Unit 1 : Principles of Optimizing Behavior

Prof. Antonio Rangel

January 2, 2016

1 Introduction

- Most models in economics are based on the assumption that economic agents optimize some objective function.
- Ex 1. Consumers decide how much to buy by maximizing utility given their wealth and market prices.
- Ex 2. Firms decide how much to sell by maximizing profits given their technological constraints and market prices.
- We begin the course by studying some key ideas regarding optimization that are at work in all economic models.
- Understanding these principles will allow us to gain deeper economic insight in later units.

2 Unconstrained optimization

2.1 Basics

- General optimization problem:

$$\max_x T(x)$$

  - x: control variable
  - $T(\cdot)$: objective function; e.g. profits of a firm

- Local maxima x^* are characterized by:
  - First-order (necessary) condition: $T'(x^*) = 0$
  - Second-order (sufficient) condition: $T''(x^*) < 0$

- Simple example:
  - $\max_x 10 - (x-5)^2$
  - FOC: $-2(x^* - 5) = 0 \implies x^* = 5$
  - SOC: $-2 < 0 \quad \checkmark$

2.2 Intuition for FOC and SOC of local maxima

- (Graphical) intutition for the FOC:
  - Suppose $T'(x) > 0$. Then $T(x + dx) > T(x)$, so x not a local maximum.
  - Suppose $T'(\hat{x}) < 0$. Then $T(\hat{x} - dx) > T(\hat{x})$, so $\hat{x}$ not a local maximum.
  - Therefore x^* a maximum $\implies T'(x^*) = 0$; i.e. $T'(x^*) = 0$ is a *necessary condition* for x^* to be a maximum.

- (Graphical) intuition for the SOC:
  - Suppose $T'(x^*) = 0$ but $T''(x^*) > 0$. Then $T'(x^* + dx) > 0$ for any $dx > 0$. Therefore,

    $$T((x^*+dx)+dx) \approx T(x^*+dx)+T^{'}(x^*+dx)dx > T(x^*+dx) \approx T(x^*)+T'(x^*)dx \approx T(x^*),$$

    and thus x^* cannot be a local maximum.
  - In contrast, suppose $T'(x^*) = 0$ and $T''(x^*) < 0$. Then $T'(x^* + dx) < 0$, for any $dx > 0$. Therefore,

    $$T((x^*+dx)+dx) \approx T(x^*+dx)+T^{'}(x^*+dx)dx < T(x^*+dx) \approx T(x^*)+T'(x^*)dx \approx T(x^*),$$

    and it follows that x^* is a local maximum.

- (Mathematical) intuition for the FOC and SOCs

- Taking a Taylor expansion at any x we have that

$$T(x+dx) \approx T(x) + T'(x)dx + \frac{1}{2}T''(x)dx^2$$

- Therefore

$$dT \approx T'(x)dx + \frac{1}{2}T''(x)dx^2$$

- But then:
- FOC $\implies T'(x)dx = 0$
- SOC $\implies \frac{1}{2}T''(x)dx^2 < 0$
- Therefore FOC & SOC $\implies dT < 0$.
- In other words, if the FOC and SOC are satisfied, small deviations around x necessarily decrease the value of the function

- Economic intuition for FOC:
  - $T(x)$: total payoff of taking action x
  - $T'(x)$: marginal payoff of increasing level of action x
  - marginal payoff of additional $x > 0 \implies$ signal to (locally) increase x
  - marginal payoff of additional $x < 0 \implies$ signal to (locally) decrease x
  - At optimum, marginal payoff of additional unit is 0.

2.3 Remarks on basic optimization

- Local maximum $\neq$ global maximum
- Global max need not exist, even if FOC & SOC satisfied at some x
- FOC necessary for local maximization, but not sufficient
- $T''(x) = 0$ not sufficient for x to be a local maximum
- $\min_x T \sim \max_x -T$
- Adding a constant to T doesn't change the value of x^* at which T is maximized.

3 Optimization in economic problems

3.1 Adding economic structure

- In many economic models the optimization problem has additional useful structure
- Assumption 1: $T(x) = B(x) - C(x) =$ benefit − cost
- Example: Firm
  - $x =$ level of output
  - $B(x) =$ revenue from selling x units
  - $C(x) =$ cost of producing x units
  - $T(x) =$ profit $=$ revenue − cost
- Example: Consumer buying a computer
  - $x =$ units of computing power ($x = 0$ denotes no computer)
  - $B(x) =$ benefit of x in dollars
  - $C(x) =$ market cost of x in dollars
  - $T(x) =$ net utility of buying $x = B(x) - C(x)$
- Assumption 2: $x \geq 0$

  Can't produce negative output, can't buy negative amount of a good, etc.
- Assumption 3: $T(x)$ is strictly concave
  - Graph of T over $x \geq 0$ looks like an inverted bowl
  - $T''(x) < 0$ for all $x \geq 0$
- Why is T strictly concave in many economic problems?
  - $T(x) = B(x) - C(x)$
  - In many economic problems we have $B' > 0, B'' \leq 0$, $C' > 0, C'' \geq 0$ (with not both $B'' = 0$ and $C'' = 0$). Then $T'' < 0$.

