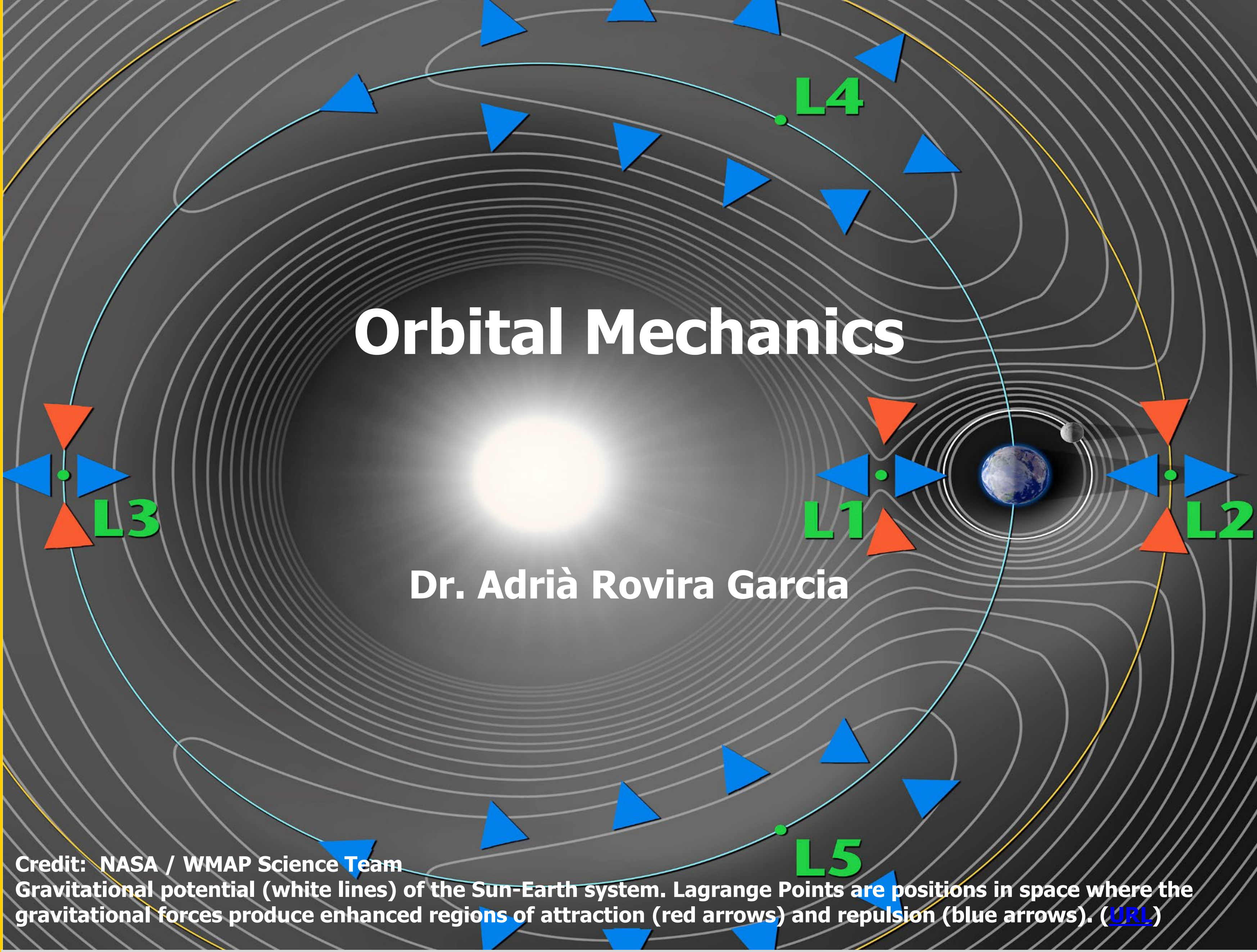



# Orbital Mechanics

Dr. Adrià Rovira Garcia

Credit: NASA / WMAP Science Team
Gravitational potential (white lines) of the Sun-Earth system. Lagrange Points are positions in space where the gravitational forces produce enhanced regions of attraction (red arrows) and repulsion (blue arrows). (URL)

Escola Superior d'Enginyeries Industrial, Aeroespacial i Audiovisual de Terrassa (**ESEIAAT**)

**AMVO**

Adrià Rovira-Garcia

Contact: **adria.rovira@upc.edu**
Campus Nord UPC Jordi Girona 1-3, 08034 Barcelona (Spain). **T: +34 93 4012531**

# Module 3 Requirements

**Objectives:**

**Reference Systems: Coordinates and Time**
**Keplerian Orbits**
**Orbital Maneuvers**
**Interplanetary Trajectories**

**Time Allocation:**

**8 hours**



Adrià Rovira-Garcia

# Module 3 Requirements

**Objectives:**

- **Reference Systems: Coordinates and Time**
  - **Interplanetary systems**
  - **Earth-based systems**
  - **Satellite-based systems**
- **Keplerian Orbits**
- **Orbital Maneuvers**
- **Interplanetary Trajectories**

**Time Allocation:**

- **8 hours**

# Reference Systems vs Frames

Coordinates must be expressed in a well defined reference. Therefore an **accurate definition and determination is essential**.

Distinction must be made between:

- **Reference System:** Theoretical definition, including models and standards for its implementation *[generally only one system]*.
  - **Space-fixed** or inertial systems, in which the positions of stars are fixed or almost fixed. ➔Suitable for satellite motion.
  - **Earth-fixed** systems, co-rotating with the Earth, in which terrestrial points can be expressed conveniently on the Earth's surface. ➔Suitable for user position.
- **Reference Frame:** Practical implementation, through observations and a set of reference coordinates (a set of fundamental stars for space-fixed frames or fiducial stations for a Earth-fixed frames) *[usually many frames]*.

**Source:**
Sanz J, Juan JM, Rovira-Garcia A (2019) “GNSS Data Processing Lectures” ISBN 978-84-93230-7-6

# Reference Systems

Table 3-1 Vallado DA (2013) "Fundamentals of Astrodynamics and Applications" 4th Edition

| System | Name | Symbol | Origin | Fundamental Plane | Principal Direction | Example Use |
|---|---|---|---|---|---|---|
| Inter-planetary | Heliocentric | XYZ | Sun | Ecliptic | Vernal equinox | Patched conic |
| | Solar system | (XYZ) ICRF | Barycenter | varies | varies | Planetary motion |
| Earth based | Geocentric | IJK | Earth | Earth equator | Vernal equinox | General |
| | Earth | (IJK) GCRF | Earth | varies | varies | Perturbations |
| | Body-fixed | (IJK) ITRF | Earth | Earth equator | Greenwich meridian | Observations |
| | Earth-Moon (synodic) | (IJK) S | Barycenter | Invariable plane | Earth | Restricted three body |
| | Topocentric horizon | SEZ | Site | Local horizon | South | Radar observations |
| | Topocentric equatorial | (IJK) t | Site | Parallel to Earth equator | Vernal equinox | Optical observations |
| Satellite based | Perifocal | PQW | Earth | Satellite orbit | Periapsis | Processing |
| | Satellite radial | RSW | Satellite | Satellite orbit | Radial vector | Relative motion, Perturbations |
| | Satellite normal | NTW | Satellite | Satellite orbit | Normal to $\vec{V}_{sat}$ vector | Perturbations |
| | Equinoctial | EQW | Satellite | Satellite orbit | Calculated vector | Perturbations |

# Module 3 Requirements

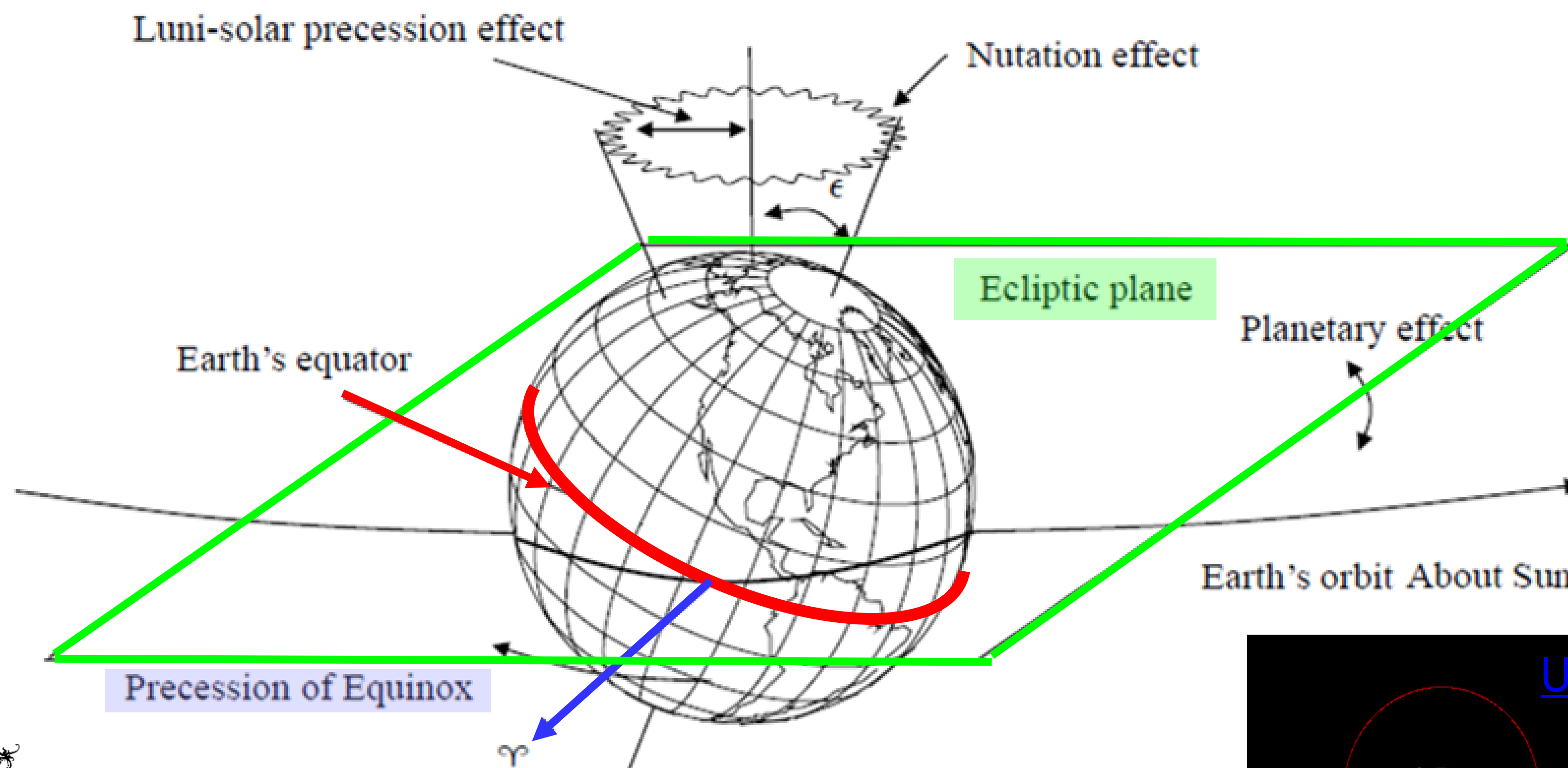
**Objectives:**

- **Reference Systems: Coordinates and Time**
  - **Interplanetary systems**
  - **Earth-based systems**
  - **Satellite-based systems**
- **Keplerian Orbits**
- **Orbital Maneuvers**
- **Interplanetary Trajectories**

**Time Allocation:**

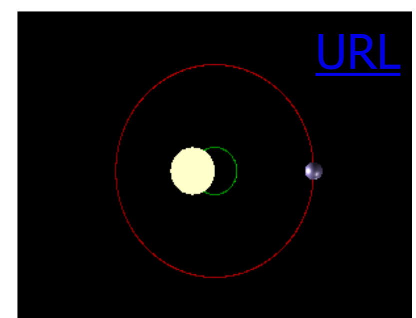
- **8 hours**

| Motion | Period | Comment |
|---|---|---|
| Precession | 25,765 years | Caused by Earth oblateness and gravities of moon and sun |
| Nutation | 18.6 years | Periodic and near-periodic effects < 300 years duration |
| Rotation | 23 h 56 m 4 s | Revolution time measured with respect fixed reference star |
| Translation | 365 d 6 h 9 m | Orbital period around the Sun |





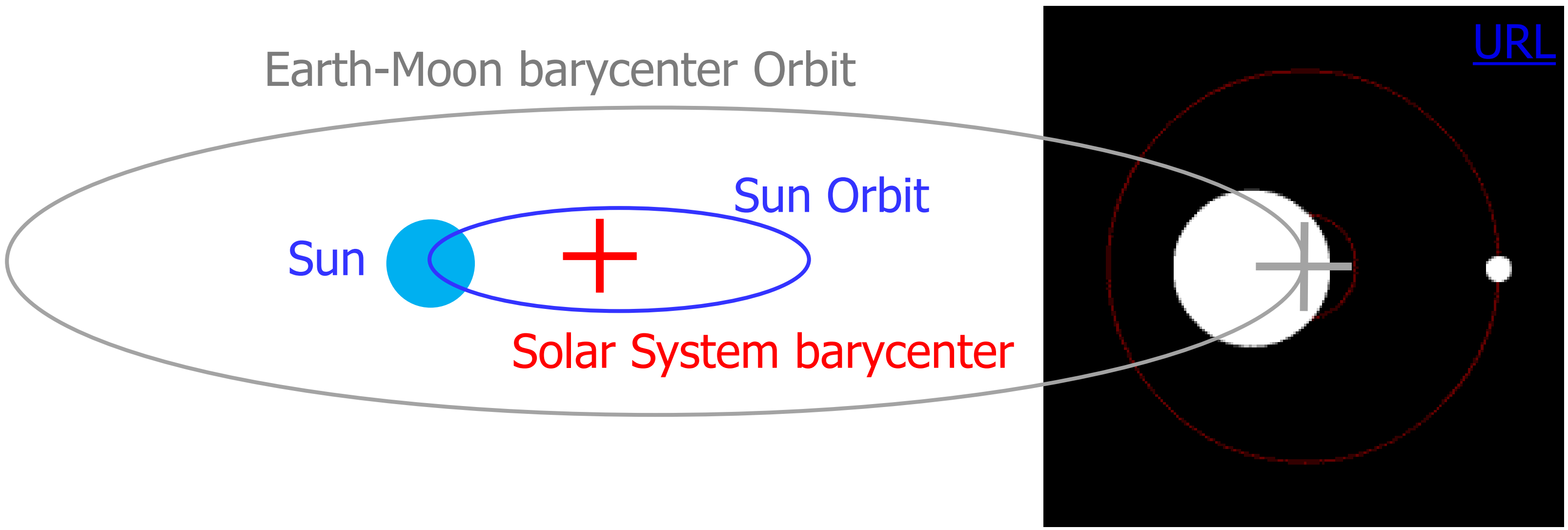
*Note: Ram symbol ♈ designates the direction of the spring Vernal Equinox (VE), often referred to as the first point of Aries.*
*VE pointed to the constellation Aries during Christ's lifetime.*
*VE points today to the constellation Pisces, 56° western, because precession.*




**Source:** Fig 3-25 Vallado DA (2013) "Fundamentals of Astrodynamics and Applications" 4th Ed Space Tech Library.

# Ecliptic

The ecliptic is the plane perpendicular explicitly defined by the mean **orbital angular momentum vector** of the Earth-Moon barycenter.




The simple diagram below illustrates, to scale, the sizes of the Earth, the Moon, and the distance between them. The Earth/Moon barycenter lies within the Earth at a distance of approximately 4670 km from the Earth's center toward the Moon. The current best estimate of the Earth-Moon mass ratio is 81.300570±0.000005 [2].


URL

**Source:**
International Astronomical Union (2006) "Resolution B1 - Adoption of the P03 Precession Theory and Definition of the Ecliptic" XXVI International Astronomical Union General Assembly

Escola Superior d'Enginyeries Industrial, Aeroespacial i Audiovisual de Terrassa (ESEIAAT)

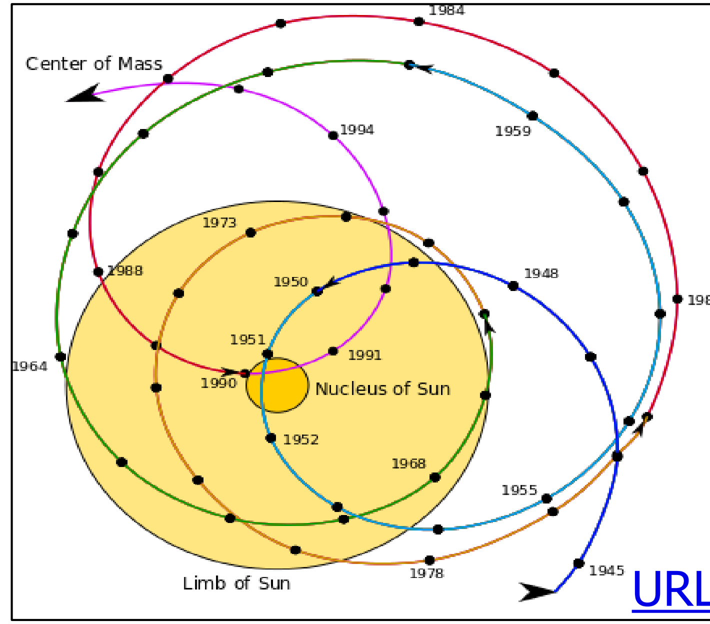
AMVO

Adrià Rovira-Garcia

# International Celestial Reference System (ICRS)

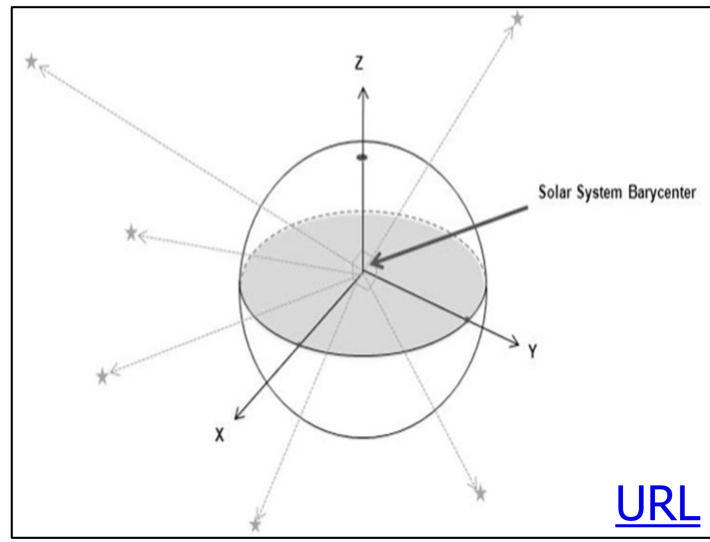


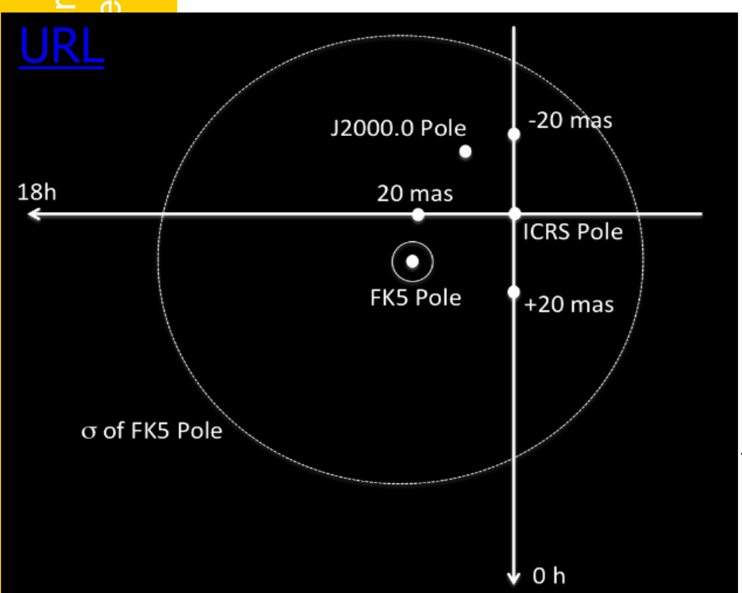
**Origin:** barycenter of the solar system.
**Rotation: NO.** ICRS is fixed with respect to distant objects of the Universe.




ICRS pole and right ascensions origin are maintained fixed relative to quasars within +/- 20 microarcseconds:

- **X-axis:** quasar 3C273B the optically brightest quasar in our sky. Origin of right ascensions, details in Arias et al. (1995).
- **Base-plane (normal to z-axis):** equatorial plane at 12h January 1st 2000 for continuity with previous fundamental reference systems




**Source:** International Earth Rotation and Reference Systems Service (IERS)
Arias et al. (1995) "The extragalactic reference system of the International Earth Rotation Service, ICRS" Astronomy and Astrophysics 303:604-608

# Module 3 Requirements

**Objectives:**

- **Reference Systems: Coordinates and Time**
  - **Interplanetary systems**
  - **Earth-based systems**
  - **Satellite-based systems**
- **Keplerian Orbits**
- **Orbital Maneuvers**
- **Interplanetary Trajectories**

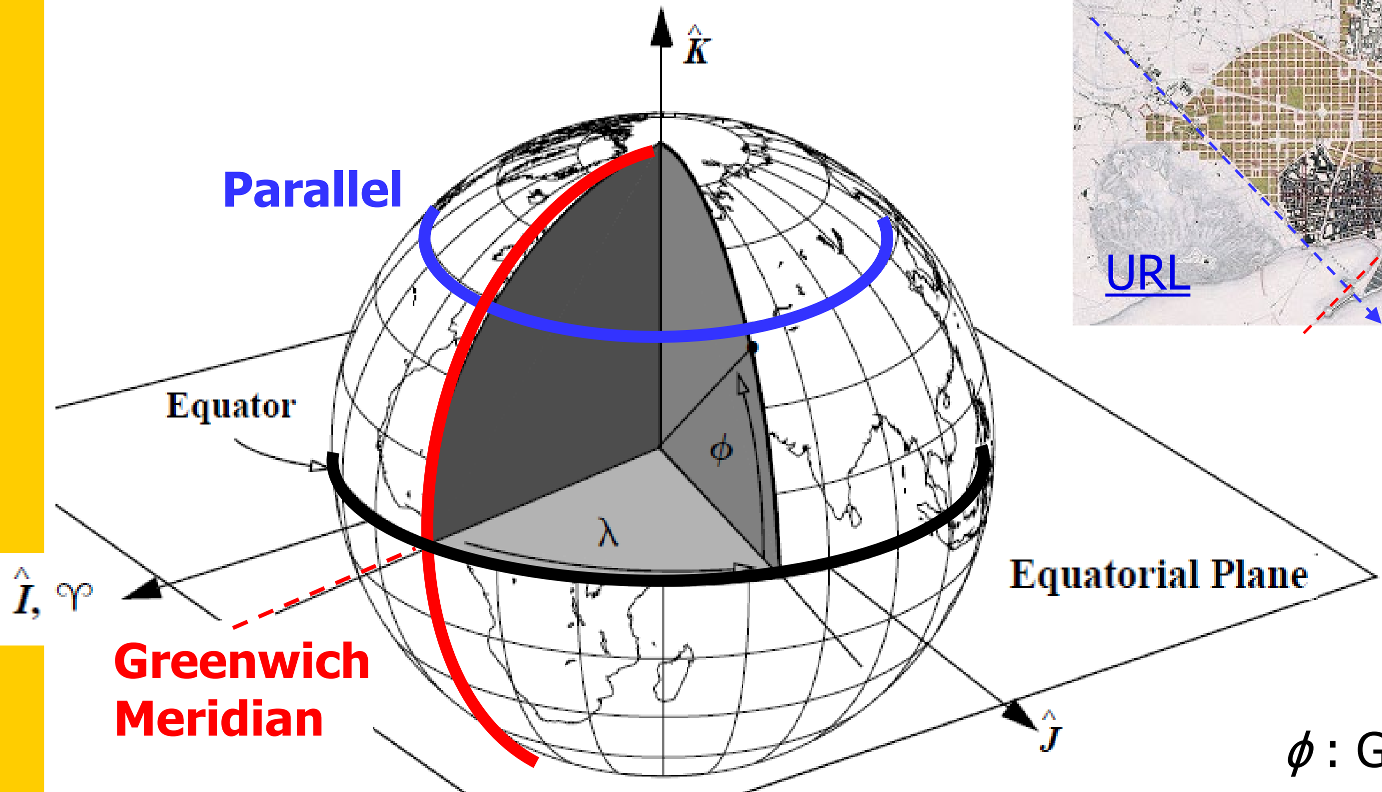
**Time Allocation:**

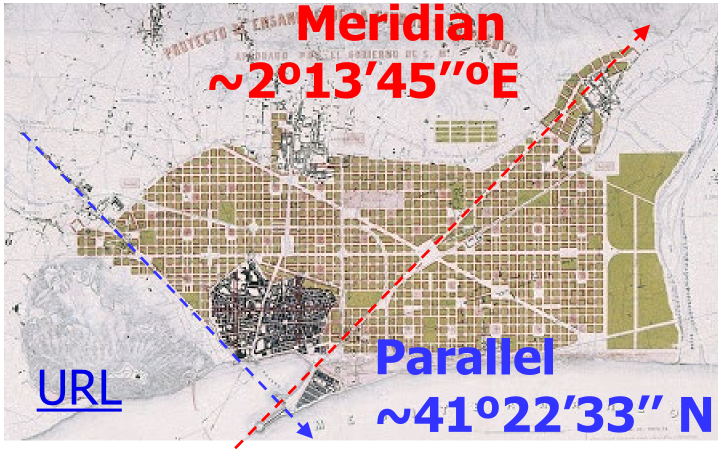
- **8 hours**

# Geocentric Equatorial Coordinate System

**Origin:** center of the Earth
**Rotation:** only around the solar system barycenter and Earth-Moon barycenter
**Notation:** index IJK
**Fundamental plane:** Earth equator
**Axis Orientation:**

- I points towards the (precessing) vernal equinox
- K Earth rotation axis through the North Pole
- J points 90° to the east in the equatorial plane




λ: Longitude
$\phi$ : Geocentric latitude

Figure 3-11 Vallado DA (2013) "Fundamentals of Astrodynamics and Applications" 4th Edition, Space Technology Library
Dèlambre M (1792) "[Base du système métrique décimal ou Mesure de l'arc du méridien compris entre les parallèles de Dunkerque et Barcelone]"

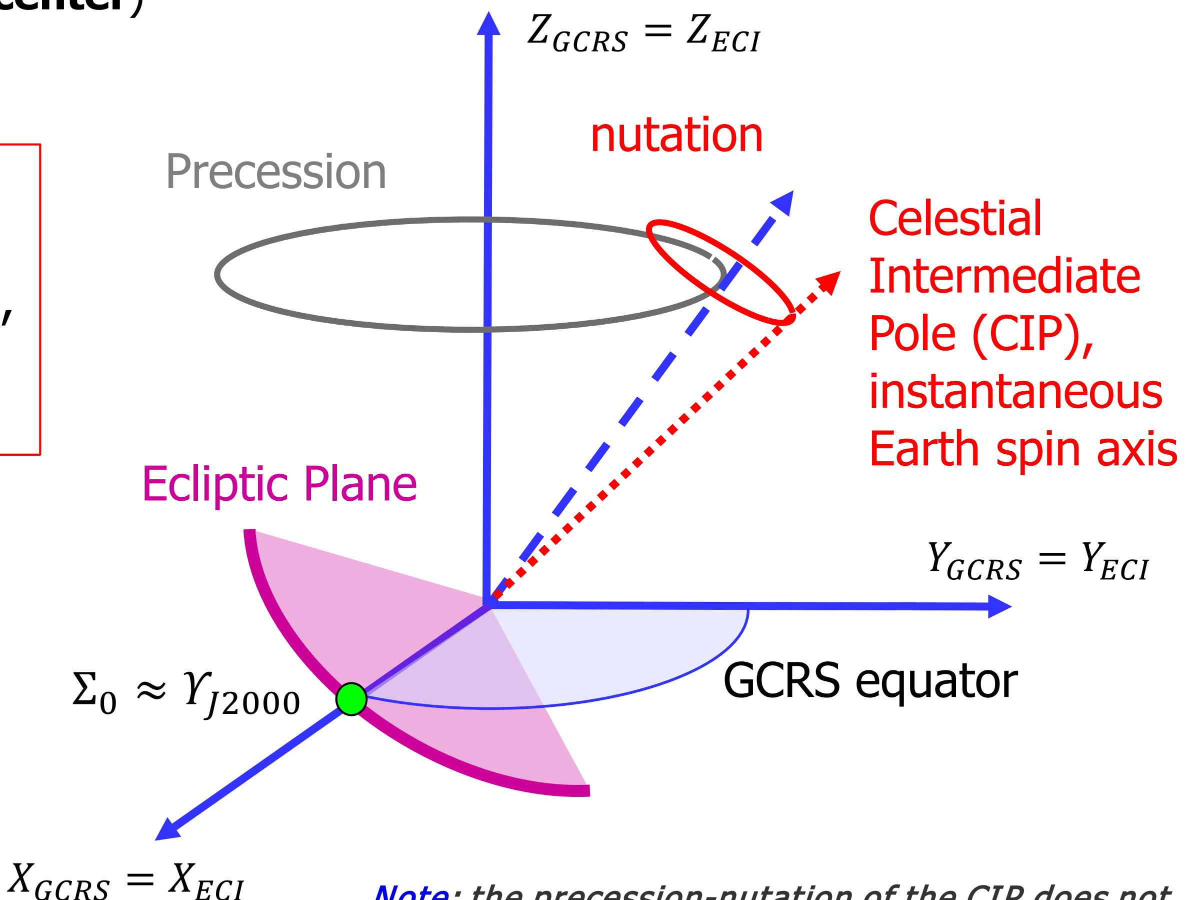
# Geocentric Celestial Reference System (GCRS)

**Origin:** center of the Earth
**Rotation:** only around the solar system barycenter and Earth-Moon barycenter

**Standard Earth Centered Inertial (ECI) coordinate system**

**Fundamental plane and axis orientation:**
- Same as ICRS (**SS barycenter**)

Precession and Nutation transformation is based in analytical expressions, which are valid for long time intervals




Adopted as celestial reference frame for the IERS since 1/1/1997
New designation of Conventional Terrestrial Reference System (CTRS)
IERS TN#36

*Note: the precession-nutation of the CIP does not have spectral components smaller than 2 days.*

# International Terrestrial Reference System (ITRS)

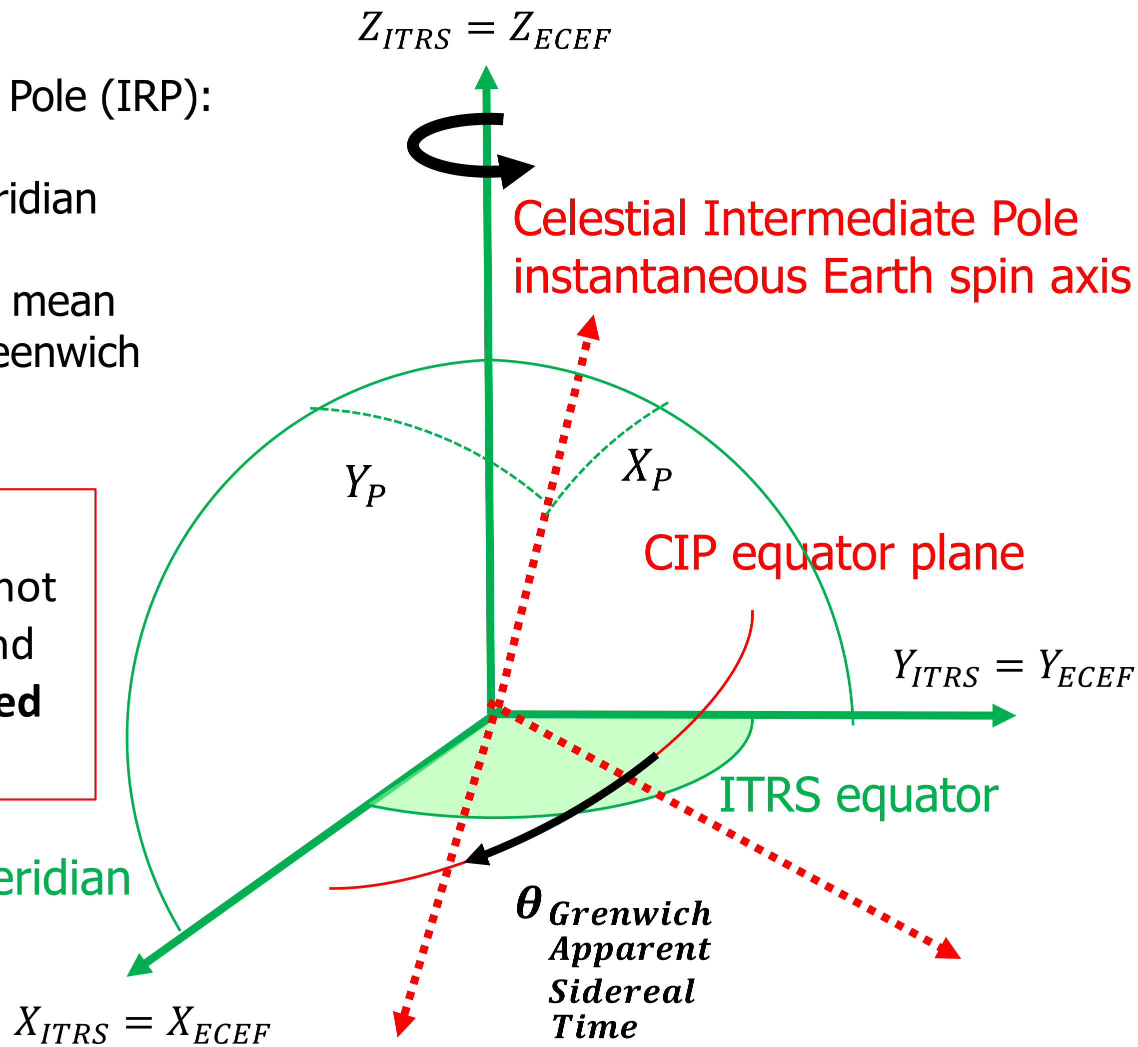
**Origin:** Geocenter (center of mass of the whole Earth, including oceans and atmosphere)
**Rotation: Earth,** around the solar system barycenter and Earth-Moon barycenter

**ITRS is the standard Earth Centered Earth Fixed (ECEF) System**

**Axis Orientation:**

- **Z-axis:** the IERS Reference Pole (IRP): the Earth rotational pole.
- **X-axis:** IERS Reference Meridian (IRM): Greenwich meridian
- First alignment: close to the mean equator of 1900 and the Greenwich meridian.

the Earth's rotation and orientation **parameters** cannot be modelled analytically and must be periodically **updated using observations**



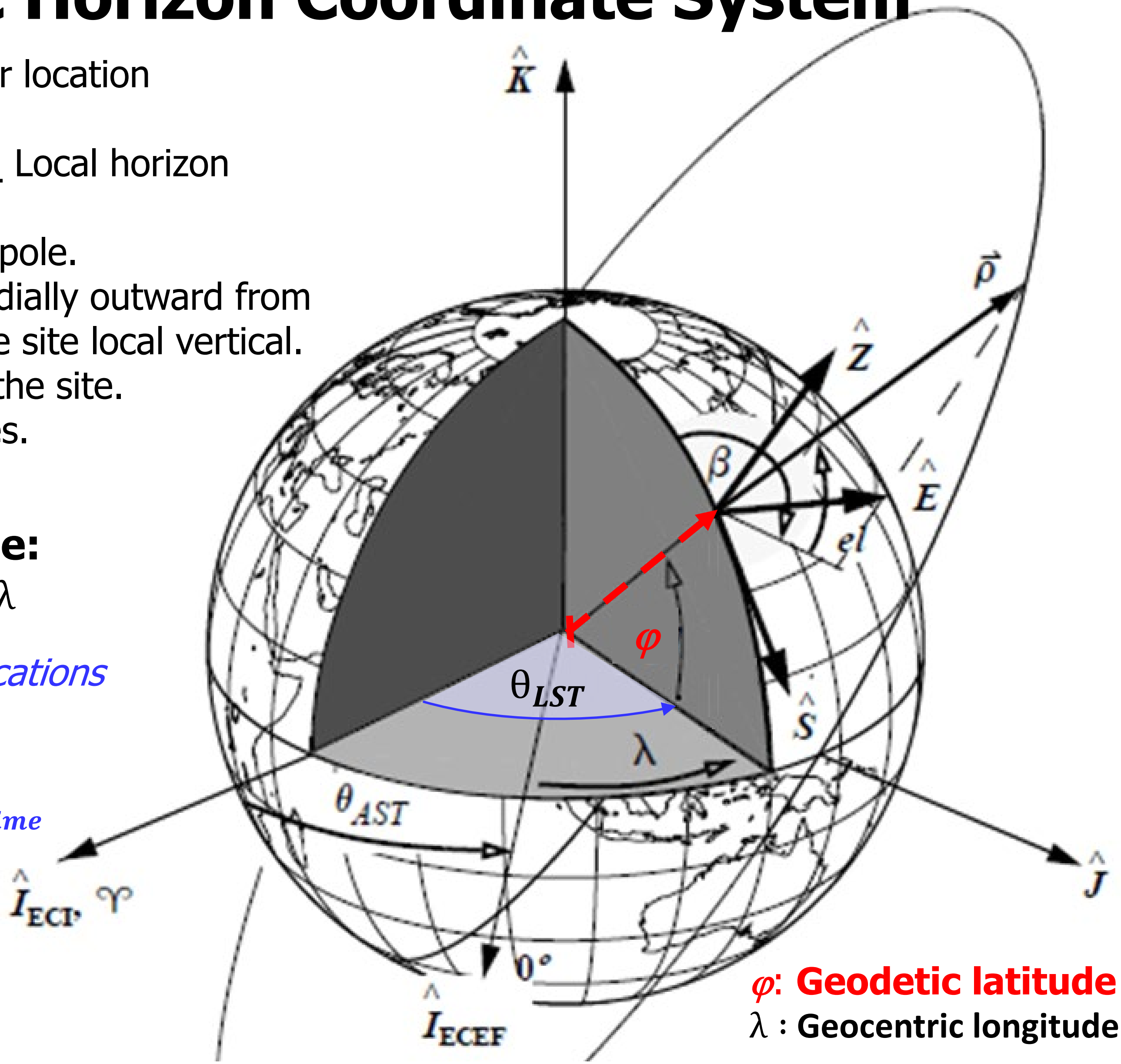



Pisacane VL (2005) Fundamentals of space systems. 2nd ed. Oxford: Oxford University Press

# Topocentric Horizon Coordinate System

**Origin:** observing/user location
**Notation:** SEZ
**Fundamental Plane:** Local horizon
**Axis Orientation:**

- S points due south pole.
- Z (zenith) points radially outward from the site along the site local vertical.
- E points east from the site. Undefined at Poles.




