Solution to Practice Problems for Preparing the Final Exam

MAT1322, Summer 2016

1. Let R be the region under the graph of $y = 2 - x^2$ and above the graph of $y = 2x^2 + 2x - 3$. Construct a definite integral that calculates the area of R.

Solution. Let $2 - x^2 = 2x^2 + 2x - 3$. $3x^2 + 2x - 5 = 0$. $x = -\frac{5}{3}$, 1. The intersection points of these two graphs are at $x = -\frac{5}{3}$ and $x = 1$.

The area of this region is

$$A = \int_{-5/3}^{1} (2 - x^2 - 2x^2 - 2x + 3)dx = \int_{-5/3}^{1} (-3x^2 - 2x + 5)dx = \frac{256}{27}.$$

2. Let R be the region between the graph of $y = 2 - x^2$ and the graph of $y = 2x^2 + 2x - 3$ in the interval [0, 2]. Find the area of R.

Solution. Since the graphs has an intersect at $x = 1$, when $0 < x < 1$, $2 - x^2 > 2x^2 + 2x - 3$, and, when $1 < x < 2$, $2x^2 + 2x - 3 > 2 - x^2$.

The area of the region is

$$\int_0^1 (2 - x^2 - 2x^2 - 2x + 3)dx + \int_1^2 (2x^2 + x - 3 - 2 + x^2)dx = \int_0^1 (-3x^2 - 2x + 5)dx + \int_1^2 (3x^2 + 2x - 5)dx$$

$= 3 + 5 = 8$.

3. Let R be the region under the graph of $y = 2 - x^2$ and above the graph of $y = 2x^2 + 2x - 3$, $0 \le x \le 1$. Solid S has R as its base, and the cross sections perpendicular to x-axis are squares. Write an integral that calculates the volume of S.

Solution. At a given value of x, the cross section is a square with side-length $(2 - x^2) - (2x^2 + 2x - 3) = 5 - 2x - 3x^2$. Then the area of the cross section at a given x is $A(x) = (5 - 2x - 3x^2)^2$. The volume is

$$V = \int_0^1 (5 - 2x - 3x^2)^2 dx = \frac{167}{15}.$$

4. Let R be the region under the graph of $y = 4 - x^2$ and above the line $y = 4 - 2x$ in the first quadrant. Construct a definite integral that calculates the volume of the solid obtained by revolving R about the line $y = -1$ using cross section perpendicular to the y-axis.

Solution. Since $y = 4 - x^2$, $x = \sqrt{4-y}$. Then $r_{\text{outer}} = \sqrt{4-y} + 1$. Since $y = 4 - 2x$, $x = 2 - y / 2$. Then $r_{\text{inner}} = 2 - y / 2 + 1 = 3 - y / 2$. The volume of the solid is

$$V = \pi\int_0^4\left((\sqrt{4-y}+1)^2 - \left(3-\frac{y}{2}\right)^2\right)dy = \pi\int_0^4\left(2\sqrt{4-y}+2y-\frac{1}{4}y^2-4\right)dy = \frac{16}{3}\pi .$$

5. Construct a definite integral that calculates the volume of a solid obtained by revolving the region under the graph of $y = 2 - x - x^3$ in the first quadrant about the y-axis.

Solution. Since the inverse of this function cannot be found, the method of cylindrical shell is used.

$$V = 2\pi\int_0^1 x(2-x-x^3)dx = \frac{14}{15}\pi .$$

6. The lower half of a sphere with radius 2 m is filled with water. Construct a definite integral that finds the work needed to pump the water out to a point 1 m above the center of the sphere. Let ρ be the density of water and let g be the acceleration of gravity.

Solution. The answer depends on the choice of the variable.

