On Rosenbluth potential form of relativistic collision operator

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Outline

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Introduction and Motivation

- A weakly-relativistic Fokker-Planck operator for electron-electron collision was first used by Karney and Fisch to calculate lower-hybrid and electron cyclotron current drive efficiencies [C.F.F. Karney and N.J. Fisch. *Physics of Fluids*, 28, 116(1985)].
- The present work extends Karney and Fisch's work by expressing the weakly-relativistic collision operator in terms of a pair of Rosenbluth potential functions, and working out a general Legendre expansion of these two potentials.
- This general Legendre expansion reproduces the results in Karney and Fisch's paper and is useful in implementing the weakly-relativistic operator in Fokker-Planck codes.

Collision operator

- Collision term of species a off species b is expressed as an operator, $C(f_a, f_b)$, where f_a and f_b are respectively the distribution functions of test and background particles.

$$\frac{\partial f_a}{\partial t} + \boldsymbol{v} \cdot \nabla f_a + \frac{q_a}{m_a}\left(\boldsymbol{E} + \frac{\boldsymbol{v}}{c} \times \boldsymbol{B}\right) \cdot \nabla_u f_a = \sum_b C(f_a, f_b)$$

- Because collisions in plasmas are mainly due to small-angle scattering, the collision term can be written as the divergence of a flux in velocity/momentum space

$$C(f_a, f_b) = -\nabla_u \cdot \boldsymbol{S}^{a/b}$$

Landau collision integral

- Collision flux is given by Landau collision integral

$$\boldsymbol{S}^{a/b} \propto \int \boldsymbol{U} \cdot \left(\frac{f_b(\boldsymbol{u}')}{m_a} \frac{\partial f_a(\boldsymbol{u})}{\partial \boldsymbol{u}} - \frac{f_a(\boldsymbol{u})}{m_b} \frac{\partial f_b(\boldsymbol{u}')}{\partial \boldsymbol{u}'} \right) d^3\boldsymbol{u}'$$

- Simple Landau collision kernel

$$\boldsymbol{U} = \frac{\boldsymbol{I}}{|\boldsymbol{v} - \boldsymbol{v}'|} - \frac{(\boldsymbol{v} - \boldsymbol{v}')(\boldsymbol{v} - \boldsymbol{v}')}{|\boldsymbol{v} - \boldsymbol{v}'|^3}$$

- Landau kernel is actually weakly-relativistic approximation to a more complicated fully-relativistic kernel
- Weakly relativistic collision operator

Fokker-Planck form

The collision flux can also be written in Fokker-Planck form

$$\boldsymbol{S}^{a/b} = -\,\boldsymbol{D}^{a/b} \cdot \frac{\partial f_a(\boldsymbol{u})}{\partial \boldsymbol{u}} + \boldsymbol{F}^{a/b} f_a(\boldsymbol{u}),$$

with the diffusion tensor $\boldsymbol{D}^{a/b}$ and friction vector $\boldsymbol{F}^{a/b}$ given respectively by,

$$\boldsymbol{D}^{a/b} \propto \int \boldsymbol{U} f_b(\boldsymbol{u}') d^3\boldsymbol{u}'$$

$$\boldsymbol{F}^{a/b} \propto \int \left(\frac{\partial}{\partial \boldsymbol{u}'} \cdot \boldsymbol{U} \right) f_b(\boldsymbol{u}') d^3\boldsymbol{u}'$$

Non-relativistic case

- For non-relativistic case, gradient in momentum space reduces to one in velocity space

$$\boldsymbol{S}^{a/b} \propto \int \boldsymbol{U} \cdot \left(\frac{f_b(\boldsymbol{v}')}{m_a} \frac{\partial f_a(\boldsymbol{v})}{\partial \boldsymbol{v}} - \frac{f_a(\boldsymbol{u})}{m_b} \frac{\partial f_b(\boldsymbol{v}')}{\partial \boldsymbol{v}'} \right) d^3\boldsymbol{v}'$$

$$C(f_a, f_b) = \nabla_v \cdot \boldsymbol{S}^{a/b}$$

- Collision kernel remain unchanged

$$\boldsymbol{U} = \frac{\boldsymbol{I}}{|\boldsymbol{v} - \boldsymbol{v}'|} - \frac{(\boldsymbol{v} - \boldsymbol{v}')(\boldsymbol{v} - \boldsymbol{v}')}{|\boldsymbol{v} - \boldsymbol{v}'|^3}$$