**Local Sidereal Time:**

$$\theta_{LST} = \theta_{GMST} + \lambda$$

*Note: For precise applications*
$\theta_{Apparent\ Sidereal\ Time}$
*replaces*
$\theta_{Grenwich\ Mean\ Sidereal\ Time}$

$\varphi$: **Geodetic latitude**
$\lambda$ : **Geocentric longitude**

$\beta$: ***azimuth,*** the angle measured from north, clockwise to the location beneath the object of interest.
***el: elevation***, measured from the local horizon, positive up to the object of interest

Fig 3-12 Vallado DA (2013) "Fundamentals of Astrodynamics and Applications" 4th Ed. Space Technology Library

Escola Superior d'Enginyeries Industrial, Aeroespacial i Audiovisual de Terrassa (ESEIAAT)

# Module 3 Requirements

**Objectives:**

- **Reference Systems: Coordinates and Time**
  - **Interplanetary systems**
  - **Earth-based systems**
  - **Satellite-based systems**
- **Keplerian Orbits**
- **Orbital Maneuvers**
- **Interplanetary Trajectories**

**Time Allocation:**

- **8 hours**

Escola Superior d'Enginyeries Industrial, Aeroespacial i Audiovisual de Terrassa (ESEIAAT)

# Perifocal Coordinate System

**Origin:** Non-empty ellipse focus (e.g. Earth center)
**Notation:** PQW
**Fundamental Plane:** Satellite Orbit (PQ plane)
**Axis Orientation:**

P axis points towards orbit Periapsis
Q axis is 90° from the P axis in the direction of satellite motion
W axis is normal to the orbit plane

The PQW best suited for orbits with a well-defined eccentricity.
However, circular orbits do not have periapsis

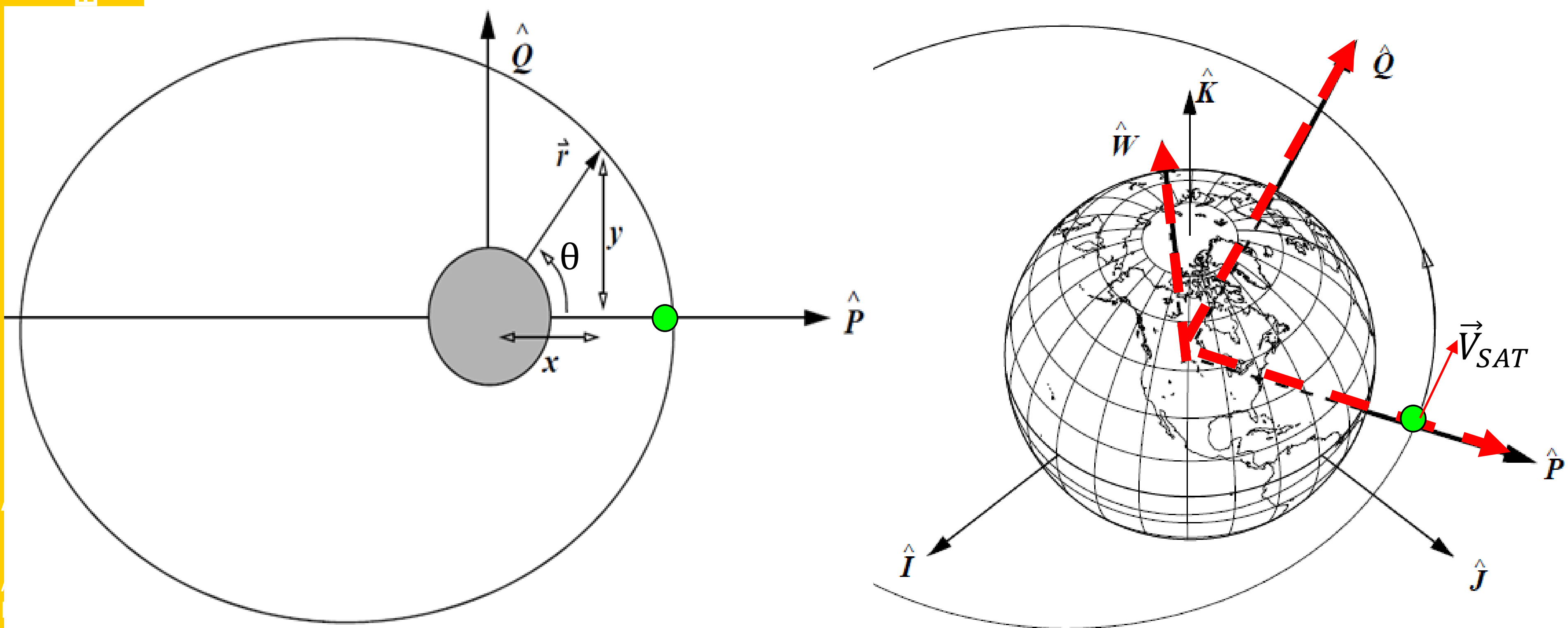
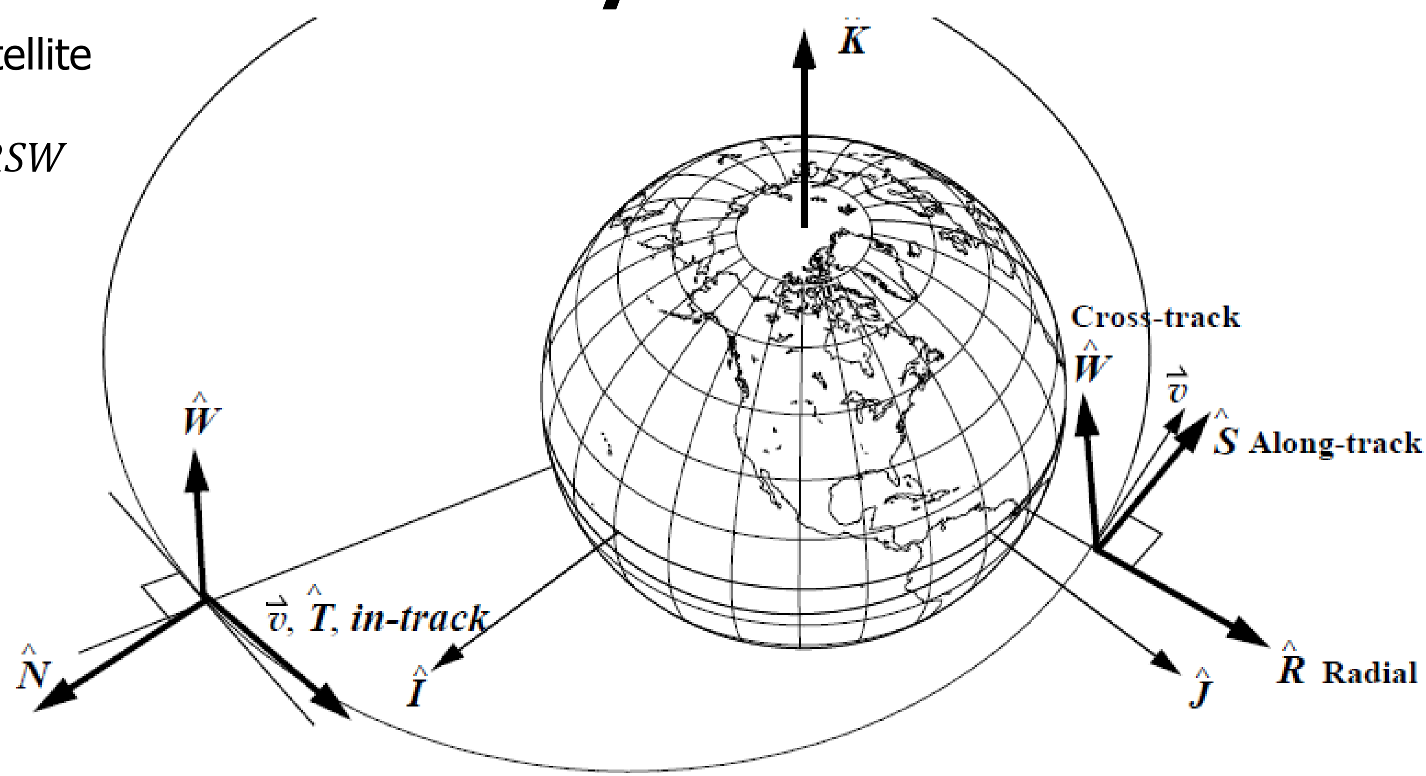



Figure 3-14 Vallado DA (2013) "Fundamentals of Astrodynamics and Applications" 4th Edition, Space Technology Library

# Satellite Coordinate Systems

**Origin:** Satellite
**Notation:**
$NTW$ and $RSW$




- **Normal** axis is normal to the velocity vector
  - Note: N only aligns with the radius:
    - circular orbits
    - elliptical orbits at both apses
- **In-track** axis is always parallel to velocity
- **Cross-track** axis is normal to the orbital plane (not usually aligned with the K axis)

- **Radial** axis points out from the satellite along the geocentric radius vector
- **Along-Track** axis is normal to the position vector and positive in the direction of the velocity vector.
  - Note: S axis only aligns with the velocity
    - circular orbits
    - elliptical orbits at both apses
- **Cross-track** axis is normal to the orbital plane (not usually aligned with the K axis)



Figure 3-15 Vallado DA (2013) "Fundamentals of Astrodynamics and Applications" 4th Edition, Space Technology Library

Escola Superior d'Enginyeries Industrial, Aeroespacial i Audiovisual de Terrassa (ESEIAAT)

# Module 3 Requirements

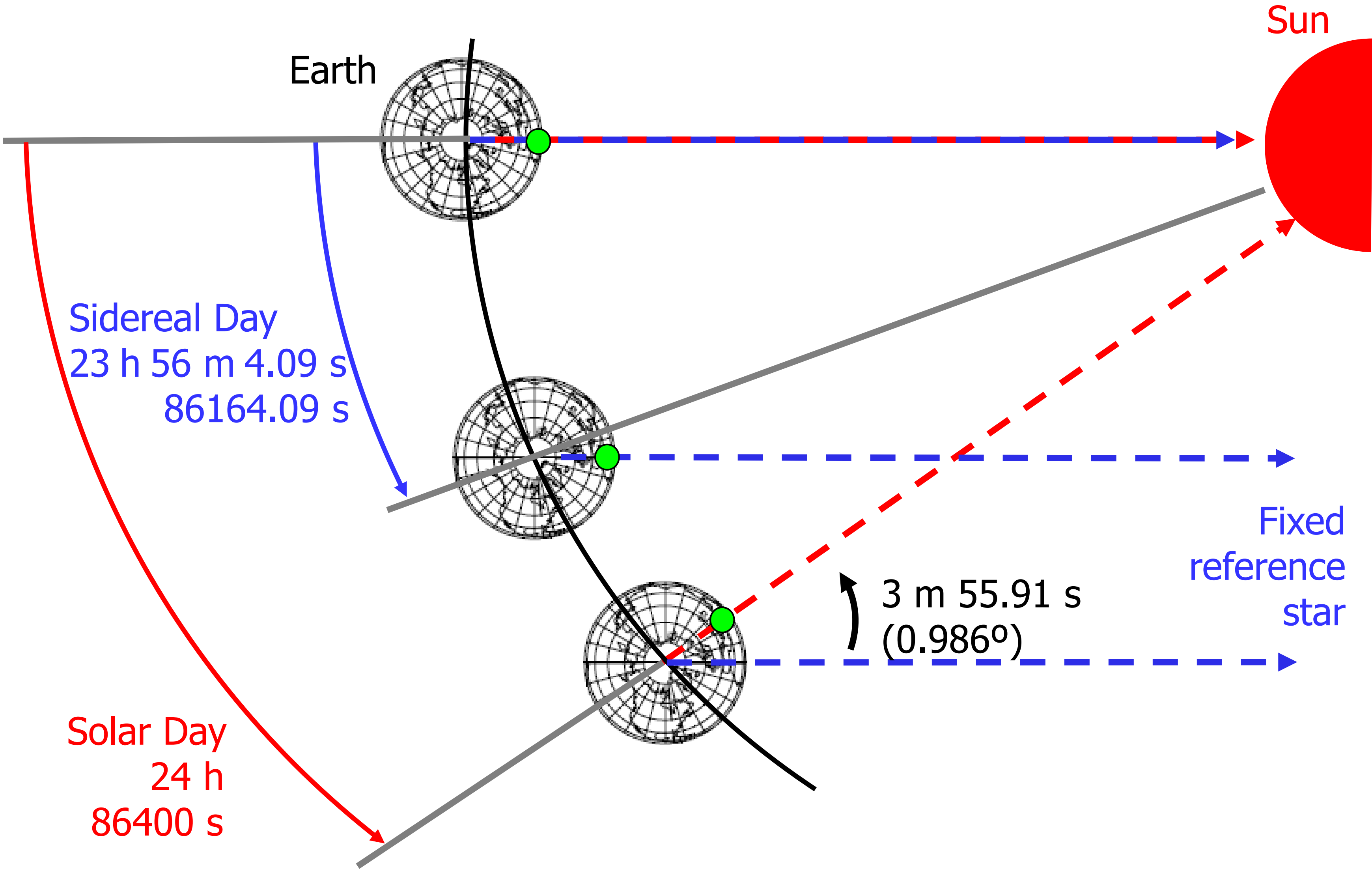
**Objectives:**

- **Reference Systems: Coordinates and Time**
- **Keplerian Orbits**
- **Orbital Maneuvers**
- **Interplanetary Trajectories**

**Time Allocation:**

- **8 hours**

# Solar vs Sidereal Time




$$\frac{\text{solar day}}{\text{sidereal day}} = \frac{86400}{86164.09} = 1.00273791$$

$$3 \text{ m } 55.91 \text{ s} \frac{360°}{23 \text{ h } 56 \text{ m } 4.09 \text{ s}} = 0.986°$$

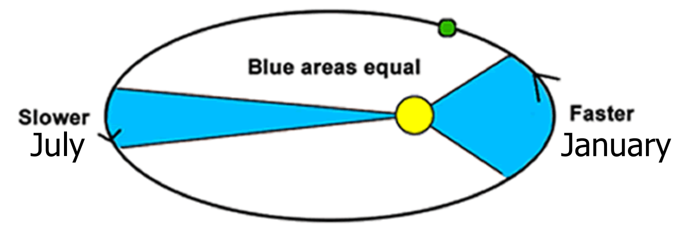
Figure 3-21 Vallado DA (2013) "Fundamentals of Astrodynamics and Applications" 4th Edition, Space Technology Library

Escola Superior d'Enginyeries Industrial, Aeroespacial i Audiovisual de Terrassa (ESEIAAT)

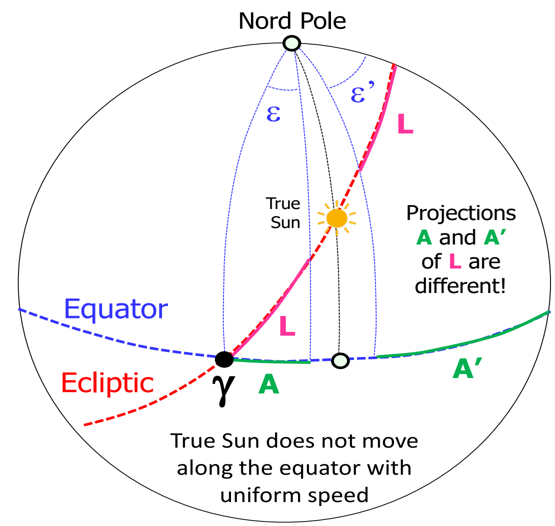
# Earth’s Rotation Times

Our time keeping was initially based on the **motion of the Sun**, but the way that this time flows is affected by two main causes:

- **The orbit of Earth is elliptic**:
  Thus, according to Kepler's second law, the orbital speed is not constant.




- **Earth's axis of rotation is not perpendicular** to the plane of Earth's orbit around the Sun (the ecliptic), hence the angular rate is not constant. It moves fastest at the end of December and slowest in mid-September.



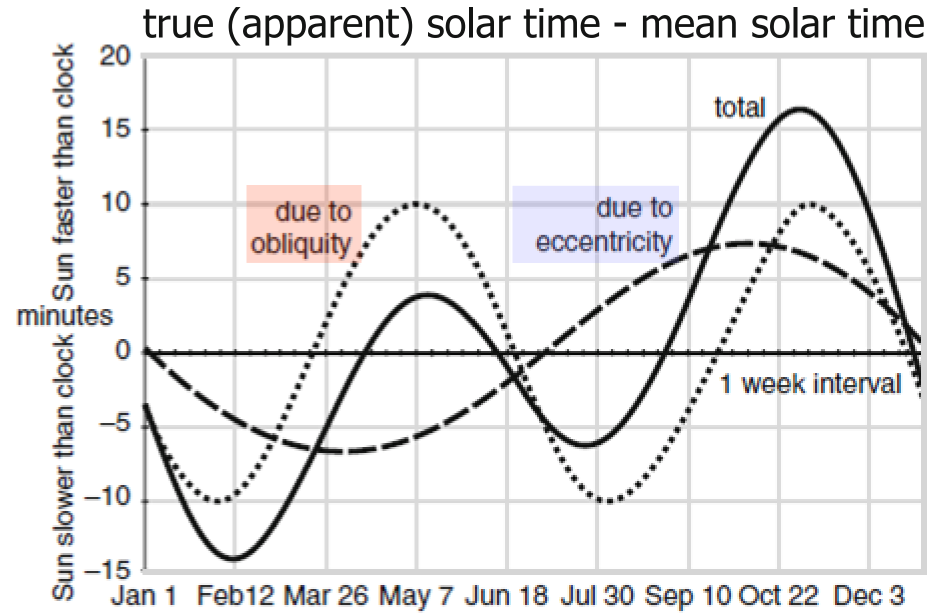

To get a more uniform time:
- we define a **fictitious mean Sun**, which moves along the (celestial) equator of Earth with uniform speed.
- **Mean Solar Time** as the hour angle of the mean **fictitious** Sun

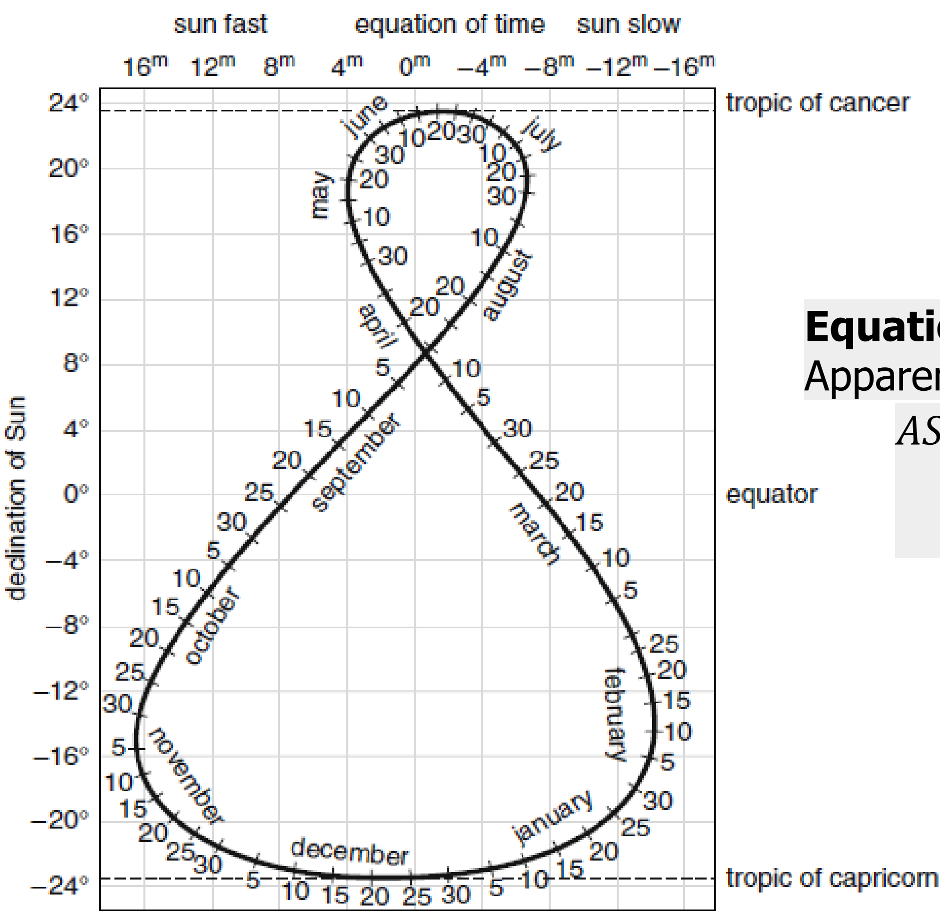
Universal Second = 1/86400 of one rotational day

**Source:** Sanz J, Juan JM, Rovira-Garcia A (2019) “GNSS Data Processing Lectures” ISBN 978-84-93230-7-6

# Equation of time

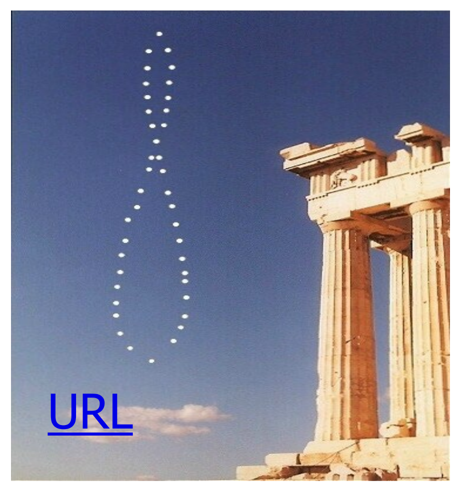
Obliquity of the ecliptic and eccentricity of the Earth orbit contributions for year 2000



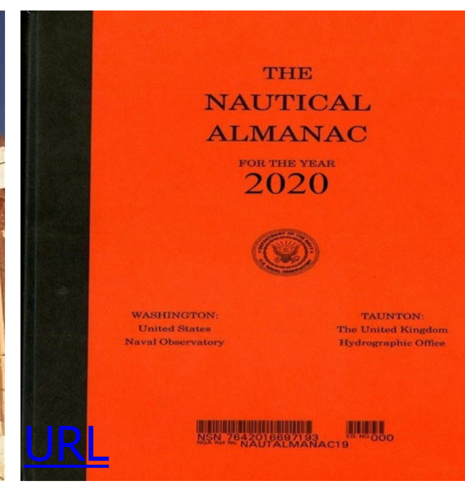




**Equation of time** is the difference between the true Apparent Solar Time (AST) and the Mean Solar Time (MST):

$$AST - MST = \color{red}{595^s \cdot \sin(198° + 1.9713° \cdot d)}$$
$$+ \color{blue}{442^s \cdot sin(175° + 0.9856° \cdot d)}$$
$$d = JD - JD(31.12.2019) = JD - 2458848.5$$



URL




URL

Analemmas are found on sundials to convert true (apparent) solar time to mean solar time.

**Source:** Figs 9.7/9.8 Soffel M, Langhans R (2013) "Space-Time Reference Systems" Astronomy and astrophysics library, Springer

# Julian Day (JD)

Calculations for long time intervals use Julian Date (after Julio Scalier).

Days are counted from 1st of January of 4713 BC in a correlative manner.

The Julian Day (JD) starts at noon 12 hours of the corresponding civil day (e.g. 6th January 1980 = JD 2 444 244.5).

The following expression gives the JD for a civil date (YY/MM/DD UT):

$$JD = int\,[365.25 \cdot y] + int\,[30.6001 \cdot (m+1)] + DD + \frac{UT(h)}{24} + 1\,720\,981.5$$

where:

$$\begin{cases} if\ MM \leq 2 & y = YY - 1 \ \ and \ \ m = MM + 12 \\ if\ MM > 2 & y = YY \qquad and \quad m = MM \end{cases}$$

The **Modified Julian Day (MJD)** is used by subtracting JD(17/11/1858 00h):

$$MJD = JD - 2\,400\,000.5$$

**Source:**

Sanz J, Juan JM, Hernández-Pajares M (2013) "GNSS Data Processing, Vol. 1: Fundamentals and Algorithms"

# Greenwich Sidereal Times (GSTs)

Greenwich Apparent Sidereal Time (GAST), "Apparent" means "True":
$\theta_{\mathrm{GAST}} = \theta_{\mathrm{GMST}} + \alpha_{\mathrm{E}}$

- $\theta_{\mathrm{GMST}}$ is the **Greenwich Mean Sidereal Time** (GMST), in seconds, computed as:
  $\theta_{\mathrm{GMST}} = 1.002737909350795 \cdot \mathrm{UT1} + \theta_{\mathrm{GMST}_0}$

  - $1.002737909350795$ converts from solar 24h to sidereal day 3m56s shorter $\approx 86400/86164$
  - $\mathrm{UT1} = \mathrm{UTC} + \Delta(\mathrm{UT1} - \mathrm{UTC})$ ($\Delta$ from the IERS Bulletin A)

  - $\theta_{\mathrm{GMST}_0}$ is the GMST, in seconds, at 00h00 UT computed as:
    $\theta_{\mathrm{GMST}_0} = 6\mathrm{h}41\mathrm{m}50\mathrm{s}54841 + 8640184\mathrm{s}812866 \cdot \mathrm{T_{JC}} + 0\mathrm{s}093104 \cdot \mathrm{T_{JC}^2} - 6\mathrm{s}2 \cdot 10^{-6} \cdot \mathrm{T_{JC}^3}$
    $\theta_{\mathrm{GMST}_0} = 24110.54841 + 8640184.812866 \cdot \mathrm{T_{JC}} + 0.093104 \cdot \mathrm{T_{JC}^2} - 6.2 \cdot 10^{-6} \cdot \mathrm{T_{JC}^3}$

    - where $\mathrm{T_{JC}}$ are the Julian Centuries:

$$\mathrm{T_{JC}} = \frac{\mathrm{JD(UT1)} - \mathrm{JD_{2000}}}{100 \cdot \mathrm{JY}} = \frac{\mathrm{JD(UT1)} - 2451545.0}{36525}$$

  From which the **Local Mean Sidereal Time** can be derived as $\mathrm{LMST} = \mathrm{GMST} + \lambda_{\mathrm{observer}}$

  *Note: seconds to degrees conversion by* $\theta_{\mathrm{GMST}} = \frac{360°}{86400\ \mathrm{s}} \mathrm{mod}(\theta_{\mathrm{GMST}}, 86400)$

- $\alpha_{\mathrm{E}}$ is the equation of the equinoxes: the right ascension component of nutation:
  $\alpha_{\mathrm{E}} = arc\tan\left(\frac{N_{12}}{N_{11}}\right)$ where $N_{ij}$ are the elements of the matrix from IAU Nutation Model



**Source:** Soffel M, Langhans R (2013) "Space-Time Reference Systems" Astronomy and astrophysics library, Springer

# Module 3 Requirements

**Objectives:**

- **Reference Systems: Coordinates and Time**
- **Keplerian Orbits**
  - **Two body problem**
  - **Circles, Ellipses, Parabolas and Hyperbolas**
  - **Keplerian Orbital Elements**
- **Orbital Maneuvers**
- **Interplanetary Trajectories**

**Time Allocation:**

- **8 hours**

# Three laws of Kepler

**First Law:** If two objects in space interact gravitationally, each will describe an orbit that is a conic section with the center of mass at one focus.

If the bodies are permanently associated, their orbits will be ellipses; if not, their orbits will be hyperbolas.

**Second Law:** If two objects in space interact gravitationally (whether or not they move in closed elliptical orbits), a line joining them sweeps out equal areas in equal intervals of time.

**Third Law:** If two objects in space revolve around each other due to their mutual gravitational attraction, the sum of their masses multiplied by the square of their period of mutual revolution is proportional to the cube of the mean distance between them;

$$(m+M)T^2 = \frac{4\pi^2}{G}a^3$$

Where G is Newton's gravitational constant $\mathrm{G} = 6.673\ 10^{-11}\ \frac{m^3}{kg \cdot s^2}$

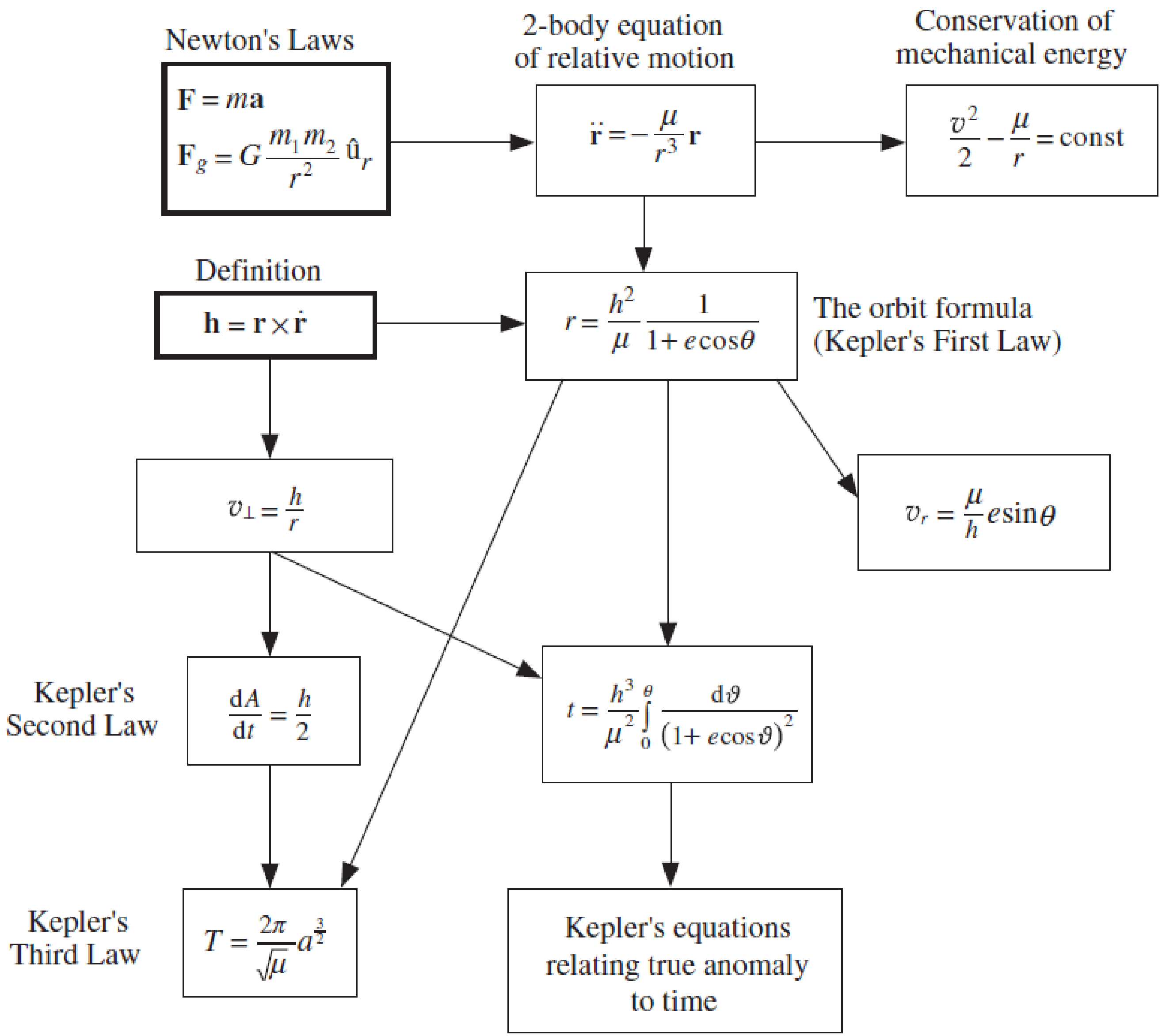
**Source:**
Wertz JR et al (2009) "Orbit and constellation design and management" Space Technology Series

Escola Superior d'Enginyeries Industrial, Aeroespacial i Audiovisual de Terrassa (ESEIAAT)

AMVO

Adrià Rovira-Garcia

# Keplerian Orbits: Layout




**Source:** Fig B.1 Curtis HD (2014) “Orbital Mechanics for Engineering Students” (3rd Edition) Elsevier.

# Orbit Equation (first law of Kepler)

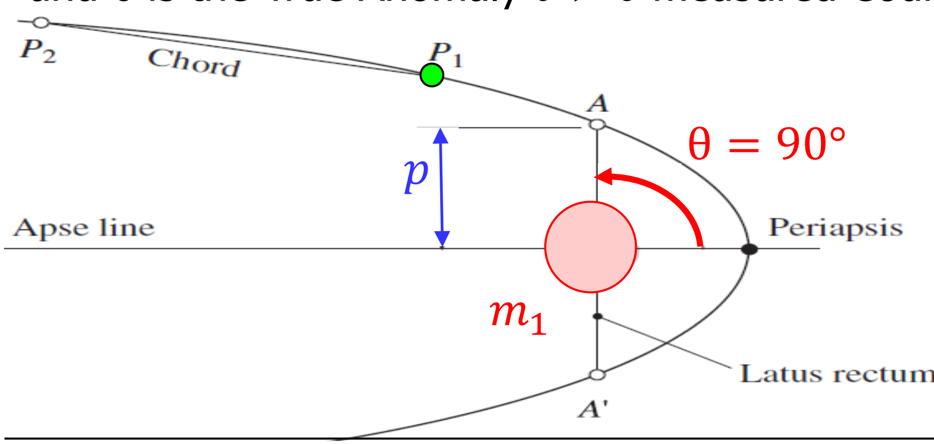
The path of the body $m_2$ around $m_1$, relative to $m_1$:

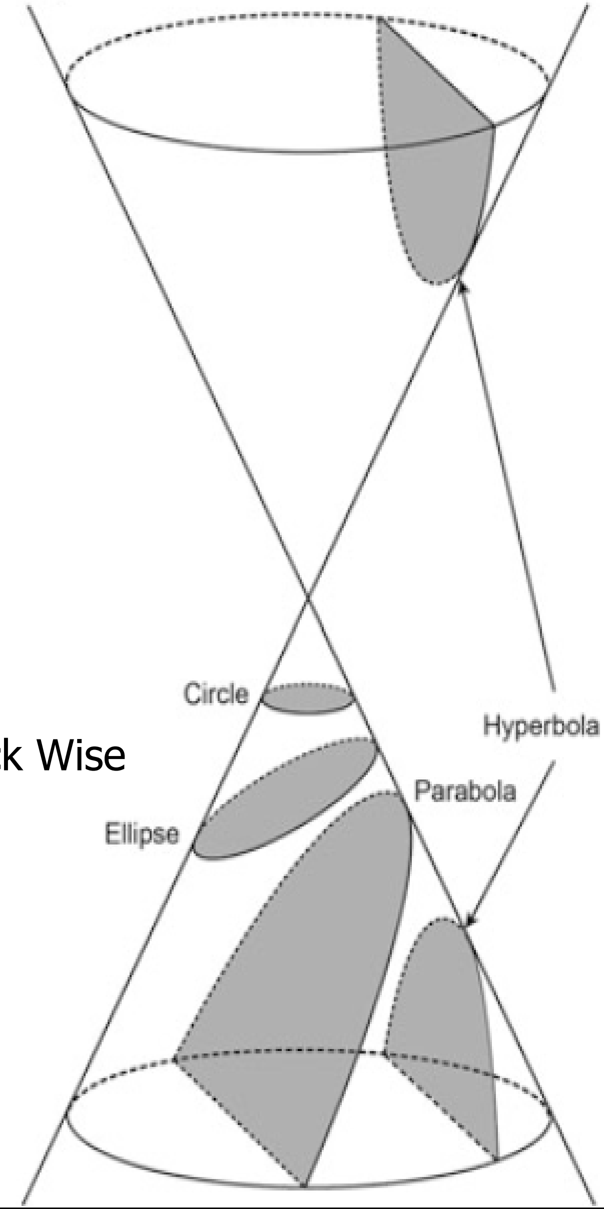
$$\boxed{r = \frac{h^2}{\mu}\frac{1}{1 + e \cdot \cos\theta} = \frac{p}{1 + e \cdot \cos\theta}}$$

where p, h, and e are constants:

- eccentricity $e \geq 0$
  - $e = 0$ circle
  - $0 < e < 1$ elipse
  - $e = 1$ parabola
  - $e > 1$ hyperbola
- Gravitational parameter $\mu = G(m_1 + m_2)$
- Angular momentum $h = |\mathbf{h}| = |\mathbf{r} \times \dot{\mathbf{r}}|$
- Orbit Parameter $p = \frac{h^2}{\mu}$ (semilatus rectum or width)

and θ is the True Anomaly $\theta > 0$ measured Counter Clock Wise




**Source:** Fig. 2.13 Curtis HD (2014) "Orbital Mechanics for Engineering Students" (3rd Edition) Elsevier.
Fig. 7.8 of Walter U (2019) "Astronautics. The Physics of Space Flight" (3rd Edition) Springer.

