A

MAT 1320 B Fall 2011 November 16th, 8:30 Prof. Desjardins

TEST #2

Max = 20

Student Number: $\boxed{\text{Solutions}}$

- Time: 80 min.
- Only basic scientific calculators are permitted: non-programmable, non-graphing, no differentiation or integration capability. Notes or books are not permitted.
- Work all problems in the space provided. Use the backs of the pages for rough work if necessary. Do not use any other paper.
- Write *only* in non-erasable ink (ball-point or pen), not in pencil. Cross out, if necessary, but do not erase or overwrite. Graphs and sketches may be drawn in pencil.
- Problems require complete and clearly presented solutions and carry part marks if there is substantial correct work toward the solution.

A

1. [1 point] Differentiate $g(\theta) = 2\sin(\tan(e^{\sec(\theta)}))$.

$$\boxed{g'(\theta) = 2\cos(\tan(e^{\sec\theta}))\sec^2(e^{\sec\theta})\, e^{\sec\theta} \sec\theta\tan\theta}$$

2. [3 points] Use logarithmic differentiation to find the derivative of $f(x) = \dfrac{\cos^3 x \arcsin x}{(x+3)^4}$.

$$\ln(f(x)) = \ln\left(\frac{\cos^3 x \arcsin x}{(x+3)^4}\right) = 3\ln(\cos x) + \ln(\arcsin x) - 4\ln(x+3)$$

so $$\frac{d}{dx}(\ln(f(x))) = \frac{d}{dx}(3\ln(\cos x) + \ln(\arcsin x) - 4\ln(x+3))$$

$$\frac{1}{f(x)} f'(x) = \frac{-3\sin x}{\cos x} + \frac{1}{\arcsin x}\frac{1}{\sqrt{1-x^2}} - \frac{4}{x+3}$$

thus $$\boxed{f'(x) = \frac{\cos^3 x \arcsin x}{(x+3)^4}\left[-3\tan x + \frac{1}{\arcsin x\sqrt{1-x^2}} - \frac{4}{x+3}\right]}$$

3. [2 points] Find the equation of the tangent line to the curve $x^2y^3 - 2e^x\cos y = 2x$ at the point $(0, \pi/2)$.

$$\frac{d}{dx}\left(x^2y^3 - 2e^x\cos y\right) = \frac{d}{dx}(2x)$$

$$2xy^3 + 3x^2y^2\,y' - 2e^x\cos y + 2e^x\sin y\,y' = 2$$

so $$y' = \frac{2 - 2xy^3 + 2e^x\cos y}{3x^2y^2 + 2e^x\sin y}$$

when $x = 0,\ y = \pi/2,\ y' = \frac{2}{2} = 1$

so $y - \pi/2 = (1)(x - 0)$

or $\boxed{y = x + \pi/2}$

4. [3 points] If $f''(x) = 6e^{3x} + 9x^2 - 3\sin x$, $f(0) = 1$ and $f'(0) = 1$, what is f?

$$f'(x) = 2e^{3x} + 3x^3 + 3\cos x + C$$

$$f'(0) = 2 + 3 + C = 1 \Rightarrow C = -4$$

so $f'(x) = 2e^{3x} + 3x^3 + 3\cos x - 4$

then $f(x) = \frac{2}{3}e^{3x} + \frac{3}{4}x^4 + 3\sin x - 4x + K$

but $f(0) = \frac{2}{3} + K = 1 \Rightarrow K = 1/3$

$$\therefore\ \boxed{f(x) = \frac{2}{3}e^{3x} + \frac{3}{4}x^4 + 3\sin x - 4x + 1/3}$$

A

5. [8 points] Evaluate the following:

(a) $\displaystyle\int \frac{4x}{1+x^4}\,dx$ (let $u = x^2$, then $du = 2x\,dx$)

$$= \int \frac{2}{1+u^2}\,du$$

$$= 2\arctan(u) + C$$

$$= \boxed{2\arctan(x^2) + C}$$

(b) $\displaystyle\int x^2 \sin(2x)\,dx$ $\left(\begin{array}{ll} u = x^2 & dv = \sin(2x)\,dx \\ du = 2x\,dx & v = -\frac{1}{2}\cos(2x) \end{array}\right)$

$$= -\frac{1}{2}x^2\cos(2x) + \int x\cos(2x)\,dx \qquad \left(\begin{array}{ll} u = x & dv = \cos(2x)\,dx \\ du = dx & v = \frac{1}{2}\sin(2x) \end{array}\right)$$

$$= -\frac{1}{2}x^2\cos(2x) + \frac{1}{2}x\sin(2x) - \frac{1}{2}\int \sin(2x)\,dx$$

$$= \boxed{-\frac{1}{2}x^2\cos(2x) + \frac{1}{2}x\sin(2x) + \frac{1}{4}\cos(2x) + C}$$

