On Robust Controlled Invariants for Continuous-time Monotone Systems

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Abstract: This paper delves into the problem of computing robust controlled invariants for monotone continuous-time systems, with a specific focus on lower-closed specifications. We consider the classes of state monotone (SM) and control-state monotone (CSM) systems, we provide the structural properties of robust controlled invariants for these classes of systems and show how these classes significantly impact the computation of invariants. Additionally, we introduce a notion of feasible points, demonstrating that their existence is sufficient to characterize robust controlled invariants for the considered class of systems. The study further investigates the necessity of reducing the feasibility condition for CSM and Lipschitz systems, unveiling conditions that guide this reduction. Leveraging these insights, we construct an algorithm for the computation of robust controlled invariants. To demonstrate the practicality of our approach, we applied the developed algorithm to the coupled tank problem.

Keywords: Controlled-invariant, Continuous-time monotones systems, Safety

1. INTRODUCTION

Within the domain of dynamical system theory, controlled invariants, also denoted as viable sets in Aubin (2009), are pivotal. They are sets wherein trajectories initiated within them, subject to corresponding controls, remain within the defined set. Their applications span various problems in system analysis, notably the exploration of attractor existence, system performance, robustness, and practical stability, as shown in Blanchini and Miani (2008). Practical applications are prominent in safety-critical systems such as autonomous cars, robotics, biology, and medicine via meticulous verification of safety constraints.

Linking this theoretical foundation to empirical research, Blanchini and Miani (2008) relates the study of controlled invariants to Lyapunov stability. Various methodological approaches have been proposed, ranging from employing linear programming techniques by Sassi and Girard (2012) and semi-definite programming by Korda et al. (2014) for polynomial systems, to the computation of interval-controlled invariants tailored to a general class of nonlinear-systems in Saoud and Sanfelice (2021). Furthermore, the exploration of symbolic control techniques in Tabuada (2009) and Saoud (2019) is undertaken to augment both the comprehension and computational efficiency of controlled invariant sets. This intricate scholarly landscape underscores the foundational role of controlled invariant sets in the systematic analysis and design of dynamic systems. In Dórea and Hennet (1999), authors provide necessary and sufficient conditions for invariance in polyhedral sets for linear continuous-time systems, with applications to the controllability of systems subject to trajectory constraints. The authors also provide extensions to constrained and additively disturbed systems. In Du et al. (2020), the authors outline conditions for the existence of a positively invariant polyhedron in positive linear systems with bounded disturbances, expressed as solvable linear programming inequalities. The paper also establishes a connection between Lyapunov stability and positively invariant polyhedra. In Choi et al. (2023), the authors explore the interconnections between reachability, controlled invariance, and control barrier functions. They introduce the concept of "Inevitable Forward Reachable Tube" as a tool to analyze controlled invariant sets and establish a strong link between these concepts.

Related work: The computational investigation of controlled invariants within continuous-time systems has primarily focused on sets defined by multidimensional intervals. Within this domain, the treatment of continuous-time monotone autonomous multi-affine systems has been addressed in a seminal work by Abate et al. (2009), with subsequent extensions to systems incorporating inputs detailed in the study by Meyer et al. (2016), and discrete-time monotone systems in Saoud and Arcak (2024). Expanding upon the foundation in Saoud and Arcak (2024), our work extends the proposed approach to continuous-time systems. In contrast to the discrete case, which focused on closed-loop robust invariance, our emphasis centers predominantly on open-loop robust invariance. We also define two less classes of monotone systems. Then, using feasible trajectories, we deduce an algorithmic procedure to compute robust controlled invariants.

Due to space constraints, the proofs are omitted and will be published elsewhere.

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2. PRELIMINARIES

2.1 Notation

The symbols $\mathbb{N}$, $\mathbb{N}_{>0}$, $\mathbb{R}$ and $\mathbb{R}_{>0}$ denote the set of positive integers, non-negative integers, real and non-negative real numbers, respectively. Given $n \in \mathbb{N}_{>0}$ and a set $Y \subseteq \mathbb{R}^n$, denotes the set of functions from $\mathbb{R}_{\geq 0}$ to Y. Given a nonempty set K, $\mathrm{Int}(K)$ denotes its interior, $\mathrm{cl}(K)$ denotes its closure, ∂K denotes its boundary and $\overline{K}$ is its complement. The Euclidean norm is denoted by $\|.\|$. For $x \in \mathbb{R}^n$ and for $\varepsilon \geq 0$, $\mathcal{B}_\varepsilon(x) = \{z \in \mathbb{R}^n \mid \|z - x\| \leq \varepsilon\}$ and for a set $K \subseteq \mathbb{R}^n$, $\mathcal{B}_\varepsilon(K) = \cup_{x \in K} \mathcal{B}_\varepsilon(x)$. For $f \in Y^{\mathbb{R}}$, $\|f\|_\infty$ is defined by $\|f\|_\infty = \sup_{t \geq 0} \|f(t)\| \in \mathbb{R}_{\geq 0} \cup \{+\infty\}$. If K is a closed subset of $\mathbb{B}$, we denote by $\pi_K(x) = \{y \in K | ||x - y|| = \inf_{z \in K} ||z - x||\}$. We denote by $x_i \underset{X}{\rightarrow} x$ that for all $i \in \mathbb{N}_{\geq 0}$, $x_i \in X$ and $x_i \underset{n \to \infty}{\rightarrow} x$. We denote by $t_i \downarrow l$ that t_i is a strictly decreasing sequence that converges to l. For $x \in \mathbb{R}$ we define $\lfloor x \rfloor$ and $\lceil x \rceil$, the consecutive integers such that $\lfloor x \rfloor \leq x < \lceil x \rceil$. Given a proposition P, $\neg P$ denotes its negation.

