$$r^{(t+1)} = Mr^{(t)}$$

Markov chains

- Set of states **X**
- Transition matrix **P** where $P_{ij} = P(X_t{=}i \mid X_{t-1}{=}j)$
- **π** specifying the stationary probability of being at each state $x \in X$
- Goal is to find **π** such that **π = P π**

Why is This Analogy Useful?

- **Theory of Markov chains**

- **Fact:** For **any start vector**, the power method applied to a Markov transition matrix **P** will **converge** to a **unique** positive stationary vector as long as **P** is **stochastic**, **irreducible** and **aperiodic**.

Make M Stochastic

- **Stochastic:** Every column sums to **1**
- **Solution:** Add **green** links

$$A = M + a^T \left(\frac{1}{n} e\right)$$

- a_i =1 if node *i* has out deg 0, =0 else
- **e** vector of all 1s
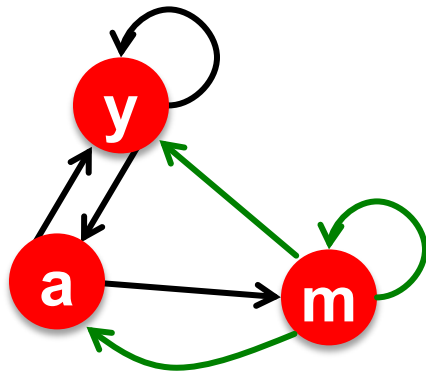
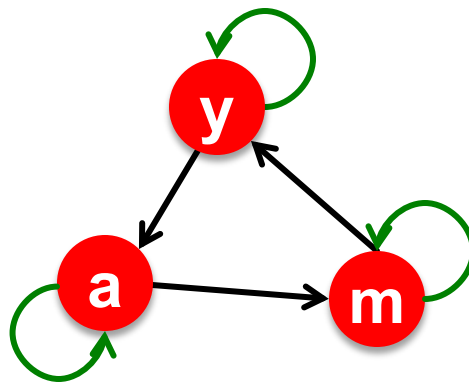|	y	a	m
y	½	½	**1/3**
a	½	0	**1/3**
m	0	½	**1/3**

$r_y = r_y/2 + r_a/2 + r_m/3$

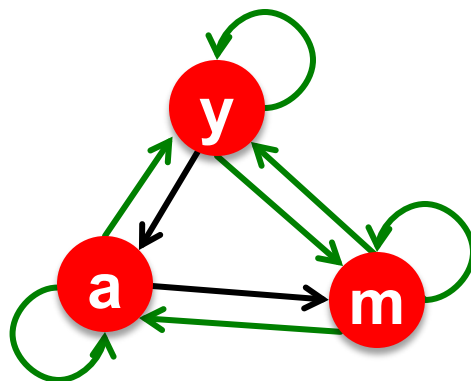$r_a = r_y/2 + r_m/3$

$r_m = r_a/2 + r_m/3$

Make M Aperiodic

- A chain is **periodic** if there exists $k > 1$ such that the interval between two visits to some state s is always a multiple of k

- **Solution:** Add **green** links

Make *M* Irreducible

- From any state, there is a non-zero probability of going from any one state to any another

- **Solution:** Add **green** links

Solution: Random Jumps

- **Google's solution that does it all:**
  - Makes **M stochastic, aperiodic, irreducible**
- **At each step, random surfer has two options:**
  - With probability β, follow a link at random
  - With probability $1-\beta$, jump to some random page
- **PageRank equation** [Brin-Page, 98]

$$r_j = \sum_{i \to j} \beta \, \frac{r_i}{d_i} + (1 - \beta)\frac{1}{n}$$

d_i ... out-degree of node i

The above formulation assumes that **M** has no dead ends. We can either preprocess matrix **M (bad!)** or explicitly follow random teleport links with probability 1.0 from dead-ends.