Escola Superior d'Enginyeries Industrial, Aeroespacial i Audiovisual de Terrassa (**ESEIAAT**)

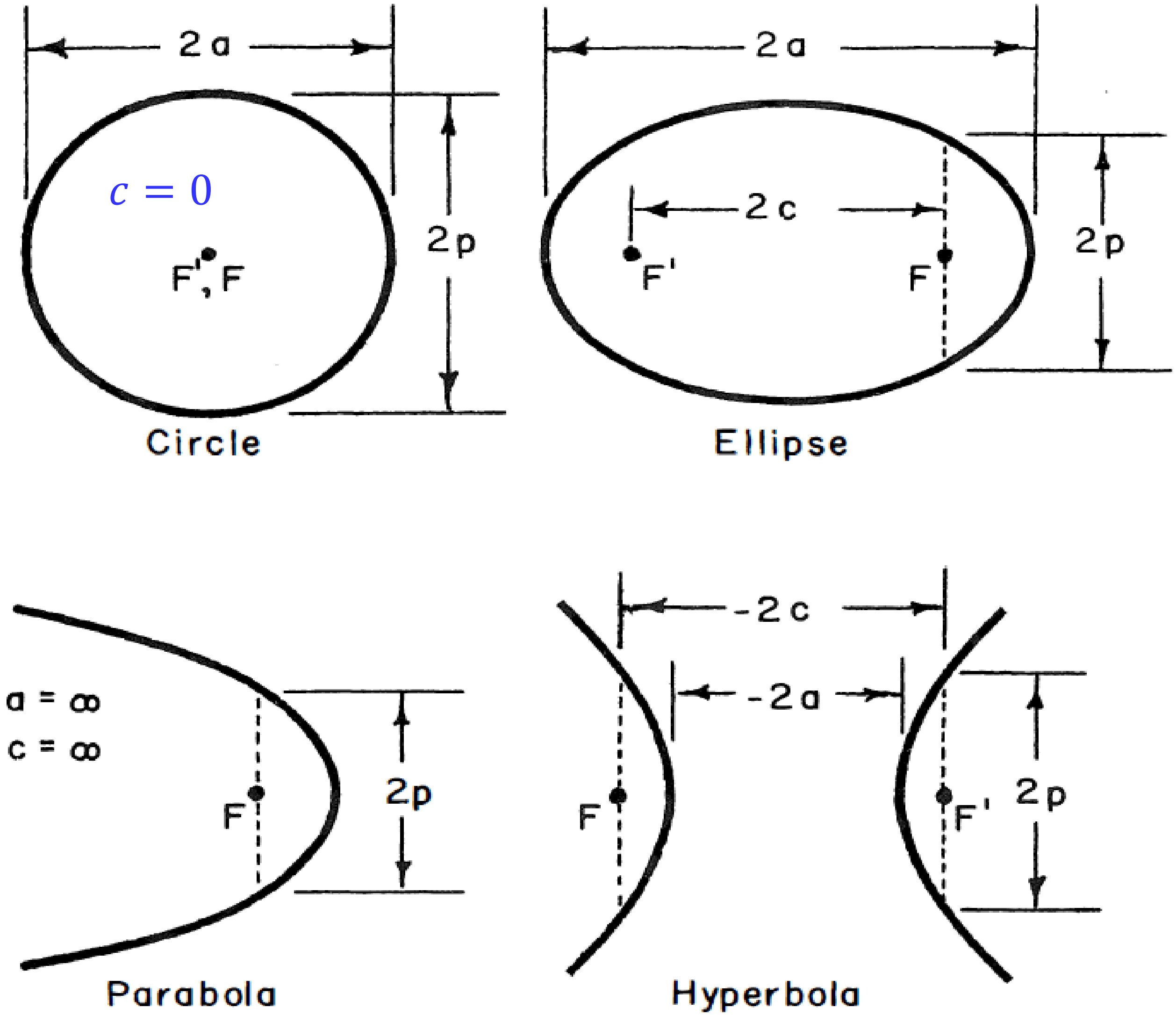
**AMVO**

Adrià Rovira-Garcia

# Reminder!

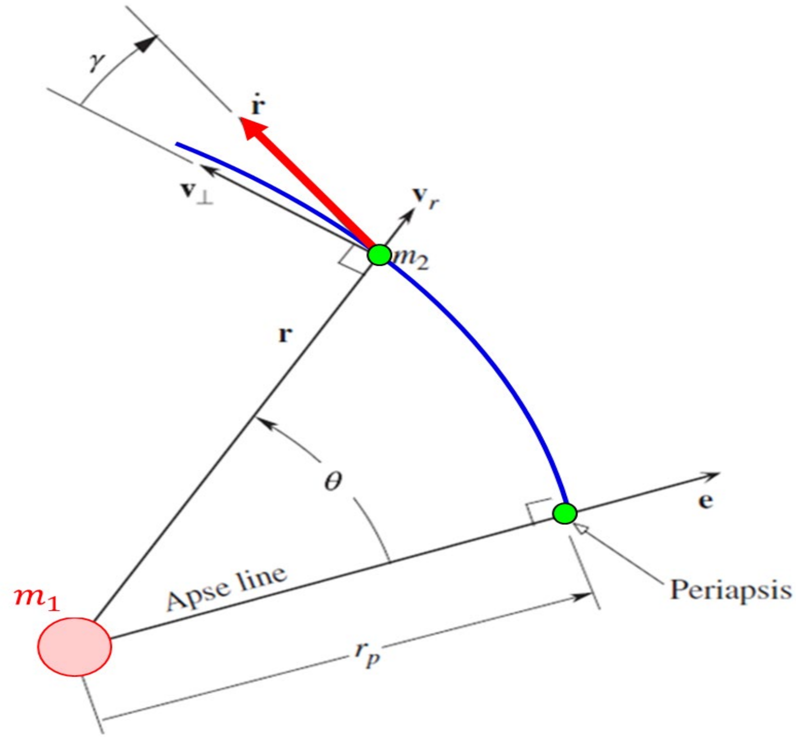
$$p = \frac{h^2}{\mu}$$

Geometrical dimensions common to all conic sections




**Source:** Fig. 1.5-3 Bate RR, Mueller DD, White JE (1976) "Fundamentals of Astrodynamics" Dover Publications

# Orbit Velocity




## Velocity components

Radial:

$$v_r = \frac{d}{\mathrm{dt}} r = \frac{\mu}{h} e\ sin\theta$$

Along-track:

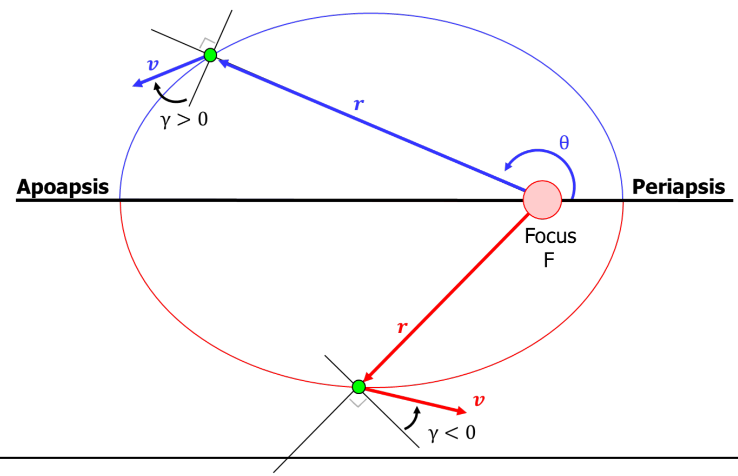
$$v_\perp = r\dot{\theta} = \frac{\mu}{h}(1 + e\ cos\theta)$$

since $h = r \cdot v_\perp = r \cdot (\dot{\theta} \cdot r)$

## Flight path angle (wrt local horizon):

$$\tan\gamma = \frac{v_r}{v_\perp} = \frac{e\ sin\theta}{1 + e\cos\theta} = \frac{e\ r}{p}\sin\theta$$

$$\gamma \begin{cases} > 0 & \text{when moving away from periapsis} \\ = 0 & \text{at apsides (periapsis and apoapsis)} \\ < 0 & \text{when moving towards periapsis} \end{cases}$$




**Source:** Fig 2.12 Curtis HD (2014) "Orbital Mechanics for Engineering Students" (3rd Edition) Elsevier.

# Orbit Velocity in different systems

**RSW:** Radial $\hat{\mathbf{u}}_\mathbf{r}$, Along-Track $\hat{\mathbf{u}}_\perp$, Cross-Track $\hat{\mathbf{u}}_\mathbf{w}$

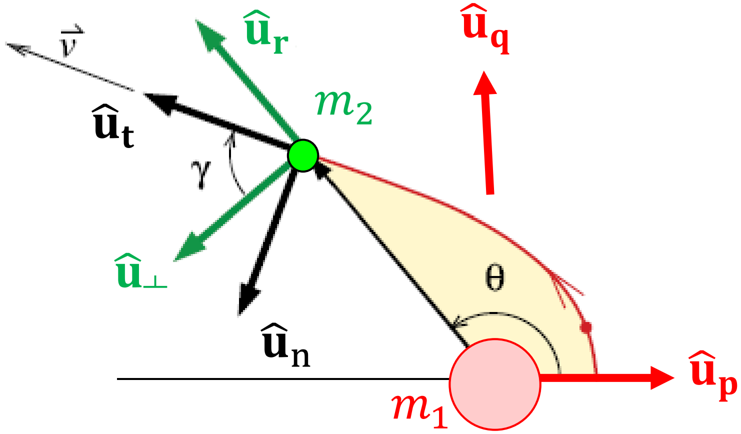
$$\dot{\mathbf{r}} = \dot{\mathrm{r}}\,\hat{\mathbf{u}}_\mathbf{r} + \frac{\mathrm{h}}{r}\hat{\mathbf{u}}_\perp = \frac{\mu}{\mathrm{h}}\begin{pmatrix} \mathrm{e}\cdot\sin\theta \\ 1+\mathrm{e}\cdot\cos\theta \\ 0 \end{pmatrix}_{RSW} = v\begin{pmatrix} \sin\gamma \\ \cos\gamma \\ 0 \end{pmatrix}_{RSW}$$

**NTW**: Normal $\hat{\mathbf{u}}_\mathbf{n}$, In-track $\hat{\mathbf{u}}_\mathbf{t}$, Cross-Track $\hat{\mathbf{u}}_\mathbf{w}$

$$\dot{\mathbf{r}} = \dot{\mathrm{r}}\,\hat{\mathbf{u}}_\mathbf{t} = \begin{pmatrix} 0 \\ v \\ 0 \end{pmatrix}_{NTW} = \begin{pmatrix} \cos\gamma & -\sin\gamma & 0 \\ \sin\gamma & \cos\gamma & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} v_\mathrm{r} \\ v_s \\ 0 \end{pmatrix}_{RSW}$$

**PQW:** Periapsis $\hat{\mathbf{u}}_\mathbf{p}$, Semi-latus rectum $\hat{\mathbf{u}}_\mathbf{q}$, Cross-Track $\hat{\mathbf{u}}_\mathbf{w}$

$$\dot{\mathbf{r}} = \dot{\mathrm{x}}\,\hat{\mathbf{u}}_\mathbf{p} + \dot{\mathrm{y}}\,\hat{\mathbf{u}}_\mathbf{q} == \frac{\mu}{\mathrm{h}}\begin{pmatrix} -\sin\theta \\ \mathrm{e}+\cos\theta \\ 0 \end{pmatrix}_{PQW}$$




**Source:** Fig 2.12 Curtis HD (2014) "Orbital Mechanics for Engineering Students" (3rd Edition) Elsevier.

Escola Superior d'Enginyeries Industrial, Aeroespacial i Audiovisual de Terrassa (ESEIAAT)

AMVO

Adrià Rovira-Garcia

# Module 3 Requirements

**Objectives:**

- **Reference Systems: Coordinates and Time**
- **Keplerian Orbits**
  - **Two body problem**
  - **Circles, Ellipses, Parabolas and Hyperbolas**
  - **Keplerian Orbital Elements**
- **Orbital Maneuvers**
- **Interplanetary Trajectories**

**Time Allocation:**

- **8 hours**



Adrià Rovira-Garcia

# Law of Areas (second law of Kepler)

**The angular momentum of $m_2$ relative to $m_1$ is conserved:**

$$\mathbf{H}_{m_2} = \mathbf{r} \times m_2\, \dot{\mathbf{r}}$$

$$\mathbf{h} = \frac{\mathbf{H}_{m_2}}{m_2} = \mathbf{r} \times \dot{\mathbf{r}}$$

$$\frac{d\mathbf{h}}{dt} = \dot{\mathbf{r}} \times \dot{\mathbf{r}} + \mathbf{r} \times \ddot{\mathbf{r}} = 0 + \mathbf{r} \times \left(-\frac{G m_1}{r^3}\boldsymbol{r}\right) = 0$$

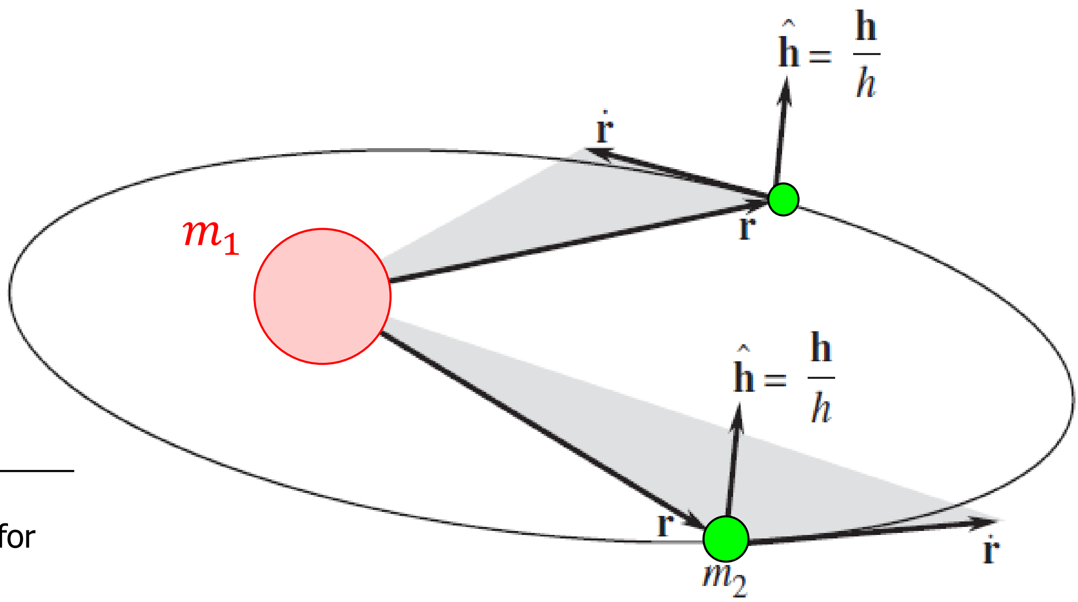
$$G\frac{m_1 m_2}{r^3}\mathbf{r} = m_2 \ddot{\mathbf{r}}$$

Therefore, $\mathbf{h} = \mathbf{r} \times \dot{\mathbf{r}} = \textbf{constant}$:

- Straight line (hence $\mathbf{h} = 0$)

or

- $\mathbf{r}$ and $\dot{\mathbf{r}}$ remain in the same plane

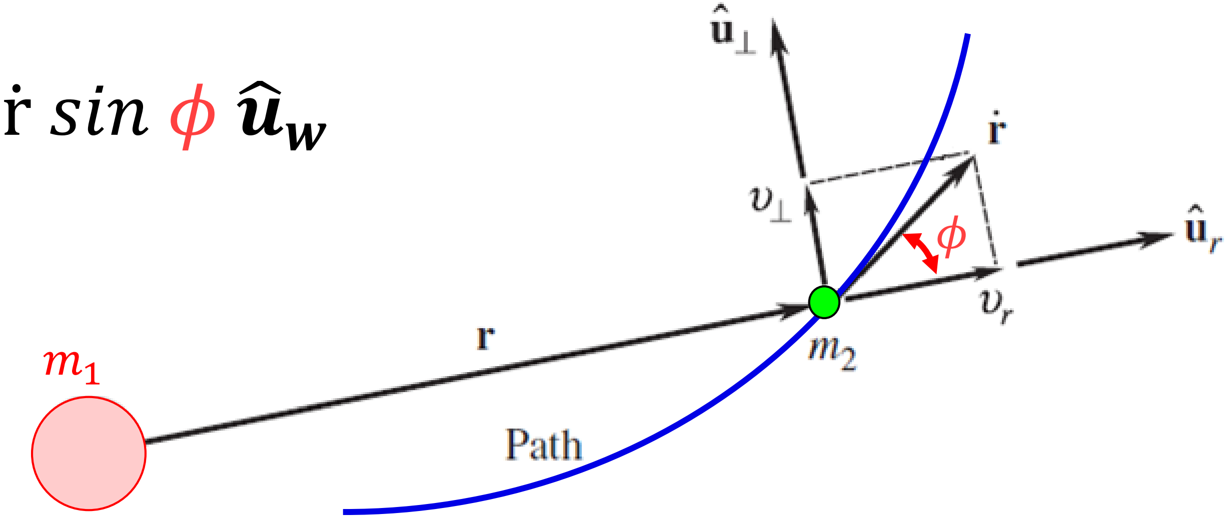



**Source:**
Fig 2.8 Curtis HD (2014) "Orbital Mechanics for Engineering Students" (3rd Edition) Elsevier.

# Law of Areas (second law of Kepler)

$$\mathbf{h} = \mathbf{r} \times \dot{\mathbf{r}} = \mathrm{r}\,\dot{\mathrm{r}}\,sin\,\phi\,\hat{\boldsymbol{u}}_w$$

| RSW | |
|---|---|
| $\hat{\mathbf{u}}_\mathbf{r}$ | Radial |
| $\hat{\mathbf{u}}_\perp$ | Along-Track |
| $\hat{\mathbf{u}}_\mathbf{w}$ | Cross-Track |

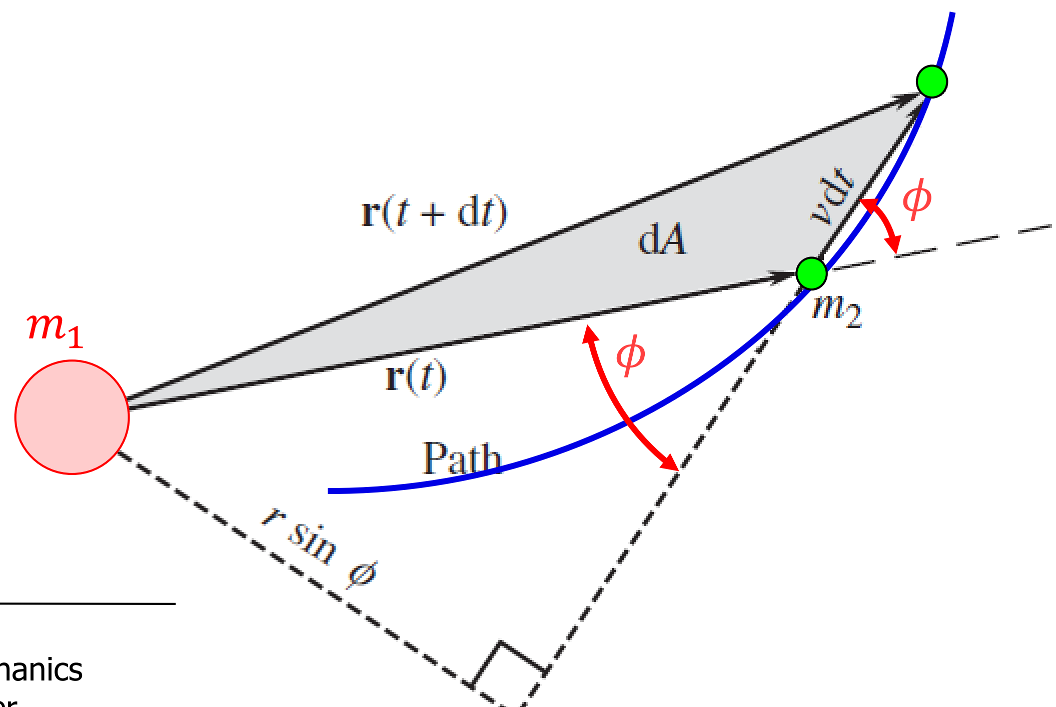



$$\mathrm{dA} = \frac{1}{2} \cdot base \cdot altitude$$

$$\mathrm{dA} = \frac{1}{2} \cdot r\ sin\ \phi \cdot \mathrm{v\ dt}$$

$$\frac{\mathrm{dA}}{dt} = \frac{1}{2} \cdot (\mathrm{r} \cdot v\ )sin\ \phi$$

$$\frac{\mathrm{dA}}{dt} = \frac{1}{2} h = constant$$




**Source:**
Figs 2.9 & 2.10 Curtis HD (2014) "Orbital Mechanics for Engineering Students" (3rd Edition) Elsevier.

# Time since periapsis

$$\boxed{\begin{aligned} r &= \frac{h^2}{\mu}\frac{1}{1+e\cdot\cos\theta} \\ h &= r\cdot v_\perp = r\cdot(r\cdot\dot{\theta}) \end{aligned}} \longrightarrow$$

$$h = \frac{h^4}{\mu^2}\frac{1}{(1+e\cdot\cos\theta)^2}\frac{d\theta}{dt}$$

$$\frac{\mu^2}{h^3}dt = \frac{d\theta}{(1+e\cdot\cos\theta)^2}$$

$$\frac{\mu^2}{h^3}\int_{t_p=0}^{t}dt = \int_0^{\theta}\frac{d\theta}{(1+e\cdot\cos\theta)^2}$$

$$\int\frac{dx}{(a+b\cdot\cos x)^2} \begin{cases} b<a & \dfrac{1}{(a^2-b^2)^{3/2}}\left(2a\tan^{-1}\sqrt{\dfrac{a-b}{a+b}}\tan\left(\dfrac{x}{2}\right)-\dfrac{b\sqrt{a^2-b^2}\sin x}{a+b\cdot\cos x}\right) \\ b=a & \dfrac{1}{a^2}\left(\dfrac{1}{2}\tan\left(\dfrac{x}{2}\right)+\dfrac{1}{6}\tan^3\left(\dfrac{x}{2}\right)\right) \\ b>a & \dfrac{1}{(b^2-a^2)^{3/2}}\left(\dfrac{b\sqrt{b^2-a^2}\sin x}{a+b\cdot\cos x}-a\ln\left(\dfrac{\sqrt{b+a}+\sqrt{b-a}\tan\left(\frac{x}{2}\right)}{\sqrt{b+a}-\sqrt{b-a}\tan\left(\frac{x}{2}\right)}\right)\right) \end{cases}$$

**Source:** Curtis HD (2014) "Orbital Mechanics for Engineering Students" (3rd Edition) Elsevier.

# Circular Orbit (e=0)

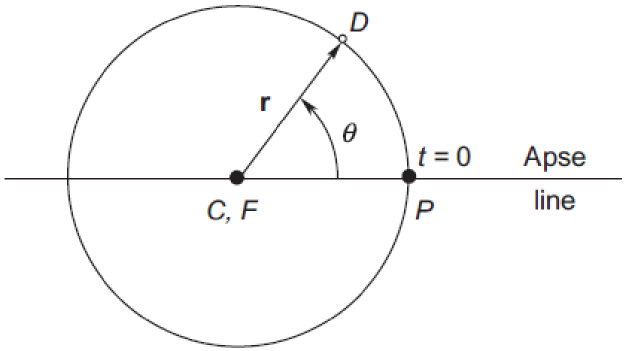
$$\mathrm{r} = \frac{h^2}{\mu}\frac{1}{1+\mathrm{e}\cdot\cos\theta} = \frac{h^2}{\mu} \longrightarrow h^2 = \mu\, r$$

$$\left.\begin{aligned} v_r &= \frac{d}{\mathrm{dt}} r = \frac{\mu}{h} e\, sin\theta = 0 \\ v_\perp &= r\dot{\theta} = \frac{\mu}{h}(1 + e\, cos\theta) = \frac{\mu}{\sqrt{\mu\, r}} = \sqrt{\frac{\mu}{r}} \end{aligned}\right\} \mathrm{v}_{circ} = \sqrt{\frac{\mu}{r}}$$

$$T = \frac{2\,\pi}{\omega} = \frac{2\,\pi}{v_\perp / r} = 2\,\pi\sqrt{\frac{r^3}{\mu}}$$

$$\varepsilon = \frac{\mathrm{v}^2}{2} - \frac{\mu}{r} = \frac{\mu}{r}\left(\frac{1}{2} - 1\right) = -\frac{1}{2}\frac{\mu}{r}$$

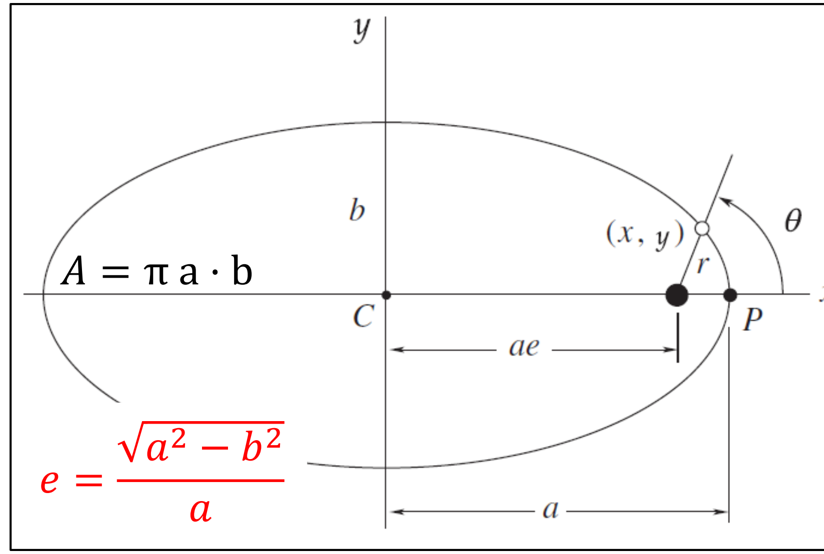
$$\frac{\mu^2}{\mathrm{h}^3}\int_{\mathrm{t_p}=0}^{\mathrm{t}} \mathrm{dt} = \int_0^{\theta} \frac{\mathrm{dx}}{(1+\mathrm{e}\cdot\cos\mathrm{x})^2} \longrightarrow \frac{\mu^2}{\mathrm{h}^3}\mathrm{t} = \theta$$




**Source:** Fig. 3.1 Curtis HD (2014) "Orbital Mechanics for Engineering Students" (3rd Edition) Elsevier.

# Elliptical Orbit (third law of Kepler)

$$r = \frac{h^2}{\mu}\frac{1}{1 + e \cdot \cos\theta} \begin{cases} \theta = 0^\circ & r_p = \dfrac{h^2}{\mu}\dfrac{1}{1+e} \\ \theta = 180^\circ & r_a = \dfrac{h^2}{\mu}\dfrac{1}{1-e} \end{cases}$$




$$r_p + r_a = 2a$$

$$\frac{h^2}{\mu}\frac{1}{1 - e^2} = a$$

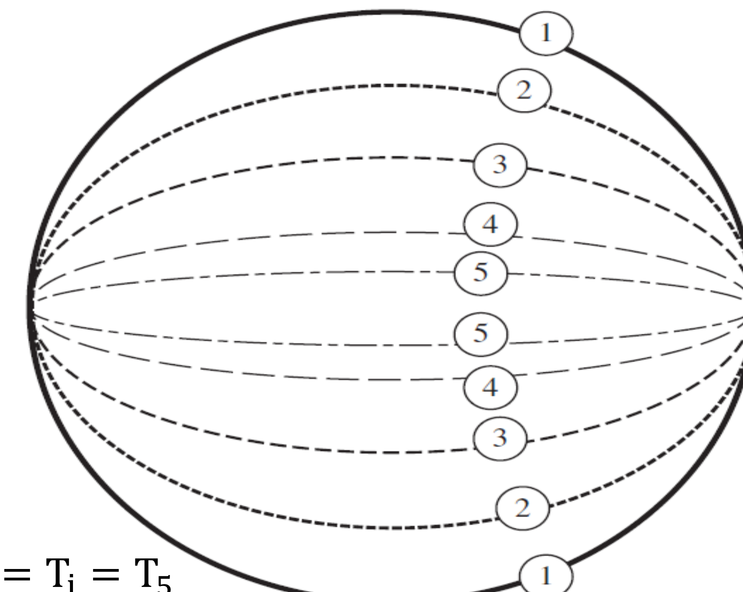
$$h^2 = \mu \cdot a \cdot (1 - e^2)$$

*Using the second Kepler law:*

$$\frac{dA}{dt} = \frac{1}{2}h$$

$$T = \frac{A}{\frac{1}{2}h} = \frac{\pi\, a \cdot b}{\frac{1}{2}h} = 2\pi \frac{a \cdot a\sqrt{1 - e^2}}{\sqrt{\mu \cdot a \cdot (1 - e^2)}}$$

$$\boxed{T = 2\pi\sqrt{\frac{a^3}{\mu}}}$$




$$T_1 = T_i = T_5$$

All five ellipses have the same major axis, then, their periods and energies are identical

**Source:** Figs 2.12 and 2.20 Curtis HD (2014) "Orbital Mechanics for Engineering Students" (3rd Edition) Elsevier.

Escola Superior d'Enginyeries Industrial, Aeroespacial i Audiovisual de Terrassa (**ESEIAAT**)

**AMVO**

Adrià Rovira-Garcia

# Elliptical Orbit (0<e<1)

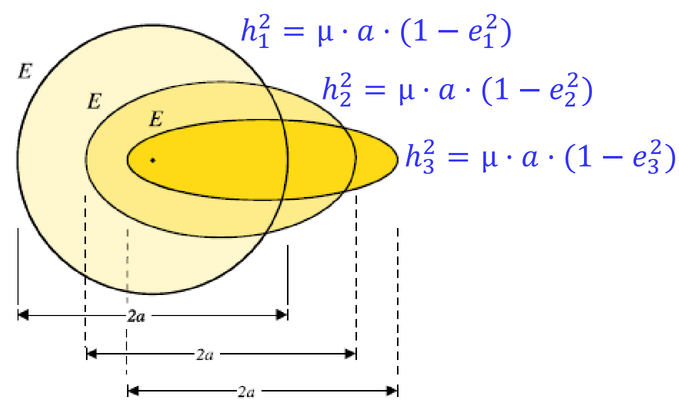
$$r = \frac{h^2}{\mu}\frac{1}{1 + e \cdot \cos\theta} \qquad h^2 = \mu \cdot a \cdot (1 - e^2)$$

$$v_r = \frac{d}{dt}r = \frac{\mu}{h} e\, sin\theta = \frac{\mu}{\sqrt{\mu \cdot a \cdot (1 - e^2)}} e\, sin\theta = \sqrt{\frac{\mu \cdot e^2}{a \cdot (1 - e^2)}}\, sin\theta$$

$$v_\perp = r\dot{\theta} = \frac{\mu}{h}(1 + e\, cos\theta) = \frac{\mu}{\sqrt{\mu \cdot a \cdot (1 - e^2)}}(1 + e\, cos\theta) = \sqrt{\frac{\mu}{a \cdot (1 - e^2)}}(1 + e\, cos\theta)$$

$$\varepsilon = \frac{v^2}{2} - \frac{\mu}{r}$$

$$\varepsilon_p = \frac{{v_p}^2}{2} - \frac{\mu}{r_p} = \frac{1}{2}\frac{\mu}{a \cdot (1 - e^2)} \cdot (1 + e)^2 - \frac{\mu}{\frac{h^2}{\mu}\frac{1}{1 + e}}$$




$$\varepsilon_p = \frac{1}{2}\frac{\mu}{a \cdot (1 - e^2)} \cdot (1 + e)^2 - \frac{\mu}{\frac{\mu \cdot a \cdot (1 - e^2)}{\mu}\frac{1}{1 + e}}$$

$$\varepsilon_p = \frac{\mu}{2\, a \cdot (1 - e^2)}(1 + e^2 + 2e - 2 - 2e) = \frac{\mu}{2\, a \cdot (1 - e^2)}(e^2 - 1) = -\frac{\mu}{2\, a}$$

**Source:** Curtis HD (2014) “Orbital Mechanics for Engineering Students” (3rd Edition) Elsevier.

# Elliptical Orbit (0<e<1)

$$\frac{\mu^2}{h^3}\int_{t_p=0}^{t} dt = \int_0^{\theta} \frac{dx}{(1+e\cdot\cos x)^2}$$

$$\frac{\mu^2}{h^3}t = \frac{1}{(1-e^2)^{3/2}}\left(2\tan^{-1}\left(\sqrt{\frac{1-e}{1+e}}\tan\left(\frac{\theta}{2}\right)\right) - \frac{e\sqrt{1-e^2}\sin\theta}{1+e\cdot\cos\theta}\right)$$

$$\frac{\mu^2}{h^3}t = \frac{1}{(1-e^2)^{3/2}}M$$

Where M is the **Mean anomaly M**, that can be related with the orbital period T as:

$$M = \frac{2\pi}{T}t = n\,t$$

That can be related with an **Eccentric anomaly $\boldsymbol{E}$** using Kepler equation:

$$M = E - e\sin E$$

and back to the **True anomaly $\boldsymbol{\theta}$**

$$\tan\left(\frac{\theta}{2}\right) = \sqrt{\frac{1+e}{1-e}}\tan\left(\frac{E}{2}\right)$$

*<u>Note</u>: anomaly is an angle that cannot be traced back to a true circular motion*

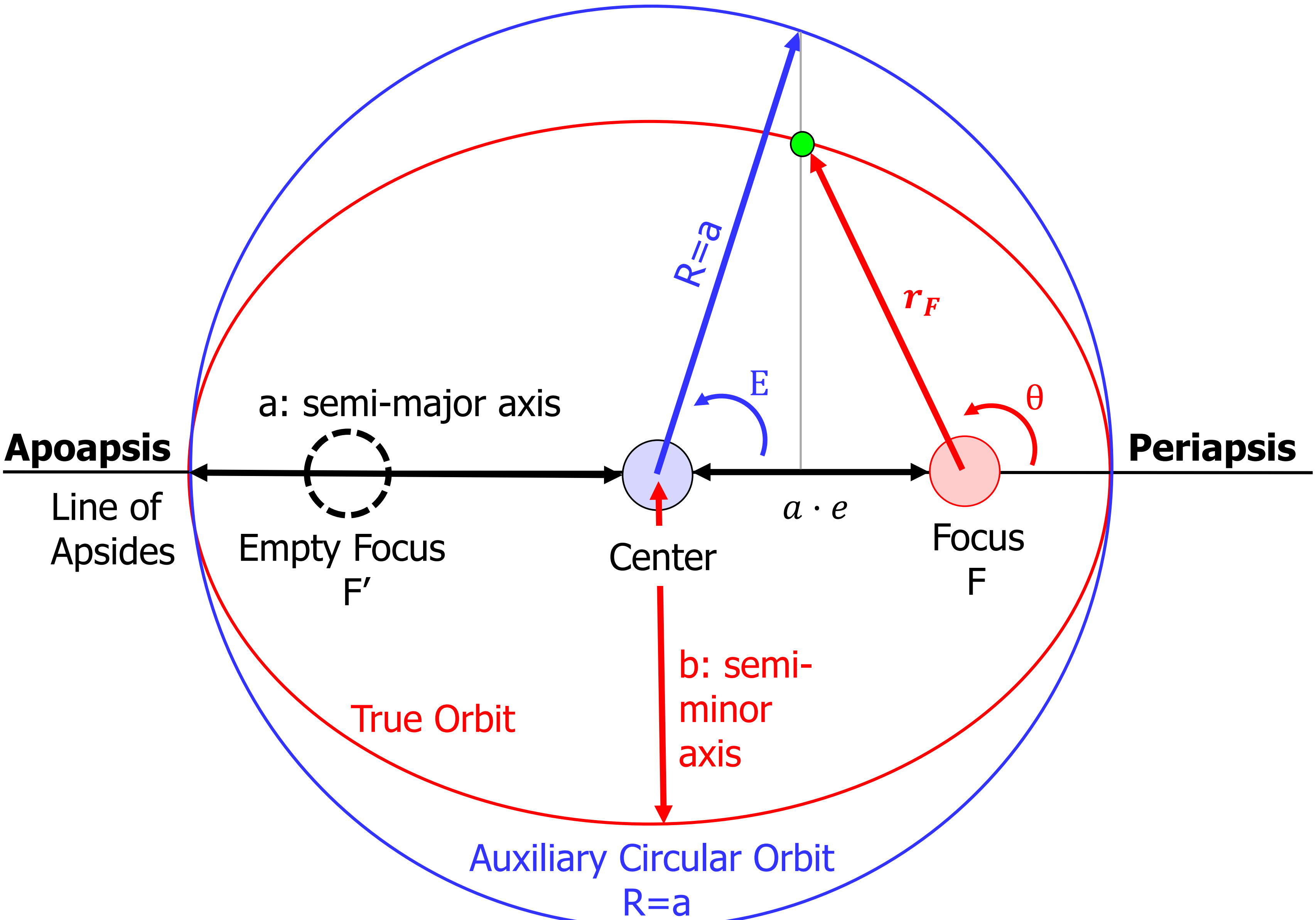
**Source:** Curtis HD (2014) "Orbital Mechanics for Engineering Students" (3rd Edition) Elsevier.