2.2 Partial orders

A partially ordered set $\mathcal{L}$ has an associated binary relation $\leq_{\mathcal{L}}$ where for all $l_1, l_2, l_3 \in \mathcal{L}$, the binary relation satisfies: (i) $l_1 \leq_{\mathcal{L}} l_1$, (ii) if $l_1 \leq_{\mathcal{L}} l_2$ and $l_2 \leq_{\mathcal{L}} l_1$ then $l_1 =_{\mathcal{L}} l_2$ and, (iii) if $l_1 \leq_{\mathcal{L}} l_2$ and $l_2 \leq_{\mathcal{L}} l_3$ then $l_1 \leq_{\mathcal{L}} l_3$. If neither $l_1 \leq_{\mathcal{L}} l_2$ nor $l_2 \leq_{\mathcal{L}} l_1$ holds, we say that l_1 and l_2 are incomparable. The set of all incomparable couples in $\mathcal{L}$ is denoted by $\mathrm{Inc}_{\mathcal{L}}$. We say that $l_1 <_{\mathcal{L}} l_2$ iff $l_1 \leq_{\mathcal{L}} l_2$ and $l_1 \neq_{\mathcal{L}} l_2$. Finally, a partial ordering $m \leq_{\mathcal{L}^{\mathbb{R}}} n$ between a pair of functions of a (real variable) with values in $\mathcal{L}$ holds if and only if $m(t) \leq_{\mathcal{L}} n(t)$. For a partially ordered set $\mathcal{L}$, closed intervals are $[x, y]_{\mathcal{L}} := \{z \mid x \leq_{\mathcal{L}} z \leq_{\mathcal{L}} y\}$.

Definition 1. Given a partially ordered set $\mathcal{L}$, for $a \in \mathcal{L}$, we define, $\downarrow a := \{x \in \mathcal{L} \mid x \leq_{\mathcal{L}} a\}$ and $\uparrow a := \{x \in \mathcal{L} \mid a \leq_{\mathcal{L}} x\}$. When $A \subseteq \mathcal{L}$ then its lower closure (respectively upper closure) is $\downarrow A := \bigcup_{a \in A} \downarrow a$ (respectively $\uparrow A := \bigcup_{a \in A} \uparrow a$). A subset $A \subseteq \mathcal{L}$ is said to be *lower-closed* (respectively *upper-closed*) if $\downarrow A = A$ (respectively $\uparrow A = A$).

We have the following definitions relative to partially ordered sets.

Definition 2. Let $\mathcal{L}$ be a partially ordered set and $A \subseteq \mathcal{L}$. The set A is said to be *bounded below* (in $\mathcal{L}$) if there exists a compact set $B \subseteq \mathcal{L}$ such that $A \subseteq \uparrow B$. Similarly, the set A is said to be *bounded above* (in $\mathcal{L}$) if there exists a compact set $B \subseteq \mathcal{L}$ such that $A \subseteq \downarrow B$.

Definition 3. Let $\mathcal{L}$ be a partially ordered set and consider a closed subset $A \subseteq \mathcal{L}$. If the set A is bounded below then the set of *minimal elements* of A is defined as $\min(A) := \{x \in A \mid \forall x_1 \in A,\, x \leq_{\mathcal{L}} x_1 \text{ or } (x, x_1) \in \mathrm{Inc}_{\mathcal{L}}\}$. Similarly, if the set A is bounded above then the set of *maximal elements* of A is defined as $\max(A) := \{x \in A \mid \forall x_1 \in A,\, x \geq_{\mathcal{L}} x_1 \text{ or } (x, x_1) \in \mathrm{Inc}_{\mathcal{L}}\}$.

In the rest of the paper, we will focus on lower-closed sets; analogous results can be formulated for upper-closed sets.

