2}\int_{\lambda}^{1}\frac{1}{\left\langle\sqrt{1-\lambda'\frac{B}{B_{\max}}}\right\rangle}d\lambda' \tag{94}$$

This equation agrees with Eq. (29) in Lin-Liu's paper[1]. Note that, for uniform magnetic field, λ reduces to $\sin^2\theta$, $H(\lambda)$ reduces to

$$H(\lambda)=\frac{1}{2}\int_{\lambda}^{1}\frac{1}{\sqrt{1-\lambda'}}d\lambda'=\sqrt{1-\lambda}=|\cos\theta| \tag{95}$$

and $\text{sgn}(u_\parallel)H(\lambda)$ reduces to $\cos\theta$.

The equation for $F(u)$ [Eq. (91)] can be written as

$$-\text{sgn}(u_\parallel)\frac{u}{u_e}\frac{\nu_{e0}}{\gamma}+\text{sgn}(u_\parallel)\lambda_s(u)f_c\frac{\partial F(u)}{\partial u}=-[\nu_{ei}(u)+\nu_D(u)]\text{sgn}(u_\parallel)F(u) \tag{96}$$

$$\Longrightarrow\lambda_s(u)f_c\frac{\partial F(u)}{\partial u}+[\nu_{ei}(u)+\nu_D(u)]F(u)=\frac{u}{u_e}\frac{\nu_{e0}}{\gamma} \tag{97}$$

$$\Longrightarrow\frac{\gamma^2}{u^2}\frac{\partial F(u)}{\partial u}+\frac{Z_{\text{eff}}+1}{f_c}\frac{\gamma}{u^3}F(u)=\frac{u}{u_e}\frac{1}{\gamma}\frac{1}{f_c u_e^3} \tag{98}$$

This equation agrees with Eq. (31) in Lin-Liu's paper[1]. If using the normalization: $\overline{u}=u/c$, $\overline{F}=F/F_0$, $F_0=c^4/u_e^4$, Eq. (98) can be written as

$$\frac{\partial\overline{F}}{\partial\overline{u}}+\frac{Z_{\text{eff}}+1}{f_c}\frac{1}{\gamma\overline{u}}\overline{F}=\frac{\overline{u}^3}{\gamma^3}\frac{1}{f_c} \tag{99}$$

where $\gamma=\sqrt{1+\overline{u}^2}$. Note that Eq. (99) is independent of temperature parameter Θ.

The quantity defined in Eq. (86) can be further written as

$$\begin{aligned}
f_c &= \left\langle\frac{B^2}{B_{\max}^2}\right\rangle\frac{3}{2}\int_0^1 d\lambda' H(\lambda') && (100)\\
&= \left\langle\frac{B^2}{B_{\max}^2}\right\rangle\frac{3}{2}\int_0^1 \{d[H(\lambda')\lambda']-\lambda' dH(\lambda')\}\\
&= \left\langle\frac{B^2}{B_{\max}^2}\right\rangle\frac{3}{2}\int_0^1 [-\lambda' dH(\lambda')]\\
&= \left\langle\frac{B^2}{B_{\max}^2}\right\rangle\frac{3}{4}\int_0^1 \frac{\lambda' d\lambda'}{\left\langle\sqrt{1-\lambda'\frac{B}{B_{\max}}}\right\rangle} && (101)
\end{aligned}$$

This expression agrees with Eq. (32) in Lin-Liu's paper[1]. To get Eq. (101), one needs to use the technique of integration by parts. (Thank Yemin for letting me know this.) Note that f_c is a quantity determined by the magnetic equilibrium. This quantity is the well-known effective circulating particle fraction in the neoclassical transport theory. Note that for uniform magnetic field, the integration in Eq. (101) can be performed to give $f_c=1$.

10 Wave induced flux

Cohen's simplified quasilinear diffusion operator for EC waves[1, 2] is given by

$$S_w(f)=\delta(\boldsymbol{x}-\boldsymbol{x}_R)\tilde{\Lambda}D_0\delta\left(\omega-k_\parallel v_\parallel-l\frac{\omega_c}{\gamma}\right)\tilde{\Lambda}f \tag{102}$$

where $\tilde{\Lambda}$ is a differential operator in velocity space

$$\tilde{\Lambda}=\frac{\partial}{\partial\varepsilon}+\frac{k_\parallel}{\omega}\frac{1}{m_e}\frac{\partial}{\partial u_\parallel}, \tag{103}$$

and

$$D_0 \propto \left(\frac{k_\perp u_\perp}{2\omega_c}\right)^{2l-2} \frac{u_\perp^2}{\gamma^2} \tag{104}$$

Using Eq. (102) in Eq. (38) (i.e., Eq. (11) in Lin-Liu's paper[1]) and the fact $\tilde{\Lambda}(f_m) = -f_m/T_e$, $\tilde{\Lambda}(\varepsilon) = 1$ and performing the flux average and an integration by parts in velocity space, one gets

$$\begin{aligned}
\zeta^\star &= -\frac{4}{\ln\Lambda}\left\langle\frac{B}{B_{\max}}\right\rangle \frac{\langle \int d\Gamma \overline{\chi} S_w(f_m)\rangle}{\langle \int d\Gamma \varepsilon/m_e v_e^2 S_w(f_m)\rangle}. \\
&= -\frac{4}{\ln\Lambda}\left\langle\frac{B}{B_{\max}}\right\rangle \frac{\int d\Gamma \overline{\chi}\, \tilde{\Lambda} D_0 \delta\left(\omega - k_\parallel v_\parallel - l\frac{\omega_c}{\gamma}\right)\tilde{\Lambda} f_m}{\int d\Gamma \varepsilon/m_e v_e^2\, \tilde{\Lambda} D_0 \delta\left(\omega - k_\parallel v_\parallel - l\frac{\omega_c}{\gamma}\right)\tilde{\Lambda} f_m} \\
&= -\frac{4}{\ln\Lambda}\left\langle\frac{B}{B_{\max}}\right\rangle \frac{\int d\Gamma \overline{\chi}\, \tilde{\Lambda} D_0 \delta\left(\omega - k_\parallel v_\parallel - l\frac{\omega_c}{\gamma}\right) f_m}{\int d\Gamma \varepsilon/m_e v_e^2\, \tilde{\Lambda} D_0 \delta\left(\omega - k_\parallel v_\parallel - l\frac{\omega_c}{\gamma}\right) f_m} \\
&= -\frac{4}{\ln\Lambda}\left\langle\frac{B}{B_{\max}}\right\rangle m_e v_e^2 \frac{\int d\Gamma D_0 \delta\left(\omega - k_\parallel v_\parallel - l\frac{\omega_c}{\gamma}\right) f_m \tilde{\Lambda}\overline{\chi}}{\int d\Gamma D_0 \delta\left(\omega - k_\parallel v_\parallel - l\frac{\omega_c}{\gamma}\right) f_m}
\end{aligned} \tag{105}$$

