Mathematical methods frequently used in plasma physics

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When reading papers on theoretic plasma physics, I found that there are several mathematical methods frequently used to obtain approximate solutions to the physical problems. Since these mathematical methods are usually mixed with various physical details, it is not easy for beginners to identify these methods at first glance. Therefore it is helpful to write a note to discuss exclusively these mathematical methods, which is the motivation of writting this article.

1 Perturbation method

Almost all analytical techinques used to abtain approximate solutions in plasma physics can be categorized to the categrory of perturbation methods. I learnt the perturbation method (from the matematical respect) from Hinch's book[1].

1.1 Regular perturbation theory

1.2 Singular perturbation theory

2 Multiple-scale analysis

In mathematics and physics, multiple-scale analysis (also called the method of multiple scales) comprises techniques used to construct uniformly valid approximations to the solutions of perturbation problems, both for small as well as large values of the independent variables. This is done by introducing fast-scale and slow-scale variables for an independent variable, and subsequently treating these variables, fast and slow, as if they are independent. An example of the application of the multiple-scale analysis in plasma physics is the method of matched asymptotic expansions used in treating tearing modes problem.

2.1 Method of matched asymptotic expansions

In a large class of singularly perturbed problems, the domain is divided into two or more subdomains. In one of these, often the largest, the solution is accurately approximated by an asymptotic series found by treating the problem as a *regular* perturbation (i.e., by setting a relatively small parameter to zero). The other subdomains consist of one or more small areas in which that approximation is inaccurate, generally because the perturbation terms in the problem are not negligible there. These areas are referred to as transition layers, and as boundary or interior layers depending on whether they occur at the domain boundary or inside the domain.

An approximation in the form of an asymptotic series is obtained in the transition layer(s) by treating that part of the domain as a separate perturbation problem. The approximated solution in the transition layer(s) is called the "inner solution" and the other is the "outer solution" named for their relationship to the transition layer(s). The outer and inner solutions are then combined through a process called "matching" in such a way that an approximate solution for the whole domain is obtained.

2.1.1 A simple example

Consider the boundary value problem

$$\epsilon\frac{d^2y}{dt^2}+(1+\epsilon)\frac{dy}{dt}+y=0, \tag{1}$$

with boundary condition $y(0)=0$ and $y(1)=1$, where ϵ is a small parameter $0<\epsilon\ll 1$.

Let us assume the terms d^2y/dt^2, dy/dt, and y are of the same order, and treat Eq. (1) as a regular perturbation problem about the small parameter ε. Then the leading order balance is obtained by setting $\epsilon=0$, giving

$$\frac{dy}{dt}+y=0. \tag{2}$$

The solution of the above equation is

$$y(t)=Ae^{-t}. \tag{3}$$

To satisfy the left boundary condition $y(0)=0$, the constant A in the solution should satisfy that $A=0$. On the other hand, to satisfy the right-hand boundary condition $y(1)=1$, we obtain that $A=e$, which does not agree with the value determined by the left boundary condition. This fact indicates Eq. (1) is not a regular perturbation problem in the whole domain $0\leqslant t\leqslant 1$. We now suppose that Eq. (1) is still a regular perturbation problem in the subdomain $t\sim O(1)$ (we call this subdomain "outer region"), but singular in the subdomain $t\sim\epsilon\ll 1$ (we call this subdomain "inner region"). Then, in the outer region, the solution satisfying the right boundary condition is written as

$$y(t)=e^{1-t}. \tag{4}$$

Using Eq. (4), we have $y\sim e$ at the right boundary of the inner region. Also note that $y=0$ at the left boundary of the inner region. Therefore, the variation of y across the inner region is $\Delta y\sim e$. Note that the width of the inner region is ε. Using these, in the inner region, the differential terms are estimated as

$$\frac{dy}{dt}\sim\frac{1}{\epsilon}, \tag{5}$$

and

$$\frac{d^2y}{dt^2}\sim\frac{1}{\epsilon^2}. \tag{6}$$

Using these orderings, the leading order balance of Eq. (1) is given by the first two terms, i.e.,

$$\epsilon\frac{d^2y}{dt^2}+\frac{dy}{dt}=0. \tag{7}$$

The solution to the above equation is given by

$$y(t)=B-Ce^{-t/\epsilon}. \tag{8}$$

To satisfy the boundary condition $y(0)=0$, we obtain that $B=C$. Then the solution is written as