A. If let x be the distance between a layer of water and the top of the half sphere, then the radius of the sphere $r(x) = \sqrt{4-x^2}$, and the area of the cross section is $\pi r(x)^2 = \pi(4 - x^2)$. If the thickness of the layer is dx, then the volume of the layer is $dV = \pi(4 - x^2)dx$, and the weight of this layer is $dw = \rho g dV = \pi\rho g(4 - x^2)dx$. The distance that this layer of water has to be lifted is $x + 1$. Hence, the work needed to lift this layer is

$$dW = (x + 1)dw = \pi\rho g(4 - x^2)(x + 1)dx.$$

The top layer has $x = 0$, and the bottom layer has $x = 2$. Hence, the total work is

$$W = \pi\rho g\int_0^2 (4-x^2)(x+1)dx = \frac{28}{3}\pi\rho g .$$

B. If let x be the distance between a layer of water and the bottom of the half sphere, then the radius of the sphere $r(x) = \sqrt{4-(2-x)^2}$, and the area of the cross section is $\pi r(x)^2$ $= \pi(4 - (2 - x)^2)) = \pi(4x - x^2)$. If the thickness of the layer is dx, then the volume of the layer is $dV = \pi(4x - x^2)dx$, and the weight of this layer is $dw = \rho g dV = \pi\rho g(4x - x^2)dx$. The distance that this layer of water has to be lifted is $3 - x$. Hence, the work needed to lift this layer is

$$dW = (3 - x)dw = \pi\rho g(4x - x^2)(3 - x)dx.$$

The top layer has $x = 2$, and the bottom layer has $x = 0$. Hence, the total work is

$$W = \pi\rho g\int_0^2 (4x - x^2)(3 - x)dx = \frac{28}{3}\pi\rho g\ .$$

C. If let x be the distance between a layer of water and 1 meter above the half sphere, then the radius of the sphere $r(x) = \sqrt{4-(x-1)^2}$, and the area of the cross section is $\pi r(x)^2$ $= \pi(4 - (x - 1)^2)) = \pi(3 + 2x - x^2)$. If the thickness of the layer is dx, then the volume of the layer is $dV = \pi(3 + 2x - x^2)dx$, and the weight of this layer is $dw = \rho g dV = \pi\rho g(3 + 2x - x^2)dx$. The distance that this layer of water has to be lifted is x. Hence, the work needed to lift this layer is

$$dW = (3 - x)dw = \pi\rho g x(3 + 2x - x^2)dx.$$

The top layer has $x = 1$, and the bottom layer has $x = 3$. Hence, the total work is

$$W = \pi\rho g\int_1^3 x(3 + 2x - x^2)dx = \frac{28}{3}\pi\rho g\ .$$

7. Find the value of the improper integral $\int_1^e \frac{1}{x\sqrt{\ln x}}\,dx$ by the definition of the improper integral.

Solution. Use variable substitution $u = \ln x$. Then $u' = 1 / x$.

$$\int_1^e \frac{1}{x\sqrt{\ln x}}\,dx = \lim_{a\to 1^+}\int_a^e \frac{1}{x\sqrt{\ln x}}\,dx = \lim_{a\to 1^+}\int_{\ln a}^1 \frac{1}{\sqrt{u}}\,du = \lim_{a\to 1^+}\left[2\sqrt{u}\,\right]_{u=\ln a}^1 = 2\lim_{a\to 1^+}(1-\sqrt{\ln a}) = 2\ .$$

8. Use the comparison test to determine whether the improper integral $\int_0^1 \frac{1+\sqrt{x}}{2\sqrt{x}-x}\,dx$ is convergent or divergent.

Solution. Guess: When x is near zero, x is much less than $\sqrt{x}$. Hence, $2\sqrt{x}$ behaves line $2\sqrt{x}$. When x is close to zero, $\sqrt{x}$ is much less than 1. Hence, $1 + \sqrt{x}$ behaves like 1. When x is close to zero, the integrand behaves like $\frac{1}{2\sqrt{x}}$. Since the integral $\int_0^1 \frac{1}{2\sqrt{x}}dx$ converges, we may guess that this integral converges.

Justify by comparison test: To show that this improper integral converges, we need a function bigger than $\frac{1+\sqrt{x}}{2\sqrt{x}-x}$, whose integral from 0 to 1 converges. To find a bigger function, we need a bigger numerator and a smaller denominator. Since $1 + \sqrt{x} < 2$, and $2\sqrt{x} - x = \sqrt{x} + (\sqrt{x} - x) > \sqrt{x}$, $\frac{1+\sqrt{x}}{2\sqrt{x}-x} > \frac{2}{\sqrt{x}}$. Since $\int_0^1 \frac{2}{\sqrt{x}}dx = 2\int_0^1 \frac{1}{\sqrt{x}}dx$ converges, this integral converges.