A

(c) $\int \frac{3x-24}{x^2-2x-8}\,dx$

$$\left(\frac{3x-24}{(x-4)(x+2)} = \frac{A}{x-4} + \frac{B}{x+2} \qquad \begin{array}{ll} A+B=3 & 6A=-12 \\ 2A-4B=-24 & A=-2 \\ 4A+4B=12 & B=5 \end{array} \right)$$

$$= \int \left(\frac{5}{x+2} - \frac{2}{x-4} \right) dx$$

$$= \boxed{5 \ln|x+2| - 2\ln|x-4| + C}$$

(d) $\int \cos^2(t)\,\sin^5(t)\,dt$

$$= \int \cos^2 t \sin^4 t \sin t \, dt$$

$$= \int \cos^2 t \, (1-\cos^2 t)^2 \sin t \, dt$$

$$= \int \cos^2 t \, (1 - 2\cos^2 t + \cos^4 t) \sin t \, dt$$

$$= \int (\cos^2 t - 2\cos^4 t + \cos^6 t) \sin t \, dt \qquad \left(\begin{array}{l} \text{let } u = \cos t \\ du = -\sin t \, dt \end{array} \right)$$

$$= \int (-u^2 + 2u^4 - u^6) \, du$$

$$= -\frac{1}{3}u^3 + \frac{2}{5}u^5 - \frac{1}{7}u^7 + C$$

$$= \boxed{-\frac{1}{3}\cos^3 t + \frac{2}{5}\cos^5 t - \frac{1}{7}\cos^7 t + C}$$

A

6. [3 points] Use Simpson's Rule with $n = 6$ to approximate $\int_0^3 \sqrt{1+x^2}\,dx$ to 4 decimal places.

$n = 6 \Rightarrow \Delta x = \frac{3-0}{6} = 0.5$

so $x_0 = 0$, $x_1 = 0.5$, $x_2 = 1$, $x_3 = 1.5$, $x_4 = 2$, $x_5 = 2.5$, $x_6 = 3$

then $\int_0^3 \sqrt{1+x^2}\,dx \approx \frac{\Delta x}{3}\Big[\sqrt{1+x_0^2} + 4\sqrt{1+x_1^2} + 2\sqrt{1+x_2^2} + 4\sqrt{1+x_3^2} + 2\sqrt{1+x_4^2} + 4\sqrt{1+x_5^2} + \sqrt{1+x_6^2}\Big]$

$= \frac{0.5}{3}\Big[\sqrt{1+(0)^2} + 4\sqrt{1+(0.5)^2} + 2\sqrt{1+(1)^2} + 4\sqrt{1+(1.5)^2} + 2\sqrt{1+(2)^2} + 4\sqrt{1+(2.5)^2} + \sqrt{1+(3)^2}\Big]$

$= \frac{0.5}{3}\Big[1 + 4.472136 + 2.8284271 + 7.2111026 + 4.472136 + 10.77033 + 3.1622777\Big]$

$\simeq \boxed{5.6527}$

B

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• Problems require complete and clearly presented solutions and carry part marks if there is substantial correct work toward the solution.

B

1. [1 point] Differentiate $g(\theta) = 3\cos(\sec(e^{\tan(\theta)}))$.

$$\boxed{g'(\theta) = -3\sin(\sec(e^{\tan\theta}))\sec(e^{\tan\theta})\tan(e^{\tan\theta})e^{\tan\theta}\sec^2\theta}$$

2. [3 points] Use logarithmic differentiation to find the derivative of $f(x) = \dfrac{\sin^3 x \arctan x}{(x+4)^3}$.

