1 Solving the Spitzer-Harm equation by using variational principle

The Spitzer-Harm equation is

$$\frac{1}{f_{em}}C^l(f_{e1})=ev_\parallel \tag{1}$$

where $C^l(f_{e1})\equiv C^{e/e}(f_{e1},f_{em})+C^{e/e}(f_{em},f_{e1})+C^{e/i}(f_{e1},f_i)$ is the linearised collision operator, $v_\parallel$ is the velocity parallel to equilibrium magnetic field, f_{em} is the equilibrium Maxwelian distribution. Define the linear operator

$$L(...)\equiv\frac{1}{f_{em}}C^l(...). \tag{2}$$

It is easy to prove that the operator $L(...)$ defined this way is self-adjoint, i.e.,

$$\int\psi_1 L(\psi_2)d^3\boldsymbol{u}=\int\psi_2 L(\psi_1)d^3\boldsymbol{u}, \tag{3}$$

where ψ_1 and ψ_2 are two arbitrary distribution functions, $d^3\boldsymbol{u}$ is the volume element in momentum-per-unit-rest-mass space. The integration is over the whole momentum space. Define $g(\boldsymbol{v})=ev_\parallel$, then the Spitzer-Harm equation (1) is written as

$$L(f_{e1})=g. \tag{4}$$

Define a functional

$$\dot{S}_3=\int f_{e1}[L(f_{e1})-2g]d^3\boldsymbol{u}, \tag{5}$$

then it is easy to prove that $\delta\dot{S}_3=0$ is equivalent to Eq. (4) (we need the fact that $L(\chi)$ is self-adjoint to prove the equivalence), i.e., the solution to Eq. (1) is the function that can maximise (or minimise) the functional $\dot{S}_3$. Note that the solution to $\delta\dot{S}_3=0$ satisfies Eq. (4), thus, it can also satisfies the following equation

$$\int 2\frac{v_\parallel}{c}f_{em}[L(f_{e1})-g]d^3\boldsymbol{u}=0. \tag{6}$$

However becuase we can only obtain the approximate solution to $\delta\dot{S}_3=0$ (instead of the exact solution), Eq. (6) will only be satified to some tolerance. We prefer to make Eq. (6) be satisfied better (why do we prefer this?), thus we include Eq. (6) as a constraint to the approximate solution. Presenting this in the language of constraint varation is that we want to obtain the function that can maximise $\dot{S}_3$ and at the same time satisfies the constraint of Eq. (6). Recalling the constraint variation we learnt in textbooks, the problem reduces to the following: First construct a functional S with a Lagrangian multiplier λ, (I have questions here)

$$S=\dot{S}_3+\lambda S_b, \tag{7}$$

with

$$S_b=\int 2\frac{v_\parallel}{c}f_{em}[L(f_{e1})-g]d^3\boldsymbol{u}. \tag{8}$$

Then the solution to $\delta S=0$ can make $\delta\dot{S}_3$ be equal zero and at the same time satisfies Eq. (6). (However, from my numerical resuts, I found that the solution to $\delta\dot{S}_3=0$ with no constraint of Eq. (6) is almost identical to $\delta\dot{S}_3=0$ with the constraint. Thus, for the sake of simplicity, I prefer to remove the constraint. I also found that the difference between the two approaches is still small even for the cases where we use collision operators that does not conserve momentum.)

We know the solution to Eq. (1) consists of only the first Legerndre harmonic, i.e., $f_{e1}(\boldsymbol{u})=\tilde{f}_{e1}^{(1)}(u)\cos\theta$, where θ is the pitch-angle. Approximating $\tilde{f}_{e1}^{(1)}(u)$ by an expansion in terms of some basis functions, f_{e1} is written as

$$f_{e1}(\boldsymbol{u})=\left[\sum_{j=1}^{n}d_jb_j(u)\right]f_{em}(u)\cos\theta. \tag{9}$$

Then maximizing or minimising the functional S requires (I have questions here)

$$\frac{\partial S}{\partial d_i}=0,\ \text{For } i=1,2,\ldots n \tag{10}$$

and

$$\frac{\partial S}{\partial \lambda}=0. \tag{11}$$

The left-hand side of Eq. (10) is written

$$\begin{aligned}\frac{\partial S}{\partial c_i} &= \int\left\{\frac{\partial f_{e1}}{\partial c_i}L(f_{e1})+f_{e1}\frac{\partial L(f_{e1})}{\partial c_i}-2g\frac{\partial f_{e1}}{\partial c_i}+2\lambda\frac{v_\|}{c}f_{em}\frac{\partial L(f_{e1})}{\partial c_i}\right\}d^3\boldsymbol{u}\\ &= \int\left\{b_i f_{em}\cos\theta\ L(f_{e1})+f_{e1}\ L(b_i f_{em}\cos\theta)-2g\ b_i\ f_{em}\cos\theta+\right.\\ &\quad\left.2\lambda\frac{v_\|}{c}f_{em}L(b_i f_{em}\cos\theta)\right\}d^3\boldsymbol{u}\end{aligned} \tag{12}$$