2.3 Continuous-time control systems

In this paper, we consider the class of continuous-time control systems Σ of the form:

$$\dot{x} = f(x, u, d) \tag{1}$$

where $x \in \mathcal{X}$ is a state, $u \in \mathcal{U}$ is a control input and $d \in \mathcal{D}$ is a disturbance input. The trajectories of (1) are denoted by $\Phi(., x_0, \mathbf{u}, \mathbf{d})$ where $\Phi(t, x_0, \mathbf{u}, \mathbf{d})$ is the state reached at time $t \in \mathbb{R}_{\geq 0}$ from the initial state x_0 under the control input $\mathbf{u} : \mathbb{R}_{\geq 0} \to \mathcal{U}$ and the disturbance input $\mathbf{d} : \mathbb{R}_{\geq 0} \to \mathcal{D}$. For $X \subseteq \mathcal{X}$, $U \subseteq \mathcal{U}$, $D \subseteq \mathcal{D}$, and $t \geq 0$ we use the notation $\Phi(t, X, U, D) = \{\Phi(t, x, \mathbf{u}, \mathbf{d}) \mid x \in X, \mathbf{u} \in U^{\mathbb{R}}, \mathbf{d} \in D^{\mathbb{R}}\}$ to denote the reachable set of the system Σ at time t from an initial condition $x \in X$ under a control input $\mathbf{u} : \mathbb{R}_{\geq 0} \to U$ and a disturbance input $\mathbf{d} : \mathbb{R}_{\geq 0} \to D$.

In the rest of the paper, we make the following assumption:

Assumption 1. The system Σ in (1) is well-posed, i.e, for all $x \in \mathcal{X}$, for all $\mathbf{u} \in U^{\mathbb{R}}$, for all $\mathbf{d} \in D^{\mathbb{R}}$, we have a unique solution of the system Σ, $\phi(t, x, \mathbf{u}, \mathbf{d})$ defined for all $t \geq 0$. See Khalil (2002), Angeli and Sontag (1999).

3. MONOTONES SYSTEMS AND PRELIMINARIES ON INVARIANCE

3.1 Monotones systems

In this section, we describe classes of monotone continuous-time control systems, emphasizing their capacity to maintain order on their states and control inputs. Subsequently, we furnish characterizations of the considered classes of systems.

Definition 4. Consider the continuous-time control system Σ in (1). The system Σ is said to be:

- *State monotone (SM)* if its sets of states and disturbance inputs are equipped with partial orders $\leq_{\mathcal{X}}$ and $\leq_{\mathcal{D}^{\mathbb{R}}}$, respectively, and for all $x_1, x_2 \in \mathcal{X}$, for all $\mathbf{u} \in \mathcal{U}^{\mathbb{R}}$ and for all $\mathbf{d}_1, \mathbf{d}_2 \in \mathcal{D}^{\mathbb{R}}$, if $x_1 \leq_{\mathcal{X}} x_2$ and $\mathbf{d}_1 \leq_{\mathcal{D}^{\mathbb{R}}} \mathbf{d}_2$ then $\Phi(t, x_1, \mathbf{u}, \mathbf{d}_1) \leq_{\mathcal{X}} \Phi(t, x_2, \mathbf{u}, \mathbf{d}_2)$, for all $t \geq 0$;
- *Control-state monotone (CSM)* if its sets of states, inputs and disturbances are equipped with partial orders, $\leq_{\mathcal{X}}$, $\leq_{\mathcal{U}^{\mathbb{R}}}$ and $\leq_{\mathcal{D}^{\mathbb{R}}}$, respectively, and for all $x_1, x_2 \in \mathcal{X}$, $\mathbf{u}_1, \mathbf{u}_2 \in \mathcal{U}^{\mathbb{R}}$ and for all $\mathbf{d}_1, \mathbf{d}_2 \in \mathcal{D}^{\mathbb{R}}$, if $x_1 \leq_{\mathcal{X}} x_2$, $\mathbf{u}_1 \leq_{\mathcal{U}^{\mathbb{R}}} \mathbf{u}_2$ and $\mathbf{d}_1 \leq_{\mathcal{D}^{\mathbb{R}}} \mathbf{d}_2$ then $\Phi(t, x_1, \mathbf{u}, \mathbf{d}_1) \leq_{\mathcal{X}} \Phi(t, x_2, \mathbf{u}_2, \mathbf{d}_2)$, for all $t \geq 0$.