$$y(t)=B(1-e^{-t/\epsilon}). \tag{9}$$

Eq. (9) is the "inner solution". To determine the constant B, we evlaue Eq. (9) in the intermediate region, $\epsilon\ll t\ll 1$, which gives

$$y(t)=B. \tag{10}$$

In the region $t\ll 1$, the "outer solution", Eq. (4), reduces to

$$y(t)=e. \tag{11}$$

To match the "inner" and "outer" solution, we obtain

$$\Rightarrow B=e. \tag{12}$$

Thus the "inner solution" is written as

$$y(t)=e\left(1-e^{-t/\epsilon}\right) \tag{13}$$

The solution valid for the whole region is written as the sum of the "inner solution" and the "outer solution", but excluding the value in the intermediate region, i.e.,

$$\begin{aligned} y &= y_I(t) + y_O(t) - e \\ &= e - e^{1-t/\epsilon} + e^{1-t} - e \\ &= e\Big(e^{-t} - e^{-t/\epsilon}\Big). \end{aligned} \tag{14}$$

By substituting the above solution to the original equation (1), it is found that the above approximate solution happens to exactly sovle Eq. (1). However, the boundary condition at $t=1$ is not satisfied exactly by the solution in Eq. (14). It happens that the solution to Eq. (1) that satisfies the boundary condition $y(0)=0$ and $y(1)=1$ can be obtained analytically, which is given by (obtained by using mathematical software)

$$\begin{aligned} y(t) &= \frac{e^{1+1/\epsilon-t-t/\epsilon}\Big(e^t - e^{t/\epsilon}\Big)}{e - e^{1/\epsilon}} \\ &= \frac{e^{-t} - e^{-t/\epsilon}}{e^{-1} - e^{-1/\epsilon}}. \end{aligned} \tag{15}$$

Comparing Eqs. (14) and (15), we find the difference is only a multiplying constant. Fig. 1 compares the solution obtained by the the method of matched asymptotic expansion with the "outer solution".

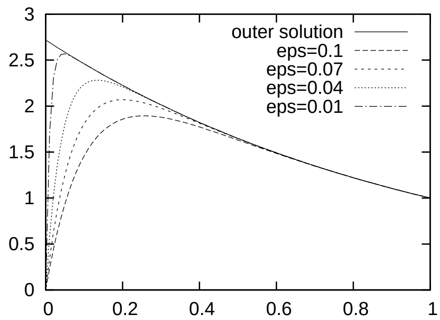

Figure 1. Comparison of the outer solution Eq. (4) with the asymptotic matched solution Eq. (14) for different values of ε. The exact analytical solution (15) is also plotted which are indistinguishable from the asymptotic matched solution (14) at this scale. Note that, with the decreasing of the value of ε, the boundary layer becomes narrower and the asymptotic matched solution converges to the outer solution pointwise everywhere except for the boundary laryer (thus this convergence is not uniformly).

2.2 Tearing mode(**not finished**)

The linearised mhd system consists of three equations, nameley, the linearised equation of state

$$\frac{\partial p_1}{\partial t} = -\gamma p_0 \nabla \cdot \boldsymbol{u}_1 - \boldsymbol{u}_1 \cdot \nabla p_0, \tag{16}$$

the linearised induction equation

$$\frac{\partial \boldsymbol{B}_1}{\partial t} = \nabla \times (\boldsymbol{u} \times \boldsymbol{B}_0) + \eta \nabla^2 \boldsymbol{B}_1 / \mu_0, \tag{17}$$

and the linearized momentum equation

$$\rho_{m0} \frac{\partial \boldsymbol{u}_1}{\partial t} = -\nabla p_1 + (\nabla \times \boldsymbol{B}_1)/\mu_0 \times \boldsymbol{B}_0 + (\nabla \times \boldsymbol{B}_0)/\mu_0 \times \boldsymbol{B}_1. \tag{18}$$