In the process of obtaining the above, we assume that

$$\int d\Gamma \tilde{\Lambda}(g) = 0,$$

which is easy to prove (to be proved by me later). Eq. (105) agrees with Eq. (38) in Liu-Liu's paper[1]. Substituting the expression of D_0 into Eq. (105), one obtains

$$\zeta^\star = -\frac{4}{\ln\Lambda}\left\langle\frac{B}{B_{\max}}\right\rangle m_e v_e^2 \frac{\int d\Gamma (u_\perp)^{2l} \gamma^{-2} \delta\left(\omega - k_\parallel v_\parallel - l\frac{\omega_c}{\gamma}\right) f_m \tilde{\Lambda}\overline{\chi}}{\int d\Gamma\, (u_\perp)^{2l} \gamma^{-2} \delta\left(\omega - k_\parallel v_\parallel - l\frac{\omega_c}{\gamma}\right) f_m} \tag{106}$$

Using the transform rule for the volume element (using Jacobi), one can get the velocity volume element in $(\gamma, u_\parallel)$ coordinates,

$$d\Gamma = 2\pi c^2 \gamma d\gamma d u_\parallel. \tag{107}$$

Using this and the fact $\delta(\omega - k_\parallel v_\parallel - l\omega_c/\gamma) = \gamma\delta(\gamma\omega - k_\parallel u_\parallel - l\omega_c)$ in Eq. (106), one gets

$$\zeta^\star = -\frac{4}{\ln\Lambda}\left\langle\frac{B}{B_{\max}}\right\rangle m_e v_e^2 \frac{\int d\gamma d u_\parallel (u_\perp)^{2l} \delta(\gamma\omega - k_\parallel u_\parallel - l\omega_c) f_m \tilde{\Lambda}\overline{\chi}}{\int d\gamma d u_\parallel (u_\perp)^{2l} \delta(\gamma\omega - k_\parallel u_\parallel - l\omega_c) f_m} \tag{108}$$

Performing the integration over $u_\parallel$, one gets

$$\zeta^\star = -\frac{4}{\ln\Lambda}\left\langle\frac{B}{B_{\max}}\right\rangle m_e v_e^2 \frac{\int d\gamma (u_\perp)^{2l} f_m \tilde{\Lambda}\overline{\chi}}{\int d\gamma (u_\perp)^{2l} f_m} \tag{109}$$

Note that $u_\parallel$ in the above equation has already been replaced by $(\gamma\omega - l\omega_c)/k_\parallel$. And the integrand of Eq. (109) is expressed as a function of γ.

$$u_\perp^2 = c^2(\gamma^2 - 1) - u_\parallel^2 = c^2\left[\gamma^2 - 1 - \left(\frac{\gamma - y}{n_\parallel}\right)^2\right], \tag{110}$$

where $y = l\omega_c/\omega$, $n_\parallel = k_\parallel c/\omega$. The zeroes of $u_\perp^2$ are determined by the quadratic equation

$$\gamma^2 - 1 - \left(\frac{\gamma - y}{n_\parallel}\right)^2 = 0 \tag{111}$$

$$\Longrightarrow \left(1 - \frac{1}{n_\parallel^2}\right)\gamma^2 + \left(2\frac{y}{n_\parallel^2}\right)\gamma - \left(1 + \frac{y^2}{n_\parallel^2}\right) = 0 \tag{112}$$

For EC wave, we have $n_{\|} < 1$, so the coefficient before γ^2 is negative. To keep $u_{\perp}^2$ positive, one has to limit γ to the range $\gamma_{\min} < \gamma < \gamma_{\max}$, with

$$\gamma_{\min} = \frac{-y + n_{\|}\sqrt{n_{\|}^2 + y^2 - 1}}{n_{\|}^2 - 1} \tag{113}$$

$$\gamma_{\max} = \frac{-y - n_{\|}\sqrt{n_{\|}^2 + y^2 - 1}}{n_{\|}^2 - 1} \tag{114}$$

In order to ensure the quantity in the square root, $n_{\|}^2 + y^2 - 1$ is positive, it is required that

$$n_{\|}^2 > 1 - y^2 \tag{115}$$

Note that the r.h.s of Eq. (113) is positive when $n_{\|} < 1$.

Using Mathematic function "Plot", it is shown that $\gamma_{\min}(n_{\|})$ is a monotonically decreasing function while $\gamma_{\max}(n_{\|})$ is monotonically increasing, thus the width of the integration region, $W = \gamma_{\max}(n_{\|}) - \gamma_{\min}(n_{\|})$, increases with increasing $n_{\|}$. For big $n_{\|}$, it is not easy to numerically calculate the accurate value of the integration in Eq. (109).

11 Transform operator $\tilde{\Lambda}$ from $(\varepsilon, u_{\|})$ to (u, λ)

In relativistic case, the kinetic energy is given by

$$\varepsilon = (\gamma - 1) m_e c^2 \tag{116}$$

Using Eq.(116), one gets

$$\begin{aligned} d\varepsilon &= m_e c^2 \frac{d\gamma}{du} du \\ &= m_e v du \end{aligned} \tag{117}$$

Now we want to transform the operator $\tilde{\Lambda}$ defined in Eq.(103) from $(\varepsilon, u_{\|})$ space to (u, λ) space.

$$(u, \lambda) \longrightarrow (\varepsilon, u_{\|})$$

We first need to express u and λ as function of ε and $u_{\|}$. From Eq.(116), we know u is only a function of ε,

$$u = u(\varepsilon) = c\sqrt{\left(\frac{\varepsilon}{mc^2} + 1\right)^2 - 1} \tag{118}$$

The definition of λ is given by

$$\lambda \equiv \frac{B_{\max}}{B} \frac{u_{\perp}^2}{u^2}. \tag{119}$$

We need to express λ in terms of ε and $u_{\|}$, which gives

$$\lambda = \frac{B_{\max}}{B}\left(1 - \frac{u_{\|}^2}{u^2(\varepsilon)}\right) = \lambda(\varepsilon, u_{\|}) \tag{120}$$

which is a function of both ε and $u_{\|}$. Using results in Eqs.(118)(120), one gets,

$$\begin{aligned} \frac{\partial}{\partial \varepsilon} &= \frac{\partial}{\partial u}\frac{\partial u}{\partial \varepsilon} + \frac{\partial}{\partial \lambda}\frac{\partial \lambda}{\partial \varepsilon} \\ &= \frac{1}{mv}\frac{\partial}{\partial u} + \left(-\frac{B_{\max}}{B}\right) u_{\|}^2 \left(-\frac{2}{u^3}\frac{du}{d\varepsilon}\right)\frac{\partial}{\partial \lambda} \\ &= \frac{1}{mv}\frac{\partial}{\partial u} + \left(\frac{B_{\max}}{B}\right) u_{\|}^2 \left(\frac{2}{u^3}\frac{1}{mv}\right)\frac{\partial}{\partial \lambda} \end{aligned}$$

and

$$\begin{aligned}\frac{\partial}{\partial u_\parallel} &= \frac{\partial}{\partial u}\frac{\partial u}{\partial u_\parallel} + \frac{\partial}{\partial \lambda}\frac{\partial \lambda}{\partial u_\parallel}\\ &= 0 + \left(-\frac{B_{\max}}{B}\frac{1}{u^2}2u_\parallel\right)\frac{\partial}{\partial \lambda}\end{aligned}$$