Proposition 1. Consider the control system Σ in (1). We have the following properties:

(i) The system Σ is SM if and only if for all $x \in X$, $\mathbf{u} \in U^{\mathbb{R}}$ and $\mathbf{d} \in D^{\mathbb{R}}$ we have for all $t \geq 0$
$$\Phi(t, \downarrow x, \mathbf{u}, \downarrow \mathbf{d}) \subseteq \downarrow \Phi(t, x, \mathbf{u}, \mathbf{d})$$
(ii) The system Σ is CSM if and only if for all $x \in X$, $\mathbf{u} \in U^{\mathbb{R}}$ and $\mathbf{d} \in D^{\mathbb{R}}$ we have for all $t \geq 0$
$$\Phi(t, \downarrow x, \downarrow \mathbf{u}, \downarrow \mathbf{d}) \subseteq \downarrow \Phi(t, x, \mathbf{u}, \mathbf{d})$$

3.2 Preliminaries on invariance

We introduce the concept of robust controlled invariant for continuous-time dynamical systems consistent with Dórea and Hennet (1999) and Korda et al. (2014).

Definition 5. Consider the system Σ in (1) and let $X \subseteq \mathcal{X}$, $U \subseteq \mathcal{U}$ and $D \subseteq \mathcal{D}$ be the constraints sets on the states, inputs and disturbances, respectively. The set $K \subseteq \mathcal{X}$ is a robust controlled invariant for the system Σ and constraint set (X, U, D) if $K \subseteq X$ and the following holds:

$$\forall x \in K,\ \exists \mathbf{u} \in U^{\mathbb{R}} \ \text{ s.t } \ \Phi(t, x, \mathbf{u}, D) \subseteq K, \forall t \geq 0 \qquad (2)$$

From Definition 5, one can readily see that the robust controlled invariance property is closed under union. Hence, there exists a unique robust controlled invariant that is maximal, in the sense that it contains all the robust controlled invariants.

Definition 6. Consider the system Σ in (1) and let $X \subseteq \mathcal{X}$, $U \subseteq \mathcal{U}$ and $D \subseteq \mathcal{D}$ be the constraints sets on the states, inputs and disturbances, respectively. The set $K \subseteq \mathcal{X}$ is the *maximal* robust controlled invariant set for the system Σ and constraint set (X, U, D) if:

- $K \subseteq \mathcal{X}$ is a robust controlled invariant for the system Σ and constraint set (X, U, D);
- K contains any robust controlled invariant for the system Σ and constraint set (X, U, D).

4. STRUCTURAL PROPERTIES AND COMPUTATION OF ROBUST CONTROLLED INVARIANTS

4.1 Structural properties of robust controlled invariants

We first propose the following general characterization of the topological structure of robust controlled invariants for nonlinear systems under more regularity on the dynamics of the system.

Proposition 2. Consider the system Σ in (1) and let $X \subseteq \mathcal{X}$, $U \subseteq \mathcal{U}$ and $D \subseteq \mathcal{D}$ be constraints sets on the states, inputs, and disturbances, respectively. Suppose that the set of state X is closed, and the set of control inputs U and disturbance inputs D are compact. Suppose that the dynamics $f : \mathcal{X} \times \mathcal{U} \times \mathcal{D} \to \mathcal{X}$ of the system Σ is Lipschitz over its first argument [1], continuous over its second and third arguments. Then the following properties hold:

(i) If the set $K \subseteq X$, is a robust controlled invariant for the system Σ and constraint set (X, U, D), then the set $\mathrm{cl}(K)$ is a robust controlled invariant for the system Σ and constraint set (X, U, D);
(ii) If the set $K \subseteq X$ is the maximal robust controlled invariant for the system Σ and constraint set (X, U, D), then the set K is closed.

In the following, we provide different characterizations of robust controlled invariants when dealing with monotone dynamical systems and lower-closed safety specifications (i.e. a lower closed set of constraints $\mathcal{X}$ on the state-space).

Theorem 3. Consider the system Σ in (1) and let $X \subseteq \mathcal{X}$, $U \subseteq \mathcal{U}$ and $D \subseteq \mathcal{D}$ be the constraints sets on the states, inputs and disturbances, respectively, where the set X is lower closed. The following properties hold:

(i) If the system Σ is SM and if a set K is a robust controlled invariant of the system Σ and constraint set (X, U, D), then its lower closure is also a robust controlled invariant for the system Σ and constraint set (X, U, D);
(ii) If the system Σ is SM then the maximal robust controlled invariant K for the system Σ and constraint set (X, U, D) is lower closed;
(iii) If the system Σ is SM and the set of disturbance inputs D is closed and bounded above then the maximal robust controlled invariant for the system Σ and constraint set (X, U, D) is the maximal robust controlled invariant for the system Σ and the constraint set $(X, U, D_{\max})$, where $D_{\max} = \max(D)$;
(iv) If the system Σ is CSM, the set of control inputs U is closed and bounded below and the set of disturbance inputs D is closed and bounded above, then the maximal robust controlled invariant for the system Σ and constraint set (X, U, D) is the maximal robust controlled invariant for the system Σ and constraint set $(X, U_{\min}, D_{\max})$, where $U_{\min} = \min(U)$ and $D_{\max} = \max(D)$.