Using the self-adjoint property of the operator L, the above equation is written as

$$\frac{\partial S}{\partial c_i}=\int\left\{2b_i f_{em}\cos\theta L(f_{e1})-2gb_i\,f_{em}\cos\theta+2\lambda\frac{v_\|}{c}f_{em}L(b_i f_{em}\cos\theta)\right\}d^3\boldsymbol{u} \tag{13}$$

Using this, Eq. (10) is written as

$$\int\left\{2b_i f_{em}\cos\theta L(f_{e1})-2gb_i\,f_{em}\cos\theta+2\lambda\frac{v_\|}{c}f_{em}L(b_i f_{em}\cos\theta)\right\}d^3\boldsymbol{u}=0 \tag{14}$$

Using the expansion expression of f_{e1} in the above equation yields

$$\int\left\{2b_i f_{em}\cos\theta\sum_{j=1}^{n}d_j\,L(b_j f_{em}\cos\theta)-2gb_i f_{em}\cos\theta+2\lambda\frac{v_\|}{c}f_{em}L(b_i f_{em}\cos\theta)\right\}d^3\boldsymbol{u}=0 \tag{15}$$

Define matrix elements

$$m_{i,j}=2\int b_i f_{em}\cos\theta L(b_j f_{em}\cos\theta)d^3\boldsymbol{u}, \tag{16}$$

with $i=1,2,\ldots n$ and $j=1,2,\ldots n$,

$$m_{i,n+1}=2\int\frac{v_\|}{c}f_{em}L(b_i f_{em}\cos\theta)d^3\boldsymbol{u}, \tag{17}$$

with $i=1,2,\ldots n$, and

$$\delta_i=2\int b_i f_{em}\cos\theta g d^3\boldsymbol{u}, \tag{18}$$

with $i=1,2,\ldots n$, then Eq. (15) is written as

$$m_{i,j}d_j+m_{i,n+1}\lambda=\delta_i,\ \text{For } i=1,2,\ldots,n \tag{19}$$

where Einstein's summation rule over the repeated subscript is adopted. Now consider the second condition to maximize or minimise the functional S, Eq. (11), which is written as

$$\int 2\frac{v_\|}{c}f_{em}[L(f_{e1})-g]d^3\boldsymbol{u}=0. \tag{20}$$

Using the expansion expression of f_{e1} in the above equation yields

$$\int 2\frac{v_\|}{c}f_{em}\left[\sum_{j=1}^{n}d_j L(b_j f_{em}\cos\theta)-g\right]d^3\boldsymbol{u}=0, \tag{21}$$

which can be further written as

$$m_{n+1,j}d_j+0\cdot\lambda=\delta_{n+1}, \tag{22}$$

with m_{n+1},j and δ_{n+1} given by

$$m_{n+1,j}=2\int\frac{v_\|}{c}f_{em}L(b_j f_{em}\cos\theta)d^3\boldsymbol{u},\ \text{For } j=1,2,\ldots,n \tag{23}$$

and

$$\delta_{n+1} = 2\int \frac{v_\|}{c} f_{em} g d^3\boldsymbol{u} \tag{24}$$

Equations (19) and (22) constitute a closed system to determine the $n+1$ unknown coefficients, $d_1, d_2, d_3, ...d_n$, and λ.

1.1 Expression of matrix elements

Using

$$\frac{1}{f_{em}} C^{e/e}(f_{em}\chi, f_{em}) = \frac{1}{u^2}\frac{\partial}{\partial u}\left(u^2 D^{e/e}_{cuu}\frac{\partial \chi}{\partial u}\right) + F^{e/e}_u \frac{\partial \chi}{\partial u} + \frac{D_{\theta\theta}}{u^2}\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left[\sin\theta\left(\frac{\partial\chi}{\partial\theta}\right)\right], \tag{25}$$

the differential part of the $L(...)$ operator, denoted by $L_d(f_{e1})$, is written as