Then the operator $\tilde{\Lambda}$ can be expressed as

$$\begin{aligned}m_e u_e^2 \tilde{\Lambda}\overline{\chi} &= m_e u_e^2 \frac{\partial \overline{\chi}}{\partial \varepsilon} + u_e^2 \frac{k_\parallel}{\omega}\frac{\partial \overline{\chi}}{\partial u_\parallel}\\ &= m_e u_e^2 \left[\frac{1}{mv}\frac{\partial \overline{\chi}}{\partial u} + \left(\frac{B_{\max}}{B}\right) u_\parallel^2 \left(\frac{2}{u^3}\frac{1}{mv}\right)\frac{\partial \overline{\chi}}{\partial \lambda}\right]\\ &+ u_e^2 \frac{k_\parallel}{\omega}\left(-\frac{B_{\max}}{B}\frac{1}{u^2}2u_\parallel\right)\frac{\partial \overline{\chi}}{\partial \lambda}\\ &= \gamma \frac{u_e^2}{u}\frac{\partial \overline{\chi}}{\partial u} + 2\frac{B_{\max}}{B}\frac{u_e^2 u_\parallel}{u^3}\left(\frac{u_\parallel \gamma}{u} - \frac{k_\parallel c}{\omega}\frac{u}{c}\right)\frac{\partial \overline{\chi}}{\partial \lambda}\end{aligned} \tag{121}$$

Substituting the expression of $\overline{\chi}$ into Eq.(121), one gets

$$m_e u_e^2 \tilde{\Lambda}\overline{\chi} = \mathrm{sgn}(u_\parallel)\left\{\gamma \frac{u_e^2}{u}\frac{dF(u)}{du}H(\lambda) + 2\frac{B_{\max}}{B}\frac{u_e^2 u_\parallel}{u^3}\left(\frac{u_\parallel \gamma}{u} - \frac{n_\parallel u}{c}\right)F(u)\frac{dH(\lambda)}{d\lambda}\right\}, \tag{122}$$

where λ and u are given respectively by

$$\lambda = \frac{B_{\max}}{B}\frac{u_\perp^2}{u^2} = \frac{B_{\max}}{B}\frac{\gamma^2 - 1 - \left(\frac{\gamma - y}{n_\parallel}\right)^2}{\gamma^2 - 1} = \frac{B_{\max}}{B}\left(1 - \frac{(\gamma - y)^2}{(\gamma^2 - 1)n_\parallel^2}\right) \tag{123}$$

$$u = c\sqrt{\gamma^2 - 1} \tag{124}$$

12 Simple magnetic field configuration: concentric circular flux surfaces

$$B_t(\varepsilon, \theta_p) = \frac{B_{t0}}{1 + \epsilon \cos\theta_p} \tag{125}$$

$$B_p(\varepsilon, \theta_p) = \frac{B_{p0}}{1 + \epsilon \cos\theta_p} \tag{126}$$

where $\epsilon = r/R_0$ which labels different flux surface, θ_p is the poloidal angle. Then

$$B(\varepsilon, \theta_p) = \sqrt{B_t^2 + B_p^2} = \frac{B_0}{1 + \epsilon \cos\theta_p}, \tag{127}$$

where $B_0 = \sqrt{B_{t0}^2 + B_{p0}^2}$. The local major radius is given by $R = R_0(1 + \epsilon \cos\theta_p)$. It can be proved that $\nabla \cdot \boldsymbol{B} = 0$ is satisfied. [Proof:

$$B_t = \frac{B_0}{R} \tag{128}$$

$$\tan\theta = \frac{z}{R - R_0}$$

$$\sin\theta = \frac{z}{\sqrt{z^2 + (R - R_0)^2}}$$

$$\cos\theta = \frac{R - R_0}{\sqrt{z^2 + (R - R_0)^2}}$$

$$B_z = -\frac{B_{p0}}{R}\cos\theta = -\frac{B_{p0}}{R}\frac{R-R_0}{\sqrt{z^2+(R-R_0)^2}} \tag{129}$$

$$B_R = \frac{B_{p0}}{R}\sin\theta = \frac{B_{p0}}{R}\frac{z}{\sqrt{z^2+(R-R_0)^2}} \tag{130}$$

] I now wonder whether this kind of magnetic field satisfies G-S equation. The answer is no.

We choose

$$\psi = -B_{p0}\sqrt{z^2+(R-R_0)^2} \tag{131}$$

then magnetic field is expressed as

$$B_R = -\frac{1}{R}\frac{\partial\psi}{\partial z},\ B_z = \frac{1}{R}\frac{\partial\psi}{\partial R} \tag{132}$$

Will ψ satisfies G-S equation?

$$\Rightarrow \frac{\partial^2\psi}{\partial z^2} + R\frac{\partial}{\partial R}\left(\frac{1}{R}\frac{\partial\psi}{\partial R}\right) = -\mu_0 R^2\frac{dP}{d\psi} - \frac{dF}{d\psi}F(\psi) \tag{133}$$

$$F \equiv RB_\phi(R,z) = \text{const},$$

$$\Longrightarrow \frac{\partial^2\psi}{\partial z^2} + R\frac{\partial}{\partial R}\left(\frac{1}{R}\frac{\partial\psi}{\partial R}\right) = -\mu_0 R^2\frac{dP}{d\psi} - 0 \tag{134}$$

Using Eq.(131), the left hand side of Eq.(133) is reduced to (Using Mathematic)

$$\frac{\partial^2\psi}{\partial z^2} + R\frac{\partial}{\partial R}\left(\frac{1}{R}\frac{\partial\psi}{\partial R}\right) = \frac{-R_0/R}{\sqrt{z^2+(R-R_0)^2}}$$

Then G-S equation, Eq.(134), is reduced to

$$\mu_0\frac{dP}{d\psi} = \frac{R_0/R^3}{\sqrt{z^2+(R-R_0)^2}} \tag{135}$$

$$\Longrightarrow \mu_0\frac{dP}{d\psi} = \frac{R_0}{\psi R^3} \tag{136}$$

Eq.(136) tells us that P now can not be expressed as a function of only ψ. Thus this kind of magnetic field is not a solution to G-S equation. (This kind of magnetic field is the zero order approximate solution to G-S equation, in other words, it is a solution in the cylindric geometry approximation (I will check this later.). (Thank Yemin and Xiaotao for letting me know this.))

The safety factor of the flux surface ε can be written as

$$\begin{aligned} q(\varepsilon) &= \frac{1}{2\pi}\oint\frac{1}{R}\frac{B_t}{B_p}dl_p \\ &= \frac{1}{2\pi R_0}\frac{B_{t0}}{B_{p0}}\oint\frac{dl_p}{1+\epsilon\cos\theta_p} \\ &= \frac{r}{2\pi R_0}\frac{B_{t0}}{B_{p0}}\int_0^{2\pi}\frac{d\theta_p}{1+\epsilon\cos\theta_p} \\ &= \frac{B_{t0}}{B_{p0}}\frac{\epsilon}{\sqrt{1-\epsilon^2}} \end{aligned} \tag{137}$$

In obtaining the last equality, use was made of

$$\int_0^{2\pi}\frac{d\theta_p}{1+a\cos\theta_p} = \frac{2\pi}{\sqrt{1-a^2}},\ \text{for } -1<a<1. \tag{138}$$

Eq. (137) agrees with expressions in Karney's paper[3].

