This note is a review of Karney's paper[1].

1 Basic equation

In the banana regime, the leading order distribution function is a function of only constants of the motion, and satisfies the following equation

$$\frac{\partial f}{\partial t} = \langle C(f) \rangle_b - \langle \nabla_u \cdot \boldsymbol{S} \rangle_b, \tag{1}$$

where $f = f(u, \mu, \sigma, t)$ in the passing region; in the trapped region, $f = f(u, \mu, t)$, is independent of σ since σ is not a constant of motion for trapped particles. That is

$$f = \begin{cases} f(u, \mu, \sigma, t), & \text{in passing region} \\ f(u, \mu, t), & \text{in trapped region} \end{cases} \tag{2}$$

$C(f)$ is the local collision term including electron-electron and electron-ion collisions:

$$C(f) = C^{e/e}(f, f_m) + C^{e/e}(f_m, f) + C^{e/i}(f, f_i). \tag{3}$$

Here $\langle ... \rangle_b$ is the bounce average (orbit integration) operator

$$\langle ... \rangle_b \equiv \frac{1}{\tau_b} \int (...) \frac{dl}{v_\parallel}, \tag{4}$$

where

$$\tau_b = \int \frac{dl}{v_\parallel}. \tag{5}$$

The integration in Eqs. (4) and (5) is evaluated along the unperturbed particle's orbit in phase space. In the approximation adopted here, (1) the energy and magnetic momentum are assumed to be the constants of motion and (2) the spatial path of the orbit is assumed to be along the magnetic field line. Therefore, the integration is evaluated by keeping energy and magnetic momentum constant, and the dl integration in Eqs. (4) and (5) is along the magnetic field line. In a more accurate treatment of the orbit average, the second assumption in the above is discarded, a finite-width guiding center orbit is used to perform the integration in Eqs. (4) and (5). I will discuss this issue later.

2 Transform to middle-plane coordinates

The distribution function in terms of energy, magnetic moment and σ

$$f = f(u, \mu, \sigma), \tag{6}$$

is independent of poloidal location, thus is independent of l, where l is the lenghth of magnetic field line. Then in terms of u and pitch angle coordinates θ, the distribution function

$$g(u, \theta, l) = f\left(u, \frac{m u^2 \sin^2\theta}{2B(l)}, \mathrm{sgn}(\cos\theta)\right) \tag{7}$$

will depend on l. Define distribution function at the middle-plane (i.e., $l = 0$) as $G(u_0, \theta_0)$, i.e.,

$$G(u_0, \theta_0) = g(u_0, \theta_0, l = 0), \tag{8}$$

then it is obvious that

$$g(u, \theta, l) = G(u_0, \theta_0) \tag{9}$$

provided (u_0, θ_0) and (u, θ, l) are related by

$$\begin{cases} u_0 = u \\ \dfrac{\sin\theta_0}{B(0)} = \dfrac{\sin\theta}{B(l)} \\ \mathrm{sgn}(\cos\theta_0) = \mathrm{sgn}(\cos\theta) \end{cases} \tag{10}$$

Note that for passing particles, the transforming rule is exactly identical with the constants of energy, magnetic moment and σ. For trapped particles, since σ is not the constant of motion, so the last transforming rule in Eq. (10) is not consistent with trapped particles' motion. However, since f in the trapped region is independent of σ, the last transforming rule in Eq. (10) does not come to play a role in the transforming, thus can be ignored for trapped particles.

For later use, we define

$$b(l) = \frac{B(l)}{B(0)}. \tag{11}$$

3 Parallel current

The current density at the mid-plane (i.e., $l = 0$) is

$$J_{0\parallel} = q \int G(u_0, \theta_0) v_0 \cos\theta_0 d^3\boldsymbol{u}_0. \tag{12}$$

The current density at arbitrary poloidal location is

$$\begin{aligned} J_{\parallel} &= q \int g(u, \theta, l) v \cos\theta d^3\boldsymbol{u} \\ &= q 2\pi \int g(u, \theta, l) v \cos\theta \sin\theta u^2 du d\theta \end{aligned} \tag{13}$$

Now we transform Eq. (13) from (u, θ) to (u_0, θ_0) coordinates. Using

$$\sin^2\theta_0 = \sin^2\theta \frac{B(0)}{B(\theta_p)} \tag{14}$$

one gets

$$\sin\theta_0 \cos\theta_0 d\theta_0 = \sin\theta \cos\theta d\theta / b(l), \tag{15}$$

Using Eq. (15) and the relation between $G(u_0, \theta_0)$ and $g(u, \theta, l)$, Eq. (13) is written as

$$\begin{aligned} J_{\parallel} &= q b(l) 2\pi \int G(u_0, \theta_0) v_0 \cos\theta_0 \sin\theta_0 u_0^2 d\theta_0 du_0 \\ &= q b(l) \int d^3 u_0 G(u_0, \theta_0) v_0 \cos\theta_0 \\ &= b(l) J_{0\parallel} \end{aligned} \tag{16}$$

This relates current at arbitrary poloidal location with the current at the middle-plane. This relation can also be obtained by noticing that

$$\nabla \cdot \left(j_{\parallel} \frac{\boldsymbol{B}}{B} \right) = 0,$$

which leads to

$$\boldsymbol{B} \cdot \nabla \left(\frac{j_{\parallel}}{B} \right) = 0,$$

which further indicates that $j_{\parallel}/B$ is a function of flux surface, thus

$$\frac{j_{\parallel}}{B} = \frac{j_{\parallel 0}}{B(0)} \Longrightarrow j_{\parallel} = b(l) j_{0\parallel}. \tag{17}$$

4 Adjoint equation

In the toroidal geometry and in the banana regime, the adjoint equation is given by

$$\langle C(\chi f_{em})\rangle_b = e\frac{L}{\tau_b}\Theta f_{em}, \tag{18}$$

(Eq. (18) agrees with Eq. (6) in Karney's paper[1].) where L is the length of the magnetic field line when it travels one circle around the poloidal direction, $\Theta = 1$ for passing particles and $\Theta = 0$ for trapped particles. It is instructive to compare this equation with its counterpart in homogeneous magnetic field, which is given by

$$C(\chi f_{em}) = e v_{\parallel} f_{em} \tag{19}$$

(Eq. (19) agrees with Eq. (95) in Karney's 1986 report[2].) The diffences are that (1) the bounce-average operator appears in case of toroidal geometry, and (2) the averaged parallel velocity, L/τ_b, replaces the $v_{\parallel}$ in the homogeneous case, (3) of course, there is trapped-particles effect in toroidal geometry, reflected by $\Theta = 0$ for trapped particles, in Eq. (18).

Note that in Karney's paper[1], $\hat{C}(...)$ is defined as

$$\hat{C}(\chi) = \frac{1}{f_m}[C^{e/e}(f_m\chi, f_m) + C^{e/e}(f_m, f_m\chi) + C^{e/i}(f_m\chi, f_i)], \tag{20}$$

which is related with $C^l(...)$ by

$$\hat{C}(\chi) = \frac{1}{f_m} C(\chi f_m) \tag{21}$$

5 Bounce-averaged collision operator

5.1 To bounce-average $C^{a/b}(f_m\chi, f_m)$ term

We have

$$C^{a/b}(f_a, f_m) = \frac{1}{u^2}\frac{\partial}{\partial u}u^2\left(D^{a/b}_{cuu}\frac{\partial f_a}{\partial u} - F^{a/b}_{cu} f_a\right) + \frac{1}{u\sin\theta}\frac{\partial}{\partial\theta}\left[\sin\theta\left(D^{a/b}_{c\theta\theta}\frac{1}{u}\frac{\partial f_a}{\partial\theta}\right)\right] \tag{22}$$

Using this, the first term of Eq. (20) is written as

$$\begin{aligned}
C^{a/b}(f_m\chi, f_m) &= \frac{1}{u^2}\frac{\partial}{\partial u}u^2\left(D^{a/b}_{cuu}\frac{\partial f_m\chi}{\partial u} - F^{a/b}_{cu} f_m\chi\right) + \frac{D_{\theta\theta}}{u^2\sin\theta}\frac{\partial}{\partial\theta}\left[\sin\theta\left(\frac{\partial f_m\chi}{\partial\theta}\right)\right] \\
&= \frac{1}{u^2}\frac{\partial}{\partial u}u^2\left(D^{a/b}_{cuu}\left(\frac{\partial\chi}{\partial u}f_m + \chi\left(-\frac{mv}{T}f_m\right)\right) - F^{a/b}_{cu} f_m\chi\right) + \\
&\quad \frac{D_{\theta\theta}}{u^2\sin\theta}\frac{\partial}{\partial\theta}\left[\sin\theta\left(\frac{\partial f_m\chi}{\partial\theta}\right)\right] \\
&= \frac{1}{u^2}\frac{\partial}{\partial u}u^2\left(D^{a/b}_{cuu}\frac{\partial\chi}{\partial u}f_m + D^{a/b}_{cuu}\left(-\frac{mv}{T}\right)f_m\chi - F^{a/b}_{cu} f_m\chi\right) + \\
&\quad f_m\frac{D_{\theta\theta}}{u^2\sin\theta}\frac{\partial}{\partial\theta}\left[\sin\theta\left(\frac{\partial\chi}{\partial\theta}\right)\right]
\end{aligned} \tag{23}$$

In obtaining the second equality, use was made of

$$\frac{\partial f_m(u)}{\partial u} = f_m(u)\left(-\frac{mv}{T}\right). \tag{24}$$