Escola Superior d'Enginyeries Industrial, Aeroespacial i Audiovisual de Terrassa (**ESEIAAT**)

**AMVO**

Adrià Rovira-Garcia

# Elliptical Orbit (0<e<1)




**Source:** Curtis HD (2014) "Orbital Mechanics for Engineering Students" (3rd Edition) Elsevier.

Escola Superior d'Enginyeries Industrial, Aeroespacial i Audiovisual de Terrassa (ESEIAAT)
AMVO
Adrià Rovira-Garcia

# Solving the Kepler Equation

Note that for ellipse orbits, typically we have the Mean Anomaly M instead of the True Anomaly θ

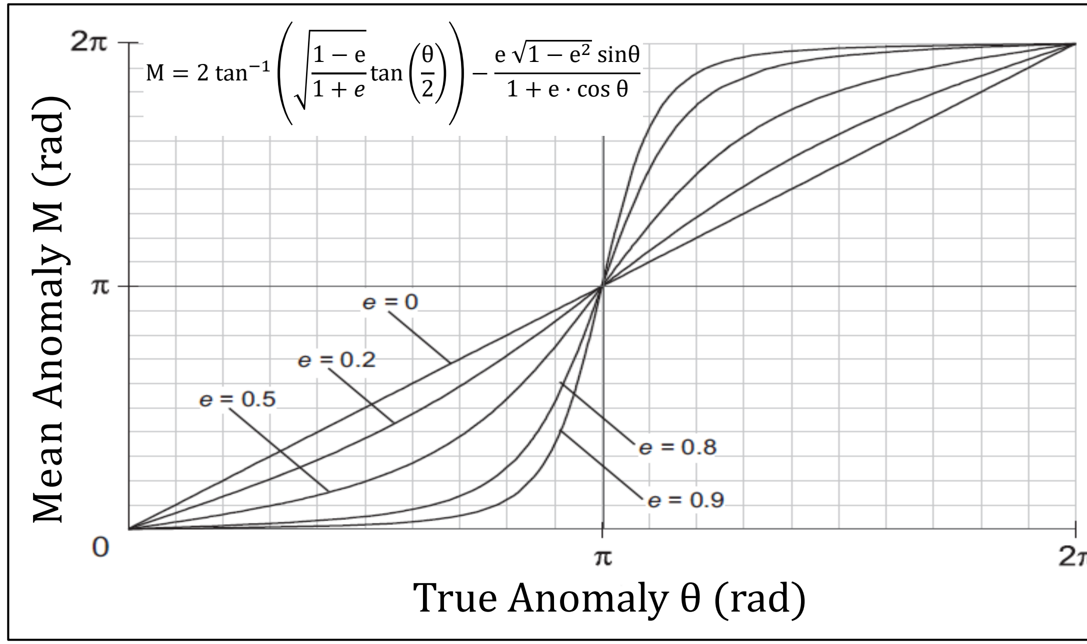
$$n = \frac{2\pi}{T} = \sqrt{\mu/a^3}$$

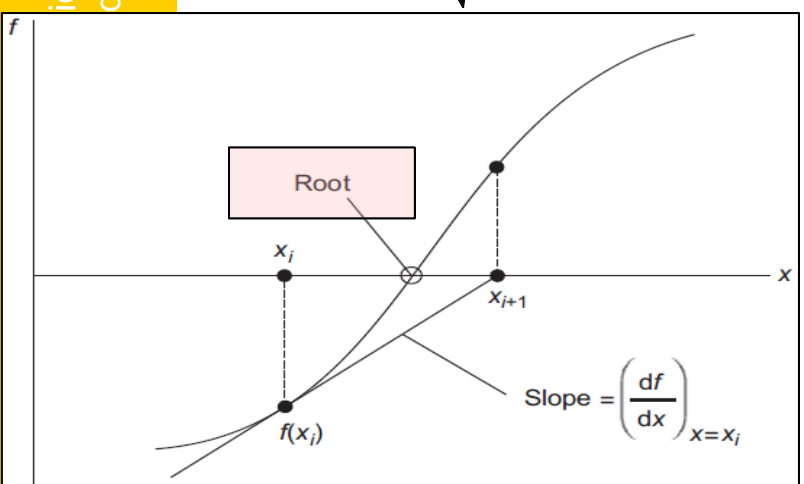
$$M = nt$$

$$M = E - e\sin E$$

$$f(E) = E - e\sin E - M$$
$$f'(E) = 1 - e\cos E$$

$$\tan\left(\frac{\theta}{2}\right) = \sqrt{\frac{1+e}{1-e}}\tan\left(\frac{E}{2}\right)$$




We can solve this (transcendental) equation using Newton iterative method

$$f'(x_i) = \frac{f(x_{i+1}) - f(x_i)}{x_{i+1} - x_i}$$

$$f'(x_i) = \frac{0 - f(x_i)}{x_{i+1} - x_i}$$

$$x_{i+1} = x_i - \frac{f(x_i)}{f'(x_i)} \longrightarrow E_{i+1} = E_i - \frac{E_i - e\sin E_i - M}{1 - e\cos E_i}$$



**Source:** Figures 3.6 and 3.7 of Curtis HD (2010) "Orbital Mechanics for Engineering Students" (3rd Edition) Elsevier.

# Parabolic Orbit (e=1)

$$r = \frac{h^2}{\mu}\frac{1}{1+\cos\theta} \longrightarrow r_a(\theta = 180°) = \infty \longrightarrow a = \infty$$

$$v_r = \frac{d}{dt}r = \frac{\mu}{h}e\ sin\theta = \frac{\mu}{h}sin\theta$$

$$v_\perp = r\dot{\theta} = \frac{\mu}{h}(1 + e\ cos\theta) = \frac{\mu}{h}(1 + cos\theta)$$

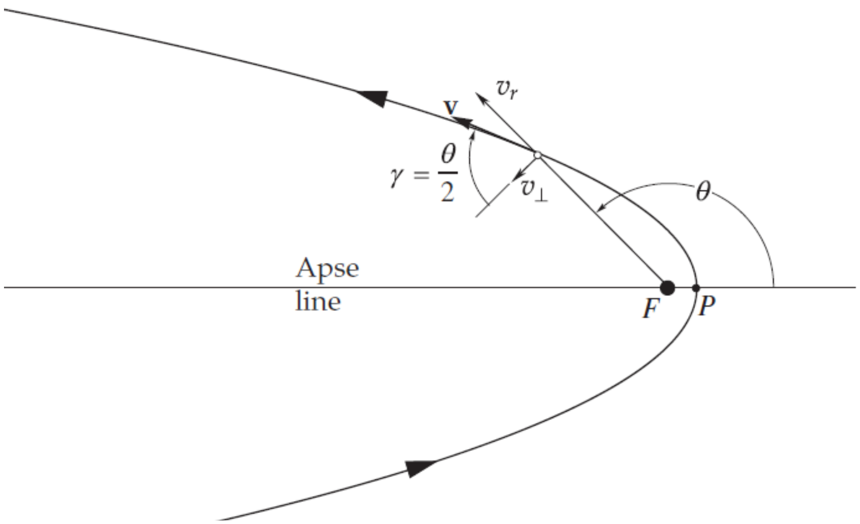
$$\longrightarrow \tan\gamma = \frac{v_r}{v_\perp} = \frac{sin\theta}{1+cos\theta} = \tan\frac{\theta}{2}$$

$$\varepsilon_p = \frac{{v_p}^2}{2} - \frac{\mu}{r_p} = \frac{1}{2}\left(\frac{\mu}{h}\cdot 2\right)^2 - \frac{\mu}{\frac{h^2}{\mu}\frac{1}{1+1}} = 2\left(\frac{\mu}{h}\right)^2 - 2\left(\frac{\mu}{h}\right)^2 = 0$$

$$\varepsilon_\infty = \frac{{v_\infty}^2}{2} - \frac{\mu}{r_\infty} \longrightarrow v_\infty = 0$$

$m_2$ will arrive to infinity with zero relative velocity to $m_1$, without returning.

Therefore, parabolic paths are called escape trajectories




$$v_{esc} = \sqrt{\frac{2\mu}{r}}$$

$$v_{esc} = \sqrt{2}\cdot v_{circ}$$

$$\text{Earth: } v_{esc} \approx 11.2\ km/s \approx 32.9\ M$$

**Source:** Fig 2.22 Curtis HD (2014) "Orbital Mechanics for Engineering Students" (3rd Edition) Elsevier.

# Parabolic Orbit (e=1)

$$\frac{\mu^2}{h^3}\int_{t_p=0}^{t} dt = \int_0^{\theta} \frac{dx}{(1+1\cdot\cos x)^2}$$

$$\frac{\mu^2}{h^3}t = \frac{1}{2}\tan\left(\frac{\theta}{2}\right) + \frac{1}{6}\tan^3\left(\frac{\theta}{2}\right)$$

$$\frac{\mu^2}{h^3}t = M$$

M is dimensionless, and it may be thought of as the "**mean anomaly**" for the parabola. M can be related with **True anomaly θ:**

$$M = \frac{1}{2}\tan\left(\frac{\theta}{2}\right) + \frac{1}{6}\tan^3\left(\frac{\theta}{2}\right)$$ (known as Baker's equation)

Solving the cubic equation

$$\frac{1}{2}\tan\left(\frac{\theta}{2}\right) + \frac{1}{6}\tan^3\left(\frac{\theta}{2}\right) - M = 0$$

We obtain the only real root:

$$\tan\left(\frac{\theta}{2}\right) = \left(3M + \sqrt{(3\,M)^2+1}\right)^{1/3} - \left(3M + \sqrt{(3\,M)^2+1}\right)^{-1/3}$$

**Source:** Curtis HD (2014) "Orbital Mechanics for Engineering Students" (3rd Edition) Elsevier.

# Hyperbolic Orbit (e>1)

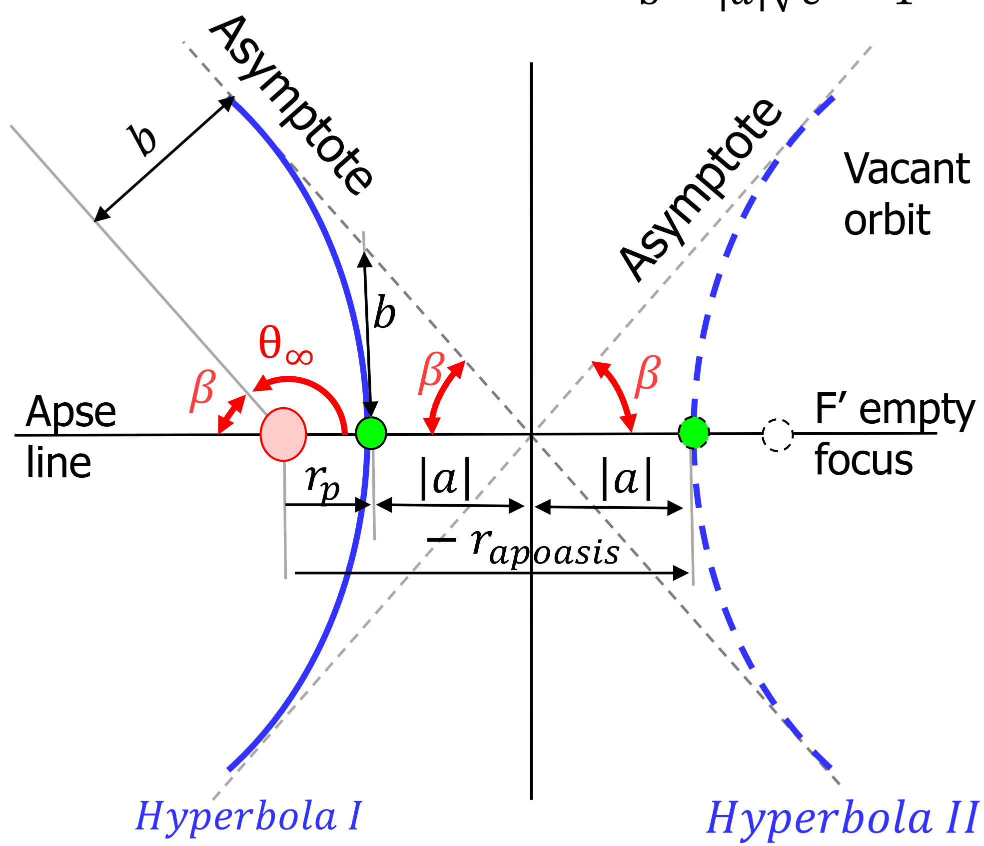
$$r = \frac{h^2}{\mu}\frac{1}{1+e\cos\theta} \longrightarrow \theta_\infty = \mathrm{acos}(-1/e) \longrightarrow r_\infty = \infty \longrightarrow \beta = 180° - \theta_\infty$$

$$\left.\begin{aligned} r_p &= \frac{h^2}{\mu}\frac{1}{1+e} \\ r_a &= \frac{h^2}{\mu}\frac{1}{1-e} \end{aligned}\right] \longrightarrow 2|a| = -r_a - r_p \longrightarrow |a| = \frac{h^2}{\mu}\frac{1}{e^2-1} \longrightarrow \left[\begin{aligned} r &= |a|\frac{e^2-1}{1+e\cos\theta} \\ e^2 &= 1 + \frac{h^2}{\mu\cdot|a|} > 1 \\ b &= |a|\sqrt{e^2-1} \end{aligned}\right.$$

$$\begin{aligned} \varepsilon_p &= \frac{{v_p}^2}{2} - \frac{\mu}{r_p} \\ &= \frac{1}{2}\frac{\mu^2}{h^2}(1+e)^2 - \frac{\mu}{\frac{h^2}{\mu}\frac{1}{1+e}} \\ &= \frac{1}{2}\frac{\mu^2}{h^2}(1+e)[(1+e)-2] \\ &= \frac{1}{2}\frac{\mu^2}{h^2}(e^2-1) = \frac{\mu}{2|a|} > 0 \end{aligned}$$

$$\varepsilon = \frac{v^2}{2} - \frac{\mu}{r} = \frac{\mu}{2|a|} \longrightarrow v_\infty = \sqrt{\frac{\mu}{|a|}}$$

$m_2$ will arrive to infinity with an excess velocity $v_\infty$




**Source:** Fig 2.25 Curtis HD (2014) "Orbital Mechanics for Engineering Students" (3rd Ed) Elsevier.

# Hyperbolic Orbit (e>1)

$$\frac{\mu^2}{h^3}\int_{t_p=0}^{t} dt = \int_0^{\theta} \frac{dx}{(1 + e \cdot \cos x)^2}$$

$$\frac{\mu^2}{h^3} t = \frac{1}{(e^2-1)^{3/2}}\left(\frac{e\sqrt{e^2-1}\sin\theta}{1+e\cdot\cos\theta} - \ln\left(\frac{\sqrt{e+1}+\sqrt{e-1}\tan\left(\frac{\theta}{2}\right)}{\sqrt{e+1}-\sqrt{e-1}\tan\left(\frac{\theta}{2}\right)}\right)\right)$$

$$\frac{\mu^2}{h^3}(e^2-1)^{3/2}\, t = \frac{e\sqrt{e^2-1}\sin\theta}{1+e\cdot\cos\theta} - \ln\left(\frac{\sqrt{e+1}+\sqrt{e-1}\tan\left(\frac{\theta}{2}\right)}{\sqrt{e+1}-\sqrt{e-1}\tan\left(\frac{\theta}{2}\right)}\right)$$

$$\frac{\mu^2}{h^3}(e^2-1)^{3/2}\, t = M$$

where M is the **Mean anomaly**, that can be related with an **Eccentric anomaly H:**

$$M = e \cdot \sinh H - H$$

and back to the **True anomaly $\theta$:**

$$\tan\left(\frac{\theta}{2}\right) = \frac{\sqrt{e+1}}{\sqrt{e-1}}\tanh\left(\frac{H}{2}\right)$$

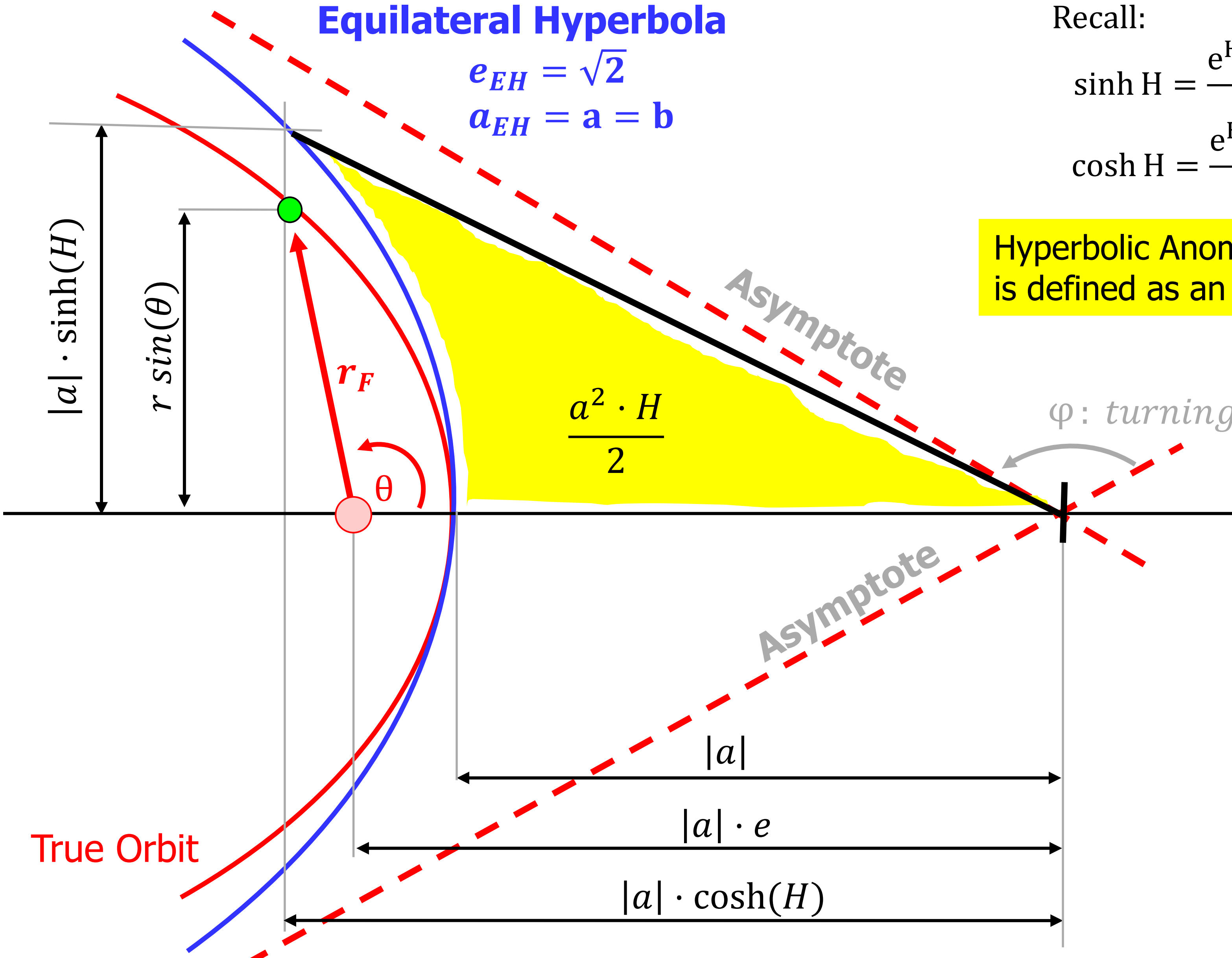
**Source:** Curtis HD (2014) "Orbital Mechanics for Engineering Students" (3rd Edition) Elsevier.

Escola Superior d'Enginyeries Industrial, Aeroespacial i Audiovisual de Terrassa (ESEIAAT)

AMVO

Adrià Rovira-Garcia

# Hyperbolic Orbit (e>1)




Recall:

$$\sinh \mathrm{H} = \frac{e^{\mathrm{H}} - e^{-\mathrm{H}}}{2}$$

$$\cosh \mathrm{H} = \frac{e^{\mathrm{H}} + e^{-\mathrm{H}}}{2}$$

Hyperbolic Anomaly H is defined as an area

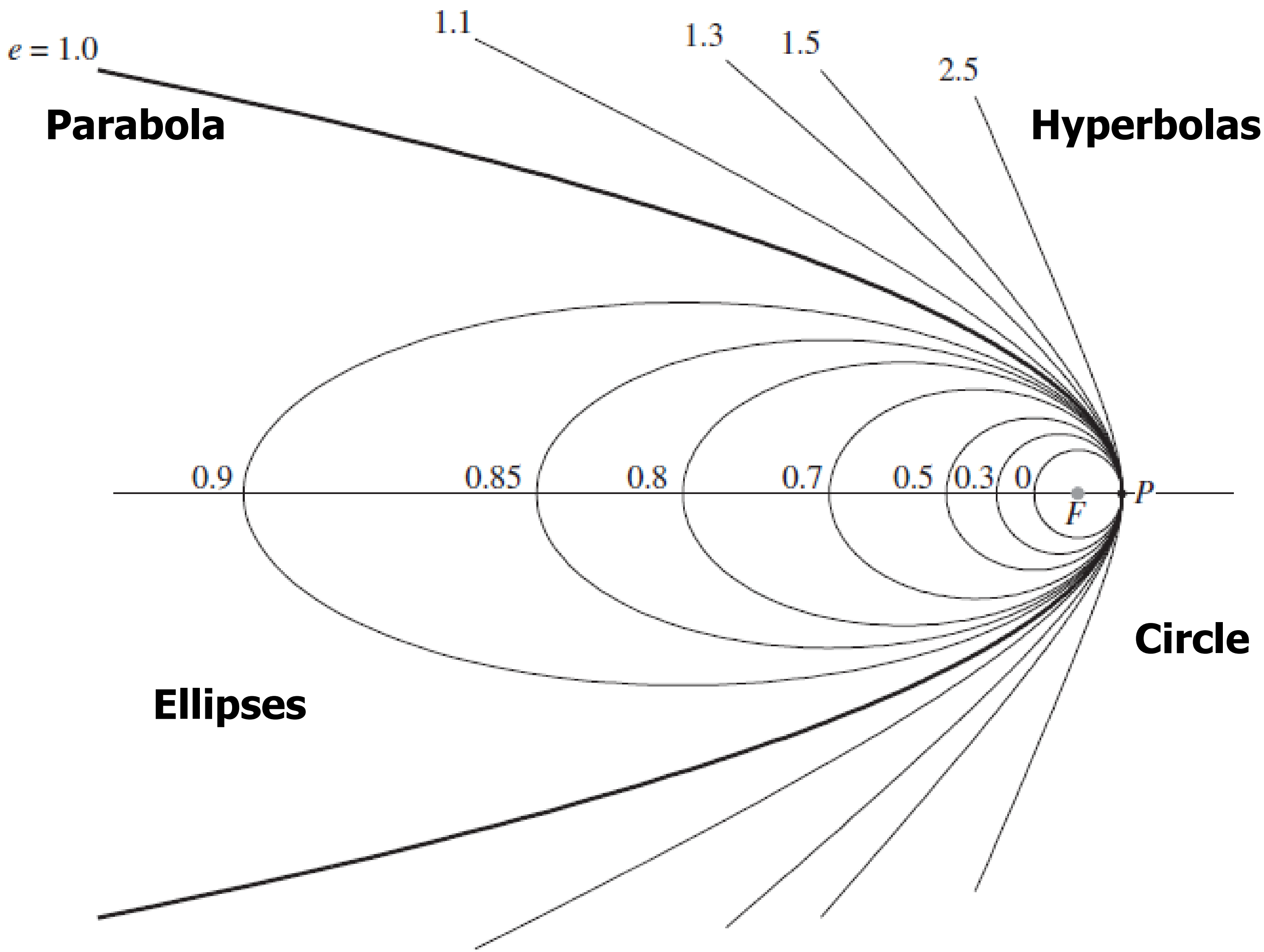
**Source:** Fig 2-4 Vallado DA (2013) "Fundamentals of Astrodynamics and Applications" 4th Edition

Escola Superior d'Enginyeries Industrial, Aeroespacial i Audiovisual de Terrassa (ESEIAAT)



Adrià Rovira-Garcia

# Keplerian Orbits: Summary




**Source:** Fig 2.27 Curtis HD (2014) "Orbital Mechanics for Engineering Students" (3rd Edition) Elsevier.

# Keplerian Orbits: Summary

| Quantity | Circle | Ellipse | Parabola | Hyperbola |
|---|---|---|---|---|
| Defining Parameters | a= semimajor axis = radius | a = semimajor axis<br>b = semiminor axis | $p = h^2/\mu$ = semi-latus rectum<br>$r_p$ = q = perifocal distance | a = semi-transverse axis (a<0)<br>b = semi conjugate axis |
| Parametric Equation | $x^2 + y^2 = a^2$ | $x^2/a^2 + y^2/b^2 = 1$ | $x^2 = 4qy$ | $x^2/a^2 - y^2/b^2 = 1$ |
| Eccentricity, e | $e = 0$ | $e = \dfrac{\sqrt{a^2 - b^2}}{a} \quad 0 < e < 1$ | $e = 1$ | $e = \dfrac{\sqrt{a^2+b^2}}{\|a\|} \quad e > 1$ |
| Periapsis Distance, $r_p$ | $r_p = a$ | $r_p = a(1-e)$ | $r_p = p/2$ | $r_p = \|a\|(1-e)$ |
| Velocity V, at Distance, r, from Focus | $v = \sqrt{\mu/r}$ | $v = \sqrt{\mu\,(2/r - 1/a)}$ | $v = \sqrt{2\mu/r}$ | $v = \sqrt{\mu\,(2/r + 1/\|a\|)}$ |
| Total Energy Per Unit Mass | $\varepsilon = -\mu/2a < 0$ | $\varepsilon = -\mu/2a < 0$ | $\varepsilon = 0$ | $\varepsilon = \mu/2\|a\| > 0$ |
| Mean Angular Motion | $n = \sqrt{\mu/a^3}$ | $n = \sqrt{\mu/a^3}$ | $n = \sqrt{\mu/p^3}$ | $n = \sqrt{\mu/\|a\|^3}$ |
| Period | $T = 2\pi/n$ | $T = 2\pi/n$ | $T = \infty$ | $T = \infty$ |
| Anomaly | $\theta = M = E$ | $\tan\left(\dfrac{\theta}{2}\right) = \sqrt{\dfrac{1+e}{1-e}}\tan\left(\dfrac{E}{2}\right)$ | $\tan\left(\dfrac{\theta}{2}\right) = \left(3M + \sqrt{(3M)^2+1}\right)^{1/3} - \left(3M + \sqrt{(3M)^2+1}\right)^{-1/3}$ | $\tan\left(\dfrac{\theta}{2}\right) = \sqrt{\dfrac{e+1}{e-1}}\tanh\left(\dfrac{H}{2}\right)$ |
| Mean Anomaly | $M = M_0 + nt$ | $M = E - e\sin E$ | $M = qD + D^3/6$<br>$D = \sqrt{2 \cdot q}\,\tan(\theta/2)$ | $M = (e\sinh H) - H$ |
| Distance from Focus | $r = a$ | $r = a(1 - e\cos E)$ | $r = p/(1 + e\cos\theta)$ | $r = \|a\|(1 - e\cosh H)$ |



**Source:** Table 2-4 Wertz JR et al (2009) “Orbit and constellation design and management” Space Technology Series

# Module 3 Requirements

**Objectives:**

- Reference Systems: Coordinates and Time
- Keplerian Orbits
  - Two body problem
  - Circles, Ellipses, Parabolas and Hyperbolas
  - Keplerian Orbital Elements
- Orbital Maneuvers
- Interplanetary Trajectories

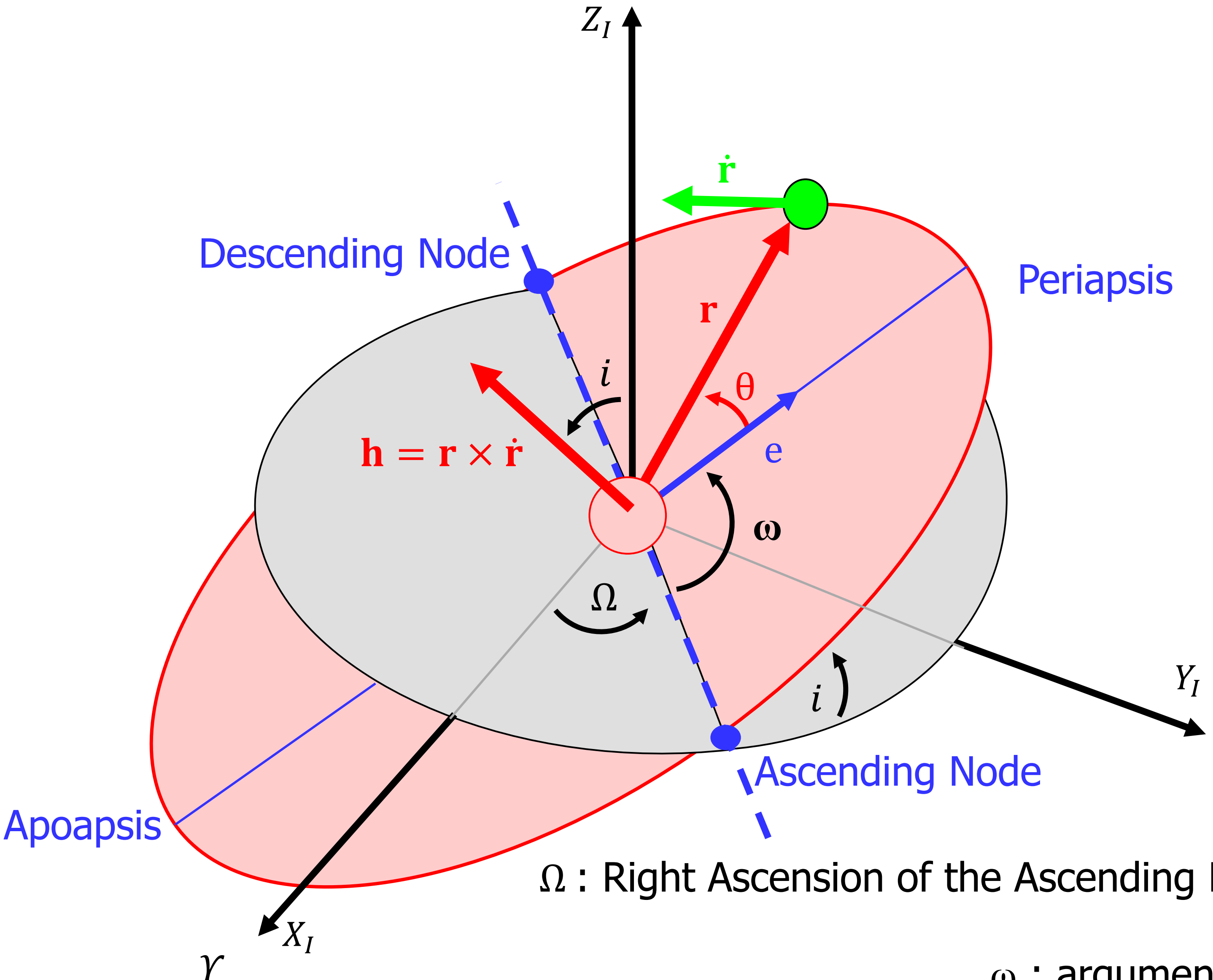
**Time Allocation:**

- 8 hours



Adrià Rovira-Garcia

# Keplerian Orbital Elements




$\Omega$ : Right Ascension of the Ascending Node (RAAN)

$i$ : inclination

$\omega$ : argument of periapsis

$\theta$: true anomaly

**Source:** Fig. 7.10 of Walter U (2019) "Astronautics. The Physics of Space Flight" (3rd Edition) Springer.

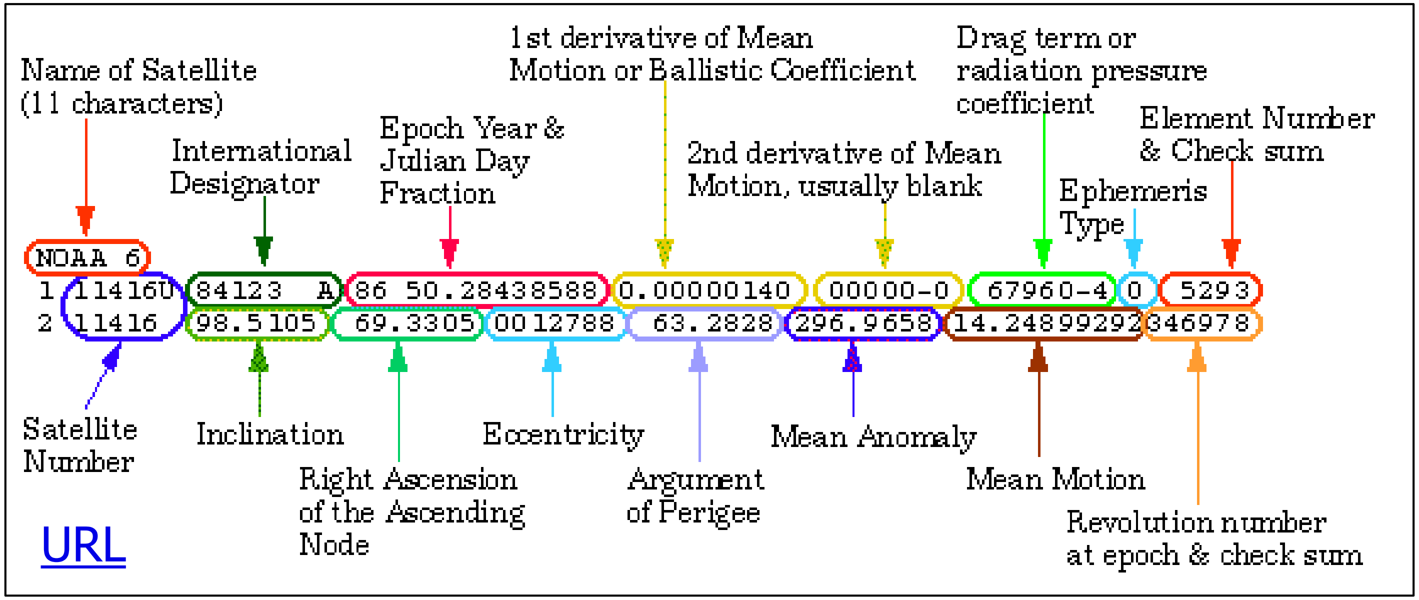
# Keplerian Orbital Elements

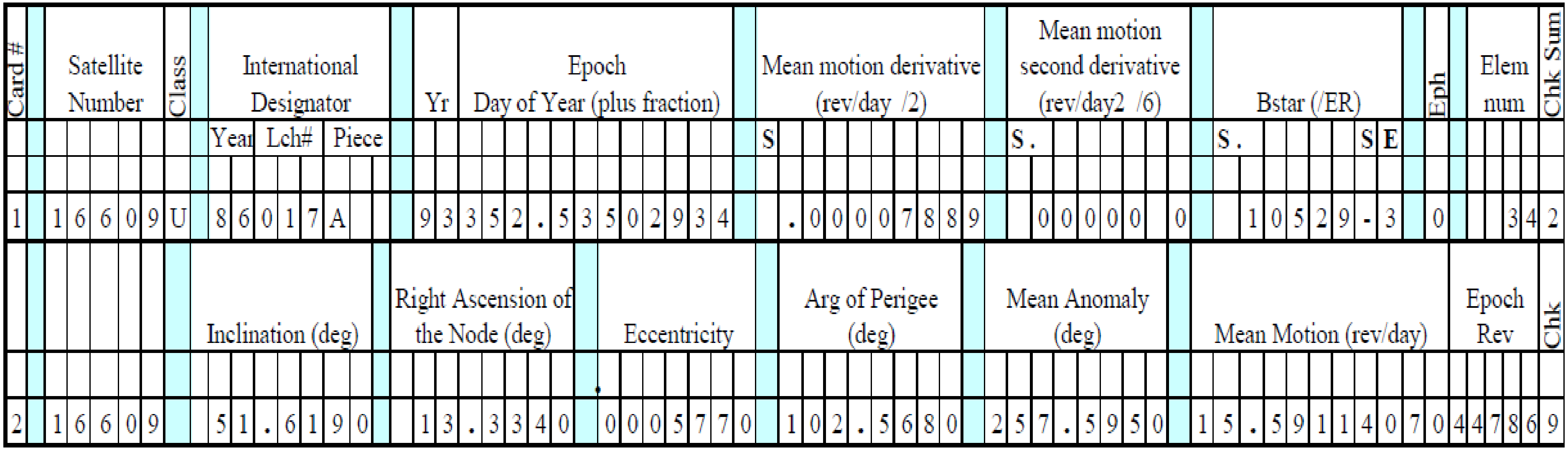
| element | | description | range |
|---|---|---|---|
| $a$ | semi-major axis | orbit size | $\lvert a \rvert > 0$ |
| $e$ | eccentricity | orbit shape | $e \geq 0$ |
| $\Omega$ | Right Ascension Ascending Node (RAAN) | orbit direction: the angle from the vernal equinox to ascending node, where satellite moves from the southern into the northern hemisphere | $0 \leq \Omega < 360°$ |
| $i$ | inclination | orbit tilt: angle between the angular momentum vector and the unit vector in the Z-direction | $0 \leq i < 180°$ |
| $\omega$ | argument of periapsis | orbit rotation: the angle measured in the direction of motion between the line of nodes and the vector e, which points to the periapsis | $0 \leq \omega < 360°$ |
| $\theta$ | true anomaly | satellite position in its orbit, related to a time-stamp | $0 \leq \theta < 360°$ |

Note circular orbits ($e = 0$), the argument of latitude $u$ can replace the argument of periapsis $\omega$ and true anomaly $\theta$ as $u = \omega + \theta$.