The minimum magnetic field on a flux surface ε is given by

$$B(\varepsilon,0)=\frac{B_0}{1+\epsilon} \tag{139}$$

From Eqs. (127) and (139), one gets

$$b(l)\equiv\frac{B(l)}{B(0)}=\frac{B}{B_{\min}}=\frac{1+\epsilon}{1+\epsilon\cos\theta_p} \tag{140}$$

Now calculate the length of the field line when it travels a full circle around poloidal direction. Magnetic field lines satisfy the equation

$$dl=\frac{B}{B_p}dl_p=\frac{B_0}{B_{p0}}dl_p \tag{141}$$

$$\begin{aligned} L &= \oint dl \\ &= \oint \frac{B_0}{B_{p0}}dl_p \\ &= \frac{B_0}{B_{p0}}\oint dl_p \\ &= \frac{B_0}{B_{p0}}2\pi r \end{aligned} \tag{142}$$

This result does not agree with the result in Karney's paper[3]. [Karney's result:

$$\begin{aligned} L=2\pi R_0 Q\sqrt{1+B_{p0}^2/B_{t0}^2} &= 2\pi R_0\frac{B_{t0}}{B_{p0}}\frac{\epsilon}{\sqrt{1-\epsilon^2}}\frac{B_0}{B_{t0}} \\ &= 2\pi r\frac{B_0}{B_{p0}}\frac{1}{\sqrt{1-\epsilon^2}} \end{aligned} \tag{143}$$

]

$$\begin{aligned} \langle A(\theta_p)\rangle &= \frac{\oint A(\theta_p)\frac{dl_p}{B_p}}{\oint \frac{dl_p}{B_p}} \\ &= \frac{\int_0^{2\pi} A(\theta_p)\frac{r\,d\theta_p}{B_p(\theta_p)}}{\int_0^{2\pi}\frac{r\,d\theta_p}{B_p(\theta_p)}} \\ &= \frac{\int_0^{2\pi} A(\theta_p)\,(1+\epsilon\cos\theta_p)d\theta_p}{\int_0^{2\pi}(1+\epsilon\cos\theta_p)d\theta_p} \\ &= \frac{1}{2\pi}\int_0^{2\pi} A(\theta_p)\,(1+\epsilon\cos\theta_p)d\theta_p \end{aligned} \tag{144}$$

$$\begin{aligned} \langle b\rangle &= \frac{1}{2\pi}\int_0^{2\pi}\frac{1+\epsilon}{1+\epsilon\cos\theta_p}(1+\epsilon\cos\theta_p)d\theta_p \\ &= (1+\epsilon) \end{aligned} \tag{145}$$

$$B=B_0/(1+\epsilon\cos\theta_p)$$

$$\Longrightarrow B_{\max}=B_0/(1-\epsilon)$$

$$h\equiv\frac{B}{B_{\max}}$$

$$\begin{aligned}\langle h\rangle &= \frac{1}{2\pi}\int_0^{2\pi}\frac{B}{B_{\max}}(1+\epsilon\cos\theta_p)d\theta_p\\ &= B_0/B_{\max}\\ &= 1-\varepsilon\end{aligned} \tag{146}$$

$$\begin{aligned}\left\langle h^2\right\rangle &= \frac{1}{2\pi}\int_0^{2\pi}\frac{B^2}{B_{\max}^2}(1+\epsilon\cos\theta_p)d\theta_p\\ &= (1-\varepsilon)^2\frac{1}{2\pi}\int_0^{2\pi}1/(1+\epsilon\cos\theta_p)d\theta_p\end{aligned}$$

Using integration formula

$$\int_0^{2\pi}\frac{d\theta_p}{1+a\cos\theta_p}=\frac{2\pi}{\sqrt{1-a^2}},\ \text{for } -1<a<1$$

one gets

$$\left\langle h^2\right\rangle=\frac{(1-\varepsilon)^2}{\sqrt{1-\varepsilon^2}} \tag{147}$$

$$\begin{aligned}\left\langle\sqrt{1-h}\right\rangle &= \frac{1}{2\pi}\int_0^{2\pi}\sqrt{1-\frac{B}{B_{\max}}}(1+\epsilon\cos\theta_p)d\theta_p\\ &= \frac{1}{2\pi}\int_0^{2\pi}\sqrt{1-\frac{1-\varepsilon}{1+\epsilon\cos\theta_p}}(1+\epsilon\cos\theta_p)d\theta_p\\ &= \frac{1}{\pi}\int_0^{\pi}\sqrt{1-\frac{1-\varepsilon}{1+\epsilon\cos\theta_p}}(1+\epsilon\cos\theta_p)d\theta_p\\ &= \frac{1}{\pi}\left[(1+\varepsilon)\mathrm{Arcsin}\left(\sqrt{\frac{2\varepsilon}{1+\varepsilon}}\right)+\sqrt{2\varepsilon(1-\varepsilon)}\right]\end{aligned} \tag{148}$$

The last equality is obtained by using Mathematica. This equation corresponds to Eq. (A13) in Liu-Liu's paper[1]. However, Lin-Liu's Eq. (A13) has an error in the last term. (The plus sign should be changed to minus sign.)

$$\begin{aligned}\left\langle\sqrt{1-\lambda h}\right\rangle &= \frac{1}{2\pi}\int_0^{2\pi}\sqrt{1-\frac{B}{B_{\max}}\lambda}(1+\epsilon\cos\theta_p)d\theta_p\\ &= \frac{1}{2\pi}\int_0^{2\pi}\sqrt{1-\frac{1-\varepsilon}{1+\epsilon\cos\theta_p}\lambda}(1+\epsilon\cos\theta_p)d\theta_p\\ &= \frac{1}{\pi}\int_0^{\pi}\sqrt{1-\frac{1-\varepsilon}{1+\epsilon\cos\theta_p}\lambda}(1+\epsilon\cos\theta_p)d\theta_p.\end{aligned} \tag{149}$$

The above integration can not be analytically evaluated. Using this, the effective circulating fraction f_c can be written as

$$\begin{aligned}f_c &= \left\langle h^2\right\rangle\frac{3}{4}\int_0^1\frac{\lambda' d\lambda'}{\left\langle\sqrt{1-\lambda' h}\right\rangle}\\ &= \frac{(1-\varepsilon)^2}{\sqrt{1-\varepsilon^2}}\frac{3}{4}\int_0^1\frac{\pi\lambda' d\lambda'}{\int_0^{\pi}\sqrt{1-\frac{1-\varepsilon}{1+\epsilon\cos\theta_p}\lambda}(1+\epsilon\cos\theta_p)d\theta_p},\end{aligned} \tag{150}$$

I evaluate Eq. (150) numerically in the code.

13 Numerical results

Using the above formula, I wrote a code to calculate ECCD efficiencies in toroidal geometry using relativistic high-velocity-limit collision model. In the code, Lin-Liu's analytic solution to the adjoint equation is used. The task of the code is to perform the integration in the formula for calculating the efficiencies. Fig. 2 compares the efficiencies calculated by my code with the one calculated by Lin-Liu's code. Good agreement is observed.