For Maxwellian background distribution, one has

$$F^{a/b}_{cu} = -\frac{m_b v}{T_b}D^{a/b}_{cuu}.\ \text{check this for relativistic case!} \tag{25}$$

Using this in Eq. (23) gives

$$
\begin{aligned}
C^{a/b}(f_m\chi, f_m) &= \frac{1}{u^2}\frac{\partial}{\partial u}u^2\bigg(D^{a/b}_{cuu}\frac{\partial\chi}{\partial u}f_m\bigg) + f_m\frac{D_{\theta\theta}}{u^2\mathrm{sin}\theta}\frac{\partial}{\partial\theta}\bigg[\mathrm{sin}\theta\bigg(\frac{\partial\chi}{\partial\theta}\bigg)\bigg] \qquad (26)\\
&= f_m\frac{1}{u^2}\frac{\partial}{\partial u}\bigg(u^2D^{a/b}_{cuu}\frac{\partial\chi}{\partial u}\bigg) + u^2D^{a/b}_{cuu}\frac{\partial\chi}{\partial u}\frac{1}{u^2}\frac{\partial}{\partial u}(f_m) + f_m\frac{D_{\theta\theta}}{u^2\mathrm{sin}\theta}\frac{\partial}{\partial\theta}\bigg[\mathrm{sin}\theta\bigg(\frac{\partial\chi}{\partial\theta}\bigg)\bigg]\\
&= f_m\frac{1}{u^2}\frac{\partial}{\partial u}\bigg(u^2D^{a/b}_{cuu}\frac{\partial\chi}{\partial u}\bigg) + D^{a/b}_{cuu}\frac{\partial\chi}{\partial u}\bigg(-\frac{mv}{T}\bigg)f_m + f_m\frac{D_{\theta\theta}}{u^2\mathrm{sin}\theta}\frac{\partial}{\partial\theta}\bigg[\mathrm{sin}\theta\bigg(\frac{\partial\chi}{\partial\theta}\bigg)\bigg]\\
&= f_m\frac{1}{u^2}\frac{\partial}{\partial u}\bigg(u^2D^{a/b}_{cuu}\frac{\partial\chi}{\partial u}\bigg) + F^{a/b}_u\frac{\partial\chi}{\partial u}f_m + f_m\frac{D_{\theta\theta}}{u^2}\frac{1}{\mathrm{sin}\theta}\frac{\partial}{\partial\theta}\bigg[\mathrm{sin}\theta\bigg(\frac{\partial\chi}{\partial\theta}\bigg)\bigg] \qquad (27)
\end{aligned}
$$

Dividing Eq. (27) by f_m gives

$$
\frac{1}{f_m}C^{e/e}(f_m\chi, f_m) = \frac{1}{u^2}\frac{\partial}{\partial u}\bigg(u^2D^{a/b}_{cuu}\frac{\partial\chi}{\partial u}\bigg) + F^{a/b}_u\frac{\partial\chi}{\partial u} + \frac{D_{\theta\theta}}{u^2}\frac{1}{\mathrm{sin}\theta}\frac{\partial}{\partial\theta}\bigg[\mathrm{sin}\theta\bigg(\frac{\partial\chi}{\partial\theta}\bigg)\bigg] \qquad (28)
$$

Before bounce-averaging the above collision operator, we want to transform it to the "midplane coordinates". The collision operator in terms of midplane coordinates makes it convenient to perform the orbit integration, since these coordinates are kept constant when evaluating the orbit integration (since we assume the orbit conserves energy and magnetic momentum).

Define the form of distribution $\chi(u,\theta,l)$ in (u,μ,σ) coordinators as $f(u,\mu,\sigma)$, then $\chi(u,\theta,l)$ is related to $f(u,\mu,\sigma)$ by

$$
\chi(u,\theta,l) = f\bigg(u, \frac{mu^2\,\mathrm{sin}^2\theta}{2B(l)}, \mathrm{sgn}(\mathrm{cos}\theta)\bigg) \qquad (29)
$$

Define

$$
\chi_0(u_0,\theta_0) = \chi(u_0,\theta_0,l=0) \qquad (30)
$$

Following the procedure in the first section, it can be proved that

$$
\chi(u,\theta,l) = \chi_0(u_0,\theta_0) \qquad (31)
$$

provided that

$$
u_0 = u, \qquad (32)
$$

and

$$
\frac{\mathrm{sin}^2\theta_0}{B_0} = \frac{\mathrm{sin}^2\theta}{B(l)} \qquad (33)
$$

Here (u_0, θ_0) will be called "midplane coordinates" in the following. It is obvious that u_0 and θ_0 should be kept constant when evaluating the orbit integration since they can be considered as the initial conditions of the orbit.

It is simple to transform (using Eq. (32) and (134) as transforming rule) the energy diffusion and slowing-down part (the first and second terms in Eq. (28)) to midplane coordinates since these terms do not involve pitch angle variable: simply replace the u by u_0, then the job are done. The last term in Eq. (28) is the famous pitch angle scattering operator,

$$
\frac{D_{\theta\theta}}{u^2}\frac{1}{\mathrm{sin}\theta}\frac{\partial}{\partial\theta}\bigg(\mathrm{sin}\theta\frac{\partial}{\partial\theta}\bigg). \qquad (34)
$$

We now consider transforming this operator to midplane coordinates. Define $\xi = \mathrm{cos}\theta$, then the above operator (omitting the $D_{\theta\theta}/u^2$ factor) is written as

$$
\frac{\partial}{\partial\xi}\bigg[(1-\xi^2)\frac{\partial}{\partial\xi}\bigg] \qquad (35)
$$

Now we want to transform from (u,ξ) to (u_0,ξ_0) coordinate, where $\xi_0 \equiv \mathrm{cos}\theta_0$. Eq. (33) gives

$$
\frac{1-\xi_0^2}{B_0} = \frac{1-\xi^2}{B} \Longrightarrow \frac{\partial\xi_0}{\partial\xi} = \frac{B_0}{B}\frac{\xi}{\xi_0} \qquad (36)
$$

$$\Longrightarrow \frac{\partial}{\partial \xi} = \frac{\partial}{\partial \xi_0}\frac{\partial \xi_0}{\partial \xi} = \frac{B_0}{B}\frac{\xi}{\xi_0}\frac{\partial}{\partial \xi_0} \tag{37}$$

The bounce average (orbit integration) operator is

$$\langle \ldots \rangle_b \equiv \frac{1}{\tau_b}\int (\ldots)\frac{dl}{v_\parallel}, \tag{38}$$

where

$$\tau_b = \int \frac{dl}{v_\parallel}. \tag{39}$$

The integration in Eqs. (38) and (39) is evaluated along the unperturbed particle's orbit in phase space. In the approximation adopted here, the energy and magnetic momentum are constants of the particle's motion. Therefore, the integration is evaluated by keeping energy and magnetic momentum constant. In (v_0, ξ_0) coordinates, this is to keep v_0 and ξ_0 constant. The orbit in spatial space is assumed to be along the magnetic field. For passing particles, the spatial path is a full poloidal circle when projected to poloidal plane; for trapped particles, this is a forward and then backward path. Using the above results, the bounce averaged pitch angle operator is written as

$$\begin{aligned}
\left\langle \frac{\partial}{\partial \xi}(1-\xi^2)\frac{\partial}{\partial \xi}\right\rangle_b &= \frac{1}{\tau_b}\int \frac{dl}{v_\parallel}\left[\frac{\partial}{\partial \xi}(1-\xi^2)\frac{\partial}{\partial \xi}\right] \\
&= \frac{1}{\tau_b}\int \frac{dl}{v_\parallel}\left[\frac{\partial}{\partial \xi}(1-\xi^2)\frac{B_0}{B}\frac{\xi}{\xi_0}\frac{\partial}{\partial \xi_0}\right] \\
&= \frac{1}{\tau_b}\int \frac{dl}{v_\parallel}\left[\frac{\partial}{\partial \xi}(1-\xi_0^2)\frac{\xi}{\xi_0}\frac{\partial}{\partial \xi_0}\right] \qquad (40) \\
&= \frac{1}{\tau_b}\int \frac{dl}{v_\parallel}\left[\frac{B_0}{B}\frac{\xi}{\xi_0}\frac{\partial}{\partial \xi_0}(1-\xi_0^2)\frac{\xi}{\xi_0}\frac{\partial}{\partial \xi_0}\right] \\
&= \frac{1}{\tau_b}\int \frac{dl}{v}\left[\frac{B_0}{B}\frac{1}{\xi_0}\frac{\partial}{\partial \xi_0}(1-\xi_0^2)\frac{\xi}{\xi_0}\frac{\partial}{\partial \xi_0}\right] \\
&= \frac{1}{\tau_b v_0}\int \frac{dl}{B}\left[B_0\frac{1}{\xi_0}\frac{\partial}{\partial \xi_0}(1-\xi_0^2)\frac{\xi}{\xi_0}\frac{\partial}{\partial \xi_0}\right]. \qquad (41)
\end{aligned}$$

Note that ξ_0 is the starting value (initial condition) of the orbit, thus the orbit integration can be exchanged with the drivative with respect to ξ_0, yielding

$$\left\langle \frac{\partial}{\partial \xi}(1-\xi^2)\frac{\partial}{\partial \xi}\right\rangle_b = B_0\frac{1}{\tau_b v_0}\frac{1}{\xi_0}\frac{\partial}{\partial \xi_0}\left[(1-\xi_0^2)\frac{1}{\xi_0}\left(\int \frac{dl}{B}\xi\right)\left(\frac{\partial}{\partial \xi_0}\right)\right] \tag{42}$$