Derived from conditions (ii) and (iii), maximal robust controlled invariants can be characterized through the utilization of maximal elements of the state space X, and maximal disturbance elements in $D_{\max}$. Conversely, for CSM systems detailed in (iv) and (v), the invariance property can be ensured using only the minimal control elements in $U_{\min}$. Let us note that since the maximal robust controlled invariant set is lower closed, the boundary of this set can be taught as a Pareto Front. As such, Multidimensional binary search techniques can be applied to compute this set Legriel et al. (2010).

Proposition 4. Consider the system Σ in (1) and let $X \subseteq \mathcal{X}$, $U \subseteq \mathcal{U}$ and $D \subseteq \mathcal{D}$ be the constraints sets on the states, inputs and disturbances, respectively, where the set X is lower closed. Consider a closed and lower closed set $K \subseteq X$. The following properties hold:

(i) If the system Σ is SM and the set of disturbance inputs D is closed and bounded above then the set K is a robust controlled invariant of the system Σ and constraint set (X, U, D), if and only if $\forall x \in \max(K)$ there exist $\mathbf{u} \in U^{\mathbb{R}}$, such that for any disturbance input $\mathbf{d} : \mathbb{R}_{\geq 0} \to D_{\max}$, the solution of the open-loop system $\Phi(., x, \mathbf{u}, \mathbf{d}) : \mathbb{R}_{\geq 0} \to \mathcal{X}$ satisfies
$$\Phi(t, x, \mathbf{u}, \mathbf{d}) \in K,\ \forall t \geq 0 \qquad (3)$$
where $D_{\max} = \max(D)$;
(ii) If the system Σ is CSM, the set of control inputs U is closed and bounded below and the set of disturbance inputs D is closed and bounded above then the set K is a robust controlled invariant of the system Σ and constraint set (X, U, D), if and only if $\forall x \in \max(K)$ there exist $\mathbf{u} : \mathbb{R}_{\geq 0} \to U_{\min}$, such that such that for any disturbance input $\mathbf{d} : \mathbb{R}_{\geq 0} \to D_{\max}$, the solution of the open-loop system $\Phi(., x, \mathbf{u}, \mathbf{d}) : \mathbb{R}_{\geq 0} \to \mathcal{X}$ satisfies
$$\Phi(t, x, \mathbf{u}, \mathbf{d}) \in K,\ \forall t \geq 0 \qquad (4)$$
where $U_{\min} = \min(U)$ and $D_{\max} = \max(D)$.

Proposition 4 introduces significant simplifications to the task of verifying whether a lower closed set qualifies as a robust controlled invariant. Instead of examining the

[1] The map $f : \mathcal{X} \times \mathcal{U} \times \mathcal{D} \to \mathcal{X}$ is Lipschitz continuous over its first argument if there exists a constant $\lambda \geq 0$ such that for all $x_1, x_2 \in \mathcal{X}$, for all $u \in U$ and for all $d \in D$, we have $\|f(x_1, u, d) - f(x_2, u, d)\| \leq \lambda \|x_1 - x_2\|$.

robust controlled invariance condition (see equation (2)) for all elements $x \in K$, $\mathbf{u} \in U^{\mathbb{R}}$, and $\mathbf{d} \in D^{\mathbb{R}}$ in the context of general nonlinear systems, the verification process can be confined to the following cases:

- $x \in \max(K)$, $\mathbf{u} \in U^{\mathbb{R}}$, and $\mathbf{d} \in D^{\mathbb{R}}_{\max}$ for SM systems;
- $x \in \max(K)$, $\mathbf{u} \in U^{\mathbb{R}}_{\min}$, and $\mathbf{d} \in D^{\mathbb{R}}_{\max}$ for CSM systems.

Additionally, this property proves particularly advantageous in practical applications, especially when $\max(K)$ is finite, $D_{\max}$ and $U_{\min}$ are singletons, while K, D, and U are infinite.

The preceding characterizations of robust controlled invariants are global criteria. While exact, these criteria pose challenges in practical verification. In the following section, using the notion of feasibility, we introduce a characterization of Robust controlled invariants.

4.2 Feasibility and robust controlled invariance

In this section, We introduce the concept of feasibility. This Idea has been introduced for discrete systems in Saoud and Arcak (2024); Sadraddini and Belta (2018).

Definition 7. A point $x_0 \in X$ is said to be *feasible* with respect to the constraint set (X, U, D) if there exists an input trajectory $\mathbf{u} : \mathbb{R}_{\geq 0} \to \mathcal{U}$ and $T > 0$ such that

$$\Phi(t, x_0, \mathbf{u}, D) \subseteq X, \quad \forall\, 0 \leq t < T \tag{5}$$

and

$$\Phi(T, x_0, \mathbf{u}, D) \subseteq\downarrow \bigcup_{0 \leq t < T} \Phi(t, x_0, \mathbf{u}, D). \tag{6}$$

In the following, we characterize the concept of feasibility for SM systems.