**Source:** Table 6-2 Larson WJ, Wertz JR (2005) "Space Mission Analysis and Design" Microcosm Press

# Two Line Elements (TLEs)




**Source:** Hoots FR, Roehrich RL (1980) "SPACETRACK REPORT NO. 3 - Models for Propagation of NORAD Element Sets"
Figure 2-19 Vallado DA (2013) "Fundamentals of Astrodynamics and Applications" 4th Edition, Space Technology Library

# Module 3 Requirements

**Objectives:**

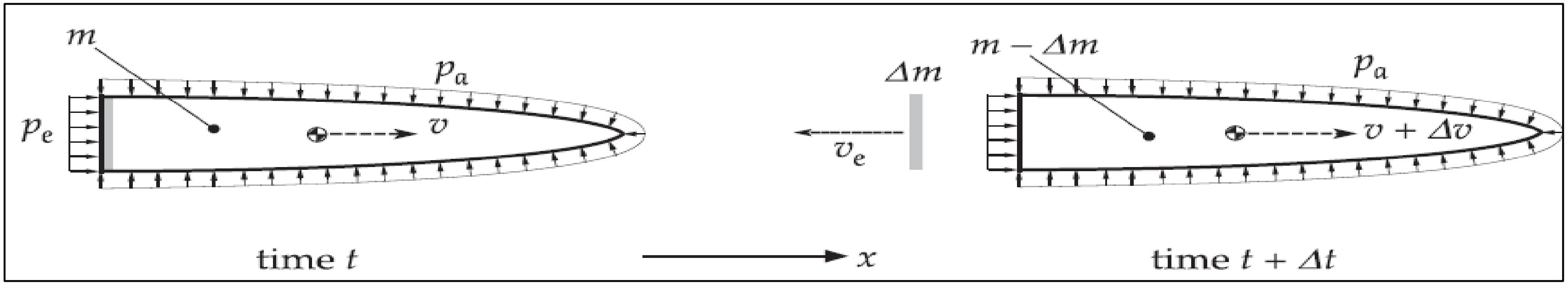
- Reference Systems: Coordinates and Time
- Keplerian Orbits
- Orbital Maneuvers
  - Introduction to Impulsive Maneuvers
  - Maneuvers Types
- Interplanetary Trajectories

**Time Allocation:**

- 8 hours

# Impulsive Maneuvers

Assumption: $\Delta V$ takes place in zero time. $\dot{\boldsymbol{r}}$ changes but $\boldsymbol{r}$ not




The Impulse of the system: $\mathbf{P} = \mathrm{m} \cdot \mathbf{v}$

Its variation is due to a Force: $\mathbf{F} = \dfrac{\mathrm{d}\mathbf{P}}{\mathrm{dt}}$

Hence

$$\mathbf{F} \cdot \mathrm{dt} = \mathrm{d}\mathbf{P}$$
$$(\mathrm{P_e} - \mathrm{P_a}) \cdot \mathrm{A_e} \cdot \mathrm{dt} = \mathrm{P}(\mathrm{t} + \Delta\mathrm{t}) - \mathrm{P}(\mathrm{t})$$
$$(\mathrm{P_e} - \mathrm{P_a}) \cdot \mathrm{A_e} \cdot \mathrm{dt} = (\mathrm{m} - \Delta\mathrm{m})\,(\mathrm{v} + \mathrm{dv}) + \Delta\mathrm{m}(\mathrm{v} - \mathrm{v_e}) - \mathrm{m\,v}$$
$$(\mathrm{P_e} - \mathrm{P_a}) \cdot \mathrm{A_e} = \mathrm{m}\frac{\mathrm{dv}}{\mathrm{dt}} - \mathrm{v_e}\frac{\Delta\mathrm{m}}{\mathrm{dt}}$$

In the case of adapted nozzle, $P_e = P_a$
and noting that our control volume is losing mass (i.e. mass is ejected): $\Delta m = -\mathrm{dm}$

$$\mathrm{m}\frac{\mathrm{dv}}{\mathrm{dt}} = -\mathrm{v_e}\frac{\mathrm{dm}}{\mathrm{dt}}$$

$$\mathrm{dv} = -\mathrm{v_e} \cdot \frac{\mathrm{dm}}{\mathrm{m}} \longrightarrow \boxed{\Delta\mathrm{V} = \mathrm{v_e} \cdot \ln\frac{\mathrm{m_0}}{\mathrm{m_f}}}$$

**Source:** Figure 11-2 Curtis HD (2014) “Orbital Mechanics for Engineering Students” (3rd Edition).

# Impulsive Maneuvers

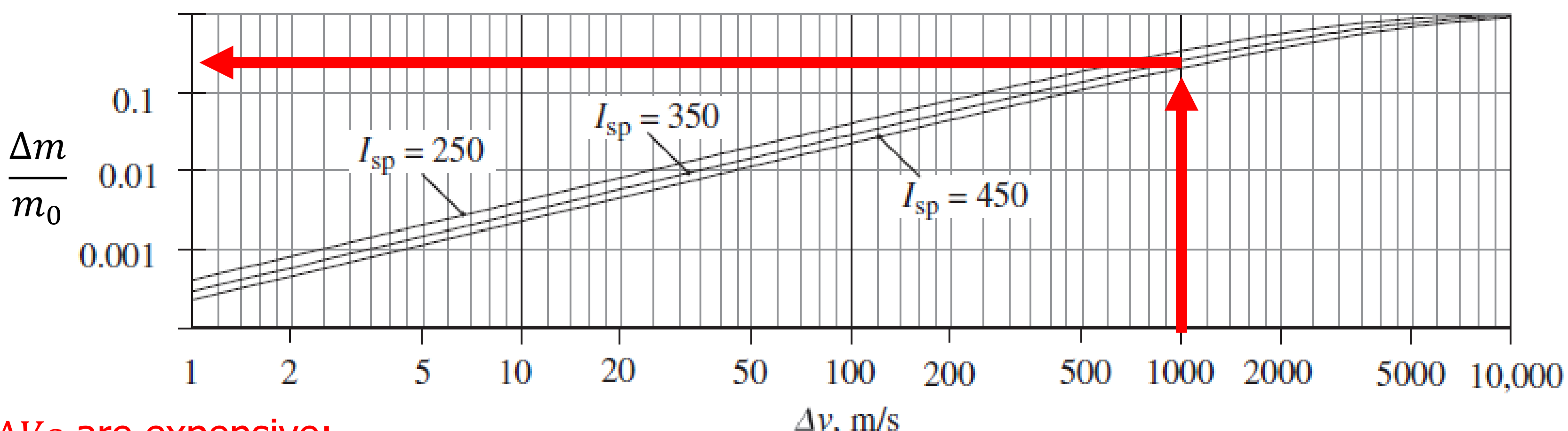
The ideal Rocket equation

$$\frac{\Delta \mathrm{m}}{\mathrm{m}_0} = 1 - e^{-\frac{\Delta V}{I_{sp} \cdot g_{SL}}}$$

Where the specific impulse $I_{sp}$ is defined as:

$$I_{sp} = \frac{\text{Thrust}}{g_{SL} \cdot \dot{m}_{SL}} = \frac{v_e}{g_{SL}}$$

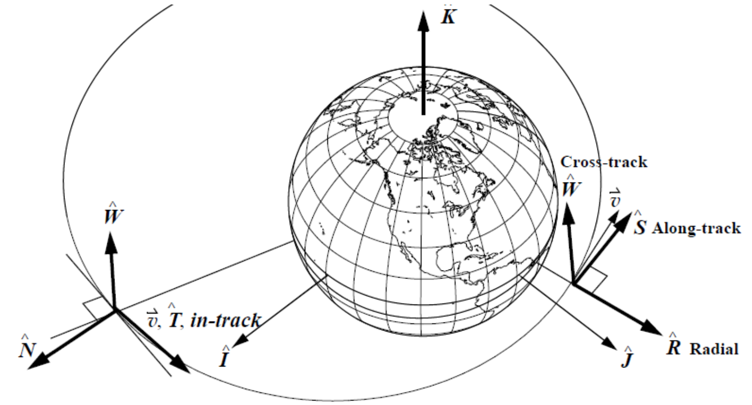
| Propellant | $I_{sp}$ (s) |
|---|---|
| Cold gas | 50 |
| Monopropellant hydrazine | 230 |
| Solid propellant | 290 |
| Nitric acid/monomethyl hydrazine | 310 |
| Liquid oxygen/liquid hydrogen | 455 |
| Ion propulsion | >3000 |



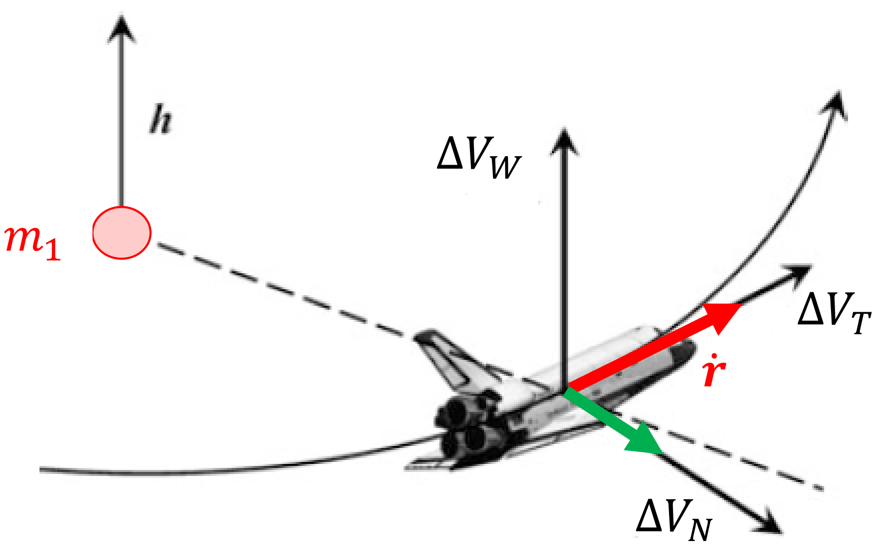

$\Delta V s$ are expensive:
Note that $\Delta V$ of ~1 km/s require propellant ~25% of the spacecraft mass before the burn

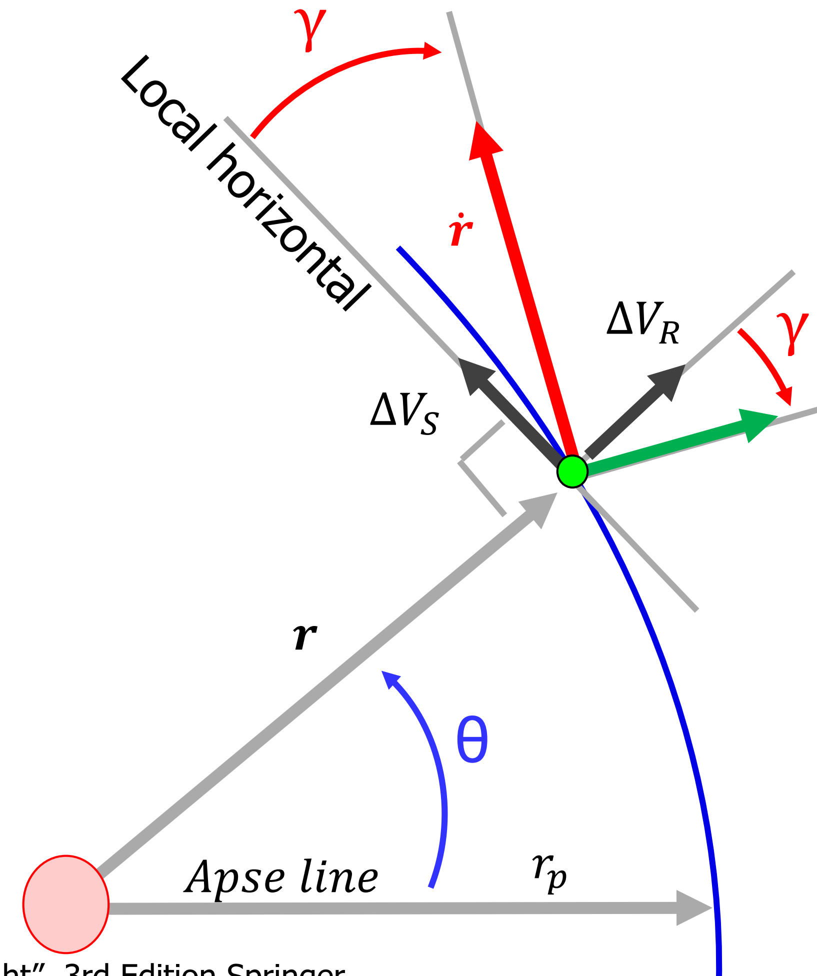
**Source:**
Table 6-1 and Figure 6-1 Curtis HD (2014) "Orbital Mechanics for Engineering Students" (3rd Edition).

# Impulsive Orbital Maneuvers




$$\Delta V_R = \Delta V_T \cdot \sin\gamma + \Delta V_N \cdot \cos\gamma$$
$$\Delta V_S = \Delta V_T \cdot \cos\gamma - \Delta V_N \cdot \sin\gamma$$
$$\Delta V_W = \Delta V_W$$

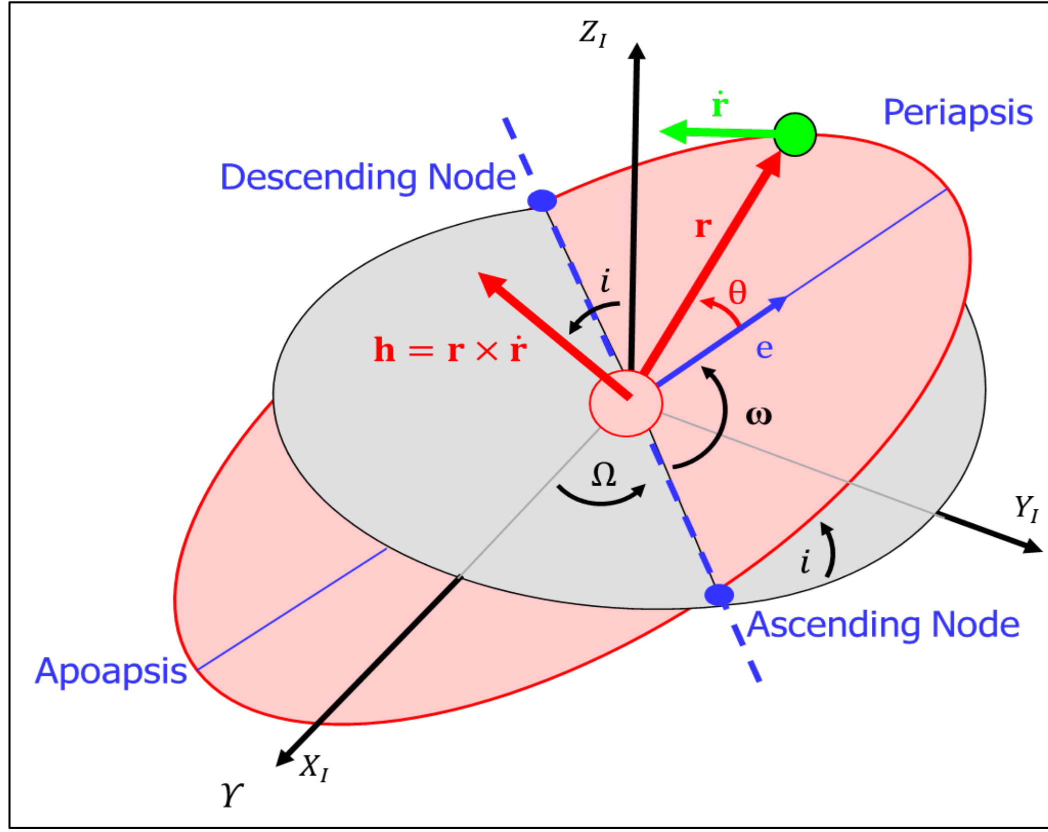



**Source:**
Fig 8-1 Walter U (2019) “Astronautics. The Physics of Space Flight” 3rd Edition Springer

# Impulsive Orbital Maneuvers

Assuming $\frac{\Delta V}{V} \ll 1$
the orbital elements $\Delta\vartheta$ is:




$$\Delta a = 2 \cdot \sqrt{\frac{a^3}{\mu} \cdot \frac{1 + 2 \cdot e \cdot \cos\theta + e^2}{1 - e^2}} \cdot \Delta V_T$$

$$\Delta i = \frac{\cos(\theta + \omega)}{1 + e \cdot \cos\theta} \sqrt{\frac{a\,(1 - e^2)}{\mu}} \Delta V_W$$

$$\Delta\Omega = \frac{\sin\,(\theta + \omega)}{(1 + e\cos\theta)\sin i} \cdot \sqrt{\frac{a\,(1 - e^2)}{\mu}} \Delta V_W$$

$$\Delta e = 2\sqrt{\frac{a}{\mu} \cdot \frac{(1 - e^2) \cdot (e + \cos\theta)^2}{1 + 2 \cdot e \cdot \cos\theta + e^2}} \cdot \Delta V_T + \frac{\sin\theta}{1 + e \cdot \cos\theta} \sqrt{\frac{a\,(1 - e^2)^3}{\mu}} \Delta V_N$$

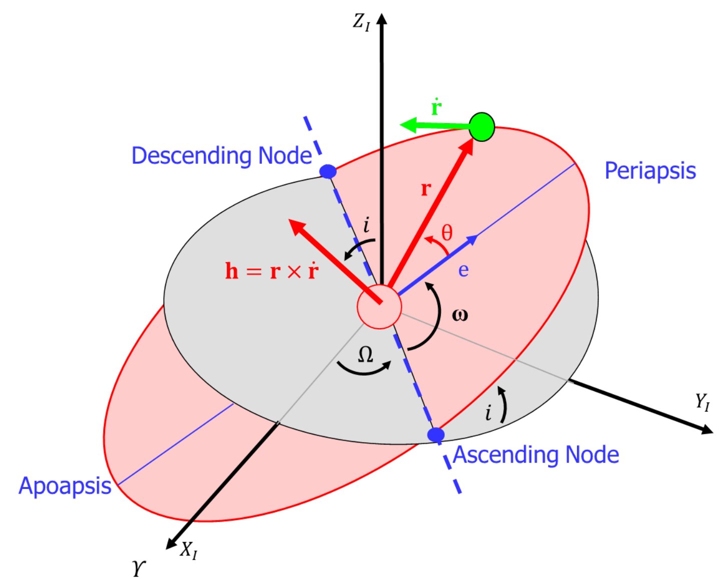
$$\Delta\omega = \frac{2\sin\theta}{e} \sqrt{\frac{a}{\mu} \cdot \frac{1 - e^2}{1 + 2 \cdot e \cdot \cos\theta + e^2}} \cdot \Delta V_T - \left(1 + \frac{1}{e} \frac{e + \cos\theta}{1 + e\cos\theta}\right) \sqrt{\frac{a}{\mu} \cdot \frac{1 - e^2}{1 + 2 \cdot e \cdot \cos\theta + e^2}} \Delta V_N$$
$$- \frac{\sin(\theta + \omega)}{(1 + e \cdot \cos\theta)\tan i} \sqrt{\frac{a\,(1 - e^2)}{\mu}} \Delta V_W$$

**Source:** Walter U (2019) "Astronautics. The Physics of Space Flight" 3rd Edition Springer

# Impulsive Orbital Maneuvers

$$\begin{pmatrix} \Delta a \\ \Delta e \\ \Delta i \\ \Delta \Omega \\ \Delta \omega \end{pmatrix} = \begin{pmatrix} f(\theta) & 0 & 0 \\ f(\theta) & f(\theta) & 0 \\ 0 & 0 & f(\theta+\omega) \\ 0 & 0 & f(\theta+\omega, i) \\ f(\theta) & -f(\theta) & -f(\theta+\omega, i) \end{pmatrix} \begin{pmatrix} \Delta V_T \\ \Delta V_N \\ \Delta V_W \end{pmatrix}$$

- $\Delta V_T$ and $\Delta V_N$ are contained in the orbital plane → can only affect $a$, $e$, $\omega$ but not the orientation of the orbital plane $i$, $\Omega$
- $\Delta V_W$ can only change the direction of orbital motion $i$, $\Omega$, $\omega$, but not $a$, $e$ associated with the orbit energy (speed) or size
- 6 optimal positions for $\Delta V$:
    - Regarding True Anomaly
        - Periapsis $\theta = 0°$
        - Apoapsis $\theta = 180°$
    - Nodes
        - Ascending Node: $\omega + \theta = 0°$
        - Descending Node: $\omega + \theta = 180°$
    - Orthogonal to Nodes
        - $\omega + \theta = 90°$
        - $\omega + \theta = 270°$




**Source:**
Walter U (2019) "Astronautics. The Physics of Space Flight" 3rd Edition Springer

# Module 3 Requirements

**Objectives:**

- **Reference Systems: Coordinates and Time**
- **Keplerian Orbits**
- **Orbital Maneuvers**
  - **Introduction to Impulsive Maneuvers**
  - **Maneuvers Types**
    - **Intercepting Orbits:**
      - **One Impulse maneuver**
        - **Inclination change**
        - **RAAN change**
        - **Apse line rotation**
    - **Non-Intercepting Orbits**
      - **Two Impulse maneuver**
        - **Hohmann Transfer**
        - **Circular Rendezvous different orbits**
- **Interplanetary Trajectories**

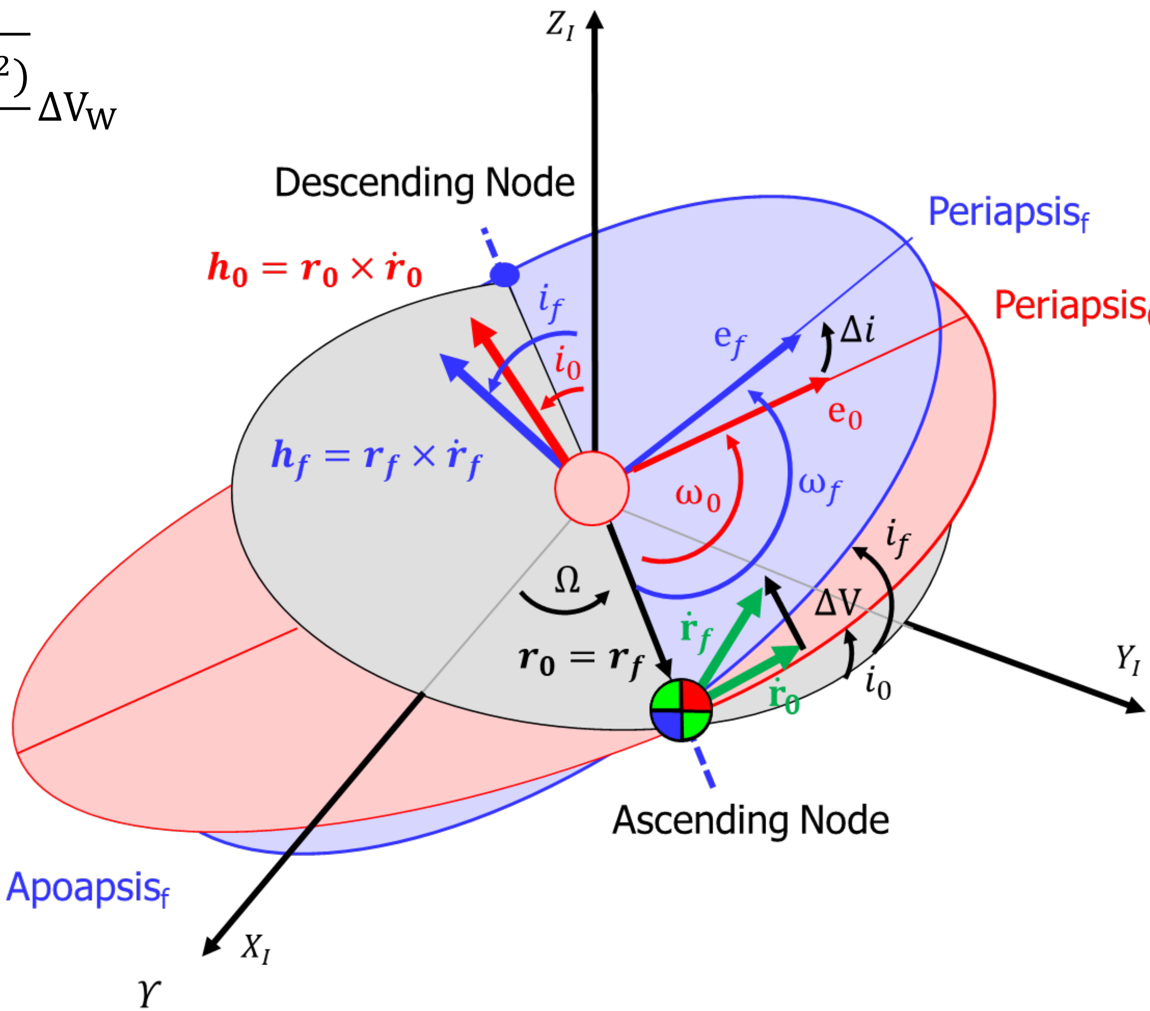
**Time Allocation:**

**8 hours**

# Inclination Change at Nodes

$\Delta V = (0,0,\Delta V_W)$ applied at, for instance, $\theta + \omega = 0$:

$$\Delta a = 0$$
$$\Delta e = 0$$
$$\Delta i = \frac{1}{1 + e \cdot \cos\theta} \sqrt{\frac{a\,(1 - e^2)}{\mu}}\,\Delta V_W$$
$$\Delta\Omega = 0$$
$$\Delta\omega = 0$$




UPC
Escola Superior d'Enginyeries Industrial, Aeroespacial i Audiovisual de Terrassa (ESEIAAT)
AMVO
Adrià Rovira-Garcia



**Source:** Walter U (2019) “Astronautics. The Physics of Space Flight” 3rd Edition Springer

# Inclination Change at Nodes

$\Delta V = (0, 0, \Delta V_W)$ at, for instance, $\theta + \omega = 0$:

From the law of cosines:

$$\Delta V^2 = V_f^{\,2} + V_0^{\,2} - 2 \cdot V_f V_0 \cos \Delta i$$

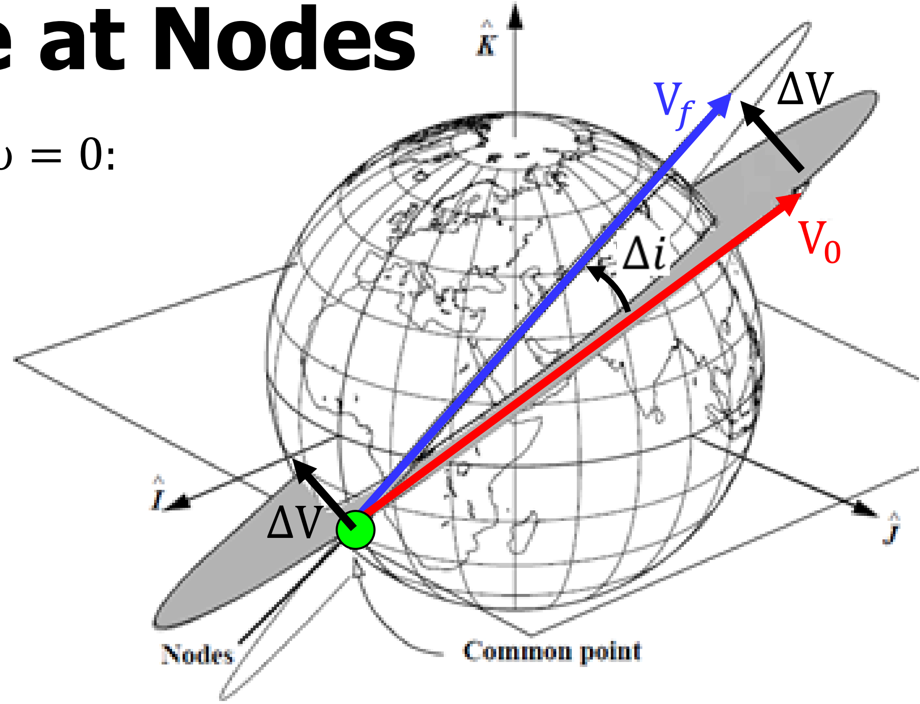
Equal velocity module $|V_f| = |V_0|$

$$\Delta V^2 = 2\, V_0^{\,2} (1 - \cos \Delta i)$$

After some trigonometric identities:

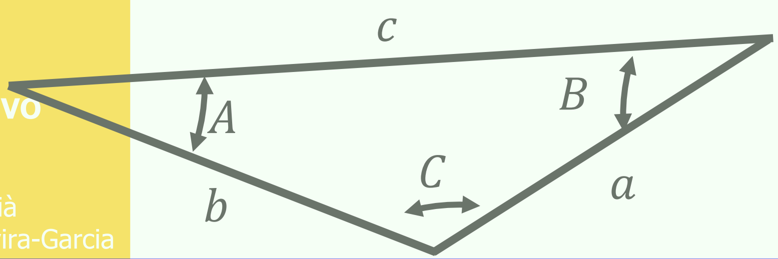
$$\Delta V^2 = 4\, V_0^{\,2} \sin^2\left(\frac{\Delta i}{2}\right)$$

$$\Delta V = 2 \cdot V_0 \cdot \left|\sin \frac{\Delta i}{2}\right|$$




- An expensive maneuver
- To be done close to apogee, $\min(V_0)$ ideal during a bielliptic transfer

$$a^2 = b^2 + c^2 - 2 \cdot bc \cos A$$
$$b^2 = a^2 + c^2 - 2 \cdot ac \cos B$$
$$c^2 = a^2 + b^2 - 2 \cdot ab \cos C$$

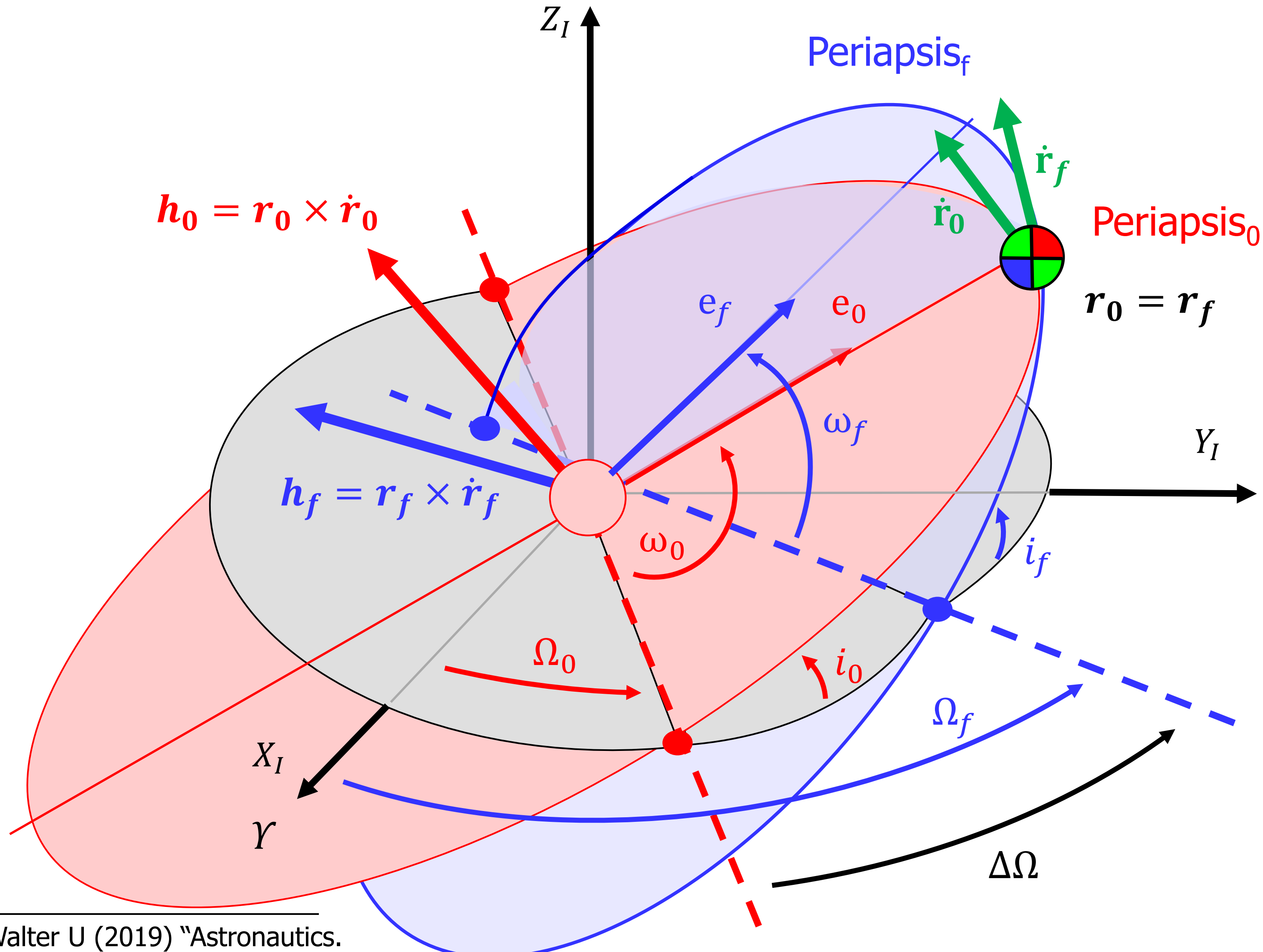



$$\cos 2\left(\frac{\varphi}{2}\right) = \cos^2\left(\frac{\varphi}{2}\right) - \sin^2\left(\frac{\varphi}{2}\right)$$
$$- \quad 1 = \cos^2\left(\frac{\varphi}{2}\right) + \sin^2\left(\frac{\varphi}{2}\right)$$
$$1 - \cos(\varphi) = 2 \cdot \sin^2\left(\frac{\varphi}{2}\right)$$