$$\ln(f(x)) = \ln\left(\frac{\sin^3 x \arctan x}{(x+4)^3}\right) = 3\ln(\sin x) + \ln(\arctan x) - 3\ln(x+4)$$

then $\dfrac{d}{dx}(\ln(f(x))) = \dfrac{d}{dx}(3\ln(\sin x) + \ln(\arctan x) - 3\ln(x+4))$

$$\frac{1}{f(x)} f'(x) = \frac{3\cos x}{\sin x} + \frac{1}{\arctan x}\frac{1}{1+x^2} - \frac{3}{x+4}$$

$$\therefore \boxed{f'(x) = \frac{\sin^3 x \arctan x}{(x+4)^3}\left[3\cot x + \frac{1}{(1+x^2)\arctan x} - \frac{3}{x+4}\right]}$$

B

3. [2 points] Find the equation of the tangent line to the curve $x^3y^2 - 3e^x \sin y = 3x$ at the point $(0, \pi)$.

$$\frac{d}{dx}\left(x^3y^2 - 3e^x \sin y \right) = \frac{d}{dx}(3x)$$

$$3x^2y^2 + 2x^3yy' - 3e^x \sin y - 3e^x \cos y\, y' = 3$$

$$\text{so} \quad y' = \frac{3 - 3x^2y^2 + 3e^x \sin y}{2x^3y - 3e^x \cos y}$$

$$\text{at } (0,\pi) \qquad y' = \frac{3}{3} = 1$$

$$\text{so} \quad y - \pi = (1)(x-0)$$

$$\text{or} \quad \boxed{y = x + \pi}$$

4. [3 points] If $f''(x) = 6e^{2x} + 12x^2 - 4\cos x$, $f(0) = 1$ and $f'(0) = 1$, what is f?

$$f'(x) = 3e^{2x} + 4x^3 - 4\sin x + C$$

$$f'(0) = 3 + C = 1 \implies C = -2$$

$$\text{so} \quad f'(x) = 3e^{2x} + 4x^3 - 4\sin x - 2$$

$$\text{then} \quad f(x) = \frac{3}{2}e^{2x} + x^4 + 4\cos x - 2x + K$$

$$\text{thus} \quad f(0) = \frac{3}{2} + 4 + K = 1 \implies K = -9/2$$

$$\text{so} \quad \boxed{f(x) = \frac{3}{2}e^{2x} + x^4 + 4\cos x - 2x - 9/2}$$

B

5. [8 points] Evaluate the following:

(a) $\int \frac{4x}{\sqrt{1-x^4}}\,dx$

$\left(\text{let } u = x^2 \quad \text{then } du = 2x\,dx\right)$

$= \int \frac{2}{\sqrt{1-u^2}}\,du$

$= 2\arcsin(u) + C$

$= \boxed{2\arcsin(x^2) + C}$

(b) $\int x^2 \cos(2x)\,dx$

$\left(\begin{array}{ll} u = x^2 & dv = \cos(2x)dx \\ du = 2x\,dx & v = \frac{1}{2}\sin(2x) \end{array}\right)$

$= \frac{1}{2}x^2\sin(2x) - \int x\sin(2x)\,dx$

$\left(\begin{array}{ll} u = x & dv = \sin(2x)dx \\ du = dx & v = -\frac{1}{2}\cos(2x) \end{array}\right)$

$= \frac{1}{2}x^2\sin(2x) + \frac{1}{2}x\cos(2x) - \frac{1}{2}\int \cos(2x)\,dx$

$= \boxed{\frac{1}{2}x^2\sin(2x) + \frac{1}{2}x\cos(2x) - \frac{1}{4}\sin(2x) + C}$

(c) $\int \frac{7x - 10}{x^2 - 2x - 8}\,dx$

$$\left(\frac{7x-10}{(x-4)(x+2)} = \frac{A}{x-4} + \frac{B}{x+2} \right.$$

$$\begin{aligned} A+B &= 7 \\ 2A-4B &= -10 \\ 4A+4B &= 28 \end{aligned} \qquad \left. \begin{aligned} 6A &= 18 \\ A &= 3 \\ B &= 4 \end{aligned} \right)$$

$$= \int \left(\frac{3}{x-4} + \frac{4}{x+2} \right) dx$$

$$= \boxed{3 \ln|x-4| + 4 \ln|x+2| + C}$$

(d) $\int \cos^5(t) \sin^2(t)\,dt$

$$= \int \cos t \cos^4 t \sin^2 t \, dt$$

$$= \int \cos t \, (1-\sin^2 t)^2 \sin^2 t \, dt$$

$$= \int \cos t \, (1 - 2\sin^2 t + \sin^4 t) \sin^2 t \, dt$$

$$= \int (\sin^2 t - 2\sin^4 t + \sin^6 t) \cos t \, dt \qquad \left(\begin{aligned} u &= \sin t \\ du &= \cos t \, dt \end{aligned} \right)$$

$$= \int (u^2 - 2u^4 + u^6) \, du$$

$$= \frac{1}{3}u^3 - \frac{2}{5}u^5 + \frac{1}{7}u^7 + C$$

$$= \boxed{\frac{1}{3}\sin^3 t - \frac{2}{5}\sin^5 t + \frac{1}{7}\sin^7 t + C}$$