$$\begin{aligned}
L_d(f_{e1}) \;\equiv\;& \frac{1}{f_{em}} C^{e/e}(f_{e1}, f_{em}) \\
=\;& \frac{1}{u^2}\frac{\partial}{\partial u}\left(u^2 D^{e/e}_{cuu}\frac{\partial f_{e1}/f_{em}}{\partial u}\right) + F^{e/e}_u \frac{\partial f_{e1}/f_{em}}{\partial u} + \frac{D_{\theta\theta}}{u^2}\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left[\sin\theta\left(\frac{\partial f_{e1}/f_{em}}{\partial\theta}\right)\right] \\
=\;& \frac{1}{u^2}\frac{\partial}{\partial u}\left(u^2 D^{e/e}_{cuu}\frac{\partial \sum_{j=1}^n d_j b_j \cos\theta}{\partial u}\right) + F^{e/e}_u \frac{\partial \sum_{j=1}^n d_j b_j \cos\theta}{\partial u} + \\
& \frac{D_{\theta\theta}}{u^2}\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left[\sin\theta\left(\frac{\partial \sum_{j=1}^n d_j b_j \cos\theta}{\partial\theta}\right)\right] \\
=\;& \frac{1}{u^2}\frac{\partial}{\partial u}\left(u^2 D^{e/e}_{cuu}\left(\sum_{j=1}^n d_j \frac{\partial b_j}{\partial u}\right)\right)\cos\theta + F^{e/e}_u\left(\sum_{j=1}^n d_j\frac{\partial b_j}{\partial u}\right)\cos\theta - \\
& 2\frac{D_{\theta\theta}}{u^2}\cos\theta\sum_{j=1}^n d_j b_j.
\end{aligned} \tag{26}$$

Thus we have

$$L_d(b_j f_{em}\cos\theta) = \frac{1}{u^2}\frac{\partial}{\partial u}\left(u^2 D^{e/e}_{cuu}\frac{\partial b_j}{\partial u}\right)\cos\theta + F^{e/e}_u\frac{\partial b_j}{\partial u}\cos\theta - 2\frac{D_{\theta\theta}}{u^2} b_j\cos\theta. \tag{27}$$

Using this, the element of the matrix, $m_{ij} = 2\int b_i f_{em}\cos\theta L(b_j f_{em}\cos\theta) d^3\boldsymbol{u}$, is written as

$$m_{ij} = 2\int b_i f_{em}\cos\theta[L_d(b_j f_{em}\cos\theta) + I_1(b_j)\cos\theta] d^3\boldsymbol{u}, \tag{28}$$

where $I_1(b_j)\cos\theta$ is the integration part (i.e., the field particle term of the collision operator) of the operator L. Using the above results, the matrix element is further written as

$$\begin{aligned}
m_{ij} \;=\;& 2\int_0^\pi \int_0^\infty b_i f_{em}\cos\theta[L_d(b_j f_{em}\cos\theta) + I_1(b_j)\cos\theta] 2\pi\sin\theta u^2 d\theta du \\
=\;& 2A\int_0^\infty b_i f_{em}\left[\frac{1}{u^2}\frac{\partial}{\partial u}\left(u^2 D^{e/e}_{cuu}\frac{\partial b_j}{\partial u}\right) + F^{e/e}_u\frac{\partial b_j}{\partial u} - 2\frac{D_{\theta\theta}}{u^2} b_j + I_1(b_j)\right]u^2 du, \qquad (29) \\
=\;& 2A\int_0^\infty b_i f_{em}\left[\left(2u D^{e/e}_{cuu} + u^2\frac{dD^{e/e}_{cuu}}{du}\right)\frac{\partial b_j}{\partial u} + u^2 D^{e/e}_{cuu}\frac{\partial^2 b_j}{\partial u^2} + u^2 F^{e/e}_u\frac{\partial b_j}{\partial u} - 2D_{\theta\theta}\, b_j + \right. \\
& \left. I_1(b_j)u^2\right] du, \qquad (30)
\end{aligned}$$

where $A = 2\pi\int_0^\pi \cos^2\theta\sin\theta d\theta$. The inhomogeneous term is written as

$$\begin{aligned}
\delta_i \;=\;& 2\int b_i f_{em}\cos\theta e v_\| d^3\boldsymbol{u} \\
=\;& 2\int_0^\pi 2\pi\int_0^\infty b_i f_{em} e v\cos^2\theta\sin\theta u^2 d\theta du \\
=\;& 2A\int_0^\infty b_i f_{em}\, e v\, u^2 du.
\end{aligned} \tag{31}$$

2 Basis functions

The basis function suggested by Lin-Liu is

$$B_i(u) = \frac{u}{\gamma^{\alpha}} u^i. \tag{32}$$

where $\gamma = \sqrt{1 + u^2/c^2}$, $\alpha = 2$. Define $x = u/c$ with c the velocity of light in vacuum and the normalized basis function

$$b_i(x) = \frac{x}{\left(\sqrt{1+x}\right)^2} x^i, \tag{33}$$

then the first order derivative of $b_i(x)$ is written as

$$b'(x) = \frac{x^i}{1+x^2} + x^i \frac{1-x^2}{(1+x^2)^2}, \tag{34}$$

and the second order derivative is written as

$$b''(x) = i \frac{i x^{i-1}(1+x^2) - 2x^{i+1}}{(1+x^2)^2} + i x^{i-1} \frac{1-x^2}{(1+x^2)^2} + x^i \frac{-6x + 2x^3}{(1+x^2)^3}. \tag{35}$$

Another kind of basis function, which is adopted by Marushkenco, is of the same form of Eq. (32) but with $\alpha = 1$.