**Source:** Figure 6-10 Vallado DA (2013) "Fundamentals of Astrodynamics and Applications" 4th Edition

# RAAN Change at orthogonal of Nodes

$\Delta V = (0,0,\Delta V_W)$ at, for instance, $\theta + \omega = 90°$:




**Source:** Walter U (2019) "Astronautics. The Physics of Space Flight" 3rd Edition Springer

# RAAN Change at orthogonal of Nodes

$\Delta V = (0,0,\Delta V_W)$ at, for instance, $\theta + \omega = 90°$:

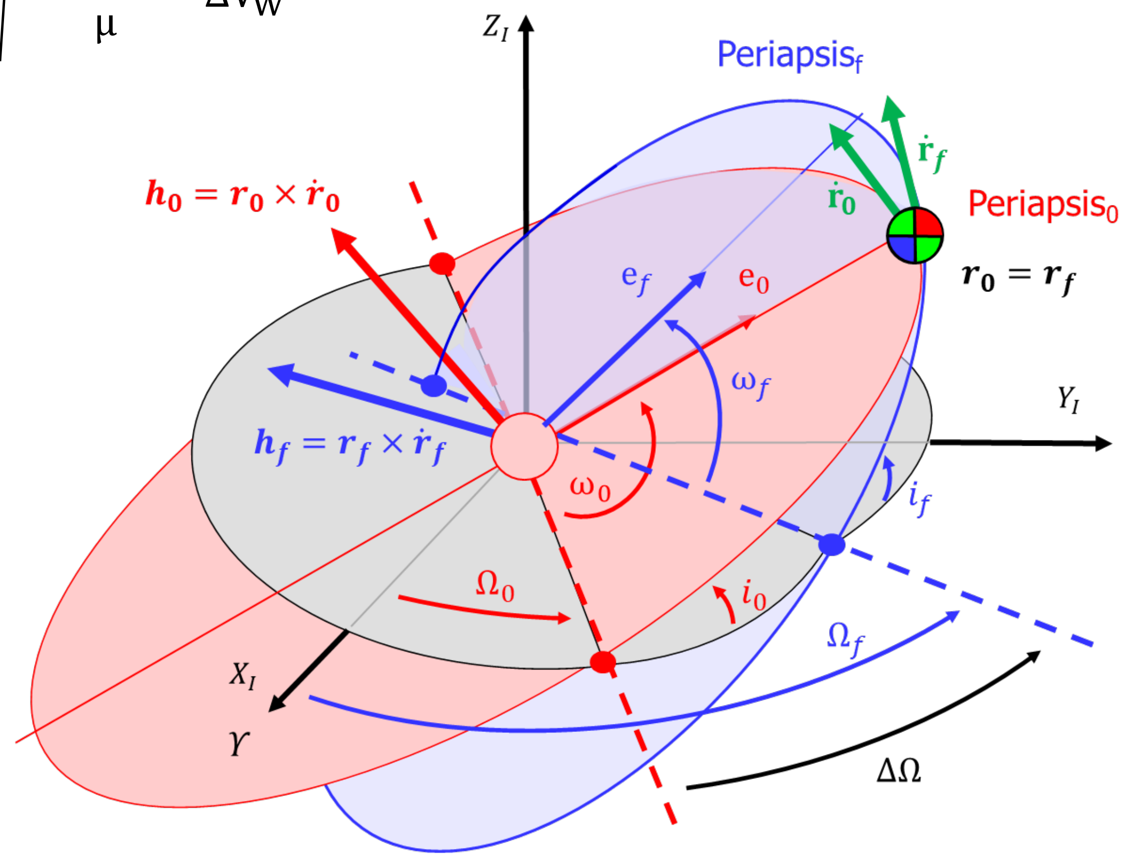
$$\Delta\Omega = \frac{1}{(1 + e\cos\theta)\sin i} \cdot \sqrt{\frac{a\,(1 - e^2)}{\mu}}\Delta V_W$$

$$\Delta\omega = -\frac{1}{(1 + e \cdot \cos\theta)\tan i}\sqrt{\frac{a\,(1 - e^2)}{\mu}}\Delta V_W$$

$\Delta a = 0$

$\Delta e = 0$

$\Delta i = 0$



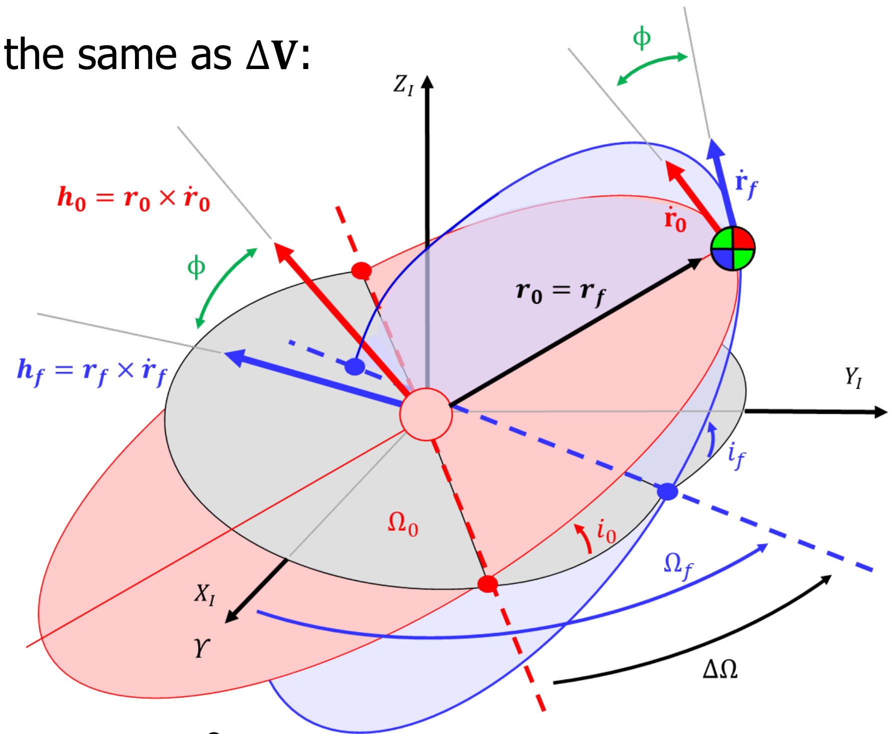

**Source:** Walter U (2019) "Astronautics. The Physics of Space Flight" 3rd Edition Springer

# RAAN Change at orthogonal of Nodes

since $\mathbf{r_0} = \mathbf{r_f}$ the angle $\phi$ between $\Delta\mathbf{h}$ is the same as $\Delta\mathbf{V}$:

$$\mathbf{h_0} = \mathbf{r_0} \times \dot{\mathbf{r}}_\mathbf{0}$$
$$\mathbf{h_f} = \mathbf{r_f} \times \dot{\mathbf{r}}_\mathbf{f}$$

$$\cos\phi = \frac{\mathbf{h_0}\cdot\mathbf{h}_f}{|\mathbf{h_0}|\cdot|\mathbf{h}_f|}$$




Computation of $\boldsymbol{h_0}$ and $\boldsymbol{h_f}$ :

$$\begin{pmatrix} h_x \\ h_y \\ h_z \end{pmatrix} = R_3(-\Omega)R_1(-i)\begin{pmatrix} 0 \\ 0 \\ h \end{pmatrix}_{RSW}$$

$$\begin{pmatrix} h_x \\ h_y \\ h_z \end{pmatrix} = \begin{pmatrix} \cos\Omega & -\sin\Omega & 0 \\ \sin\Omega & \cos\Omega & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos i & -\sin i \\ 0 & \sin i & \cos i \end{pmatrix}\begin{pmatrix} 0 \\ 0 \\ h \end{pmatrix}_{RSW}$$

$$\begin{pmatrix} h_x \\ h_y \\ h_z \end{pmatrix} = \begin{pmatrix} \cos\Omega & -\sin\Omega\cdot\cos i & \sin\Omega\cdot\sin i \\ \sin\Omega & \cos\Omega\cdot\cos i & -\cos\Omega\cdot\sin i \\ 0 & \sin i & \cos i \end{pmatrix}\begin{pmatrix} 0 \\ 0 \\ h \end{pmatrix}_{RSW}$$

$$\begin{pmatrix} h_x \\ h_y \\ h_z \end{pmatrix} = h\begin{pmatrix} \sin\Omega\cdot\sin i \\ -\cos\Omega\cdot\sin i \\ \cos i \end{pmatrix}$$

**Source:** Walter U (2019) "Astronautics. The Physics of Space Flight" 3rd Edition Springer

Escola Superior d'Enginyeries Industrial, Aeroespacial i Audiovisual de Terrassa (**ESEIAAT**)
**AMVO**
Adrià Rovira-Garcia

Escola Superior d'Enginyeries Industrial, Aeroespacial i Audiovisual de Terrassa (ESEIAAT)

# RAAN Change at orthogonal of Nodes

since $\mathbf{r_0} = \mathbf{r_f}$ the angle $\phi$ between $\Delta\mathbf{h} = \mathbf{h_f} - \mathbf{h_0}$ is the same as $\Delta\mathbf{V} = \mathbf{V_f} - \mathbf{V_0}$:

$$\cos\phi = \frac{\mathbf{h_0}\cdot\mathbf{h_f}}{|\mathbf{h_0}|\cdot|\mathbf{h_f}|} = \sin\Omega_o \sin i_o \cdot \sin\Omega_f \sin i_f + \cos\Omega_o \sin i_o \cdot \cos\Omega_f \sin i_f + \cos i_o \cdot \cos i_f$$

Noting that:

$$\cos(\Omega_f - \Omega_0) = \cos\Omega_f \cdot \cos\Omega_o + \sin\Omega_f \cdot \sin\Omega_o$$
$$\cos(i_f - i_0) = \cos i_f \cdot \cos i_o + \sin i_f \cdot \sin i_o$$

$$\cos\phi = \sin i_o \cdot \sin i_f \cdot [\cos(\Delta\Omega) + 1] + \cos(\Delta i)$$

Noting that:

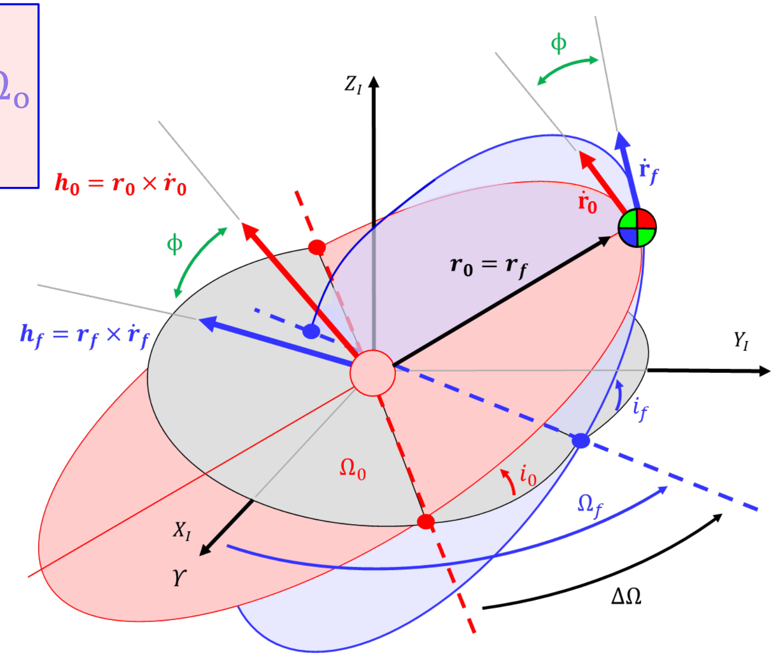
$$\cos(\varphi) = 1 - 2\cdot\sin^2\left(\frac{\varphi}{2}\right)$$

$$\sin^2\left(\frac{\phi}{2}\right) = \sin i_o \cdot \sin i_f \cdot \sin^2\left(\frac{\Delta\Omega}{2}\right) + \sin^2\left(\frac{\Delta i}{2}\right)$$

We are ready to apply the law of cosines:

$$\Delta V^2 = V_f^{\,2} + V_0^{\,2} - 2\cdot V_f \cdot V_0 \cos\phi$$

$$\Delta V^2 = V_f^{\,2} + V_0^{\,2} - 2\cdot V_f \cdot V_0 \left[1 - 2\sin i_o \cdot \sin i_f \cdot \sin^2\left(\frac{\Delta\Omega}{2}\right) - 2\sin^2\left(\frac{\Delta i}{2}\right)\right]$$



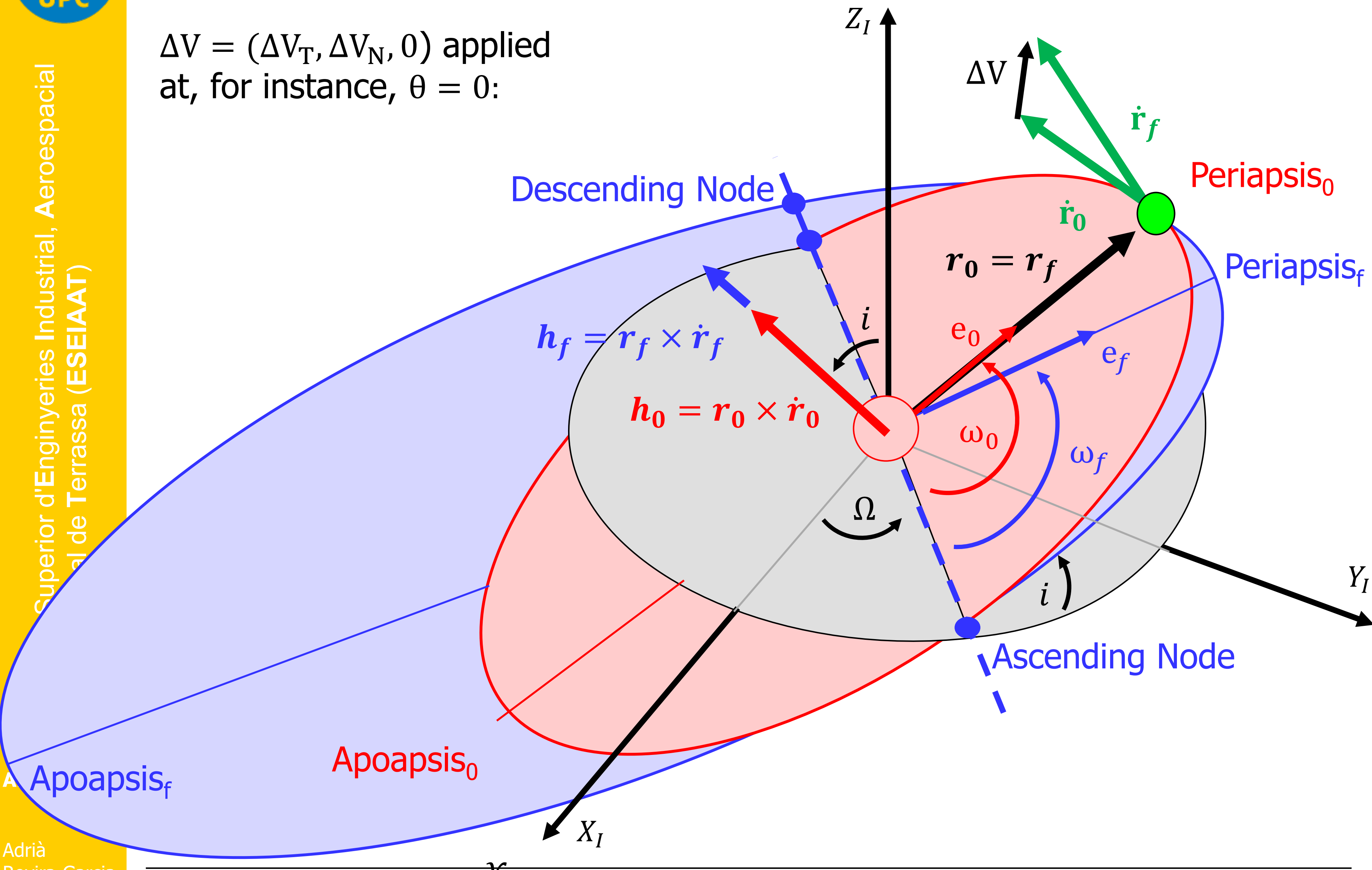

$$\Delta V = 2\cdot V_0 \sin i_o \left|\sin\left(\frac{\Delta\Omega}{2}\right)\right|$$

- An expensive maneuver
- To be performed close to apogee for $\min(V_0)$
- To be performed at minimum inclination $\min(i_o)$

**Source:** Walter U (2019) "Astronautics. The Physics of Space Flight" 3rd Edition Springer

# Rotation of Apse Line

$\Delta V = (\Delta V_T, \Delta V_N, 0)$ applied
at, for instance, $\theta = 0$:




**Source:**
Walter U (2019) "Astronautics. The Physics of Space Flight" 3rd Edition Springer

# Rotation of Apse Line

$\Delta V = (\Delta V_T, \Delta V_N, 0)$ applied at, for instance, $\theta = 0$:

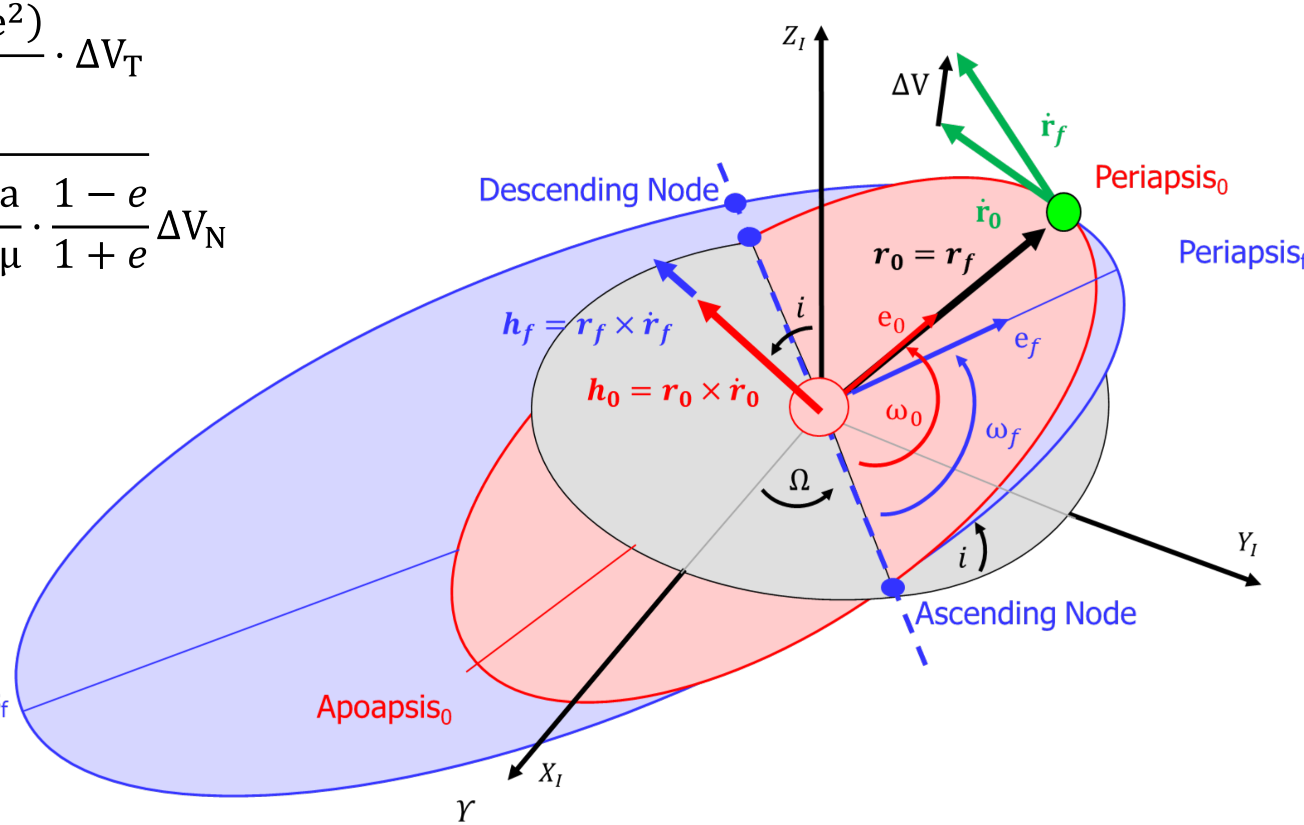
$$\Delta a = 2 \cdot \sqrt{\frac{a^3}{\mu} \cdot \frac{1+e}{1-e}} \cdot \Delta V_T$$

$$\Delta e = 2 \cdot \sqrt{\frac{a}{\mu} \cdot \frac{(1-e^2)}{1}} \cdot \Delta V_T$$

$$\Delta \omega = -\left(1 + \frac{1}{e}\right) \sqrt{\frac{a}{\mu} \cdot \frac{1-e}{1+e}} \Delta V_N$$

$$\Delta i = 0$$

$$\Delta \Omega = 0$$

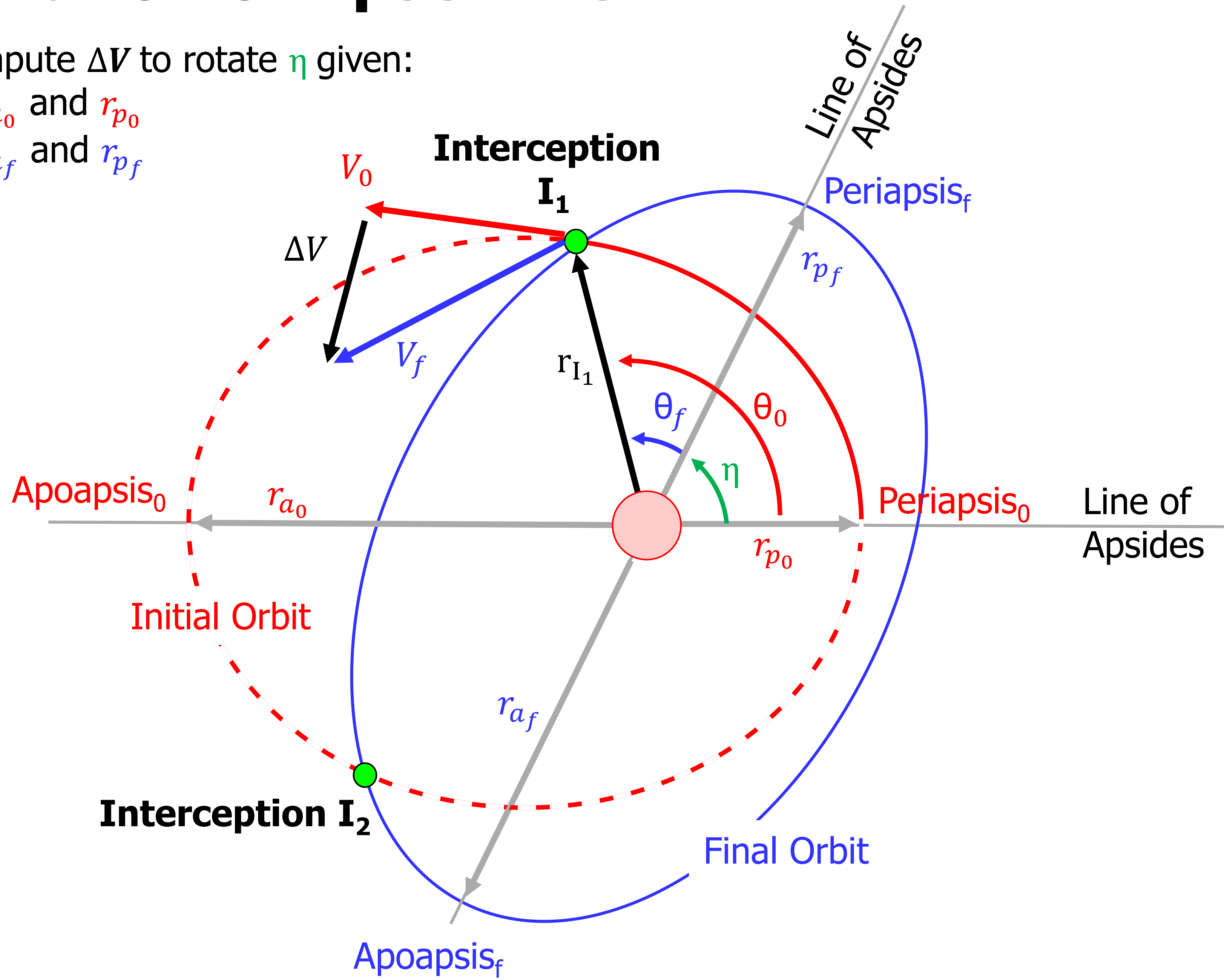



**Source:**
Walter U (2019) “Astronautics. The Physics of Space Flight” 3rd Edition Springer

# Rotation of Apse Line

Compute $\Delta V$ to rotate $\eta$ given:

- $r_{a_0}$ and $r_{p_0}$
- $r_{a_f}$ and $r_{p_f}$




**Source:** Figure 6-1 Curtis HD (2014) "Orbital Mechanics for Engineering Students" (3rd Edition) Elsevier.

# Rotation of Apse Line

$$e_0 = \frac{r_{a_0} - r_{p_0}}{r_{a_0} + r_{p_0}} \qquad h_0^2 = \mu \cdot r_{p_0}(1 + e_0 \cdot \cos 0)$$

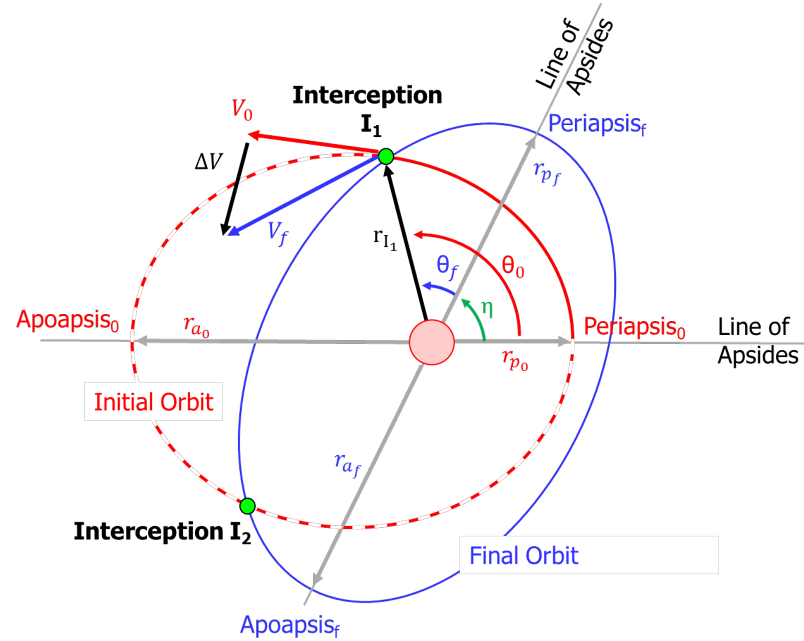
$$e_f = \frac{r_{a_f} - r_{p_f}}{r_{a_f} + r_{p_f}} \qquad h_f^2 = \mu \cdot r_{p_f}(1 + e_f \cdot \cos 0)$$

$$r_{I_0} = \frac{h_0^2}{\mu} \frac{1}{1 + e_0 \cdot \cos \theta_0}$$

$$r_{I_f} = \frac{h_f^2}{\mu} \frac{1}{1 + e_f \cdot \cos(\eta - \theta_0)}$$

$$-$$

$$0 = \frac{h_0^2}{1 + e_0 \cdot \cos \theta_0} - \frac{h_f^2}{1 + e_f \cdot (\cos \eta \cos \theta_0 + \sin \eta \sin \theta_0)}$$




$$h_0^2 + h_0^2\, e_f \cos \eta \cos \theta_0 + h_0^2\, e_f \sin \eta \sin \theta_0 = h_f^2 + h_f^2 e_0 \cdot \cos \theta_0$$

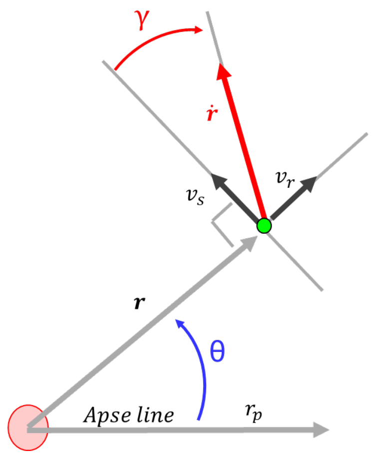
$$(h_0^2\, e_f \cos \eta - h_f^2 e_0)\cos \theta_0 + h_0^2\, e_f \sin \eta \sin \theta_0 = h_f^2 - h_0^2$$

$$a \cos \theta_0 + b \sin \theta_0 = c$$

$$\boxed{\theta_0 = \operatorname{atan}\left(\frac{b}{a}\right) \pm \operatorname{acos}\left\{\frac{c}{a} \cos\left[\operatorname{atan}\left(\frac{b}{a}\right)\right]\right\}} \left\{ \begin{array}{l} \theta_{0_{I_1}} \longrightarrow \boxed{\theta_f = \theta_0 - \eta} \quad \boxed{r_I = r_{I_1}(\theta_0)} \\ \theta_{0_{I_2}} \end{array} \right.$$

**Source:** Figure 6-1 Curtis HD (2014) "Orbital Mechanics for Engineering Students" (3rd Edition) Elsevier.

# Rotation of Apse Line



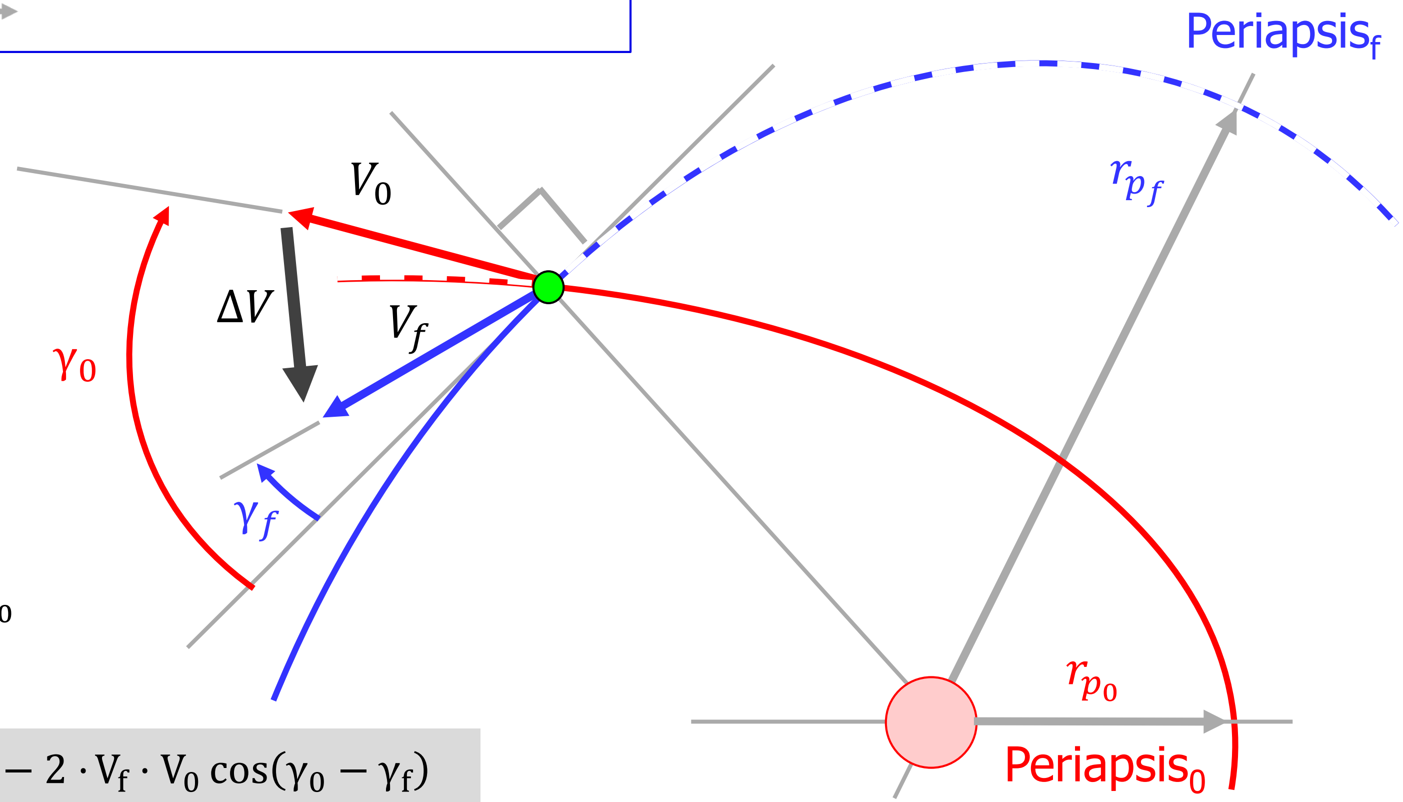

$$v_r = \frac{\mu}{h} e \sin\theta$$

$$v_\perp = v_s = \frac{\mu}{h}(1 + e\cos\theta)$$

$$\tan\gamma = \frac{v_r}{v_s} = \frac{e\sin\theta}{1 + e\cos\theta}$$

$\theta_0$: $v_{r_0}(\theta_0), v_{s_0}(\theta_0), \gamma_0$

$\theta_f$: $v_{r_f}(\theta_f), v_{s_f}(\theta_f), \gamma_f$




$$V_0{}^2 = v_{r_0}^2 + v_{s_0}^2$$

$$V_f{}^2 = v_{r_f}^2 + v_{s_f}^2$$

$$\Delta V^2 = V_f{}^2 + V_0{}^2 - 2 \cdot V_f \cdot V_0 \cos(\gamma_0 - \gamma_f)$$

**Source:** Curtis HD (2014) "Orbital Mechanics for Engineering Students" (3rd Edition) Elsevier.

# Module 3 Requirements

**Objectives:**

- Reference Systems: Coordinates and Time
- Keplerian Orbits
- Orbital Maneuvers
  - Introduction to Impulsive Maneuvers
  - Maneuvers Types
    - Intercepting Orbits:
      - One Impulse maneuver
        - Inclination change
        - RAAN change
        - Apse line rotation
    - Non-Intercepting Orbits
      - Two Impulse maneuver
        - Hohmann Transfer
        - Circular Rendezvous different orbits
- Interplanetary Trajectories