Now I want to write Eq. (42) in the form given by Karney[1]. Eq. (42) is further written as

$$\begin{aligned}
\left\langle \frac{\partial}{\partial \xi}(1-\xi^2)\frac{\partial}{\partial \xi}\right\rangle_b &= \frac{1}{\tau_b v_0}\frac{1}{\xi_0}\frac{\partial}{\partial \xi_0}\left[(1-\xi_0^2)\frac{1}{\xi_0}\left(\int B_0\frac{dl}{B}\xi\right)\left(\frac{\partial}{\partial \xi_0}\right)\right] \\
&= \frac{1}{\tau_b v_0}\frac{1}{\xi_0}\frac{\partial}{\partial \xi_0}\left[(1-\xi_0^2)\frac{v_0\cos\theta_0}{v_0\cos^2\theta_0}\left(\int B_0\frac{dl}{B}\xi\right)\left(\frac{\partial}{\partial \xi_0}\right)\right] \\
&= \frac{1}{\tau_b v_0}\frac{1}{\xi_0}\frac{\partial}{\partial \xi_0}\left[(1-\xi_0^2)\,v_0\cos\theta_0\left(\int \frac{B_0}{B}\frac{dl}{v_\parallel}\frac{\cos^2\theta}{\cos^2\theta_0}\right)\left(\frac{\partial}{\partial \xi_0}\right)\right] \\
&= \frac{1}{\tau_b v_0}\frac{1}{\xi_0}\frac{\partial}{\partial \xi_0}\left[(1-\xi_0^2)\,v_0\cos\theta_0\left(\int \frac{B_0}{B}\frac{dl}{v_\parallel}\frac{\cos^2\theta\sin^2\theta}{\cos^2\theta_0\sin^2\theta}\right)\left(\frac{\partial}{\partial \xi_0}\right)\right] \\
&= \frac{1}{\tau_b v_0}\frac{1}{\xi_0}\frac{\partial}{\partial \xi_0}\left[(1-\xi_0^2)\,v_0\cos\theta_0\left(\int \frac{B_0}{B_0}\frac{dl}{v_\parallel}\frac{\cos^2\theta\sin^2\theta_0}{\cos^2\theta_0\sin^2\theta}\right)\left(\frac{\partial}{\partial \xi_0}\right)\right] \\
&= \frac{1}{\tau_b v_0}\frac{1}{\xi_0}\frac{\partial}{\partial \xi_0}\left[(1-\xi_0^2)v_0\cos\theta_0\left(\int \frac{dl}{v_\parallel}\frac{\tan^2\theta_0}{\tan^2\theta}\right)\left(\frac{\partial}{\partial \xi_0}\right)\right] \\
&= \frac{1}{\tau_b v_0}\frac{1}{\xi_0}\frac{\partial}{\partial \xi_0}\left[(1-\xi_0^2)\,v_0\cos\theta_0\tau_b\left\langle\frac{\tan^2\theta_0}{\tan^2\theta}\right\rangle_b\left(\frac{\partial}{\partial \xi_0}\right)\right] \qquad (43)
\end{aligned}$$

Define $\lambda = v_0 \cos\theta_0 \tau_b / L$, where L is a length quantity (to be specified later), then Eq. (43) is written as

$$\left\langle \frac{\partial}{\partial\xi}(1-\xi^2)\frac{\partial}{\partial\xi} \right\rangle = \frac{1}{\lambda}\frac{1}{\sin\theta_0}\frac{\partial}{\partial\theta_0}\left[\sin\theta_0\,\lambda\left\langle \frac{\tan^2\theta_0}{\tan^2\theta} \right\rangle_b \left(\frac{\partial}{\partial\theta_0}\right)\right] \tag{44}$$

Eq. (44) agrees with the expression following Eq. (8) in Karney's paper[1]. It is instructive to compare the bounce averaged pitch angle scattering operator with the local one. The difference are that some new factors appear in the bounce averaged operator. Specifically, two factors, namely,

$$\lambda = v_0 \cos\theta_0 \tau_b / L, \tag{45}$$

and

$$\beta \equiv \left\langle \frac{\tan^2\theta_0}{\tan^2\theta} \right\rangle_b \tag{46}$$

appear in the bounce averaged operator. Consider the second factor, which involves a orbit integration of particle's pitch angle θ. In the (u, μ) phase space or equivalently (u_0, ξ_0) phase space, particle is static, remaining on the same point of the phase space, since energy and magnetic moment is (or assumed to be) constant. We now transform the phase space coordinators from (u_0, ξ_0) to (u, θ, l). Note that the spatial coordinates l appear in the new coordinator system, since the transforming rule, Eqs. (32) and (134), involves l. Examining the same particle's motion in (v, θ, l) phase space (the motion in (v, θ, l) phase space is obtained by examining the motion equation, usually approximate equation and solution are used as the orbit), particle's θ and l will usually change with time. Since the collision term $C(g(v, \theta, l))$ depends on θ, the collision term felt by the particle will change with time. The averaged time change rate of $f_0(v, \mu)$ in one period of time of particle's motion is equal to the collision term averaged in this period, $\tau_b^{-1} \int C(g(v, \theta(\tau), l(\tau))d\tau$. The factor in Eq. (46) is a result resulting from this time averaging process.

Note that the spatial variable l enter the formula through the dependence of pitch angle $\theta(l(\tau))$ on l. There is no explicit dependence of collision operator on l.

Also note that all the "particle motion" mentioned in the above is the motion of the particles' guiding-center.

5.2 To bounce-average $C^{e/e}(f_m, f_m\chi)$ term

Now consider the second term of Eq. (20)

$$\tilde{C} \equiv \frac{1}{f_m} C^{e/e}(f_m, f_m\chi) \tag{47}$$

Expand angular part of $\chi(u, \theta, l)$ in terms of Legendre harmonics

$$\chi(u, \theta, l) = \sum_{k\,\mathrm{odd}} \tilde{\chi}_k(u, l) P_k(\cos\theta), \tag{48}$$

(the sum should contain only odd order Legendre harmonics since χ is an odd function of $u_\parallel$) then the expansion coefficients are given by

$$\tilde{\chi}_k(u, l) \;=\; \frac{2k+1}{2}\int_0^\pi \chi(u, \theta, l) P_k(\cos\theta)\sin\theta d\theta \tag{49}$$

Note that $\chi(u, \theta, l)$ is an odd function about $\cos\theta$, and $P_k(-\cos\theta) = (-1)^k P_k(\cos\theta)$, thus, for odd k, $\chi(v, \theta, l) P_k(\cos\theta)$ is an even function about $\cos\theta$. Therefore Eq. (49) is reduced to

$$\tilde{\chi}_k(u, l) = (2k+1)\int_0^{\pi/2} \chi(u, \theta, l) P_k(\cos\theta)\sin\theta d\theta \tag{50}$$

Now we want to transform the above integration to (u_0, θ_0) coordinates.

$$\begin{aligned}\tilde{\chi}_k(u,l) &= (2k+1)\int_0^{\pi/2} \chi_0(u_0,\theta_0)P_k(\cos\theta)\sin\theta d\theta \\ &= (2k+1)\int_0^{\pi/2} \chi_0(u_0,\theta_0)P_k(\cos\theta)\frac{\cos\theta_0}{\cos\theta}\frac{\cos\theta}{\cos\theta_0}\sin\theta d\theta \\ &= (2k+1)\int_0^{\pi/2} \chi_0(u_0,\theta_0)\tilde{P}_k\frac{\cos\theta}{\cos\theta_0}\sin\theta d\theta \end{aligned} \tag{51}$$

where

$$\tilde{P}_k \equiv P_k(\cos\theta)\frac{\cos\theta_0}{\cos\theta}. \tag{52}$$

Using

$$\frac{\sin^2\theta_0}{B_0} = \frac{\sin^2\theta}{B(l)} \tag{53}$$

one gets

$$\frac{\cos\theta}{\cos\theta_0}\sin\theta d\theta = b(l)\sin\theta_0 d\theta_0 \tag{54}$$

Using this in Eq. (51) gives

$$\tilde{\chi}_k(u,l) = (2k+1)b(l)\int_0^{\theta_l} \chi_0(u_0,\theta_0)\tilde{P}_k\sin\theta_0 d\theta_0, \tag{55}$$

Note that the upper limit of the integration, $\theta_l = \arcsin\left(\sqrt{1/b(l)}\right)$, varies as l change. For any value of l, $\theta_l \geqslant \theta_{\rm tr} = \arcsin(\sqrt{1/b_{\max}})$. Note that $\chi_0(u_0, \theta_0)$ is zero in the trapped region (i.e., $\theta_0 > \theta_{\rm tr}$). Thus the upper limit of the integration, θ_l, can be chosen to be $\theta_{\rm tr}$ for any value of l,

$$\tilde{\chi}_k(u,l) = (2k+1)b(l)\int_0^{\theta_{\rm tr}} \chi_0(u_0,\theta_0)\tilde{P}_k\sin\theta_0 d\theta_0. \tag{56}$$