The Google Matrix

- **PageRank equation** [Brin-Page, 98]

$$r_j = \sum_{i \to j} \beta \frac{r_i}{d_i} + (1 - \beta) \frac{1}{n}$$

- **The Google Matrix A:**

$$A = \beta\, M + (1 - \beta) \frac{1}{n} e \cdot e^T$$

e   vector of all 1s
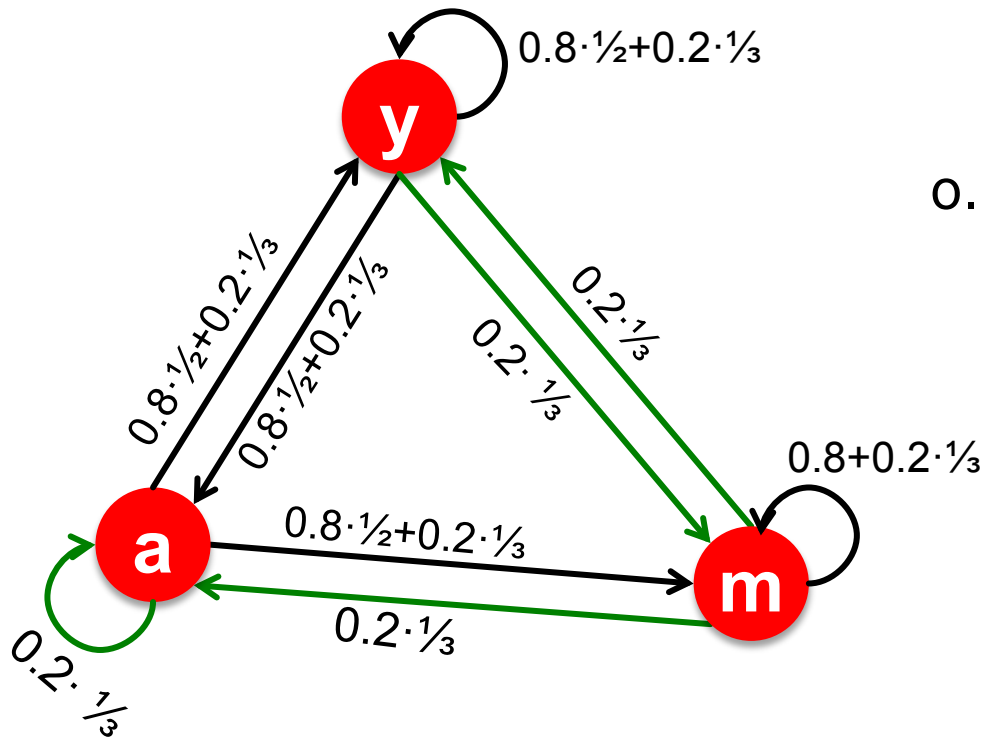
- **A is stochastic, aperiodic and irreducible, so**

$$r^{(t+1)} = A \cdot r^{(t)}$$

- **What is β?**

  - In practice $\beta = 0.8, 0.9$ (make 5 steps and jump)

Random Teleports ($\beta = 0.8$)

M

$$0.8 \begin{bmatrix} 1/2 & 1/2 & 0 \\ 1/2 & 0 & 0 \\ 0 & 1/2 & 1 \end{bmatrix}$$

1/n·1·1ᵀ

$$+ 0.2 \begin{bmatrix} 1/3 & 1/3 & 1/3 \\ 1/3 & 1/3 & 1/3 \\ 1/3 & 1/3 & 1/3 \end{bmatrix}$$

$$\begin{array}{c|ccc} y & 7/15 & 7/15 & 1/15 \\ a & 7/15 & 1/15 & 1/15 \\ m & 1/15 & 7/15 & 13/15 \end{array}$$

A

$$\begin{array}{ccccccc} y & & 1/3 & 0.33 & 0.24 & 0.26 & & 7/33 \\ a & = & 1/3 & 0.20 & 0.20 & 0.18 & \ldots & 5/33 \\ m & & 1/3 & 0.46 & 0.52 & 0.56 & & 21/33 \end{array}$$