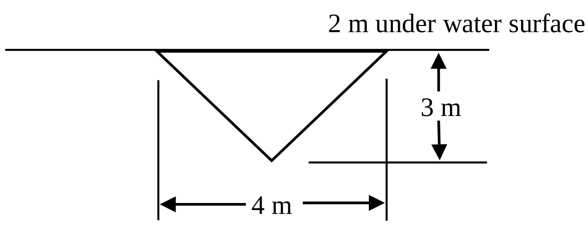
9. Find the length of the arc $y = \frac{x^2}{4} - \ln\sqrt{x}$, $1 \le x \le e$.

Solution. $y' = \frac{x}{2} - \frac{1}{2x}$. $(y')^2 = \frac{x^2}{4} - \frac{1}{2} + \frac{1}{4x^2}$, $1 + (y')^2 = \frac{x^2}{4} + \frac{1}{2} + \frac{1}{4x^2} = \left(\frac{x}{2} + \frac{1}{2x}\right)^2$.

The length of the arc is

$$L = \int_1^e \sqrt{1+(y')^2}dx = \int_1^e \left(\frac{x}{2} + \frac{1}{2x}\right)dx = \left[\frac{x^2}{4} + \frac{\ln x}{2}\right]_{x=1}^{e} = \frac{e^2}{4} + \frac{1}{2} - \frac{1}{4} = \frac{1}{4}(e^2+1).$$

10. Construct a definite integral that is used to calculate the force, in Newtons, acting on a triangular surface submerged into water as shown in the following figure:



Let ρ be the density of water and let g be the acceleration of gravity.

Solution. The solution depends on how the variable is chosen.

A. Let x be the distance between a horizontal slice of the surface and the top of the triangle. Then the length of this slice is $\frac{4}{3}(3-x)$. The area of a horizontal slice with height dx is dA = $\frac{4}{3}(3-x)\,dx$. The depth of this slice is $x + 2$. The pressure is $P(x) = \rho g(x + 2)$. The force acting on this slice is $P(x)dA$ = $\frac{4}{3}\rho g(x+2)(3-x)\,dx$. Since the top of the triangle has $x = 0$ and the bottom of the triangle has $x = 3$, the total force is $F = \frac{4}{3}\rho g\int_0^3 (x+2)(3-x)dx$.

B. Let x be the distance between a horizontal slice of the surface and the bottom of the triangle. Then the length of this slice is $\frac{4}{3}x$. The area of a horizontal slice with height dx is $dA = \frac{4}{3}x\,dx$. The depth of this slice is $5 - x$. The pressure is $P(x) = \rho g(5 - x)$. The force acting on this slice is $P(x)dA$ = $\frac{4}{3}\rho gx(5-x)\,dx$. Since the top of the triangle has $x = 3$ and he bottom of the triangle has $x = 0$, the total force is $F = \frac{4}{3}\rho g\int_0^3 x(5-x)dx$.

C. Let x be the distance between a horizontal slice of the surface and the water surface (i.e., the depth of the slice). Then the length of this slice is $\frac{4}{3}(5-x)$. The area of a horizontal slice with height dx is $dA = \frac{4}{3}(5-x)\,dx$. The depth of this slice is x. The pressure is $P(x) = \rho gx$. The force acting on this slice is $P(x)dA$ = $\frac{4}{3}\rho gx(5-x)\,dx$. Since the top of the triangle has $x = 2$ and he bottom of the triangle has $x = 5$, the total force is $F = \frac{4}{3}\rho g\int_2^5 x(5-x)dx$.

11. Consider a plate in x-y plane bounded by the graph of $y = \frac{1}{1+x}$, and the x-axis, $0 \le x \le 1$. Find the center of mass of this plate if it has the uniform unit density.