3 Converted to different normalizations

Define

$$b_j(x) = \frac{x}{\gamma^2} x^j, \tag{36}$$

where $x = u/c$, and

$$h_j(y) = \frac{y}{\gamma^2} y^j, \tag{37}$$

where $y = u/v_{te}$, then

$$b_j(x) = h_j(y) \left(\frac{v_{te}}{c}\right)^{j+1} \tag{38}$$

Define

$$\overline{\chi}_{1c} = \frac{\chi_1}{ec} \nu_c, \tag{39}$$

and

$$\overline{\chi}_{1\text{th}} = \frac{\chi_1}{e v_{te}} \nu_t, \tag{40}$$

then we obtain the relationship between $\overline{\chi}_{1c}$ and $\overline{\chi}_{1\text{th}}$,

$$\overline{\chi}_{1\text{th}} = \left(\frac{c}{v_{te}}\right)^4 \overline{\chi}_{1c} \tag{41}$$

Expanding $\overline{\chi}_{1c}$ using the basis functions $b_j(x)$

$$\overline{\chi}_{1c}(x) = \sum_{j=1}^{4} d_j b_j(x), \tag{42}$$

Using Eq. (42) in the Eq. (41), gives

$$\overline{\chi}_{1\text{th}} = \left(\frac{c}{v_{te}}\right)^4 \sum_{j=1}^{4} d_j b_j(x) \tag{43}$$

Using the relation between $b_j(x)$ and $h_j(y)$ [Eq. (38)], the above equation is written as

$$\begin{aligned}\overline{\chi}_{1\mathrm{th}} &= \left(\frac{c}{v_{te}}\right)^4 \sum_j^4 d_j h_j(y) \left(\frac{v_{te}}{c}\right)^{j+1} \\ &= \sum_j^4 d_j \left(\frac{v_{te}}{c}\right)^{j-3} h_j(y). \end{aligned} \tag{44}$$

If we write

$$\overline{\chi}_{1\mathrm{th}} = \sum_j^4 e_j h_j(y), \tag{45}$$

then Eq. (44) indicates the expansion coefficients e_j is related to d_j by

$$e_j = d_j \left(\frac{v_{te}}{c}\right)^{j-3}. \tag{46}$$

4 Numerical results

The fully relativistic collision operator with field particle term is used to model the electron-electron collision. The thermal velocity is defined by $v_{te} = \sqrt{2T_e/m_e}$. For the case of $\Theta = 0.2$, the Spitzer functions obtained by using different methods are plotted in Fig. 1.

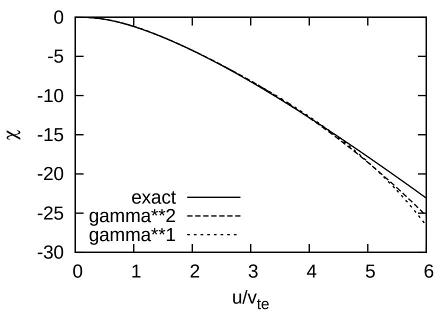

Figure 1. Comparison of Spitzer functions obtained by using different methods. Here u is the momentum per unit rest mass. The exact solution is obtained by numerically solving the Spitzer-Harm equation. The electrical conductivity calculated from the exact solution is $\sigma = 5.436715$. Other parameters are $Z_{\mathrm{eff}} = 1$ and $\Theta = 0.2$

The expansion coefficients for the basis functions with $\alpha = 1$ are given by

$$\begin{pmatrix} d_1 \\ d_2 \\ d_3 \\ d_4 \\ \lambda \end{pmatrix} = \begin{pmatrix} -0.8796 \\ -0.6229 \\ 0.13203 \\ -1.404\times 10^{-2} \\ -3.167\times 10^{-3} \end{pmatrix}, \tag{47}$$

and the corresponding electrical conductivity is $\sigma = 5.399592799$.