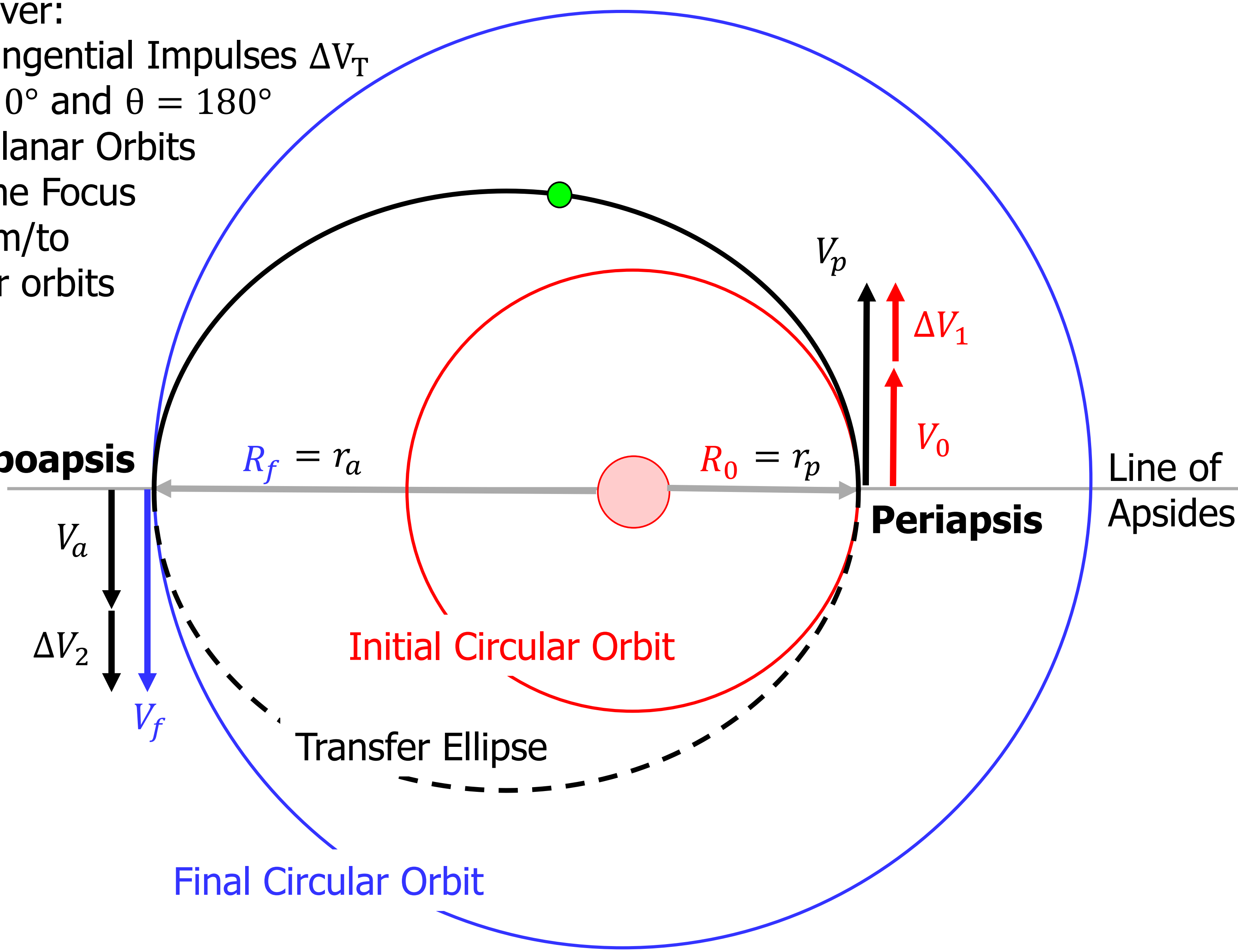
**Time Allocation:**

8 hours

# Original* Hohmann Transfer: Outward

Maneuver:

- 2 tangential Impulses $\Delta V_T$ at $\theta = 0°$ and $\theta = 180°$
- Coplanar Orbits
- Same Focus
- From/to circular orbits




**Source:** Figure 6-1 Vallado DA (2013) "Fundamentals of Astrodynamics and Applications" 4th Edition
*Hohmann W (1925) "Die Erreichbarkeit der Himmelskörper (The Attainability of Celestial Bodies)"

Escola Superior d'Enginyeries Industrial, Aeroespacial i Audiovisual de Terrassa (ESEIAAT)

AMVO

Adrià Rovira-Garcia

# Original* Hohmann Transfer: Inward

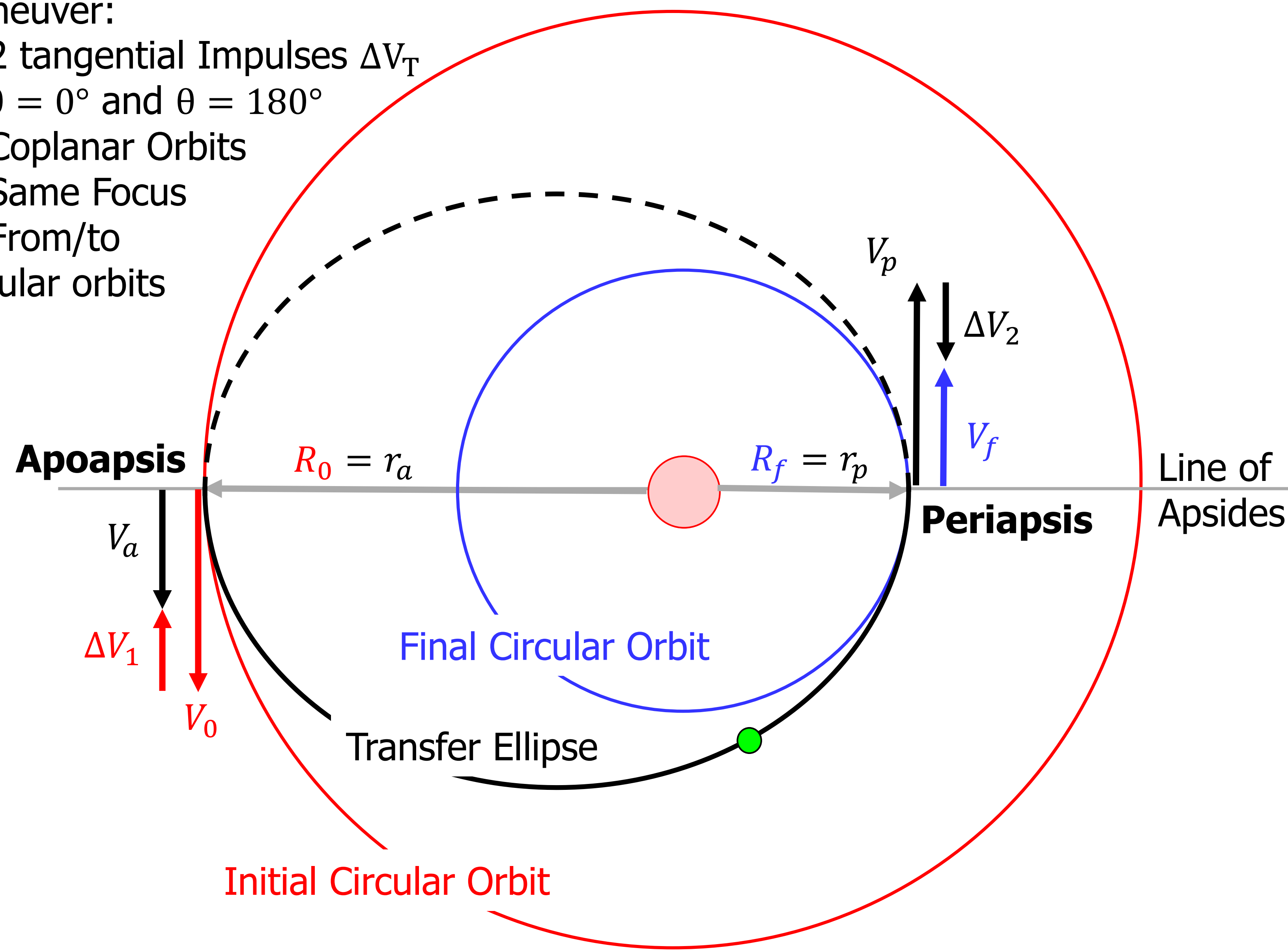
Maneuver:

- 2 tangential Impulses $\Delta V_T$

at $\theta = 0°$ and $\theta = 180°$

- Coplanar Orbits
- Same Focus
- From/to

circular orbits




**Source:** Figure 6-1 Vallado DA (2013) "Fundamentals of Astrodynamics and Applications" 4th Edition
*Hohmann W (1925) "Die Erreichbarkeit der Himmelskörper (The Attainability of Celestial Bodies)"

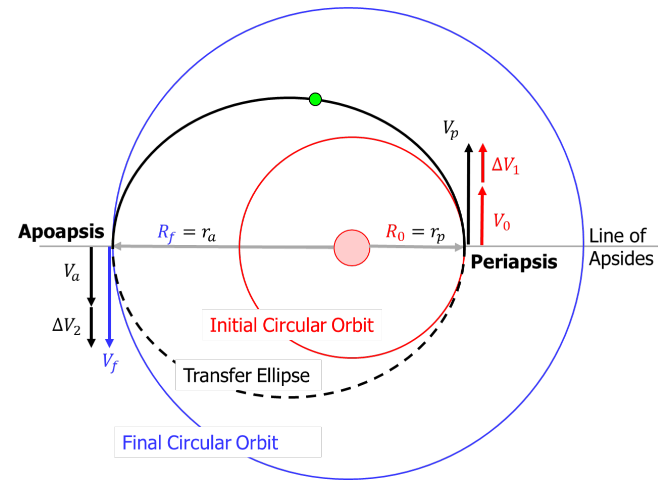
# Original Hohmann Transfer Formulae

$(r_0, r_f)\ to\ (\Delta t_{trans}, \Delta V)$

$$a_{trans} = \frac{r_0 + r_f}{2}$$

$$e_{trans} = \frac{r_0 - r_f}{r_0 + r_f}$$

$$\Delta t_{trans} = \frac{T_{trans}}{2} = \pi\sqrt{\frac{a_{trans}^3}{\mu}}$$




$$V_0 = \sqrt{\frac{\mu}{r_0}}$$

$$V_p = \sqrt{\frac{2\cdot\mu}{r_0} - \frac{\mu}{a_{trans}}}$$

$$\Delta V_1 = V_p - V_0$$

$$V_a = \sqrt{\frac{2\cdot\mu}{r_f} - \frac{\mu}{a_{trans}}}$$

$$V_f = \sqrt{\frac{\mu}{r_f}}$$

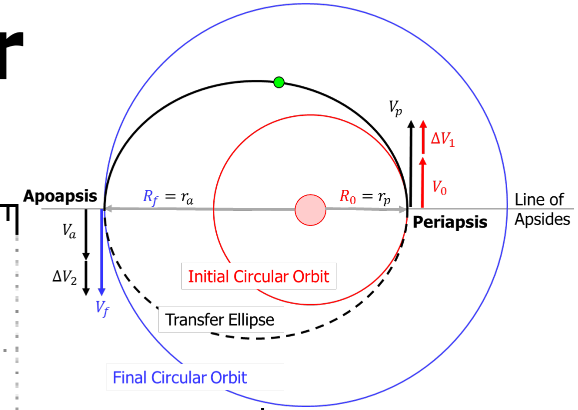
$$\Delta V_2 = V_f - V_a$$

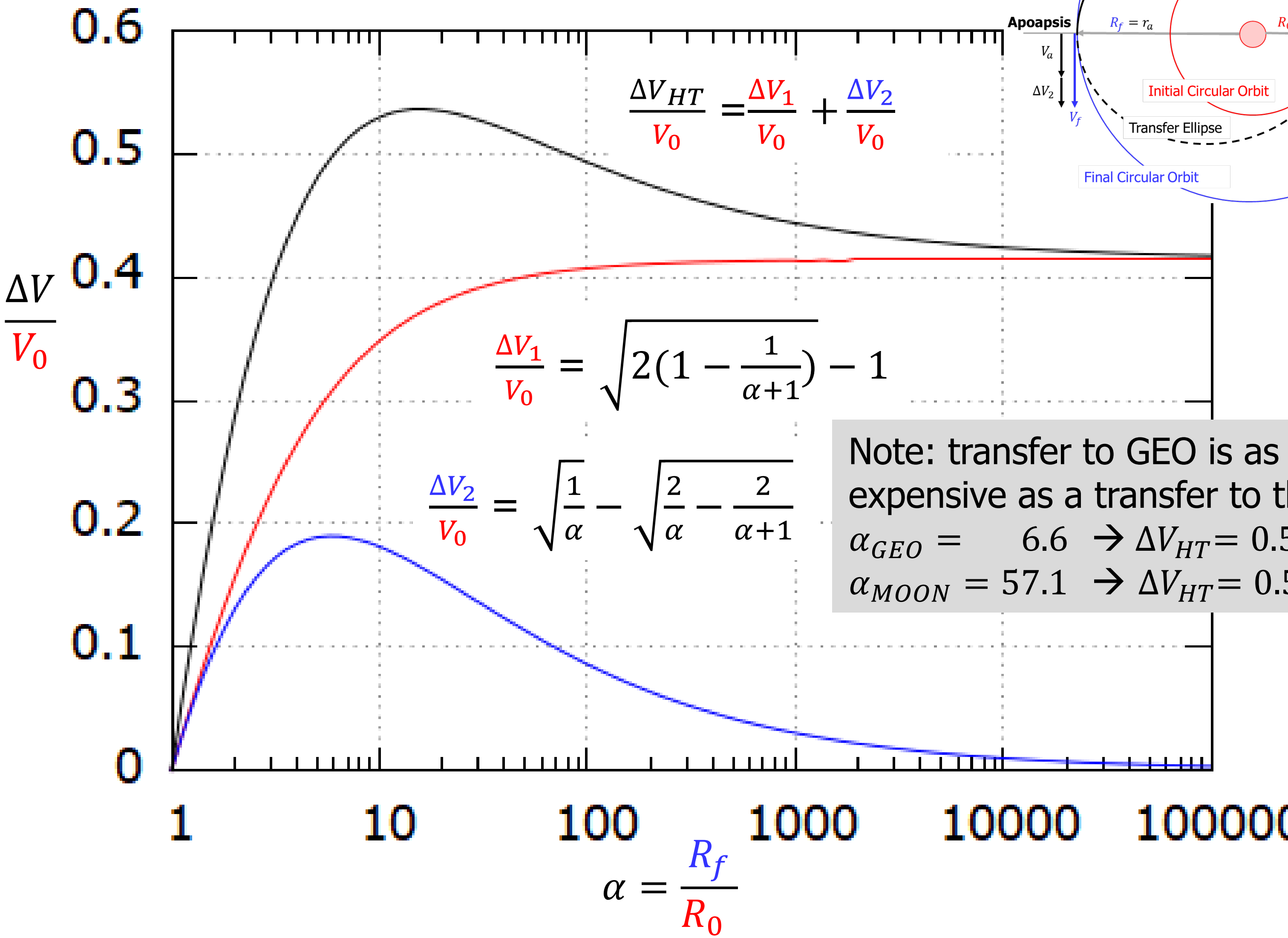
$$\Delta V = |\Delta V_1| + |\Delta V_2|$$

$$\frac{\Delta m}{m_0} = 1 - e^{-\frac{\Delta V}{I_{sp}\cdot g_{SL}}}$$

**Source:** Figure 6-1 Vallado DA (2013) "Fundamentals of Astrodynamics and Applications" 4th Edition

# Original Hohmann Transfer




$$\frac{\Delta V_{HT}}{V_0} = \frac{\Delta V_1}{V_0} + \frac{\Delta V_2}{V_0}$$

$$\frac{\Delta V_1}{V_0} = \sqrt{2\left(1 - \frac{1}{\alpha+1}\right)} - 1$$

$$\frac{\Delta V_2}{V_0} = \sqrt{\frac{1}{\alpha}} - \sqrt{\frac{2}{\alpha} - \frac{2}{\alpha+1}}$$

Note: transfer to GEO is as expensive as a transfer to the moon.

$\alpha_{GEO} = \quad 6.6 \rightarrow \Delta V_{HT} = 0.50 \cdot V_0$

$\alpha_{MOON} = 57.1 \rightarrow \Delta V_{HT} = 0.51 \cdot V_0$

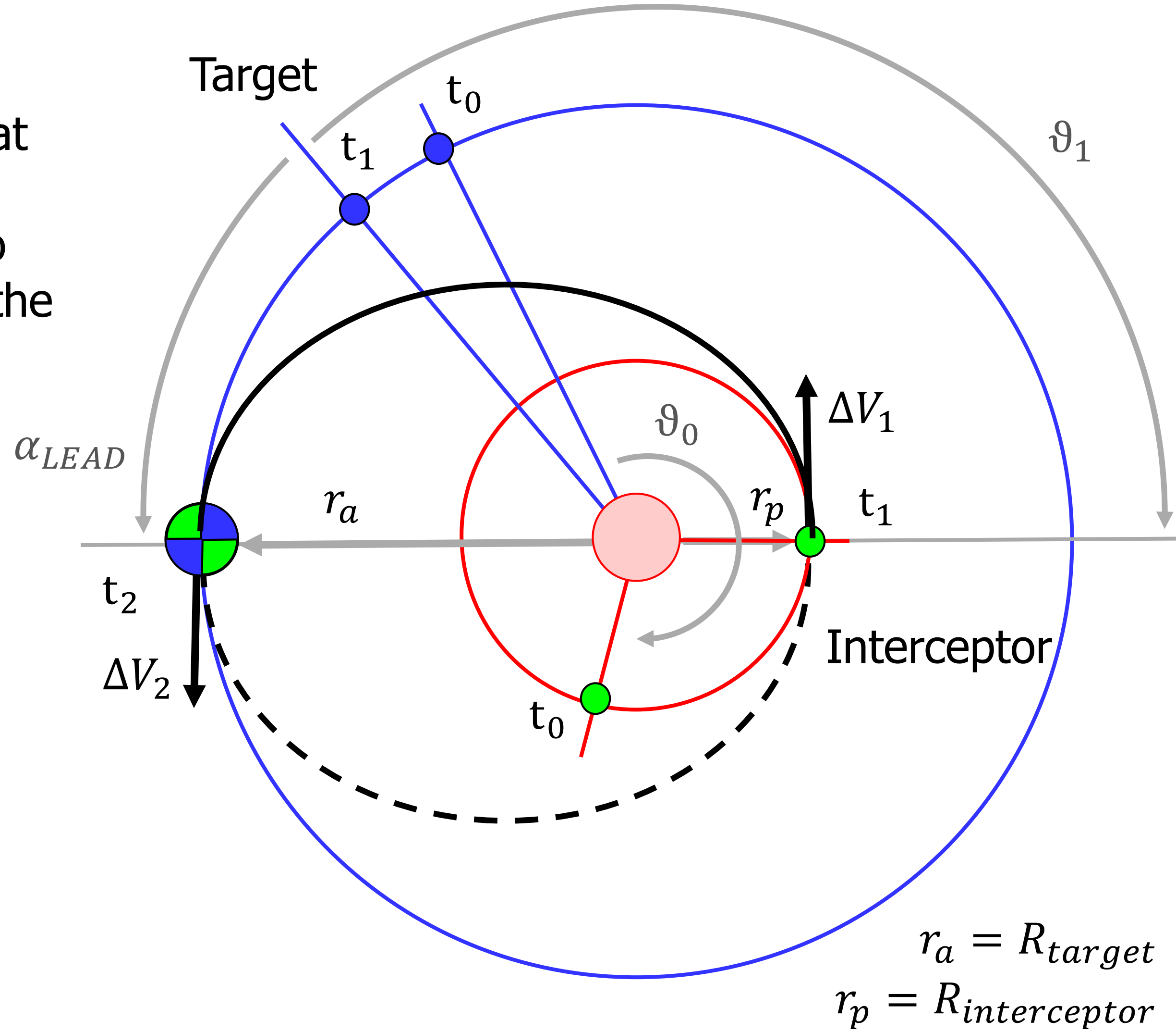
```
gnuplot set logscale x; set xtics auto; plot [1:100000] sqrt(2*(1-1/(x+1)))-1 lc "red" noti, sqrt(1/x)-
sqrt(2/x-2/(1+x)) lc "blue" noti, sqrt(2*(1-1/(x+1)))-1+sqrt(1/x) - sqrt(2/x-2/(1+x)) noti lc "black"
```

**Source:** Figure 8-12 Walter U (2019) “Astronautics. The Physics of Space Flight” (3rd Edition) Springer.

# Circular Rendezvous different orbits

Maneuver:

- Target and Interceptor are in **<u>different</u>** circular coplanar coaxial orbits
- 2 tangential Impulses $\Delta V_T$ at $\theta = 0°$ and $\theta = 180°$
- $\vartheta$ : angle from the target to the interceptor, positive in the target's motion
  - $\vartheta_0 < 0$
  - $\vartheta_1 < 0$
  - $\alpha_{LEAD} > 0$




**Source:** Figure 6-17 Vallado DA (2013) "Fundamentals of Astrodynamics and Applications" 4th Edition

# Circular Rendezvous different orbits

The mean anomaly motion of the circular orbits:

$$n_{target} = \sqrt{\mu/R_{tgt}^3} \qquad n_{int} = \sqrt{\mu/R_{int}^3}$$

The semi-major axis seizes the transfer time:

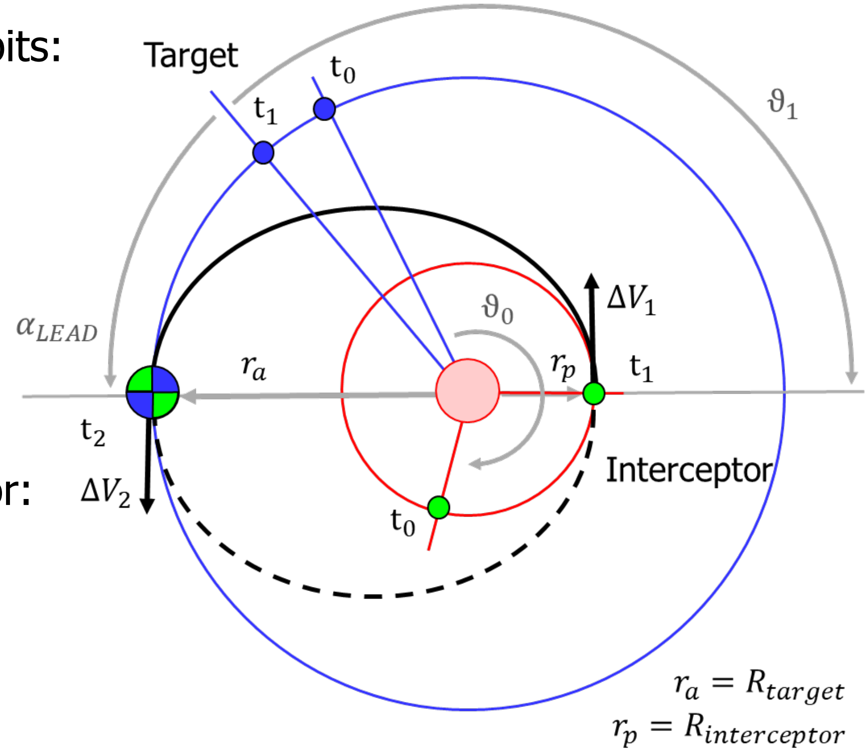
$$a_{trans} = (r_a + r_p)/2$$

$$\Delta t_{trans} = T_{trans}/2 = \pi\sqrt{a_{trans}^3/\mu}$$

The angle $\vartheta$ from the target to the interceptor:

$$\alpha_{LEAD} = n_{target}\,\Delta t_{trans}$$

$$\vartheta_1 = \alpha_{LEAD} - \pi$$




The $\Delta V$ budget:

$$v_{int} = \sqrt{\frac{\mu}{R}} \qquad v_p = \sqrt{\frac{2\mu}{a_{trans}} - \frac{\mu}{r_p}} \quad \Bigg\} \quad \Delta V_1 = V_p - V_{int}$$

$$v_a = \sqrt{\frac{2\mu}{a_{trans}} - \frac{\mu}{r_a}} \qquad v_{target} = \sqrt{\frac{\mu}{R_{target}}} \quad \Bigg\} \quad \Delta V_2 = V_{target} - V_a$$

$$\Delta V_T = |\Delta V_1| + |\Delta V_2|$$

Time of correct geometry after k revolutions:

$$T_{wait} = \frac{\vartheta_0 - \vartheta_1 + 2\pi k}{n_{int} - n_{target}}$$

# Module 3 Requirements

**Objectives:**

- **Reference Systems: Coordinates and Time**
- **Keplerian Orbits**
- **Orbital Maneuvers**
- **Interplanetary Trajectories**
  - **Sphere Of Influence**
  - **Patched Conics**

**Time Allocation:**

- **8 hours**

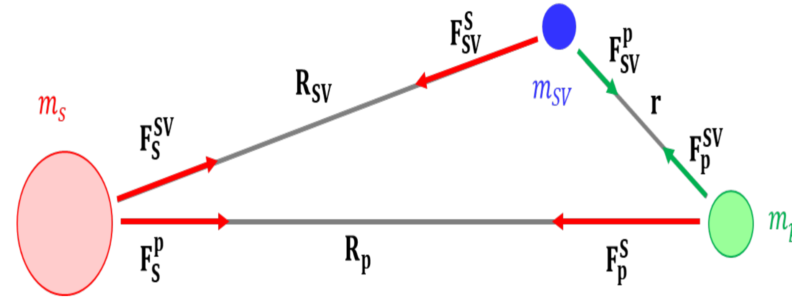


Adrià Rovira-Garcia

# Sphere of Influence (SOI)

**Inside the planet SOI**, the motion of the Space Vehicle is determined by its equations of motion relative to the planet: $\ddot{\mathbf{r}}$.

**Outside the planet SOI**, the motion of the spacecraft is computed relative to the sun: $\ddot{\mathbf{R}}_{\mathbf{SV}}$.




$$\begin{cases} \ddot{\mathbf{r}} = -G\dfrac{m_S}{R_P^3}\mathbf{r} - G\dfrac{m_P}{r^3}\mathbf{r} \\ \ddot{\mathbf{R}}_{\mathbf{SV}} = -G\dfrac{m_S}{R_{SV}^3}\mathbf{R}_{\mathbf{SV}} - G\dfrac{m_P}{r^3}\mathbf{r} \end{cases} \qquad \begin{cases} \ddot{\mathbf{r}} = \mathbf{a_S} + \mathbf{a_P} \\ \ddot{\mathbf{R}}_{\mathbf{SV}} = \mathbf{A_S} + \mathbf{A_P} \end{cases}$$

$\dfrac{\mathbf{a_S}}{\mathbf{a_P}}$ : the perturbing effect of the **sun** on the vehicle's orbit around the planet

$\dfrac{\mathbf{A_P}}{\mathbf{A_S}}$: the perturbing effect of the **planet** on the vehicle's orbit around the sun

$$\frac{\mathbf{a_S}}{\mathbf{a_P}} < \frac{\mathbf{A_P}}{\mathbf{A_S}}$$

$$\frac{G\dfrac{m_S}{R_P^3}r}{G\dfrac{m_P}{r^2}} < \frac{G\dfrac{m_P}{r^2}}{G\dfrac{m_S}{R_{SV}^2}}$$

$$\frac{m_S}{m_P}\left(\frac{r}{R_P}\right)^3 < \frac{m_P}{m_S}\left(\frac{R_{SV}}{r}\right)^2$$

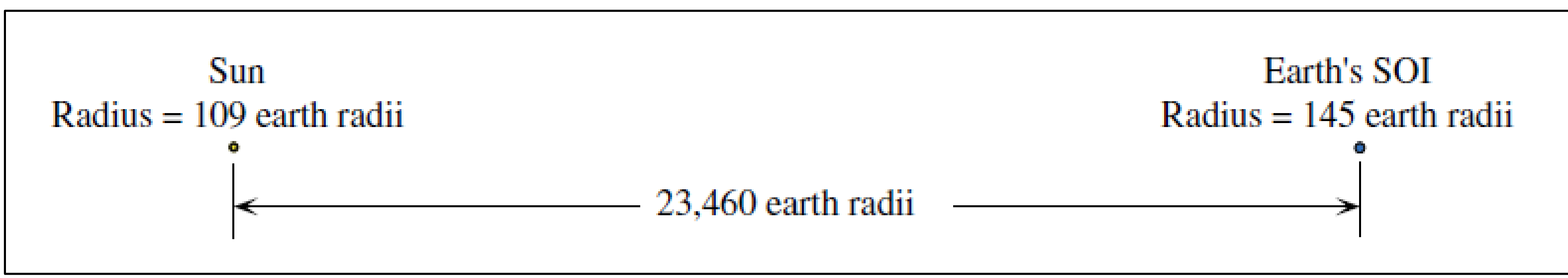
since $R_P \approx R_{SV}$

$$\boxed{\frac{r_{SOI}}{R_P} = \left(\frac{m_P}{m_S}\right)^{2/5}}$$

**Source:** Fig. 8.6 Curtis HD (2014) "Orbital Mechanics for Engineering Students" (3rd Edition) Elsevier.

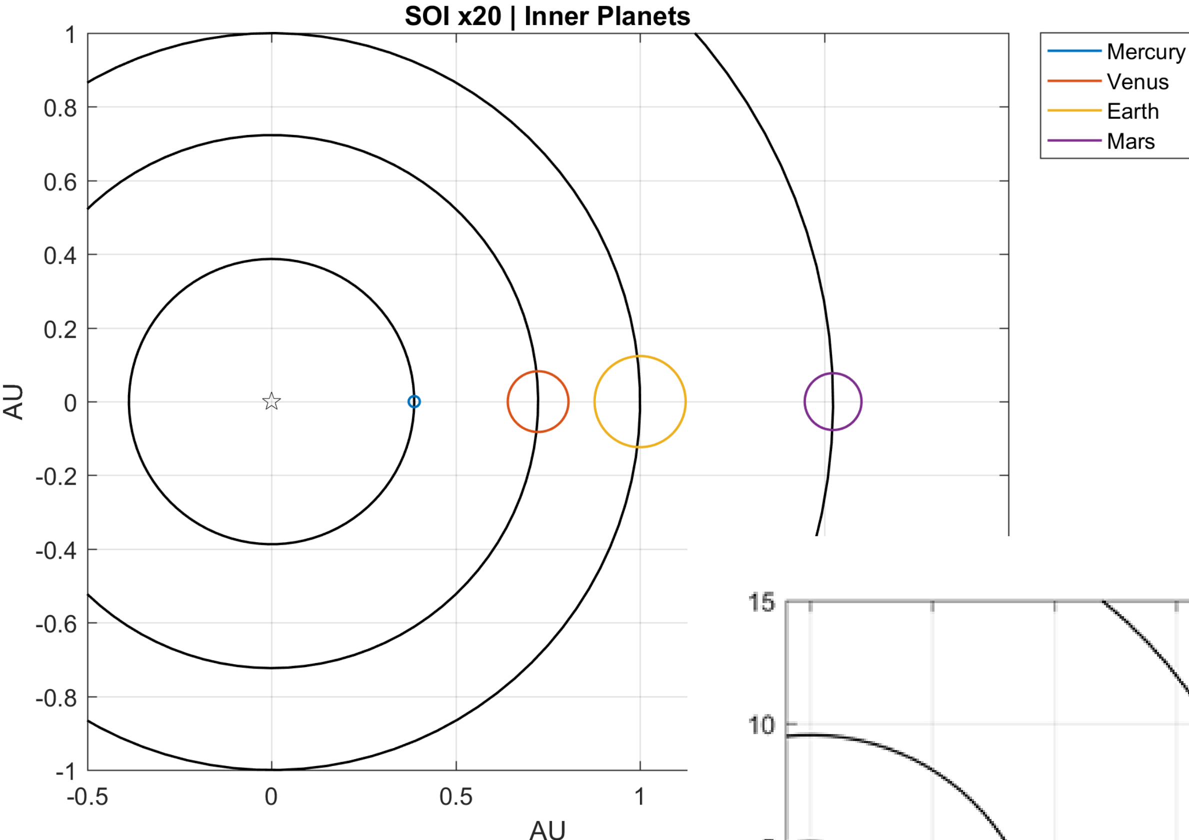
# Sphere of Influence (SOI)

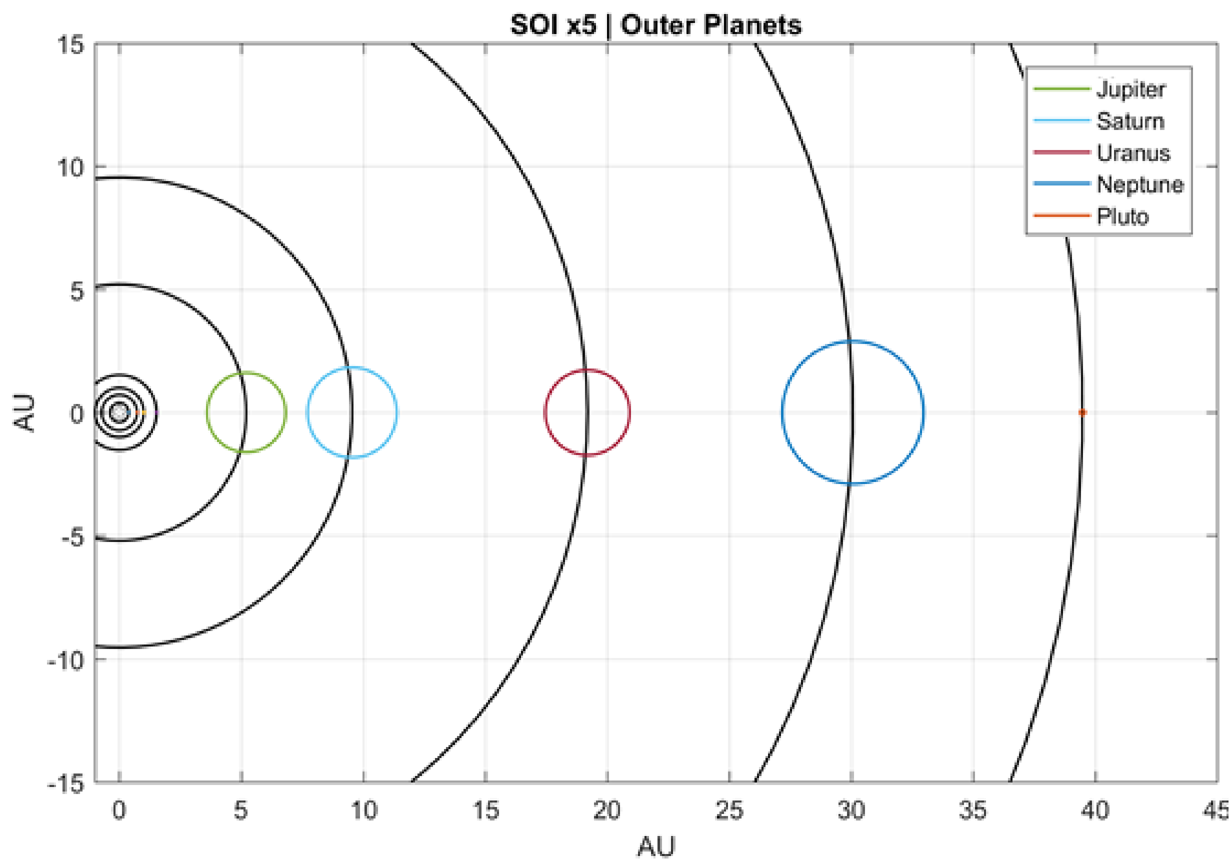
| Body | Semi-major axis (km) | Mass (kg) | sphere of influence (km) |
|---|---:|---:|---:|
| Sun | - | 1.99E+30 | - |
| Moon | 384,400 | 7.34E+22 | 66,100 |
| Mercury | 57,909,083 | 3.30E+23 | 112,000 |
| Venus | 108,208,601 | 4.87E+24 | 616,000 |
| Earth | 149,598,023 | 5.97E+24 | 925,000 |
| Mars | 227,939,186 | 6.42E+23 | 577,000 |
| Jupiter | 778,298,361 | 1.90E+27 | 48,200,000 |
| Saturn | 1,429,394,133 | 5.69E+26 | 54,800,000 |
| Uranus | 2,875,038,615 | 8.67E+25 | 51,800,000 |
| Neptune | 4,504,449,769 | 1.03E+26 | 86,600,000 |
| Pluto | 5,915,799,000 | 1.50E+22 | 3,080,000 |




**Source:**
Table D-3 Vallado DA (2013) "Fundamentals of Astrodynamics and Applications" 4th Edition
Table A.2 and Fig 8.7 Curtis HD (2014) "Orbital Mechanics for Engineering Students" (3rd Edition) Elsevier.