Solution. $A = \int_0^1 \frac{1}{1+x}dx = \left[\ln|1+x|\right]_{x=0}^1 = \ln 2.$

$$M_y = \int_0^1 \frac{x}{1+x}dx = \int_0^1\left(1-\frac{1}{1+x}\right)dx = 1-\ln 2.$$

$$M_x = \frac{1}{2}\int_0^1 \frac{1}{(1+x)^2}\,dx = -\frac{1}{2}\left[\frac{1}{1+x}\right]_{x=0}^{1} = \frac{1}{4}.$$

$$\bar{x} = \frac{M_y}{A} = \frac{1-\ln 2}{\ln 2} = \frac{1}{\ln 2} - 1 \approx 0.4427,\ \ \bar{y} = \frac{M_x}{A} = \frac{1/4}{\ln 2} = \frac{1}{4\ln 2} \approx 0.3607.$$

12. Consider differential equation $y' = y(8 - y)$. Let $y(t)$ be a solution to this equation. Answer the following questions without solving this equation:

(a) For which values of y, is $y(t)$ increasing? For which values of y, is $y(t)$ decreasing?

(b) For which value of y, does $y(t)$ has an inflection point.

Solution. (a) Since $y(8 – y) > 0$ when $0 < y < 8$, and $y(8 – y) < 0$ when $y < 0$ or $y > 8$, $y(t)$ is increasing when $0 < y < 8$, and $y(t)$ is decreasing when $y < 0$ or $y > 8$.

(b) At the value $y = 4$, $y(t)$ has an inflection point.

13. Suppose the ventilation system brings 2 m^3 of air with 0.01% carbon dioxide per minute into a room of volume 500 m^3, and the same amount of well-mixed air is taken out from the room. Let $Q(t)$ be the quantity, in m^3, of carbon dioxide in the room at time t. Find a differential equation that is satisfied by function $Q(t)$. (You don't need to solve this equation!)

Solution. $r_{\text{in}} = 2 \times 0.0001 = 0.0002$, $r_{\text{out}} = 2Q / 500 = 0.004Q$. The equation is

$Q' = 0.0002 – 0.004Q$, or $Q' = 0.004(0.05 – Q)$. $Q(0) = 2.5$.

14. Consider the initial-value problem: $y' = 2t \cos^2 y$, $y(1) = \pi / 4$. Assume $-\frac{\pi}{2} < y < \frac{\pi}{2}$.

(a) Solve this initial-value problem.
(b) Use Euler's method with step size $h = 0.25$ to find an approximation of $y(1.5)$. (Use 4 digits after the decimal point in your calculation).

Solution. (a) Separate the variables: $\int \frac{1}{\cos^2 y}\,dy = \int 2t\,dt$. $\tan y = t^2 + C$. With the initial condition, $c = 0$. Then $\tan y = t^2$, $y = \arctan t^2$.

(b) The iteration formula is $y_{i+1} = y_i + 0.25(2t_i \cos^2 y_i)$

i	t_i	y_i
0	1	0.7854
1	1.25	$0.7854 + 0.25 \times 2 \times 1 \times \cos^2(0.7854) \approx 1.0354$
2	1.50	$1.0354 + 0.25 \times 2 \times 1.25 \times \cos^2(1.0354) \approx 1.1981$

15. Suppose the population of a country grows exponentially. At the beginning of 2010, the population is 1 million, and at the begging of 2016, the population is 1.05 million. When would the population of this country reach 1.2 million?

Solution. The model is $P(t) = P(0)e^{kt}$. Take the beginning of 2010 as $t = 0$. Then $1.05 = e^{6k}$. $6k = \ln 1.05$. $k = \ln 1.05 / 6$.

Suppose, after T years, the population reaches 1.2 million. Then $1.2 = e^{kT}$, and $kT = \ln 1.2$. Hence, $T = \dfrac{\ln 1.2}{k} = \dfrac{6 \ln 1.2}{\ln 1.05} \approx 22.4$ years. The population will reach 1.2 million in the middle of 2032.

16. Consider differential equation $\dfrac{dy}{dt} = y(y + 1)$.

(a) Find equilibrium solutions of this equation.

(b) Solve this equation with initial condition $y(0) = 3$.

Solution. (a) Let $y(y + 1) = 0$. The equilibrium solutions are $y = 0$, and $y = -1$.

(b) Separating the variables: $\displaystyle\int \frac{1}{y(y+1)}\,dy = \int \left(\frac{1}{y} - \frac{1}{y+1}\right)dy = \int dt$.