The expansion coefficients for the basis functions with $\alpha = 2$ are given by

$$\begin{pmatrix} d_1 \\ d_2 \\ d_3 \\ d_4 \\ \lambda \end{pmatrix} = \begin{pmatrix} -0.7903 \\ -0.75596 \\ -9.184\times 10^{-2} \\ -1.026\times 10^{-2} \\ -2.3159\times 10^{-2} \end{pmatrix}, \tag{48}$$

and the corresponding electrical conductivity is $\sigma = 5.40591$

For the case of $\Theta = 0.1$, the results are plotted in Fig. 2.

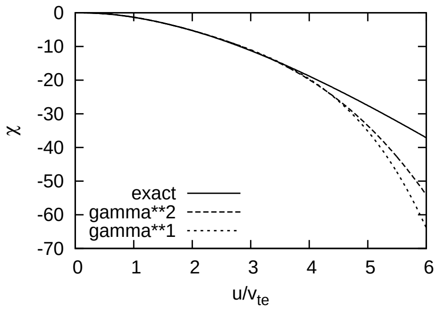

Figure 2. Comparison of Spitzer functions obtained by using different methods. Here u is the momentum per unit rest mass. The exact solution is obtained by numerically solving the Spitzer-Harm equation. The electrical conductivity calculated from the exact solution is $\sigma = 6.20962$. Other parameters are $Z_{\mathrm{eff}} = 1$ and $\Theta = 0.1$.

The expansion coefficients for basis functions with $\alpha = 1$ are given by

$$\begin{pmatrix} d_1 \\ d_2 \\ d_3 \\ d_4 \\ \lambda \end{pmatrix} = \begin{pmatrix} -0.7622 \\ -0.9089 \\ 0.28980 \\ -4.3076 \times 10^{-2} \\ -6.8590 \times 10^{-4} \end{pmatrix}, \tag{49}$$

and the corresponding electrical conductivity is $\sigma = 6.1713$

The expansion coefficients for basis functions with $\alpha = 2$ are given by

$$\begin{pmatrix} d_1 \\ d_2 \\ d_3 \\ d_4 \\ \lambda \end{pmatrix} = \begin{pmatrix} -0.69164 \\ -1.04844 \\ 0.23532 \\ -6.3977 \times 10^{-2} E - 002 \\ -5.76135 \times 10^{-4} \end{pmatrix}, \tag{50}$$

and the corresponding electrical conductivity is $\sigma = 6.173076$

For the case of $\Theta = 0.05$, the results are plotted in Fig. 3.

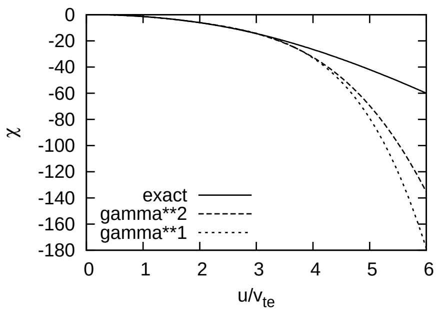

Figure 3. Comparison of Spitzer functions obtained by using different methods. Here u is the momentum per unit rest mass. The exact solution is obtained by numerically solving the Spitzer-Harm equation. The electrical conductivity calculated from the exact solution is $\sigma = 6.738248$. Other parameters are $Z_{\mathrm{eff}} = 1$ and $\Theta = 0.05$.

The expansion coefficients for the basis functions with $\alpha = 1$ are given by

$$\begin{pmatrix} d_1 \\ d_2 \\ d_3 \\ d_4 \\ \lambda \end{pmatrix} = \begin{pmatrix} -0.49372 \\ -1.517871 \\ 0.664713 \\ -0.11566 \\ 1.067 \times 10^{-4} \end{pmatrix}, \tag{51}$$

and the corresponding electrical conductivity is $\sigma = 6.7240$.

The expansion coefficients for the basis functions with $\alpha = 2$ are given by

$$\begin{pmatrix} d_1 \\ d_2 \\ d_3 \\ d_4 \\ \lambda \end{pmatrix} = \begin{pmatrix} -0.41647 \\ -1.70888 \\ 0.7367428 \\ -0.153566 \\ 1.211 \times 10^{-4} \end{pmatrix}, \tag{52}$$

and the corresponding electrical conductivity is $\sigma = 6.724212$.

For the case of $\Theta = 0.02$, the results are plotted in Fig. 4.