Using the fact that spherical harmonics are angular eigen-functions of the collision operator, one has

$$\tilde{C}_k \equiv \frac{1}{f_m(u)}C^{e/e}(f_m(u), f_m(u)\tilde{\chi}_k(u,l)P_k(\cos\theta)) = I_k(\tilde{\chi}_k(u,l))P_k(\cos\theta), \tag{57}$$

where $I_k(\tilde{\chi}_k(u,l))$ is a function independent of θ. Now we bounce-average $\tilde{C}_k$,

$$\begin{aligned}\left\langle \tilde{C}_k \right\rangle_b &= \frac{1}{\tau_b}\int \frac{dl}{v_\parallel}\tilde{C}_k \\ &= \frac{1}{\tau_b}\int \frac{dl}{v_\parallel}P_k(\cos\theta)I_k(\tilde{\chi}_k(u,l)) \\ &= \frac{1}{\tau_b}\int \frac{dl}{v_0\cos\theta}\frac{\cos\theta_0}{\cos\theta_0}P_k(\cos\theta)I_k(\tilde{\chi}_k(u,l)) \\ &= \frac{1}{\tau_b}\int \frac{dl}{v_0\cos\theta_0}\tilde{P}_k I_k(\tilde{\chi}_k(u,l)) \\ &= \frac{1}{\tau_b v_0\cos\theta_0}\int dl\tilde{P}_k I_k(\tilde{\chi}_k(u,l)) \\ &= \frac{1}{\lambda}\int \frac{dl}{L}\tilde{P}_k I_k(\tilde{\chi}_k(u,l)) \end{aligned} \tag{58}$$

where $\lambda = v_0\cos\theta_0\tau_b/L$. For $k=1$,

$$I_k(\tilde{\chi}_k(u,l)) = I_1[\tilde{\chi}_1(u,l)], \tag{59}$$

where, for weakly-relativistic case, the integration operator, $I_1(f)$, is given by Eq. (7) in Karney's paper[2]. For $k=1$, Eq. (56) gives

$$\tilde{\chi}_1(u,l) = 3b(l)\int_0^{\theta_{\rm tr}} \chi_0(u_0,\theta_0)\tilde{P}_1\sin\theta_0 d\theta_0, \tag{60}$$

where $\tilde{P}_1$ can be reduced to

$$\begin{aligned}\tilde{P}_1 &= P_1(\cos\theta)\frac{\cos\theta_0}{\cos\theta}\\ &= \cos\theta_0\end{aligned} \tag{61}$$

Then

$$\begin{aligned}\tilde{\chi}_1(u,l) &= 3b(l)\int_0^{\theta_{\rm tr}}\chi_0(u_0,\theta_0)\cos\theta_0\sin\theta_0 d\theta_0,\\ &= b(l)F_1(u_0),\end{aligned} \tag{62}$$

where

$$F_1(u_0)\equiv 3\int_0^{\theta_{\rm tr}}\chi_0(u_0,\theta_0)\cos\theta_0\sin\theta_0 d\theta_0, \tag{63}$$

is a function of only u_0. The integration in Eq. (63) is performed numerically in the code.

$$\begin{aligned}\left\langle\tilde{C}_1\right\rangle_b &= \frac{1}{\lambda}\int\frac{dl}{L}\cos\theta_0 I_1[b(l)F_1(u_0)]\\ &= \frac{\cos\theta_0}{\lambda}\left(\int\frac{dl}{L}b(l)\right)I_1[F_1(u_0)]\end{aligned} \tag{64}$$

The geometric factor in Eq. (64), for the case of circular flux surface equilibrium, can be reduced to

$$\int b(l)\frac{dl}{L}=\sqrt{\frac{1+\varepsilon}{1-\varepsilon}} \tag{65}$$

The term in Eq. (64)

$$K_1(u_0)=I_1[F_1(u_0)]$$

is identical with the corresponding term in uniform magnetic field. The code for the uniform magnetic field case is re-used here.

Note that in the above we keep only the first Legendre harmonic of $\chi(u,\theta,l)$, the result in Eq. (64) indicates the dependence of the bounce-averaged collision term, $\langle C(f_m, f_m\chi(u,\theta,l))\rangle$, on θ_0 is $\cos\theta_0/\lambda$, instead of $\cos\theta_0$. (Note that $\cos\theta_0/\lambda=1/\overline{\tau}_b$ is a complicated function of θ_0, refer to the next section for the expression of $\overline{\tau}_b$. Also note that when ε approaches zero, i.e,. for homogeneous magnetic field, $1/\overline{\tau}_b=\cos\theta_0$.)

If

$$\chi(u,\theta,l)=\sum_{k\,\rm odd}\tilde{\chi}_k(u,l)P_k(\cos\theta), \tag{66}$$

then

$$\begin{aligned}\chi_0(u_0,\theta_0) &= \chi(u,\theta,l)\\ &= \sum_{k\,\rm odd}\tilde{\chi}_k(u,l)P_k(\cos\theta)\\ &= \sum_{k\,\rm odd}\tilde{\chi}_k(u_0,l)P_k\left(\sqrt{1-b(1-\xi_0^2)}\right)\end{aligned}$$

$$\frac{1-\xi_0^2}{B_0}=\frac{1-\xi^2}{B} \tag{67}$$

$$1-\xi^2=b(1-\xi_0^2)$$

$$\xi^2=1-b(1-\xi_0^2)$$

$$\xi=\sqrt{1-b(1-\xi_0^2)}$$

$$\chi_0(u_0,\theta_0)=\sum_{j\,\rm odd}\tilde{\chi}_j(u_0)P_j(\cos\theta_0), \tag{68}$$

$$\begin{aligned}
\tilde{P}_3 &= P_3(\cos\theta)\frac{\cos\theta_0}{\cos\theta} \\
&= \frac{1}{2}\big(-3\cos\theta+5\cos^3\theta\big)\frac{\cos\theta_0}{\cos\theta} \\
&= \frac{1}{2}\big(-3+5\cos^2\theta\big)\cos\theta_0 \\
&= \frac{1}{2}\big(-3+5\big(1-b\sin^2\theta_0\big)\big)\cos\theta_0 \\
&= \frac{1}{2}\big(2-5b\sin^2\theta_0\big)\cos\theta_0 \\
&= \frac{1}{2}\big(2-5b(1-\cos^2\theta_0)\big)\cos\theta_0 \\
&= \frac{1}{2}(2\cos\theta_0-5b(\cos\theta_0-\cos^3\theta_0)) \\
&= \frac{1}{2}(2\cos\theta_0+b(-3\cos\theta_0+5\cos^3\theta_0-2\cos\theta_0)) \\
&= b\,P_3(\cos\theta_0)+\frac{1}{2}(2\cos\theta_0+b(-2\cos\theta_0)) \\
&= b\,P_3(\cos\theta_0)+(1-b)P_1(\cos\theta_0)
\end{aligned} \tag{69, 70}$$

$$\frac{\sin^2\theta_0}{B_0}=\frac{\sin^2\theta}{B(l)} \tag{71}$$

$$b\sin^2\theta_0=1-\cos^2\theta \tag{72}$$

6 Magnetic equilibrium

Simple circular flux surface magnetic equilibrium:

$$B_t(\varepsilon,\theta_p)=\frac{B_{t0}}{1+\epsilon\cos\theta_p} \tag{73}$$

$$B_p(\varepsilon,\theta_p)=\frac{B_{p0}}{1+\epsilon\cos\theta_p} \tag{74}$$

where $\epsilon=r/R_0$ which labels different flux surface, θ_p is the poloidal angle. Then

$$B(\varepsilon,\theta_p)=\sqrt{B_t^2+B_p^2}=\frac{B_{\rm axi}}{1+\epsilon\cos\theta_p}, \tag{75}$$

where $B_{\rm axi}=\sqrt{B_{t0}^2+B_{p0}^2}$. The local major radius is given by $R=R_0(1+\epsilon\cos\theta_p)$. It can be proved that $\nabla\cdot\boldsymbol{B}=0$ is satisfied.

The minimum magnetic field on a flux surface ε is given by

$$B_0\equiv B(\varepsilon,0)=\frac{B_{\rm axi}}{1+\epsilon} \tag{76}$$

From Eqs.(75)(76), one gets

$$b(l)\equiv\frac{B(l)}{B(\varepsilon,0)}=\frac{1+\epsilon}{1+\epsilon\cos\theta_p} \tag{77}$$

Now calculate the length of the field line when it travels a full circle around poloidal direction. Magnetic field lines satisfy the equation

$$dl=\frac{B}{B_p}dl_p=\frac{B_{\rm axi}}{B_{p0}}dl_p \tag{78}$$

Define

$$G=\frac{B_{\rm axi}}{B_{p0}} \tag{79}$$

then

$$
\begin{aligned}
L &= \oint dl \\
&= \oint G dl_p \\
&= G \oint dl_p \\
&= G 2\pi r
\end{aligned} \tag{80}
$$

This result does not agree with the result in Karney's paper[1]. [Karney's result:

$$
\begin{aligned}
L = 2\pi R_0 Q \sqrt{1 + B_{p0}^2 / B_{t0}^2} &= 2\pi R_0 \frac{B_{t0}}{B_{p0}} \frac{\epsilon}{\sqrt{1-\epsilon^2}} \frac{B_{\rm axi}}{B_{t0}} \\
&= 2\pi r \frac{B_{\rm axi}}{B_{p0}} \frac{1}{\sqrt{1-\epsilon^2}} = G 2\pi r \frac{1}{\sqrt{1-\epsilon^2}}
\end{aligned} \tag{81}
$$

Karney's result is incorrect.]