B

6. [3 points] Use Simpson's Rule with $n = 6$ to approximate $\int_0^3 \sqrt{2+x^2}\,dx$ to 4 decimal places.

$n = 6$ so $\Delta x = \frac{3-0}{6} = 0.5$

$x_0 = 0, \quad x_1 = 0.5, \quad x_2 = 1, \quad x_3 = 1.5, \quad x_4 = 2, \quad x_5 = 2.5, \quad x_6 = 3$

$$\int_0^3 \sqrt{2+x^2}\,dx \approx \frac{\Delta x}{3}\Big[\sqrt{2+x_0^2} + 4\sqrt{2+x_1^2} + 2\sqrt{2+x_2^2} + 4\sqrt{2+x_3^2} + 2\sqrt{2+x_4^2} + 4\sqrt{2+x_5^2} + \sqrt{2+x_6^2}\Big]$$

$$= \frac{0.5}{3}\Big[\sqrt{2+(0)^2} + 4\sqrt{2+(0.5)^2} + 2\sqrt{2+(1)^2} + 4\sqrt{2+(1.5)^2} + 2\sqrt{2+(2)^2} + 4\sqrt{2+(2.5)^2} + \sqrt{2+(3)^2}\Big]$$

$$= \frac{0.5}{3}\Big[1.4142136 + 6 + 3.4641016 + 8.2462113 + 4.8989795 + 11.489125 + 3.3166248\Big)$$

$$\simeq \boxed{6.4715}$$

C

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C

1. [1 point] Differentiate $g(\theta) = 2\tan(\sin(e^{\sec(\theta)}))$.

$$\boxed{g'(\theta) = 2\sec^2(\sin(e^{\sec\theta}))\cos(e^{\sec\theta})\,e^{\sec\theta}\sec\theta\tan\theta}$$

2. [3 points] Use logarithmic differentiation to find the derivative of $f(x) = \dfrac{\cos^4 x \arctan x}{(x+1)^2}$.

$$\ln(f(x)) = \ln\left(\frac{\cos^4 x \arctan x}{(x+1)^2}\right) = 4\ln(\cos x) + \ln(\arctan x) - 2\ln(x+1)$$

then $$\frac{d}{dx}(\ln(f(x))) = \frac{d}{dx}(4\ln(\cos x) + \ln(\arctan x) - 2\ln(x+1))$$

$$\frac{1}{f(x)}f'(x) = \frac{-4\sin x}{\cos x} + \frac{1}{\arctan x}\frac{1}{1+x^2} - \frac{2}{x+1}$$

so $$\boxed{f'(x) = \frac{\cos^4 x \arctan x}{(x+1)^2}\left[-4\tan x + \frac{1}{(1+x^2)\arctan x} - \frac{2}{x+1}\right]}$$

3. [2 points] Find the equation of the tangent line to the curve $x^3y^2 + 2e^x \sin y = 2x$ at the point $(0, \pi)$.

$$\frac{d}{dx}\left(x^3y^2 + 2e^x \sin y\right) = \frac{d}{dx}(2x)$$

$$3x^2y^2 + 2x^3yy' + 2e^x \sin y + 2e^x \cos y\, y' = 2$$

so $$y' = \frac{2 - 3x^2y^2 - 2e^x \sin y}{2x^3y + 2e^x \cos y}$$

then at $(0, \pi)$, $y' = \frac{2}{-2} = -1$

so $y - \pi = (-1)(x - 0)$

or $\boxed{y = \pi - x}$

4. [3 points] If $f''(x) = 8e^{2x} - 6x^2 + 4\cos x$, $f(0) = 1$ and $f'(0) = 1$, what is f?