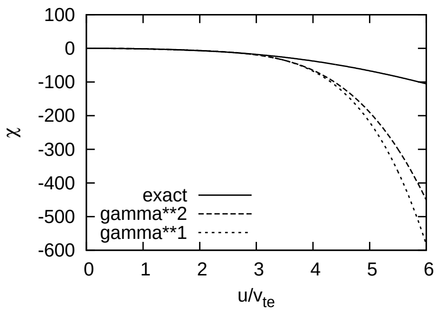

Figure 4. Comparison of Spitzer functions obtained by using different methods. Here u is the momentum per unit rest mass. The exact solution is obtained by numerically solving the Spitzer-Harm equation. The electrical conductivity calculated from the exact solution is $\sigma = 7.12791$. Other parameters are $Z_{\text{eff}} = 1$ and $\Theta = 0.02$.

The expansion coefficients for the basis functions with $\alpha = 1$ are given by

$$\begin{pmatrix} d_1 \\ d_2 \\ d_3 \\ d_4 \\ \lambda \end{pmatrix} = \begin{pmatrix} 0.1634 \\ -3.003 \\ 1.6185 \\ -0.304 \\ 2.4654 \times 10^{-4} \end{pmatrix}, \tag{53}$$

and the corresponding electrical conductivity is $\sigma = 7.2002$.

The expansion coefficients for the basis functions with $\alpha = 2$ are given by

$$\begin{pmatrix} d_1 \\ d_2 \\ d_3 \\ d_4 \\ \lambda \end{pmatrix} = \begin{pmatrix} 0.238844 \\ -3.2076 \\ 1.7512 \\ -0.3454 \\ 2.473 \times 10^{-4} \end{pmatrix}, \tag{54}$$

and the corresponding electrical conductivity is $\sigma = 7.2004$.

For the case of $\Theta = 0.01$, the results are plotted in Fig. 5.

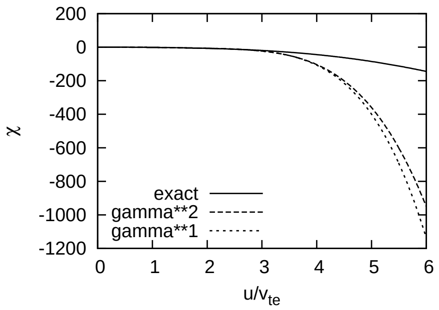

Figure 5. Comparison of Spitzer functions obtained by using different methods. Here u is the momentum per unit rest mass. The exact solution is obtained by numerically solving the Spitzer-Harm equation. The electrical conductivity calculated from the exact solution is $\sigma = 7.273785683772494$. Other parameters are $Z_{\text{eff}} = 1$ and $\Theta = 0.01$.

The expansion coefficients for the basis functions with $\alpha = 1$ are given by

$$\begin{pmatrix} d_1 \\ d_2 \\ d_3 \\ d_4 \\ \lambda \end{pmatrix} = \begin{pmatrix} 0.87451 \\ -4.6118 \\ 2.6638 \\ -0.5120 \\ 1.7172 \times 10^{-4} \end{pmatrix}, \tag{55}$$

and the corresponding electrical conductivity is $\sigma = 7.4500$.

The expansion coefficients for the basis functions with $\alpha = 2$ are given by

$$\begin{pmatrix} d_1 \\ d_2 \\ d_3 \\ d_4 \\ \lambda \end{pmatrix} = \begin{pmatrix} 0.93605 \\ -4.7827 \\ 2.7858 \\ -0.54614 \\ 1.71675 \times 10^{-4} \end{pmatrix}, \tag{56}$$

and the corresponding electrical conductivity is $\sigma = 7.4502$.

5 Solution to Spitzer-Harm equation in Lorentz limit

In the Lorentz limit, electrons only collision with stationary ions. The collision term of electron is given by

$$C^l(f_{e1}) = \frac{D_{\theta\theta}}{u^2} \frac{1}{\sin\theta} \frac{\partial}{\partial\theta}\left[\sin\theta\left(\frac{\partial f_{e1}}{\partial\theta}\right)\right], \tag{57}$$

where $D_{\theta\theta} = \Gamma^{e/e} Z_{\text{eff}}/(2v)$, Z_{eff} is the effective charge number of ions. Writing the solution to the Spitzer-Harm equation as $f_{e1} = f_{em}(u)\chi(\boldsymbol{u})$, with $\chi(\boldsymbol{u}) = \chi_1(u)\cos\theta$, then the left-hand side of the Spitzer-Harm equation (1) is written as

$$\begin{aligned} \frac{1}{f_{em}} C^l(f_{e1}) &= \frac{D_{\theta\theta}}{u^2} \frac{1}{\sin\theta} \frac{\partial}{\partial\theta}\left[\sin\theta\left(\frac{\partial f_{e1}/f_{em}}{\partial\theta}\right)\right] \\ &= \frac{D_{\theta\theta}}{u^2} \frac{1}{\sin\theta} \frac{\partial}{\partial\theta}\left[\sin\theta\left(\frac{\partial \chi_1\cos\theta}{\partial\theta}\right)\right] \\ &= -\frac{2D_{\theta\theta}}{u^2}\chi_1\cos\theta \\ &= -\Gamma^{e/e}\frac{Z_{\text{eff}}}{u^2 v}\chi_1\cos\theta. \end{aligned} \tag{58}$$