Now we consider the calculation of the following two bounce-averaged terms:

$$
\begin{aligned}
\lambda &= v_0 \cos\theta_0 \tau_b / L \\
&= \cos\theta_0 \overline{\tau}_b,
\end{aligned} \tag{82}
$$

and

$$
\beta = \left\langle \frac{\tan^2\theta_0}{\tan^2\theta} \right\rangle_b,
$$

where

$$
\begin{aligned}
\tau_b &= \int \frac{dl}{v_\parallel} \\
&= \gamma \int \frac{dl}{u_\parallel} \\
&= \gamma \int \frac{G dl_p}{u_\parallel} \\
&= G\gamma \int_0^{2\pi} \frac{r d\theta_p}{u_\parallel}
\end{aligned} \tag{83}
$$

Define a characteristic time

$$
\tau_0 \equiv \frac{L}{v_0},
$$

then use this characteristic time to normalize τ_b,

$$
\begin{aligned}
\overline{\tau_b} &= \tau_b / \tau_0 \\
&= G\gamma \int_0^{2\pi} \frac{r d\theta_p}{\frac{L}{v_0} u_\parallel} \\
&= G \int_0^{2\pi} \frac{r d\theta_p}{\frac{L}{u_0} u_\parallel} \\
&= G \frac{r}{L} \int_0^{2\pi} \frac{d\theta_p}{\cos\theta} \\
&= G \frac{r}{L} \int_0^{2\pi} \frac{d\theta_p}{\sqrt{1 - \sin^2\theta}} \\
&= G \frac{r}{L} \int_0^{2\pi} \frac{d\theta_p}{\sqrt{1 - b \sin^2\theta_0}} \\
&= \frac{1}{2\pi} \int_0^{2\pi} \frac{d\theta_p}{\sqrt{1 - b \sin^2\theta_0}}
\end{aligned} \tag{84}
$$

where

$$b(l) = \frac{1+\epsilon}{1+\epsilon\cos\theta_p} \tag{85}$$

Note that $\overline{\tau_b}$ in Eq. (84) is independent of v_0.

Now consider the second term, β.

$$\begin{aligned}\beta &= \left\langle \frac{\tan^2\theta_0}{\tan^2\theta} \right\rangle_b \\ &= \frac{1}{\tau_b}\int \frac{dl}{v_\parallel}\frac{\tan^2\theta_0}{\tan^2\theta}\end{aligned} \tag{86}$$

Using

$$\tan^2\theta = \frac{b\sin^2\theta_0}{1-b\sin^2\theta_0} \tag{87}$$

Eq.(86) is written as

$$\begin{aligned}\beta(v_0,\theta_0) &= \frac{1}{\tau_b}\int \frac{dl}{v_\parallel}\,\frac{\tan^2\theta_0}{\frac{b\sin^2\theta_0}{1-b\sin^2\theta_0}} \\ &= \frac{1}{\tau_b}\int \frac{dl}{v_\parallel}\,\frac{1}{b\cos^2\theta_0}\big(1-b\sin^2\theta_0\big) \\ &= \frac{1}{\tau_b}\gamma G r\int \frac{d\theta_p}{u_\parallel}\,\frac{1}{b\cos^2\theta_0}\big(1-b\sin^2\theta_0\big) \\ &= \frac{1}{\overline{\tau_b}}\gamma G r\frac{v_0}{L}\int \frac{d\theta_p}{u_\parallel}\,\frac{1}{b\cos^2\theta_0}\big(1-b\sin^2\theta_0\big) \\ &= \frac{1}{\overline{\tau_b}}G r\frac{1}{L}\int \frac{d\theta_p}{\sqrt{1-b\sin^2\theta_0}}\,\frac{1}{b\cos^2\theta_0}\big(1-b\sin^2\theta_0\big) \\ &= \frac{1}{\overline{\tau_b}}\frac{1}{2\pi}\int \frac{d\theta_p}{\sqrt{1-b\sin^2\theta_0}}\,\frac{1}{b\cos^2\theta_0}\big(1-b\sin^2\theta_0\big) \\ &= \frac{1}{\overline{\tau_b}}\frac{1}{2\pi}\int_0^{2\pi} \frac{d\theta_p}{\sqrt{1-b\sin^2\theta_0}}\,\frac{1}{b\cos^2\theta_0}\big(1-b\sin^2\theta_0\big) \\ &= \frac{1}{\overline{\tau_b}}\frac{1}{2\pi}\int_0^{2\pi} d\theta_p\,\sqrt{1-b\sin^2\theta_0}\frac{1}{b\cos^2\theta_0}\end{aligned} \tag{88}$$

From Eq.(88), one knows β is independent of v_0. Eqs. (84) and (88) will be used in the numerical code. Note that λ is related with $\overline{\tau_b}$ by

$$\lambda(\theta_0) = \overline{\tau_b}\,\cos\theta_0. \tag{89}$$

Since $\overline{\tau_b}$ is independent of v_0, Eq.(89) indicates λ is also independent of v_0.

$$\sin\theta_{\mathrm{tr}} = 1/b_{\max} = \frac{B_{\min}}{B_{\max}} \tag{90}$$

7 Boundary conditions

We'll solve the adjoint equation in the passing region, i.e., $0<u_0<u_{\max}$, $0<\theta_0<\theta_{\mathrm{tr}}$, where $u_{\max}$ is a maximum momentum chosen by us, which value should be much larger than the thermal momentum. Since χ is zero in the trapped region, the boundary condition set for passing region should be that χ is zero at the boundary between the passing and trapped region. In the middle-plane coordinates, this is to say

$$\chi_0(u_0,\theta_0=\theta_{\mathrm{tr}}) = 0. \tag{91}$$

The $u_0=u_{\max}$ boundary condition is set as

$$\frac{\partial^2}{\partial u_0^2}\chi_0(u_0,\theta_0)|_{u_0=u_{\max}} = 0. \tag{92}$$

The inner boundary conditions are obtained using symmetry which is discussed in another note.

8 Bounce-averaged divergence term

$$\begin{aligned}\langle\nabla_u\cdot\boldsymbol{S}\rangle_b &= \left\langle\frac{1}{u^2}\frac{\partial}{\partial u}(u^2 S_u)+\frac{1}{u\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta S_\theta)\right\rangle_b\\ &= \left\langle\frac{1}{u^2}\frac{\partial}{\partial u}(u^2 S_u)\right\rangle_b+\left\langle\frac{1}{u\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta S_\theta)\right\rangle_b\end{aligned} \tag{93}$$

To facilitate the bounce-averaged process, we first transform the terms in Eq. (93) to middle-plane coordinates. It is obvious that the first term is transformed to

$$\frac{1}{u^2}\frac{\partial}{\partial u}(u^2 S_u)=\frac{1}{u_0^2}\frac{\partial}{\partial u_0}(u_0^2 S_u) \tag{94}$$

Then its bounce-averaged form is given by

$$\left\langle\frac{1}{u^2}\frac{\partial}{\partial u}(u^2 S_u)\right\rangle_b=\frac{1}{u_0^2}\frac{\partial}{\partial u_0}(u_0^2\langle S_u\rangle_b) \tag{95}$$

The second term in Eq. (93) is transformed to middle-plane coordinates to give

$$\frac{1}{u\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta S_\theta)=-\frac{1}{u_0}\frac{B_0}{B}\frac{\xi}{\xi_0}\frac{\partial}{\partial\xi_0}(\sin\theta S_\theta) \tag{96}$$

In obtaining the above, use was made of

$$\frac{\partial}{\partial\xi}=\frac{B_0}{B}\frac{\xi}{\xi_0}\frac{\partial}{\partial\xi_0} \tag{97}$$

Then the bounce-averaged form is given by

$$\begin{aligned}\left\langle\frac{1}{u\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta S_\theta)\right\rangle_b &= -\frac{1}{u_0}\left\langle\frac{B_0}{B}\frac{\xi}{\xi_0}\frac{\partial}{\partial\xi_0}(\sin\theta S_\theta)\right\rangle_b\\ &= -\frac{1}{u_0}\frac{1}{\tau_b}\int\frac{dl}{v_\parallel}\frac{B_0}{B}\frac{\xi}{\xi_0}\frac{\partial}{\partial\xi_0}(\sin\theta S_\theta)\\ &= -\frac{1}{u_0}\frac{1}{\tau_b}\int\frac{dl}{v_0}\frac{B_0}{B}\frac{1}{\xi_0}\frac{\partial}{\partial\xi_0}(\sin\theta S_\theta)\\ &= -\frac{1}{u_0}\frac{1}{\tau_b}\frac{1}{\xi_0}\frac{\partial}{\partial\xi_0}\left(\int\frac{dl}{v_0}\frac{B_0}{B}\sin\theta S_\theta\right)\\ &= -\frac{1}{u_0}\frac{1}{\tau_b}\frac{1}{\xi_0}\frac{\partial}{\partial\xi_0}\left(\int\frac{B_0}{B}\frac{dl}{v_\parallel}\xi\sin\theta S_\theta\right)\\ &= -\frac{1}{u_0}\frac{1}{\tau_b}\frac{1}{\xi_0}\frac{\partial}{\partial\xi_0}\left(\int\frac{B_0}{B}\frac{dl}{v_\parallel}\xi\sin\theta\frac{\tan\theta}{\tan\theta_0}\frac{\tan\theta_0}{\tan\theta}S_\theta\right)\\ &= -\frac{1}{u_0}\frac{1}{\tau_b}\frac{1}{\xi_0}\frac{\partial}{\partial\xi_0}\left(\int\frac{B_0}{B}\frac{dl}{v_\parallel}\sin^2\theta\frac{\cos\theta_0}{\sin\theta_0}\frac{\tan\theta_0}{\tan\theta}S_\theta\right)\\ &= -\frac{1}{u_0}\frac{1}{\tau_b}\frac{1}{\xi_0}\frac{\partial}{\partial\xi_0}\left(\int\frac{dl}{v_\parallel}\sin^2\theta_0\frac{\cos\theta_0}{\sin\theta_0}\frac{\tan\theta_0}{\tan\theta}S_\theta\right)\\ &= -\frac{1}{u_0}\frac{1}{\tau_b}\frac{1}{\xi_0}\frac{\partial}{\partial\xi_0}\left(\sin\theta_0\cos\theta_0\int\frac{dl}{v_\parallel}\frac{\tan\theta_0}{\tan\theta}S_\theta\right)\\ &= -\frac{1}{u_0}\frac{1}{\tau_b v_0\cos\theta_0}\frac{\partial}{\partial\xi_0}\left(\sin\theta_0 v_0\cos\theta_0\int\frac{dl}{v_\parallel}\frac{\tan\theta_0}{\tan\theta}S_\theta\right)\\ &= -\frac{1}{u_0}\frac{1}{\tau_b v_0\cos\theta_0}\frac{\partial}{\partial\xi_0}\left(\sin\theta_0 v_0\cos\theta_0\tau_b\left\langle\frac{\tan\theta_0}{\tan\theta}S_\theta\right\rangle_b\right)\\ &= -\frac{1}{u_0}\frac{1}{\lambda}\frac{\partial}{\partial\xi_0}\left(\sin\theta_0\lambda\left\langle\frac{\tan\theta_0}{\tan\theta}S_\theta\right\rangle_b\right)\end{aligned} \tag{98}$$