$$f'(x) = 4e^{2x} - 2x^3 + 4\sin x + C$$

$$f'(0) = 4 + C = 1 \Rightarrow C = -3$$

thus $f'(x) = 4e^{2x} - 2x^3 + 4\sin x - 3$

so $f(x) = 2e^{2x} - \frac{1}{2}x^4 - 4\cos x - 3x + K$

but $f(0) = 2 - 4 + K = 1 \Rightarrow K = 3$

$\therefore$ $\boxed{f(x) = 2e^{2x} - \frac{1}{2}x^4 - 4\cos x - 3x + 3}$

C

5. [8 points] Evaluate the following:

(a) $\int \frac{6x}{\sqrt{1-x^4}}\,dx$

(let $u = x^2$
$du = 2x\,dx$)

$= \int \frac{3}{\sqrt{1-u^2}}\,du$

$= 3\arcsin(u) + C$

$= \boxed{3\arcsin(x^2) + C}$

(b) $\int x^2 \sin(3x)\,dx$

($u = x^2$ $\quad dv = \sin(3x)\,dx$
$du = 2x\,dx$ $\quad v = -\frac{1}{3}\cos(3x)$)

$= -\frac{1}{3}x^2\cos(3x) + \frac{2}{3}\int x\cos(3x)\,dx$

($u = x$ $\quad dv = \cos(3x)\,dx$
$du = dx$ $\quad v = \frac{1}{3}\sin(3x)$)

$= -\frac{1}{3}x^2\cos(3x) + \frac{2}{9}x\sin(3x) - \frac{2}{9}\int \sin(3x)\,dx$

$= \boxed{-\frac{1}{3}x^2\cos(3x) + \frac{2}{9}x\sin(3x) + \frac{2}{27}\cos(3x) + C}$

C

(c) $\int \frac{16-x}{x^2-2x-8}\,dx$

$$\left(\frac{16-x}{(x-4)(x+2)} = \frac{A}{x-4} + \frac{B}{x+2} \right.$$

$$\left. \begin{array}{ll} A+B=-1 & 6A=12 \\ 2A-4B=16 & A=2 \\ 4A+4B=-4 & B=-3 \end{array} \right)$$

$$= \int \left(\frac{2}{x-4} - \frac{3}{x+2} \right) dx$$

$$= \boxed{2 \ln|x-4| - 3\ln|x+2| + C}$$

(d) $\int \cos^3(t)\,\sin^4(t)\,dt$

$$= \int \cos t \cos^2 t \sin^4 t \, dt$$

$$= \int \cos t \,(1-\sin^2 t) \sin^4 t \, dt$$

$$= \int \cos t \,(\sin^4 t - \sin^6 t)\, dt \qquad \left(\begin{array}{l} u = \sin t \\ du = \cos t\, dt \end{array} \right)$$

$$= \int (u^4 - u^6)\, du$$

$$= \frac{1}{5}u^5 - \frac{1}{7}u^7 + C$$

$$= \boxed{\frac{1}{5}\sin^5 t - \frac{1}{7}\sin^7 t + C}$$

C

6. [3 points] Use Simpson's Rule with $n = 6$ to approximate $\int_0^3 \sqrt{3+x^2}\,dx$ to 4 decimal places.

$$n = 6 \Rightarrow \Delta x = \frac{3-0}{6} = 0.5$$

$x_0 = 0$, $x_1 = 0.5$, $x_2 = 1$, $x_3 = 1.5$, $x_4 = 2$, $x_5 = 2.5$, $x_6 = 3$

$$\int_0^3 \sqrt{3+x^2}\,dx \simeq \frac{\Delta x}{3}\Big[\sqrt{3+x_0^2} + 4\sqrt{3+x_1^2} + 2\sqrt{3+x_2^2} + 4\sqrt{3+x_3^2} + 2\sqrt{3+x_4^2} + 4\sqrt{3+x_5^2} + \sqrt{3+x_6^2}\Big]$$

$$= \frac{0.5}{3}\Big[\sqrt{3+(0)^2} + 4\sqrt{3+(0.5)^2} + 2\sqrt{3+(1)^2} + 4\sqrt{3+(1.5)^2} + 2\sqrt{3+(2)^2} + 4\sqrt{3+(2.5)^2} + \sqrt{3+(3)^2}\Big]$$

$$= \frac{0.5}{3}\Big[1.7320508 + 7.2111026 + 4 + 9.1651514 + 5.2915026 + 12.165525 + 3.4641016\Big]$$

$$\simeq \boxed{7.1716}$$