3.2 Additional intuition

- T is a strictly concave function iff $T''(x) < 0$ for all x
  - Graph looks like inverted bowl
- T is a weakly concave function iff $T''(x) \leq 0$ for all x
- T is a strictly convex function iff $T''(x) > 0$ for all x
  - T strictly convex iff $-T$ is a strictly concave function
  - Graph looks like a bowl
- T is a weakly convex function iff $T''(x) \geq 0$ for all x
- Why are benefits concave?
  - Example from consumption. Let x=spoonfuls of ice-cream. First spoonful of ice cream is fantastic, next spoonful is not quite as great, and eventually an additional spoonful provides almost no benefit. This implies that $B' > 0$ and $B'' < 0$, and thus B is strictly concave.
  - Example from firm. Let x = amount sold in market. $B(x) = Revenue(x) = px$, where $p > 0$ are the market prices. In this case $B' > 0$ and $B'' = 0$. Thus, revenue is a weakly concave function.
- Why are costs convex?
  - Consumer's costs are given by $C(x) = px$, which is a weakly convex function
  - Firm's costs often strictly convex. Ex: extracting rare rocks: first rock is on the surface, but need to dig deeper and deeper to find more and more rocks

3.3 Why is the additional structure useful?

- Assumptions 1-3 imply that a global optimum:
  - exists,
  - is unique, and
  - has useful economic intuition
- Concavity conditions:
  - $B' > 0, B'' \leq 0$
  - $C' > 0, C'' \geq 0$
  - $B'' = 0$ or $C'' = 0$, but not not both
- Solution looks like:
  - If $B'(0) < C'(0)$, then $x^* = 0$.
  - If $B'(0) \geq C'(0)$, then x^* is point where MB=MC, i.e. $B'(x^*) = C'(x^*)$.
- Economic inuition
  - Marginal value of dx = MT = MB − MC
  - Increase payoff by increasing x if $MT > 0$, which is true iff $MB > MC$.
- REMARK 1: Assumptions 1-3 do not guarantee the existence of a global maximum.
  - Why? MB and MC costs are not guaranteed to cross
- REMARK 2: Crossing conditions rule out this problem.
  - Crossing condition: $B' \to 0$ as $x \to \infty$, or $C' \to \infty$ as $x \to \infty$, or both
  - This condition guarangees that if $B'(0) \geq C'(0)$, then the MB and MC curves must cross

- REMARK 3: Assumptions 1-3, plus the concavity and crossing conditions, guarantee the uniqueness of a global maximum
  - Why?
  - Suppose there is a maximum at x^*.
  - Then $MB(x^*) = MC(x^*)$.
  - Then concavity conditions imply that MC > MB at every point to the right of x^*.
- KEY RESULT: Suppose that $\max_{x \geq 0} B(x) - C(x)$ satisfies both the concavity and the crossing conditions. Then there exists a unique global maximum at
  - $x^* = 0$, if $B'(0) < C'(0)$
  - solution to $B'(x) = C'(x)$ otherwise

3.4 Example

- Consider a profit-maximizing firm:
  - $B(x) = \text{Revenue} = px$
  - $C(x) = \text{Cost} = \theta x^2$
  - Problem of the firm: $\max_{x \geq 0} px - \theta x^2$
- Concavity, crossing conditions satisfied
- $B'(0) = p > C'(0) = 0$, so solution satisfies $B' = C'$.
- $B'(x) = C'(x) \implies x^* = \frac{p}{2\theta}$
- Solution makes economic sense:
  - If p higher, produce more
  - If θ higher, which increases MC, produce less

3.5 Example

- Consider again problem of the firm in previous example
- Claim: Max total payoff $\neq$ Max average total payoff
- Why?
  - Average total payoff $= p - \theta x$
  - Thus, average payoff maximal at $x = 0$, even though total payoff maximized at $x^* = \frac{p}{2\theta}$

4 Constrained optimization

4.1 More on corner solutions

- KEY RESULT: Consider the following optimization problem

$$\max T(x) \text{ s.t. } x \geq L, x \leq B$$

and assume that $T'' < 0$. Then there exists a unique global maximum given by:

  - Case 1. Corner solution at $x^* = L$ if $T'(L) < 0$
  - Case 2. Corner solution at $x^* = B$ if $T'(B) > 0$
  - Case 3. Interior solution satisfying $T'(x^*) = 0$ if T' crosses x-axis between L and B.