$\ln|y| - \ln|y+1| = t + C$. $\left|\dfrac{y}{y+1}\right| = K_1 e^t$, where $K_1 = e^C > 0$. $\dfrac{y}{y+1} = Ke^t$, where $K = \pm K_1 \neq 0$.

Use the initial condition, $\dfrac{3}{4} = K$. $\dfrac{y}{y+1} = \dfrac{3}{4}e^t$. $4y = 3(y+1)e^t$, $(4 - 3e^t)y = 3e^t$.

$$y = \frac{3e^t}{4 - 3e^t} = \frac{3}{4e^{-t} - 3}.$$

17. Find $\displaystyle\sum_{n=0}^{\infty} \frac{5^{n+1} + (-1)^n}{3^{2n}}$.

Solution. $\sum_{n=0}^{\infty}\frac{5^{n+1}+(-1)^n}{3^{2n}} = \sum_{n=0}^{\infty}\frac{5^{n+1}}{3^{2n}} + \sum_{n=0}^{\infty}\frac{(-1)^n}{3^{2n}} = \frac{5}{1-5/9} + \frac{1}{1+1/9} = \frac{45}{4} + \frac{9}{10} = \frac{486}{40} = \frac{243}{20}$.

18. Use the integral test to show that series $\sum_{n=1}^{\infty}\frac{n}{e^n}$ is convergent. List the conditions to use this test. If $S_5 = \sum_{n=1}^{5}\frac{n}{e^n} \approx 0.895$, find an upper bound and a lower bound of the series.

Solution. Since the general term $\frac{n}{e^n}$, as a function of n, is positive, decreasing and continuous, we can the integral test.

$\int_0^{\infty} xe^{-x}dx = \lim_{b\to\infty}\left[-(1+x)e^{-x}\right]_{x=0}^{b} = \lim_{b\to\infty}(1-(b+1)e^{-b}) = 1 < \infty$, this series is convergent.

The sum S of the series is between $S_5 + \int_6^{\infty} xe^{-x}dx$ and $S_5 + \int_5^{\infty} xe^{-x}dx$.

Since $\int_5^{\infty} xe^{-x}dx \approx 0.040$, and $\int_6^{\infty} xe^{-x}dx \approx 0.017$, S is in the interval (0.912, 0.935).

19. Use appropriate test method to determine each of the following series is convergent or divergent. State the condition(s) for the test method to apply.

(a) $\sum_{n=2}^{\infty}\frac{(-1)^{n-1}}{\ln n}$; (b) $\sum_{n=1}^{\infty}\frac{2n-1}{\sqrt{n^3+n}}$; (c) $\sum_{n=1}^{\infty}\frac{\sqrt{n}+\sin n}{2n^2-1}$;

(d) $\sum_{n=2}^{\infty}(-1)^{n-1}\cos\left(\frac{1}{n^2}\right)$.

Solution. (a) This is an alternating series. Since $\frac{1}{\ln x}$, as a function of x, is decreasing and approaches zero, by the alternating series test, this series is convergent.

(b) Since this is a positive series, we can use the limit comparison test.

Let $a_n = \frac{2n-1}{\sqrt{n^3+n}}$, and let $b_n = \frac{1}{\sqrt{n}}$. Then

$$\lim_{n\to\infty}\left|\frac{a_n}{b_n}\right| = \lim_{n\to\infty}\frac{2n-1}{\sqrt{n^3+n}}\sqrt{n} = \lim_{n\to\infty}\frac{\sqrt{n}(2n-1)/(n\sqrt{n})}{\sqrt{n^3+n}/(n\sqrt{n})} = \lim_{n\to\infty}\frac{2-1/n}{\sqrt{1+1/n^2}} = 2\,.$$

Since $\sum_{n=1}^{\infty}\frac{1}{\sqrt{n}}$ is a p-series with $p = 1 / 2$, it is divergent. Then series $\sum_{n=1}^{\infty}\frac{2n-1}{\sqrt{n^3+n}}$ is also divergent.

This question may also be solved by comparison test. Compare $\frac{2n-1}{\sqrt{n^3+n}}$ with $\frac{n}{\sqrt{2n^3}}$.