Thus the Spitzer-Harm equation (1) is written as

$$-\Gamma^{e/e}\frac{Z_{\text{eff}}}{u^2 v}\chi_1\cos\theta = e v \cos\theta \tag{59}$$

which gives

$$\chi_1 = -\frac{e}{\Gamma^{e/e}}\frac{u^4}{\gamma^2 Z_{\rm eff}}. \tag{60}$$

Normalise χ_1 as

$$\overline{\chi}_1 \equiv \frac{\chi_1}{e v_{\rm th}/\nu_t}, \tag{61}$$

where $v_{\rm th}$ is electron thermal velocity, $\nu_t = \Gamma^{e/e}/v_{\rm th}^3$. Then Eq. (60) is written as

$$\overline{\chi}_1 = -\frac{\overline{u}^4}{\gamma^2 Z_{\rm eff}}, \tag{62}$$

where $\overline{u} = u/v_{\rm th}$. I use Eq. (62) to benchmark the numerical code. In the numerical code, $\overline{\chi}_1$ is approximated as

$$\overline{\chi}_1 = \frac{\overline{u}}{\gamma^2}(d_1\overline{u} + d_2\overline{u}^2 + d_3\overline{u}^3 + d_4\overline{u}^4 + ...). \tag{63}$$

When the Lorentz operator is used as the collision model, I found numerically that except for the coefficient $d_3 \approx -1/\overline{Z}_i$, all the coefficients are vanishingly small (on the order of 10^{-10}).

Fig. 6 plots the numerical results for the Spitzer function of the Lorentz gas with the parameters $Z_{\rm eff} = 1$, $\Theta = T_e/m_e c^2 = 0.2$.

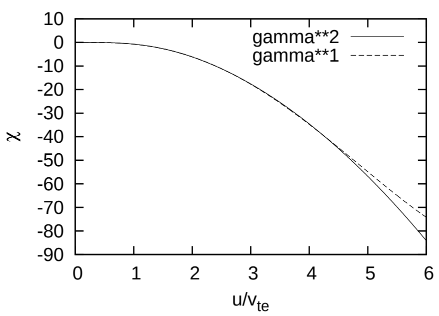

Figure 6. Spitzer function for Lorentz operator with the parameters $Z_{\rm eff} = 1$, $\Theta = T_e/m_e c^2 = 0.2$. Here $v_{te} = \sqrt{2T_e/m_e}$. This figure agrees with Lin-Liu's Fig. 1 in the email on Aug. 2.

(My results have a minus sign difference from Lin-Liu's. This difference results from the sign of the inhomogeneous term. According to Lin-Liu's formulation given in Manuscript08022012.pdf, where the parallel electrical field is $E_\parallel = \nu_{e0}T_e/(e v_{te})$, the result of χ should be negative. Tell me if you find I'm wrong.)

The expanding coefficients for the result of the dashed line in Fig. 6 is

$$\begin{pmatrix} d_1 \\ d_2 \\ d_3 \\ d_4 \\ \lambda \end{pmatrix} = \begin{pmatrix} 0.3179 \\ -0.9021 \\ -0.3252 \\ 0.04033 \\ 8.678\times 10^{-4} \end{pmatrix} \tag{64}$$

Comparing the above coefficients with Lin-Liu's result (refer to LG Check.pdf):

$$\begin{pmatrix} d_1 \\ d_2 \\ d_3 \\ d_4 \\ \lambda \end{pmatrix} = \begin{pmatrix} -0.32405 \\ 0.911369 \\ 0.32089 \\ -0.0396749 \\ -0.00232577 \end{pmatrix} \tag{65}$$

indicates that my result is in agreement with Lin-Liu's.

The expanding coefficients for the result of the solid line in Fig. 6 is

$$\begin{pmatrix} d_1 \\ d_2 \\ d_3 \\ d_4 \\ \lambda \end{pmatrix} = \begin{pmatrix} 7.46\times 10^{-16} \\ -1.54\times 10^{-15} \\ -0.9999999 \\ -1.45\times 10^{-16} \\ 3.53\times 10^{-18} \end{pmatrix}, \tag{66}$$

which indicates the numerical result reproduces the analytical result $d_3 = -1/Z_{\text{eff}} = -1.0$.