We consider a equilibrium where the magnetic field takes the form

$$\boldsymbol{B}_0 = B_{0y}(x). \tag{19}$$

We can consider the case that the perturbed quantities take the form

$$A(x,y,z,t)=A(x)\exp(iky+\gamma t). \tag{20}$$

Taking curl of both sides of Eq. (18), we obtain

$$\rho_{m0}\frac{\partial\nabla\times\boldsymbol{u}_1}{\partial t}=\nabla\times[(\nabla\times\boldsymbol{B}_1)/\mu_0\times\boldsymbol{B}_0]+\nabla\times[(\nabla\times\boldsymbol{B}_0)/\mu_0\times\boldsymbol{B}_1]. \tag{21}$$

$$\nabla\times\boldsymbol{B}_1=\begin{vmatrix} i & j & k\\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z}\\ B_{1x}(x,y) & B_{1y}(x,y) & B_{1z}(x,y)\end{vmatrix}=\hat{e}_x ikB_z-\hat{e}_y\frac{\partial B_{1z}}{\partial x}+\hat{e}_z\left(\frac{\partial B_{1y}}{\partial x}-ikB_{1x}\right) \tag{22}$$

$$\nabla\times\boldsymbol{B}_1\times\boldsymbol{B}_0=\hat{e}_z ikB_zB_{0y}-\hat{e}_xB_{0y}\left(\frac{\partial B_{1y}}{\partial x}-ikB_{1x}\right). \tag{23}$$

$$\nabla\times[(\nabla\times\boldsymbol{B}_1)\times\boldsymbol{B}_0]=\begin{vmatrix} i & j & k\\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z}\\ B_{0y}\left(\frac{\partial B_{1y}}{\partial x}-ikB_{1x}\right) & 0 & ikB_zB_{0y}\end{vmatrix}=\hat{e}_z \tag{24}$$

$$\nabla\times\boldsymbol{B}_0=\begin{vmatrix} i & j & k\\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z}\\ 0 & B_{0y}(x) & 0\end{vmatrix}=\frac{\partial B_{0y}(x)}{\partial x}\hat{e}_z. \tag{25}$$

$$\nabla\times\boldsymbol{u}_1=\begin{vmatrix} i & j & k\\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z}\\ u_{1x}(x,y) & u_{1y}(x,y) & u_{1z}(x,y)\end{vmatrix}=\hat{e}_x iku_z-\hat{e}_y\frac{\partial u_{1z}}{\partial x}+\hat{e}_z\left(\frac{\partial u_{1y}}{\partial x}-iku_{1x}\right) \tag{26}$$

$$\gamma\mu_0\rho_{m0}\left(\frac{\partial u_{1y}}{\partial x}-iku_{1x}\right)=$$

3 Manuscript

Since we assume that the equation in the "inner region" is a singular perturbation problem,

the leading order equation can not be obtained by setting $\epsilon=0$ in Eq. (1). To obtain the leading order equation, we define a new variable

$$\tau=\frac{t}{\epsilon} \tag{27}$$

and $g(\tau)=y(t)$. The terms in Eq. () are written respectively as

$$\frac{dy}{dt}=\frac{dg}{dt}=\frac{dg}{d\tau}\frac{d\tau}{dt}=\frac{dg}{d\tau}\frac{1}{\epsilon} \tag{28}$$

$$\frac{d^2y}{dt^2}=\frac{d}{dt}\left(\frac{dg}{d\tau}\frac{1}{\epsilon}\right)=\frac{1}{\epsilon^2}\frac{d^2g}{d\tau^2} \tag{29}$$

Eq. () is written as

$$\frac{1}{\epsilon}\frac{d^2g}{d\tau^2}+(1+\epsilon)\frac{dg}{d\tau}\frac{1}{\epsilon}+g=0 \tag{30}$$

The leading order equation is

$$\frac{1}{\epsilon}\frac{d^2g}{d\tau^2}+\frac{dg}{d\tau}\frac{1}{\epsilon}=0, \tag{31}$$

i.e.,

$$\frac{d^2g}{d\tau^2}+\frac{dg}{d\tau}=0. \tag{32}$$

The solution to the above equation is given by

$$g(\tau) = B - Ce^{-\tau}. \tag{33}$$

i.e.,

$$y(t) = B - Ce^{-t/\epsilon}. \tag{34}$$

Bibliography

[1] E.J. Hinch. *Perturbation methods*. Cambridge University Press, 1991.