4.2 Problems with equality constraints

- Consider the following optimization problem, which includes an equality constraint

$$\begin{aligned} \max\ & U(x) + V(y) \text{ subject to} \\ & x \geq 0 \\ & y \geq 0 \\ & px + qy = W \end{aligned}$$

- Example: x and y denote the amount consumed of two goods, $U(x)$ and $V(y)$ denote the benefits generated by consuming each good (in \$s), p and q denote the prices of each good, and W denotes total wealth/income.
- Suppose that $U', V' > 0$; $U'', V'' < 0$.
- Formally, this is a multi-variate optimization problem.
- But can solve using a simple trick.
- Use equality constraint to simplify problem:

$$px + qy = W \implies y = \frac{W - px}{q}$$

- Substituting into the optimization problem, we get the following uni-variate maximization problem:

$$\max_{x \geq 0, x \leq \frac{w}{p}} U(x) + V\left(\frac{w - px}{q}\right)$$

- Problem can be put in the "benefit - cost" framework:
  - Think of $U(x)$ as benefit from consuming x.
  - Think of $-V(\frac{W-px}{q})$ as cost of consuming x. Intuition: Cost of consuming more x is having less income for y, and thus deriving less benefit from consumption of y.
- KEY RESULT: Consider the following optimization problem

$$\begin{aligned} \max\ & U(x) + V(y) \text{ subject to} \\ & x \geq 0 \\ & y \geq 0 \\ & px + qy = W \end{aligned}$$

  and assume that $U', V' > 0$; $U'', V'' < 0$. Then there exists a unique global maximum given by:

  - Case 1. Corner solution at $x^* = \frac{W}{p}$ and $y^* = 0$ if $U'(\frac{W}{p}) > V'(0)$

- Case 2. Corner solution at $x^* = 0$ and $y^* = \frac{W}{q}$ if $U'(0) < V'(\frac{W}{q})\frac{p}{q}$
- Case 3. Interior solution satisfying

$$\frac{U'(x^*)}{V'(y^*)} = \frac{p}{q}$$

and

$$y^* = \frac{W - px^*}{q}$$

otherwise.

4.3 Example with interior solution

- Consider the problem:

$$\begin{aligned} &\max\ a\ln(x) + b\ln(y) \text{ subject to} \\ &x \geq 0 \\ &y \geq 0 \\ &px + qy = W \end{aligned}$$

- Rewrite it as

$$\begin{aligned} &\max\ a\ln(x) - (-b\ln(\frac{W - px}{q})) \text{ subject to} \\ &x \geq 0 \\ &x \leq \frac{W}{p} \end{aligned}$$

- As before, think of the first term as the benefit of consuming x, and of the second term as the cost of consuming x
- Under this interpretation we get that MB $= \frac{a}{x}$ and MC $= \frac{bp}{W-px}$
- $MB(0) > MC(0)$ and $MB(\frac{W}{p}) < MC(\frac{W}{p})$ implies that the solution is interior and given by the FOC: $MB(x^*) = MC(x^*)$.
- Doing the algebra we get that $x^* = \frac{W}{p}(\frac{a}{a+b})$ and $y^* = \frac{W}{q}(\frac{b}{a+b})$

4.4 Example with corner solution

- Consider a slightly different version of the previous problem:

$$\begin{aligned} \max\ & a\ln(x+1) + b\ln(y+1) \text{ subject to} \\ & x \geq 0 \\ & y \geq 0 \\ & px + qy = W \end{aligned}$$

- Assume $p = q = 1$
- Solution may be interior or corner, depending on a and b
- $x^* = W \iff \frac{a}{W+1} \geq b$
- $x^* = 0 \iff \frac{b}{W+1} \geq a$
- Interior solutions in between

5 Final remarks

- Characterizing global maxima in general optimization problems is quite hard
- But characterizing global maxima is quite easy and intuitive in economic problems that satisfy three key assumptions: (A1) The objective function can be written as Benefits minus Costs, (2) $x \geq 0$, and (3) the concavity and crossing conditions hold.
- In this case a unique global maximum always exists, and as shown in the key results above, it has a simple characterization
- Furthermore, it is often possible to reduce more complex optimization problems (e.g., those involving two control variables) to simpler ones (involving only one control variable) that we know how to solve.
- Advice for problem solving:

  1. Write down maximization problem
  2. Transform into simple familiar case

3. Are solutions interior or corner?
4. Characterize solutions using the formulas from the key results