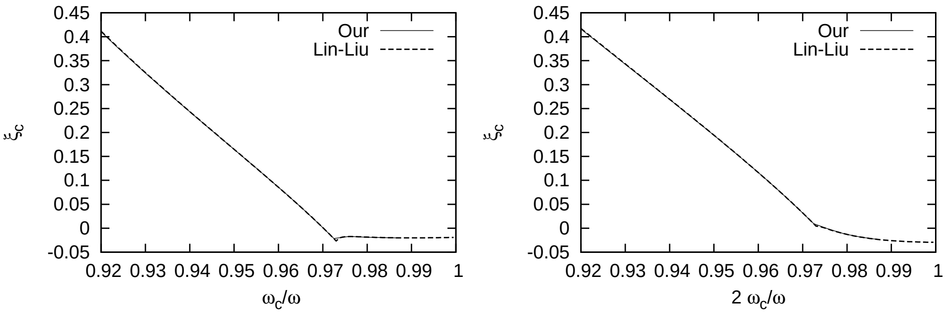

Figure 2. Left figure: $Z_i = 1.67$, $\varepsilon = 0.2$,$n_\parallel = 0.4$, $T_e = 25$keV, $\theta_p = 15°$, $l = 1$. Right figure: $l = 2$ and the other parameters are the same as the left figure.

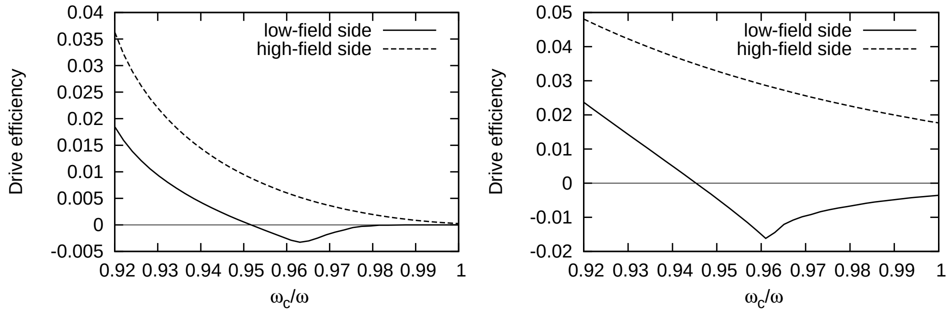

Figure 3. $\varepsilon = 0.3$, low-field-side($\theta_p = 0$), high-field-side($\theta_p = 180$), $T_e = 2$keV(Left Fig). $T_e = 25$keV(Right Fig) $n_\parallel = 0.4$, $Z_{\rm eff} = 1.67$, $l = 1$.

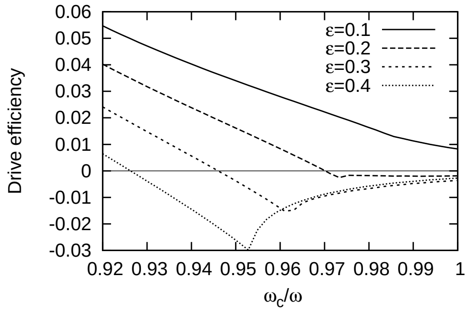

Figure 4. Drive efficiency as a function of ω_c/ω at different flux surface. Near low-field-side resonance($\theta_p = 15°$), $T_e = 25$keV, $n_\parallel = 0.4$, $l = 1$.

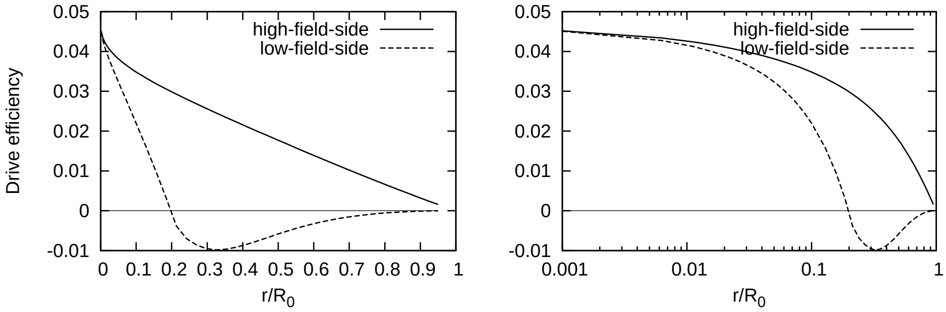

Figure 5. $T_e = 25\text{keV}$, $n_\parallel = 0.4$, $Z_{\text{eff}} = 1.67$, $l = 1$, $\omega_c/\omega = 0.97$. Uniform scale(Left figure); Logarithm scale (Right figure)

13.1 $\varepsilon \to 0$ case: Homogeneous magnetic field

We note that in the limit $\varepsilon \to 0$, the above theory of current drive in toroidal geometry reduces to the case of current drive in homogeneous magnetic field. Therefore, if very small ε (e.g. $\varepsilon = 10^{-6}$), is chosen to calculate the drive efficiency, the result should be close to the efficiency in a homogeneous plasma. Fig. 7 compares the efficiencies for the case of $\varepsilon = 10^{-6}$ and a homogeneous plasma . Good agreement is observed for the two cases. Note that the efficiency for the homogeneous plasma is calculated with a code written for the special case of homogeneous plasma.

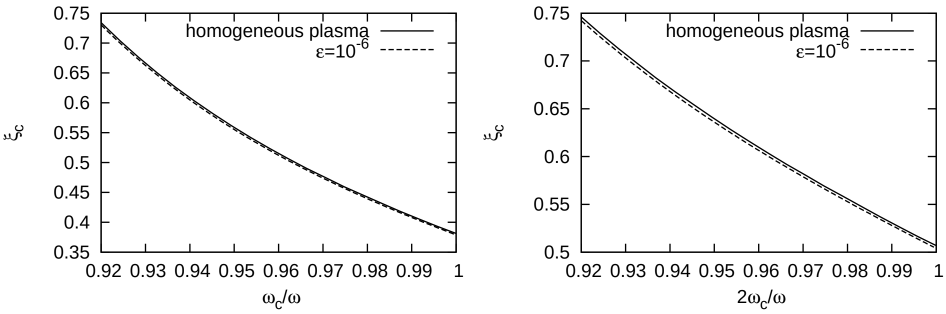

Figure 6. Left figure: $Z_i = 1.67$, $n_\parallel = 0.4$, $T_e = 25\text{keV}$, $l = 1$. Right figure: $l = 2$ and other parameters are the same as left figure. The $\varepsilon = 10^{-6}$ case is calculated using Lin-Liu's code (and using the simplified diffusion operator).

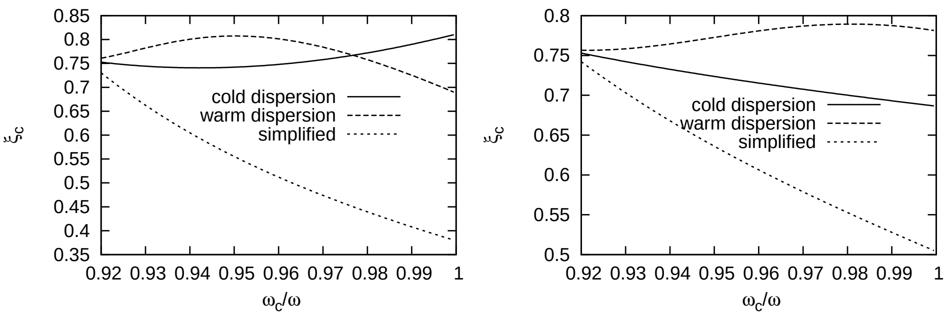

Figure 7. Left figure: $Z_i = 1.67$, $n_\parallel = 0.4$, $T_e = 25\text{keV}$, $\varepsilon = 10^{-6}$, $l = 1$. Right figure: $l = 2$ and other parameters are the same as left figure. The results are calculated using Lin-Liu's code. The poloidal location of absorption is chosen to be $\theta_p = 0$. For the small aspect ratio in this case the θ_p parameter is not important, since this case is very close to current drive in homogeneous magnetic field. Note that for the simplified wave diffusion operator, the efficiency does not depends on the dispersion relation.