Using Eqs. (95) and (98), one gets

$$\begin{aligned}\langle\nabla\cdot\boldsymbol{S}\rangle_b &= \frac{1}{u_0^2}\frac{\partial}{\partial u_0}(u_0^2\langle S_u\rangle_b)-\frac{1}{\lambda u_0}\frac{\partial}{\partial\xi_0}\left(\sin\theta_0\lambda\left\langle\frac{\tan\theta_0}{\tan\theta}S_\theta\right\rangle_b\right)\\ &= \frac{1}{u_0^2}\frac{\partial}{\partial u_0}(u_0^2\frac{1}{\lambda}\lambda\langle S_u\rangle_b)-\frac{1}{\lambda u_0}\frac{\partial}{\partial\xi_0}\left(\sin\theta_0\lambda\left\langle\frac{\tan\theta_0}{\tan\theta}S_\theta\right\rangle_b\right)\end{aligned} \tag{99}$$

Noticing that λ is independent of u_0, Eq. (99) is written as

$$\begin{aligned}\langle\nabla\cdot\boldsymbol{S}\rangle_b &= \frac{1}{\lambda}\frac{1}{u_0^2}\frac{\partial}{\partial u_0}(u_0^2\lambda\langle S_u\rangle)-\frac{1}{\lambda u_0}\frac{\partial}{\partial\xi_0}\left(\sin\theta_0\lambda\left\langle\frac{\tan\theta_0}{\tan\theta}S_\theta\right\rangle\right)\\ &= \frac{1}{\lambda}\nabla_{u_0}\cdot(\lambda\boldsymbol{S}_0)\end{aligned} \tag{100}$$

where

$$S_{0u}=\langle S_u\rangle_b,$$

$$S_{0\theta}=\left\langle\frac{\tan\theta_0}{\tan\theta}S_\theta\right\rangle_b.$$

9 Landau-damped waves

The induced flux in momentum space by Landau-damped waves is

$$\boldsymbol{S}\propto v_\parallel f_m\delta(v_\parallel-v_p)\hat{\boldsymbol{u}}_\parallel, \tag{101}$$

We further assume that the ray pierces the flux surface at a single poloidal angle ϕ' where $b=b'$, then

$$\boldsymbol{S}\propto v_\parallel f_m\delta(v_\parallel-v_p)\delta(\phi-\phi')\hat{\boldsymbol{u}}_\parallel. \tag{102}$$

The flux's component parallel to $\hat{\boldsymbol{u}}$ and $\hat{\boldsymbol{\theta}}$ is, respectively,

$$S_u\propto v_\parallel f_m\delta(v_\parallel-v_p)\delta(\phi-\phi')\cos\theta \tag{103}$$

$$S_\theta\propto -v_\parallel f_m\delta(v_\parallel-v_p)\delta(\phi-\phi')\sin\theta \tag{104}$$

Then

$$\begin{aligned}S_{0u} &= \langle S_u\rangle_b\\ &\propto \left\langle v_\parallel f_m\delta(v_\parallel-v_p)\delta(\phi-\phi')\cos\theta\right\rangle_b\\ &= \frac{1}{\tau_b}\int v_\parallel\left[f_m\delta(v_\parallel-v_p)\delta(\phi-\phi')\cos\theta\right]\frac{dl}{v_\parallel}\\ &= \frac{1}{\tau_b}f_m\int\delta(v_\parallel-v_p)\delta(\phi-\phi')\cos\theta dl\\ &= \frac{1}{\tau_b}Grf_m\int_0^{2\pi}\delta(v_0\cos\theta-v_p)\delta(\phi-\phi')\cos\theta d\phi\\ &= \begin{cases}0 & \text{For those trapped particles which can not reach } \phi'\\ \frac{1}{\tau_b}Grf_m\delta(v_0\cos\theta'-v_p)\cos\theta' & \text{Otherwise}\end{cases}\end{aligned} \tag{105}$$

where θ' is related to θ_0 by

$$\frac{\sin^2\theta'}{B(\phi')} = \frac{\sin^2\theta_0}{B_0} \Longrightarrow \sin^2\theta' = \sin^2\theta_0 b'. \tag{106}$$

Note that for those trapped particles which can not reach the poloidal location ϕ', the result in Eq.(105) is zero. For those trapped particles which can reach the poloidal location ϕ', they will pass the poloidal location ϕ' twice, one pass corresponds to $\cos\theta'$ is positive, another pass corresponds to $\cos\theta'$ is negative. Now comes the question which value should be chosen for $\cos\theta'$. We assume v_p is positive, then only the case in which the value of $\cos\theta'$ is positive can make resonant happen. For passing particles, the sign of $\cos\theta'$ will remain the same with that of $\cos\theta_0$. To make resonant possible, $\cos\theta_0$ should be positive which means only particles in one of the two passing regions can resonant with the wave.

The second branch of Eq.(105) is further written as

$$S_{0u} \propto \frac{1}{\tau_b}\frac{L}{2\pi} f_m[\delta(v\cos\theta' - v_p)]\cos\theta'$$

Using $\lambda = \tau_b v_0 \cos\theta_0 / L$, one gets

$$\begin{aligned} \lambda S_{0u} &\propto \frac{1}{2\pi} v_0 \cos\theta_0 f_m[\delta(v\cos\theta' - v_p)]\cos\theta' \\ &= \frac{1}{2\pi} v_0 \cos\theta' f_m[\delta(v\cos\theta' - v_p)]\cos\theta_0 \end{aligned} \tag{107}$$

$$\begin{aligned} S_{0\theta} &= \left\langle \frac{\tan\theta_0}{\tan\theta} S_\theta \right\rangle \\ &\propto -\frac{1}{\tau_b}\int \frac{\tan\theta_0}{\tan\theta}\big[v_\parallel f_m \delta(v_\parallel - v_p)\delta(\phi-\phi')\sin\theta\big]\frac{dl}{v_\parallel} \\ &= -\frac{1}{\tau_b} f_m \tan\theta_0 \int \cos\theta\big[\delta(v_\parallel - v_p)\delta(\phi-\phi')\big]dl \\ &= -\frac{1}{\tau_b} G r f_m \tan\theta_0 \int_0^{2\pi} \cos\theta\big[\delta(v_\parallel - v_p)\delta(\phi-\phi')\big]\,d\phi \\ &= \begin{cases} 0 & \text{For those trapped particles which can not reach } \phi' \\ -\frac{1}{\tau_b} G r f_m \tan\theta_0 \cos\theta'[\delta(v_0\cos\theta' - v_p)] & \text{Otherwise} \end{cases} \end{aligned}$$

Using $\lambda = \tau_b v_0 \cos\theta_0 / L$, the second branch of the above equation is written

$$\lambda S_{0\theta} \propto -\frac{1}{2\pi} v_0 f_m \cos\theta' [\delta(v\cos\theta' - v_p)]\sin\theta_0. \tag{108}$$

Using Eqs.(107)(108), one gets

$$\lambda \boldsymbol{S}_0 \propto \frac{1}{2\pi} v_0 f_m \cos\theta' \delta(v_0\cos\theta' - v_p)\hat{\boldsymbol{u}}_{0\parallel}. \tag{109}$$

Now calculate the power deposition

$$P_0 = m\int d^3\boldsymbol{u}_0 \lambda \boldsymbol{S}_0 \cdot \boldsymbol{v}_0 \tag{110}$$

$$\propto m\frac{1}{2\pi}\int d^3\boldsymbol{u}_0 v_0 f_m \cos\theta' \delta(v_0\cos\theta' - v_p) v_0 \cos\theta_0 \tag{111}$$