Rosenbluth potential

- Non-relativistic Rosenbluth potential

$$g_b(\boldsymbol{v}) = -\frac{1}{4\pi}\int \frac{f_b(\boldsymbol{v}')}{|\boldsymbol{v}-\boldsymbol{v}'|} d^3\boldsymbol{v}'$$

$$h_b(\boldsymbol{v}) = -\frac{1}{8\pi}\int |\boldsymbol{v}-\boldsymbol{v}'| f_b(\boldsymbol{v}') d^3\boldsymbol{v}'$$

- Diffusion and friction coefficients can be expressed in terms of these two potential functions

$$\begin{aligned} \boldsymbol{D}_c^{a/b} &\propto \nabla_v \nabla_v h_b(\boldsymbol{v}), \\ \boldsymbol{F}_c^{a/b} &\propto \nabla_v g_b(\boldsymbol{v}), \end{aligned}$$

Legendre harmonic expansion

- Assuming axially symmetry about magnetic field for the background distribution, $f_b = f_b(v, \theta)$, where θ is the included angle between velocity and magnetic field (called pitch angle).
- Expand the angular part of $f_b(v,\ \theta)$ and potential function as Legendre harmonics

$$f_b(v,\theta) = \sum_{l=0}^{\infty} f_b^l(v) P_l(\cos\theta),$$

$$g_b(v,\theta) = \sum_{l=0}^{\infty} g_b^l(v) P_l(\cos\theta)$$

$$h_b(v,\theta) = \sum_{l=0}^{\infty} h_b^l(v) P_l(\cos\theta)$$

- The expansion coefficients of background distribution and the potential function have the following relation

$$
\begin{aligned}
g_b^l(v) &= -\frac{1}{2l+1}\Bigg[\int_0^v \frac{(v')^{l+2}}{v^{l+1}} f_b^l(v')dv' \\
&+ \int_v^\infty \frac{v^l}{(v')^{l-1}} f_b^l(v')dv'\Bigg]
\end{aligned}
$$

$$
\begin{aligned}
h_b^l(v) &= \frac{1}{2(4l^2-1)} \\
&\times \Bigg[\int_0^v \frac{(v')^{l+2}}{v^{l-1}}\left(1-\frac{2l-1}{2l+3}\frac{v'^2}{v^2}\right) f_b^l(v')dv' \\
&+ \int_v^\infty \frac{v^l}{(v')^{l-3}}\left(1-\frac{2l-1}{2l+3}\frac{v^2}{v'^2}\right) f_b^l(v')dv'\Bigg].
\end{aligned}
$$

Weakly-relativistic Rosenbluth potential

- Weakly-relativistic Rosenbluth potential

$$h_b(\boldsymbol{v}) = -\frac{1}{8\pi}\int |\boldsymbol{v}-\boldsymbol{v}'|\gamma'^5 f_b(\boldsymbol{u}')d^3\boldsymbol{v}'.$$

$$\begin{aligned} g_b(\boldsymbol{v}) &= -\frac{1}{4\pi}\int \left[\frac{1}{|\boldsymbol{v}-\boldsymbol{v}'|}\left(\frac{1}{\gamma'}+\frac{1}{\gamma'^3}\right)\right. \\ &+ \left.\frac{\left(\boldsymbol{v}\cdot\boldsymbol{v}'-v'^2\right)^2}{|\boldsymbol{v}-\boldsymbol{v}'|^3\gamma'}\right]\gamma'^5 f_b(\boldsymbol{u}')d^3\boldsymbol{v}' \end{aligned}$$