Fig. 8 compares the drive efficiencies in homogeneous plasma given by three collision operators, i.e., relativistic high-velocity-limit model, semi-relativistic and full-relativistic collision model. The results indicate the semi-relativistic and full relativistic collision models gives nearly identical drive efficiency, while the high-velocity-limit model tends to underestimate the drive efficiencies, especially for the high temperature cases.

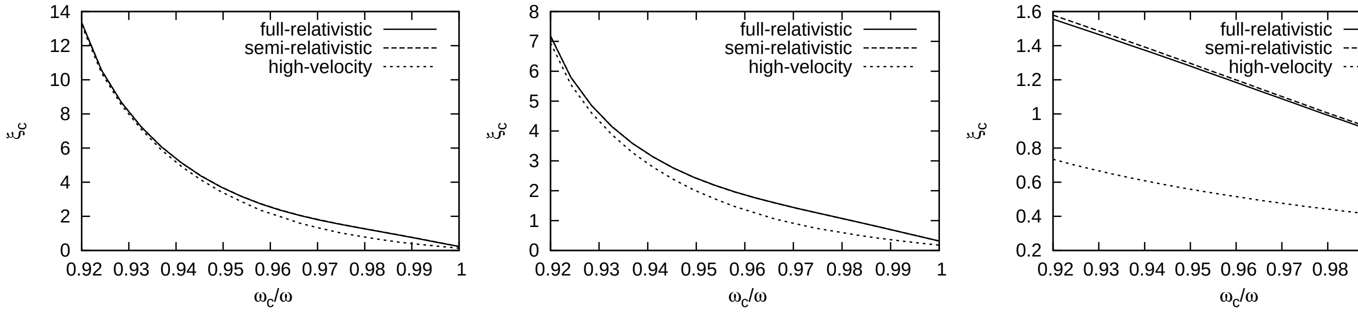

Figure 8. CD efficiencies as a function of ω_c/ω in homogeneous plasma. $T_e = 1$keV(Left figure), $T_e = 2$keV(Middle figure), T 25keV(Right figure). The other parameters are $Z_i = 1.67$, $n_\parallel = 0.4$, $l = 1$. The results are calculated using code for homogeneous pla and using simplified wave diffusion operator.

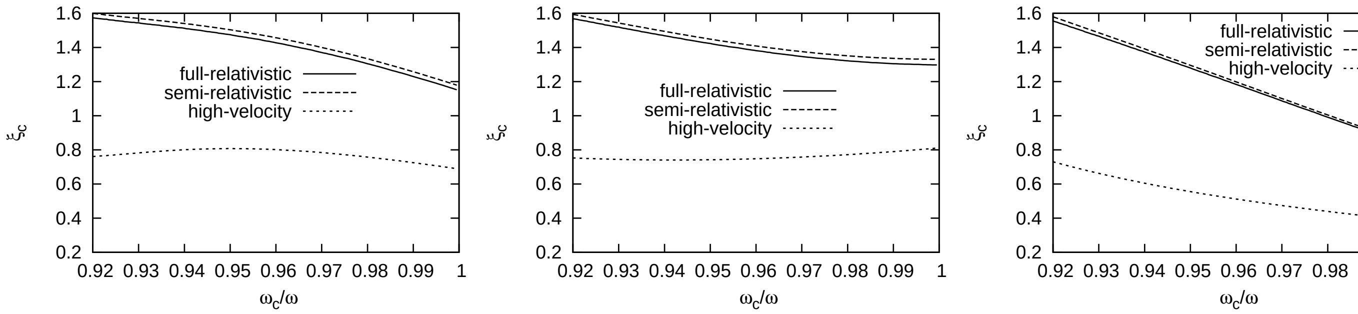

Figure 9. Drive efficiency as a function of ω_c/ω. Using warm dispersion relation (left figure); cold dispersion relation (middle figu The right figure is calculated using the simplified diffusion operator, this case one does not need dispersion relation to evaluate ciency. The other parameters are $T_e = 25$keV, $Z_i = 1.67$, $n_\parallel = 0.4$, $l = 1$, O-mode. These results are obtained through Lin-Liu's cod setting aspect ratio to 10^{-6}. These cases are very close to current drive in homogeneous plasmas. Not that the right figure is the s as the right figure in Fig. 8.

14 On the normalization of drive efficiency

Lin-Liu's definition of current drive efficiency is

$$\xi^\star = \frac{e^3}{\varepsilon_0^2} \frac{n_e}{T_e} \frac{\langle j_\parallel \rangle}{2\pi Q} \tag{151}$$

which is related to Cohen's definition of drive efficiency ξ_c by

$$\xi_c = \frac{\ln\Lambda}{4} \xi^\star \tag{152}$$

And Karney-Fisch's definition of drive efficiency[4] is given by

$$\xi_k = \frac{\langle j_\parallel \rangle}{Q} \frac{\nu_c m_e c}{e} \tag{153}$$

where

$$\nu_c = \frac{e^4 n_e \ln\Lambda}{4\pi\varepsilon_0^2 m^2 c^3} \tag{154}$$

Karney-Fisch's definition is related to ξ_c as

$$\begin{aligned} \frac{\xi_k}{\xi_c} &= \frac{\nu_c m_e c}{e} \Big/ \left(\frac{\ln\Lambda}{4} \frac{e^3}{\varepsilon_0^2} \frac{n_e}{T_e} \frac{1}{2\pi} \right) \\ &= \frac{2T_e}{mc^2} \end{aligned} \tag{155}$$

It is better to use Kareny-Fisch's definition of the normalized drive efficiency to present the results, since the efficiency is normalized to a quantity independent of temperature, thus the normalized efficiency in this case can explicitly reflect the dependence of drive efficiency on temperature. Fig. 10 is a re-plot of Fig. 8 using Karney-Fisch's normalized current drive efficiency.