Recalling the remarks made in the above, for trapped particles, $\cos\theta'$ should be chosen positive, i.e.,

$$\cos\theta' = \sqrt{1 - b'\sin^2\theta_0}. \tag{112}$$

For passing particles, since the sign of $\cos\theta'$ do not change, so to make resonance positive, the initial value should be positive, this means only one passing region ($0<\theta_0<\theta_{\rm tr}$) satisfies this condition. Particles in another region $\pi-\theta_{\rm tr}<\theta_0<\pi$ can not resonant with the wave. Using these, Eq.(111) is further written as

$$\begin{aligned}
P_0 &= m\frac{1}{2\pi}\int d^3\boldsymbol{u}_0 v_0 f_m\sqrt{1-b'\sin^2\theta_0}\,\delta(v_0\sqrt{1-b'\sin^2\theta_0}-v_p)v_0\cos\theta_0 \\
&= m\int v_0 f_m\sqrt{1-b'\sin^2\theta_0}\,\delta(v_0\sqrt{1-b'\sin^2\theta_0}-v_p)v_0\cos\theta_0\,\sin\theta_0 d\theta_0 u_0^2 du_0 \\
&= m\int du_0\int_0^{\theta_{\rm tr}} d\theta_0 v_0 f_m\sqrt{1-b'\sin^2\theta_0}\,\delta(v_0\sqrt{1-b'\sin^2\theta_0}-v_p)v_0\,\sin\theta_0\cos\theta_0 u_0^2 \\
&+ m\int du_0\int_{\theta_{\rm tr}}^{\theta_c} d\theta_0 v_0 f_m\sqrt{1-b'\sin^2\theta_0}\,\delta(v_0\sqrt{1-b'\sin^2\theta_0}-v_p)v_0\,\sin\theta_0\cos\theta_0 u_0^2 \\
&+ m\int du_0\int_{\pi-\theta_c}^{\pi-\theta_{\rm tr}} d\theta_0 v_0 f_m\sqrt{1-b'\sin^2\theta_0}\,\delta(v_0\sqrt{1-b'\sin^2\theta_0}-v_p)v_0\,\sin\theta_0\cos\theta_0 u_0^2
\end{aligned}$$

Note that the trapped particles' contribution sums to be zero. Thus P_0 reduces to

$$\begin{aligned}
P_0 &= m\int du_0\int_0^{\theta_{\rm tr}} d\theta_0 v_0 f_m\sqrt{1-b'\sin^2\theta_0}\,\delta(v_0\sqrt{1-b'\sin^2\theta_0}-v_p)v_0\,\sin\theta_0\cos\theta_0 u_0^2 \\
&= m\int du_0\int_0^{\theta_{\rm tr}} d\,(\sin\theta_0)v_0 f_m\sqrt{1-b'\sin^2\theta_0}\,\delta(v_0\sqrt{1-b'\sin^2\theta_0}-v_p)v_0\,\sin\theta_0 u_0^2 \\
&= m\int du_0\int_0^{\sqrt{1/b_{\max}}} dx\, v_0^2 u_0^2 f_m\sqrt{1-b'x^2}\,\delta(v_0\sqrt{1-b'x^2}-v_p)\,x
\end{aligned}\tag{113}$$

Now care must be taken to evaluate the integration over x (it is easy to make mistakes here since the integration involves Dirac delta function and the variable of this function is $\sqrt{1-b'x^2}$, instead of a pure x). Eq.(113) is written

$$\begin{aligned}
P_0 &= m\int du_0\int_0^{\sqrt{1/b_{\max}}} v_0^2u_0^2 f_m\sqrt{1-b'x^2}\frac{1}{v_0}\delta\left(\sqrt{1-b'x^2}-\frac{v_p}{v_0}\right)x\left(\frac{-\sqrt{1-b'x^2}}{b'x}\right)d\sqrt{1-b'x^2} \\
&= m\int du_0\int_0^{\sqrt{1/b_{\max}}} v_0u_0^2 f_m\sqrt{1-b'x^2}\,\delta\left(\sqrt{1-b'x^2}-\frac{v_p}{v_0}\right)\left(\frac{-\sqrt{1-b'x^2}}{b'}\right)d\sqrt{1-b'x^2} \\
&= m\int_{u_{0\min}}^{u_{0\max}} v_0u_0^2 f_m\frac{v_p}{v_0}\,\frac{-v_p/v_0}{b'} \\
&= -m\frac{1}{b'}\int_{u_{0\min}}^{u_{0\max}} du_0\left(v_0u_0^2 f_m\frac{v_p^2}{v_0^2}\right)
\end{aligned}\tag{114}$$

Note that $x=\sin\theta_0$ is restricted in passing region, so its value is between 0 and $\sqrt{1/b_{\max}}$. And v_0 is related to $\sin\theta_0$ by the resonant condition. Thus v_0 is limited by a condition

$$0<\frac{1}{b'}\left(1-\frac{v_p^2}{v_0^2}\right)<\frac{1}{b_{\max}},\tag{115}$$

which reduces to

$$v_p<v_0<\frac{v_p}{\sqrt{1-b'/b_{\max}}}\tag{116}$$

Therefore the limit of the integration in Eq.(114) is given by

$$\begin{cases} v_{\min}=v_p \\ v_{\max}=\dfrac{v_p}{\sqrt{1-b'/b_{\max}}} \end{cases}.\tag{117}$$

The corresponding momentum can be calculated by multiplying the above result by the Lorentz factor. Note that v_0 must have a finite upper limit (instead of infinite limit) to make resonant condition satisfied. The reason is as follows. We are dealing with passing particles which passes the poloidal location ϕ'. When they reach this location, the cosine of their pitch angle is given by $\cos\theta' = \sqrt{1 - b'\sin^2\theta_0}$, which has a nonzero (instead of zero) minimum value $\sqrt{1 - b'\sin^2\theta_{\mathrm{tr}}} = \sqrt{1 - b'/b_{\max}}$.

For $b' = b_{\max}$, Eq.(117) indicate $v_{\max}$ will be infinite. This indicate in this case there is no limit on the upper limit of the velocity to make resonance possible (it is easy to understand this). Next calculate the current.

$$\begin{aligned}
j_{0\parallel} &= \int d^3\boldsymbol{u}_0 \lambda \boldsymbol{S}_0 \cdot \frac{\partial\chi}{\partial\boldsymbol{u}_0} && (118)\\
&\propto \frac{1}{2\pi}\int d^3\boldsymbol{u}_0 v_0 f_m \cos\theta' \delta(v_0\cos\theta' - v_p)\hat{\boldsymbol{u}}_{0\parallel}\cdot\frac{\partial\chi}{\partial\boldsymbol{u}_0}\\
&= \frac{1}{2\pi}\int d^3\boldsymbol{u}_0 v_0 f_m \cos\theta' \delta(v_0\cos\theta' - v_p)\frac{\partial\chi}{\partial u_{0\parallel}}\\
&= \int du_0 u_0^2 \int v_0 f_m \sqrt{1 - b'\sin^2\theta_0}\,\delta\!\left(v_0\sqrt{1 - b'\sin^2\theta_0} - v_p\right)\frac{\partial\chi}{\partial u_{0\parallel}}\sin\theta_0 d\theta_0\\
&= \int du_0 u_0^2 \int v_0 f_m \sqrt{1 - b'\sin^2\theta_0}\frac{1}{v_0}\delta\!\left(\sqrt{1 - b'\sin^2\theta_0} - \frac{v_p}{v_0}\right)\frac{\partial\chi}{\partial u_{0\parallel}}\frac{-\sqrt{1 - b'\sin^2\theta_0}}{b'\cos\theta_0}d\sqrt{1 - b'\sin^2\theta_0}\\
&= \int_{u_{0\min}}^{u_{0\max}} du_0 u_0^2 \left(v_0 f_m \frac{v_p}{v_0}\frac{1}{v_0}\left(\frac{\partial\chi}{\partial u_{0\parallel}}\right)_{\text{on resonant line}} \frac{1}{\sqrt{1 - \frac{1}{b'}\left(1 - \frac{v_p^2}{v_0^2}\right)}}\frac{-v_p/v_0}{b'}\right)\\
&= -\frac{1}{b'}\int_{u_{0\min}}^{u_{0\max}} du_0 u_0^2 \left(f_m \left(\frac{v_p}{v_0}\right)^2 \left(\frac{\partial\chi}{\partial u_{0\parallel}}\right)_{\text{on resonant line}} \frac{1}{\sqrt{1 - \frac{1}{b'}\left(1 - \frac{v_p^2}{v_0^2}\right)}}\right) && (119)
\end{aligned}$$

where the resonant line is given by

$$\sqrt{1 - b'\sin^2\theta_0} - \frac{v_p}{v_0} = 0$$

$$\Longrightarrow \sin\theta_0 = \sqrt{\frac{1}{b'}\left(1 - \frac{v_p^2}{v_0^2}\right)} \tag{120}$$

which gives a line in (v_0, θ_0) space.