# Sphere of Influence (SOI)




**Source:** De La Torre (2017) “Astrodynamics Lectures Notes”

# Patched Conics

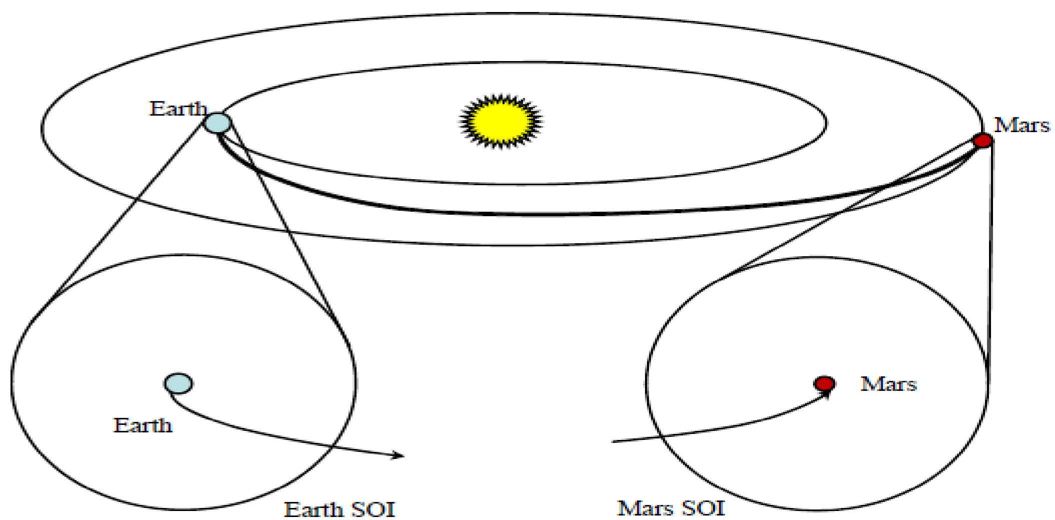
A method to estimate ballistic interplanetary trajectories (i.e. that don't require any propulsion between planetary encounters).

The **central assumption** for the patched conics method is that at one time, there is **only 1 central body** (mass) that acts on the spacecraft.

Inside every sphere of influence, we can apply the two-body equations. These orbital arcs are joined (i.e. patched) at the **edge of the sphere of influence**

Phases:

1. Departure with an escape hyperbola.
2. Interplanetary cruise, modelled as an heliocentric transfer ellipse.
3. Arrival with an approach hyperbola at the target.




**Source:** Fig. 12-2 Vallado DA (2013) "Fundamentals of Astrodynamics and Applications" 4th Edition

Escola Superior d'Enginyeries Industrial, Aeroespacial i Audiovisual de Terrassa (ESEIAAT)

AMVO

Adrià Rovira-Garcia

# Back up slides

# Coordinates Conversion: Ellipsoidal to Cartesian

The Cartesian coordinates of a point $(x, y, z)$ can be obtained from the ellipsoidal geodetic coordinates $(\varphi, \lambda, h)$ by the expressions:

$$x = (N + h)\cos\varphi\,\cos\lambda$$
$$y = (N + h)\cos\varphi\,\sin\lambda$$
$$z = \left((1 - e^2)N + h\right)\sin\varphi$$

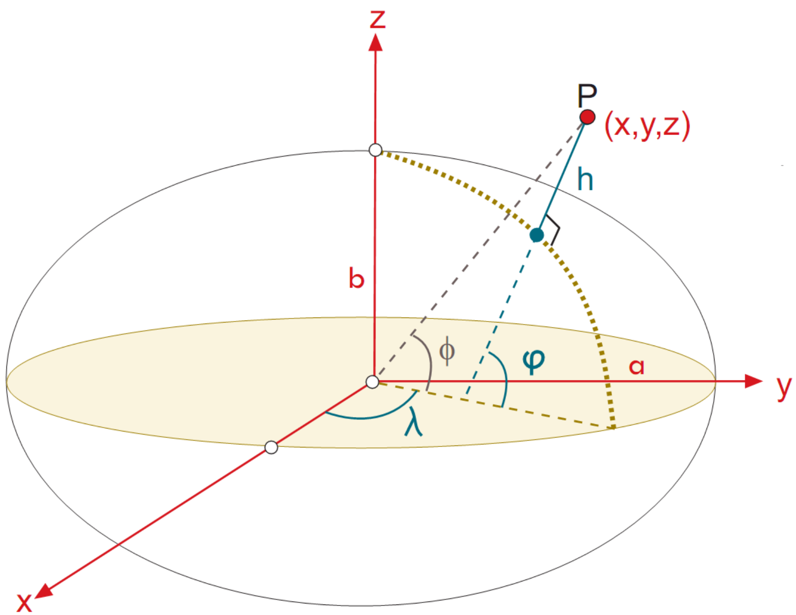
where $N$ is the radius of curvature in the prime vertical

$$N = \frac{a}{\sqrt{1 - e^2\sin^2\varphi}}$$

and where the eccentricity e is related to the semi-major axis a, the semi-minor axis b and the flattening factor by

$$f = 1 - b/a$$

$$e^2 = \frac{a^2 - b^2}{a^2} = 2f - f^2$$




**Source:**
Sanz J, Juan JM, Rovira-Garcia A (2019) “GNSS Data Processing Lectures” ISBN 978-84-93230-7-6

Escola Superior d'Enginyeries Industrial, Aeroespacial i Audiovisual de Terrassa (ESEIAAT)

# Coordinates Conversion: Cartesian to Ellipsoidal

The ellipsoidal geodetic coordinates of a point $(\varphi, \lambda, h)$ can be obtained from theCartesian coordinates $(x, y, z)$ by the expressions:

The longitude is given by:

$$\lambda = \arctan \frac{y}{x}$$

The latitude is computed by an iterative procedure. The initial value is given:

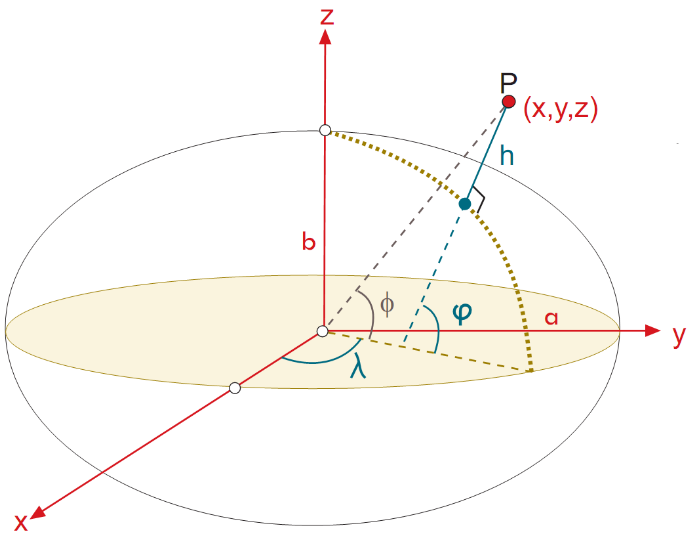
$$\varphi_{(0)} = \arctan \left[ \frac{z/p}{1 - e^2} \right] \qquad p = \sqrt{x^2 + y^2}$$

Improved values of $\varphi$, as well as the height $h$, are computed by iterating the equations:

$$N_{(i)} = \frac{a}{\sqrt{1 - e^2 \sin^2 \varphi_{(i-1)}}}$$

$$h_{(i)} = \frac{p}{\cos \varphi_{(i-1)}} - N_{(i)}$$

$$\varphi_{(i)} = \arctan \left[ \frac{z/p}{1 - \frac{N_{(i)}}{N_{(i)} + h_{(i)}} e^2} \right]$$




*The iterations are repeated until the change between two successive values of $\varphi(i)$ is smaller than the precision required.*

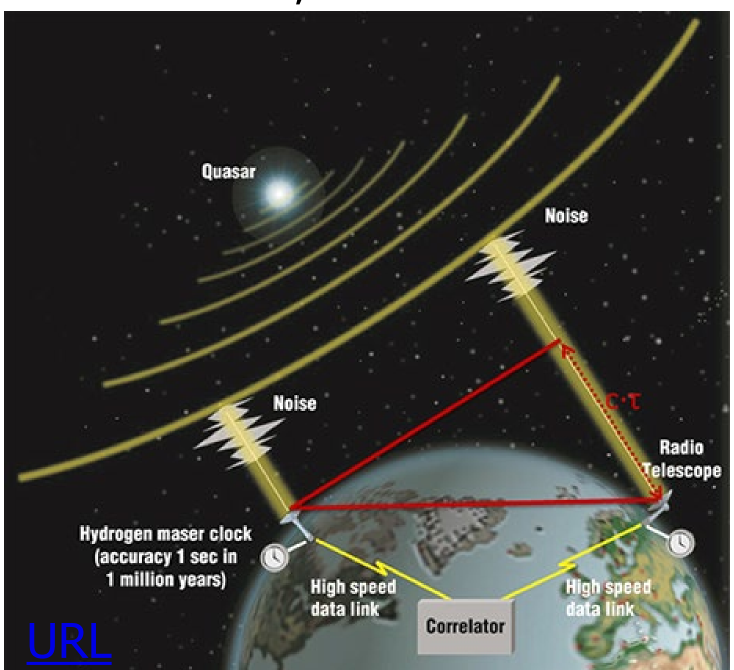
**Source:**
Sanz J, Juan JM, Rovira-Garcia A (2019) "GNSS Data Processing Lectures" ISBN 978-84-93230-7-6

# International Celestial Reference Frame (ICRF)

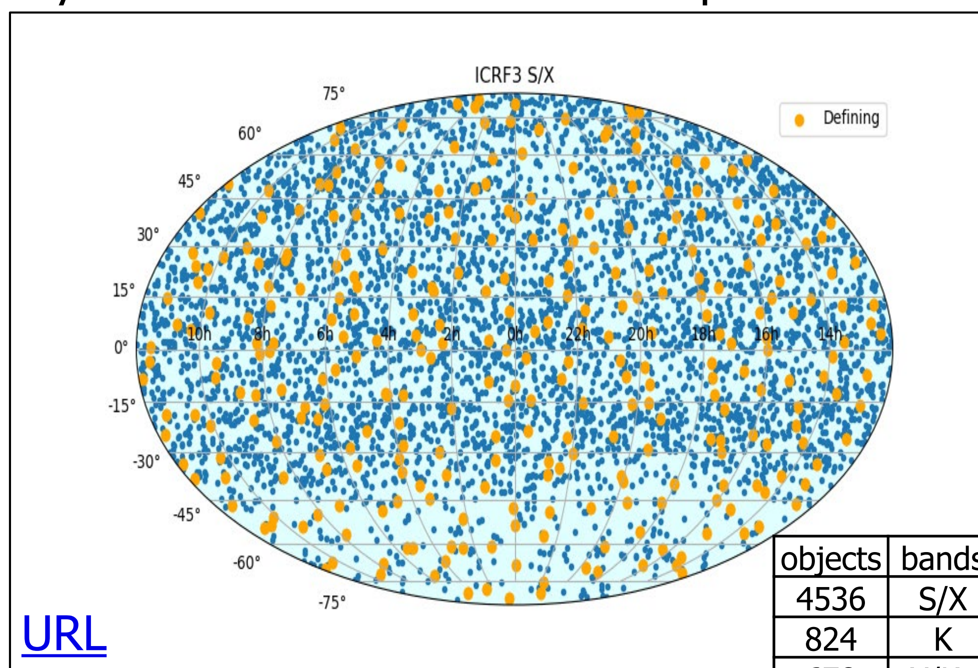
ICRS is successive realized in ICRF.
ICRF is our best estimate of an inertial frame located at Solar System barycenter
Current: ICRF#3 adopted 30/08/2018 by International Astronomy Union (IAU).

ICRS data acquired by Very Long Baseline Interferometry (VLBI)
- ~40 years of geodetic and astrometric radio frequencies (2.3 and 8.4 GHz)
- ~15 years of higher radio freqs (24 GHz and dual-frequency 8.4/32 GHz)

ICRF3 contains positions for 4536 Active Galactic Nuclei (AGN), supermassive black holes in the center of distant galaxies, uniformly distributed on the sky:
- very bright in radio frequencies
- no wobble, nor move → extremely distant and stable reference points




URL



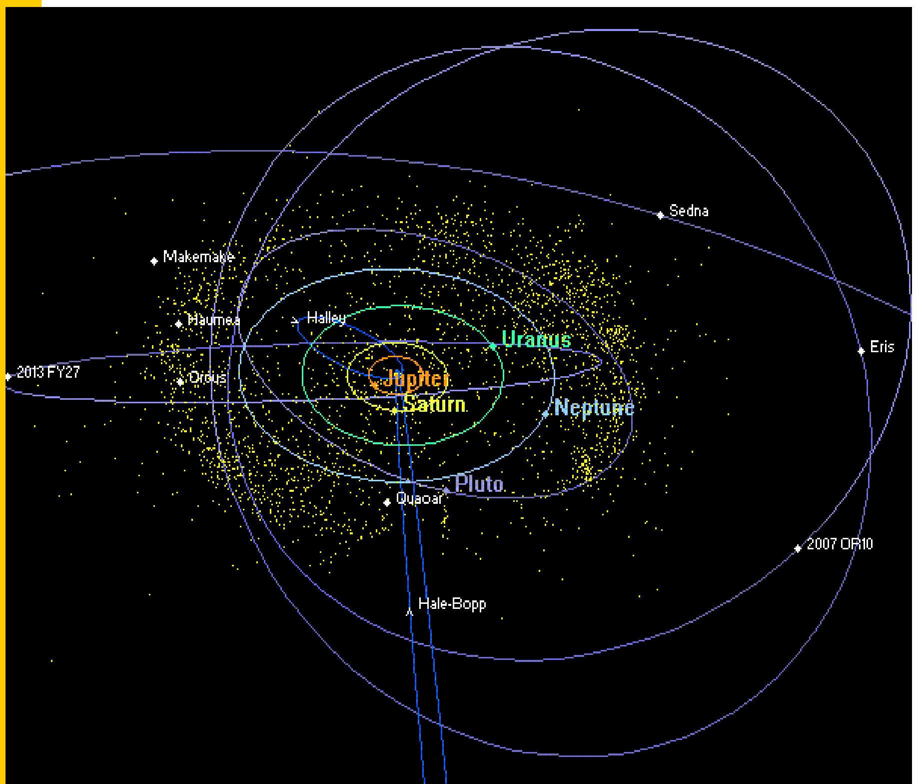

URL

| objects | bands |
|---|---|
| 4536 | S/X |
| 824 | K |
| 678 | X/Ka |

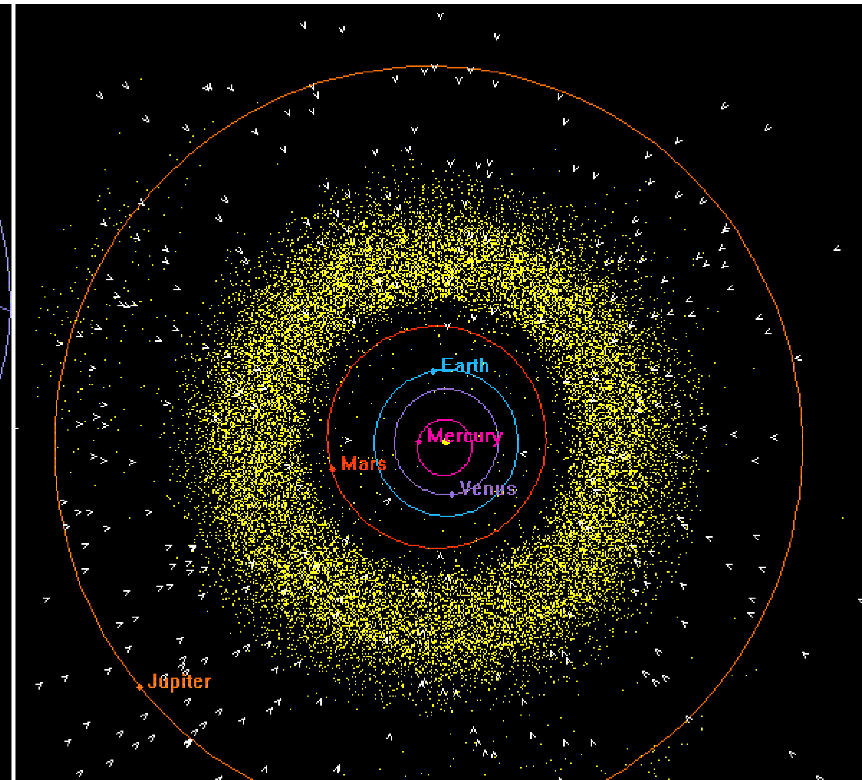
# International Celestial Reference Frame (ICRF)

ICRF usages:

- Earth-orbiting spacecraft to determine their orientation
- Ground stations tracking space probes navigating through the solar system
- To measure Earth motions in 3-D inertial space with very high precision




objects with semi-major axes greater than 6 AU [URL](URL)
(orbital periods > ~15 years) on 1/1/2018




diagrams of the inner solar system on 1/1/2018 [URL](URL)

**Source:**

US Naval Observatory (USNO) "Naval Oceanography Portal"

Jet Propulsion Laboratory (JPL) "Planetary and lunar ephemerides with respect to the solar system barycenter, in rectangular coordinates, with respect to the ICRS axes"

# International Terrestrial Reference Frame (ITRF)

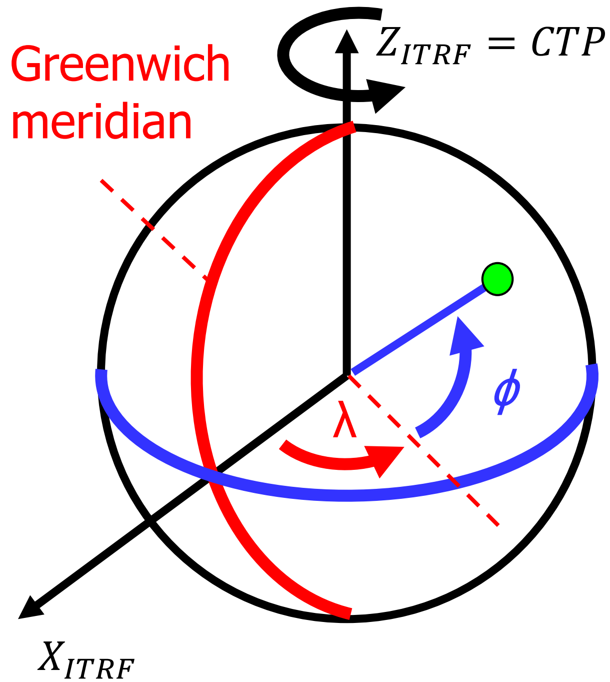
ITRF is a realization of ITRS by a set of instantaneous coordinates (and velocities) of reference points distributed on the topographic surface of the Earth (mainly space geodetic stations and related markers)

**ITRF is called Earth Centered Earth Fixed (ECEF) Frame**

ITRF Orientation initially given by the Bureau International de l'Heure (BIH) System at **1984.0** within +/- 3 milli-arcseconds

ITRF Axis:

- The *z* of the frame is the origin of the polar motion coordinate system → Conventional Terrestrial Pole (CTP) average of the poles from 1900 to 1905, by the Bureau International de l'Heure (BIH)
- *X* and *Y* axes define the terrestrial equatorial plane.
- X axis lies in what may be loosely called the Greenwich meridian.




*Note: ITRF first alignment was close to the mean equator of 1900 and the Greenwich meridian for continuity with previous terrestrial reference systems.*

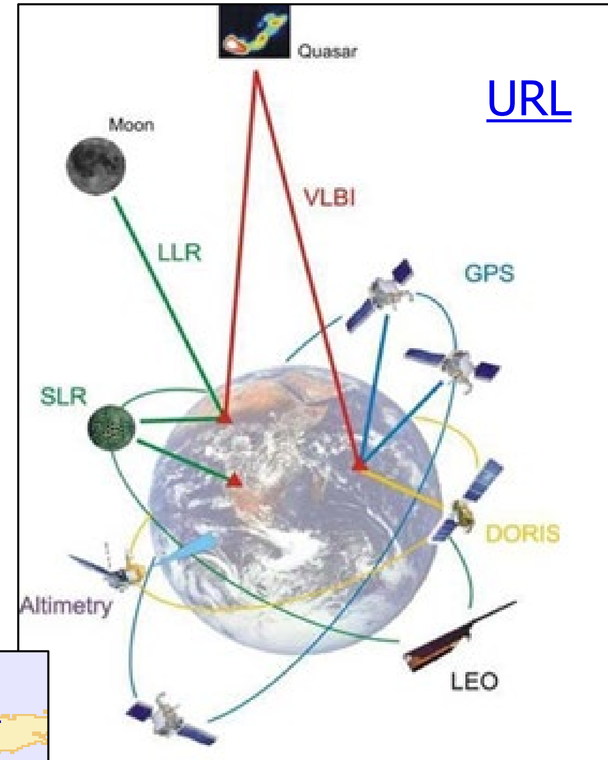

Pisacane VL (2005) Fundamentals of space systems. 2nd ed. Oxford: Oxford University Press
Leick A, Rapoport L, Tatarnikov D (2015) "GPS satellite surveying"

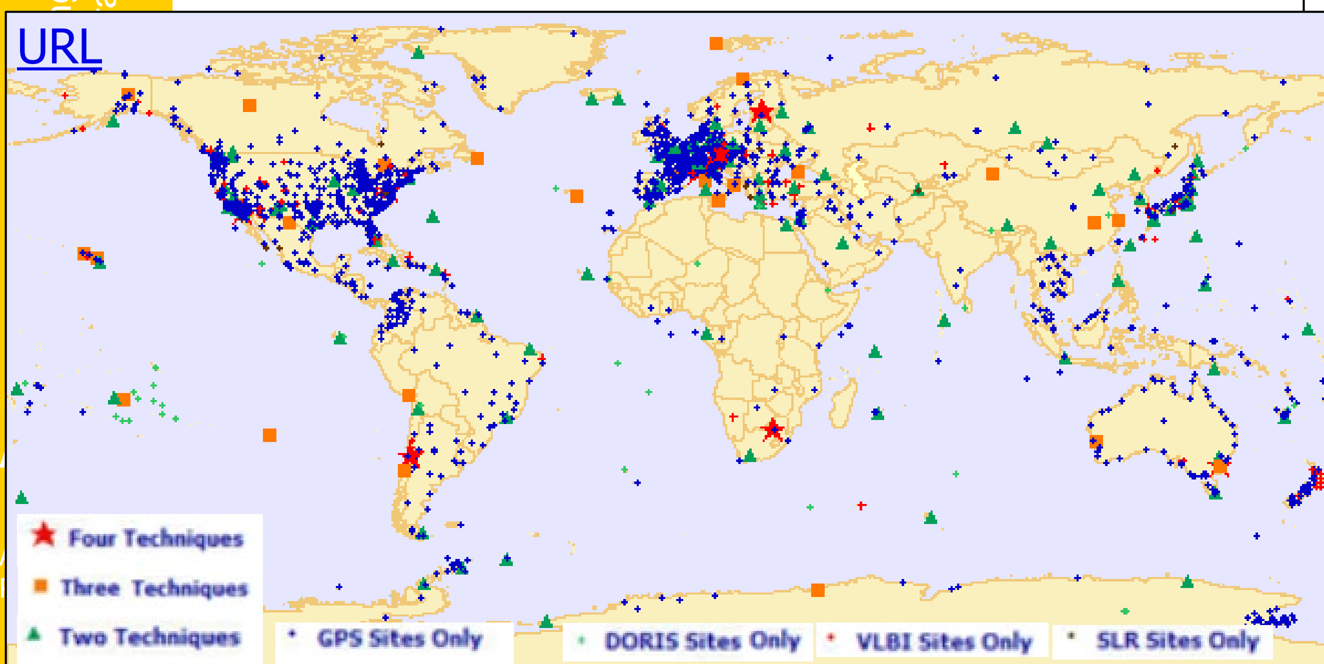
# International Terrestrial Reference Frame (ITRF)

ITRS is realized as the ITRF: with positions and velocities of a set of stations observed

- Very Long Baseline Interferometry (VLBI)
- Lunar Laser Ranging (LLR)
- Global Positioning System (GPS)
- Doppler Orbitography and Radio Positioning Integrated by Satellites (DORIS)

ITRF is a realization of ITRS by a set of instantaneous coordinates (and velocities) of reference points distributed on the topographic surface of the Earth (mainly space geodetic stations and related markers)



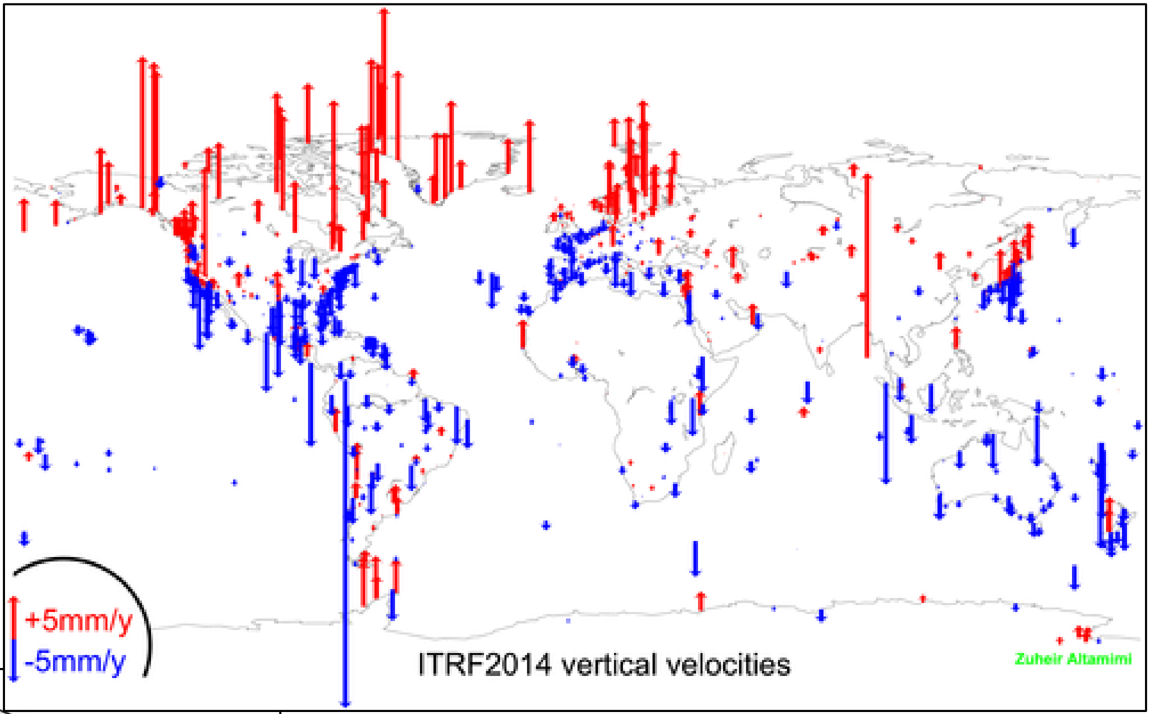

IERS Domes

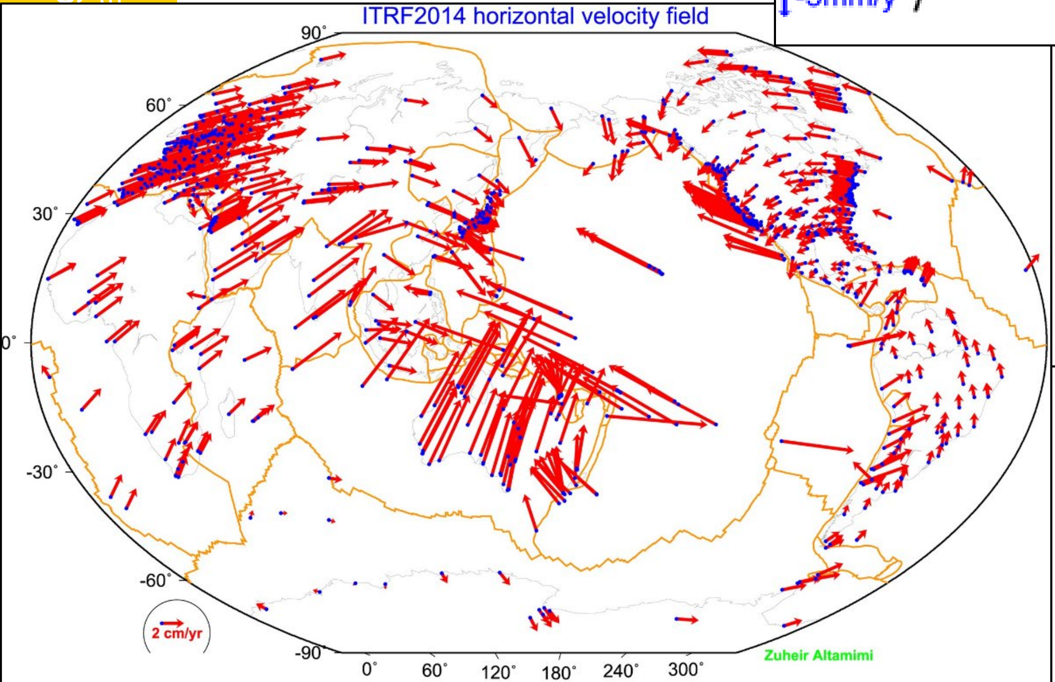
Pisacane VL (2005) Fundamentals of space systems. 2nd ed. Oxford: Oxford University Press

# International Terrestrial Reference Frame (ITRF)



Because of continental drift, positions in the ITRF are time dependent (position and velocity).




Versions:

- ITRF2014: 21/01/2016
- ITRF2008: 31/05/2010

**Source:**

Altamimi Z et al (2016) "ITRF2014: A new release of the International Terrestrial Reference Frame modeling nonlinear station motions" Journal of Geophysical Research: Solid Earth 121:6109–6131

# Calculation from the state vector (r, v) to orbital elements (a, e, Ω, *i*, ω, θ)

1. Distance: $\mathrm{r} = \sqrt{\mathbf{r} \cdot \mathbf{r}} = \sqrt{\mathrm{X}^2 + \mathrm{Y}^2 + \mathrm{X}^2}$ and radial unit vector $\hat{\boldsymbol{r}} = \boldsymbol{r}/r$

2. Velocity: $\mathrm{v} = \sqrt{\mathbf{v} \cdot \mathbf{v}} = \sqrt{\mathrm{v}_\mathrm{x}^2 + \mathrm{v}_\mathrm{y}^2 + \mathrm{v}_\mathrm{z}^2}$

3. Radial velocity: $\mathrm{v}_\mathrm{r} = \frac{\mathbf{r} \cdot \mathbf{v}}{\mathrm{r}} \begin{cases} \mathrm{v}_\mathrm{r} < 0 & \mathit{flying\ toward\ perigee} \\ \mathrm{v}_\mathrm{r} > 0 & \mathit{flying\ away\ from\ perigee} \end{cases}$

4. Orbit Eccentricity $\mathrm{e}$

$$\mathbf{e} = \frac{1}{\mu}\left[\left(\mathrm{v}^2 - \frac{\mu}{\mathrm{r}}\right)\mathbf{r} - \mathrm{r} \cdot \mathrm{v}_\mathrm{r} \cdot \mathbf{v}\right] \text{ and } \mathrm{e} = \sqrt{1 + \frac{\mathrm{h}^2}{\mu^2}\left(\mathrm{v}^2 - \frac{2 \cdot \mu}{\mathrm{r}}\right)} \begin{cases} e = 0 & \mathit{circle} \\ 0 < e < 1 & \mathit{elipse} \\ e = 1 & \mathit{parabola} \\ e > 1 & \mathit{hyperbola} \end{cases}$$

unit vector $\hat{\mathbf{e}} = \mathbf{e}/\mathrm{e}$

5. Specific angular momentum $\mathbf{h} = \mathbf{r} \times \mathbf{v}$ and $\mathrm{h} = |\mathbf{h}|$ and Orbit Parameter $\mathrm{p} = \frac{\mathrm{h}^2}{\mu}$

6. Semi-major axis $a = \frac{1}{2}(r_p + r_a) = \frac{1}{2}\left(\frac{p}{1 + e \cdot cos\ 0} + \frac{p}{1 + e \cdot cos\ 180}\right) = \frac{p}{1 - e^2}$

7. Orbit inclination

$$\mathrm{i} = \arccos\left(\frac{\mathrm{h_z}}{\mathrm{h}}\right) \begin{cases} i = 0° & \mathit{Equatorial} \\ 0° < i < 90° & \mathit{Prograde\ (direct)} \\ i = 90° & \mathit{Polar} \\ 90° < i < 180° & \mathit{Retrograde} \end{cases}$$

**Source:** Curtis HD (2014) "Orbital Mechanics for Engineering Students" (3rd Edition) Elsevier.

# Calculation from the state vector (r, v) to orbital elements (a, e, Ω, $i$, ω, θ)

8. Line of Nodes $\mathbf{N} = \hat{\boldsymbol{K}} \times \mathbf{h} = \begin{vmatrix} \mathrm{I} & \mathrm{J} & \mathrm{K} \\ 0 & 0 & 1 \\ \mathrm{h_x} & \mathrm{h_y} & \mathrm{h_z} \end{vmatrix}$ and $\mathrm{N} = |\mathbf{N}|$ and in unit form $\hat{\mathbf{N}}$

9. Right Ascension of the Ascending Node Ω

$$\Omega = \begin{cases} \mathrm{N_Y} \geq 0 & \Omega = \arccos\left(\dfrac{\mathrm{N}_x}{N}\right) \\ \mathrm{N}_Y < 0 & \Omega = 360° - \arccos\left(\dfrac{\mathrm{N}_x}{N}\right) \end{cases}$$

10. Argument of periapsis ω

$$\omega = \begin{cases} \mathrm{e_z} \geq 0 & \omega = \arccos\left(\hat{\boldsymbol{N}} \cdot \hat{\mathbf{e}}\right) \\ \mathrm{e_z} < 0 & \omega = 360° - \arccos\left(\hat{\boldsymbol{N}} \cdot \hat{\mathbf{e}}\right) \end{cases}$$

11. True Anomaly θ

$$\theta = \begin{cases} \mathrm{v_r} \geq 0 & \theta = \arccos\left(\hat{\mathbf{e}} \cdot \hat{\boldsymbol{r}}\right) = \mathit{arcos}\left(\dfrac{1}{\mathrm{e}}\left[\dfrac{\mathrm{h}^2}{\mu \cdot \mathrm{r}} - 1\right]\right) \\ \mathrm{v_r} < 0 & \theta = 360° - \arccos\left(\hat{\mathbf{e}} \cdot \hat{\boldsymbol{r}}\right) = 360° - \mathit{arcos}\left(\dfrac{1}{\mathrm{e}}\left[\dfrac{\mathrm{h}^2}{\mu \cdot \mathrm{r}} - 1\right]\right) \end{cases}$$

**Source:** Curtis HD (2014) "Orbital Mechanics for Engineering Students" (3rd Edition) Elsevier.

# Calculation from orbital elements (a, e, Ω, *i*, ω, θ) to state vector (r, v)

$$\mathbf{r} = \frac{h^2}{\mu}\frac{1}{1+e\cdot\cos\theta}\begin{pmatrix}\cos\theta\\ \sin\theta\\ 0\end{pmatrix}_{\mathbf{PQW}} = \begin{pmatrix}r\cos\theta\\ r\sin\theta\\ 0\end{pmatrix}_{\mathbf{PQW}}$$

$$\mathbf{v} = \frac{\mu}{h}\begin{pmatrix}-\sin\theta\\ e+\cos\theta\\ 0\end{pmatrix}_{\mathbf{PQW}}$$

$$\begin{pmatrix}X\\ Y\\ Z\end{pmatrix}_{\boldsymbol{IJK}} = \boldsymbol{R_3}(-\Omega)\boldsymbol{R_1}(-i)\boldsymbol{R_3}(-\omega)\begin{pmatrix}r\cos\theta\\ r\sin\theta\\ 0\end{pmatrix}_{\mathbf{PQW}}$$

$$\begin{pmatrix}V_x\\ V_y\\ V_z\end{pmatrix}_{\boldsymbol{IJK}} = \boldsymbol{R_3}(-\Omega)\boldsymbol{R_1}(-i)\boldsymbol{R_3}(-\omega)\frac{\mu}{h}\begin{pmatrix}-\sin\theta\\ e+\cos\theta\\ 0\end{pmatrix}_{\mathbf{PQW}}$$

$$R = \begin{pmatrix}\cos\Omega & -\sin\Omega & 0\\ \sin\Omega & \cos\Omega & 0\\ 0 & 0 & 1\end{pmatrix}\begin{pmatrix}1 & 0 & 0\\ 0 & \cos i & -\sin i\\ 0 & \sin i & \cos i\end{pmatrix}\begin{pmatrix}\cos\omega & -\sin\omega & 0\\ \sin\omega & \cos\omega & 0\\ 0 & 0 & 1\end{pmatrix}$$

$$R = \begin{pmatrix}\cos\Omega\cdot\cos\omega-\sin\Omega\cdot\cos i\cdot\sin\omega & -\cos\Omega\cdot\sin\omega-\sin\Omega\cdot\cos i\cdot\cos\omega & \sin\Omega\cdot\sin i\\ \sin\Omega\cdot\cos\omega+\cos\Omega\cdot\cos i\cdot\sin\omega & -\sin\omega\cdot\sin\Omega+\cos\Omega\cdot\cos i\cdot\cos\omega & -\cos\Omega\cdot\sin i\\ \sin i\cdot\sin\omega & \sin i\cdot\cos\omega & \cos i\end{pmatrix}$$