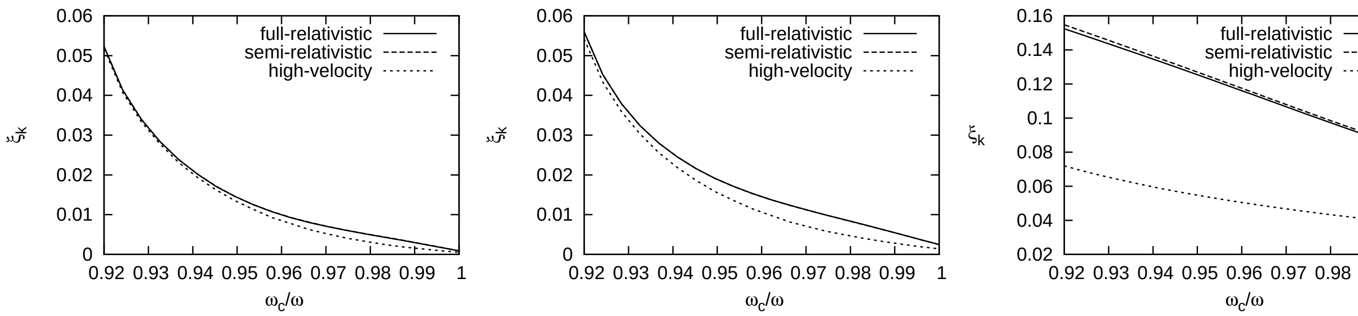

Figure 10. Karney-Fisch's normalized CD efficiencies as a function of ω_c/ω in homogeneous plasma. $T_e = 1$keV (left figure), $T_e = 2$ (middle figure), $T_e = 25$keV (right figure). The other parameters are $Z_i = 1.67$, $n_\parallel = 0.4$, $l = 1$. The results are calculated using code homogeneous plasma and using simplified wave diffusion operator.

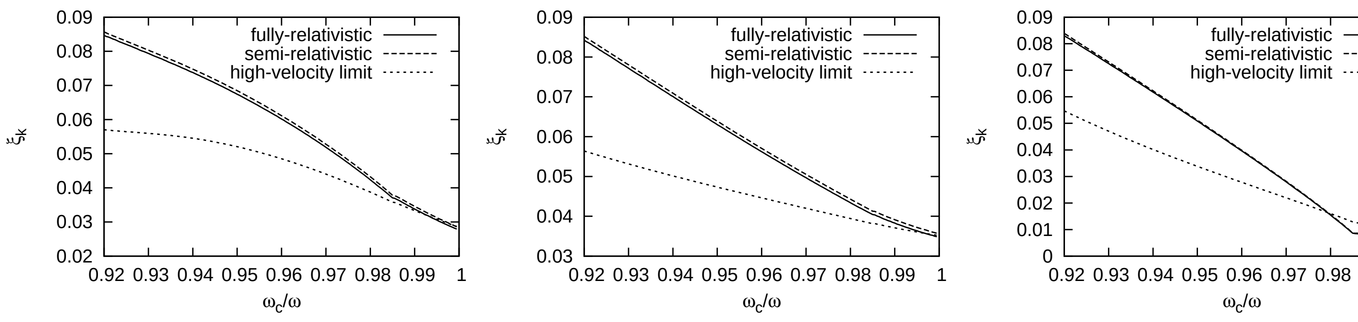

Figure 11. Warm dispersion (left figure), cold dispersion (middle figure), simplified diffusion operator (right figure). Other paramet $n_\parallel = 0.4$, $l = 1$, O-mode , $Z_i = 1.67$, $T_e = 25$keV, $\varepsilon = 0.1$, $\theta_p = 0$.

Fig. 12 plots the drive efficiency in uniform magnetic field as a function of $n_\parallel$.

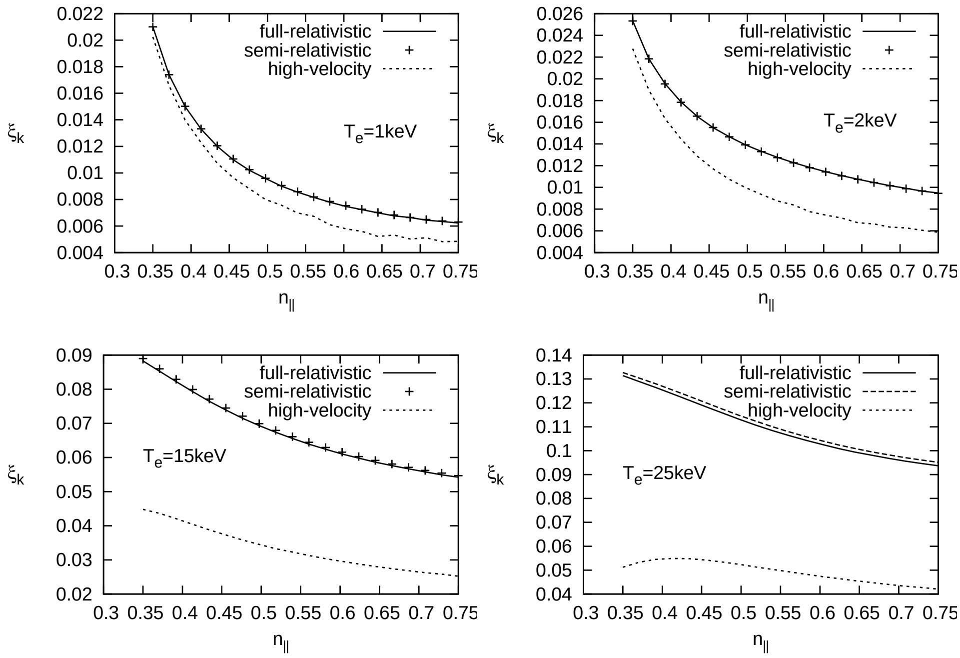

Figure 12. EC drive efficiencies as a function of parallel index of refraction $n_{\|}$. $T_e = 1$keV (top left figure), $T_e = 2$keV (top right), $T_e = 15$keV (low left), $T_e = 25$keV (low right). The other parameters are $Z_i = 1.67$, $\omega_c/\omega = 0.95$, $l = 1$.

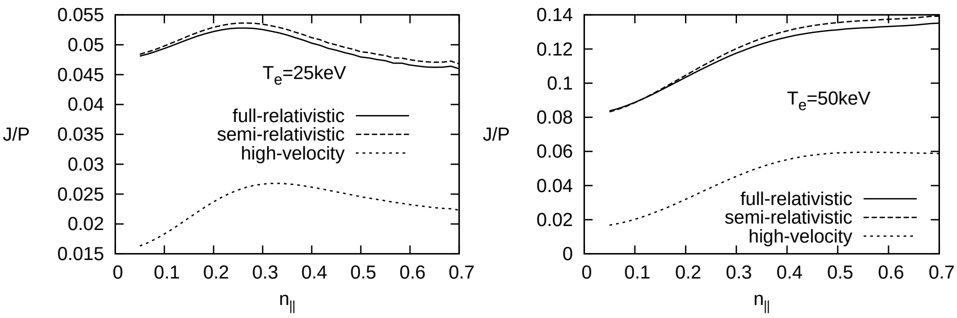

Figure 13. EC drive efficiencies in uniform magnetic field as a function of parallel index of refraction $n_{\|}$. The other parameters are $Z_i = 1.67$, $\omega_c/\omega = 1.05$, $l = 1$.

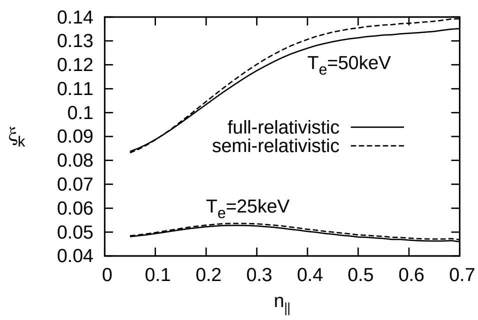

Figure 14. EC drive efficiencies in uniform magnetic field as a function of parallel index of refraction $n_{\|}$. The other parameters are $Z_i = 1.67$, $\omega_c/\omega = 1.05$, $l = 1$.