(c) Since the series is a positive series, we can use the comparison test. When n is large, $\frac{\sqrt{n}+\sin n}{2n^2-1}$ behaves like $\frac{\sqrt{n}}{2n^2} = \frac{1}{2n^{3/2}}$. Since $\sum_{n=1}^{\infty}\frac{1}{2n^{3/2}} = \frac{1}{2}\sum_{n=1}^{\infty}\frac{1}{n^{3/2}}$ is convergent, we guess this series is convergent. Since $2n^2 - 1 > n^2$, and $\sqrt{n}+\sin n < 2\sqrt{n}$, $\frac{\sqrt{n}+\sin n}{2n^2-1} < \frac{2\sqrt{n}}{n^2} = \frac{2}{n^{3/2}}$.

Since $\sum_{n=1}^{\infty}\frac{2}{n^{3/2}} = 2\sum_{n=1}^{\infty}\frac{1}{n^{3/2}}$ is convergent, this series is convergent.

(d) Since $\lim_{n\to\infty}\cos\left(\frac{1}{n^2}\right) = 1$, $\lim_{n\to\infty}(-1)^n\cos\left(\frac{1}{n^2}\right)$ does not exist. The general term does not approach zero, this series must be divergent.

20. Consider power series $\sum_{n=1}^{\infty}\frac{1}{2^n\sqrt{n^2+1}}(x+1)^n$. Determine

(a) for which value(s) of x is this series absolutely convergent?

(b) for which value(s) of x is this series convergent but not absolutely convergent?

(c) For which value(s) of x is this series divergent?

Justify all your conclusions.

Solution. The center of the series is $x = -1$. The radius of convergence is

$$\lim_{n\to\infty}\left|\frac{c_n}{c_{n+1}}\right| = \lim_{n\to\infty}\left|\left(\frac{1}{2^n\sqrt{n^2+1}}\right)\left(2^{n+1}\sqrt{(n+1)^2+1}\right)\right| = 2\lim_{n\to\infty}\sqrt{\frac{n^2+2n+2}{n^2+1}} = 2.$$

Hence, this series is absolutely convergent in interval (–1 – 2, –1 + 2) = (–3, 1), and it is divergent in (–∞, –3) and (1, ∞).

When $x = 1$, the series becomes $\sum_{n=1}^{\infty} \frac{1}{2^n\sqrt{n^2+1}} 2^n = \sum_{n=1}^{\infty} \frac{1}{\sqrt{n^2+1}}$. Since $\frac{1}{\sqrt{n^2+1}} \geq \frac{1}{\sqrt{n^2+3n^2}} = \frac{1}{2n}$, and the series $\sum_{n=1}^{\infty} \frac{1}{2n}$ diverges, this series diverges.

When $x = -3$, the series becomes $\sum_{n=1}^{\infty} \frac{1}{2^n\sqrt{n^2+1}} (-2)^n = \sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n^2+1}}$. By alternating series test, this series converges. Since series $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n^2+1}}$ diverges, this series is convergent but not absolutely convergent at $x = -3$.

Summarizing:

This series is absolutely convergent in (–3, 1); it is convergent but not absolutely convergent at $x = -3$; it is divergent in (–∞, –3) and [1, ∞).

21. Find the first four non-zero terms of the Maclaurin series of the function $y = \ln(1 + 2x^2)$.

Solution. Since $\ln(1 + x) = x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \frac{1}{4}x^4 + \ldots$,

$\ln(1 + 2x^2) = 2x^2 - \frac{1}{2}(2x^2)^2 + \frac{1}{3}(2x^2)^3 - \frac{1}{4}(2x^2)^4 + \ldots = 2x^2 - 2x^4 + \frac{8}{3}x^6 - 4x^8 + \ldots$.

22. Finding the first four non-zero terms of the Maclaurin series of the function $F(x) = \int_0^x \ln(1+2t^2)dt$.

Solution. $\ln(1 + t) = t - \frac{t^2}{2} + \frac{t^3}{3} - \frac{t^4}{4} + \ldots$. Substitute $2t^2$ for t.