Proposition 5. Consider the system Σ in (1) and let $X \subseteq \mathcal{X}$, $U \subseteq \mathcal{U}$ and $D \subseteq \mathcal{D}$ be the constraints sets on the states, inputs and disturbances, respectively. If the system Σ is SM, the set of states X is lower closed and the set of disturbance inputs D is closed and bounded above, then a point $x_0 \in X$ is feasible w.r.t the constraint set $(X, U, D_{\max})$ if and only if it is feasible w.r.t the constraint set (X, U, D), where $D_{\max} = \max(D)$.

Intuitively, the result of Proposition 5 shows that to check if a point is feasible, it is sufficient to explore the trajectories with the maximal disturbance inputs in the set $D_{\max}$.

We now show how the existence of feasible trajectories makes it possible to construct robust controlled invariants.

Theorem 6. Consider the system Σ in (1) and let $X \subseteq \mathcal{X}$, $U \subseteq \mathcal{U}$ and $D \subseteq \mathcal{D}$ be the constraints sets on the states, inputs and disturbances, respectively, where the set X is lower closed. Assume that the set of disturbances D is a multidimensional interval of the form $[d_{\min}, d_{\max}]$, with $d_{\min}, d_{\max} \in D$. If the system Σ is SM, then the following holds: If $x_0 \in X$ is feasible w.r.t the constraint set (X, U, D), then there exists an input trajectory $\mathbf{u} : \mathbb{R}_{\geq 0} \to \mathcal{U}$ and $T > 0$ such that the set

$$K =\downarrow \bigcup_{0 \leq t \leq T} \Phi(t, x_0, \mathbf{u}, D) \tag{7}$$

is a robust controlled invariant for the system Σ and constraint set (X, U, D).

We also have the following characterization of feasibility for a particular class of CSM systems.

Proposition 7. Consider the system Σ in (1) and let $X \subseteq \mathcal{X}$, $U \subseteq \mathcal{U}$ and $D \subseteq \mathcal{D}$ be the constraints sets on the states, inputs and disturbances, respectively, where X is lower closed. If the system Σ is CSM, the set of states X is lower closed, the set of control inputs U is closed and bounded below, that the set of disturbances D is a multidimensional interval of the form $[d_{\min}, d_{\max}]$, with $d_{\min}, d_{\max} \in D$, and for all $\varepsilon \in \mathbb{R}^n_{\geq 0}$, for all $x_1, x_2 \in \mathcal{X}$ and for all $u \in U$, the following condition is satisfied:

$$x_1 \geq x_2 + \varepsilon \implies \mathcal{B}_\varepsilon(\Phi(t, x_2, u, D)) \subseteq\downarrow \Phi(t, x_1, u, D), \forall t > 0 \tag{8}$$

then a point $x_0 \in X$ is feasible w.r.t the constraint set (X, U, D) if and only if it is feasible w.r.t the constraint set $(X, U_{\min}, D)$, where $U_{\min} = \min(U)$.

Moreover, we have the following result, characterizing a special case of feasibility for the particular class of monotone systems with Lipschitz dynamics.

Theorem 8. Consider the SM system Σ in (1) and let $X \subseteq \mathcal{X}$, $U \subseteq \mathcal{U}$ and $D \subseteq \mathcal{D}$ be the constraints sets on the states, inputs and disturbances, respectively. Assume that the map $f : \mathcal{X} \times \mathcal{U} \times \mathcal{D} \to \mathcal{X}$ defining the system Σ is Lipschitz on its first argument, uniformly continuous on its second and third arguments, and the sets of control inputs U and disturbance inputs D are compact. For $x_0 \in \mathcal{X}$, if the following conditions are satisfied:

(i) x_0 is feasible w.r.t the constraint set (X, U, D) and there exists $\mathbf{u} : \mathbb{R}_{\geq 0} \to \mathcal{U}$, $T > 0$ and ε_T such that

$$\mathcal{B}_{\varepsilon_T}(\Phi(T, x_0, \mathbf{u}, D)) \subseteq\downarrow \bigcup_{0 \leq t < T} \Phi(t, x_0, \mathbf{u}, D). \tag{9}$$

(ii) there exists $\gamma > 0$ such that $\mathcal{B}_\gamma(\Phi(t, x_0, \mathbf{u}, D)) \subseteq X, \quad \forall\, 0 < t < T$

then there exists $\beta > 0$ such that for any $x_1 \in \{\uparrow x_0\} \cap \mathcal{B}_\beta(x_0)$, x_1 is feasible w.r.t the constraint set (X, U, D). Moreover one can explicitly determine the value of β as a function of the parameters ε_T and γ.