15 Manuscript

The passing orbits satisfy the condition

$$w > \mu B_{\max} \tag{156}$$

which can be further written as

$$\sin^2\theta_0 < B_0/B_{\max} \tag{157}$$

where θ_0, B_0 is the pitch angle and value of magnetic field at the "midplane coordinate" where magnetic field is minimum.

The following definition does not give the adjoint operator of the high velocity operator, C^l.

$$\begin{aligned}
C^{l+}(g) &\equiv \frac{C^l(gf_m)}{f_m} \\
&= \frac{[\nu_{ei}(u)+\nu_D(u)]L(gf_m)+\frac{1}{u^2}\frac{\partial}{\partial u}u^2\lambda_s(u)gf_m}{f_m} \\
&= [\nu_{ei}(u)+\nu_D(u)]L(g)+\frac{1}{f_m}\frac{1}{u^2}\frac{\partial}{\partial u}u^2\lambda_s(u)gf_m
\end{aligned}$$

$$\begin{aligned}
\frac{1}{f_m}\frac{1}{u^2}\frac{\partial}{\partial u}u^2\lambda_s(u)gf_m &= \nu_{e0}u_e^3\frac{1}{f_m}\frac{1}{u^2}\frac{\partial}{\partial u}\gamma^2 gf_m \\
&= \nu_{e0}u_e^3\frac{1}{f_m}\frac{1}{u^2}\frac{2u}{c^2}gf_m+\nu_{e0}u_e^3\gamma^2\frac{1}{f_m}\frac{1}{u^2}\frac{\partial}{\partial u}gf_m \\
&= \nu_{e0}u_e^3\frac{2}{u}\frac{1}{c^2}g+\nu_{e0}u_e^3\gamma^2\frac{1}{f_m}\frac{1}{u^2}\frac{\partial}{\partial u}gf_m \\
&= \nu_{e0}u_e^3\frac{2}{u}\frac{1}{c^2}g+\nu_{e0}u_e^3\gamma^2\frac{1}{u^2}\frac{\partial}{\partial u}g+\nu_{e0}u_e^3\gamma^2\frac{g}{f_m}\frac{1}{u^2}\beta f_m \\
&= \nu_{e0}u_e^3\frac{2}{u}\frac{1}{c^2}g+\lambda_s\frac{\partial}{\partial u}g+\nu_{e0}u_e^3\gamma^2\frac{1}{u^2}\beta g \\
&= \nu_{e0}u_e^3\left(\frac{2}{u}\frac{1}{c^2}+\gamma^2\frac{1}{u^2}\beta\right)g+\lambda_s\frac{\partial}{\partial u}g \\
&= \nu_{e0}u_e^3\left[\frac{2}{u}\frac{1}{c^2}+\left(1+\frac{u^2}{c^2}\right)\frac{1}{u^2}\left(-\frac{2v}{u_e^2}\right)\right]g+\lambda_s\frac{\partial}{\partial u}g
\end{aligned}$$

Now consider the transformation of volume element from (u,ξ) coordinates to (u,λ,σ) coordinates.

The velocity space volume element in terms of (u,ξ) coordinates is,

$$d\Gamma = 2\pi u^2 du d\xi, \tag{158}$$

From Eq.(46), one gets

$$|\xi| = \sqrt{\frac{u_\parallel^2}{u^2}} = \sqrt{1-\lambda\frac{B}{B_{\max}}} \tag{159}$$

$$\Rightarrow \xi = \mathrm{sgn}(u_\parallel)\sqrt{1-\lambda\frac{B}{B_{\max}}} = \sigma\sqrt{1-\lambda\frac{B}{B_{\max}}} \tag{160}$$

Then one gets

$$\begin{aligned}
d\xi &= d\left(\sigma\sqrt{1-\lambda\frac{B}{B_{\max}}}\right) \\
&= \sigma\frac{1/2\left(-\frac{B}{B_{\max}}\right)}{\sqrt{1-\lambda\frac{B}{B_{\max}}}}d\lambda
\end{aligned} \tag{161}$$

Then the volume element in term of (u, λ, σ) coordinates is written as,

$$\begin{aligned} d\Gamma &= 2\pi u^2 du \sum_{\sigma=\pm 1} \sigma \frac{1/2\Big(-\frac{B}{B_{\max}}\Big)}{\sqrt{1-\lambda\frac{B}{B_{\max}}}} d\lambda \\ &= \pi u^2 du \Big(-\frac{B}{B_{\max}}\Big) \sum_{\sigma=\pm 1} \sigma \frac{1}{|\xi|} d\lambda \end{aligned}$$

$$\begin{aligned} m_e u_e^2 \tilde{\Lambda} \overline{\chi} &= \mathrm{sgn}(u_\parallel)\Big\{\gamma \frac{u_e^2}{u}\frac{c^4}{u_e^4}\frac{d\overline{F}}{du} H(\lambda) + 2\frac{B_{\max}}{B}\frac{u_e^2 u_\parallel}{u^3}\Big(\frac{u_\parallel \gamma}{u} - \frac{n_\parallel u}{c}\Big)\frac{c^4}{u_e^4}\overline{F}\frac{dH(\lambda)}{d\lambda}\Big\} \\ &= \mathrm{sgn}(u_\parallel)\Big\{\gamma \frac{u_e^2}{u}\frac{c^4}{u_e^4}\frac{1}{c}\frac{d\overline{F}}{d\overline{u}} H(\lambda) + 2\frac{B_{\max}}{B}\frac{u_e^2 u_\parallel}{u^3}\Big(\frac{u_\parallel \gamma}{u} - \frac{n_\parallel u}{c}\Big)\frac{c^4}{u_e^4}\overline{F}\frac{dH(\lambda)}{d\lambda}\Big\} \\ &= \mathrm{sgn}(u_\parallel)\Big\{\gamma \frac{1}{\overline{u}}\frac{1}{u_e^2/c^2}\frac{d\overline{F}}{d\overline{u}} H(\lambda) + 2\frac{B_{\max}}{B}\frac{\overline{u}_\parallel}{\overline{u}^3}\frac{1}{u_e^2/c^2}\Big(\frac{u_\parallel \gamma}{u} - \frac{n_\parallel u}{c}\Big)\overline{F}\frac{dH(\lambda)}{d\lambda}\Big\} \\ &= \mathrm{sgn}(u_\parallel)\frac{1}{2\Theta}\Big\{\gamma \frac{1}{\overline{u}}\frac{d\overline{F}}{d\overline{u}} H(\lambda) + 2\frac{B_{\max}}{B}\frac{\overline{u}_\parallel}{\overline{u}^3}\Big(\frac{u_\parallel \gamma}{u} - \frac{n_\parallel u}{c}\Big)\overline{F}\frac{dH(\lambda)}{d\lambda}\Big\} \end{aligned}$$

For thermal electron-electron collisions

$$\ln\Lambda = 23 - \ln(n_e^{1/2} T_e^{-3/2})$$

for $T_e \leqslant 10\,\mathrm{eV}$, and

$$\ln\Lambda = 24 - \ln(n_e^{1/2} T_e^{-1})$$

for $T_e \geqslant 10\mathrm{eV}$.

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