$\ln(1 + 2t^2) = 2t^2 - \frac{2^2t^4}{2} + \frac{2^3t^6}{3} - \frac{2^4t^8}{4} + \ldots = 2t^2 - 2t^4 + \frac{8}{3}t^6 - 4t^8 + \ldots$.

$F(x) = \int_0^x \ln(1+2t^2)dt = \int_0^x (2t^2 - 2t^4 + \frac{8}{3}t^6 - 4t^8 + ...)dt = \frac{2}{3}x^3 - \frac{2}{5}x^5 + \frac{8}{21}x^7 - \frac{4}{9}x^9 + ...$.

23. Find $F^{(7)}(0)$ of the function in question 22.

Solution. $F^{(7)}(0) / 7! = 8 / 21$. $F^{(7)}(0) = 8 \times 7! / 21 = 1920$.

24. Let $z = e^{-(x^2+y^2)}$. Find the partial derivatives z_x and z_{xy}.

Solution. $z_x = -2x\, e^{-(x^2+y^2)}$, $z_{xy} = 4xy\, e^{-(x^2+y^2)}$.

25. Consider function $z = x^2 - xy + 2y$.

(a) Find the equation of the tangent plane of the graph of this function at the point $(-1, 2)$
.

(b) If $x = st$, and $y = s^2 + t$, use the chain rule to find z_s and z_t when $s = 2$, and $t = -1$.

Solution. (a) $z_x = 2x - y$, $z_y = -x + 2$. At point $(-1, 2)$, $z_x = -4$, $z_y = 3$, and $z = 7$. The equation of the tangent plane is $z = -4(x + 1) + 3(y - 2) + 7$, or $z = -4x + 3y - 3$.

(b) $x_s = t$, $x_t = s$, $y_s = 2s$ and $y_t = 1$. At the point $(s, t) = (2, -1)$, $x = -2$, $y = 3$, $z_x = -7$, $z_y = 4$, $x_s = -1$, $x_t = 2$, $y_s = 4$, $y_t = 1$.

Hence, $z_s = z_x x_s + z_y y_s = (-7) \times (-1) + 4 \times 4 = 23$.

$z_t = z_x x_t + z_y y_t = (-7) \times 2 + 4 \times 1 = -10$.

26. Consider equation $z^3 + x^2y - xz - 2y + 5 = 0$.

(a) Find the gradient vector of an implicit function $z = f(x, y)$ defined by this equation at the point $(2, -2, 1)$.

(b) Find the equation of the tangent plane of the graph of this equation at the point $(2, -2, 1)$.

Solution. (a) Let $F(x, y, z) = z^3 + x^2y - xz - 2y + 5$. Then $F_x = 2xy - z$, $F_y = x^2 - 2$,
$F_z = 3z^2 - x$. $F_x(2, -2, 1) = -9$, $F_y(2, -2, 1) = 2$, $F_z(2, -2, 1) = 1$. $z_x = 9$, $z_y = -2$.
grad $z(2, -2, 1) = (9, -2)$.

(b) By the formula directly, we have $-9(x - 2) + 2(y + 2) + (z - 1) = 0$, or $-9x + 2y + z + 21 = 0$.

Or, use the general expression. Since $z_x = 9$, and $z_y = -2$, the equation of the tangent plane has the form $z = 9x - 2y + c$. Since $1 = 9 \times 2 - 2 \times (-2) + c$, $c = -21$. The equation of the tangent plane is $z = 9x - 2y - 21$.

27. Consider the function $z = x^2y - y^2 + 1$.

(a) Find the gradient vector of this function at point (1, 2).

(b) Find the directional derivative of this function at point (1, 2) in the direction $\mathbf{u} = 3\mathbf{i} - 4\mathbf{j}$.

Solution. (a) $z_x = 2xy$, $z_y = x^2 - 2y$. At point (1, 2), $z_x = 4$, $z_y = -3$. The gradient vector is (4, –3).

(b) The unit vector in the direction of $\mathbf{u}$ is $\frac{1}{5}(3\mathbf{i} - 4\mathbf{j})$. The directional derivative of this function at the point (1, 2) is $\frac{3}{5} \times 4 + \left(-\frac{4}{5}\right) \times (-3) = \frac{24}{5}$.