- Diffusion and friction coefficients can be expressed in terms of these two potential functions

$$\boldsymbol{D}_c^{a/b} \propto \frac{\partial^2 h_b(\boldsymbol{v})}{\partial\boldsymbol{v}\partial\boldsymbol{v}}, \quad \boldsymbol{F}_c^{a/b} \propto \frac{\partial g_b(\boldsymbol{v})}{\partial\boldsymbol{v}}$$

Legendre expansion I

$$
\begin{aligned}
h_b^l(v) &= \frac{1}{2(4l^2-1)} \times \\
&\left[\int_0^v \frac{(v')^{l+2}}{v^{l-1}} \left(1 - \frac{2l-1}{2l+3} \times \frac{(v')^2}{v^2} \right) \gamma'^5 f_b^l(u') dv' \right. \\
&+ \left. \int_v^c \frac{v^l}{(v')^{l-3}} \left(1 - \frac{2l-1}{2l+3} \times \frac{v^2}{(v')^2} \right) \gamma'^5 f_b^l(u') dv' \right]
\end{aligned}
$$

Legendre expansion II

$$
\begin{aligned}
g_b^l(v) &= -\frac{1}{2l+1} \\
&\quad \Bigg\{ \int_0^v \left[\frac{1}{c^2} \frac{{v'}^{l+2}}{v^{l-1}} {\gamma'}^4 \frac{l(l-1)}{2l-1} - \frac{1}{c^2} \frac{{v'}^{l+4}}{v^{l+1}} {\gamma'}^4 \left(\frac{l^2+l-1}{2l+3} \right) \right. \\
&+ \left. \frac{(v')^{l+2}}{v^{l+1}} \left({\gamma'}^2 + 1 \right) {\gamma'}^2 \right] f_b^l(u') dv' \\
&+ \int_v^c \left[\frac{{\gamma'}^4}{c^2} \frac{v^l}{{v'}^{l-3}} \left(\frac{l^2+l-1}{2l-1} \right) - \frac{{\gamma'}^4}{c^2} \frac{v^{l+2}}{{v'}^{l-1}} \frac{(l+1)(l+2)}{2l+3} \right. \\
&+ \left. \frac{v^l}{(v')^{l-1}} \left({\gamma'}^2 + 1 \right) {\gamma'}^2 \right] f_b^l(u') dv' \Bigg\}.
\end{aligned}
$$

- Using these formulas, we reproduced the results in the literature for low order Legendre harmonics ($l = 0, 1$).

Diffusion and friction coefficient ($l = 0$)

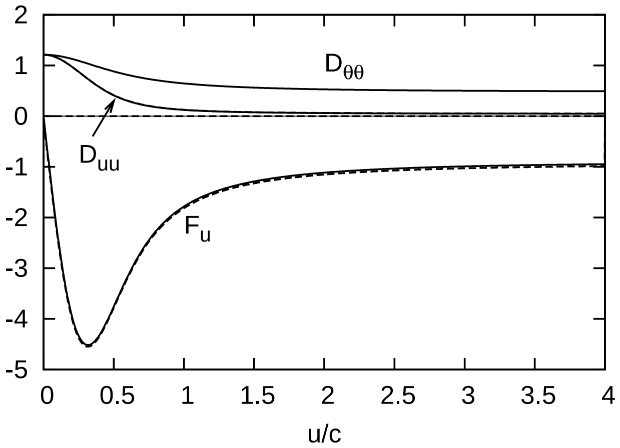

Figure 1. Diffusion and friction coefficients for collision off relativistic Maxwellian distribution as a function of momentum per unit rest mass. Solid lines are for weakly-relativistic collision model while dashed lines are for fully-relativistic collision model. These are for electron-electron collision. Electron temperature is $25\mathrm{keV}$.

Summary

- We have derived a general Legendre expansion for the Fokker-Planck coefficients of the weakly-relativistic collision operator.
- This general Legendre expansion can be used to implement the weakly-relativistic collision term efficiently in Fokker-Planck codes.

That's all. Thank you!