In Theorems 3 and 6, we derived useful characterisations of robust controlled invariant. These characterisations remain hard to track in practice. Proposition 7 introduce necessary conditions for checking feasibility for CSM systems using only minimal control inputs. Theorem 8 show that point near feasible points can also exhibit feasibility property.

4.3 Computation of Robust Controlled Invariants

This section introduces an algorithm for computing maximal robust controlled invariants, applicable to systems represented as described in (1). The algorithm operates within the constraints defined by the set (X, U, D), where X is a lower-closed constraint set, and D is a multidimensional interval $D = [d_{min}, d_{max}]$.

The algorithm, presented as Algorithm 1, is designed to handle both SM and CSM systems. A two-step process is employed, where the feasibility conditions for SM systems are initially considered, and then further simplifications are introduced for CSM systems.

The algorithm begins by initializing two sets, $\mathcal{F}_1$ and $\mathcal{F}_2$, as empty sets. The set $\mathcal{F}_1$ collects the states that belongs to the maximal robust controlled invariant set while the set $\mathcal{F}_2$ collects the remaining states. It iterates over the maximum and minimum elements of the lower-closed set X. For each element, the algorithm checks the feasibility using the command $feasible$ (line 3) by verifying conditions (5) and (6). If the conditions are met, the resulting set Z from Theorem 6 and defined as follows:

$$Z = \downarrow \bigcup_{0 \leq t \leq T} \Phi(t, x, u, D_{max}) \tag{10}$$

is included in $\mathcal{F}_1$. Conversely, if the state x is deemed unsafe, the set H defined by

$$H = \uparrow \bigcup_{0 \leq t \leq T} \Phi(t, x, u, D_{max}) \tag{11}$$

is added to $\mathcal{F}_2$.

The iterative refinement process then follows, where the algorithm refines $\mathcal{F}_1$ and $\mathcal{F}_2$ until their Hausdorff distance is below a predefined threshold ϵ. This process involves selecting elements from the complement of both sets and checking their feasibility or unsafety. The resulting refined $\mathcal{F}_1$ is returned as an approximation of the maximal robust controlled invariant set.

Algorithm 1 Computation of the maximal robust controlled invariant set

Require: A system Σ as in (1), a constraint set (X, U, D) where X is lower closed and D is a multidimensional interval $[d_{min}, d_{max}]$.
Ensure: K an approximation of the maximal robust controlled invariant set

1: $\mathcal{F}_1 = \mathcal{F}_2 = \emptyset$
2: **for** $x \in \max(X)$ **do**
3:   **if** feasible(x) **then**
4:     $\mathcal{F}_1 = \mathcal{F}_1 \bigcup Z$ ▷ Z is defined in (10)
5:   **else if** unsafe(x) **then**
6:     $\mathcal{F}_2 = \mathcal{F}_2 \bigcup H$ ▷ H is defined in (11)
7:   **end if**
8: **end for**
9: **if** $\mathcal{F}_1 = X$ **then**
10:   **return** $K = \mathcal{F}_1$
11: **end if**
12: **for** $x \in \min(X)$ **do**
13:   **if** feasible(x) **then**
14:     $\mathcal{F}_1 = \mathcal{F}_1 \bigcup Z$
15:   **else if** unsafe(x) **then**
16:     $\mathcal{F}_2 = \mathcal{F}_2 \bigcup H$
17:   **end if**
18: **end for**
19: **if** $\min(X) \subseteq \mathcal{F}_2$ **then**
20:   **return** $K = \emptyset$
21: **end if**
22: **while** $d(\mathcal{F}_1, \mathcal{F}_2) > \epsilon$ **do** ▷ Hausdorf distance
23:   Select $x \in (X \setminus \mathcal{F}_2) \cap (X \setminus \mathcal{F}_1)$
24:   **if** feasible(x) **then**
25:     $\mathcal{F}_1 = \mathcal{F}_1 \bigcup Z$
26:   **else if** unsafe(x) **then**
27:     $\mathcal{F}_2 = \mathcal{F}_2 \bigcup H$
28:   **end if**
29: **end while**
30: **return** $K = \mathcal{F}_1$

Moreover, the performance of the algorithm can be improved by considering the two following properties:

- If the system is CSM and condition from Proposition 7 are satisfied, we can limit ourselves to the constraint set (X, U_{min}, D_{max}), in this case, the trajectories with $u \in U_{\min}$ will first be explored.
- If the System Σ is Lipschitz and satisfies condition from Theorem 8, then we can modify the set Z in (10) to $Z = \downarrow \bigcup_{0 \leq t \leq T} \Phi(t, x, u, D_{max}) \bigcup \{\{\uparrow x\} \bigcap \mathcal{B}_\beta(x)\}$ with β as defined in theorem 8