The ratio of Eq. (119) to Eq. (114) gives the drive efficiency

$$\frac{j_{0\parallel}}{P_0} = \frac{\int_{u_{0\min}}^{u_{0\max}} du_0 u_0^2 \left(f_m \left(\frac{v_p}{v_0}\right)^2 \left(\frac{\partial\chi}{\partial u_{0\parallel}}\right)_{\text{on resonant line}} \frac{1}{\sqrt{1 - \frac{1}{b'}\left(1 - \frac{v_p^2}{v_0^2}\right)}}\right)}{m\int_{u_{0\min}}^{u_{0\max}} du_0 u_0^2 \left(v_0 f_m \frac{v_p^2}{v_0^2}\right)} \tag{121}$$

$$\frac{j_{0\parallel}}{P_0} = \frac{\int_{u_{0\min}}^{u_{0\max}} du_0 u_0^2 \left(f_m \left(\frac{1}{v_0}\right)^2 \left(\frac{\partial\chi}{\partial u_{0\parallel}}\right)_{\text{on resonant line}} \frac{1}{\sqrt{1 - \frac{1}{b'}\left(1 - \frac{v_p^2}{v_0^2}\right)}}\right)}{m\int_{u_{0\min}}^{u_{0\max}} du_0 u_0^2 \left(v_0 f_m \frac{1}{v_0^2}\right)} \tag{122}$$

$$\begin{aligned}
\hat{\boldsymbol{u}}_{0\parallel}\cdot\frac{\partial\chi}{\partial\boldsymbol{u}_0} &= \left(\cos\theta\hat{\boldsymbol{u}} - \sin\theta\hat{\boldsymbol{\theta}}\right)\cdot\left(\frac{\partial\chi}{\partial u}\hat{\boldsymbol{u}} + \frac{1}{u}\frac{\partial\chi}{\partial\theta}\hat{\boldsymbol{\theta}}\right)\\
&= \cos\theta\frac{\partial\chi}{\partial u} - \frac{1}{u}\sin\theta\frac{\partial\chi}{\partial\theta} && (123)
\end{aligned}$$

10 Numerical results

10.1 Contour of $\chi(u_0, \theta_0)$

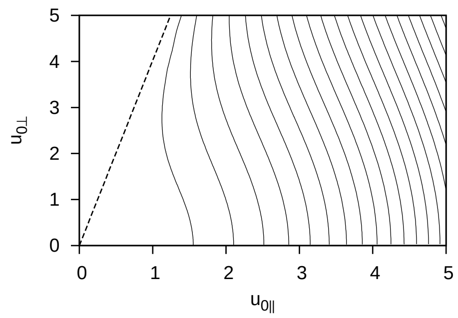

Figure 1. Contour plot of $\chi(u_0, \theta_0)$ (normalized to eu_t/ν_t) for $Z_i = 1$, $T_e = 25$keV, $\varepsilon = 0.03$. The levels of the contours are given by $\chi = 5j$ for $j = 0, 1, 2, ..., 20$ increasing from left to right. The dashed line in the figure is the boundary between the passing and trapped regions, which is given by $u_{0\perp}/u_{0\|} = 4.02$. On this boundary χ is zero.

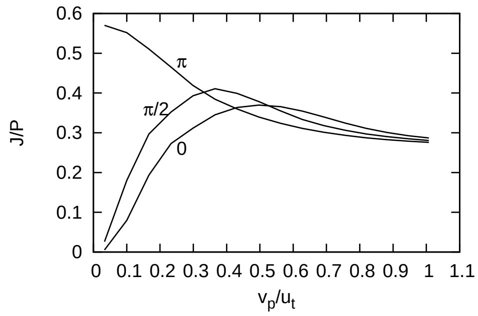

Figure 2. Current drive efficiencies (normalized to $e/m_ec\nu_c$) for landau-damped waves as a function of phase velocity (normalized by $u_t = \sqrt{T_e/m_e}$). The labels on the lines correspond to the poloidal locations of the wave absorption. The plasma parameters are $Z_i = 1$, $T_e = 25$keV, $\varepsilon = 0.03$. The weakly relativistic collision operator is used to model the electron-electron collisions.

11 Manuscript

$$\langle \nabla \cdot \boldsymbol{S} \rangle_b = \frac{1}{\lambda} \nabla_{u_0} \cdot (\lambda \boldsymbol{S}_0) \tag{124}$$

$$\langle C(f) \rangle_b = \frac{1}{\lambda} \nabla_{u_0} \cdot (\lambda \boldsymbol{S}_0) \tag{125}$$

$$\lambda \langle C(f) \rangle_b = \nabla_{u_0} \cdot (\lambda \boldsymbol{S}_0) \tag{126}$$

$$j_{0\|} = q \int d^3\boldsymbol{u}_0 v_0 \cos\theta_0 f \tag{127}$$

$$\frac{1}{f_m} \langle C(\chi f_m) \rangle_b = -q \frac{v_0 \cos\theta_0}{\lambda} \Theta \tag{128}$$

$$\Longrightarrow \frac{1}{f_m} \lambda \langle C(\chi f_m) \rangle_b = -q v_0 \cos\theta_0 \Theta \tag{129}$$

$$\int d^3\boldsymbol{u}_0 \frac{1}{f_m} f\lambda \langle C(\chi f_m)\rangle_b = -q\int d^3\boldsymbol{u}_0\, v_0 \cos\theta_0 f\Theta$$

$$\int d^3\boldsymbol{u}_0 \lambda f \frac{1}{f_m} \langle C(\chi f_m)\rangle_b = -j_{0\|}$$

Using the property of bounce-averaging collision operator

$$\int d^3\boldsymbol{u}_0 \lambda g \langle C(h f_m)\rangle_b = \int d^3\boldsymbol{u}_0 \lambda h \langle C(g f_m)\rangle_b \tag{130}$$

Please prove Eq.(130)!

$$\int d^3\boldsymbol{u}_0 \lambda \chi \langle C(f)\rangle_b = -j_{0\|}$$

$$\int d^3\boldsymbol{u}_0 \chi \nabla_{u_0} \cdot (\lambda \boldsymbol{S}_0) = -j_{0\|}$$

$$\int d^3\boldsymbol{u}_0 \lambda \boldsymbol{S}_0 \cdot \frac{\partial \chi}{\partial \boldsymbol{u}_0} = j_{0\|} \tag{131}$$

$$\begin{aligned} P_0 &= -\int d^3\boldsymbol{u}_0 w \langle \nabla_u \cdot \boldsymbol{S}\rangle_b \\ &= -\int d^3\boldsymbol{u}_0 w \frac{1}{\lambda} \nabla_{u_0} \cdot (\lambda \boldsymbol{S}_0) \\ &= \int d^3\boldsymbol{u}_0 (\lambda \boldsymbol{S}_0) \cdot \nabla_{u_0} \frac{w}{\lambda} \\ &= \end{aligned}$$

$$\nabla_{u_0} \frac{1}{\lambda} = \frac{1}{\lambda^2} \nabla_{u_0} \lambda = \frac{1}{\lambda^2} \frac{1}{u_0} \frac{\partial \lambda}{\partial \theta_0} \hat{\boldsymbol{\theta}}$$

$$\begin{aligned} \nabla_{u_0} \frac{w}{\lambda} &= \frac{1}{\lambda} \nabla_{u_0} w + w \nabla_{u_0} \frac{1}{\lambda} \\ &= \end{aligned}$$

$$\int d^3\boldsymbol{u}\, [\gamma_a m_a \boldsymbol{v} C(f_a, f_b) + \gamma_b m_b \boldsymbol{v} C(f_b, f_a)] = 0$$

$$D_{\theta\theta}^{e/i} = \Gamma^{e/e} \frac{Z_i}{2v} \tag{132}$$

It can be proved that $\nabla \cdot \boldsymbol{B} = 0$ is satisfied.

[

$$\tan\theta = \frac{z}{R - R_0}$$

$$\sin\theta = \frac{z}{\sqrt{z^2 + (R - R_0)^2}}$$

$$\cos\theta = \frac{R - R_0}{\sqrt{z^2 + (R - R_0)^2}}$$

$$B_z = -\frac{B_{p0}}{R}\cos\theta = -\frac{B_{p0}}{R}\frac{R-R_0}{\sqrt{z^2+(R-R_0)^2}}$$

$$B_R = \frac{B_{p0}}{R}\sin\theta = \frac{B_{p0}}{R}\frac{z}{\sqrt{z^2+(R-R_0)^2}}$$

]

Define the transforming rule, (where (u_0, θ_0) is called 'middle-plane coordinates'.)

$$u_0 = u \tag{133}$$

$$\frac{\sin^2\theta_0}{B_0} = \frac{\sin^2\theta}{B(l)} \tag{134}$$

then for passing particles (i.e., $\sin^2\theta_0 < B_0/B_{\max}$)

$$\chi(u,\theta,l) = f\left(mc^2(\gamma_0 - 1), \frac{mu_0^2\sin^2\theta_0}{B_0}, \mathrm{sgn}(\cos\theta_0)\right) \equiv \chi_0(u_0, \theta_0) \tag{135}$$

So we have

$$\chi(u,\theta,l) = \chi_0(u_0,\theta_0) \tag{136}$$

Note that for trapped particles, σ can not be expressed in terms of middle-plane coordinates. However, for trapped particle $f(\varepsilon, \mu, \sigma)$ is actually independent of σ.

χ is odd about $u_\parallel$ in passing region, and is even about $u_\parallel$ in trapped region. Therefore χ must be zero at the boundary separating passing and trapped region.

Bibliography

[1] C. F. F. Karney, N. J. Fisch, and A. H. Reiman. Green's function for rf-driven current in a toroidal plasma. In R. McWilliams, editor, *Radio-Frequency Power in Plasmas*, volume 190, pages 430–433, New York, 1989.

[2] Charles F. F. Karney and Nathaniel J. Fisch. Efficiency of current drive by fast waves. *Physics of Fluids*, 28(1):116–126, 1985.