5.1 **tmp*wrong**Another approach (a direct and simple method)

The differential equation

$$Ly - g(x) = 0, \tag{67}$$

is equivalent to the following "weak form"

$$\int_0^\infty \delta y[Ly - g(x)]dx = 0. \tag{68}$$

Now, if y is approximated by an expansion in a finite function space,

$$y(x) = \sum_{j=1}^{n} c_j b_j(x), \tag{69}$$

where $b_i(x)$ is the basis function, then from Eq. (68), we obtain

$$\int_0^\infty b_i(x)\left[L\left(\sum_{j=1}^{n} c_j b_j(x)\right) - g(x)\right]dx = 0 \tag{70}$$

$$\Rightarrow \int_0^\infty b_i(x)\left[\sum_{j=1}^{n} c_j L(b_j(x)) - g(x)\right]dx = 0 \tag{71}$$

Inspecting the above equation, it is obvious that we can define the following matrix elements

$$m_{ij} = \int_0^\infty b_i\, L(b_j)dx, \tag{72}$$

and

$$\delta_i = \int_0^\infty b_i(x)g(x)dx, \tag{73}$$

then write Eq. (71) as

$$m_{ij}c_j = \delta_i,\ \text{For}\, i = 1, 2, \ldots, n \tag{74}$$

where Einsteins' summation rule over the repeated index is adopted.

6 tmp

$$\int \frac{v_\parallel}{v_{te}} f_{em}[L(f_{e1}) - g]d^3\boldsymbol{u} = 0. \tag{75}$$

$$\int \left[f_{e1}L\left(\frac{v_\parallel}{v_{te}}f_{em}\right) - g\frac{v_\parallel}{v_{te}}f_{em}\right]d^3\boldsymbol{u} = 0. \tag{76}$$

$$\int \left\{f_{e1}\frac{1}{f_{em}}\left[C^{e/e}\left(\frac{v_\parallel}{v_{te}}f_{em},\ f_{em}\right) + C^{e/e}\left(f_{em},\ \frac{v_\parallel}{v_{te}}f_{em}\right) + C^{e/i}\left(\frac{v_\parallel}{v_{te}}f_{em},\ f_i\right)\right] - g\frac{v_\parallel}{v_{te}}f_{em}\right\}d^3\boldsymbol{u} = 0. \tag{77}$$

If $C^{e/e}(\frac{v_\parallel}{v_{te}} f_{em}, f_{em}) + C^{e/e}(f_{em}, \frac{v_\parallel}{v_{te}} f_{em}) = 0$ then the above equation reduces to

$$\int \left\{ \frac{f_{e1}}{f_{em}} C^{e/i}(\frac{v_\parallel}{v_{te}} f_{em}, f_i) - g \frac{v_\parallel}{v_{te}} f_{em} \right\} d^3\boldsymbol{u} = 0. \tag{78}$$

Using

$$C^{e/i}(f_{e1}, f_i) = \Gamma^{e/e} \frac{Z_i}{2vu^2} \frac{1}{\sin\theta} \frac{\partial}{\partial\theta} \left(\sin\theta \frac{\partial f_{e1}}{\partial\theta} \right) \tag{79}$$

Eq. () is written as

$$\int \left\{ f_{e1} \frac{v}{v_{te}} \Gamma^{e/e} \frac{Z_i}{2vu^2} \frac{1}{\sin\theta} \frac{\partial}{\partial\theta} \left(\sin\theta \frac{\partial P_1(\cos\theta)}{\partial\theta} \right) - g \frac{v_\parallel}{v_{te}} f_{em} \right\} d^3\boldsymbol{u} = 0. \tag{80}$$

$$\int \left\{ f_{e1} \frac{v}{v_{te}} \Gamma^{e/e} \frac{Z_i}{2vu^2} \left(-2\cos\theta \right) - g \frac{v_\parallel}{v_{te}} f_{em} \right\} d^3\boldsymbol{u} = 0. \tag{81}$$

$$\int \left\{ -f_{e1} \frac{v_\parallel}{v_{te}} \Gamma^{e/e} \frac{Z_i}{vu^2} - e \frac{v_\parallel^2}{v_{te}} f_{em} \right\} d^3\boldsymbol{u} = 0. \tag{82}$$

6.1 second kind of basis function

$$b(u) = \log(\sqrt{1+u^2}), \tag{83}$$

$$b'(u) = \frac{1}{\sqrt{1+u^2}} \frac{u}{\sqrt{1+u^2}} = \frac{u}{1+u^2} \tag{84}$$

$$b''(u) = \frac{1+u^2 - u2u}{(1+u^2)^2} = \frac{1-u^2}{(1+u^2)^2} \tag{85}$$

Bibliography

[1] *Analytical Classical Dynamics*. Richard Fitzpatrick, 2004.