5. NUMERICAL APPLICATIONS

In this section, we consider the non-linear model of coupled tanks described in Apkarian et al. (2012). The system is described by the following differential equation:

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \frac{a\sqrt{2g}}{A} \begin{pmatrix} -1 & 0 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} \sqrt{x_1} \\ \sqrt{x_2} \end{pmatrix} + \frac{1}{A} \begin{pmatrix} K_1 & 0 \\ 0 & K_2 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} + \frac{1}{A} \begin{pmatrix} 0 \\ 1 \end{pmatrix} d$$

Where x_1 and x_2 are the water height in tank 1 and 2 respectively, A is the cross-sectional of both tanks, a is the cross-sectional area of the orifice of the two tanks. Both tanks are supplied with two pumps with constants: K_1 and K_2. $u_1, u_2 \in U = [u_{\min}, u_{\max}]$ are voltages of the pump and $d \in D = [d_{\min}, d_{\max}]$ represents leakage in tank 2. One can easily check that the considered system is CSM.

In this example we impose the following safety constraints: $X = \{(x_1, x_2) \mid 0 \leq x_1 \leq 30 \text{ and } 0 \leq x_2 \leq 20\}$. Since the height of the water is always positive, this set is effectively lower-closed. We use Algorithm 1 to compute a robust controlled invariant. The parameters model are taken from Meyer (2015). Figures 1 and 2 represent the computed robust controlled invariant set for two different precisions $\epsilon = 1$cm and $\epsilon = 0.5$cm using Algorithm 1 along with feasible trajectories. In both cases the robust controlled invariant in characterized using feasible trajectories. For the first scenario with $\epsilon = 1$cm, the computation time is less than 40 ms. For the scenario with $\epsilon = 0.5$cm, the computation time is less than 145 ms. The implementations have been done in Python, on a DELL Lattitude 5430 using an Intel core i7-1265U. Files for this simulation can be found at this link.

6. CONCLUSION

In this study, we have presented characterizations of robust controlled invariants for monotone continuous-time systems. Leveraging these characterizations, we have developed an algorithmic procedure for computing maximal robust controlled invariants. The practical application of our algorithm in an illustrative example highlights the effectiveness of our approach. Moving forward, our future endeavours will delve into the realm of closed-loop robust controlled invariance (while only open-loop controllers have been considered in the current version of the paper), this will be conducted by exploring the potential utilization of tangent cones-based characterizations of invariance. Additionally, we aim to investigate other types

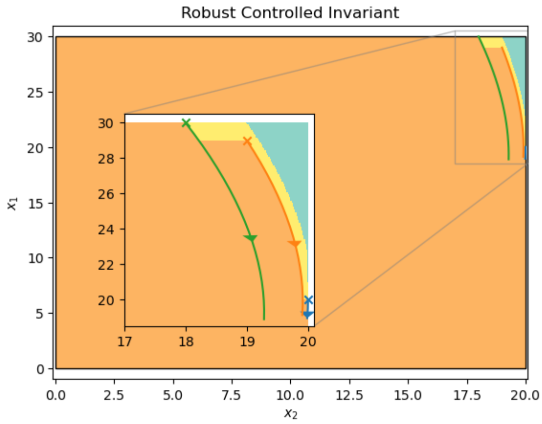

Fig. 1. The Robust controlled invariant is in brown. The Unsafe set is in teal. The non-explored region of the state constraint set is in yellow. Three feasible trajectories initiated at: [30, 18] in green, [29, 19] in brown and [20, 20] in light blue, are shown. Feasible trajectories are determined for minimal control inputs.

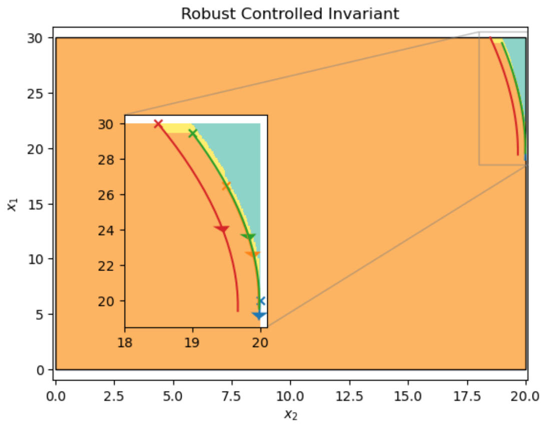

Fig. 2. The Robust controlled invariant is in brown. The non-explored region of the state constraint set is in yellow. Four feasible trajectories from the following initial conditions: [30, 18.5] in red, [29.5, 19] in green, [26.5, 19.5] in brown, [20, 20] in blue . All feasible trajectories are determined for minimal control inputs.

of specifications, particularly those related to stabilization and more general properties described by signal temporal logic formulas.

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